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Unitary Representations Induced from Maximal Parabolic Subgroups

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It is known that the problem of classifying the irreducible unitary representations of a linear connected semisimple Lie group G comes down to deciding which Langlands quotients $J(MAN, \sigma, \nu)$ are infinitesimally unitary. Here MAN is any cuspidal parabolic subgroup, σ is any discrete series or nondegenerate limit of discrete series representation of M, and ν is any complex-valued linear functional on the Lie algebra of A satisfying certain positivity and symmetry properties. The authors determine which Langlands quotients are infinitesimally unitary under the conditions that G is simple, that $\dim A = 1$, and that G is neither split F_4 nor split G_2.

For a linear connected simple Lie group G other than split F_4 or split G_2, we determine the contribution to the unitary dual of G by all Langlands quotients $J(MAN, \sigma, \nu)$ for which MAN is a cuspidal parabolic subgroup with $\dim A = 1$. For fixed MAN and σ, the contribution from $\operatorname{Re} \nu$ positive turns out always to come either from an interval of ν or from an interval together with one isolated point. The parameter of the extreme unitary point ν is given by a simple formula, and the gap, when there is one, is always of one or two sizes (except in the case that G is nonsplit F_4 and $\sigma = 1$).

For background on determining the unitary dual of G, one can consult [12], which will place our main theorem in perspective.

We state the main theorem precisely as Theorem 1.1. Let us summarize the statement when G has a compact Cartan subgroup. The theorem says that the normal situation is that the unitary points form an interval extending from the origin for a distance given as the minimum of two num-

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bers ν_0^+ and ν_0^-. But there are six kinds of exceptions, all but one of which arise only when the Dynkin diagram of G has a double line. Four kinds of exceptions say that there is a gap in the unitary points, with an isolated representation at $\min(\nu_0^+, \nu_0^-)$; prototypes of these situations occur in $Sp(n, 1)$, nonsplit F_4, $\widetilde{SO}(2n, 3)$, and $\widetilde{SO}(5, 4)$. Two further kinds of exceptions say that the unitary points form an interval but that the interval is shorter than expected; prototypes of these situations occur in $\widetilde{SO}(2n, 2)$ and $\widetilde{SO}(2n+1, 2)$.

One of the four kinds of gaps provides us with an interesting example concerning conjectures of Vogan [23] on the preservation of unitarity under cohomological induction. We give this example explicitly in Section 15.

The proof of the main theorem is in two distinct parts. The first part of the proof is to make use of cut-offs that exclude certain representations as nonunitary. Statements of most of the cut-offs are assembled in Section 3 and will not be proved in this paper. Some are variants of those announced in [2] and [3], and others are new; the idea of the proof for all of them appears in [3]. We do, however, include the proof that the cut-offs apply; this occupies Sections 4–7 and part of Section 14.

The second part of the proof is to show irreducibility of the standard induced representations at certain points. A main tool here is a theorem implicit in Speh–Vogan [20] and stated explicitly here as Theorem 8.2. However, this theorem does not handle certain cases that we assemble afterward in Table 8.1. In Lemma 8.6 we reduce these difficult cases to a small number that we handle in another paper [4].

Let K be a maximal compact subgroup of G. The case rank $G >$ rank K is distinctly different from (and much easier than) the case rank $G =$ rank K. Most of the paper is concerned with the equal rank case. When rank $G =$ rank K, the case of a Dynkin diagram with only single lines on the one hand requires the most extensive analysis but on the other hand does not use the classification of simple real groups. By contrast we do make use of the classification of simple real groups to handle double-line diagrams; use of this kind of classification in the double-line groups is not surprising since most of the exceptional cases for Theorem 1.1 arise in such groups.

The paper makes considerable use of "basic cases" and "special basic cases," as introduced in [13] and [3]. The paper [13] conjectures a relationship between unitary representations in G and unitary representations in subgroups L. We shall see in Section 15 that this conjecture fails in some of the double-line cases, particularly in $\widetilde{SO}(\text{odd, even})$. However, it is almost true in all cases, and it is true enough to help in the bookkeeping that is necessary in the proof of the main theorem.

We are indebted to D. A. Vogan for a number of helpful conversations during our work. His suggestions and methods of computation were of decisive help in addressing questions of irreducibility and isolated representations. We are grateful also to Barbasch and Vogan for sharing with us details of their work [5] before their publication.

Contents.

1. Statement of Theorem

Let G be a linear connected simple Lie group other than split F_4 or split G_2. We may assume that G is contained in a simply connected complexification $G^{\mathbb{C}}$. Let θ be a Cartan involution, let K be the corresponding maximal compact subgroup, and let MAN be the corresponding Langlands decomposition of a parabolic subgroup. We shall assume that rank $M = \operatorname{rank}(K \cap M)$, so that M has discrete series (Harish-Chandra [7]). We shall assume moreover that $\dim A = 1$. We denote corresponding Lie algebras by lower case German letters.

Let σ be a discrete series representation of M or a nondegenerate limit of discrete series [17], and let ν be a complex-valued linear functional on $\mathfrak{a}$. Then the *standard induced representation* $U(MAN, \sigma, \nu)$ is given by normalized induction as

$$U(MAN, \sigma, \nu) = \operatorname{ind}_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes 1).$$

If $\operatorname{Re} \nu \geqslant 0$ (with positivity defined relative to N) and if $\nu \neq 0$, then $U(MAN, \sigma, \nu)$ has a unique irreducible quotient $J(MAN, \sigma, \nu)$, the *Langlands quotient*. In addition, $J(MAN, \sigma, 0)$ makes sense [16] whenever the *R-group* $R_{\sigma, 0}$ is trivial. (See [17] for $R_{\sigma, 0}$ in full generality.) The problem is to decide when $J(MAN, \sigma, \nu)$ is infinitesimally unitary.

If ν is imaginary, then $J(MAN, \sigma, \nu)$ is trivially unitary. If $\operatorname{Re} \nu > 0$, then $J(MAN, \sigma, \nu)$ cannot admit a nonzero invariant Hermitian form unless the Weyl group $W(A: G)$ has a nontrivial element w and w fixes the class $[\sigma]$ of σ; moreover, ν must be real. Conversely these conditions give the existence of a nonzero invariant Hermitian form. (See [16]). Thus the problem is to decide which real parameters $\nu \geqslant 0$ are such that this form is semidefinite. If $R_{\sigma,0}$ is nontrivial, there are no such parameters ν, by Proposition 16.8 of [11]; thus we may assume $R_{\sigma,0}$ is trivial.

The solution to this problem involves counting the number of roots with certain properties and depends on having a particular kind of ordering, which in turn depends on parameters that define σ. To describe these matters, we distinguish the cases rank $G >$ rank K and rank $G =$ rank K.

First, suppose rank $G >$ rank K. (Actually this means $\mathfrak{g} \cong \mathfrak{sl}(3, \mathbb{R})$ or $\mathfrak{g} \cong \mathfrak{so}(\text{odd}, \text{odd})$.) Let $\mathfrak{b} \subseteq \mathfrak{k} \cap \mathfrak{m}$ be a compact Cartan subalgebra of $\mathfrak{m}$, so that $\mathfrak{b} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Let σ_0 be an irreducible constituent of the restriction of σ to the identity component M_0 of M; then Lemma 2.1 of [17] gives $\sigma \cong \operatorname{ind}_{M_0}^{M} \sigma_0$, and thus σ_0 determines σ. Hence σ is determined by a Harish–Chandra parameter (λ_0, Δ_-^+) for σ. Here Δ_-^+ is a positive system for the roots $\Delta_- = \Delta(\mathfrak{m}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$, and λ_0 is dominant relative to Δ_-^+. Regarding Δ_- as a subset of $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, (\mathfrak{b} \oplus \mathfrak{a})^{\mathbb{C}})$, we introduce a positive system Δ^+ for Δ such that λ_0 is Δ^+ dominant and $\theta\Delta^+ = \Delta^+$. (The condition $\theta\Delta^+ = \Delta^+$ means $i\mathfrak{b}$ comes before $\mathfrak{a}$.)

Let α_R be the (unique) positive root of $(\mathfrak{g}, \mathfrak{a})$. We may assume that Δ_-^+ is defined by a lexicographic ordering of $(i\mathfrak{b})'$, and we let α_I be the least positive element such that $\alpha_I + \alpha_R$ is a root. The element w of $W(A{:}\,G)$ exists in $\mathfrak{so}(\text{odd}, \text{odd})$ but not in $\mathfrak{sl}(3, \mathbb{R})$, and in the case of $\mathfrak{so}(\text{odd}, \text{odd})$, Lemma 10.3 of [17] shows that $w[\sigma] = [\sigma]$ if and only if $\langle \lambda_0, \alpha_I \rangle = 0$.

According to [9], $J(MAN, \sigma, \nu)$ has a unique minimal K-type Λ (in the sense of Vogan [21]) given by

$$\Lambda = \lambda_0 + \delta - 2\delta_K, \tag{1.1}$$

where δ and δ_K are the half sums of positive roots for Δ^+ and $\Delta^+(\mathfrak{k}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$, respectively. (See Sect. 14 for the nature of the roots of $\mathfrak{k}$.) We define

$$\nu_0 = 2\#\{\beta \in \Delta^+ \mid \beta|_{\mathfrak{a}} > 0 \text{ and } \langle \Lambda, \beta \rangle = 0\}. \tag{1.2}$$

Next suppose rank $G =$ rank K. Let $\mathfrak{b} \subseteq \mathfrak{g}$ be a compact Cartan subalgebra of $\mathfrak{g}$. We may assume that $\mathfrak{a}$ is built by Cayley transform relative to some noncompact root α in $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$. Then $\mathfrak{b}_- = \ker \alpha$ is a compact Cartan subalgebra of $\mathfrak{m}$, and the root system $\Delta_- = \Delta(\mathfrak{m}^{\mathbb{C}}, \mathfrak{b}_-^{\mathbb{C}})$ is given by the members of Δ orthogonal to α. Let Δ_K and Δ_n be the subsets of compact and noncompact members of Δ. Corresponding to the root α is a nontrivial homomorphism $SL(2, \mathbb{R}) \to G$, and we let γ_α be the image in G of $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ under this homomorphism. It will be convenient to identify α with its Cayley transform, so that we can write ν as a multiple of α.

Let σ_0 be an irreducible constituent of the restriction of σ to the identity component M_0 of M, and let χ be the scalar restriction of σ to the subgroup $\{1, \gamma_\alpha\}$. By Lemma 2.1 of [17], σ is induced from $\sigma_0 \otimes \chi$ on the subgroup of M generated by M_0 and $\{1, \gamma_\alpha\}$, and thus (σ_0, χ) determines σ.

Hence σ is determined by χ and a Harish–Chandra parameter (λ_0, Δ_-^+) for σ. Here Δ_-^+ is a positive system for Δ_-, and λ_0 is dominant relative to Δ_-^+. We introduce a positive system Δ^+ for Δ containing Δ_-^+ such that λ_0 is Δ^+ dominant and α is simple. Let $\Delta_K^+ = \Delta_K \cap \Delta^+$ and $\Delta_n^+ = \Delta_n \cap \Delta^+$. It is automatically true that the nontrivial element w of $W(A: G)$ exists and fixes $[\sigma]$.

If ρ_α is half the sum of the roots having positive inner product with α, then we say that σ is a *cotangent case* if

$$\chi(\gamma_\alpha) = (-1)^{2\langle \rho_\alpha, \alpha \rangle / |\alpha|^2}$$

and otherwise is a *tangent case* (for the Plancherel formula of G). According to [9], $J(MAN, \sigma, \nu)$ has one or two minimal K-types with highest weights given by the formula

$$\Lambda = \lambda_0 + \delta - 2\delta_K - \tfrac{1}{2}\alpha + \mu. \tag{1.3}$$

Here μ is 0 in a tangent case, and $\mu = \pm\frac{1}{2}\alpha$ in a cotangent case. In a tangent case, it is to be understood that $\mu = 0$ produces a Δ_K^+ dominant Λ. In a cotangent case, at least one choice of μ gives a Δ_K^+ dominant Λ, and the Δ_K^+ dominant Λ or Λ's give the minimal K-type(s). We define

$$\nu_0^+ = 1 + \frac{2\langle \mu, \alpha \rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta - \alpha \in \Delta \quad \text{and} \quad \langle \Lambda, \beta - \alpha \rangle = 0\}$$
$$+ \#\left\{\beta \in \Delta_n^+ \mid \beta - \alpha \in \Delta, |\beta|^2 < |\alpha|^2, \frac{2\langle \Lambda, \beta - \alpha \rangle}{|\beta - \alpha|^2} = +1\right\}, \tag{1.4a}$$

$$\nu_0^- = 1 - \frac{2\langle \mu, \alpha \rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta + \alpha \in \Delta \quad \text{and} \quad \langle \Lambda, \beta + \alpha \rangle = 0\}$$
$$+ \#\left\{\beta \in \Delta_n^+ \mid \beta + \alpha \in \Delta, |\beta|^2 < |\alpha|^2, \frac{2\langle \Lambda, \beta + \alpha \rangle}{|\beta + \alpha|^2} = +1\right\}. \tag{1.4b}$$

Given σ, we form λ_0 and χ as above, and we fix a choice of μ for which Λ is Δ_K^+ dominant. We say that a simple root $\beta \in \Delta^+$ is *basic* if $2\langle \lambda_0, \beta \rangle / |\beta|^2$ is as small as possible among Harish-Chandra parameters that are consistent with Δ^+ and χ and have a Δ_K^+ dominant corresponding form Λ (given by (1.3)). (A formula for this minimum value will be recalled in Sect. 2.) The root system generated by the basic simple roots will be called the *basic case* associated to λ_0.

Define

$$\Delta_{K,\perp} = \{\gamma \in \Delta_K \mid \langle \Lambda, \gamma \rangle = 0\}. \tag{1.5}$$

The *special basic case* associated to λ_0 is the group or root system generated by α and all simple roots of Δ^+ needed for the expansion of members of $\Delta_{K,\perp}$. This root system will be denoted Δ_S.

The special basic case turns out to be contained in the basic case. Although the special basic case can be computed directly from λ_0, it is easier in practice to read off the basic case and then to determine the special basic case within it. (See Table 2.1 and Lemma 2.2. An example will be given in Sect. 15.)

THEOREM 1.1 (Main theorem). (a) *Suppose* rank $G >$ rank K *and* $\langle \lambda_0, \alpha_I \rangle = 0$. *Then for* $c > 0$, $J(MAN, \sigma, \frac{1}{2}c\alpha_R)$ *is infinitesimally unitary exactly when*

$$0 < c \leqslant \nu_0.$$

(b) *Suppose* rank $G =$ rank K. *Then for* $c > 0$, $J(MAN, \sigma, \frac{1}{2}c\alpha)$ *with six exceptions is infinitesimally unitary exactly when*

$$0 < c \leqslant \min(\nu_0^+, \nu_0^-).$$

The exceptions occur when the component of α *in the associated basic case or special basic case is of one of the following forms*:

(i) *The component of* α *in the special basic case is* $\mathfrak{sp}(n, 1)$ *with* $n \geqslant 2$, *with* $\mu = 0$, *and with* α *adjacent to the long simple root. Then* $J(MAN, \sigma, \frac{1}{2}c\alpha)$ *is infinitesimally unitary exactly when*

$$0 < c \leqslant \min(\nu_0^+, \nu_0^-) - 2 \qquad \text{or} \qquad c = \min(\nu_0^+, \nu_0^-).$$

(ii) *The algebra* $\mathfrak{g}$ *is nonsplit* F_4, *and* σ *is trivial. Then* $J(MAN, \sigma, \frac{1}{2}c\alpha)$ *is infinitesimally unitary exactly when*

$$0 < c \leqslant \min(\nu_0^+, \nu_0^-) - 6 \qquad \text{or} \qquad c = \min(\nu_0^+, \nu_0^-).$$

(iii) *The component of* α *in the special basic case is* $\mathfrak{su}(n, 1)$ *with* $n \geqslant 2$ *and with* α *long, and there is an adjacent basic short simple root* ε *such that* $\mathfrak{su}(n, 1)$ *and* ε *generate an algebra* $\mathfrak{so}(2n, 3)$. *In this case, let* ζ *be the sum of the simple roots strictly between* α *and* ε *in the Dynkin diagram. If* ζ *is noncompact, then* $J(MAN, \sigma, \frac{1}{2}c\alpha)$ *is infinitesimally unitary exactly when*

$$\begin{cases} 0 < c \leqslant \min(\nu_0^+, \nu_0^- - 1) \quad \text{or} \quad c = \min(\nu_0^+, \nu_0^-) & \text{if } \nu_0^- \geqslant 2 \\ 0 < c \leqslant \min(\nu_0^+, \nu_0^-) & \text{if } \nu_0^- \leqslant 1. \end{cases}$$

If ζ is compact or 0, then $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary exactly when

$$\begin{cases} 0 < c \leqslant \min(v_0^+ - 1, v_0^-) \quad \textit{or} \quad c = \min(v_0^+, v_0^-) & \textit{if} \quad v_0^+ \geqslant 2 \\ 0 < c \leqslant \min(v_0^+, v_0^-) & \textit{if} \quad v_0^+ \leqslant 1. \end{cases}$$

(iv) *The component of α in the special basic case is $\mathfrak{so}(2n, 2)$ with $n \geqslant 2$. In this case let $v_{0,L}^+$ and $v_{0,L}^-$ be computed from* (1.4) *within an $\mathfrak{su}(n, 1)$ subsystem Δ_L of the special basic case containing α and generated by simple roots. Let β_0 be the unique positive noncompact root in the $\mathfrak{so}(2n, 2)$ that is orthogonal to α. Then exactly one of α and $-\alpha$ is conjugate by K within $\mathfrak{so}(2n, 2)$ to the root β_0. Moreover, $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary exactly when*

$$\begin{cases} 0 < c \leqslant \min(v_{0,L}^+, v_0^-) & \textit{if} \quad \beta_0 \textit{ conjugate to } \alpha \\ 0 < c \leqslant \min(v_0^+, v_{0,L}^-) & \textit{if} \quad \beta_0 \textit{ conjugate to } -\alpha. \end{cases}$$

(v) *The component of α in the special basic case is $\mathfrak{so}(2n+1, 2)$ with $n \geqslant 2$ and with α long, but the situation is not imbedded as in* (vi). *In this case, let $v_{0,L}^+$ and $v_{0,L}^-$ be computed from* (1.4) *within the $\mathfrak{su}(n, 1)$ subsystem Δ_L of the special basic case containing α and generated by simple roots. Let β_0 be the unique positive noncompact root in the $\mathfrak{so}(2n+1, 2)$ that is orthogonal to α. Then exactly one of α and $-\alpha$ is conjugate by K within $\mathfrak{so}(2n+1, 2)$ to the root β_0. Moreover, $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary exactly when*

$$\begin{cases} 0 < c \leqslant \min(v_{0,L}^+ + 1, v_0^-) & \textit{if} \quad \beta_0 \textit{ conjugate to } \alpha \\ 0 < c \leqslant \min(v_0^+, v_{0,L}^- + 1) & \textit{if} \quad \beta_0 \textit{ conjugate to } -\alpha. \end{cases}$$

(vi) *The component of α in the special basic case is $\mathfrak{so}(5, 2)$ with α long, α is the middle of the three simple roots in the component, $\mu = 0$, and there exists a Δ^+ simple noncompact basic root next to the long node of the component. Then $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary exactly when*

$$0 < c \leqslant 2 \quad \textit{or} \quad c = 3.$$

In all cases, $U(MAN, \sigma, v)$ is irreducible on the interior of any interval of v where $J(MAN, \sigma, v)$ is infinitesimally unitary.

Some comments are in order about the case rank $G =$ rank K before we proceed. The last term in the definition of v_0^+ or v_0^- should be regarded as exceptional. It can be nonzero only when there are roots of two lengths and α is long (possible only in $\mathfrak{so}$(odd, even), $\mathfrak{sp}(n, \mathbb{R})$, and split F_4). When the exceptional terms are 0, the roots that contribute to ν_0^+ and ν_0^- all lie within the special basic case; thus ν_0^+ and ν_0^- are the same whether computed in G, in the associated basic case, or in the associated special basic case. Theorem 1.1 therefore proves the conjecture in [13] that unitarity in the basic case corresponds to unitarity in G, under the assumptions that $\dim A = 1$ and that the exceptional terms of ν_0^+ and ν_0^- are 0. The conjecture can fail when an exceptional term is nonzero; we give an example of this failure in Section 15.

Situations (i), (ii), (iii), and (vi) in the theorem are the cases where there is a gap in the unitary points. For (i) and (ii), this gap is generated by the corresponding gap with the trivial representation σ in some $\mathrm{Sp}(n, 1)$ or nonsplit F_4 (cf. Kostant [18]). The gaps noted in (iii) and (vi) are new. The simplest example for (iii) is in $\widetilde{SO}(4, 3)$ with all simple roots noncompact, with α equal to one of the long simple roots, and with $\lambda_0 = 0$; the gap occurs on the interval $1 < c < 2$. The simplest example for (vi) is in $\widetilde{SO}(5, 4)$. The gap in (iii) provides an interesting example concerning conjectures of Vogan [23] on the preservation of unitarity under cohomological induction; we discuss the example in Section 15.

Situations (iv) and (v) represent a reduction in the length of the interval below what is expected. This reduction is related to the existence of lines of unitary points in the two-dimensional pictures of representations induced from a minimal parabolic subgroup of $\widetilde{SO}(N, 2)$. Curiously there is no corresponding reduction when the special basic case is an E-type diagram containing $\mathfrak{so}(2n, 2)$ as a subdiagram.

Possibly Theorem 1.1 requires no change to be valid also for split F_4. We simply have not examined all the possibilities. We have handled completely the case that α is short, but we omit the proof for that case. When α is long, situations (iii) and (v) in the theorem do occur.

2. Basic Cases and Special Basic Cases

From now through Section 11, we assume $\operatorname{rank} G = \operatorname{rank} K$. Put $\mu_\alpha = 2\langle \mu, \alpha \rangle / |\alpha|^2$.

We first dispose completely of the case of two minimal K-types by means of

Lemma 2.1. *The following conditions are equivalent.*

(a) *$J(MAN, \sigma, \nu)$ has one minimal K-type.*

(b) *$U(MAN, \sigma, 0)$ is irreducible.*

(c) *The R-group $R_{\sigma, 0}$ is trivial.*

(d) *$J(MAN, \sigma, \nu)$ is infinitesimally unitary for all ν near* 0.

(e) *$J(MAN, \sigma, \nu)$ is infinitesimally unitary for some positive ν.*

(f) $\min(\nu_0^+, \nu_0^-)$ *is not* 0.

Proof. (a)$\Leftrightarrow$(b) by Vogan [22], while (b)$\Leftrightarrow$(c) by [17]. The equivalence of (c) with (d) and (e) is explained in Chap. 16 of [11]. We prove (a)$\Leftrightarrow$(f).

First suppose that Λ is given as one minimal K-type and that $\nu_0^- = 0$. We show there is a second minimal K-type. In fact, $\nu_0^- = 0$ forces $1 - \mu_\alpha = 0$, and thus $\mu = +\frac{1}{2}\alpha$. Thus it is enough to prove that $\Lambda - \alpha$ is Δ_K^+ dominant. Assuming the contrary, let γ in Δ_K^+ have $2\langle \Lambda - \alpha, \gamma\rangle/|\gamma|^2 < 0$. Then

$$\frac{2\langle \Lambda, \gamma\rangle}{|\gamma|^2} - \frac{2\langle \alpha, \gamma\rangle}{|\gamma|^2} < 0.$$

The first term is $\geqslant 0$, and thus $2\langle \alpha, \gamma\rangle/|\gamma|^2$ must be 1 or 2. Consequently either $2\langle \Lambda, \gamma\rangle/|\gamma|^2 = 0$ and $2\langle \alpha, \gamma\rangle/|\gamma|^2 > 0$, or $2\langle \Lambda, \gamma\rangle/|\gamma|^2 = +1$ and $2\langle \alpha, \gamma\rangle/|\gamma|^2 = +2$. Put $\beta = \gamma - \alpha$. In the first case, β contributes to the term $2\#\{\beta \in \Delta_n^+ \mid \beta + \alpha \in \Delta,\ \langle \Lambda, \beta + \alpha\rangle = 0\}$, while in the second case, β contributes to the term

$$\#\{\beta \in \Delta_n^+ \mid \beta + \alpha \in \Delta,\ |\beta|^2 < |\alpha|^2,\ 2\langle \Lambda, \beta + \alpha\rangle/|\beta + \alpha|^2 = +1\}.$$

In either case, we get a contradiction to the relation $\nu_0^- = 0$.

Similarly if Λ is given and $\nu_0^+ = 0$, then we find that $\Lambda + \alpha$ is a second minimal K-type. Conversely suppose that Λ is given with $\mu = +\frac{1}{2}\alpha$, and suppose that $\Lambda - \alpha$ is Δ_K^+ dominant, thus giving a second minimal K-type. We show $\nu_0^- = 0$. First suppose that β contributes to the term

$$\#\{\beta \in \Delta_n^+ \mid \beta + \alpha \in \Delta,\ |\beta|^2 < |\alpha|^2,\ 2\langle \Lambda, \beta + \alpha\rangle/|\beta + \alpha|^2 = +1\}.$$

Then we have

$$\frac{2\langle \Lambda - \alpha, \beta + \alpha\rangle}{|\beta + \alpha|^2} = \frac{2\langle \Lambda, \beta + \alpha\rangle}{|\beta + \alpha|^2} - \frac{2\langle \alpha, \beta + \alpha\rangle}{|\beta + \alpha|^2} = 1 - 2 < 0,$$

in contradiction to the Δ_K^+ dominance of $\Lambda - \alpha$. Next suppose that β contributes to the term

$$2\#\{\beta \in \Delta_n^+ \mid \beta + \alpha \in \Delta,\ \langle \Lambda, \beta + \alpha\rangle = 0\}. \tag{2.1}$$

We may assume that β is minimal with respect to this property. We show that $\beta + \alpha$ is Δ_K^+ simple. In fact, otherwise write $\beta + \alpha = \gamma_1 + \gamma_2$ with γ_1 and γ_2 in Δ_K^+. We must have $\langle \Lambda, \gamma_1\rangle = \langle \Lambda, \gamma_2\rangle = 0$. Since $\Lambda - \alpha$ is Δ_K^+ dominant, $0 \leqslant \langle \Lambda - \alpha, \gamma_1\rangle = \langle \Lambda, \gamma_1\rangle - \langle \alpha, \gamma_1\rangle = -\langle \alpha, \gamma_1\rangle$. Thus

$\langle \alpha, \gamma_1 \rangle \leqslant 0$ and similarly $\langle \alpha, \gamma_2 \rangle \leqslant 0$. Adding, we obtain $\langle \alpha, \beta+\alpha \rangle \leqslant 0$, from which we conclude $\langle \alpha, \beta+\alpha \rangle = 0$ and therefore $\langle \alpha, \gamma_1 \rangle = \langle \alpha, \gamma_2 \rangle = 0$ and $\langle \beta, \beta+\alpha \rangle > 0$. Now $\langle \beta, \gamma_1+\gamma_2 \rangle = \langle \beta, \beta+\alpha \rangle > 0$, and we may thus assume $\langle \beta, \gamma_1 \rangle > 0$. Hence $\beta' = \beta - \gamma_1$ is in Δ_n. The equation

$$\beta' = \beta - \gamma_1 = \gamma_2 - \alpha$$

shows that β' is positive and that $\beta' + \alpha$ is the root γ_2, which is orthogonal to Λ. Thus β' exhibits β as not being appropriately minimal, contradiction. We conclude $\beta+\alpha$ is Δ_K^+ simple. Since $\mu = \frac{1}{2}\alpha$, (1.3) gives $\Lambda = \lambda_0 + \delta - 2\delta_K$ and hence

$$\begin{aligned} 0 &= \frac{2\langle \Lambda, \beta+\alpha \rangle}{|\beta+\alpha|^2} = \frac{2\langle \lambda_0, \beta+\alpha \rangle}{|\beta+\alpha|^2} + \frac{2\langle \delta, \beta+\alpha \rangle}{|\beta+\alpha|^2} - \frac{2\langle 2\delta_K, \beta+\alpha \rangle}{|\beta+\alpha|^2} \\ &\geqslant 0 + \frac{2\langle \delta, \beta+\alpha \rangle}{|\beta+\alpha|^2} - 2. \end{aligned}$$

However,

$$\frac{2\langle \delta, \beta+\alpha \rangle}{|\beta+\alpha|^2} = \frac{2\langle \delta, \beta \rangle}{(1/2)|\beta|^2} + \frac{2\langle \delta, \alpha \rangle}{|\alpha|^2} \geqslant 3,$$

and we have a contradiction. We conclude that (2.1) is 0 and thus that $v_0^- = 0$.

Similarly if Λ is given with $\mu = -\frac{1}{2}\alpha$ and if $\Lambda + \alpha$ is Δ_K^+ dominant, then $v_0^+ = 0$. This proves the lemma.

In view of Lemma 2.1, we may assume henceforth that $v_0^+ > 0$, that $v_0^- > 0$, and that the invariant Hermitian form on $J(MAN, \sigma, v)$ is positive for all v near 0.

We now show how to calculate easily the basic case and special basic case associated to λ_0. First we treat basic cases. We define a form $\lambda_{0,b}$ on $\mathfrak{b}^{\mathbb{C}}$ as follows: for β simple, $2\langle \lambda_{0,b}, \beta \rangle / |\beta|^2$ is the smallest possible value of $2\langle \lambda_0', \beta \rangle / |\beta|^2$ among Harish-Chandra parameters λ_0' that are consistent with Δ^+ and χ, lead to a nonzero representation of M, and have a Δ_K^+ dominant form Λ (for the same μ as λ_0). From Theorem 3.1 of [13] and Corollary 2.3 of [10], it follows that

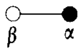
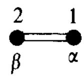
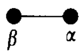
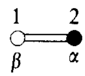
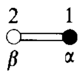
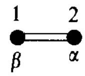
$$\frac{2\langle \lambda_{0,b}, \beta \rangle}{|\beta|^2} = \begin{cases} 1 & \text{if } \beta \perp \alpha \text{ and } \beta \text{ is compact} \\ 0 & \text{if } \beta \perp \alpha \text{ and } \beta \text{ is noncompact} \end{cases} \tag{2.2}$$

and that $2\langle \lambda_{0,b}, \beta \rangle / |\beta|^2$ is given by Table 2.1 when $\beta \not\perp \alpha$. (In the table, the noncompact roots are the black roots.)

The table allows us to obtain by inspection the Dynkin diagram of the basic case associated to λ_0. The next lemma allows us to obtain by inspec-

TABLE 2.1

Values of $2\langle\lambda_{0,b},\beta\rangle/|\beta|^2$

Roots	$2\langle\lambda_{0,b},\beta\rangle/\|\beta\|^2$	Roots	$2\langle\lambda_{0,b},\beta\rangle/\|\beta\|^2$
	$\frac{1}{2}(1+\mu_\alpha)$		$\frac{1}{2}(1-\|\mu_\alpha\|)$
	$\frac{1}{2}(1-\mu_\alpha)$		$\|\mu_\alpha+\frac{1}{2}\|-\frac{1}{2}$
	$\frac{1}{2}(1+\|\mu_\alpha\|)$		$\|\mu_\alpha-\frac{1}{2}\|-\frac{1}{2}$

tion the Dynkin diagram of the special basic case. Recall the definition of $\Delta_{K,\perp}$ in (1.5); since Λ is Δ_K^+ dominant, $\Delta_{K,\perp}^+$ is generated by a subset of the simple roots of Δ_K^+.

LEMMA 2.2. *Let γ be a simple root for Δ_K^+. Then γ is in $\Delta_{K,\perp}^+$ if and only if γ lies within the basic case and is of one of the following forms*:

(a) *γ is Δ^+ simple, and $\gamma\perp\alpha$.*

(b) *γ is Δ^+ simple, $\gamma\not\perp\alpha$,* and $|\gamma|=|\alpha|$.

(c) *γ is Δ^+ simple, $\gamma\not\perp\alpha$, $|\gamma|\neq|\alpha|$, and $\mu\neq-\frac{1}{2}\alpha$.*

(d) *γ is the sum of α and a noncompact neighbor of α, and $|\gamma|=|\alpha|$.*

(e) *$s_\alpha\gamma$ is Δ^+ simple, $\gamma\not\perp\alpha$, $|\gamma|\neq|\alpha|$, and $\mu\neq+\frac{1}{2}\alpha$.*

(f) *γ is the sum of a noncompact neighbor β of α with a noncompact neighbor β' of β having $|\beta'|=|\beta|$ and $\beta'\neq\alpha$, and $\mu=+\frac{1}{2}\alpha$.*

(g) *$s_\alpha\gamma$ is the sum of a neighbor β of α with a noncompact neighbor β' of β having $|\beta'|=|\beta|$ and $\beta'\neq\alpha$, and $\mu=-\frac{1}{2}\alpha$.*

Remarks. (1) Nondegeneracy of σ plays a role in the lemma, there being another case when σ is degenerate. Also if a root of type (f) occurs and $|\alpha|=|\beta|=|\beta'|$, then any other neighbor of α is compact, by nondegeneracy. We shall use this fact frequently without reference. Analogously in (g) if $|\alpha|=|\beta|=|\beta'|$, then any other neighbor of α is noncompact.

(2) The proof will use a handy device for reducing the case $\mu=-\frac{1}{2}\alpha$ to the case $\mu=+\frac{1}{2}\alpha$. Namely we replace Δ^+ by $s_\alpha\Delta^+$ and define $\alpha'=-\alpha$. Then Λ is unchanged, but $\mu=-\frac{1}{2}\alpha$ has been replaced by $\mu=+\frac{1}{2}\alpha'$. This device, called *reflection in* α, will be used frequently in later sections.

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(3) The special basic case is therefore generated by the simple roots mentioned in (a)–(g). These consist of α, all compact Δ^+ simple roots in the basic case except as in (c), and certain noncompact Δ^+ simple roots at distance $\leqslant 2$ from α in the Dynkin diagram.

Proof. Let γ be Δ_K^+ simple. Writing $\mu - \frac{1}{2}\alpha = \frac{1}{2}(\mu_\alpha - 1)\,\alpha$, we obtain

$$\begin{aligned}\frac{2\langle \Lambda, \gamma\rangle}{|\gamma|^2} &= \frac{2\langle \lambda_0, \gamma\rangle}{|\gamma|^2} + \frac{2\langle \delta, \gamma\rangle}{|\gamma|^2} - 2 + \tfrac{1}{2}(\mu_\alpha - 1)\,\frac{2\langle \alpha, \gamma\rangle}{|\gamma|^2}\\ &= \frac{2\langle \lambda_0 - \lambda_{0,b}, \gamma\rangle}{|\gamma|^2} + \frac{2\langle \lambda_{0,b}, \gamma\rangle}{|\gamma|^2}\\ &\quad + \frac{2\langle \delta, \gamma\rangle}{|\gamma|^2} - 2 + \frac{1}{2}(\mu_\alpha - 1)\,\frac{2\langle \alpha, \gamma\rangle}{|\gamma|^2}\end{aligned} \tag{2.3}$$

from (1.3).

First suppose γ is Δ^+ simple. If $\gamma \perp \alpha$, then

$$\frac{2\langle \lambda_{0,b}, \gamma\rangle}{|\gamma|^2} = \frac{2\langle \delta, \gamma\rangle}{|\gamma|^2} = 1$$

and

$$\frac{2\langle \Lambda, \gamma\rangle}{|\gamma|^2} = \frac{2\langle \lambda_0 - \lambda_{0,b}, \gamma\rangle}{|\gamma|^2}.$$

This handles (a). If $\gamma \not\perp \alpha$, then $2\langle \delta, \gamma\rangle/|\gamma|^2 = 1$ and

$$\frac{2\langle \Lambda, \gamma\rangle}{|\gamma|^2} \geqslant \frac{2\langle \lambda_{0,b}, \gamma\rangle}{|\gamma|^2} - 1 + \frac{1}{2}(\mu_\alpha - 1)\,\frac{2\langle \alpha, \gamma\rangle}{|\gamma|^2}$$

with equality only if γ is basic. Using Table 2.1, we see that the right side is

$$= \begin{cases} \tfrac{1}{2}(1 + \mu_\alpha) - 1 - \tfrac{1}{2}(\mu_\alpha - 1) = 0 & \text{when} \quad |\alpha| = |\gamma| \\ |\mu_\alpha + \tfrac{1}{2}| - \tfrac{1}{2} - 1 - (\mu_\alpha - 1) = |\mu_\alpha + \tfrac{1}{2}| - (\mu_\alpha + \tfrac{1}{2}) & \text{when} \quad |\alpha| > |\gamma| \\ \tfrac{1}{2}(1 + |\mu_\alpha|) - 1 - \tfrac{1}{2}(\mu_\alpha - 1) = \tfrac{1}{2}(|\mu_\alpha| - \mu_\alpha) & \text{when} \quad |\alpha| < |\gamma|, \end{cases}$$

and then (b) and (c) follow.

Next suppose that $\mu = +\frac{1}{2}\alpha$ and that γ is not Δ^+ simple. Then

$$\frac{2\langle \Lambda, \gamma\rangle}{|\gamma|^2} = \frac{2\langle \lambda_0 - \lambda_{0,b}, \gamma\rangle}{|\gamma|^2} + \frac{2\langle \lambda_{0,b}, \gamma\rangle}{|\gamma|^2} + \left(\frac{2\langle \delta, \gamma\rangle}{|\gamma|^2} - 2\right)$$

with each term on the right $\geqslant 0$. Hence γ is in $\Delta_{K,\perp}^+$ if and only if each term on the right is 0. Let us suppose this happens. Expand $\gamma = \gamma_1 + \cdots + \gamma_n$ as the sum of Δ^+ simple roots. Since $\lambda_0 - \lambda_{0,b}$ and $\lambda_{0,b}$ are dominant, each γ_j is basic and $\langle \lambda_{0,b}, \gamma_j \rangle = 0$. Since $\mu = +\frac{1}{2}\alpha$, (2.2) and Table 2.1 show that each γ_j is noncompact. Since γ is compact, the number of γ_j's is even. Also at least one γ_j has $|\gamma_j| = |\gamma|$. Since

$$2 = \frac{2\langle \delta, \gamma \rangle}{|\gamma|^2} = \sum_j \frac{2\langle \delta, \gamma_j \rangle}{|\gamma|^2} = \sum_j \frac{2\langle \delta, \gamma_j \rangle}{|\gamma_j|^2} \frac{|\gamma_j|^2}{|\gamma|^2} = \sum_j \frac{|\gamma_j|^2}{|\gamma|^2},$$

it follows that the number of γ_j is $\leqslant 3$. Therefore $\gamma = \gamma_1 + \gamma_2$ and $|\gamma_1| = |\gamma_2| = |\gamma|$. Now γ_1 and γ_2 cannot both be orthogonal to α, since otherwise γ would be a compact root strongly orthogonal to α such that $\langle \lambda_0, \gamma \rangle = 0$, in contradiction to nondegeneracy. Moreover, γ_1 and γ_2 must be adjacent. We conclude that (d) or (f) is necessary when $\mu = +\frac{1}{2}\alpha$ and γ is not Δ^+ simple. Reversing the steps, we see that (d) or (f) is sufficient for γ to be in $\Delta^+_{K,\perp}$ when $\mu = +\frac{1}{2}\alpha$.

Next suppose that $\mu = -\frac{1}{2}\alpha$ and that γ is not Δ^+ simple. We reflect in α as in Remark 2. Then $\mu = +\frac{1}{2}\alpha'$ and we are to consider γ in $s_\alpha \Delta^+$. First suppose γ is $s_\alpha \Delta^+$ simple. If $\gamma \perp \alpha'$, then $\gamma \perp \alpha$ and γ was simple for Δ^+, contradiction. So $\gamma \not\perp \alpha'$. If $|\gamma| = |\alpha|$, then (b) applies in the system $s_\alpha \Delta^+$ and yields the $\mu = -\frac{1}{2}\alpha$ part of condition (d) in the system Δ^+. If $|\gamma| \neq |\alpha|$, then (c) applies in the system $s_\alpha \Delta^+$ (since $\mu \neq -\frac{1}{2}\alpha'$) and yields the $\mu = -\frac{1}{2}\alpha$ part of condition (e) in the system Δ^+.

Still with $\mu = -\frac{1}{2}\alpha$ and γ not Δ^+ simple, suppose γ is not $s_\alpha \Delta^+$ simple. Since $\mu = +\frac{1}{2}\alpha'$, the applicable conditions in $s_\alpha \Delta^+$ are (d) and (f). However, (d) would make γ simple for Δ^+, which is not the case. Thus the condition relative to $s_\alpha \Delta^+$ for γ to be in $\Delta^+_{K,\perp}$ is (f), and the corresponding condition relative to Δ^+ is (g).

Finally suppose that $\mu = 0$ and that γ is not Δ^+ simple. Then we have

$$\frac{2\langle \Lambda, \gamma \rangle}{|\gamma|^2} = \frac{2\langle \lambda_0 - \lambda_{0,b}, \gamma \rangle}{|\gamma|^2} + \frac{2\langle \lambda_{0,b}, \gamma \rangle}{|\gamma|^2} + \left(\frac{2\langle \delta, \gamma \rangle}{|\gamma|^2} - 2 \right) - \frac{1}{2} \left(\frac{2\langle \alpha, \gamma \rangle}{|\gamma|^2} \right).$$

If $\langle \alpha, \gamma \rangle < 0$, then all terms on the right side are $\geqslant 0$, and the last one is > 0. Hence γ is not in $\Delta^+_{K,\perp}$. If $\langle \alpha, \gamma \rangle = 0$, then all terms on the right side are $\geqslant 0$, and γ is in $\Delta^+_{K,\perp}$ only if $\langle \lambda_0, \gamma \rangle = 0$. If γ is strongly orthogonal to α, this condition contradicts nondegeneracy. Otherwise $\gamma - \alpha$ is a root and

$$\frac{2\langle \delta, \gamma \rangle}{|\gamma|^2} - 2 = 2 \cdot \frac{2\langle \delta, \gamma - \alpha \rangle}{|\gamma - \alpha|^2} + \frac{2\langle \delta, \alpha \rangle}{|\alpha|^2} - 2 > 0.$$

Hence γ is not in $\Delta^+_{K,\perp}$.

Thus the only possibility is that $\langle \alpha, \gamma \rangle > 0$. Considering matters in $s_\alpha \Delta^+$

with $\alpha' = -\alpha$, we see that γ must be $s_\alpha \Delta^+$ simple if γ is in $\Delta^+_{K,\perp}$. So condition (b) or (c) applies in $s_\alpha \Delta^+$. Therefore (d) or (e) applies in Δ^+, and we have the $\mu = 0$ part of conditions (d) and (e).

3. Cut-Offs for Unitarity

If μ_0 is an integral form on $\mathfrak{b}^{\mathbb{C}}$, we let $\mu_0^\vee$ be the Δ_K^+ dominant Weyl group transform of μ_0.

Let Λ' be an integral form that is Δ_K^+ dominant, and let β be a noncompact root. We describe how to obtain $(\Lambda' + \beta)^\vee$ constructively; the result will always be of the form $\Lambda' + \beta'$ with β' a noncompact root. Let $\Delta^+_{K,\Lambda'}$ be the subset of members of Δ_K^+ orthogonal to Λ'; $\Delta^+_{K,\Lambda'}$ is generated by simple roots of Δ_K^+. The first step in the process is to make β dominant with respect to $\Delta^+_{K,\Lambda'}$, say with result β_1. If $\Lambda' + \beta_1$ is Δ_K^+ dominant, then $\beta' = \beta_1$ and we are done. Otherwise there will be a Δ_K^+ simple root γ with $2\langle \Lambda', \gamma \rangle / |\gamma|^2 = +1$ and $2\langle \beta_1, \gamma \rangle / |\gamma|^2 = -2$. Let β_2 be the short noncompact root $\beta_1 + \gamma$. Then β' is obtained by making β_2 dominant with respect to $\Delta^+_{K,\Lambda'}$. Note that the process stops with $(\Lambda' + \beta)^\vee = \Lambda' + \beta_1$ if all noncompact roots are short.

Denote by $\tau_{\Lambda'}$ an irreducible representation of K with highest weight Λ'. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. We shall denote the adjoint representation of K on $\mathfrak{p}^{\mathbb{C}}$ simply by $\mathfrak{p}^{\mathbb{C}}$. It is well known that the irreducible constituents (under K) of $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$ occur with multiplicity one, under our assumption that rank G = rank K. The proof of the following proposition will be given in another paper. (See [19] for other results in this direction.)

PROPOSITION 3.1. *Let Λ' be integral and Δ_K^+ dominant, let β be a noncompact root, and suppose $\Lambda' + \beta$ is Δ_K^+ dominant. Then $\tau_{\Lambda'+\beta}$ fails to occur in $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$ if and only if there exists a (short) Δ_K^+ simple root γ such that γ is orthogonal to Λ' and γ is orthogonal but not strongly orthogonal to β.*

Proposition 3.1 addresses one of the hypotheses of Theorem 3.2 below, which is a variant of results in Section 2 of [3] and will be proved in another paper. We return to the notation λ_0, μ, Λ, etc., used earlier.

THEOREM 3.2. *In terms of the minimal K-type Λ, let $\Lambda' = (\Lambda + \alpha)^\vee$.* Suppose that either (a), (b), and (c) or (a), (b′), and (c) hold:

(a) *$\tau_{\Lambda'}$ occurs in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$.*

(b) *$\Lambda - \alpha$ is not a weight of $\tau_{\Lambda'}$.*

(b′) *$\Lambda - \alpha$ is conjugate to $\Lambda + \alpha$ by the Weyl group of Δ_K.*

(c) *There exists a root system* $\Delta_L \subseteq \Delta$ *generated by* Δ^+ *simple roots such that* α *is in* Δ_L, Δ_L *has real rank one, and* $\Lambda' - \Lambda$ *is an integral linear combination of roots in* Δ_L.

Then $\tau_{\Lambda'}$ *occurs in* $U(MAN, \sigma, \nu)|_K$, *and the pair of K-types* $\{\Lambda, (\Lambda+\alpha)\check{}\}$ *exhibits* $J(MAN, \sigma, \frac{1}{2}c\alpha)$ *as not infinitesimally unitary for* $c > \nu_0^+$.

Remarks. (1) When all noncompact roots are short (necessarily the case in a single-line diagram, e.g.) then it follows from Proposition 3.1 and the second paragraph of this section that hypothesis (a) is satisfied.

(2) In certain double-line cases with α long, we will have to get by with the following weakening of hypothesis (c):

(c′) Let $\Delta^+_{-,n}$ be the set of positive $\mathfrak{m}$-noncompact members of Δ_-. Then every solution to the equation

$$\Lambda' - \Lambda = c\alpha + \sum_{\beta \in \Delta^+_{-,n}} n_\beta \beta + \sum_{\gamma \in \Delta_K^+} k_\gamma \gamma$$

with $c \in \mathbb{Z}$, $n_\beta \in \mathbb{Z}$, $k_\gamma \in \mathbb{Z}$, $n_\beta \geqslant 0$, $k_\gamma \geqslant 0$ has $\sum n_\beta \beta = 0$.

The dual result obtained by reflection in α is as follows.

THEOREM 3.2′. *In terms of the minimal K-type* Λ, *let* $\Lambda' = (\Lambda - \alpha)\check{}$. *Suppose that* (a), (b), *and* (c) *hold*:

(a) $\tau_{\Lambda'}$ *occurs in* $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$.

(b) $\Lambda + \alpha$ *is not a weight of* $\tau_{\Lambda'}$.

(c) *There exists a root system* $\Delta_L \subseteq \Delta$ *generated by* Δ^+ *simple roots such that* α *is in* Δ_L, Δ_L *has real rank one, and* $\Lambda' - \Lambda$ *is an integral linear combination of roots in* Δ_L.

Then $\tau_{\Lambda'}$ *occurs in* $U(MAN, \sigma, \nu)|_K$, *and the pair of K-types* $\{\Lambda, (\Lambda+\alpha)\check{}\}$ *exhibits* $J(MAN, \sigma, \frac{1}{2}c\alpha)$ *as not infinitesimally unitary for* $c > \nu_0^+$.

Remark. The same two remarks as for Theorem 3.2 apply here. A statement here with (b′) in place of (b) would be contained already in Theorem 3.2.

Let δ^+ and δ^- be the results of making α and $-\alpha$, respectively, dominant for $\Delta^+_{K,\perp}$. (See (1.5).) The δ^+ subsystem of Δ is the root system generated by α and all simple roots needed for the expansion of δ^+, and the δ^- subsystem is defined similarly. These subsystems are necessarily contained in the special basic case associated to λ_0. If α is short, then we know from the beginning of this section that $(\Lambda+\alpha)\check{} = \Lambda + \delta^+$ and $(\Lambda - \alpha)\check{} = \Lambda + \delta^-$.

COROLLARY 3.3. *Suppose that all noncompact roots are short. Every* *root β occurring in the formula for ν_0^+ lies in the δ^+ subsystem. If the δ^+ subsystem has real rank one, then hypotheses* (a) *and* (c) *are satisfied in Theorem* 3.2. *If, in addition, the δ^+ subsystem is of type A as a Dynkin diagram, then hypothesis* (b) *is satisfied. Consequently $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is not infinitesimally unitary for $c > \nu_0^+$. Similar conclusions are valid for the δ^- subsystem, Theorem* 3.2′, *and ν_0^-.*

Proof. Since α is short, the exceptional term in ν_0^+ is 0. Let W_K be the Weyl group of Δ_K. We know that $(\Lambda+\alpha)^\vee = \Lambda + \delta^+$. Hence $\Lambda+\alpha$ and $\Lambda+\delta^+$ are conjugate via W_K. If β contributes to ν_0^+, then the equality $s_{\beta-\alpha}(\alpha) = \beta$ makes $s_{\beta-\alpha}(\Lambda+\alpha) = \Lambda+\beta$. Hence $\Lambda+\beta$ is conjugate to $\Lambda+\alpha$ and therefore to $\Lambda+\delta^+$. Since $\Lambda+\delta^+$ is Δ_K^+ dominant,

$$\delta^+ - \beta = (\Lambda+\delta^+) - (\Lambda+\beta) = \sum_{\gamma\in\Delta_K^+} k_\gamma \gamma$$

and $\delta^+ = \beta + \sum k_\gamma\gamma$. Since β is positive, it follows that β is in the δ^+ subsystem.

Suppose the δ^+ subsystem has real rank one. Remark 1 after Theorem 3.2 points out that (a) holds. For (c) we choose Δ_L to be the δ^+ subsystem. Since $\Lambda' - \Lambda = \delta^+ - \alpha$, it follows that (c) holds. Finally if $\Lambda - \alpha$ is a weight of $\tau_{(\Lambda+\alpha)^\vee}$, then

$$\delta^+ + \alpha = (\Lambda+\delta^+) - (\Lambda-\alpha) = \sum_{\gamma\in\Delta_K^+} k_\gamma \gamma$$

and

$$\delta^+ - \alpha = (\Lambda+\delta^+) - (\Lambda+\alpha) = \sum_{\gamma\in\Delta_K^+} k'_\gamma \gamma,$$

when subtracted, show that 2α is the sum of compact roots in the δ^+ subsystem. If the δ^+ subsystem is of type A (as well as of real rank one), then it is of type $\mathfrak{su}(n, 1)$ and 2α is not the sum of compact roots; thus (b) must hold. Applying Theorem 3.2, we obtain the desired results for the δ^+ subsystem and for ν_0^+. The results for δ^- and ν_0^- follow similarly from Theorem 3.2′. This completes the proof.

Our remaining cut-off results could be expressed in absolute terms as in Theorem 3.2, but we prefer to express them in relative terms, giving certain prototypes and a way of embedding them in results about G. The tool for embedding results is the Vogan Signature Theorem (Theorem 3.4), which is implicit in Vogan [25]. We formulate it in language appropriate to our situation: Let Δ_L be a subsystem of Δ generated by simple roots and containing α. Let

$$\Delta(\mathfrak{u}) = \{\beta \in \Delta^+ \mid \beta \notin \Delta_L\}, \tag{3.1a}$$

and let $\delta(\mathfrak{u})$ and $\delta(\mathfrak{u}\cap\mathfrak{p})$ be the half sums of the members of $\Delta(\mathfrak{u})$ and the noncompact members of $\Delta(\mathfrak{u})$, respectively. Define

$$\lambda_{0,L}=\lambda_0-\delta(\mathfrak{u}), \tag{3.1b}$$

$$\mu_L=\mu,$$

$$\chi_L(\gamma_\alpha) \qquad \text{consistently with } \mu_L,$$

$$\Lambda_L=\Lambda-2\delta(\mathfrak{u}\cap\mathfrak{p}).$$

Then $(\lambda_{0,L}, \Delta^+\cap\Delta_L, \chi_L)$ leads to a well-defined standard induced series of representations $U^L(M_LAN_L, \sigma_L, \nu)$ of the group L corresponding to Δ_L, by §4 of [13], with Λ_L as a minimal K-type.

THEOREM 3.4 (Vogan Signature Theorem). *With the conventions above, suppose that Λ' is Δ_K^+ dominant and that $\Lambda'-\Lambda$ is the sum of members of Δ_L. Put $\Lambda'_L=\Lambda'-2\delta(\mathfrak{u}\cap\mathfrak{p})$. Then the multiplicity of $\tau_{\Lambda'}$ in $U(MAN, \sigma, \nu)$ equals the multiplicity of $\tau_{\Lambda'_L}$ in $U^L(MAN_L, \sigma_L, \nu)$. Moreover, if the standard invariant Hermitian forms for these induced representations are normalized to be positive on the τ_Λ and τ_{Λ_L} subspaces, respectively, then the signatures of these forms on the $\tau_{\Lambda'}$ and $\tau_{\Lambda'_L}$ respective subspaces are the same.*

Remarks. (1) The multiplicity result is in Speh–Vogan [20, Theorem 4.17 and pp. 267–268].

(2) The equality of the signatures requires no additional inequalities on ν.

(3) We shall use the theorem as follows. We start from Λ, pass to Λ_L, construct Λ'_L by adding some roots of Δ_L to Λ_L and by making the result dominant for $\Delta_K^+\cap\Delta_L$, and set $\Lambda'=\Lambda'_L+2\delta(\mathfrak{u}\cap\mathfrak{p})$. If Λ' is Δ_K^+ dominant, then the theorem assures us of equality of multiplicities and signatures for $\tau_{\Lambda'}$ and $\tau_{\Lambda'_L}$.

(4) Propositions 3.5–3.8 below will give us results about subgroups L that we can lift to G by means of Theorem 3.4, and their proofs will be given in a later paper.

(5) If Theorems 3.2 and 3.2′ are known for real rank one groups, then Theorem 3.4 implies them in general. However, we shall need Theorems 3.2 and 3.2′ with hypothesis (c) weakened to hypothesis (c′), and then Theorem 3.4 does not help as much.

PROPOSITION 3.5 [2, Theorem 2]. *Suppose $n\geqslant 2$ and $\mathfrak{g}=\mathfrak{sp}(n,1)$, possibly with abelian and compact factors, and suppose that the special basic case for λ_0 is all of Δ. Suppose that $\mu=0$, that α is adjacent to the long simple root, and that α is the only noncompact simple root. Put $\Lambda'=(\Lambda+\alpha)^{\vee}$*

and $\Lambda'' = (\Lambda' + \alpha)^\vee$. *Then* $\tau_{\Lambda'}$ *and* $\tau_{\Lambda''}$ *have multiplicity one in* $U(MAN, \sigma, \frac{1}{2}c\alpha)$, *the signature of the standard form on* $\tau_{\Lambda'}$ *is* $\operatorname{sgn}(v_0^+ - c) = \operatorname{sgn}(v_0^- - c)$, *and the signature of the standard form on* $\tau_{\Lambda''}$ *is* $\operatorname{sgn}(v_0^+ - c)(v_0^+ - c - 2)$.

PROPOSITION 3.6. *Suppose* $n \geqslant 2$ *and* $\mathfrak{g} = \mathfrak{so}(2n, 2)$, *possibly with abelian and compact factors, and suppose that the special basic case for* λ_0 *is all of* Δ. *Then there is a choice* $\pm$ *of sign so that* $\pm\alpha$ *is conjugate by the Weyl group of* Δ_K *to the unique positive noncompact root* β_0 *orthogonal to* α; *fix this choice of sign. Put* $\Lambda'' = (\Lambda \pm \alpha + \beta_0)^\vee = \Lambda \pm \alpha + \beta_0$. *Then* $\tau_{\Lambda''}$ *has multiplicity one in* $U(MAN, \sigma, \frac{1}{2}c\alpha)$, *and the signature of the standard form on* $\tau_{\Lambda''}$ *is* $\operatorname{sgn}(v_{0,L}^{\pm} - c)$, *where* $v_{0,L}^+$ *and* $v_{0,L}^-$ *are the quantities* v_0^+ *and* v_0^- *computed in an* $\mathfrak{su}(n, 1)$ *subdiagram containing* α *and generated by simple roots of* Δ^+.

PROPOSITION 3.7. *Suppose* $n \geqslant 2$ *and* $\mathfrak{g} = \mathfrak{so}(2n+1, 2)$, *possibly with abelian and compact factors, suppose that* α *is long, and suppose that the special basic case for* λ_0 *is all of* Δ. *Then there is a choice* $\pm$ *of sign so that* $\pm\alpha$ *is conjugate by the Weyl group of* Δ_K *to the unique positive noncompact root* β_0 *orthogonal to* α; *fix this choice of sign. Put* $\Lambda'' = (\Lambda \pm \alpha + \beta_0)^\vee = \Lambda \pm \alpha + \beta_0$. *Then* $\tau_{\Lambda''}$ *has multiplicity one in* $U(MAN, \sigma, \frac{1}{2}c\alpha)$, *and the signature of the standard form on* $\tau_{\Lambda''}$ *is* $\operatorname{sgn}(v_{0,L}^{\pm} + 1 - c)$, *where* $v_{0,L}^+$ *and* $v_{0,L}^-$ *are the quantities* v_0^+ *and* v_0^- *computed in the maximal* $\mathfrak{su}(n, 1)$ *subdiagram containing* α *and generated by simple roots of* Δ^+.

PROPOSITION 3.8. *Suppose* $n \geqslant 2$ *and* $\mathfrak{g} = \mathfrak{so}(2n, 3)$, *suppose that* α *is long, suppose that the short* Δ^+ *simple root* ε *is basic, and suppose that the special basic case for* λ_0 *is the maximal* $\mathfrak{su}(n, 1)$ *subdiagram containing* α *and generated by simple roots of* Δ^+. *Let* ζ *be the sum of simple roots strictly between* α *and* ε *in the Dynkin diagram, and suppose* ζ *is (nonzero and) noncompact. Put* $\Lambda'' = (\Lambda + (\zeta + \varepsilon))^\vee = \Lambda + \zeta + \varepsilon$. *Then* $\tau_{\Lambda''}$ *has multiplicity one in* $U(MAN, \sigma, \frac{1}{2}c\alpha)$, *and the signature of the standard form on* $\tau_{\Lambda''}$ *is* $\operatorname{sgn}(v_0^- - c)(v_0^- - c - 1)$.

Propositions 3.6–3.8 are new and will be proved in another paper.

4. Validity of Cut-Offs in Special Basic Cases, Single-Line Diagrams

Our goal for this section is to prove

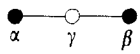
LEMMA 4.1. *Suppose that* rank G = rank K, *that the Dynkin diagram of* Δ^+ *is a single line diagram, and that the special basic case associated to* λ_0 *is all of* Δ. *Then at least one of the following happens*:

(a) *the* δ^+ *subsystem has real rank one, and* $\nu_0^+ \leqslant \nu_0^-$,

(b) *the* δ^- *subsystem has real rank one, and* $\nu_0^- \leqslant \nu_0^+$.

In view of Corollary 3.3, this lemma will prove that $\min(\nu_0^+, \nu_0^-)$ is a cutoff for unitarity, and it will do so in a way that will allow us to embed this result in larger groups. The proof uses the normalization $|\alpha|^2 = 2$ and distinguishes several cases and subcases.

(I) We first suppose there is a simple root γ_0 of $\Delta_{K,\perp}^+$ of the form (f) or (g) in Lemma 2.2. Possibly by reflecting in α, we may assume the form is (g). Then $\mu = -\frac{1}{2}\alpha$, and γ_0 is the sum of three Δ^+ simple roots α, γ, and β as in the diagram



(4.1)

(Here and elsewhere, the black roots are the noncompact ones.) We are going to compute δ^+, which is defined before Corollary 3.3, and we are going to show that $\nu_0^+ \leqslant \nu_0^-$. Before doing so, let us observe that any neighbor ε of α other than γ is necessarily noncompact. We remarked on this briefly in Remark 1 for Lemma 2.2. The reason is this: otherwise Table 2.1 shows that $\varepsilon + \alpha + \gamma + \beta$ is a compact root of $\mathfrak{m}$ that is orthogonal to λ_0, in contradiction to nondegeneracy.

(I.1) Suppose γ is not a triple point of Δ^+. We claim that $\delta^+ = \alpha + \gamma$. We form the Dynkin diagram of $\Delta_{K,\perp}^+$, labeling each simple root with its normalized inner product with α. Since $\mu = -\frac{1}{2}\alpha$ and γ is not a triple point and all other neighbors of α are noncompact, the only simple root for $\Delta_{K,\perp}^+$ of type (f) or (g) (in Lemma 2.2) is $\gamma_0 = \alpha + \gamma + \beta$, and it has label $+1$. According to Lemma 2.2, the other simple roots for $\Delta_{K,\perp}^+$ are $\alpha + \varepsilon$ with label $+1$ (for at most two noncompact neighbors ε of α), γ with label -1, and various compact Δ^+ simple roots that are orthogonal to α and have label 0.

Let us note that every $\Delta_{K,\perp}^+$ simple neighbor γ' of γ in $\Delta_{K,\perp}^+$ has label $+1$. The root γ' cannot be one of the above roots with label 0 since γ is not a triple point in Δ^+, and all other possibilities have label $+1$.

Now we can show that $\delta^+ = \gamma + \alpha$. In fact, it is clear that $\gamma + \alpha$ is conjugate to α by the Weyl group of $\Delta_{K,\perp}$. We show δ^+ is $\Delta_{K,\perp}^+$ dominant. Assuming the contrary, let γ' be $\Delta_{K,\perp}^+$ simple with $\langle \gamma + \alpha, \gamma' \rangle < 0$, i.e., $\langle \gamma + \alpha, \gamma' \rangle = -1$. Then $\langle \gamma, \gamma' \rangle = -1$ or $\langle \alpha, \gamma' \rangle = -1$. If $\langle \gamma, \gamma' \rangle = -1$, then γ' is a neighbor of γ and must have label $+1$; so $\langle \alpha, \gamma' \rangle = +1$ and $\langle \gamma + \alpha, \gamma' \rangle = 0$, contradiction. So $\langle \alpha, \gamma' \rangle = -1$, γ' has label -1, $\gamma' = \gamma$, and $\langle \gamma + \alpha, \gamma' \rangle = \langle \gamma + \alpha, \gamma \rangle > 0$, contradiction. Thus $\delta^+ = \gamma + \alpha$.

The δ^+ subsystem is of type $\mathfrak{su}(2, 1)$, which is of real rank one, and Corollary 3.3 gives $\nu_0^+ = 2$. Since $\mu = -\frac{1}{2}\alpha$, we certainly have $\nu_0^- \geqslant 2$.

(I.2) Suppose γ is a triple point of Δ^+ and the other neighbor β' of γ is noncompact. Again we claim that $\delta^+ = \alpha + \gamma$. We proceed as in (I.1). The simple roots of $\Delta^+_{K,\perp}$ with labels are $\gamma_0 = \alpha + \gamma + \beta$ with label $+1$, $\gamma'_0 = \alpha + \gamma + \beta'$ with label $+1$, γ with label -1, a root $\alpha + \varepsilon$ with label $+1$ (if α has a neighbor ε other than γ), and various compact Δ^+ simple roots with label 0. The only possible $\Delta^+_{K,\perp}$ simple neighbor γ' of γ in $\Delta^+_{K,\perp}$ is $\alpha + \varepsilon$ and has label $+1$. Thus the same argument as in (I.1) shows that $\delta^+ = \gamma + \alpha$, the δ^+ subsystem is of real rank one, $\nu_0^+ = 2$, and $\nu_0^- \geqslant 2$.

(I.3) Suppose γ is a triple point of Δ^+ and the other neighbor γ_1 of γ is compact. Let the (compact) roots extending beyond γ_1 be $\gamma_2, ..., \gamma_n$. We claim that $\delta^+ = \alpha + \gamma + \gamma_1 + \cdots + \gamma_n$. We proceed as in (I.1) and (I.2). The simple roots of $\Delta^+_{K,\perp}$ with labels are $\gamma_0 = \alpha + \gamma + \beta$ with label $+1$, γ with label -1, a root $\alpha + \varepsilon$ with label $+1$ (if α has a neighbor ε other than γ), and various compact Δ^+ simple roots with label 0 (including $\gamma_1, ..., \gamma_n$). Assuming by way of contradiction that $\alpha + \gamma + \gamma_1 + \cdots + \gamma_n$ is not $\Delta^+_{K,\perp}$ dominant, let γ' be $\Delta^+_{K,\perp}$ simple with $\langle \alpha + \cdots + \gamma_n, \gamma' \rangle = -1$. Then $\langle \gamma + \gamma_1 + \cdots + \gamma_n, \gamma' \rangle = -1$ or $\langle \alpha, \gamma' \rangle = -1$. If $\langle \gamma + \gamma_1 + \cdots + \gamma_n, \gamma' \rangle = -1$, then γ' is a neighbor of one of $\gamma, \gamma_1, ..., \gamma_n$ but is not one of these roots. Hence γ' is $\alpha + \gamma + \beta$ or $\alpha + \varepsilon$, both of which have label $+1$; so $\langle \alpha, \gamma' \rangle = +1$ and $\langle \alpha + \gamma + \gamma_1 + \cdots + \gamma_n, \gamma' \rangle = 0$, contradiction. So $\langle \alpha, \gamma' \rangle = -1$, γ' has label -1, $\gamma' = \gamma$, and

$$\langle \alpha + \gamma + \gamma_1 + \cdots + \gamma_n, \gamma' \rangle = \langle \alpha + \gamma + \gamma_1 + \cdots + \gamma_n, \gamma \rangle = 0,$$

contradiction. Thus $\delta^+ = \alpha + \gamma + \gamma_1 + \cdots + \gamma_n$.

The δ^+ subsystem is of type $\mathfrak{su}(n+2, 1)$, which is of real rank one, and Corollary 3.3 gives $\nu_0^+ = 2(n+1)$. Each root $\beta + \gamma + \gamma_1 + \cdots + \gamma_j$ for $0 \leqslant j \leqslant n$ contributes to ν_0^-, and thus $\nu_0^- \geqslant 2(n+1)$.

(II) Next we suppose that there is no simple root of $\Delta^+_{K,\perp}$ of type (f) or (g) in Lemma 2.2 and that α is a triple point. Possibly by reflecting in α, we may assume that at most one of the neighbors β_1, β_2, β_3 of α is compact; say that β_2 and β_3 are noncompact.

(II.1) If β_1 is noncompact, then we claim that $\delta^+ = \alpha$. In fact, the simple roots of $\Delta^+_{K,\perp}$ with labels are $\alpha + \beta_j$ for $1 \leqslant j \leqslant 3$ with label $+1$ and various compact Δ^+ simple roots with label 0. Since all labels are $\geqslant 0$, $\langle \alpha, \gamma' \rangle \geqslant 0$ for all $\Delta^+_{K,\perp}$ simple γ'. Thus $\delta^+ = \alpha$.

The δ^+ subsystem is of type $\mathfrak{sl}(2, \mathbb{R})$, and Corollary 3.3 gives $\nu_0^+ = 1 + \mu_\alpha \leqslant 2$. Since β_2 and β_3 contribute to ν_0^-, we have $\nu_0^- \geqslant 4$.

(II.2) If β_1 is compact, let $\gamma_1, ..., \gamma_n$ be the (compact) roots extending beyond β_1. We claim that $\delta^+ = \alpha + \beta_1 + \gamma_1 + \cdots + \gamma_n$. In fact, the simple roots of $\Delta^+_{K,\perp}$ with labels are $\alpha + \beta_2$ and $\alpha + \beta_3$ with label $+1$, β_1 with label -1, and various compact Δ^+ simple roots with label 0 (including $\gamma_1,\ldots,\gamma_n$). Arguing as in (I.3), we see that $\delta^+ = \alpha + \beta_1 + \gamma_1 + \cdots + \gamma_n$.

The δ^+ subsystem is of type $\mathfrak{su}(n+2, 1)$, which is of real rank one, and Corollary 3.3 gives $v_0^+ = 1 + \mu_\alpha + 2(n+1)$. Each root $\beta_2 + \alpha + \beta_3 + \beta_1 + \gamma_1 + \cdots + \gamma_j$ for $0 \leqslant j \leqslant n$ contributes to v_0^-, and so do β_2 and β_3; thus $v_0^- \geqslant 2(n+3) > v_0^+$.

(III) Next we suppose that there is no simple root of $\Delta_{K,\perp}^+$ of type (f) or (g) in Lemma 2.2 and that α is not a triple point.

(III.1) Suppose further that all neighbors of α are of the same type, compact or noncompact. Possibly reflecting in α, we may assume that all neighbors are noncompact. Arguing as in (II.1), we see that $\delta^+ = \alpha$, the δ^+ subsystem is of type $\mathfrak{sl}(2, \mathbb{R})$, and $v_0^+ = 1 + \mu_\alpha \leqslant 2$. If α has no neighbors at all, then the δ^- subsystem is of type $\mathfrak{sl}(2, \mathbb{R})$ and $v_0^- = 1 - \mu_\alpha$; hence we are done. Otherwise α has a noncompact neighbor β, which contributes to v_0^-, and thus $v_0^- \geqslant 2$.

(III.2) Alternatively suppose that α has two neighbors, one compact and one noncompact. If Δ^+ has no triple point, it follows that Δ^+ is of real rank one, and Lemma 4.1 is automatic. Thus we may assume that there is a triple point. Possibly by reflecting in α, we may assume that the root on the side of α toward the triple point is noncompact. Call this root β. Let the compact neighbor be γ, and let $\gamma, \gamma_1,\ldots,\gamma_n$ be the connected chain of compact roots ending in the node γ_n. We claim that $\delta^+ = \alpha + \gamma + \gamma_1 + \cdots + \gamma_n$. In fact, the simple roots of $\Delta_{K,\perp}^+$ with labels are $\alpha + \beta$ with label $+1$, γ with label -1, and various Δ^+ simple roots with label 0 (including $\gamma_1,\ldots,\gamma_n$). Arguing as in (I.3), we see that $\delta^+ = \alpha + \gamma + \gamma_1 + \cdots + \gamma_n$. The δ^+ subsystem is of type $\mathfrak{su}(n+2, 1)$, which is of real rank one, and Corollary 3.3 gives $v_0^+ = 1 + \mu_\alpha + 2(n+1)$.

We shall find a lower bound for v_0^-. Let $\varepsilon_1,\ldots,\varepsilon_k$ (with $k \geqslant 0$) be the (compact) roots from β to the triple point. Here we take ε_k to be the triple point, with ε_0 understood to be β. Let ξ_1 and ξ_2 be the other two neighbors of the triple point. Then

$$\xi_1 + \xi_2 + 2\varepsilon_k + \cdots + 2\varepsilon_1 + 2\beta + \alpha + \gamma + \gamma_1 + \cdots + \gamma_j$$

for $0 \leqslant j \leqslant n$ is a noncompact root contributing to v_0^-, as is β, and it follows that $v_0^- \geqslant 2(n+2)$. Thus $v_0^- \geqslant v_0^+$, and the proof of Lemma 4.1 is complete.

5. Validity of Cut-Offs in General, Single-Line Diagrams

To pass from special basic cases to general cases of single-line diagrams, we use the following lemma.

LEMMA 5.1. *Suppose that* rank $G=$ rank K *and that the Dynkin diagram of* Δ^+ *is a single-line diagram. Then at least one of the following things happens*:

(a) *the* δ^+ *subsystem has real rank one, and* $\nu_0^+ \leqslant \nu_0^-$,

(b) *the* δ^- *subsystem has real rank one, and* $\nu_0^- \leqslant \nu_0^-$.

Consequently $J(MAN, \sigma, \frac{1}{2}c\alpha)$ *is not infinitesimally unitary for* $c > \min(\nu_0^+, \nu_0^-)$. *Moreover, if the component of* α *in the special basic case associated to* λ_0 *is of type* $\mathfrak{so}(2n, 2)$ *with* $n \geqslant 2$ *and if* $\nu_{0,L}^{\pm}$ *(for the appropriate choice of sign) is defined within the special basic case as in Theorem* 1.1 *(situation* (iv)), *then* $\nu_{0,L}^{\pm} \leqslant \nu_0^{\pm}$ *and* $J(MAN, \sigma, \frac{1}{2}c\alpha)$ *is not infinitesimally unitary for* $c > \min(\nu_{0,L}^{\pm}, \nu_0^{\mp})$.

The statements here about ν_0^+ and ν_0^- follow immediately from Lemma 4.1 and Corollary 3.3, since the δ^+ and δ^- subsystems lie within the special basic case. Thus suppose the component of α of the associated special basic case Δ_S is of type $\mathfrak{so}(2n, 2)$, $n \geqslant 2$. Possibly by reflecting in α, we may assume that α is conjugate by the Weyl group of $\Delta_K \cap \Delta_S$ to the unique positive noncompact root β_0 orthogonal to $+\alpha$ and lying in the $\mathfrak{so}(2n, 2)$. Let $(\vee, S)$ mean "made dominant with respect to $\Delta_K^+ \cap \Delta_S$." Then Proposition 3.6 says within Δ_S that the $(K\cap S)$-type $(\Lambda_S + \alpha + \beta_0)^{(\vee, S)}$ cuts off unitarity of the Langlands quotients in S at the point $\nu_{0,L}^+$. Here

$$(\Lambda_S + \alpha + \beta_0)^{(\vee, S)} = \Lambda_S + \alpha + \beta_0 = \Lambda + \alpha + \beta_0 - 2\delta(\mathfrak{u} \cap \mathfrak{p})$$

if $\mathfrak{u}$ is built from the positive roots outside S. Suppose $(\Lambda + \alpha + \beta_0)^\vee = \Lambda + \alpha + \beta_0$. Then $(\Lambda + \alpha + \beta_0)^\vee - \Lambda = \alpha + \beta_0$ is the sum of members of Δ_S, and

$$((\Lambda + \alpha + \beta_0)^\vee)_S = \Lambda + \alpha + \beta_0 - 2\delta(\mathfrak{u} \cap \mathfrak{p}) = (\Lambda_S + \alpha + \beta_0)^{(\vee, S)}.$$

Hence the Vogan Signature Theorem (Theorem 3.4) says within G that the K-type $(\Lambda + \alpha + \beta_0)^\vee$ cuts off unitarity of the Langlands quotients in G at the point $\nu_{0,L}^+$. Therefore, to prove Lemma 5.1 (when β_0 is conjugate as above to $+\alpha$), it is enough to prove that $\Lambda + \alpha + \beta_0$ is Δ_K^+ dominant.

This strong a statement is not quite true. But we need consider only cases where $\nu_{0,L}^+$ gives a smaller cut-off than $\min(\nu_0^+, \nu_0^-)$. Thus we may assume $\nu_{0,L}^+ < \min(\nu_0^+, \nu_0^-)$. In this situation we shall be able to prove that $\Lambda + \alpha + \beta_0$ is Δ_K^+ dominant. The tool is

LEMMA 5.2. *Suppose that* rank $G=$ rank K *and that the Dynkin diagram of* Δ^+ *is a single-line diagram. Let* Δ_S *be the associated special basic case. Suppose that* β *is a noncompact root such that* $\Lambda + \alpha + \beta$ *is dominant for*

$\Delta_K^+ \cap \Delta_S$. If $\Lambda+\alpha+\beta$ is not dominant for Δ_K^+, then there is a Δ_K^+ simple root γ of one of the following forms:

(a) *γ is Δ^+ simple, is adjacent to α, and is not basic,*

(b) *γ is the sum $\gamma_1+\gamma_2$ of noncompact Δ^+ simple roots with γ_1 orthogonal to α, γ_2 adjacent to α, and $\mu \neq -\frac{1}{2}\alpha$.*

Proof. Failure of Δ_K^+ dominance means we can find γ simple for Δ_K^+ such that $2\langle \Lambda+\alpha+\beta, \gamma\rangle/|\gamma|^2<0$. Since $\Delta_{K,\perp}^+ \subseteq \Delta_K^+ \cap \Delta_S$, this γ will not be in $\Delta_{K,\perp}^+$. Thus $2\langle \Lambda, \gamma\rangle/|\gamma|^2 \geqslant 1$. Then it follows that

$$\frac{2\langle \Lambda, \gamma\rangle}{|\gamma|^2}=+1, \qquad \frac{2\langle \alpha, \gamma\rangle}{|\gamma|^2}=-1, \qquad \frac{2\langle \beta, \gamma\rangle}{|\gamma|^2}=-1. \tag{5.1}$$

From (2.3),

$$1=\frac{2\langle \lambda_0-\lambda_{0,b}, \gamma\rangle}{|\gamma|^2}+\frac{2\langle \lambda_{0,b}, \gamma\rangle}{|\gamma|^2}+\left(\frac{2\langle \delta, \gamma\rangle}{|\gamma|^2}-2\right)+\frac{1}{2}(1-\mu_\alpha). \tag{5.2}$$

If γ is Δ^+ simple, then (5.1) shows that γ is adjacent to α, and Table 2.1 shows that $2\langle \lambda_{0,b}, \gamma\rangle/|\gamma|^2=\frac{1}{2}(1+\mu_\alpha)$. Then it follows that

$$\frac{2\langle \lambda_0-\lambda_{0,b}, \gamma\rangle}{|\gamma|^2}=1$$

and γ is not basic. This is possibility (a).

If $2\langle \delta, \gamma\rangle/|\gamma|^2=2$, then $\gamma=\gamma_1+\gamma_2$ with γ_1 and γ_2 both noncompact and simple. Since $(\alpha, \gamma)<0$, we may assume that γ_1 is orthogonal to α and γ_2 is adjacent to α. Table 2.1 shows that $2\langle \lambda_{0,b}, \gamma\rangle/|\gamma|^2=\frac{1}{2}(1-\mu_\alpha)$, and thus (5.2) gives

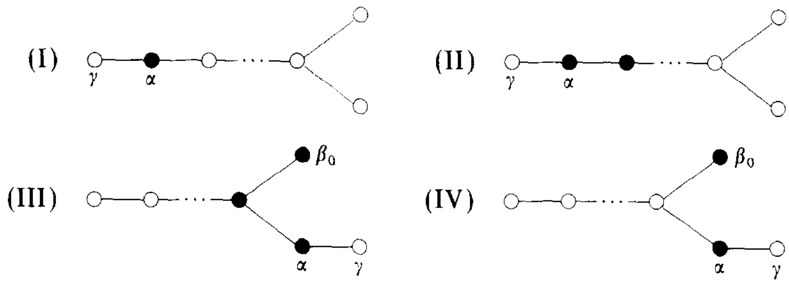
$$1=\frac{2\langle \lambda_0-\lambda_{0,b}, \gamma\rangle}{|\gamma|^2}+\frac{1}{2}(1-\mu_\alpha)+\frac{1}{2}(1-\mu_\alpha).$$

Hence $\mu_\alpha=2\langle \lambda_0-\lambda_{0,b}, \gamma\rangle/|\gamma|^2$, and μ cannot be $-\frac{1}{2}\alpha$. This is possibility (b).

Certainly (5.2) shows $2\langle \delta, \gamma\rangle/|\gamma|^2 \leqslant 3$. Thus suppose that $2\langle \delta, \gamma\rangle/|\gamma|^2=3$. Then $\gamma=\gamma_1+\gamma_2+\gamma_3$ with γ_1, γ_2, and γ_3 simple, and (5.2) shows that $2\langle \lambda_{0,b}, \gamma\rangle/|\gamma|^2=0$ and that $\mu=\frac{1}{2}\alpha$. Then it follows from Table 2.1 that γ_1, γ_2, and γ_3 are all noncompact. This is a contradiction since their sum γ is compact. Thus (a) and (b) are the only possibilities, and Lemma 5.2 is proved.

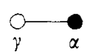
Let us return to Lemma 5.1 and the consideration of $\Lambda+\alpha+\beta_0$. Here the component of α in Δ_S is assumed to be $\mathfrak{so}(2n, 2)$, $n \geqslant 2$. Now $\alpha+\beta_0$ is orthogonal to $\Delta_K \cap \Delta_S$, and thus $\Lambda + \alpha + \beta_0$ is dominant for $\Delta_K^+ \cap \Delta_S$. Suppose it fails to be dominant for Δ_K^+. Then Lemma 5.2 produces a Δ_K^+ simple root γ of one of the two types (a) and (b).

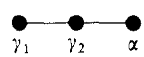
First suppose $n \geqslant 4$. If γ is of type (a), the possibilities for $\{\gamma\} \cup$ (component of α in Δ_S) initially appear to be



A little computation shows in cases (II) and (IV) that it is $-\alpha$, not $+\alpha$, that is conjugate to β_0; thus these cases are ruled out. Referring to (5.1), we see that we must have $2\langle\beta_0, \gamma\rangle/|\gamma|^2 = -1$; this rules out (III). Finally in (I) we calculate ν_0^- by the techniques of Section 4 to be $1 - \mu_\alpha$. But $\nu_{0,L}^+ = 1 + \mu_\alpha + 2(n-1)$, and thus our assumption $\nu_{0,L}^+ < \nu_0^-$ is not satisfied; this rules out (I).

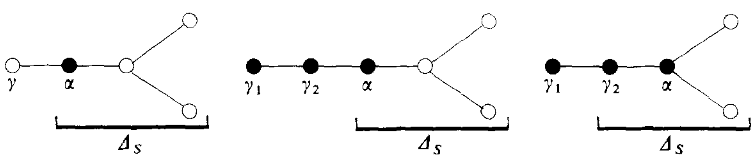
If γ is of type (b), we may assume that γ is not in Δ_S. Then the possibilities are similar to those above, except that



is replaced by


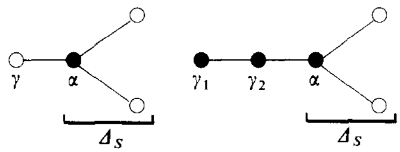
Cases (II), (III), and (IV) are ruled out for the same reasons as above. In (I), the root γ_2 may be in the same component of α in Δ_S, but γ_1 is not. If γ_2 is not in the component, the above argument applies. Otherwise there is one less root between α and the triple point, and we calculate $\nu_0^- = 3 - \mu_\alpha$ and $\nu_{0,L}^+ = 1 + \mu_\alpha + 2(n-2)$. Hence $\nu_{0,L}^+ < \nu_0^-$ is not satisfied, and (I) is ruled out.

Next suppose $n = 3$. The only real possibilities are of type (I):



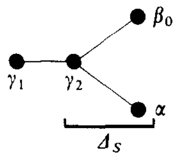
In the first case we have $\nu_0^- = 1 - \mu_\alpha$ and $\nu_{0,L}^+ = 5 + \mu_\alpha$, in the second case we have $\nu_0^- = 1 - \mu_\alpha$ and $\nu_{0,L}^+ = 5 + \mu_\alpha$, and in the third case we have $\nu_0^- = 3 - \mu_\alpha$ and $\nu_{0,L}^+ = 3 + \mu_\alpha$. Since Lemma 5.2 gives $\mu \neq -\frac{1}{2}\alpha$ in the third case, the inequality $\nu_{0,L}^+ < \nu_0^-$ fails each time, and all the cases are ruled out.

Finally suppose $n=2$; then the component of Δ_S consists of three roots in a row. Suppose α is in the middle. For α to be conjugate to β_0, the other two simple roots in Δ_S must be compact. Then the only possibilities are



In each we have $\nu_0^- = 1-\mu_\alpha$ and $\nu_{0,L}^+ = 3+\mu_\alpha$. Hence $\nu_{0,L}^+ < \nu_0^-$ fails, and these cases are ruled out.

Suppose α is at one end. Then β_0 is at the other end, and all three roots of the diagram of the component of Δ_S are noncompact. To have $\langle \gamma, \alpha \rangle = \langle \gamma, \beta_0 \rangle = -1$ as in (5.1), we must have



for a diagram. Since β_0 is a noncompact root at distance two in the special basic case, we see from Lemma 2.2 that $\mu = \frac{1}{2}\alpha$. Then $\nu_0^+ = \nu_{0,L}^+ = 2$ and $\nu_{0,L}^+ < \min(\nu_0^+, \nu_0^-)$ fails, so that this case is ruled out. This completes the proof of Lemma 5.1.

6. Validity of Cut-Offs for α Short, Double-Line Diagrams

The algebras $\mathfrak{g}$ in question are $\mathfrak{sp}(p,q)$, $\mathfrak{sp}(n,\mathbb{R})$, $\mathfrak{so}(\text{odd}, \text{even})$, split F_4, and nonsplit F_4. Nonsplit F_4 is easily handled by [2] and [1], and we shall not consider it further. The goal of this section is to obtain the lemma below for the remaining algebras, proceeding case by case. Recall that α short implies that $\Lambda+\delta^+$ and $\Lambda+\delta^-$ are Δ_K^+ dominant, hence that $(\Lambda+\alpha)^\vee = \Lambda+\delta^+$ and $(\Lambda-\alpha)^\vee = \Lambda+\delta^-$. However, it is not immediately apparent whether $\tau_{\Lambda+\delta^+}$ and $\tau_{\Lambda+\delta^-}$ occur in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$.

Lemma 6.1. *Suppose that the Dynkin diagram of Δ^+ is a double-line diagram and that α is short. Then at least one of the following things happens:*

(a) *the δ^+ subsystem has real rank one, and $\nu_0^+ \leqslant \nu_0^-$,*

(b) *the δ^- subsystem has real rank one, and $\nu_0^- \leqslant \nu_0^+$.*

Moreover, $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is not infinitesimally unitary for $c > \min(\nu_0^+, \nu_0^-)$. In addition, if the component of α in the special basic case associated to λ_0 is of type $\mathfrak{sp}(n, 1)$ with $n \geqslant 2$ and if $\mu = 0$ and α is adjacent to the long simple root, then $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is not infinitesimally unitary for

$$\min(\nu_0^+, \nu_0^-) - 2 < c < \min(\nu_0^+, \nu_0^-).$$

Proof for $\mathfrak{sp}(p, q)$. The Dynkin diagram is of type C, and the long roots are compact. Corollary 3.3 is applicable. Thus, except for the last sentence of the lemma, we can pass automatically from statements in the special basic case to statements in general. We consider the following possibilities for the special basic case Δ_S.

(I) Suppose there is a simple root γ_0 of $\Delta_{K,\perp}^+$ of the form (f) or (g) in Lemma 2.2. Possibly by reflecting in α, we may assume the form is (g). Then $\mu = -\frac{1}{2}\alpha$, and $\gamma_0 = \alpha + \gamma + \beta$ as in (4.1). We shall use the methods of Section 4 to show that $\delta^+ = \alpha + \gamma$. Then it follows that the δ^+ subsystem is of real rank one and type A and that $\nu_0^+ = 2 \leqslant \nu_0^-$. In addition, Corollary 3.3 gives $\min(\nu_0^+, \nu_0^-)$ as a cut-off for unitarity. Let ε be the neighbor of α in Δ_S other than γ (if α has two neighbors in Δ_S).

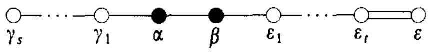
(I.1) Suppose ε is short (or nonexistent). By Lemma 2.2, the simple roots ξ of $\Delta_{K,\perp}^+$ with labels $2\langle \xi, \alpha\rangle/|\alpha|^2$ are $\alpha + \gamma + \beta$ with label $+1$, γ with label -1, $\alpha + \varepsilon$ (if ε exists) with label $+1$, and various compact Δ^+ simple roots with label 0. The only possible $\Delta_{K,\perp}^+$ simple neighbor γ' of γ in $\Delta_{K,\perp}^+$ is $\alpha + \varepsilon$ and has label $+1$. Thus the same argument as in (I.1) of Section 4 shows $\delta^+ = \gamma + \alpha$.

(I.2) Suppose ε is long. The simple roots of $\Delta_{K,\perp}^+$ with labels are $\alpha + \gamma + \beta$ with label $+1$, γ with label -1, $2\alpha + \varepsilon$ with label $+2$, and various compact Δ^+ simple roots with label 0. The only possible $\Delta_{K,\perp}^+$ simple neighbor γ' of γ in $\Delta_{K,\perp}^+$ is $2\alpha + \varepsilon$ and has label $+2 > 0$. Thus again $\delta^+ = \gamma + \alpha$.

(II) Suppose that there is no simple root of $\Delta_{K,\perp}^+$ of type (f) or (g) in Lemma 2.2. If the component of α in the special basic case is of type A, then we can appeal to (III) in Section 4. Thus we may assume that this component contains the long simple root ε of Δ.

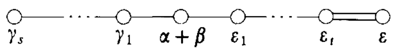
(II.1) Suppose further that the only neighbors of α are connected to α by single lines and that they are all of the same type, compact or noncompact. Possibly reflecting in α, we may assume that the neighbors are noncompact. Arguing as in (III.1) of Section 4, we find that $\delta^+ = \alpha$ and that $\nu_0^+ = 1 + \mu_\alpha \leqslant \nu_0^-$. So this case is no problem.

(II.2) Suppose that α is not as in (II.1) and that the neighbor β of α in the direction of the long root ε has $|\beta| = |\alpha|$. Possibly reflecting in α, we may assume that β is noncompact. The component of α in the special basic case is then



(6.1)

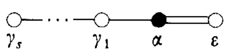
with $s \geqslant 0$ and $t \geqslant 0$. Lemma 2.2 shows that $\Delta^+_{K,\perp}$ consists of



and possible other components orthogonal to α. We readily see that $\delta^+ = \gamma_s + \cdots + \gamma_1 + \alpha$, that the δ^+ subgroup is then of type $\mathfrak{su}(s+1, 1)$, that $v_0^+ = 1 + \mu_\alpha + 2s$ (with contributions from $\gamma_j + \cdots + \gamma_1 + \alpha$, $j \geqslant 1$), and that $v_0^- \geqslant 1 - \mu_\alpha + 2(s+1) \geqslant v_0^+$ (with contributions from

$$\gamma_j + \cdots + \gamma_1 + \alpha + 2(\beta + \varepsilon_1 + \cdots + \varepsilon_t) + \varepsilon, j \geqslant 0).$$

(II.3) Suppose that α is neither as in (II.1) above nor as in (II.2). Then α is adjacent to the long simple root ε. If there is a neighbor γ_1 of α other than ε, then we may reflect in α if necessary so that γ_1 is compact. The diagram of the component of α is of the following form with $s \geqslant 0$:



(6.2)

(II.3a) Suppose $\mu = +\frac{1}{2}\alpha$. The simple roots of $\Delta^+_{K,\perp}$ with labels are γ_1 with label -1, ε with label -2, and various compact Δ^+ simple roots with label 0. Therefore $\delta^- = -\alpha$ and it follows that $v_0^- = 1 - \mu_\alpha = 0$. By Lemma 2.1 this estimate is sharp.

(II.3b) Suppose $\mu = -\frac{1}{2}\alpha$. The simple roots of $\Delta^+_{K,\perp}$ with labels are γ_1 with label -1, $2\alpha + \varepsilon$ with label $+2$, and various compact Δ^+ simple roots with label 0. Arguing as in (I.3) of Section 4, we see that $\delta^+ = \gamma_s + \cdots + \gamma_1 + \alpha$, that the δ^+ subgroup is of type $\mathfrak{su}(s+1, 1)$, that $v_0^+ = 1 + \mu_\alpha + 2s = 2s$ (with contributions from $\gamma_j + \cdots + \gamma_1 + \alpha$, $j \geqslant 1$), and that $v_0^- \geqslant 1 - \mu_\alpha + 2(s+1) = 2s + 4 \geqslant v_0^+$ (with contributions from $\gamma_j + \cdots + \gamma_1 + \alpha + \varepsilon$ for $j \geqslant 0$).

(II.3c) Suppose $\mu = 0$. The simple roots of $\Delta^+_{K,\perp}$ with labels are γ_1 with label -1, $2\alpha + \varepsilon$ with label $+2$, ε with label -2, and various compact Δ^+ simple roots with label 0. Since $s_\varepsilon s_{2\alpha+\varepsilon}(\alpha) = -\alpha$, α and $-\alpha$ are conjugate by the Weyl group of $\Delta_{K,\perp}$. Hence $\Lambda + \alpha$ and $\Lambda - \alpha$ are conjugate by the Weyl group of Δ_K. Then also $\delta^+ = \delta^-$. It is easy to see that the δ^+ subsystem is all of (6.2), and we find that $\nu_0^+ = \nu_0^- = 1 + 2(s+1)$, ν_0^+ having contributions from $\gamma_j + \cdots + \gamma_1 + \alpha$ for $j \geqslant 1$ and from $\varepsilon + \alpha$. Since the δ^+ subsystem is of real rank one and since $\Lambda + \alpha$ is conjugate to $\Lambda - \alpha$, Corollary 3.3 and Theorem 3.2 tell us that unitarity does not extend beyond $\nu_0^+ = \min(\nu_0^+, \nu_0^-)$.

Moreover, when $s \geqslant 1$, Proposition 3.5 is applicable within the special basic case Δ_S to rule out unitarity between $\nu_0^+ - 2$ and ν_0^+. Following the procedure described before the statement of Lemma 5.2, we shall apply the Vogan Signature Theorem (Theorem 3.4) to extend this conclusion from S to G. Referring to that procedure and to the statement of Proposition 3.5, we see that it is enough to prove that $(\Lambda + \delta^+ + \alpha)^{(\vee, S)}$ is Δ_K^+ dominant.

To this end, write $(\Lambda + \delta^+ + \alpha)^{(\vee, S)} = \Lambda + \delta^+ + \delta_1$. If this is not Δ_K^+ dominant, then we can proceed as in Lemma 5.2 to find a Δ_K^+ simple root γ with

$$\frac{2\langle \Lambda, \gamma\rangle}{|\gamma|^2} = 1, \qquad \frac{2\langle \delta^+, \gamma\rangle}{|\gamma|^2} = -1, \quad \text{and} \quad \frac{2\langle \delta_1, \gamma\rangle}{|\gamma|^2} = -1. \tag{6.3}$$

Now γ has to be short since ε and $2\alpha + \varepsilon$ are in $\Delta_{K,\perp}$. In standard notation for the Dynkin diagram of Δ of type C, let $\gamma = e_i \pm e_j$ with $i < j$. We can check that $\langle \delta^+, \varepsilon\rangle > 0$ while $\langle \delta_1, \varepsilon\rangle < 0$. Thus (6.3) shows that j is not the last index of the diagram (the one corresponding to ε). Since the only index common to δ and δ_1 is the last index of the diagram, it follows that either $\langle \delta^+, e_i\rangle \neq 0$ and $\langle \delta_1, e_j\rangle \neq 0$ or else $\langle \delta^+, e_j\rangle \neq 0$ and $\langle \delta_1, e_i\rangle \neq 0$ in order for (6.3) to hold. But then we see that γ lies in Δ_S, contradiction. We conclude that $(\Lambda + \delta^+ + \alpha)^{(\vee, S)}$ is Δ_K^+ dominant, as required.

Proof for $\mathfrak{sp}(n, \mathbb{R})$. The Dynkin diagram is of type C, and the long roots are noncompact. Referring to Lemma 2.2, we see that no simple root of $\Delta_{K,\perp}^+$ requires the long Δ^+ simple root for its expansion. Thus the special basic case Δ_S is contained in the diagram Δ_L containing the short Δ^+ simple roots.

Let us write $\nu_{0,L}^+$ and $\nu_{0,L}^-$ for the ν_0^+ and ν_0^- of Δ_L. Lemma 4.1 says that either the δ^+ group in Δ_L is of real rank one and $\nu_{0,L}^+ \leqslant \nu_{0,L}^-$ or the δ^- group in Δ_L is of real rank one and $\nu_{0,L}^- \leqslant \nu_{0,L}^+$. Moreover, in either case, there is no unitarity in L beyond $\min(\nu_{0,L}^+, \nu_{0,L}^+)$. Since α is short and $\Delta_{K,\perp} \subseteq \Delta_L$, we have $(\Lambda + \alpha)^{(\vee, L)} = (\Lambda + \alpha)^\vee$ and $(\Lambda - \alpha)^{(\vee, L)} = (\Lambda - \alpha)^\vee$. Thus the Vogan Signature Theorem (Theorem 3.4) says that there is no unitarity beyond $\min(\nu_{0,L}^+, \nu_{0,L}^-)$. Again since $\Delta_{K,\perp} \subseteq \Delta_L$, we have $\nu_0^+ = \nu_{0,L}^+$ and $\nu_0^- = \nu_{0,L}^-$. Therefore all the assertions in Lemma 6.1 follow for this group.

Proof for $\mathfrak{so}(odd, even)$. The Dynkin diagram is of type B, and α is the unique short simple root, which we denote e_n. We consider the following possibilities.

(I) Suppose $\mu = +\frac{1}{2}\alpha$. Then we claim that $\delta^- = -\alpha$. It follows that $\nu_0^- = 0$, and this we know is automatically a sharp cut-off for unitarity by Lemma 2.1.

To show that $\delta^- = -\alpha$, it is enough to show that $\langle -\alpha, \gamma' \rangle \geqslant 0$ for every simple root γ' of $\Delta_{K,\perp}^+$. From Lemma 2.2, the only γ' for which $\langle -\alpha, \gamma' \rangle$ is nonzero is of type (c) or (f), necessarily then $e_{n-1} - e_n$ or $e_{n-2} - e_n$. These roots have inner product $\geqslant 0$ with $-\alpha = -e_n$. Hence $\delta^- = -\alpha$.

(II) Suppose $\mu = -\frac{1}{2}\alpha$. Then similarly $\delta^+ = \alpha$ and $\nu_0^+ = 0$, which is a sharp cut-off for unitarity.

(III) Suppose $\mu = 0$. The simple roots γ' of $\Delta_{K,\perp}^+$ consist of various compact Δ^+ simple roots and also possibly $e_{n-1} \pm e_n$, by Lemma 2.2. None of these roots is short. Referring to Proposition 3.1, we see that $(\Lambda + \alpha)^\vee$ and $(\Lambda - \alpha)^\vee$ necessarily occur in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$; thus (a) holds in Theorems 3.2 and 3.2′.

Let t be the smallest index such that $e_j - e_{j+1}$ is in $\Delta_{K,\perp}^+$ for all j with $t \leqslant j < n$. Then it is easy to see that $\delta^+ = e_t$ and that

$$\delta^- = \begin{cases} -e_n & \text{if } t = n \\ e_t & \text{if } t < n. \end{cases}$$

In either case, the δ^+ and δ^- subsystems are both of type $\mathfrak{so}(2(n-t+1), 1)$, hence of real rank one. Thus (c) holds in Theorems 3.2 and 3.2′. If $t = n$, then the δ^+ and δ^- subsystems are of type A_1, and (b) holds in Theorems 3.2 and 3.2′. If $t < n$, then $\delta^+ = \delta^-$ and (b′) holds. In either case the theorems apply and show that unitarity is cut off at $\min(\nu_0^+, \nu_0^-) = \nu_0^+ = \nu_0^-$.

Proof for split F_4. No new ideas are needed, and the proof is omitted.

7. Validity of Cut-Offs for α Long, Double-Line Diagrams

The algebras $\mathfrak{g}$ in question are $\mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{so}(\text{odd}, \text{even})$, and split F_4. The goal of this section is to prove Lemma 7.1 for $\mathfrak{sp}(n, \mathbb{R})$ and $\mathfrak{so}(\text{odd}, \text{even})$.

Lemma 7.1. *Suppose that the Dynkin diagram of Δ^+ is a classical double-line diagram and that α is long. Then at least one of the following things happens*:

(a) $\nu_0^+ \leqslant \nu_0^-$, *and hypotheses* (a), (b) *and* (c′) *are satisfied in Theorem* 3.2

(b) $\nu_0^- \leqslant \nu_0^+$, *and hypotheses* (a), (b), *and* (*c'*) *are satisfied in Theorem* 3.2'.

Consequently $J(MAN, \sigma, \frac{1}{2}c\alpha)$ *is not infinitesimally unitary for* $c > \min(\nu_0^+, \nu_0^-)$. *Moreover,*

(i) *if the basic case associated to* λ_0 *satisfies the conditions of* (iii) *in Theorem* 1.1 (*which refers to* $\mathfrak{so}(2n, 3)$) *and if* ζ *is the root defined there, then* $J(MAN, \sigma, \frac{1}{2}c\alpha)$ *is not infinitesimally unitary for*

$$\min(\nu_0^+, \nu_0^- - 1) < c < \min(\nu_0^+, \nu_0^-) \qquad \text{if } \zeta \text{ is noncompact and } \nu_0^- \geqslant 2,$$

$$\min(\nu_0^+ - 1, \nu_0^-) < c < \min(\nu_0^+, \nu_0^-) \qquad \text{if } \zeta \text{ is compact or } 0 \text{ and } \nu_0^+ \geqslant 2.$$

(ii) *if the special basic case associated to* λ_0 *satisfies the conditions of* (v) *in Theorem* 1.1 (*which refers to* $\mathfrak{so}(2n+1, 2)$) *and if* $\nu_{0,L}^{\pm}$ *and* β_0 *are as defined there, then* $J(MAN, \sigma, \frac{1}{2}c\alpha)$ *is not infinitesimally unitary for*

$$c > \min(\nu_{0,L}^+ + 1, \nu_0^-) \qquad \text{if } \beta_0 \text{ conjugate to } \alpha \text{ via } K \text{ in } \mathfrak{so}(2n+1, 2),$$

$$c > \min(\nu_0^+, \nu_{0,L}^- + 1) \qquad \text{if } \beta_0 \text{ conjugate to } -\alpha \text{ via } K \text{ in } \mathfrak{so}(2n+1, 2).$$

Before coming to the individual algebras $\mathfrak{g}$ in question, we give a general result helpful in computing ν_0^+ and ν_0^- and in checking the hypotheses of Theorems 3.2 and 3.2'.

LEMMA 7.2. *Suppose that the Dynkin diagram of* Δ^+ *is a double-line diagram and that* α *is long. Whether or not* $\Lambda + \delta^+$ *is* Δ_K^+ *dominant, every root* β *contributing to the term*

$$2\#\{\beta \in \Delta_n^+ \mid \beta - \alpha \in \Delta \quad \text{and} \quad \langle \Lambda, \beta - \alpha \rangle = 0\} \tag{7.1}$$

lies in the δ^+ *subgroup. If* $\Lambda + \delta^+$ *is* Δ_K^+ *dominant, then* $\tau_{\Lambda+\delta^+}$ *occurs in* $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$ *and the exceptional term*

$$\#\{\beta \in \Delta_n^+ \mid \beta - \alpha \in \Delta, |\beta| < |\alpha|, 2\langle \Lambda, \beta - \alpha \rangle / |\beta - \alpha|^2 = 1\} \tag{7.2}$$

of ν_0^+ *is* 0. *Conversely if the exceptional term is* 0, *then* $\Lambda + \delta^+$ *is* Δ_K^+ *dominant.*

Remark. An analogous statement is valid for $\Lambda + \delta^-$ and ν_0^-.

Proof. We can regard α as an extreme weight of a compact group built from $\Delta_{K,\perp}$, and then δ^+ is the highest weight. Suppose β contributes to (7.1). If β is long, then $s_{\beta-\alpha}(\alpha) = \beta$. So β is another extreme weight and

$$\delta^+ - \beta = \sum_{\gamma \in \Delta_{K,\perp}^+} n_\gamma \gamma.$$

Therefore β is in the δ^+ subgroup. On the other hand, if β is short, then $s_{\beta-\alpha}(\alpha)=2\beta-\alpha$. So $2\beta-\alpha$ is another extreme weight and

$$\delta^+ - 2\beta + \alpha = \sum_{\gamma\in\Delta^+_{K,\perp}} n_\gamma \gamma.$$

Therefore β is in the δ^+ subgroup in this case, too.

Now suppose that $\Lambda+\delta^+$ is Δ^+_K dominant. Since δ^+ is long, it follows immediately from Proposition 3.1 that $\tau_{\Lambda+\delta^+}$ occurs in $\tau_\Lambda\otimes\mathfrak{p}^{\mathbb{C}}$. Now, arguing by contradiction, suppose β contributes to (7.2). Choose w in the Weyl group of $\Delta_{K,\perp}$ with $w\delta^+=\alpha$, and put $\gamma=w^{-1}(\beta-\alpha)$. Then

$$\frac{2\langle\Lambda,\gamma\rangle}{|\gamma|^2}=\frac{2\langle w\Lambda,\beta-\alpha\rangle}{|\beta-\alpha|^2}=\frac{2\langle\Lambda,\beta-\alpha\rangle}{|\beta-\alpha|^2}=+1$$

shows γ is positive, while

$$\frac{2\langle\Lambda+\delta^+,\gamma\rangle}{|\gamma|^2}=\frac{2\langle w\Lambda+w\delta^+,\beta-\alpha\rangle}{|\beta-\alpha|^2}=\frac{2\langle\Lambda+\alpha,\beta-\alpha\rangle}{|\beta-\alpha|^2}=+1-2<0$$

contradicts Δ^+_K dominance of $\Lambda+\delta^+$. This contradiction shows that (7.2) is 0.

Conversely suppose (7.2) is 0. Arguing by contradiction, suppose $\Lambda+\delta^+$ is not Δ^+_K dominant. Then we can find γ simple for Δ^+_K with $2\langle\Lambda,\gamma\rangle/|\gamma|^2=+1$ and $2\langle\delta^+,\gamma\rangle/|\gamma|^2=-2$. With w chosen as above, we find $2\langle\Lambda,w\gamma\rangle/|w\gamma|^2=+1$, so that $w\gamma$ is positive, and $2\langle\alpha,w\gamma\rangle/|w\gamma|^2=-2$, so that $\beta=\alpha+w\gamma$ is a root. Then β is in Δ^+_n and contributes to (7.2), contradiction.

Proof of Lemma 7.1 *for* $\mathfrak{sp}(n,\mathbb{R})$. The Dynkin diagram is of type C with the long roots noncompact. The root α is the unique long simple root, which we denote $2e_n$. Possibly reflecting in α, we may assume that the adjacent simple root $\gamma_{n-1}=e_{n-1}-e_n$ is compact.

In checking the hypotheses of Theorem 3.2 or 3.2′, let us note that (b) is automatic. For example, $\Lambda-\alpha$ cannot be a weight of $(\Lambda+\alpha)^\vee$ since the difference 2α of the weights $\Lambda-\alpha$ and $\Lambda+\alpha$ is not the sum of compact roots in this group.

For this group we shall use Lemma 7.2 (or its reflection in α) only when $\delta^+=\alpha$ or $\delta^-=-\alpha$. The lemma implies that if $\delta^+=\alpha$ and if $\Lambda+\alpha$ is Δ^+_K dominant, then (a) and (c) hold in Theorem 3.2 and $\nu_0^+=1+\mu_\alpha$. (Hypothesis (c) implies hypothesis (c′).)

To check on the Δ^+_K dominance, we shall need to know what Δ^+_K simple roots are nonorthogonal to α. In the Dynkin diagram of Δ^+, let there be k compact roots. Then there are $n-k$ noncompact roots, and we can use these to form $n-k-1$ obvious compact Δ_K^+ simple roots (each one the sum of a connected segment of Δ^+ simple roots containing noncompact roots at the ends and only there). The result is $k+(n-k-1)=n-1$ compact Δ_K^+ simple roots. Since we are working with $\mathfrak{sp}(n, \mathbb{R})$, we know that there are no other Δ_K^+ simple roots. Therefore the only Δ_K^+ simple roots that are nonorthogonal to α are

$$e_t + e_n \qquad \text{and} \qquad e_{n-1} - e_n,$$

where t is chosen so that the only noncompact root among $e_t - e_{t+1}$, $e_{t+1} - e_{t+2}, ..., e_{n-1} - e_n$ is $e_t - e_{t+1}$. We have $n - t \geqslant 2$. (Note: If no short Δ^+ simple root is noncompact, then $e_{n-1} - e_n$ is the only Δ_K^+ simple root nonorthogonal to α.)

Suppose we know that $\delta^+ = \alpha$. For $\Lambda + \alpha$ to fail to be Δ_K^+ dominant, we know from the beginning of Section 2 that there must be a Δ_K^+ simple root γ with $2\langle \Lambda, \gamma \rangle / |\gamma|^2 = 1$ and $2\langle \alpha, \gamma \rangle / |\gamma|^2 = -2$. Thus the previous paragraph forces

$$\frac{2\langle \Lambda, \gamma_{n-1} \rangle}{|\gamma_{n-1}|^2} = 1. \tag{7.3}$$

Similarly if $\delta^- = -\alpha$, then $\Lambda - \alpha$ is Δ_K^+ dominant unless

$$\frac{2\langle \Lambda, e_t + e_n \rangle}{|e_t + e_n|^2} = 1 \tag{7.4}$$

(with t existing). We now divide matters into cases.

(I) Suppose $\mu = +\frac{1}{2}\alpha$. We claim that $\delta^- = -\alpha$ and $\Lambda - \alpha$ is Δ_K^+ dominant, so that $v_0^- = 0$. In fact, the simple roots ξ of $\Delta_{K,\perp}^+$ with labels $2\langle \xi, \alpha \rangle / |\alpha|^2$ are a possible γ_{n-1} with label -1 and various compact Δ^+ simple roots with label 0, by Lemma 2.2. Hence $-\alpha$ is $\Delta_{K,\perp}^+$ dominant and $\delta^- = -\alpha$. For $\Lambda - \alpha$ to fail to be Δ_K^+ dominant, (7.4) must hold. Put $\gamma = e_t + e_n$. Then (2.3) gives

$$\begin{aligned} 1 &= \frac{2\langle \Lambda, \gamma \rangle}{|\gamma|^2} \\ &= \frac{2\langle \lambda_0 - \lambda_{0,b}, \gamma \rangle}{|\gamma|^2} + \frac{2\langle \lambda_{0,b}, \gamma \rangle}{|\gamma|^2} \\ &\quad + \left(\frac{2\langle \delta, \gamma \rangle}{|\gamma|^2} - 2 \right) - \frac{1}{2}(1 - \mu_\alpha) \frac{2\langle \alpha, \gamma \rangle}{|\gamma|^2} \\ &\geqslant \frac{2\langle \lambda_0 - \lambda_{0,b}, \gamma \rangle}{|\gamma|^2} + \frac{2\langle \lambda_{0,b}, \gamma \rangle}{|\gamma|^2} + (n-t) - (1 - \mu_\alpha). \end{aligned} \tag{7.5}$$

Since $n-t\geqslant 2$ and $\mu_\alpha=1$, this equation contradicts the Δ^+ dominance of λ_0. Our assertions follow.

(II) Suppose $\mu=0$. We claim that $\delta^-=-\alpha$ and $\Lambda-\alpha$ is Δ_K^+ dominant, so that Lemma 7.2 gives $\nu_0^-=1\leqslant\nu_0^+$ and shows that Theorem 3.2′ applies. The argument starts as in (I), and we come to (7.5). Since $\mu=0$, we are led to conclude that $t=n-2$ and that $e_{n-2}-e_{n-1}$ and $\gamma_{n-1}=e_{n-1}-e_n$ are orthogonal to λ_0. Then

$$e_{n-2}+e_{n-1}=(e_{n-2}-e_{n-1})+2\gamma_{n-1}+\alpha$$

is a compact root orthogonal to λ_0 and to α, in contradiction to non-degeneracy. Our assertions follow.

(III) Suppose $\mu=-\frac{1}{2}\alpha$. From Lemma 2.2, the only possible $\Delta_{K,\perp}^+$ simple root having nonzero inner product with α is $e_{n-2}+e_n$. Therefore $\delta^+=\alpha$, and also $\delta^-=-\alpha$ if $e_{n-2}-e_{n-1}$ is compact. In all cases, (2.3) and a little computation give

$$\frac{2\langle\Lambda,\gamma_{n-1}\rangle}{|\gamma_{n-1}|^2}=\frac{2\langle\lambda_0-\lambda_{0,\mathfrak{b}},\gamma_{n-1}\rangle}{|\gamma_{n-1}|^2}+1. \tag{7.6}$$

(III.1) Suppose $\gamma_{n-1}=e_{n-1}-e_n$ is not basic. We claim that $\Lambda+\delta^+=\Lambda+\alpha$ is Δ_K^+ dominant, so that $\nu_0^+=0$. In fact, failure of Δ_K^+ dominance means that (7.3) holds, and this contradicts (7.6), since γ_{n-1} is assumed not basic.

(III.2) Suppose γ_{n-1} is basic but $e_{n-2}-e_{n-1}$ either does not exist or is not basic. This is the most difficult case. We shall show that $\nu_0^+=1\leqslant\nu_0^-$ and $J(MAN,\sigma,\frac{1}{2}c\alpha)$ is not infinitesimally unitary for $c>\nu_0^+$.

Since $\delta^+=\alpha$, Lemma 7.2 shows that the main term (7.1) of ν_0^+ is 0. Formula (7.6) shows that $\beta=e_{n-1}+e_n$ contributes to the exceptional term (7.2) for ν_0^+. If $\beta'=e_j+e_n$ contributes also, then it follows that e_j-e_{n-1} is in $\Delta_{K,\perp}^+$. Hence there is a $\Delta_{K,\perp}$ simple root of the form $e_{j'}-e_{n-1}$. However, Lemma 2.2 shows that the only candidate for such a root is $e_{n-2}-e_{n-1}$, which cannot be in $\Delta_{K,\perp}$ since it is not basic. Thus e_j-e_{n-1} cannot be in $\Delta_{K,\perp}^+$, and there is no such β'. We conclude that $\nu_0^+=1$, and we have $2\leqslant\nu_0^-$ since $\mu=-\frac{1}{2}\alpha$.

We are left with showing that the hypotheses of Theorem 3.2 are satisfied; we prove (a) and (c′). Referring to the beginning of Section 3 and noting from (7.3) that γ_{n-1} is a Δ_K^+ simple root with $2\langle\Lambda,\gamma_{n-1}\rangle/|\gamma_{n-1}|^2=+1$ and $2\langle\alpha,\gamma_{n-1}\rangle/|\gamma_{n-1}|^2=-2$, we see that $(\Lambda+\alpha)^\vee=(\Lambda+\alpha+\gamma_{n-1})^\vee=(\Lambda+e_{n-1}+e_n)^\vee$. Since $e_{n-1}+e_n$ is $\Delta_{K,\perp}^+$ dominant, we conclude that $(\Lambda+\alpha)^\vee=\Lambda+e_{n-1}+e_n$. For $\tau_{(\Lambda+\alpha)^\vee}$ to fail to occur in $\tau_\Lambda\otimes\mathfrak{p}^{\mathbb{C}}$, we need a Δ_K^+ simple root γ such that γ is orthogonal but not strongly orthogonal to $e_{n-1}+e_n$ and such that $\langle \Lambda, \gamma \rangle = 0$. The first condition forces $\gamma = \gamma_{n-1}$. However, $\langle \Lambda, \gamma_{n-1} \rangle$ is not 0, by (7.6), and there is no such γ. By Proposition 3.1, (a) holds in Theorem 3.2.

Let us check (c′). We seek solutions to

$$e_{n-1}+e_n = (\Lambda+\alpha)^\vee - \Lambda = c\alpha + \sum_{\substack{\beta \perp 2e_n \\ \beta \in \Delta_n^+}} n_\beta \beta + \sum_{\gamma \in \Delta_K^+} k_\gamma \gamma.$$

The only roots β and γ that can contribute are $ae_{n-1}+be_n$, with each $a \geqslant 0$ and with the sum of the a's equal to 1. No such β's can contribute, and thus (c′) holds. Since we already know (b), the hypotheses of Theorem 3.2 are satisfied.

(III.3) Suppose γ_{n-1} and $e_{n-2}-e_{n-1}$ are both basic. Then $e_{n-2}-e_{n-1}$ must be compact (to avoid degeneracy with respect to $e_{n-2}+e_{n-1}$). Thus $\beta = e_{n-1}+e_n$ and $\beta = e_{n-2}+e_n$ both contribute to the exceptional term (7.2) of v_0^+, and so $v_0^+ \geqslant 2$. We claim that $\Lambda+\delta^- = \Lambda - \alpha$ is Δ_K^+ dominant, so that Lemma 7.2 gives $v_0^- = 2 \leqslant v_0^+$ and shows that Theorem 3.2′ applies.

For $\Lambda - \alpha$ to fail to be Δ_K^+ dominant, we must have (7.4) and hence (7.5). Here $t \leqslant n-3$ because $e_t - e_{t+1}$ is noncompact. Since $\mu_\alpha = -1$, we conclude from (7.5) that

$$0 = \langle \lambda_{0,b}, e_t + e_n \rangle \geqslant \langle \lambda_{0,b}, e_{n-2} - e_{n-1} \rangle > 0,$$

contradiction. Thus $\Lambda - \alpha$ is Δ_K^+ dominant.

Proof of Lemma 7.1 *for* $\mathfrak{so}$(*odd, even*). The Dynkin diagram is of type B. We use standard notation for the simple roots, taking e_n to be the short simple root. Let α be $e_j - e_{j+1}$. Possibly reflecting in α, we may assume that the next simple root after α ($e_{j+1}-e_{j+2}$ or e_n) is noncompact.

Since $e_j - e_{j+1}$ is noncompact, one of e_j and e_{j+1} is compact. Let e_k be the short root with the largest index such that e_k is compact. Then $k \geqslant j$. The root e_k is Δ_K^+ simple. Moreover, $\mathfrak{k} = \mathfrak{so}(\text{odd}) \oplus \mathfrak{so}(\text{even})$, and hence there is no other short Δ_K^+ simple root.

(I) Suppose the exceptional term (7.2) of v_0^+ or v_0^- is not 0. We shall classify the situations where this happens. Lemma 7.2 says that it is necessary and sufficient that $\Lambda+\delta^+$ or $\Lambda+\delta^-$, respectively, fails to be Δ_K^+ dominant. The remarks at the beginning of Section 3 then say that $2\langle \Lambda, e_k \rangle / |e_k|^2 = 1$ and that $\langle \delta^+, e_k \rangle < 0$ or $\langle \delta^-, e_k \rangle < 0$, respectively.

If the exceptional term of v_0^+ is not 0, then there exists a short noncompact root β with $\beta - \alpha$ in Δ^+. The only possibility is $\beta = e_j$, and then $\beta - \alpha = e_{j+1}$ is compact with $2\langle \Lambda, e_{j+1} \rangle / |e_{j+1}|^2 = +1$. Consequently

$e_{j+1}-e_k$ is in $\Delta^+_{K,\perp}\cup\{0\}$. Moreover, the exceptional term for v_0^+ is no more than one.

Similarly if the exceptional term of v_0^- is not 0, then it is one and the root e_j-e_k is in $\Delta^+_{K,\perp}\cup\{0\}$. Since e_j-e_k and $e_{j+1}-e_k$ cannot both be in $\Delta_K\cup\{0\}$, the exceptional term for only one of v_0^+ and v_0^- can be nonzero.

Conversely if $2\langle\Lambda, e_k\rangle/|e_k|^2=1$ and $e_{j+1}-e_k$ is in $\Delta^+_{K,\perp}\cup\{0\}$, then $\beta=e_j$ exhibits the exceptional term of v_0^+ as nonzero. If $2\langle\Lambda, e_k\rangle/|e_k|^2=1$ and e_j-e_k is in $\Delta^+_{K,\perp}\cup\{0\}$, then $\beta=e_{j+1}$ exhibits the exceptional term of v_0^- as nonzero.

We now limit the possibilities for k by applying (2.3) with $\gamma=e_k$. We shall show that $k=n-1$ or $k=n$ and that $k=n-1$ implies $j=n-2$ or $j=n-1$. Substituting for δ in (2.3), we have

$$1=\frac{2\langle\lambda_0-\lambda_{0,b}, e_k\rangle}{|e_k|^2}+\frac{2\langle\lambda_{0,b}, e_k\rangle}{|e_k|^2}$$
$$+(2(n-k)-1)-\frac{1}{2}(1-\mu_\alpha)\frac{2\langle\alpha, e_k\rangle}{|e_k|^2}. \tag{7.7}$$

Suppose $k\leqslant n-2$. The third term on the right is $\geqslant 3$ and can be offset by the last term only if $k=j+1$. Then α will be orthogonal to $e_{n-1}-e_n$. However, e_n and e_{n-1} noncompact implies $e_{n-1}-e_n$ compact, and thus the second term on the right will be at least 2. We conclude $k\geqslant n-1$.

Suppose $k=n-1$, so that the third term on the right of (7.7) is 1. If $j<n-2$, then the fourth term is 0; hence the sum $2\langle\lambda_0, e_{n-1}\rangle/|e_{n-1}|^2$ of the first two terms is 0, in contradiction to nondegeneracy.

If $k=n-1$ and $j=n-2$, we can substitute from Table 2.1 to see that (7.7) holds if and only if $\mu=+\frac{1}{2}\alpha$ and both $e_{n-1}-e_n$ and e_n are basic. In this case the exceptional term of v_0^+ is nonzero.

If $k=n-1$ and $j=n-1$, then Table 2.1 shows that (7.7) becomes

$$1=\frac{2\langle\lambda_0-\lambda_{0,b}, e_n\rangle}{|e_n|^2}+\left|\mu_\alpha-\frac{1}{2}\right|-\frac{1}{2}+1-(1-\mu_\alpha).$$

The necessary and sufficient conditions, classified by μ, are as follows:

$$\begin{aligned} &\mu=\tfrac{1}{2}\alpha \quad &&\text{and} \quad &&e_n \text{ basic} \\ &\mu\neq\tfrac{1}{2}\alpha \quad &&\text{and} \quad &&e_n \text{ one step removed from basic.}\end{aligned} \tag{7.8}$$

In these cases the exceptional term of v_0^- is nonzero.

Now suppose $k=n$. Our normalization of the root next to α then forces $j\leqslant n-2$. We know that the exceptional term of v_0^+ is nonzero if and only if $2\langle\Lambda, e_n\rangle/|e_n|^2=1$ and $e_{j+1}-e_n$ is in $\Delta^+_{K,\perp}$. Since $e_{j+1}-e_{j+2}$ is noncompact, the latter condition happens if and only if $\mu=\frac{1}{2}\alpha$, $e_{j+2}-e_{j+3}$ is noncompact, and $e_j - e_{j+1}, e_{j+1} - e_{j+2}, \ldots, e_{n-1} - e_n$ all lie in the special basic case. Similarly the exceptional term of ν_0^- is nonzero if and only if $2\langle \Lambda, e_n \rangle / |e_n|^2 = 1$ and $e_j - e_n$ is in $\Delta_{K,\perp}^+$. Since $e_{j+1} - e_{j+2}$ is noncompact, the latter condition happens if and only if all subsequent roots are compact and $e_j - e_{j+1}, e_{j+1} - e_{j+2}, \ldots, e_{n-1} - e_n$ all lie in the special basic case.

Next we show that Theorem 3.2 or 3.2′ is applicable in each of these situations to $\min(\nu_0^+, \nu_0^-)$.

(I.1) $k = n-1$, $j = n-2$, $\mu = \frac{1}{2}\alpha$, $e_{n-1} - e_n$ and e_n both basic. The exceptional term of ν_0^+ is 1. The root $e_{j-1} - e_j$ (if it exists) cannot be noncompact basic since otherwise e_{j-1} would contradict nondegeneracy. It follows that the $\Delta_{K,\perp}^+$ simple roots γ' with labels $2\langle \gamma', \alpha \rangle / |\alpha|^2$ are $\alpha + \beta$ (for $\beta = e_{n-1} - e_n$) with label $+1$, as well as some roots orthogonal to β with label $\leqslant 0$. Consequently we have $\delta^- = \beta$ and $\nu_0^- = 2 < 3 \leqslant \nu_0^+$. Lemma 7.2 says that $\Lambda + \delta^-$ is Δ_K^+ dominant and that $\tau_{\Lambda+\delta^-}$ occurs in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$. Moreover, the δ^- subsystem is of type $\mathfrak{su}(2,1)$, and the proof of Corollary 3.3 shows that (b) and (c) hold in Theorem 3.2′. So assertion (b) in Lemma 7.1 is proved.

However, in this situation the conditions of (iii) in Theorem 1.1 are satisfied, and Lemma 7.1 thus asserts more. The root ζ is $e_{n-1} - e_n$ and is noncompact. Since $\nu_0^- = 2 < \nu_0^+$, we are to rule out unitarity strictly between $\nu_0^- - 1 = 1$ and $\nu_0^- = 2$. For the system Δ_L generated by α, ζ, and e_n, Proposition 3.8 does exactly this, by means of the $(K \cap L)$-type $\Lambda_L + e_{n-1}$. To extend the conclusion to G, it is enough, by Theorem 3.4, to show that $\Lambda + e_{n-1}$ is Δ_K^+ dominant. Let γ be Δ_K^+ simple. If γ is in $\Delta_{K,\perp}^+$ and $\langle e_{n-1}, \gamma \rangle < 0$, then Lemma 2.2 shows that $e_{n-3} - e_{n-2}$ exists and is noncompact basic, in contradiction to nondegeneracy. Hence $\langle \Lambda + e_{n-1}, \gamma \rangle \geqslant 0$ for γ in $\Delta_{K,\perp}^+$. For all other Δ_K^+ simple γ, we have $2\langle \Lambda, \gamma \rangle / |\gamma|^2 \geqslant 1$ and $2\langle e_{n-1}, \gamma \rangle / |\gamma|^2 \geqslant -1$. Hence $\langle \Lambda + e_{n-1}, \gamma \rangle \geqslant 0$. Consequently (i) holds in Lemma 7.1.

(I.2) $k = n-1$, $j = n-1$, e_n as in (7.8). The exceptional term of ν_0^- is 1. From Lemma 2.2, no $\Delta_{K,\perp}^+$ simple root involves e_n in its expansion in terms of single roots.

(I.2a) Suppose $e_{n-2} - e_{n-1}$, if it exists, is not compact basic. Then every $\Delta_{K,\perp}^+$ simple root not of type (f) in Lemma 2.2 has label $2\langle \gamma', \alpha \rangle / |\alpha|^2$ nonnegative. Hence $\mu \neq \frac{1}{2}\alpha$ implies $\delta^+ = \alpha$; in these cases $\nu_0^+ = 1 + \mu_\alpha < 2 \leqslant \nu_0^-$, and the argument in (I.1) shows that (a), (b), and (c) hold in Theorem 3.2.

Suppose $\mu = \frac{1}{2}\alpha$. Then we are assuming e_n is basic; so there cannot be a $\Delta_{K,\perp}^+$ simple root of type (f) because otherwise e_{n-3} would exhibit a degeneracy. Thus every $\Delta_{K,\perp}^+$ simple root has label nonnegative, and we have $\delta^+ = \alpha$. That is, $\nu_0^+ = 1 + \mu_\alpha = 2$, and the argument in (I.1) shows that (a), (b), and (c) hold in Theorem 3.2. If $e_{n-2}-e_{n-1}$ does not exist or is not noncompact basic, then every $\Delta^+_{K,\perp}$ root has label 0, and we have $\delta^- = -\alpha$. By Lemma 7.2, $\nu_0^- = (1-\mu_\alpha)+2\cdot 0+1=1$. Let us show in this situation that $(\Lambda-\alpha)^\vee$ satisfies (a), (b), and (c′) in Theorem 3.2, so that ν_0^- gives a sharper cut-off for unitarity. From the first part of Section 3, we have $(\Lambda-\alpha)^\vee = (\Lambda-\alpha+e_{n-1})^\vee = (\Lambda+e_n)^\vee$, and this is $\Lambda+e_n$ since e_n is $\Delta^+_{K,\perp}$ dominant. Since $\langle \Lambda, e_{n-1}\rangle \neq 0$, Proposition 3.1 shows that (a) holds. For (b), we note that $\Lambda+\alpha$ cannot be a weight unless $(\Lambda+e_n)-(\Lambda+\alpha) = -e_{n-1}+2e_n$ is a sum of members of Δ^+_K, which it is not. For (c′), we are to examine solutions of

$$e_n = (\Lambda+e_n)-\Lambda = c\alpha + \sum_{\beta\in\Delta^+_{-,n}} n_\beta\beta + \sum_{\gamma\in\Delta^+_K} k_\gamma\gamma.$$

The only β that could possibly contribute is $\beta = e_{n-1}+e_n$, and we would need

$$e_n = c(e_{n-1}-e_n)+a(e_{n-1}+e_n)+be_{n-1}$$

with $a \geqslant 0$, $b\geqslant 0$. The coefficient of e_n forces $c=a-1$, and then the coefficient of e_{n-1} forces $2a+b=1$. Hence $a=0$ and $\sum n_\beta\beta=0$. In short, (c′) holds. Thus $\nu_0^-=1$ is a cut-off for unitarity when $e_{n-2}-e_{n-1}$ does not exist or is not noncompact basic.

If $e_{n-2}-e_{n-1}$ is noncompact basic, then $\beta=e_{n-2}-e_{n-1}$ contributes to ν_0^- and we have $\nu_0^- \geqslant 3 > 2 = \nu_0^+$. So the assertion (a) in Lemma 7.1 is already proved. However, in this case the conditions of (iii) in Theorem 1.1 are satisfied, and Lemma 7.1 asserts more. The sum ζ is 0 here. Since $\nu_0^+ = 2 < \nu_0^-$, we are to rule out unitarity strictly between $\nu_0^+-1=1$ and $\nu_0^+=2$. This is easy; renumbering our indices in the order $n-3$, $n-1$, n, $n-2$ and reflecting in α, we do not change Δ^+_- and we reduce matters to (I.1), where we know there is no unitarity between 1 and 2. Consequently (i) holds in Lemma 7.1.

(7.1b) Suppose $e_{n-2}-e_{n-1}$ is compact basic. This root is simple for $\Delta^+_{K,\perp}$ and has label -1. If $\mu\neq -\frac{1}{2}\alpha$, then all other $\Delta^+_{K,\perp}$ simple roots have label 0. In this case all labels are $\leqslant 0$, and hence $\delta^- = -\alpha$ and $\nu_0^- = (1-\mu_\alpha)+2\cdot 0+1 = 2-\mu_\alpha$. Since $\beta=e_{n-2}-e_n$ contributes to ν_0^+, we have $\nu_0^-\leqslant 2\leqslant \nu_0^+$. To see that hypotheses (a), (b), and (c′) of Theorem 3.2′ are satisfied, we apply the argument in (I.2a) word for word.

Now suppose $\mu = -\frac{1}{2}\alpha$. If there is a $\Delta^+_{K,\perp}$ simple root of type (g), then it must be $e_{n-3}-e_n$ and we obtain $\delta^+ = e_{n-2}-e_n$, so that $\nu_0^+=2$ and the hypotheses of Theorem 3.2 are satisfied for this estimate. Since $\nu_0^- \geqslant (1-\mu_\alpha)+2\cdot 0+1=3$, the assertion of Lemma 7.1 is proved in this case.

If $\mu=-\frac{1}{2}\alpha$ and there is no $\Delta^+_{K,\perp}$ simple root of type (g), then all labels are $\leqslant 0$. So $\delta^- = -\alpha$ and $v_0^- = (1-\mu_\alpha) + 2\cdot 0 + 1 = 3$; the argument in (I.2a) shows that hypotheses (a), (b), and (c′) of Theorem 3.2′ are satisfied. If $e_{n-3} - e_{n-2}$ does not exist or is not compact basic, then $\delta^+ = e_{n-2} - e_n$, the δ^+ subsystem is of type $\mathfrak{su}(2, 1)$, $v_0^+ = 2$, and familiar arguments as in Corollary 3.3 show that (a), (b), and (c) in Theorem 3.2 are satisfied. If $e_{n-3} - e_{n-2}$ is compact basic, then $v_0^+ \geqslant 4 > 3 = v_0^-$, and the assertion of Lemma 7.1 is therefore already proved in this case.

(I.3) $k = n, j \leqslant n-2$.

(I.3a) Exceptional term of v_0^+ nonzero. Then $2\langle \Lambda, e_n\rangle/|e_n|^2 = 1$, $\mu = \frac{1}{2}\alpha$, $\beta' = e_{j+2} - e_{j+3}$ is noncompact, and $e_j - e_{j+1}$, $e_{j+1} - e_{j+2}, ..., e_{n-1} - e_n$ all lie in the special basic case. Let $\beta = e_{j+1} - e_{j+2}$. The root $\beta + \beta'$ is $\Delta^+_{K,\perp}$ simple of type (f), and its label is -1. The root $\alpha + \beta$ is $\Delta^+_{K,\perp}$ simple with label $+1$. All other $\Delta^+_{K,\perp}$ simple roots have label $\leqslant 0$, since $e_{j-1} - e_j$ (if it exists) cannot be noncompact basic, by nondegeneracy. Hence $\delta^- = \beta$. Then $v_0^- = (1-\mu_\alpha) + 2 + 0 = 2$, and the techniques of Corollary 3.3 show that hypotheses (a), (b) and (c) of Theorem 3.2′ are satisfied. Since $\mu_\alpha = 1$, v_0^+ is $\geqslant 2$ and hence is $\geqslant v_0^-$.

(I.3b) Exceptional term of v_0^- nonzero. Then $2\langle \Lambda, e_n\rangle/|e_n|^2 = 1$, all roots beyond $e_{j+1} - e_{j+2}$ are compact, and $e_j - e_{j+1}$, $e_{j+1} - e_{j+2}, ..., e_{n-1} - e_n$ all lie in the special basic case. Since $e_{j+1} - e_{j+2}$ contributes to v_0^- and since the exceptional term of v_0^- is nonzero, we have $v_0^- \geqslant 3$.

Let us dispose of two preliminary subcases. First suppose that $e_{j-2} - e_{j+1}$ is a $\Delta^+_{K,\perp}$ simple root of type (g). Then $\mu = -\frac{1}{2}\alpha$, and we can check that $\delta^+ = e_{j-1} - e_{j+1}$. Hence $v_0^+ = 2 < v_0^-$, and the techniques of Corollary 3.3 show that hypotheses (a), (b), and (c) of Theorem 3.2 are satisfied.

Next suppose that $e_{j-1} - e_j$, if it exists, is noncompact. Then $\delta^+ = \alpha$, $v_0^+ = 1 + \mu_\alpha < v_0^-$, and again the techniques of Corollary 3.3 show that hypotheses (a), (b), and (c) of Theorem 3.2 are satisfied.

The remaining subcase is that the component of α in the special basic case has real rank one (with α and $e_{j+1} - e_{j+2}$ as the only noncompact simple roots). Here we claim that the hypotheses of Theorem 3.2 are satisfied for $(\Lambda + \alpha)^\vee = \Lambda + \delta^+$ and the hypotheses of Theorem 3.2′ are satisfied for $(\Lambda - \alpha)^\vee = (\Lambda + \delta^-)^\vee = (\Lambda + \delta^- + e_n)^\vee$. There is no problem for $(\Lambda + \alpha)^\vee$, since we can use Lemma 7.2 and the techniques of Corollary 3.3. For $(\Lambda - \alpha)^\vee$, we check that $\delta^- + e_n = e_{j+1}$ and that e_{j+1} is $\Delta^+_{K,\perp}$ dominant. Hence $(\Lambda - \alpha)^\vee = \Lambda + e_{j+1}$. Since $\langle \Lambda, e_n\rangle \neq 0$, Proposition 3.1 shows that (a) holds. For (b) we have

$$(\Lambda - \alpha)^\vee - (\Lambda + \alpha) = e_{j+1} - (e_j - e_{j+1}),$$

and this is not a sum of compact positive roots; hence $\Lambda+\alpha$ is not a weight. Finally, for (c′), we check solutions of the equation

$$e_{j+1}=(\Lambda-\alpha)^{\vee}-\Lambda=c\alpha+\sum_{\beta\in\Delta^{+}_{-,n}}n_\beta\beta+\sum_{\gamma\in\Delta_K^+}k_\gamma\gamma.$$

The only possible β that can contribute is $\beta=e_j+e_{j+1}$, and we would need

$$e_{j+1}=c(e_j-e_{j+1})+a(e_j+e_{j+1})+be_j$$

with $a\geqslant 0$, $b\geqslant 0$. As in (I.2a), this forces $\sum n_\beta\beta=0$. Thus (c′) holds.

(II) Suppose that α or $-\alpha$ is conjugate by the Weyl group of $\Delta^+_{K,\perp}$ to $\beta_0=e_j+e_{j+1}$. (We shall see that this case is disjoint from (I).) Since $\Delta_{K,\perp}$ is contained in the special basic case and since an A type group has only permutations in its Weyl group, e_n must lie in the component of α in the special basic case.

(II.1) Suppose $j<n-1$. Then e_n is compact and has $2\langle\Lambda,e_n\rangle/|e_n|^2=0$. Hence $k=n$ in the notation earlier; since $2\langle\Lambda,e_k\rangle/|e_k|^2\neq 1$, the exceptional terms for ν_0^+ and ν_0^- are zero. Lemma 7.2 and the techniques of Corollary 3.3 show that the hypotheses (a), (b), and (c) of Theorem 3.2 are satisfied if the δ^+ subsystem is of real rank one (then necessarily of A type) and that the hypotheses (a), (b), and (c) of Theorem 3.2′ are satisfied if the δ^- subsystem is of real rank one.

If $e_{j+1}-e_{j+3}$ is a $\Delta^+_{K,\perp}$ simple root of type (f) in Lemma 2.2, then we check that $\delta^-=e_{j+1}-e_{j+2}$. The δ^- subgroup is then of type $\mathfrak{su}(2,1)$, so that the remarks above apply. We have $\nu_0^-=2\leqslant\nu_0^+$.

Thus assume $e_{j+1}-e_{j+2}$ is not part of a $\Delta^+_{K,\perp}$ simple root of type (f). If $e_{j-2}-e_{j+1}$ is a $\Delta^+_{K,\perp}$ simple root of type (g), then we check that $\delta^+=e_{j-1}-e_{j+1}$. We have $\nu_0^+=2\leqslant\nu_0^-$, and again the remarks above apply. By nondegeneracy, there is no other possibility for a $\Delta^+_{K,\perp}$ simple root of type (f) or (g).

Next suppose that $e_{j-1}-e_j$ exists and is noncompact and basic. Then $\delta^+=\alpha$, $\nu_0^+=1+\mu_\alpha\leqslant 2\leqslant\nu_0^-$, and again the remarks above apply.

The remaining alternative is that the component of α in the special basic case Δ_S is of type $\mathfrak{so}(\text{odd},2)$. (This was not true previously during (II.1).) Let e_l-e_{l+1}, $l\leqslant j$, be the first simple root in this component. We have $\delta^+=e_l-e_{j+1}$, and the δ^+ subsystem is of type $\mathfrak{su}(j-l+1,1)$. So the remarks above apply. The corresponding cut-off for unitarity is

$$\nu_0^+=1+\mu_\alpha+2(j-l),\tag{7.9}$$

with contributions from $\beta=e_i-e_{j+1}$ for $l\leqslant i\leqslant j-1$. We have $\nu_0^+\leqslant\nu_0^-$ since ν_0^- has contributions from $\beta=e_i+e_{j+1}$ for $l\leqslant i\leqslant j-1$ and from $\beta=e_{j+1}$.

Within the special basic case, Proposition 3.7 applies and gives us another cut-off for unitarity, namely $\nu_{0,L}^{-}+1$, where

$$\nu_{0,L}^{-}=1-\mu_\alpha+2(n-j-1)\geqslant 3-\mu_\alpha. \tag{7.10}$$

This cut-off comes from consideration of the $(K\cap S)$-type

$$(\Lambda-\alpha+\beta_0)^{(\vee,S)}=(\Lambda+2e_{j+1})^{(\vee,S)}=\Lambda+2e_{j+1}.$$

When $\nu_{0,L}^{-}<\nu_0^{+}$, we want this estimate to persist for G. As usual, the Vogan Signature Theorem (Theorem 3.4) shows that it is enough to prove that $\Lambda+2e_{j+1}$ is Δ_K^{+} dominant.

Failure of $\Lambda+2e_{j+1}$ to be Δ_K^{+} dominant means that there is some $i<l$ with e_i-e_{j+1} simple in Δ_K^{+} (but not in $\Delta_{K,\perp}^{+}$) with $\langle\Lambda+2e_{j+1}, e_i-e_{j+1}\rangle<0$. Then $2\langle\Lambda, e_i-e_{j+1}\rangle/|e_i-e_{j+1}|^2=1$. Suppose $l\leqslant j-1$. With $\gamma=e_i-e_{j+1}$, (2.3) and Table 2.1 give

$$\begin{aligned}1&\geqslant\frac{2\langle\lambda_{0,b}, e_l-e_{j+1}\rangle}{|\gamma|^2}+\left(\frac{2\langle\delta, e_{l-1}-e_{j+1}\rangle}{|\gamma|^2}-2\right)-\frac{1}{2}(1-\mu_\alpha)\\&=2(j-l)-1+\mu_\alpha.\end{aligned} \tag{7.11}$$

This inequality fails if $l\leqslant j-2$ or if both $l=j-1$ and $\mu_\alpha=1$. If $l=j-1$ and $\mu_\alpha\leqslant 0$, then (7.9) and (7.10) give

$$\nu_0^{+}=3+\mu_\alpha\leqslant 3-\mu_\alpha\leqslant\nu_{0,L}^{-},$$

so that we do not care whether dominance persists.

We also must consider $l=j$. In this case, (7.9) and (7.10) give

$$\nu_0^{+}=1+\mu_\alpha\leqslant 3-\mu_\alpha\leqslant\nu_{0,L}^{-},$$

so that again we do not care whether dominance persists.

(II.2) Suppose $j=n-1$. Then e_n is noncompact, and we must have $2\langle\Lambda, e_{n-1}\rangle/|e_{n-1}|^2=0$. Hence $k=n-1$ in the notation earlier, and again the exceptional terms for ν_0^{+} and ν_0^{-} are zero. We can again show that (a), (b), and (c) in Theorem 3.2 or 3.2′ are satisfied by showing that the corresponding subsystem is of real rank one.

The condition $2\langle\Lambda, e_{n-1}\rangle/|e_{n-1}|^2=0$ forces $\mu\neq+\frac{1}{2}\alpha$, by Lemma 2.2. As in (I), we first eliminate all situations but those where the component of α in the special basic case Δ_S is of type $\mathfrak{so}(\text{odd}, 2)$, but not $\mathfrak{so}(3, 2)$.

If $e_{n-2}-e_{n-1}$ does not exist or is not compact basic, then $\delta^{+}=\alpha$ and $\nu_0^{+}=1+\mu_\alpha\leqslant 2\leqslant\nu_0^{-}$, since ν_0^{-} gets a contribution from $\beta=e_n$. (There can be no $\Delta_{K,\perp}^{+}$ simple root of type (f) in Lemma 2.2 since $\mu\neq+\frac{1}{2}\alpha$.)

If $\gamma = e_{n-2} - e_{n-1}$ exists and is compact basic, first suppose $e_{n-3} - e_n$ is a $\Delta^+_{K,\perp}$ simple root of type (g) in Lemma 2.2. Then $\mu = -\frac{1}{2}\alpha$, $\delta^+ = \alpha + \gamma$, and $\nu_0^+ = 2 \leqslant \nu_0^-$. So there is no problem in this case.

Now suppose that $e_{n-2} - e_{n-1}$ is compact basic and no root of type (g) is present. Then the component of α in Δ_S has compact simple roots $e_l - e_{l+1}, \ldots, e_{n-2} - e_{n-1}$ and noncompact simple roots $e_{n-1} - e_n$ and e_n. It is of the form $\mathfrak{so}(2(n-l)+1, 2)$, and we may assume $l \leqslant n-2$. Then $\delta^+ = e_l - e_n$, and the δ^+ subsystem is of type $\mathfrak{su}(n-l, 1)$. The corresponding cut-off for unitarity is

$$\nu_0^+ = 1 + \mu_\alpha + 2(n-l-1) \tag{7.12}$$

with contributions from $\beta = e_i - e_n$ for $l \leqslant i \leqslant n-2$. We have $\nu_0^+ \leqslant \nu_0^-$ since ν_0^- has contributions from $\beta = e_i + e_n$ for $l \leqslant i \leqslant n-2$ and from $\beta = e_n$.

Within Δ_S, Proposition 3.7 applies and gives us another cut-off for unitarity, namely $\nu_{0,L}^- - 1$, where

$$\nu_{0,L}^- = 1 - \mu_\alpha. \tag{7.13}$$

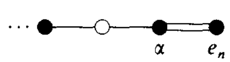
This cut-off comes from consideration of the $(K \cap S)$-type

$$(\Lambda - \alpha + \beta_0)^{(\vee, S)} = (\Lambda + 2e_n)^{(\vee, S)} = \Lambda + 2e_n.$$

When $\nu_{0,L}^- < \nu_0^+$, we want this estimate to persist for G. As in (II.1), it is enough to show that $\Lambda + 2e_n$ is Δ_K^+ dominant. Failure of Δ_K^+ dominance would lead to (7.11) with $j = n-1$, and we conclude that $l = j - 1 = n-2$. In this case (7.12) and (7.13) imply

$$\nu_0^+ = 3 + \mu_\alpha \qquad \text{while} \quad \nu_{0,L}^- = 1 - \mu_\alpha.$$

Since we are not allowing $\mu = +\frac{1}{2}\alpha$, we have a problem only when $\mu = 0$ and the diagram is



with all the illustrated roots basic. This is the situation in (vi) of Theorem 1.1, and condition (v) assumes we are not attempting to treat this case. (Here we can show that unitarity continues beyond $\nu_{0,L}^- = 1$ to the point 2 and that 3 is a unitarity point, but the proof that there is a gap in unitarity from 2 to 3 uses different methods and is postponed to Section 13.)

(III) Suppose that neither α nor $-\alpha$ is conjugate by the Weyl group of $\Delta^+_{K,\perp}$ to $\beta_0 = e_j + e_{j+1}$ and that the exceptional terms of ν_0^+ and ν_0^- are 0. Lemma 7.2 and the techniques of Corollary 3.3 show that the hypotheses (a), (b), and (c) of Theorem 3.2 are satisfied if the δ^+ subsystem is of real rank one (then necessarily of A type) and that the hypotheses (a), (b), and (c) of Theorem 3.2′ are satisfied if the δ^- subsystem is of real rank one.

(III.1) Suppose that e_n is not in the component of α within the special basic case. Then the computation of δ^+ and δ^- takes place in a single-line diagram, and Lemma 4.1 tells us either that the δ^+ subsystem has real rank one and $\nu_0^+ \leqslant \nu_0^-$ or that the δ^- subsystem has real rank one and $\nu_0^- \leqslant \nu_0^+$. Hence the remarks above complete the proof of (a) or (b) of Lemma 7.1 in this case.

The situation of (III.1) may meet the conditions of (iii) in Theorem 1.1, and then Lemma 7.1 requires more. In this situation, α cannot be adjacent to e_n, since otherwise e_n basic leads to (I) if $\mu = +\frac{1}{2}\alpha$ and to (II) if $\mu \neq +\frac{1}{2}\alpha$. Thus (iii) requires the following: e_n must be noncompact basic, and the component of α in the special basic case must be of real rank one and must be adjacent to e_n. Taking Δ_L to be generated by this component and e_n, we prepare to apply Proposition 3.8 to Δ_L. The root ζ is $e_{j+1} - e_n$ and is noncompact. Proposition 3.8 uses the $(K \cap L)$-type $\Lambda_L'' = \Lambda_L + \zeta + e_n = \Lambda_L + e_{j+1}$ to assert nonunitarity within L for $\nu_0^- - 1 < c < \nu_0^-$, hence for

$$\min(\nu_0^+, \nu_0^- - 1) < c < \min(\nu_0^+, \nu_0^-).$$

(Note that ν_0^+ and ν_0^- are the same in L as in G since Δ_L includes the component of α in the special basic case and since the exceptional terms of ν_0^+ and ν_0^- are 0.) To extend this conclusion to G, it is enough, by Theorem 3.4, to show that $\Lambda + e_{j+1}$ is Δ_K^+ dominant. Let γ be Δ_K^+ simple. If γ is in $\Delta_{K,\perp}^+$ and $\langle e_{j+1}, \gamma \rangle < 0$, then γ is in $\Delta_K^+ \cap \Delta_L$ and we know $\langle \Lambda + e_{j+1}, \gamma \rangle \geqslant 0$. For all other Δ_K^+ simple γ with $\langle e_{j+1}, \gamma \rangle < 0$, we have $2\langle \Lambda, \gamma \rangle / |\gamma|^2 \geqslant 1$ and $2\langle e_{j+1}, \gamma \rangle / |\gamma|^2 \geqslant -1$. Hence $\langle \Lambda + e_{j+1}, \gamma \rangle \geqslant 0$. Consequently (i) holds in Lemma 7.1.

(III.2) Suppose that e_n is in the component of α within the special basic case. If $j < n-1$, then $2\langle \Lambda, e_n \rangle / |e_n|^2 = 0$, and we see that we are in case (II.1), contradiction. So $j = n-1$. Since e_n is in the special basic case, Lemma 2.2 shows that $2\langle \Lambda, e_{n-1} \rangle / |e_{n-1}|^2 = 0$, and we see that we are in case (II.2), contradiction. So case (III.2) does not occur.

8. Tools for Proving Irreducibility

When $\min(\nu_0^+, \nu_0^-) > 0$, we know from Lemma 2.1 that $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary for small positive c. It then follows from a general continuity argument (cf. [14, Sect. 14]) that $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary for c in any interval $[0, c_0]$ such that

$U(MAN, \sigma, \frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0$. In every case where Theorem 1.1 asserts unitarity on an interval, we shall prove the assertion by establishing the corresponding irreducibility.

Two of our tools will be the following results specialized to our situation with $\dim A = 1$ from Speh–Vogan [20]. We say that α *satisfies the parity condition* for $U(MAN, \sigma, \frac{1}{2}c\alpha)$ if either σ is a cotangent case and c is an even integer or σ is a tangent case and c is an odd integer. (See Sect. 1 for "cotangent case" and "tangent case.")

THEOREM 8.1 [20, p. 292]. *Fix $c > 0$. Then $U(MAN, \sigma, \frac{1}{2}c\alpha)$ can be reducible only when either*

(a) *α satisfies the parity condition for $U(MAN, \sigma, \frac{1}{2}c\alpha)$, or*

(b) *there is a root $\beta \neq \pm\alpha$ with $\langle \lambda_0 + \frac{1}{2}c\alpha, \beta\rangle > 0$ and $\langle \lambda_0 - \frac{1}{2}c\alpha, \beta\rangle < 0$ such that $2\langle \lambda_0 + \frac{1}{2}c\alpha, \beta\rangle/|\beta|^2$ is an integer.*

THEOREM 8.2 [20, Sects. 4, 5]. *Let $U(MAN, \sigma, \nu)$ be given with $\nu = \frac{1}{2}c\alpha$ and $c > 0$. Let Δ_L be a subsystem of Δ generated by simple roots and containing α. If the representation $U^L(M_L AN_L, \sigma_L, \nu)$ defined by (3.1) is irreducible and the set $\Delta(\mathfrak{u})$ of roots in (3.1a) satisfies*

$$\langle \lambda_0 + \nu, \beta\rangle \geqslant 0 \tag{8.1}$$

for all β in $\Delta(\mathfrak{u})$, then $U(MAN, \sigma, \nu)$ is irreducible.

We shall use Theorem 8.1 normally in the following form.

COROLLARY 8.3. *For $c > 0$, $U(MAN, \sigma, \frac{1}{2}c\alpha)$ can be reducible only if c is an integer. Moreover, if α is short or if all roots have the same length, then reducibility forces α to satisfy the parity condition for $U(MAN, \sigma, \frac{1}{2}c\alpha)$.*

Proof. For any root β, formula (1.3) and the integrality of Λ show that $2\langle \lambda_0 + \mu - \frac{1}{2}\alpha, \beta\rangle/|\beta|^2$ is an integer. Thus $\lambda_0 + \frac{1}{2}c_0\alpha$ is integral, where $c_0 = 0$ if σ is a cotangent case and $c_0 = 1$ if σ is a tangent case. Suppose $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is reducible and α does not satisfy the parity condition. Choose β as in (b) of Theorem 8.1. Since $2\langle \lambda_0 + \frac{1}{2}c\alpha, \beta\rangle/|\beta|^2$ is an integer, so is $2\langle \frac{1}{2}(c - c_0)\alpha, \beta\rangle/|\beta|^2$. Whether α is long or short, this condition forces $c - c_0$ to be an integer. Hence c is an integer. If α is short, the same condition forces $c - c_0$ to be an even integer. Thus α satisfies the parity condition, contradiction. This proves the corollary.

Let Δ_L be a subsystem of Δ generated by simple roots and containing α. We say that *(SV) holds* if the inequality (8.1) holds for all β in $\Delta(\mathfrak{u})$. In this case Theorem 8.2 allows us to infer irreducibility in G from irreducibility in L. The starting points using Theorem 8.2 are Proposition 8.4 below and the observation (evident from Table 2.1) that any Δ^+ simple root that is basic in G and lies in Δ_L is basic in L.

PROPOSITION 8.4. *For* $\mathfrak{g}$ *equal to* $\mathfrak{so}(2n, 1)$ *or* $\mathfrak{su}(n, 1)$ *and the basic case equal to all of* Δ, $U(MAN, \sigma, \frac{1}{2}c\alpha)$ *is irreducible for* $0 < c < \min(\nu_0^+, \nu_0^-)$. *The same conclusion applies to* $\mathfrak{g} = \mathfrak{sp}(n, 1)$ *when* $\sigma \neq 1$; *if* $\sigma = 1$ *and* $n \geqslant 2$, *then irreducibility extends for* $0 < c < \min(\nu_0^+, \nu_0^-) - 2$.

Remark. This is a reinterpretation of well-known results. See, e.g., [1].

In a classical group, it is easy to check directly whether (SV) holds, but in an E-type diagram we need some simplification of the condition such as in

LEMMA 8.5. *Suppose that the Dynkin diagram of* Δ^+ *has only single lines. Let* Δ_{L_1} *be a subsystem of* Δ *generated by simple roots and containing* α, *and let* Δ_L *be the component of* α *in* Δ_{L_1}. *Let* γ_i *be the (simple) neighbors of* Δ_L^+ *in* $\Delta^+ - \Delta_L^+$. *For each* γ_i, *let* β_i *be the sum (with multiplicity one apiece) of the simple roots from* γ_i *to* α, *including* γ_i *but not including* α. *If* $\langle \lambda_0 + \nu, \beta_i \rangle \geqslant 0$ *for each* i, *then* (SV) *holds for* Δ_{L_1} *and* $\lambda_0 + \nu$.

Proof. For this proof, let us normalize all root lengths squared to be 2. Without loss of generality we can shrink Δ_{L_1} to equal Δ_L. Let β in $\Delta(\mathfrak{u})$ be a root to be checked. We may assume $\langle \beta, \alpha \rangle = -1$. Let β' be the sum of the simple roots contributing to β (each counted just once in β'). Then β' is in $\Delta(\mathfrak{u})$. Let Π be the set of simple roots contributing to β'. We distinguish four cases.

(1) Suppose α is not in Π. Since Π is connected and $\Pi \cup \{\alpha\}$ has no loops, α has just one neighbor γ_0 in Π. Hence $\langle \beta', \alpha \rangle = -1$. Moreover $\beta - \beta' \in \sum \Delta^+$ implies $\langle \lambda_0, \beta - \beta' \rangle \geqslant 0$, and $\langle \beta - \beta', \alpha \rangle = 0$ implies $\langle \nu, \beta - \beta' \rangle = 0$. Hence $\langle \lambda_0 + \nu, \beta \rangle \geqslant \langle \lambda_0 + \nu, \beta' \rangle$, and it is enough to handle β'. Since β' is in $\Delta(\mathfrak{u})$ and α is in Δ_L^+, Π contains a root γ that is in $\Delta(\mathfrak{u})$ but is adjacent to Δ_L^+, i.e., one of the roots γ_i. The corresponding β_i is the sum of the roots from γ_0 to $\gamma = \gamma_i$. Then $\langle \beta_i, \alpha \rangle = -1$. From $\beta' - \beta_i \in \sum \Delta^+$ we obtain $\langle \lambda_0, \beta' - \beta_i \rangle \geqslant 0$, and from $\langle \beta' - \beta_i, \alpha \rangle = 0$ we obtain $\langle \nu, \beta' - \beta_i \rangle = 0$. Thus $\langle \lambda_0 + \nu, \beta' \rangle \geqslant \langle \lambda_0 + \nu, \beta_i \rangle$, and it is enough to handle β_i, as asserted.

(2) Suppose α is in Π and is a node of Π. We have $\langle \beta, \alpha \rangle = -1$ and $\langle \beta', \alpha \rangle = +1$. Let γ be the unique neighbor of α in Π. Then we can write

$$\beta = a\alpha + b\gamma + \eta,$$

$$\beta' = \alpha + \gamma + \eta',$$

where $2a-b=-1$, $b\geqslant 1$, η and η' are sums of simple roots orthogonal to α, and $\eta-\eta'$ is a sum of simple roots. Hence

$$\begin{aligned}\langle\lambda_0+\tfrac{1}{2}c\alpha,\beta\rangle &= b\langle\lambda_0,\gamma\rangle+\langle\lambda_0,\eta\rangle+ca-\tfrac{1}{2}bc\\ &= b\langle\lambda_0,\gamma\rangle+\langle\lambda_0,\eta\rangle-\tfrac{1}{2}c\\ &\geqslant \langle\lambda_0,\gamma\rangle+\langle\lambda_0,\eta'\rangle-\tfrac{1}{2}c\\ &= \langle\lambda_0+\tfrac{1}{2}c\alpha,\beta'-\alpha\rangle.\end{aligned}$$

It is therefore enough to handle the root $\beta'-\alpha$, which is handled by (1).

(3) Suppose α is in Π and has two neighbors in Π. We have $\langle\beta',\alpha\rangle=0$. Removing α from Π, we obtain two components, and we let β'_1 and β'_2 be the sums of the simple roots in the components, so that $\beta'=\beta'_1+\alpha+\beta'_2$. Since β' is in $\Delta(\mathfrak{u})$, at least one of β'_1 and β'_2, say β'_i, is in $\Delta(\mathfrak{u})$. Then $\langle\beta'_i,\alpha\rangle=-1=\langle\beta,\alpha\rangle$, $\beta-\beta'_i$ is in $\sum\Delta^+$, and it is enough to handle the root β'_i, which is handled by (1).

(4) Suppose α is a triple point in Π. We have $\langle\beta',\alpha\rangle=-1=\langle\beta,\alpha\rangle$. Since $\beta-\beta'$ is in $\sum\Delta^+$ and $\langle\beta-\beta',\alpha\rangle=0$, it is enough to handle β'. Removing α from Π, we obtain three components, and we let β'_1, β'_2, β'_3 be the sums of the simple roots in the components. Since β' is in $\Delta(\mathfrak{u})$, at least one component, say the one for β'_1, must extend outside Δ_L^+. We write $\beta'=\beta'_1+(\beta'_2+\alpha+\beta'_3)$ and

$$\langle\lambda_0+\nu,\beta'\rangle=\langle\lambda_0+\nu,\beta'_1\rangle+\langle\lambda_0+\nu,\beta'_2+\alpha+\beta'_3\rangle.$$

Here $\beta'_2+\alpha+\beta'_3$ is a root orthogonal to α; hence we can drop ν from the second term on the right. We have $\beta'-\beta'_1\in\sum\Delta^+$ and $\langle\beta',\alpha\rangle=-1=\langle\beta'_1,\alpha\rangle$. Hence it is enough to handle β'_1, which is handled by (1). This proves the lemma.

To get started with (SV), we need some other irreducibility beyond that in Proposition 8.4. We assemble in Lemma 8.6 the additional information that we need.

LEMMA 8.6. *In the 27 configurations* (a)–(aa) *of Table* 8.1, $U(MAN,\sigma,\tfrac{1}{2}c\alpha)$ *is irreducible for* $0<c<\min(\nu_0^+,\nu_0^-)$. *In configurations* (bb) *and* (cc), $U(MAN,\sigma,\tfrac{1}{2}c\alpha)$ *is irreducible for* $0<c<\min(\nu_0^+,\nu_{0,L}^-+1)$.

Proof of Lemma 8.6. The idea is to combine Proposition 8.4 and the use of (SV) with some special irreducibility results proved by Vogan's composition series algorithm and assembled in our paper [4]. However, the configurations in Table 8.1 include a certain amount of duplication (with

TABLE 8.1

Configurations Addressed by Lemma 8.6

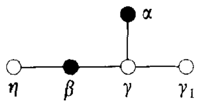
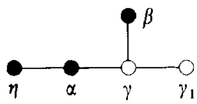
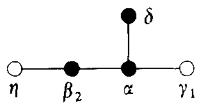
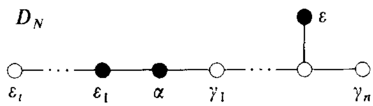
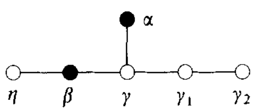
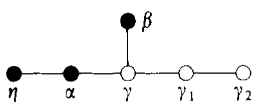
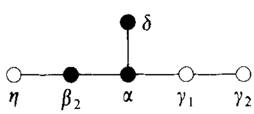
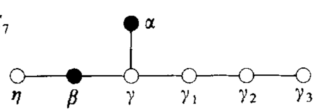
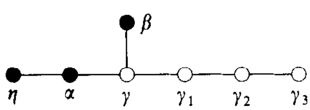
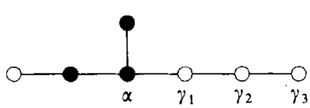
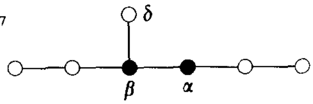
(a)	D_5		All roots basic $\mu = -\frac{1}{2}\alpha$ $\nu_0^+ = 4 \leqslant \nu_0^-$
(b)	D_5		All roots basic $\mu = -\frac{1}{2}\alpha$ $\nu_0^+ = 4 \leqslant \nu_0^-$
(c)	D_5		All roots basic $\mu = +\frac{1}{2}\alpha$ $\nu_0^+ = 4 \leqslant \nu_0^-$
(d)	D_N		All roots basic μ arbitrary $n \geqslant 2$, $t \geqslant 1$, $n > 2$ or $\mu \neq -\frac{1}{2}\alpha$ $\nu_0^+ = 1 + \mu_\alpha + 2n$ $\nu_0^- = 1 - \mu_\alpha + 2t$
(e)	E_6		All roots basic $\mu = -\frac{1}{2}\alpha$ $\nu_0^+ = 6 \leqslant \nu_0^-$
(f)	E_6		All roots basic $\mu = -\frac{1}{2}\alpha$ $\nu_0^+ = 6 \leqslant \nu_0^-$
(g)	E_6		All roots basic μ arbitrary $\nu_0^+ = 5 + \mu_\alpha \leqslant \nu_0^-$
(h)	E_7		All roots basic $\mu = -\frac{1}{2}\alpha$ $\nu_0^+ = 8 \leqslant \nu_0^-$
(i)	E_7		All roots basic $\mu = -\frac{1}{2}\alpha$ $\nu_0^+ = 8 \leqslant \nu_0^-$
(j)	E_7		All roots basic μ arbitrary $\nu_0^+ = 7 + \mu_\alpha \leqslant \nu_0^-$

Table continued

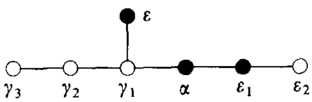
TABLE 8.1—*Continued*

(k) E_7



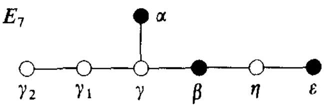
All roots basic
μ arbitrary
$\nu_0^+ = 5 + \mu_\alpha \leqslant \nu_0^-$

(l) E_7



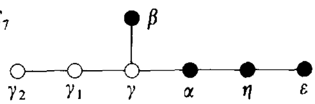
All roots basic
$\mu = 0$
$\nu_0^- = 5,\ \nu_0^+ = 7$

(m) E_7



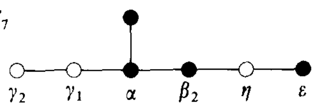
All roots basic
$\mu = -\frac{1}{2}\alpha$
$\nu_0^+ = 6 \leqslant \nu_0^-$

(n) E_7



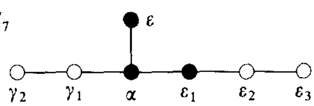
All roots basic
$\mu = -\frac{1}{2}\alpha$
$\nu_0^+ = 6 \leqslant \nu_0^-$

(o) E_7



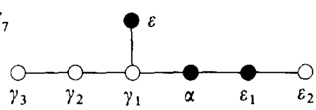
All roots basic
$\mu = +\frac{1}{2}\alpha$
$\nu_0^+ = 6 \leqslant \nu_0^-$

(p) E_7



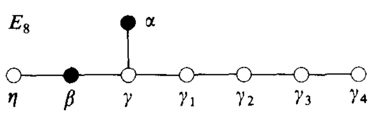
ε one step removed from basic
All other roots basic
$\mu = +\frac{1}{2}\alpha$
$\nu_0^+ = 6 = \nu_0^-$

(q) E_7



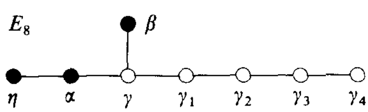
ε one step removed from basic
All other roots basic
$\mu = -\frac{1}{2}\alpha$
$\nu_0^+ = 6 = \nu_0^-$

(r) E_8



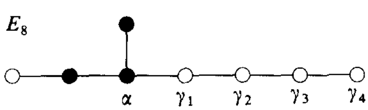
All roots basic
$\mu = -\frac{1}{2}\alpha$
$\nu_0^+ = 10 \leqslant \nu_0^-$

(s) E_8



All roots basic
$\mu = -\frac{1}{2}\alpha$
$\nu_0^+ = 10 \leqslant \nu_0^-$

(t) E_8

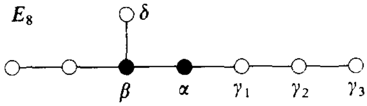


All roots basic
μ arbitrary
$\nu_0^+ = 9 + \mu_\alpha \leqslant \nu_0^-$

Table continued

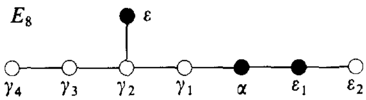
TABLE 8.1—*Continued*

(u) E_8



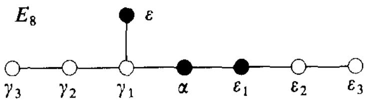
All roots basic
μ arbitrary
$v_0^+ = 7 + \mu_\alpha \leqslant v_0^-$

(v) E_8



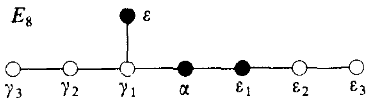
All roots basic
$\mu = -\frac{1}{2}\alpha$
$v_0^- = 6$, $v_0^+ = 8$

(w) E_8



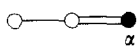
All roots basic
$\mu \neq -\frac{1}{2}\alpha$
$v_0^+ = 7 + \mu_\alpha$
$v_0^- = 7 - \mu_\alpha$

(x) E_8



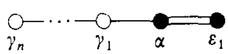
ε one step removed from basic
All other roots basic
$\mu \neq +\frac{1}{2}\alpha$
$v_0^+ = 7 + \mu_\alpha$
$v_0^- = 7 - \mu_\alpha$

(y) C_3



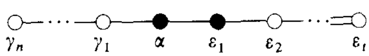
All roots basic
$\mu = -\frac{1}{2}\alpha$
$v_0^- = 2 = v_0^+$

(z) B_N



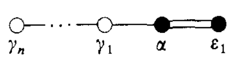
ε_1 one step removed from basic
All other roots basic
$\mu \neq +\frac{1}{2}\alpha$
$n \geqslant 1$
$v_0^+ = 1 + \mu_\alpha + 2n$
$v_0^- = 2 - \mu_\alpha$

(aa) B_N



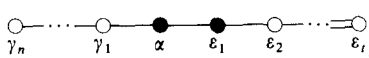
ε_t one step removed from basic
All other roots basic
μ arbitrary
$n \geqslant 0$, $t \geqslant 2$
$v_0^+ = 1 + \mu_\alpha + 2n$
$v_0^- = 2 - \mu_\alpha + 2(t-1)$

(bb) B_N



All roots basic
$\mu \neq +\frac{1}{2}\alpha$
$n \geqslant 1$
$v_0^+ = 1 + \mu_\alpha + 2n$
$v_{0,L}^- = 1 - \mu_\alpha$

(cc) B_N



All roots basic
μ arbitrary
$n \geqslant 0$, $t \geqslant 2$
$v_0^+ = 1 + \mu_\alpha + 2n$
$v_{0,L}^- = 1 - \mu_\alpha + 2(t-1)$

Δ_-^+ imbedded in Δ^+ in distinct ways), and we must sort out the duplication in order to reduce matters to [4].

Suppose β is a simple root with $\langle\beta, \alpha\rangle \neq 0$ such that $\langle\lambda_0, \beta\rangle = 0$. If we replace Δ^+ by $s_\beta\Delta^+$, then the members of Δ_-^+ remain positive, and λ_0 remains dominant for G. The only difficulty is that α does not remain simple. However, we can repeat the process with an $s_\beta\Delta^+$ simple root β' with $\langle\beta', \alpha\rangle \neq 0$ such that $\langle\lambda_0, \beta'\rangle = 0$, and continue in this way. After several steps, α may again be simple, and then we have a new valid way of imbedding Δ_-^+ in the positive roots for G.

Under this change of imbedding, the tangent–cotangent decision is preserved, but $\mu = +\frac{1}{2}\alpha$ may get changed into $\mu = -\frac{1}{2}\alpha$. The important thing is to follow the patterns of $2\langle\lambda_0, \beta\rangle/|\beta|^2$ for the successive systems of simple roots and then to interpret the new pattern as involving roots basic for $\mu = +\frac{1}{2}\alpha$ or $\mu = -\frac{1}{2}\alpha$ (see Table 2.1).

There is a second way of changing the imbedding: reflection in α. We have used this device extensively already; it involves replacing α by $-\alpha$ and v and $-v$.

For an example in detail, consider configurations (a), (b), and (c) in Table 8.1. If we apply s_γ to (a) and $s_{\alpha+\gamma}$ to the result, then (a) is transformed into (c) (with new letters for the simple roots other than α). If we apply s_{β_2} to (c) and $s_{\alpha+\beta_2}$ to the result, then (c) is transformed into (b). So (a), (b), and (c) are really the same.

Now let us come to the proof. In the single-line diagrams, Corollary 8.3 shows that it is enough to prove irreducibility at $v = \frac{1}{2}c\alpha$ when c is an integer and $c \equiv 1 + \mu_\alpha \bmod 2$ and $0 \leqslant c < \min(v_0^+, v_0^-)$. The latter condition we can rewrite as $0 \leqslant c \leqslant \min(v_0^+, v_0^-) - 2$. Moreover, $c = 0$ is handled automatically by Lemma 2.1. In all single-line diagrams, let us normalize all root lengths squared to be 2.

In (a), (b), and (c), which we know now to be equivalent, we have to prove irreducibility only for $c = 2$. In the case of (b), this irreducibility is asserted at the end of Section 1 and in Section 5 of [4]. Thus we are done with (a), (b), and (c).

Consider (d). We divide matters into subcases, first supposing that $v_0^+ > v_0^-$. In this circumstance let Δ_L be the horizontal subdiagram. The numbers v_0^+ and v_0^- are the same in Δ_L as in Δ, and Proposition 8.4 says we have irreducibility in Δ_L. To use (SV) and Theorem 8.2 to pass to irreducibility in Δ, Lemma 8.5 says that it is enough to show that $c \leqslant \min(v_0^+, v_0^-) - 2$ implies

$$\langle\lambda_0 + \tfrac{1}{2}c\alpha, \gamma_1 + \cdots + \gamma_{n-1} + \varepsilon\rangle \geqslant 0. \tag{8.2}$$

Since $v_0^+ > v_0^-$, we have $v_0^+ - 2 \geqslant v_0^-$ and $c \leqslant v_0^+ - 4$. Since all roots are basic, Table 2.1 gives

$$\begin{aligned}\langle \lambda_0 + \tfrac{1}{2}c\alpha, \gamma_1 + \cdots + \gamma_{n-1} + \varepsilon \rangle &= \tfrac{1}{2}(1+\mu_\alpha) + (n-2) - \tfrac{1}{2}c \\ &\geqslant \tfrac{1}{2}(1+\mu_\alpha) + (n-2) - \tfrac{1}{2}v_0^+ + 2 \\ &= \tfrac{1}{2}(1+\mu_\alpha) + n - \tfrac{1}{2}(1+\mu_\alpha + 2n) = 0.\end{aligned}$$

Thus (8.2) holds, and we have irreducibility in Δ.

Next suppose that $v_0^+ = v_0^-$. The above computation anyway proves irreducibility for $c \leqslant v_0^+ - 4$, and we have to handle $c = v_0^+ - 2$. If $\mu = +\frac{1}{2}\alpha$, this irreducibility is asserted by the cotangent cases in Section 6 of [4], while if $\mu = 0$, this irreducibility is asserted by the tangent cases in Section 6 of [4]. If $\mu = -\frac{1}{2}\alpha$, then we apply the reflections s_{γ_1} and $s_{\alpha+\gamma_1}$ to see that the diagrams for $\mu = +\frac{1}{2}\alpha$ and $\mu = -\frac{1}{2}\alpha$ are equivalent if n and t are adjusted suitably.

Finally suppose that $v_0^+ < v_0^-$. Then we let Δ_L be the diagram with $\varepsilon_t, ..., \varepsilon_{j+1}$ chopped off in such a way that $v_0^+ = v_0^-$ in the subdiagram. We have just seen irreducibility in Δ_L. If $c \leqslant v_0^+ - 2$, then

$$\begin{aligned}\langle \lambda_0 + \tfrac{1}{2}c\alpha, \varepsilon_{j+1} + \cdots + \varepsilon_1 \rangle &= \tfrac{1}{2}(1-\mu_\alpha) + j - \tfrac{1}{2}c \\ &= \tfrac{1}{2}v_{0,L}^- - \tfrac{1}{2}c \\ &= \tfrac{1}{2}v_0^+ - \tfrac{1}{2}c \geqslant 0.\end{aligned}$$

Thus Lemma 8.5 says that (SV) holds, and hence we have irreducibility in Δ.

We turn to E_6. Two reflections of (e) leads us to (g) with $\mu = +\frac{1}{2}\alpha$, and two reflections of (f) leads us to (g) with $\mu = +\frac{1}{2}\alpha$. Thus we need only consider (g). If $\mu = -\frac{1}{2}\alpha$, we can take Δ_L to be Δ with the node δ deleted, and Proposition 8.4 gives us irreducibility at $c = 2$ in Δ_L. Since Table 2.1 gives $\langle \lambda_0 + \alpha, \delta \rangle = 0$, Lemma 8.5 shows that (SV) holds at $v = \alpha$, hence that we have irreducibility in Δ for $c \leqslant \min(v_0^+, v_0^-) - 2$.

If $\mu = 0$, we can still take Δ_L to be Δ with δ deleted, and Proposition 8.4 gives us irreducibility at $c = 1$ and $c = 3$ in Δ_L. Here $\langle \lambda_0 + \frac{1}{2}c\alpha, \delta \rangle = \frac{1}{2}(1-c)$, and only the irreducibility at $c = 1$ extends to Δ in this way. For $c = 3$, we appeal directly to Section 7a of [4].

If $\mu = +\frac{1}{2}\alpha$, we take Δ_L to be Δ with γ_2 deleted, and case (c) gives us irreducibility at $c = 2$ in Δ_L. Since $\langle \lambda_0 + \alpha, \gamma_1 + \gamma_2 \rangle = 1 \geqslant 0$, Lemma 8.5 shows that (SV) holds at $v = \alpha$, hence that we have irreducibility in Δ for $c = 2$. For $c = 4$, we appeal directly to Section 7b of [4].

Next we consider E_7. Two reflections of (h) leads us to (j) with $\mu = +\frac{1}{2}\alpha$, and two reflections of (i) leads us to (j) with $\mu = +\frac{1}{2}\alpha$. Let us therefore consider (j). For $\mu = -\frac{1}{2}\alpha$, we can reflect twice and pass to (k) with $\mu = +\frac{1}{2}\alpha$; so we handle this case by handling (k) shortly. For $\mu = 0$ and $\mu = +\frac{1}{2}\alpha$, we take Δ_L to be the E_6 subdiagram given in (g); in Δ_L we have irreducibility for $c \leqslant 3+\mu_\alpha$. Since

$$\langle \lambda_0 + \tfrac{1}{2}(3+\mu_\alpha)\,\alpha, \gamma_1+\gamma_2+\gamma_3 \rangle = \tfrac{1}{2}(1+\mu_\alpha) + 2 - \tfrac{1}{2}(3+\mu_\alpha) = 1 \geqslant 0,$$

this irreducibility extends to Δ. For $c = 5+\mu_\alpha$, we appeal directly to Sections 7e and g of [4] for the irreducibility.

In (k), we can let Δ_L be the horizontal subdiagram. Proposition 8.4 gives us irreducibility in Δ_L. Since $v_0^+ = 5+\mu_\alpha$, the expression

$$\langle \lambda_0 + \tfrac{1}{2}c\alpha, \delta+\beta \rangle = 1 + \tfrac{1}{2}(1-\mu_\alpha) - \tfrac{1}{2}c$$

is $\geqslant 0$ for $c \leqslant 3+\mu_\alpha$ if $\mu = -\frac{1}{2}\alpha$ or $\mu = 0$, and it is $\geqslant 0$ for $c=2$ if $\mu = +\frac{1}{2}\alpha$. Thus we obtain the desired irreducibility in Δ except when $\mu = +\frac{1}{2}\alpha$ and $c=4$; in this case we appeal directly to Section 7c of [4].

In (l), we take Δ_L to be the D_6 subdiagram obtained by deleting γ_3. The D_6 subdiagram is the case $n=t=2$ of (d), and (d) says we have irreducibility there for $c \leqslant 3$. Since

$$\langle \lambda_0 + \tfrac{3}{2}\alpha, \gamma_3+\gamma_2+\gamma_1 \rangle = 1 \geqslant 0,$$

this irreducibility extends to Δ and handles (l).

Two reflections of (m) leads us to (o), and two reflections of (o) leads us to (n). Let us therefore consider (n). We take Δ_L to be the E_6 subdiagram in which ε has been deleted. Case (f) gives us irreducibility in Δ_L at $c=2$. Since $\langle \lambda_0+\alpha, \varepsilon+\eta \rangle = 0$, this irreducibility extends to Δ. For $c=4$, we appeal directly to Section 7f of [4] for the irreducibility.

Two reflections of (q) leads us to (p), which we consider now. We take Δ_L to be the A_6 subdiagram in which ε has been deleted. Proposition 8.4 gives us irreducibility in Δ_L for $c \leqslant 4$, hence at $c=2$. Since ε is one step removed from basic, we have $\langle \lambda_0+\alpha, \varepsilon \rangle = 0$, and thus the irreducibility at $c=2$ extends to Δ. For $c=4$, we appeal directly to Section 7d of [4] for the irreducibility.

Finally we consider E_8. Two reflections of (r) leads us to (t) with $\mu = +\frac{1}{2}\alpha$, and two reflections of (s) leads us to (t) with $\mu = +\frac{1}{2}\alpha$. Let us therefore consider (t). For $\mu = -\frac{1}{2}\alpha$, we can reflect twice and pass to (u) with $\mu = +\frac{1}{2}\alpha$; so we handle this case by handling (u) shortly. For $\mu = 0$ and $\mu = +\frac{1}{2}\alpha$, we take Δ_L to be the E_7 subdiagram given in (j); in Δ_L we have irreducibility for $c \leqslant 5+\mu_\alpha$. Since

$$\langle \lambda_0 + \tfrac{1}{2}(5+\mu_\alpha)\,\alpha, \gamma_1+\gamma_2+\gamma_3+\gamma_4 \rangle = \tfrac{1}{2}(1+\mu_\alpha) + 3 - \tfrac{1}{2}(5+\mu_\alpha) = 1 \geqslant 0,$$

this irreducibility extends to Δ. For $c = 7+\mu_\alpha$, we appeal directly to Sections 7l and m of [4] for the irreducibility.

In (u), first let $\mu = -\frac{1}{2}\alpha$. We take Δ_L to be the horizontal subdiagram, and Proposition 8.4 gives us irreducibility for $c \leqslant 4$. Since $\langle \lambda_0 + 2\alpha, \delta + \beta \rangle = 0$, this irreducibility extends to the required irreducibility in Δ. For $\mu = 0$ and $\mu = +\frac{1}{2}\alpha$, we take Δ_L to be the E_7 subdiagram given in (k); in Δ_L we have irreducibility for $c \leqslant 3 + \mu_\alpha$. Since

$$\langle \lambda_0 + \tfrac{1}{2}(3 + \mu_\alpha)\, \alpha, \gamma_1 + \gamma_2 + \gamma_3 \rangle = 1 \geqslant 0,$$

this irreducibility extends to Δ. For $c = 5 + \mu_\alpha$, we appeal directly to Sections 7h and i of [4] for the irreducibility.

In (v), we let Δ_L be the D_7 subdiagram obtained by deleting γ_4. The D_7 subdiagram is the case $n = 3$ and $t = 2$ of (d), and (d) says we have irreducibility there for $c \leqslant 4$. Since

$$\langle \lambda_0 + 2\alpha, \gamma_4 + \gamma_3 + \gamma_2 + \gamma_1 \rangle = 1 \geqslant 0,$$

this irreducibility extends to Δ and handles (v).

In (w), where $\mu \neq -\frac{1}{2}\alpha$, let Δ_L be the D_7 subdiagram obtained by deleting γ_3. The D_7 subdiagram is the case $n = 2$ and $t = 3$ of (d), and (d) says we have irreducibility there for $c \leqslant 3 + \mu_\alpha$. Since

$$\langle \lambda_0 + 2\alpha, \gamma_3 + \gamma_2 + \gamma_1 \rangle = \tfrac{1}{2}(1 + \mu_\alpha) \geqslant 0,$$

this irreducibility extends to Δ. This handles $\mu = +\frac{1}{2}\alpha$ completely and handles $c \leqslant 3$ when $\mu = 0$. For $\mu = 0$ and $c = 5$, we appeal directly to Section 7k of [4] for the irreducibility.

In (x), we first suppose $\mu = -\frac{1}{2}\alpha$. Let Δ_L be the E_7 subdiagram given in (q); in Δ_L we have irreducibility for $c \leqslant 4$. Since

$$\langle \lambda_0 + 2\alpha, \varepsilon_3 + \varepsilon_2 + \varepsilon_1 \rangle = 1 \geqslant 0,$$

this irreducibility extends to the required irreducibility in Δ. Now suppose $\mu = 0$. Let Δ_L be the A_7 horizontal subdiagram; in Δ_L, Proposition 8.4 gives us irreducibility for $c \leqslant 5$. The irreducibility for $c = 1$ and $c = 3$ extends to Δ_L since $\langle \lambda_0 + \frac{3}{2}\alpha, \varepsilon + \gamma_1 \rangle = 0$. For $c = 5$, we appeal directly to Section 7j of [4] for the irreducibility.

For configuration (y), we appeal directly to Section 4 of [4] for the irreducibility.

Consider (z). For $\mu = -\frac{1}{2}\alpha$, we can reflect twice and pass to (aa) with $\mu = +\frac{1}{2}\alpha$; so we handle this case by handling (aa) shortly. Thus let $\mu = 0$. When $n = 1$, we can appeal directly to Section 3a of [4] for the irreducibility. For $n > 1$, we take Δ_L to be the subsystem generated by γ_1, α, and ε_1, and we shall show that (SV) holds. Since the diagram has a double line, Lemma 8.5 does not apply, but we can see directly that $\beta=\gamma_2+\gamma_1$ gives the worst possible situation. Since

$$\frac{2\langle\lambda_0+\frac{1}{2}\alpha, \gamma_2+\gamma_1\rangle}{|\alpha|^2}=1\geqslant 0,$$

(SV) holds, and the irreducibility in Δ_L extends to Δ. (Here we have used Corollary 8.3 to reduce matters to c an integer, but we do not restrict the parity of c.)

Consider (aa). We divide matters into subcases, first supposing that $\nu_0^+<\nu_0^-$. In this circumstance let Δ_L be the A_{N-1} subdiagram obtained by deleting ε_t. In Δ_L we have $\nu_{0,L}^+=\nu_0^+$ and $\nu_{0,L}^-=\nu_0^- -1$, and Proposition 8.4 says we have irreducibility in Δ_L. Again we show that (SV) holds. This time the root to check is $\beta=\varepsilon_1+\cdots+\varepsilon_t$, and $c\leqslant\min(\nu_0^+,\nu_0^-)-1\leqslant \nu_0^- -2$ implies

$$\begin{aligned}
&2\langle\lambda_0+\tfrac{1}{2}c\alpha, \varepsilon_1+\cdots+\varepsilon_t\rangle/|\alpha|^2\\
&\quad=(\tfrac{1}{2}(1-\mu_\alpha)+(t-2)+1)-\tfrac{1}{2}c\\
&\quad\geqslant\tfrac{1}{2}(1-\mu_\alpha)+(t-2)+1-\tfrac{1}{2}(2-\mu_\alpha+2(t-1)-2)\\
&\quad=\tfrac{1}{2}>0.
\end{aligned}\tag{8.3}$$

Hence (SV) holds, and the irreducibility extends to Δ.

Next suppose that $\nu_0^+=\nu_0^-+1$. The above computation anyway proves irreducibility for $c\leqslant\nu_0^+-3$, and we have to handle $c=\nu_0^+-2$. If $\mu=+\frac{1}{2}\alpha$, this irreducibility is asserted by the cotangent cases in Section 3a of [4], while if $\mu=0$, this irreducibility is asserted by the tangent cases in Section 3a of [4]. If $\mu=-\frac{1}{2}\alpha$, then we apply the reflections s_{γ_1} and $s_{\alpha+\gamma_1}$ to see that the diagrams for $\mu=+\frac{1}{2}\alpha$ and $\mu=-\frac{1}{2}\alpha$ are equivalent if n and t are adjusted suitably.

Finally suppose that $\nu_0^+>\nu_0^-+1$. Then we let Δ_L be the diagram with $\gamma_n,\ldots,\gamma_{j+1}$ dropped off in such a way that $\nu_0^+=\nu_0^-+1$ in the subdiagram. We have just seen irreducibility in Δ_L. The worst case for (SV) is $\beta=\gamma_{j+1}+\cdots+\gamma_1$, and $c\leqslant\nu_0^+-2$ implies

$$\begin{aligned}
2\langle\lambda_0+\tfrac{1}{2}c\alpha, \gamma_{j+1}+\cdots+\gamma_1\rangle/|\alpha|^2&=\tfrac{1}{2}(1+\mu_\alpha)+j-\tfrac{1}{2}c\\
&=\tfrac{1}{2}\nu_0^+-\tfrac{1}{2}c\geqslant 0.
\end{aligned}$$

Thus (SV) holds, and the irreducibility extends to Δ.

In (bb) and (cc), we are to prove irreducibility for $0<c<\min(\nu_0^+,\nu_{0,L}^-+1)$. Configuration (bb) is handled in the same way as (z): For $\mu=-\frac{1}{2}\alpha$, we reflect twice and pass to (cc). For $\mu=0$, we appeal directly to Section 3b of [4] when $n=1$, and we apply (SV) with Δ_L as the $n=1$ system to pass to general n.

Consider (cc). We divide matters into subcases, first supposing that $\nu_0^+ \leqslant \nu_{0,L}^-$. In this circumstance let Δ_L be the A_{N-1} subdiagram obtained by deleting ε_t. Then ν_0^+ and $\nu_{0,L}^-$ are the ν_0^+ and ν_0^- for Δ_L, and Proposition 8.4 says we have irreducibility in Δ_L for $c \leqslant \min(\nu_0^+, \nu_{0,L}^-) - 1 = \nu_0^+ - 1 \leqslant \nu_{0,L}^- - 1$. We imitate the calculation in (8.3), finding that $c \leqslant \nu_{0,L}^- - 1$ implies

$$\begin{aligned} 2\langle \lambda_0 + \tfrac{1}{2}c\alpha, \varepsilon_1 + \cdots + \varepsilon_t \rangle / |\alpha|^2 &= (\tfrac{1}{2}(1-\mu_\alpha) + (t-2) + \tfrac{1}{2}) - \tfrac{1}{2}c \\ &\geqslant \tfrac{1}{2}(1-\mu_\alpha) + (t-2) + \tfrac{1}{2} - \tfrac{1}{2}(1 - \mu_\alpha + 2(t-1) - 1) = 0. \end{aligned}$$

Hence (SV) holds, and the irreducibility extends to Δ.

Next suppose that $\nu_0^+ = \nu_{0,L}^- + 2$. The above computation anyway proves irreducibility for $c \leqslant \nu_0^+ - 3$, and we have to handle $c = \nu_0^+ - 2$. This is done by reference to Section 3b of [4] in the same way that configuration (aa) referred to Section 3a of [4].

Finally suppose that $\nu_0^+ > \nu_{0,L}^- + 2$. Then we let Δ_L be the diagram with $\gamma_n, \ldots, \gamma_{j+1}$ dropped off in such a way that $\nu_0^+ = \nu_{0,L}^- + 2$ in the subdiagram. Then we can argue as for configuration (aa) to see that (SV) holds for $c \leqslant \nu_0^+ - 2$, and the irreducibility for $c \leqslant \nu_{0,L}^-$ therefore extends to Δ.

9. Irreducibility in Special Basic Cases, Single-Line Diagrams

In this section we shall apply the results of Section 8 to prove Lemma 9.1. In Section 10 we shall extend this lemma suitably to all λ_0 for single-line diagrams. The extended lemma, when combined with Lemma 5.1, will complete the proof of Theorem 1.1b for single-line diagrams, in view of the remarks at the beginning of Section 8.

Lemma 9.1. *Suppose that* rank G = rank K, *that the Dynkin diagram of* Δ^+ *is a single-line diagram, and that the special basic case associated to* λ_0 *is all of* Δ. *If* $\mathfrak{g} \neq \mathfrak{so}(2n, 2)$ *with* $n \geqslant 2$, *then* $U(MAN, \sigma, \tfrac{1}{2}c\alpha)$ *is irreducible for* $0 \leqslant c < \min(\nu_0^+, \nu_0^-)$. *If* $\mathfrak{g} = \mathfrak{so}(2n, 2)$ *with* $n \geqslant 2$ *and if* β_0 *is the unique positive noncompact root orthogonal to* α, *then* $U(MAN, \sigma, \tfrac{1}{2}c\alpha)$ *is irreducible for*

$$0 \leqslant c < \begin{cases} \min(\nu_{0,L}^+, \nu_0^-) & \text{if } \beta_0 \text{ conjugate to } \alpha \text{ via } K \\ \min(\nu_0^+, \nu_{0,L}^-) & \text{if } \beta_0 \text{ conjugate to } -\alpha \text{ via } K; \end{cases}$$

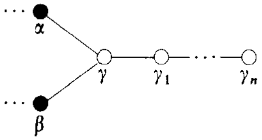
here $\nu_{0,L}^+$ *and* $\nu_{0,L}^-$ *are the* ν_0^+ *and* ν_0^- *for a maximal* $\mathfrak{su}(n,1)$ *subdiagram containing* α.

According to Corollary 8.3, we need check irreducibility at $\frac{1}{2}c\alpha$ only for c an integer of the correct parity and less than the bound stated in the lemma. The parity in question is this: we are to check even c if $\mu = \pm\frac{1}{2}\alpha$ and odd c if $\mu = 0$. Moreover, Lemma 2.1 implies that we can disregard $c = 0$.

We proceed by following the division into cases given in Section 4. We take all root lengths squared to be 2. Evidently there is nothing to prove unless $\min(\nu_0^+, \nu_0^-) > 2$.

(I) Suppose there is a simple root γ_0 of $\Delta_{K,\perp}^+$ of the form (f) or (g) in Lemma 2.2. Possibly by reflecting in α, we may assume that the form is (g). Then $\mu = -\frac{1}{2}\alpha$, and γ_0 is the sum of three Δ^+ simple roots α, γ, and β as in the diagram (4.1). From Section 4, we know that $\nu_0^+ \leqslant \nu_0^-$. Moreover, $\nu_0^+ = 2$ (and hence there is nothing to prove) unless we are in case

(I.3) Suppose γ is a triple point of Δ^+ and the other neighbor γ_1 of γ is compact. Let the (compact) roots extending beyond γ_1 be $\gamma_2, \ldots, \gamma_n$. The diagram is



and $\nu_0^+ = 2n + 2$. When α and β are both nodes, the diagram is $\mathfrak{so}(2n, 2)$ with $n \geqslant 2$, and $-\alpha$ is conjugate to $\beta = \beta_0$ via K. Thus Lemma 9.1 asserts irreducibility only for $0 < c < \nu_{0,L}^- = 2$, and there is nothing to prove. So we may assume that α and β are not both nodes, and we are to prove irreducibility for c equal to any even integer with $0 < c < 2n + 2$. If $n \geqslant 2$, the diagram is of type E, and we must have $n \leqslant 4$.

If $n = 4$, the diagram is of type E_8, and Lemma 9.1 follows from Lemma 8.6, part (r) or (s). If $n = 3$, the diagram is of type E_7, and Lemma 9.1 follows from Lemma 8.6, part (h) or (i).

If $n = 2$, we can assert only that the diagram contains E_6 as in (e) or (f) of Table 8.1. If the diagram is merely E_6, then Lemma 9.1 follows from Lemma 8.6. If the diagram is E_7 or E_8, then η in (e) or (f) has a second neighbor η', necessarily compact, and we have to prove irreducibility at $c = 2$ and $c = 4$. Let Δ_L^+ be the E_6 subdiagram, in which we know there is irreducibility at $c = 2$ and $c = 4$. To pass to Δ^+, we show that (SV) holds. By Lemma 8.5 and Theorem 8.2, it is enough to show for $c = 4$ that

$$\langle \lambda_0 + \tfrac{1}{2}c\alpha, \eta' + \eta + \beta + \gamma \rangle \geqslant 0 \qquad \text{in the case of (e)}$$

$$\langle \lambda_0 + \tfrac{1}{2}c\alpha, \eta' + \eta \rangle \geqslant 0 \qquad \text{in the case of (f).}$$

Since η' is compact, Table 2.1 shows that the left sides in both cases are 0; hence (SV) holds.

Finally suppose $n=1$. We have to prove irreducibility at $c=2$. If α is a node, then Δ^+ contains the D_5 subdiagram in (a) of Table 8.1. Lemma 8.6 tells us there is irreducibility in this subdiagram, which we denote Δ_L^+. To pass to Δ^+, we show that (SV) holds. Letting η' be a (necessarily compact) neighbor of η other than β, we see from Lemma 8.5 and Theorem 8.2 that it is enough to show that

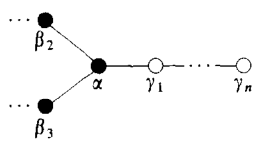
$$\langle \lambda_0 + \tfrac{1}{2}(2\alpha), \eta' + \eta + \beta + \gamma \rangle \geqslant 0.$$

From Table 2.1 we see that the left side is 1; hence (SV) holds.

If α is not a node, then Δ^+ contains the D_5 subdiagram in (b) in Table 8.1. Let Δ_L^+ be this subdiagram. Lemma 8.6 tells us we have irreducibility at $c=2$ in the subdiagram. To pass to Δ^+, we have to check (by Lemma 8.5 and Theorem 8.2) that $\langle \lambda_0 + \alpha, \delta \rangle \geqslant 0$ for two possible roots δ. If η has a neighbor $\eta' \neq \alpha$, then we must check $\delta = \eta' + \eta$; however, $\langle \lambda_0 + \alpha, \eta' + \eta \rangle \geqslant 1$. If β has a neighbor $\eta' \neq \alpha$, then we must check $\delta = \eta' + \beta + \gamma$; however, $\langle \lambda_0 + \alpha, \eta' + \beta + \gamma \rangle = 0$. Hence (SV) holds.

(II) Suppose that there is no simple root of $\Delta_{K,\perp}^+$ of type (f) or (g) in Lemma 2.2 and that α is a triple point. Possibly by reflecting in α, we may assume that at most one of the neighbors β_1, β_2, β_3 of α is compact; say that β_2 and β_3 are noncompact. From Section 4 we know that $\nu_0^+ \leqslant \nu_0^-$. Moreover, $\nu_0^+ \leqslant 2$ (and hence there is nothing to prove) unless we are in case

(II.2) Suppose β_1 is compact. We write γ_1 for β_1. Let the (compact) roots extending beyond γ_1 be $\gamma_2, \ldots, \gamma_n$. The diagram is



and $\nu_0^+ = 1 + \mu_\alpha + 2n$. When β_2 and β_3 are both nodes, the diagram is $\mathfrak{so}(2n, 2)$ with $n \geqslant 2$, and $-\alpha$ is conjugate to $\beta_2 + \alpha + \beta_3 = \beta_0$ via K. Thus Lemma 9.1 asserts irreducibility only for $0 < c < \nu_{0,L}^- = 3 - \mu_\alpha$; here we may take Δ_L^+ to be the A_{n+2} subdiagram obtained by deleting β_3. We know that there is irreducibility in Δ_L^+ for $c \leqslant 1 - \mu_\alpha$. Since

$$\langle \lambda_0 + \tfrac{1}{2}(1-\mu_\alpha)\,\alpha, \beta_3 \rangle = \tfrac{1}{2}(1-\mu_\alpha) - \tfrac{1}{2}(1-\mu_\alpha) = 0,$$

Lemma 8.5 shows that (SV) holds at $c = 1 - \mu_\alpha$. Hence we have the required irreducibility in this case. So we may assume that β_2 and β_3 are not both nodes, and we are to prove irreducibility for c equal to any positive integer satisfying $0 < c < 1 + \mu_\alpha + 2n$ and $c \equiv 1 + \mu_\alpha \bmod 2$.

If $n \geqslant 2$, the diagram is of type E, and we must have $n \leqslant 4$. The cases $n = 3$ and $n = 4$ are then handled by Lemma 8.6 (j) and (t).

Suppose $n = 2$. We have to prove irreducibility for $c \leqslant 3 + \mu_\alpha$. The diagram certainly contains E_6 as in (g) of Table 8.1. If the diagram is merely E_6, then Lemma 9.1 follows from Lemma 8.6. If the diagram is E_7 or E_8, then η in (g) has a second neighbor η', necessarily compact. Let Δ_L^+ be the E_6 subdiagram, in which we know there is irreducibility for $c \leqslant 3 + \mu_\alpha$. Since

$$\langle \lambda_0 + \tfrac{1}{2}(3 + \mu_\alpha)\,\alpha,\, \eta' + \eta + \beta_2 \rangle = (1 + 1 + \tfrac{1}{2}(1 - \mu_\alpha)) - \tfrac{1}{2}(3 + \mu_\alpha) = 1 - \mu_\alpha \geqslant 0,$$

Lemma 8.5 shows that (SV) holds at $c = 3 + \mu_\alpha$. Hence we have irreducibility in Δ^+ when $n = 2$.

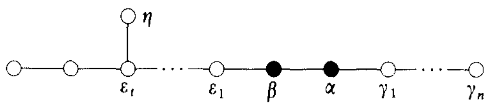
Finally suppose $n = 1$. We have to prove irreducibility for $c \leqslant 1 + \mu_\alpha$. If $\mu = -\tfrac{1}{2}\alpha$, there is nothing to prove. If $\mu = 0$, let Δ_L^+ be the D_4 subdiagram containing β_2, β_3, α, and γ_1. We saw at the start of (II.2) that irreducibility occurs in Δ_L for $c = 1$ when $\mu = 0$. If η' is a second neighbor of β_2 or β_3, then we easily see that $\langle \lambda_0 + \tfrac{1}{2}\alpha,\ \eta' + \beta_2 \rangle$ or $\langle \lambda_0 + \tfrac{1}{2}\alpha,\ \eta' + \beta_3 \rangle$ is $\geqslant 0$. Lemma 8.5 shows that (SV) holds at $c = 1$, and we have the required irreducibility. Finally if $\mu = +\tfrac{1}{2}\alpha$, then Lemma 9.1 follows from part (c) of Lemma 8.6.

(III) Suppose there is no simple root of $\Delta_{K,\perp}^+$ of type (f) or (g) in Lemma 2.2 and that α is not a triple point. If all neighbors of α are of the same type, compact or noncompact, then $\min(\nu_0^+, \nu_0^-) \leqslant 2$ and there is nothing to prove. Thus we may assume we are in case

(III.2) Suppose that α has two neighbors, one compact and one noncompact. If Δ^+ has no triple point, then Δ^+ is of real rank one and Proposition 8.4 applies. Thus we may assume there is a triple point. Possibly by reflecting in α, we may assume that the root β on the side of α toward the triple point is noncompact. Let the compact neighbor be γ_1, and let $\gamma_1, \ldots, \gamma_n$ be the connected chain of compact roots ending in the node γ_n. We know from Section 4 that $\nu_0^+ = 1 + \mu_\alpha + 2n \leqslant \nu_0^-$.

If the diagram is of type D, then it is of type $\mathfrak{so}(2N, 2)$ with $N \geqslant 2$, and Lemma 9.1 insists on irreducibility only up to $\min(\nu_0^+, \nu_{0,L}^-)$, where Δ_L^+ is either of the maximal A_N subdiagrams. The required irreducibility holds in Δ_L^+ by Proposition 8.4, and (just as at the start of case (II.2)) we can show that (SV) holds for $c \leqslant \nu_{0,L}^- - 2$. Hence the required irreducibility holds in Δ^+.

Thus we may assume the diagram is of type E. If β is not a triple point, then the diagram is of the form



with $n \geqslant 1$ and $t \geqslant 1$. Also $t+n \leqslant 3$ since there are at most eight simple roots. We are to prove irreducibility for $c \leqslant 1+\mu_\alpha+2(n-1)$. Let Δ_L^+ be the subdiagram obtained by deleting η. Since $n-t \leqslant 1$, we have

$$\nu_0^+ = \nu_{0,L}^+ = 1+\mu_\alpha+2n \leqslant 1-\mu_\alpha+2(t+3) = \nu_{0,L}^-.$$

Thus Proposition 8.4 shows that there is irreducibility in Δ_L^+ for $c \leqslant 1+\mu_\alpha+2(n-1)$. Since

$$\begin{aligned}
&\langle \lambda_0 + \tfrac{1}{2}(1+\mu_\alpha+2(n-1))\,\alpha, \eta+\varepsilon_t+\cdots+\varepsilon_1+\beta\rangle \\
&\quad = (t+1+\tfrac{1}{2}(1-\mu_\alpha)) - \tfrac{1}{2}(1+\mu_\alpha+2(n-1)) \\
&\quad = t-n+2-\mu_\alpha \geqslant 0,
\end{aligned}$$

Lemma 8.5 says that (SV) holds and that we have irreducibility in Δ^+.

If β is a triple point, then $t=0$ in the above analysis and the argument still works for $n=1$ (even if the unlabeled branch in the diagram has more than 2 roots). For $n=2$ and $n=3$, we appeal directly to parts (k) and (u) of Lemma 8.6, and then the proof of Lemma 9.1 is complete.

10. Irreducibility in General, Single-Line Diagrams

We can now complete the proof of Thereom 1.1 for single-line diagrams with rank G = rank K. The theorem follows immediately from Lemmas 5.1 and 10.1, in view of the remarks at the beginning of Section 8.

Lemma 10.1. *Suppose that* rank G = rank K *and that the Dynkin diagram of* Δ^+ *is a single-line diagram. If the component of* α *in the special basic case is not* $\mathfrak{so}(2n, 2)$ *with* $n \geqslant 2$, *then* $U(MAN, \sigma, \tfrac{1}{2}c\alpha)$ *is irreducible for* $0 \leqslant c < \min(\nu_0^+, \nu_0^-)$. *If the component* Δ_C *of* α *in the special basic case is* $\mathfrak{so}(2n, 2)$ *with* $n \geqslant 2$, *let* β_0 *be the unique positive noncompact root in* Δ_C *orthogonal to* α, *and let* ν_{0,L_1}^+ *and* ν_{0,L_1}^- *be the* ν_0^+ *and* ν_0^- *for a maximal*

$\mathfrak{su}(n, 1)$ *subdiagram* Δ_{L_1} *of* Δ_C *containing* α; *in this case* $U(MAN, \sigma, \frac{1}{2}c\alpha)$ *is irreducible for*

$$0 \leqslant c < \begin{cases} \min(\nu_{0,L_1}^+, \nu_0^-) & \textit{if } \beta_0 \textit{ conjugate to } \alpha \textit{ via } K \cap C \\ \min(\nu_0^+, \nu_{0,L_1}^-) & \textit{if } \beta_0 \textit{ conjugate to } -\alpha \textit{ via } K \cap C. \end{cases}$$

Let $\Delta_{L'}$ be a maximal subsystem of Δ that either is a special basic case or is one of the configurations of Table 8.1. If one is available, choose $\Delta_{L'}$ to be a maximal one such that the component of α has a triple point. (More precisely, we order the Dynkin subdiagrams of Δ under inclusion, insisting that compactness/noncompactness be preserved under inclusion, that α map to α, that μ be preserved, and that $2\langle\lambda_0, \beta\rangle/|\beta|^2$ be the same for each β as for the image of β under the inclusion. With respect to this notion of inclusion, we have a finite partially ordered set, and $\Delta_{L'}$ is to be maximal in this ordering and, if possible, is to have a triple point within the component of α.) Let Δ_L be the component of α in $\Delta_{L'}$. The idea will be to show that (SV) holds for the passage from irreducibility in L to irreducibility in G, for the required range of ν; then Theorem 8.2 will prove Lemma 10.1.

Irreducibility in L is a consequence of Lemmas 8.6 and 9.1. In checking that (SV) holds, it is enough to check that

$$\langle\lambda_0 + \nu, \beta_i\rangle \geqslant 0 \tag{10.1}$$

for the special roots β_i described in Lemma 8.5. As in Sections 8 and 9, the only ν's that need checking are points $\frac{1}{2}c\alpha$ with c an integer in the correct range with $c \equiv 1 + \mu_\alpha \bmod 2$; moreover, we can disregard $c = 0$. Consequently there is nothing to prove unless $\min(\nu_0^+, \nu_0^-) \geqslant 2$.

Let ε be the neighbor of Δ_L that we adjoin and test in Lemma 8.5. Since we have to check whether $\langle\lambda_0 + \frac{1}{2}c\alpha, \varepsilon + \cdots\rangle$ is $\geqslant 0$, the worst case will often be where $\langle\lambda_0, \varepsilon\rangle = 0$. In particular, if ε is orthogonal to α, the worst case will be that ε is noncompact and basic (unless we want to take into account some degeneracy that arises).

We divide matters into cases according to the nature of Δ_L. We normalize all root lengths squared to be 2.

(I) Suppose that Δ_L contains a simple root of $\Delta_{K,\perp}^+$ of the form (f) or (g) in Lemma 2.2, with Δ^+ simple constiuents α, γ, β and with γ not a triple point. By (I) of Section 4, we have $\min(\nu_0^+, \nu_0^-) \leqslant 2$, and hence there is nothing to prove.

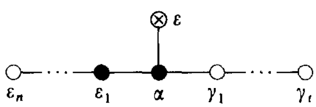
Henceforth we assume that (I) is not the case.

(II) Suppose that Δ_L contains no simple root at all of $\Delta_{K,\perp}^+$ of the form (f) or (g), and suppose that every neighbor of α in Δ_L is compact, or else that every neighbor of α in Δ_L is noncompact. By (II.1) and (III.1) of Section 4, we have $\min(\nu_0^+, \nu_0^-) \leqslant 2$, and hence there is nothing to prove.

Henceforth we assume that (II) is not the case.

(III) Suppose that Δ_L has a Dynkin diagram of type A. Since neither (I) nor (II) holds, Δ_L is of real rank one and α is not a node. We divide matters into subcases according to the placement of ε.

(III.1) Suppose that ε is a neighbor of α. Then the root β_i of Lemma 8.5 is just ε itself. The root ε cannot be basic since otherwise Lemma 2.2 would force it to be in the special basic case, hence in Δ_L. Exactly one neighbor of α in Δ_L is noncompact, and, possibly reflecting in α, we take it to be on the long branch of $\Delta_L - \{\alpha\}$. The root ε may be compact or noncompact; we write $\otimes$ for it and define $s = +1$ if ε is compact, $s = -1$ if ε is noncompact. The diagram of $\Delta_L \cup \{\varepsilon\}$ is



with $n \geqslant t \geqslant 1$, and classification requires $t \leqslant 2$. Then

$$v_0^+ = 1 + \mu_\alpha + 2t \qquad \text{and} \qquad v_0^- = 1 - \mu_\alpha + 2n.$$

For $c \leqslant v_0^+ - 2$, we have

$$\begin{aligned} \langle \lambda_0 + \tfrac{1}{2}c\alpha, \varepsilon \rangle &\geqslant 1 + \langle \lambda_{0,b}, \varepsilon \rangle - \tfrac{1}{2}(v_0^+ - 2) \\ &= 1 + \tfrac{1}{2}(1 + s\mu_\alpha) - \tfrac{1}{2}(1 + \mu_\alpha) - (t-1) \qquad (10.2) \\ &= 2 - t + \tfrac{1}{2}(s-1)\,\mu_\alpha. \end{aligned}$$

If $\mu_\alpha \leqslant 0$, (10.2) is $\geqslant 0$ for $t \leqslant 2$, i.e., in all circumstances. So suppose $\mu_\alpha = 1$. Then (10.2) is $\geqslant 0$ for $t = 1$ and also for $t = 2$ if ε is compact. Moreover, there is no difficulty unless ε is only one step removed from basic.

So suppose $\mu_\alpha = 1$, $t = 2$, ε is noncompact, and ε is only one step removed from basic. Then $c \leqslant v_0^- - 2$ gives

$$\langle \lambda_0 + \tfrac{1}{2}c\alpha, \varepsilon \rangle = 1 - \tfrac{1}{2}c \geqslant 1 - (n-1) = 2 - n.$$

Hence (10.1) holds for $n = 2$. If $n \geqslant 3$, then $\Delta_L \cup \{\varepsilon\}$ contains the diagram (p) in Table 8.1, in contradiction to the requirement that Δ_L contain a triple point if possible.

(III.2) Suppose that ε is not a neighbor of α. Let $\varepsilon_1, \ldots, \varepsilon_t$ and $\gamma_1, \ldots, \gamma_n$ be the roots extending from α, with ε_t and γ_n nodes in Δ_L and with γ_n no farther from ε than ε_t is. Possibly by reflecting in α, we may assume ε_1 is noncompact. Then the diagram of Δ_L is

with $t \geqslant 1$ and $n \geqslant 1$, and ε (by classification) is attached at most two roots from the γ_n end of the diagram.

Suppose ε is attached to γ_n. Then the root to check is $\beta_i = \gamma_1 + \cdots + \gamma_n + \varepsilon$, and $v_0^+ = 1 + \mu_\alpha + 2n$; hence $c \leqslant v_0^+ - 2$ implies

$$\begin{aligned}\langle \lambda_0 + \tfrac{1}{2}c\alpha, \gamma_1 + \cdots + \gamma_n + \varepsilon \rangle &= \langle \lambda_0, \varepsilon \rangle + (\tfrac{1}{2}(1 + \mu_\alpha) + (n-1)) - \tfrac{1}{2}c \\ &\geqslant \langle \lambda_0, \varepsilon \rangle + \tfrac{1}{2}(1 + \mu_\alpha) + (n-1) - \tfrac{1}{2}(1 + \mu_\alpha + 2n - 2) \\ &= \langle \lambda_0, \varepsilon \rangle. \end{aligned} \tag{10.3}$$

Hence (10.1) holds.

Suppose ε is attached one root from the end, necessarily to γ_{n-1}. Then the root to check is $\beta_i = \gamma_1 + \cdots + \gamma_{n-1} + \varepsilon$. A computation analogous to (10.3) shows that (10.1) holds unless $\langle \lambda_0, \varepsilon \rangle = 0$, i.e., unless ε is noncompact basic. But if ε is noncompact basic, we have a contradiction to the construction of Δ_L: If $n > 2$ or $\mu \neq -\frac{1}{2}\alpha$, then $\Delta_L \cup \{\varepsilon\}$ is the diagram (d) in Table 8.1, while if $n = 2$ and $\mu = -\frac{1}{2}\alpha$, then $\Delta_L \cup \{\varepsilon\}$ contains the diagram (b) in the table; in either case a choice of Δ_L with a triple point was available, contradiction. Thus (10.1) holds when ε is attached one root from the end.

Suppose ε is attached two roots from the end. Either $n \geqslant 3$, or $n = 1$ and ε is attached to ε_1. We first suppose that $n \geqslant 3$, so that ε is attached to γ_{n-2}. Then the root to check is $\beta_i = \gamma_1 + \cdots + \gamma_{n-2} + \varepsilon$. A computation analogous to (10.3) shows that (10.1) holds if ε is noncompact and at least two removed from basic or if ε is compact and nonbasic. On the other hand, if ε were compact basic, ε would already be part of Δ_L, contradiction. So we may assume that ε is noncompact and either is basic or is one step removed from basic. Meanwhile we have $v_0^- = 1 - \mu_\alpha + 2t$; hence $c \leqslant v_0^- - 2$ implies

$$\begin{aligned}&\langle \lambda_0 + \tfrac{1}{2}c\alpha, \gamma_1 + \cdots + \gamma_{n-2} + \varepsilon \rangle \\ &\quad \geqslant \langle \lambda_0 - \lambda_{0,b}, \varepsilon \rangle + \tfrac{1}{2}(1 + \mu_\alpha) + (n-3) - \tfrac{1}{2}c \\ &\quad \geqslant \langle \lambda_0 - \lambda_{0,b}, \varepsilon \rangle + \tfrac{1}{2}(1 + \mu_\alpha) + (n-3) - \tfrac{1}{2}(1 - \mu_\alpha + 2t - 2) \\ &\quad = \langle \lambda_0 - \lambda_{0,b}, \varepsilon \rangle + n - t - 2 + \mu_\alpha. \end{aligned} \tag{10.4}$$

Hence (10.1) holds if $n > t + 2$ or if $n = t + 2$ and $\mu_\alpha \neq -1$. The classification implies $n + t + 2 \leqslant 8$; hence $n + t \leqslant 6$. If $n = 5$, then $t \leqslant 1$ and $n > t + 2$, so that (10.1) holds. If $n = 4$, then $t \leqslant 2$ and (10.1) holds unless $t = 2$ and

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$\mu_\alpha = -1$. For (10.1) to fail here, ε must be basic, in which case $\Delta_L \cup \{\varepsilon\}$ would be just the diagram (v) in Table 8.1, in contradiction to maximality.

Now suppose $n=3$, so that $t \leqslant 3$. First, assume ε is one step removed from basic. Then (10.4) shows that (10.1) holds if $2 \geqslant t - \mu_\alpha$. Hence (10.1) holds if $t=1$ or if $t=2$ and $\mu_\alpha \geqslant 0$ or if $t=3$ and $\mu_\alpha = +1$. If $t=2$ and $\mu_\alpha = -1$, then $\Delta_L \cup \{\varepsilon\}$ is just the diagram (q) in Table 8.1, in contradiction to maximality, while if $t=3$ and $\mu_\alpha \neq +1$, then $\Delta_L \cup \{\varepsilon\}$ is just the diagram (x), again a contradiction.

Next assume ε is basic. Then (10.4) shows that (10.1) holds if $1 \geqslant t - \mu_\alpha$. On the other hand, Lemma 2.2 says that ε already is in Δ_L if $\mu_\alpha = -1$; thus we may assume $\mu_\alpha = +1$ or $\mu_\alpha = 0$. Hence (10.1) holds if $t=1$ or if $t=2$ and $\mu_\alpha = +1$. If $t=2$ and $\mu_\alpha = 0$, then $\Delta_L \cup \{\varepsilon\}$ is just the diagram (l) in Table 8.1, in contradiction to maximality, while if $t=3$, then $\Delta_L \cup \{\varepsilon\}$ is just the diagram (w), again a contradiction.

Finally suppose ε is attached two roots from the end and that $n=1$, so that the neighbor of ε in Δ_L is ε_1. Since we have already handled cases where ε is attached zero or one root from the end, as well as some cases where ε is attached two roots from the end, we may assume $t \geqslant 4$. The root to check is $\beta_i = \varepsilon + \varepsilon_1$. We have $v_0^+ = 3 + \mu_\alpha$. If $c \leqslant v_0^+ - 2$, then

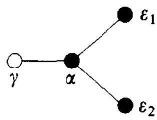
$$\begin{aligned}
&\langle \lambda_0 + \tfrac{1}{2}c\alpha, \varepsilon + \varepsilon_1 \rangle \\
&\quad \geqslant \langle \lambda_0 - \lambda_{0,b}, \varepsilon \rangle + \langle \lambda_{0,b}, \varepsilon \rangle + \tfrac{1}{2}(1 - \mu_\alpha) - \tfrac{1}{2}(1 + \mu_\alpha) \\
&\quad = \langle \lambda_0 - \lambda_{0,b}, \varepsilon \rangle + \langle \lambda_{0,b}, \varepsilon \rangle - \mu_\alpha. \qquad (10.5)
\end{aligned}$$

From (10.5) we see that (10.1) holds unless ε is noncompact basic and $\mu_\alpha = +1$. In this case Lemma 2.2 shows that ε is already in Δ_L, contradiction. Hence (10.1) holds in all cases.

(IV) Suppose that Δ_L has a Dynkin diagram of type D_4.

(IV.1) Suppose (also) that Δ_L contains a simple root of $\Delta_{K,\perp}^+$ of the form (f) or (g) in Lemma 2.2. Possibly by reflecting in α, we may assume that the form is (g). Let α, γ, and β be the simple constituents as in (4.1); the root γ is compact and we have $\mu = -\tfrac{1}{2}\alpha$. The root γ has to be the triple point in D_4, and we let δ be the remaining node. If δ is noncompact, then $v_0^+ = 2$ and there is nothing to prove. If δ is compact, then the diagram is $\mathfrak{so}(6, 2)$, and we have a valid estimate $v_{0,L_1}^- = 2$, where Δ_{L_1} is the A_3 diagram containing α, γ, and δ; thus again there is nothing to prove.

(IV.2) Suppose that Δ_L contains no simple root of $\Delta_{K,\perp}^+$ of the form (f) or (g) in Lemma 2.2. Then α is not a node, since by assumption we are not in case (II). Thus α is the triple point. Since we are not in case (II), we may assume, possibly by reflecting in α, that two of the nodes are noncompact and one is compact. The diagram of Δ_L is then of the form



and is of type $\mathfrak{so}(6,2)$. We have

$$\nu_0^+ = 3 + \mu_\alpha \qquad \text{and} \qquad \nu_{0,L_1}^- = 3 - \mu_\alpha,$$

where Δ_{L_1} is the A_3 diagram containing γ, α, and ε_1. If $\mu_\alpha \neq 0$, then one of these estimates is 2, and there is nothing to prove. So assume $\mu_\alpha = 0$. We adjoin ε to Δ_L, necessarily to one of the nodes. Since $\mu = 0$, $c \leqslant 1$ implies

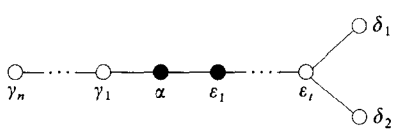
$$\langle \lambda_0 + \tfrac{1}{2}c\alpha, \varepsilon + \text{node} \rangle \leqslant \langle \lambda_0, \text{node} \rangle - \tfrac{1}{2}c \geqslant \tfrac{1}{2} - \tfrac{1}{2} = 0.$$

Thus (10.1) holds.

(V) Suppose that Δ_L has a Dynkin diagram of type D_N, $N \geqslant 5$, and is $\mathfrak{so}$(even, even).

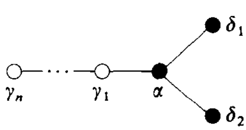
(V.1) Suppose (also) that Δ_L contains a simple root of $\Delta_{K,\perp}^+$ of the form (f) or (g) in Lemma 2.2. Possibly by reflecting in α, we may assume that the form is (g), given as in (4.1) as the sum of α, γ, and β with γ compact and with $\mu = -\tfrac{1}{2}\alpha$. Since we are not in case (I), γ is the triple point. We may assume that the third neighbor of γ is compact, since otherwise $\min(\nu_0^+, \nu_0^-) = 2$. Then α and β are nodes. Also Δ_L is of the form $\mathfrak{so}(2N-2, 2)$, and we have a valid estimate $\nu_{0,L_1}^- = 2$. Thus there is nothing to prove.

(V.2) Suppose that Δ_L contains no simple root of $\Delta_{K,\perp}^+$ of the form (f) or (g) in Lemma 2.2. Since we are not in case (II), we may assume, possibly by reflecting in α, that the diagram is



with $n \geqslant 1$, $t \geqslant 1$

or



with $n \geqslant 1$, $t = 0$.

We have valid estimates

$$\nu_0^+ = 1 + \mu_\alpha + 2n \qquad \text{and} \qquad \nu_{0,L_1}^- = 1 - \mu_\alpha + 2(t+1).$$

If the root ε is adjoined to γ_n, then the root to check is $\beta_i = \varepsilon + \gamma_n + \cdots + \gamma_1$. If $c \leqslant \nu_0^+ - 2$, then

$$\begin{aligned}\langle \lambda_0 + \tfrac{1}{2}c\alpha, \varepsilon + \gamma_n + \cdots + \gamma_1 \rangle &= \langle \lambda_0, \varepsilon \rangle + \tfrac{1}{2}(1 + \mu_\alpha) + (n-1) - \tfrac{1}{2}c \\ &\geqslant \langle \lambda_0, \varepsilon \rangle,\end{aligned}$$

and hence (10.1) holds. If the root ε is adjoined to δ_j ($j = 1$ or 2), then the root to check is $\beta_i = \varepsilon + \delta_j + \varepsilon_t + \cdots + \varepsilon_1$. If $c \leqslant \nu_{0,L_1}^- - 2$, then

$$\begin{aligned}\langle \lambda_0 + \tfrac{1}{2}c\alpha, \varepsilon + \delta_j + \varepsilon_t + \cdots + \varepsilon_1 \rangle &= \langle \lambda_0, \varepsilon \rangle + \tfrac{1}{2}(1 - \mu_\alpha) + t - \tfrac{1}{2}c \\ &\geqslant \langle \lambda_0, \varepsilon \rangle,\end{aligned}$$

and hence (10.1) holds.

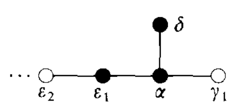
(VI) Suppose that Δ_L has a Dynkin diagram of type D_N, $N \geqslant 5$, and is $\mathfrak{so}^*(2N)$. Referring to Table 8.1, we see that we must consider diagram (d) as Δ_L, in addition to all possible special basic cases.

(VI.1) Suppose (also) that Δ_L contains a simple root of $\Delta_{K,\perp}^+$ of the form (f) or (g) in Lemma 2.2. Possibly by reflecting in α, we may assume that the form is (g), given as in (4.1) as the sum of α, γ, and β with γ compact and with $\mu = -\tfrac{1}{2}\alpha$. Since we are not in case (I), γ is the triple point. Then exactly one of α and β is a node. Also $\nu_0^+ = 4 \leqslant \nu_0^-$, and the diagram Δ_L contains either (a) or (b) in Table 8.1; we let $\Delta_{L'}$ be this D_5 subdiagram. It is enough to test the adjoining of ε to $\Delta_{L'}$. To have (10.1), we need $\langle \lambda_0, \beta_i \rangle \geqslant 1$.

Suppose $\Delta_{L'}$ is as in (a). If ε is adjoined to α, it is not basic (because it is not in Δ_L), and thus $\langle \lambda_0, \varepsilon \rangle \geqslant 1$. If ε is adjoined to γ_1, then $\langle \lambda_{0,b}, \gamma_1 \rangle = 1$ handles matters. If ε is adjoined to η, then $\langle \lambda_{0,b}, \eta \rangle = 1$ handles matters. So (10.1) holds.

Suppose $\Delta_{L'}$ is as in (b). If ε is adjoined to β, nondegeneracy says ε cannot be noncompact basic, and thus $\langle \lambda_0, \varepsilon \rangle \geqslant 1$. If ε is adjoined to γ_1, then $\langle \lambda_{0,b}, \gamma_1 \rangle = 1$ handles matters. If ε is adjoined to η, then $\langle \lambda_{0,b}, \eta \rangle = 1$ (valid since $\mu = -\tfrac{1}{2}\alpha$) handles matters. So (10.1) holds.

(VI.2) Suppose that Δ_L contains no simple root of $\Delta_{K,\perp}^+$ of the form (f) or (g) in Lemma 2.2. Since we are not in case (II), we may assume, possibly by reflecting in α, that Δ_L is as in (d) in Table 8.1 or is special basic of the form



(10.6)

First, suppose Δ_L is as in (d) in Table 8.1 but with the noncompact node called δ. Let ε be adjoined to Δ_L. If ε is adjoined to γ_n, then $\beta_i = \gamma_1 + \cdots + \gamma_n + \varepsilon$ and thus $c \leqslant v_0^+ - 2$ routinely gives

$$\langle \lambda_0 + \tfrac{1}{2}c\alpha, \gamma_1 + \cdots + \gamma_n + \varepsilon \rangle \geqslant \langle \lambda_0, \varepsilon \rangle \geqslant 0.$$

Similarly if ε is adjoined to ε_t, then $\beta_i = \varepsilon + \varepsilon_t + \cdots + \varepsilon_1$ and thus $c \leqslant v_0^- - 2$ routinely gives

$$\langle \lambda_0 + \tfrac{1}{2}c\alpha,\ \varepsilon + \varepsilon_t + \cdots + \varepsilon_1 \rangle \geqslant \langle \lambda_0, \varepsilon \rangle \geqslant 0.$$

If ε is adjoined to the noncompact node δ, then $\beta_i = \gamma_1 + \cdots + \gamma_{n-1} + \delta + \varepsilon$, and $c \leqslant v_0^+ - 2$ gives us only

$$\langle \lambda_0 + \tfrac{1}{2}c\alpha, \gamma_1 + \cdots + \gamma_{n-1} + \delta + \varepsilon \rangle \geqslant \langle \lambda_0, \varepsilon \rangle - 1.$$

However, ε cannot be noncompact basic, since otherwise $\delta + \varepsilon$ would exhibit degeneracy, and this expression is therefore $\geqslant 0$. Hence (10.1) holds no matter how ε is placed.

Now suppose Δ_L is special basic of the form (10.6). Here $v_0^+ = 3 + \mu_\alpha \leqslant v_0^-$. Let $\Delta_{L'}$ be the system generated by the five simple roots pictured in (10.6). With ε adjoined to $\Delta_{L'}$ (instead of Δ_L), it is enough to prove that the root β_i defined in Lemma 8.5 satisfies

$$\langle \lambda_0 + \tfrac{1}{2}(1 + \mu_\alpha)\,\alpha, \beta_i \rangle \geqslant 0. \tag{10.7}$$

If ε is adjoined to ε_2, then $\beta_i = \varepsilon + \varepsilon_2 + \varepsilon_1$ and we have

$$\langle \lambda_0 + \tfrac{1}{2}(1 + \mu_\alpha)\,\alpha, \varepsilon + \varepsilon_2 + \varepsilon_1 \rangle = \langle \lambda_0, \varepsilon \rangle + 1 + \tfrac{1}{2}(1 - \mu_\alpha) - \tfrac{1}{2}(1 + \mu_\alpha) \geqslant 0.$$

If ε is adjoined to γ_1, then $\beta_i = \gamma_1 + \varepsilon$ and we have

$$\langle \lambda_0 + \tfrac{1}{2}(1 + \mu_\alpha)\,\alpha, \gamma_1 + \varepsilon \rangle = \langle \lambda_0, \varepsilon \rangle + \tfrac{1}{2}(1 + \mu_\alpha) - \tfrac{1}{2}(1 + \mu_\alpha) \geqslant 0.$$

Finally if ε is adjoined to δ, then $\beta_i = \delta + \varepsilon$ and we have

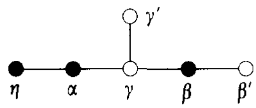
$$\langle \lambda_0 + \tfrac{1}{2}(1 + \mu_\alpha)\,\alpha, \delta + \varepsilon \rangle = \langle \lambda_0, \varepsilon \rangle + \tfrac{1}{2}(1 - \mu_\alpha) - \tfrac{1}{2}(1 + \mu_\alpha) = \langle \lambda_0, \varepsilon \rangle - \mu_\alpha.$$

This expression is $\geqslant 0$ unless ε is noncompact basic and $\mu_\alpha = +1$, in which case $\delta + \varepsilon$ is a root of type (f) in Lemma 2.2 and ε is already in Δ_L. Thus (10.7) is valid, and (10.1) holds in all cases.

(VII) Suppose that Δ_L has a Dynkin diagram of type E_6. Referring to Table 8.1, we see that we may assume that Δ_L is a special basic case.

(VII.1) Suppose (also) that Δ_L contains a simple root of $\Delta^+_{K,\perp}$ of the form (f) or (g) in Lemma 2.2. Possibly by reflecting in α, we may assume that the form is (g), given as in (4.1) as the sum of α, γ, and β with γ compact and with $\mu = -\frac{1}{2}\alpha$. Since we are not in case (I), γ is the triple point. If the third neighbor of γ (other than α and β) is noncompact, then $\nu_0^+ = 2 \leqslant \nu_0^-$ from Section 4, and there is nothing to prove. So we assume this neighbor is compact. One of the three neighbors of γ must be a node in Δ_L, and we divide into cases accordingly.

(VII.1a) Suppose the third neighbor is a node. Then $\nu_0^+ = 4 \leqslant \nu_0^-$, and the diagram is



Whether ε is placed next to η or to β', the equalities $\langle \lambda_0, \eta \rangle = \langle \lambda_0, \beta' \rangle = 1$ force $\langle \lambda_0 + \alpha, \beta_i \rangle \geqslant 0$. Thus (10.1) holds whatever the placement of ε.

(VII.1b) Suppose α is a node. Then $\nu_0^+ = 6 \leqslant \nu_0^-$, and the diagram is (e) in Table 8.1. If ε is placed next to γ_2, then $\beta_i = \gamma + \gamma_1 + \gamma_2 + \varepsilon$ and we have

$$\langle \lambda_0 + 2\alpha, \gamma + \gamma_1 + \gamma_2 + \varepsilon \rangle = \langle \lambda_0, \varepsilon \rangle;$$

thus (10.1) holds in this case. If ε is placed next to η, then $\beta_i = \varepsilon + \eta + \beta + \gamma$ and we have

$$\langle \lambda_0 + 2\alpha, \varepsilon + \eta + \beta + \gamma \rangle = \langle \lambda_0, \varepsilon \rangle - 1.$$

Thus (10.1) holds in this case unless ε is noncompact basic. But when ε is noncompact basic, $\Delta_L \cup \{\varepsilon\}$ is just the diagram (m) in Table 8.1, in contradiction to maximality. Thus (10.1) holds in all cases.

(VII.1c) Suppose β is a node. Then $\nu_0^+ = 6 \leqslant \nu_0^-$, and the diagram is (f) in Table 8.1. If ε is placed next to γ_2, then $\beta_i = \gamma + \gamma_1 + \gamma_2 + \varepsilon$ and we have

$$\langle \lambda_0 + 2\alpha, \gamma + \gamma_1 + \gamma_2 + \varepsilon \rangle = \langle \lambda_0, \varepsilon \rangle;$$

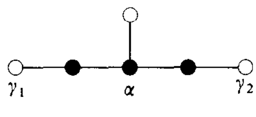
thus (10.1) holds in this case. If ε is placed next to η, then $\beta_i = \varepsilon + \eta$ and we have

$$\langle \lambda_0 + 2\alpha, \varepsilon + \eta \rangle = \langle \lambda_0, \varepsilon \rangle - 1.$$

Thus (10.1) holds in this case unless ε is noncompact basic. But when ε is noncompact basic, $\Delta_L \cup \{\varepsilon\}$ is just the diagram (n) in Table 8.1, in contradiction to maximality. Thus (10.1) holds in all cases.

(VII.2) Suppose that Δ_L contains no simple root of $\Delta^+_{K,\perp}$ of the form (f) or (g) in Lemma 2.2. Since we are not in case (II), α is the triple point or α is a non-node next to the triple point.

(VII.2a) Suppose α is the triple point. Possibly by reflecting in α, we may assume that two of the neighbors are noncompact and the other neighbor is compact, since we are not in case (II). If the diagram is



(10.8)

then $\nu_0^+ = 3 + \mu_\alpha \leqslant \nu_0^-$ and the equalities $\langle \lambda_0, \gamma_1 \rangle = \langle \lambda_0, \gamma_2 \rangle = 1$ force $\langle \lambda_0 + \frac{1}{2}(1 + \mu_\alpha)\,\alpha, \beta_i \rangle \geqslant 0$ whether ε is placed next to γ_1 or to γ_2. Thus (10.1) holds whatever the placement of ε.

The alternative is for the diagram to be (g) in Table 8.1, with $\nu_0^+ = 5 + \mu_\alpha \leqslant \nu_0^-$. If ε is placed next to γ_2, then $\beta_i = \gamma_1 + \gamma_2 + \varepsilon$ and we have

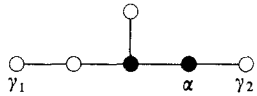
$$\langle \lambda_0 + \tfrac{1}{2}(3 + \mu_\alpha)\,\alpha, \gamma_1 + \gamma_2 + \varepsilon \rangle = \langle \lambda_0, \varepsilon \rangle;$$

thus (10.1) holds in this case. If ε is placed next to η, then $\beta_i = \varepsilon + \eta + \beta_2$ and we have

$$\langle \lambda_0 + \tfrac{1}{2}(3 + \mu_\alpha)\,\alpha, \varepsilon + \eta + \beta_2 \rangle = \langle \lambda_0, \varepsilon \rangle - \mu_\alpha.$$

Thus (10.1) holds in this case unless ε is noncompact basic and $\mu_\alpha = +1$. But when ε is noncompact basic and $\mu_\alpha = +1$, $\Delta_L \cup \{\varepsilon\}$ is just the diagram (o) in Table 8.1, in contradiction to maximality. Thus (10.1) holds in all cases.

(VII.2b) Suppose α is a non-node next to the triple point. Possibly by reflecting in α, we may assume, since we are not in case (II), that the diagram is



with $\nu_0^+ = 3 + \mu_\alpha$. Whether ε is placed next to γ_1 or to γ_2, the equalities $\langle \lambda_0, \gamma_1 \rangle = 1$ and $\langle \lambda_0, \gamma_2 \rangle = \frac{1}{2}(1 + \mu_\alpha)$ force $\langle \lambda_0 + \frac{1}{2}(1 + \mu_\alpha)\,\alpha, \beta_i \rangle \geqslant 0$. Thus (10.1) holds in all cases.

(VIII) Suppose that Δ_L has a Dynkin diagram of type E_7. Referring to Table 8.1, we see that we have to consider special basic cases and a number of other configurations.

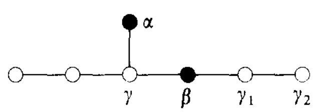
(VIII.1) Suppose (also) that Δ_L contains a simple root of $\Delta^+_{K,\perp}$ of the form (f) or (g) in Lemma 2.2. Possibly by reflecting in α, we may assume that the form is (g), given as in (4.1) as the sum of α, γ, and β with γ compact and with $\mu = -\frac{1}{2}\alpha$. Since we are not in case (II), γ is the triple point. If the third neighbor of γ (other than α and β) is noncompact, then $\nu_0^+ = 2 \leqslant \nu_0^-$ from Section 4, and there is nothing to prove. So we assume this neighbor is compact. One of the three neighbors of γ must be a node in Δ_L, and we divide into cases accordingly.

(VIII.1a) If the third neighbor is a node, then the E_6 subdiagram $\Delta_{L'}$ is as in (VII.1a) and the argument given there handles matters.

(VIII.1b) Suppose α is a node. Then Δ_L is special basic or is of the form (m) in Table 8.1. First suppose Δ_L is special basic. If Δ_L is of the form (h) in Table 8.1, then $\nu_0^+ = 8 \leqslant \nu_0^-$ and ε must be placed next to γ_3. Since

$$\langle \lambda_0 + 3\alpha, \gamma + \gamma_1 + \gamma_2 + \gamma_3 + \varepsilon \rangle = \langle \lambda_0, \varepsilon \rangle,$$

(10.1) holds. The other possibility for Δ_L special basic is the diagram



with $\nu_0^+ = 6 \leqslant \nu_0^-$. Here ε must be placed next to γ_2, and

$$\langle \lambda_0 + 2\alpha, \gamma + \beta + \gamma_1 + \gamma_2 + \varepsilon \rangle = \langle \lambda_0, \varepsilon \rangle;$$

thus (10.1) holds.

Next suppose Δ_L is of the form (m); let us denote the root ε in that diagram by δ. The root ε that we adjoin to Δ_L must be adjoined to δ and cannot be noncompact basic (to avoid a degeneracy from $\varepsilon + \delta$). Thus

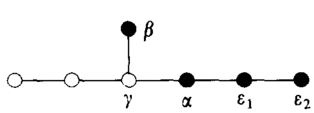
$$\langle \lambda_0 + 2\alpha, \delta + \varepsilon + \eta + \beta + \gamma \rangle = \langle \lambda_0, \varepsilon \rangle - 1 \geqslant 0,$$

and (10.1) holds.

(VIII.1c) Suppose β is a node. Then Δ_L is special basic or is of the form (n) in Table 8.1. First suppose Δ_L is special basic. If Δ_L is of the form (i) in Table 8.1, then $\nu_0^+ = 8 \leqslant \nu_0^-$ and ε must be placed next to γ_3. Since

$$\langle \lambda_0 + 3\alpha, \gamma + \gamma_1 + \gamma_2 + \gamma_3 + \varepsilon \rangle = \langle \lambda_0, \varepsilon \rangle,$$

(10.1) holds. The other possibility for Δ_L special basic is the diagram



with $\nu_0^+ = 6 \leqslant \nu_0^-$. Here ε must be placed next to ε_2, and

$$\langle \lambda_0 + 2\alpha, \varepsilon_1 + \varepsilon_2 + \varepsilon \rangle = \langle \lambda_0, \varepsilon \rangle;$$

thus (10.1) holds.

Next suppose Δ_L is of the form (n); let us denote the root ε in that diagram by δ. The root ε that we adjoin to Δ_L must be adjoined to δ and cannot be noncompact basic (to avoid a degeneracy from $\varepsilon + \delta$). Thus

$$\langle \lambda_0 + 2\alpha, \varepsilon + \delta + \eta \rangle = \langle \lambda_0, \varepsilon \rangle - 1 \geqslant 0,$$

and (10.1) holds.

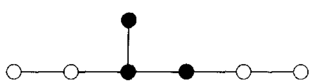
(VIII.2) Suppose that Δ_L contains no simple root of $\Delta_{K,\perp}^+$ of the form (f) or (g) in Lemma 2.2. Since we are not in case (II), we may assume α is not a node in Δ_L.

(VIII.2a) Suppose α is the triple point. Possibly by reflecting in α, we may assume that two of the neighbors are noncompact and the other is compact. As in (VII.2a) there is no difficulty if the neighbor of α that is a node is compact.

Suppose Δ_L is special basic. If Δ_L is of the form (j) in Table 8.1, then $\nu_0^+ = 7 + \mu_\alpha \leqslant \nu_0^-$ and the root ε must be adjoined to γ_3. Then

$$\langle \lambda_0 + \tfrac{1}{2}(5 + \mu_\alpha)\,\alpha, \gamma_1 + \gamma_2 + \gamma_3 + \varepsilon \rangle = \langle \lambda_0, \varepsilon \rangle$$

and (10.1) holds. The alternative is for Δ_L to be of the form



with $\nu_0^+ = 5 + \mu_\alpha$. Here ε must be placed next to ε_3, and

$$\langle \lambda_0 + \tfrac{1}{2}(3 + \mu_\alpha)\,\alpha, \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon \rangle = \langle \lambda_0, \varepsilon \rangle + 1 - \mu_\alpha \geqslant 0;$$

thus (10.1) holds.

If Δ_L is not special basic, it is of one of the forms (p) and (o) in Table 8.1; let us denote the root ε in those diagrams by δ. In (p) we have $\mu = \frac{1}{2}\alpha$ and $\nu_0^+ = 6 = \nu_0^-$. The root ε must be placed next to ε_3, and we have

$$\langle \lambda_0 + 2\alpha, \varepsilon + \varepsilon_3 + \varepsilon_2 + \varepsilon_1 \rangle = \langle \lambda_0, \varepsilon \rangle;$$

thus (10.1) holds. In (o) we have $\mu = \frac{1}{2}\alpha$ and $\nu_0^+ = 6 \leqslant \nu_0^-$. The root ε that we adjoin to Δ_L must be adjoined to δ and cannot be noncompact basic (to avoid a degeneracy from $\varepsilon+\delta$). Thus

$$\langle \lambda_0+2\alpha, \varepsilon+\delta+\eta+\beta_2\rangle=\langle \lambda_0, \varepsilon\rangle-1\geqslant 0,$$

and (10.1) holds.

(VIII.2b) Suppose α is adjacent to the triple point on the medium-length branch. Possibly reflecting in α, we may assume the triple point is noncompact. Then $\nu_0^+=3+\mu_\alpha\leqslant \nu_0^-$. Referring to Table 8.1, we see that Δ_L is special basic. Thus the roots $\gamma_1, \gamma_2, \gamma_3$ on the long branch are all compact, and we have

$$\langle \lambda_0+\tfrac{1}{2}(1+\mu_\alpha)\,\alpha, \beta_i\rangle\geqslant 3-\tfrac{1}{2}(1+\mu_\alpha)\geqslant 0.$$

Thus (10.1) holds.

(VIII.2c) Suppose α is on the long branch from the triple point. Possibly reflecting in α, we may assume its neighbor that is closer to the triple point is noncompact. Then Δ_L is special basic or is the reflection in α of (q) or (l) in Table 8.1. If Δ_L is special basic, it is of the form (k) or has α one root closer to the end. In the respective cases, we have $\nu_0^+=5+\mu_\alpha\leqslant \nu_0^-$ and $\nu_0^+=3+\mu_\alpha\leqslant \nu_0^-$. We find $\langle \lambda_0+\frac{1}{2}(\nu_0^+-2)\,\alpha, \beta_i\rangle=\langle \lambda_0, \varepsilon\rangle$, and hence (10.1) holds. If Δ_L is the reflection in α of (q) or (l), then $\nu_0^+=6=\nu_0^-$ and $\nu_0^+=5<\nu_0^-$ in the respective cases. Again we find $\langle \lambda_0+\frac{1}{2}(\nu_0^+-2)\,\alpha, \beta_i\rangle=\langle \lambda_0, \varepsilon\rangle$, and hence (10.1) holds.

(IX) Suppose that Δ_L has a Dynkin diagram of type E_8. Then $\Delta_L=\Delta$, and there is nothing to prove. This completes the proof of Lemma 10.1.

11. Irreducibility in Double-Line Diagrams

Now we take up the irreducibility problem in double-line diagrams. Lemmas 11.1 and 11.2, in combination with Lemmas 6.1 and 7.1, will complete the proof of Theorem 1.1 for double-line diagrams except for the unitarity of the isolated representations and the nonunitarity of the gap in Theorem 1.1b(vi).

Lemma 11.1. *Suppose that the Dynkin diagram of Δ^+ is a classical double-line diagram and that α is short. Then $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is irreducible for $0\leqslant c<\min(\nu_0^+, \nu_0^-)$ unless the component of α in the special basic case associated to λ_0 is of type $\mathfrak{sp}(n, 1)$ with $n\geqslant 2$, with $\mu=0$, and with α adjacent* *to the long simple root. In this case* $U(MAN, \sigma, \frac{1}{2}c\alpha)$ *is irreducible for* $0 \leqslant c < \min(v_0^+, v_0^-) - 2$.

Proof. Corollary 8.3 shows that it is enough to prove irreducibility when c is an integer with $c \equiv 1 + \mu_\alpha \bmod 2$ and with c in the above range. Lemma 2.1 implies that we may disregard $c = 0$. We shall follow the division into cases in Section 6, using the notation introduced there. Evidently there is nothing to prove unless $\min(v_0^+, v_0^-) > 2$.

Proof for $\mathfrak{sp}(p, q)$. Normalize so that the short roots β have $|\beta|^2 = 2$.

(I) Suppose there is a simple root of $\Delta_{K,\perp}^+$ of the form (f) or (g) in Lemma 2.2. Then $\min(v_0^+, v_0^-) = 2$, and there is nothing to prove.

(II) Suppose there is no simple root of $\Delta_{K,\perp}^+$ of type (f) or (g) in Lemma 2.2.

If the component of α in the special basic case is a Dynkin diagram of type A, we let Δ_L be that subdiagram. Unless α has two neighbors in Δ_L, one compact and one noncompact, case (III) of Section 4 shows that $\min(v_0^+, v_0^-) \leqslant 2$, and there is nothing to prove. Thus we may assume that Δ_L is of real rank one and α is not a node. Proposition 8.4 gives us irreducibility in Δ_L for $c \leqslant \min(v_0^+, v_0^-) - 2$; we shall show that (SV) holds, so that the irreducibility extends to Δ. Let Δ_L have simple roots $e_i - e_{i+1}, \ldots, e_j - e_{j+1}$ with $i \leqslant j$, and let $\alpha = e_k - e_{k+1}$, $i + 1 \leqslant k \leqslant j - 1$. For β in $\Delta(\mathfrak{u})$ to give $\langle \lambda_0 + \frac{1}{2}c\alpha, \beta\rangle < 0$, we must have $\langle \beta, \alpha\rangle < 0$. Thus $\beta = e_s - e_k$ or $\beta = e_{k+1} \pm e_t$ or $\beta = 2e_{k+1}$. The worst cases are $e_{i-1} - e_k$, $e_{k+1} - e_{j+2}$, and $2e_{k+1}$. Possibly by reflecting in α, we may assume $e_{k+1} - e_{k+2}$ is noncompact. Then $c \leqslant v_0^+ - 2$ implies

$$\begin{aligned}&\langle \lambda_0 + \tfrac{1}{2}c\alpha, e_{i-1} - e_k\rangle \\ &\quad = \langle \lambda_0, e_{i-1} - e_i\rangle + \tfrac{1}{2}(1 + \mu_\alpha) + (k - i - 1) - \tfrac{1}{2}c \\ &\quad \geqslant \langle \lambda_0, e_{i-1} - e_i\rangle \geqslant 0,\end{aligned}$$

while $c \leqslant v_0^- - 2$ implies

$$\begin{aligned}&\langle \lambda_0 + \tfrac{1}{2}c\alpha, e_{k+1} - e_{j+2}\rangle \\ &\quad = \langle \lambda_0, e_{j+1} - e_{j+2}\rangle + \tfrac{1}{2}(1 - \mu_\alpha) + (j - k - 1) - \tfrac{1}{2}c \\ &\quad \geqslant \langle \lambda_0, e_{j+1} - e_{j+2}\rangle \geqslant 0.\end{aligned}$$

Since $k \leqslant j - 1$, $2e_{k+1}$ is not simple. Thus we have

$$\begin{aligned}&\langle \lambda_0 + \tfrac{1}{2}c\alpha, 2e_{k+1}\rangle \\ &\quad = \langle \lambda_0, e_{k+1} - e_{j+2}\rangle + \langle \lambda_0, e_{k+1} + e_{j+2}\rangle + 2\langle \tfrac{1}{2}c\alpha, e_{k+1} - e_{j+2}\rangle \\ &\quad \geqslant 2\langle \lambda_0 + \tfrac{1}{2}c\alpha, e_{k+1} - e_{j+2}\rangle,\end{aligned}$$

and this we have seen is $\geqslant 0$ for $c \leqslant \min(\nu_0^+, \nu_0^-) - 2$. Thus (SV) holds, and we obtain the desired irreducibility.

Now suppose that the component of α in the special basic case is of type C. Let Δ_L be that subdiagram.

(II.1) Suppose that the only neighbors of α are connected to α by single lines and that they are all of the same type, compact or noncompact. Then $\min(\nu_0^+, \nu_0^-) \leqslant 2$, and there is nothing to prove.

For the remainder of the proof for $\mathfrak{sp}(p, q)$, we assume that (II.1) is not the case.

(II.2) Suppose α is not adjacent to the long root ε. Possibly by reflecting in α, we may assume that Δ_L is of the form (6.1) with $s \geqslant 1$ and $t \geqslant 0$. Then $\nu_0^+ = 1 + \mu_\alpha + 2s \leqslant \nu_0^-$, and Proposition 8.4 gives us irreducibility for $c \leqslant \nu_0^+ - 2$. For applying (SV), the worst root to test is $\eta + \gamma_s + \cdots + \gamma_1$, where η is a second simple neighbor of γ_s. Then we find that $c \leqslant \nu_0^+ - 2$ implies

$$\langle \lambda_0 + \tfrac{1}{2}c\alpha, \eta + \gamma_s + \cdots + \gamma_1 \rangle \geqslant \langle \lambda_0, \eta \rangle \geqslant 0, \tag{11.1}$$

and the irreducibility extends to Δ.

(II.3) Suppose α is adjacent to ε. Possibly by reflecting in α, we may assume that Δ_L is of the form (6.2) with $s \geqslant 0$. If $\mu = +\frac{1}{2}\alpha$, then $\nu_0^- = 0$ and there is nothing to prove. If $\mu = -\frac{1}{2}\alpha$, we have $\nu_0^+ = 2s \leqslant \nu_0^-$, and Propostion 8.4 gives us irreducibility in Δ_L for $c \leqslant \nu_0^+ - 2$. For applying (SV), the worst root to test is $\eta + \gamma_s + \cdots + \gamma_1$, where η is a second simple neighbor of γ_s, and we have nothing to prove unless $s > 0$. If $s > 0$ and $c \leqslant \nu_0^+ - 2$, then (11.1) holds, and the irreducibility extends to Δ.

Finally suppose $\mu = 0$. If $s > 0$, then $\nu_0^+ = \nu_0^- = 1 + 2(s+1)$, and Proposition 8.4 gives us irreducibility in Δ_L for $c \leqslant \nu_0^+ - 4$. For applying (SV), the worst root to test is $\eta + \gamma_s + \cdots + \gamma_1$, where η is a second simple neighbor of γ_s. For $c \leqslant \nu_0^+ - 4$, we obtain

$$\langle \lambda_0 + \tfrac{1}{2}c\alpha, \eta + \gamma_s + \cdots + \gamma_1 \rangle \geqslant \langle \lambda_0, \eta \rangle \geqslant 0,$$

and the irreducibility extends to Δ. If $s = 0$, we have $\nu_0^+ = \nu_0^- = 3$, and Proposition 8.4 gives us irreducibility for $c \leqslant \nu_0^+ - 2$. For applying (SV), the worst root to test is η, a second simple neighbor of α. For $c = 1$, we have

$$\langle \lambda_0 + \tfrac{1}{2}\alpha, \eta \rangle = \langle \lambda_0 - \lambda_{0,b}, \eta \rangle \geqslant 0$$

since $\langle \lambda_{0,b}, \eta \rangle = \frac{1}{2}$. Thus the irreducibility extends to Δ.

Proof for $\mathfrak{sp}(n, \mathbb{R})$. The special basic case is necessarily contained in the

A_{n-1} subdiagram of Δ. If there is a simple root of $\Delta^+_{K,\perp}$ of the form (f) or (g) in Lemma 2.2, then $\min(\nu_0^+, \nu_0^-) = 2$, and there is nothing to prove. Otherwise we denote by Δ_L the component of α in the special basic case; Δ_L is an A type diagram. Then the same argument as at the start of (II) for $\mathfrak{sp}(p, q)$ gives the required irreducibility.

Proof for $\mathfrak{so}$(*odd, even*). We may suppose that $\mu = 0$, since otherwise $\min(\nu_0^+, \nu_0^-) = 0$ and there is nothing to prove. Let Δ_L be the component of α in the special basic case. In the notation of Section 6, Δ_L is of type $\mathfrak{so}(2(n-t+1), 1)$, and we can compute that $\nu_0^+ = \nu_0^- = 1 + 2(n-t)$. Proposition 8.4 says that there is irreducibility in Δ_L for $c \leqslant \nu_0^+ - 2$. For applying (*SV*), the worst root to check is $e_{t-1} - e_n$. With $|e_n|^2 = 1$, we have

$$\begin{aligned}
&\langle \lambda_0 + \tfrac{1}{2}(\nu_0^+ - 2)\,\alpha, e_{t-1} - e_n \rangle \\
&\quad = \langle \lambda_0, e_{t-1} - e_t \rangle + [(n-t-1) + \tfrac{1}{2}] - \tfrac{1}{2}(\nu_0^+ - 2) \\
&\quad = \langle \lambda_0, e_{t-1} - e_t \rangle \geqslant 0,
\end{aligned}$$

and therefore the irreducibility for $c \leqslant \nu_0^+ - 2$ extends to Δ. This completes the proof of Lemma 11.1.

LEMMA 11.2. *Suppose that the Dynkin diagram of Δ^+ is a classical double-line diagram and that α is long. Then $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is irreducible for $0 \leqslant c < \min(\nu_0^+, \nu_0^-)$ except in the following situations*:

(i) *if the basic case associated to λ_0 satisfies the conditions of* (iii) *in Theorem* 1.1 (*which refers to* $\mathfrak{so}(2n, 3)$) *and if ζ is the root defined there, then irreducibility extends for*

$$0 \leqslant c < \min(\nu_0^+, \nu_0^- - 1) \qquad \text{if } \zeta \text{ is noncompact and } \nu_0^- \geqslant 2,$$

$$0 \leqslant c < \min(\nu_0^+ - 1, \nu_0^-) \qquad \text{if } \zeta \text{ is compact or } 0 \text{ and } \nu_0^+ \geqslant 2.$$

(ii) *if the special basic case associated to λ_0 satisfies the conditions of* (v) *in Theorem* 1,1 (*which refers to* $\mathfrak{so}(2n+1, 2)$) *or the conditions of* (vi) *in Theorem* 1.1 (*which refers to an entended version of* $\mathfrak{so}(5, 2)$) *and if $\nu_{0,L}^\pm$ and β_0 are as defined there, then irreducibility extends for*

$$0 \leqslant c < \min(\nu_{0,L}^+ + 1, \nu_0^-) \quad \text{if } \beta_0 \text{ conjugate to } \alpha \text{ via } K \text{ in } \mathfrak{so}(2n+1, 2),$$

$$0 \leqslant c < \min(\nu_0^+, \nu_{0,L}^- + 1) \quad \text{if } \beta_0 \text{ conjugate to } -\alpha \text{ via } K \text{ in } \mathfrak{so}(2n+1, 2).$$

Remark. In situation (vi) of Theorem 1.1, conclusion (ii) here gives irreducibility for $0 \leqslant c < 2$, which is the correct interval for Theorem 1.1.

Proof. Corollary 8.3 shows that it is enough to prove irreducibility when c is an integer in the above range. (Unfortunately there is no longer a restriction on the parity of c.) Lemma 2.1 implies that we may disregard $c=0$. We shall follow the division into cases in Section 7, using the notation introduced there. Evidently there is nothing to prove unless $\min(\nu_0^+, \nu_0^-) > 1$.

Proof for $\mathfrak{sp}(n, \mathbb{R})$. The root α is the unique long simple root $2e_n$. Possibly by reflecting in α, we may assume that the adjacent simple root $\gamma_{n-1} = e_{n-1} - e_n$ is compact. Then $\min(\nu_0^+, \nu_0^-) \leqslant 1$ (and there is nothing to prove) unless we are in case

(III.3) $\mu = -\frac{1}{2}\alpha$, γ_{n-1} and $e_{n-2} - e_{n-1}$ both compact basic. Then $\nu_0^- = 2 \leqslant \nu_0^+$, and we are to prove irreducibility at $c=1$. Let Δ_L be the subsystem generated by $e_{n-2} - e_{n-1}$, γ_{n-1}, and α. This is of type $\mathfrak{sp}(3, \mathbb{R})$, and we have irreducibility in Δ_L by part (y) of Lemma 8.6. To pass to Δ, we show that (SV) holds. If β is in $\Delta(\mathfrak{u})$ with $\langle \beta, \alpha \rangle < 0$, then $\beta = e_i - e_n$, and the worst case is evidently $\beta = e_{n-3} - e_n$. Normalizing so that $|\beta|^2 = 2$, we compute that

$$\begin{aligned}\langle \lambda_0 + \tfrac{1}{2}\alpha, e_{n-3} - e_n \rangle &= \langle \lambda_0, e_{n-3} - e_{n-2} \rangle + (1+0) - \langle \tfrac{1}{2}\alpha, e_{n-3} - e_n \rangle \\ &= \langle \lambda_0, e_{n-3} - e_{n-2} \rangle \geqslant 0.\end{aligned}$$

Hence (SV) holds, and we have irreducibility in Δ, by Theorem 8.2.

Proof for $\mathfrak{so}(odd, even)$. Let $\alpha = e_j - e_{j+1}$. Possibly by reflecting in α, we can arrange that the next simple root from α toward e_n is noncompact. Let e_k be the short Δ_K^+ simple root, and normalize root lengths so that $|e_k|^2 = 1$.

(I) Suppose that the exceptional term (7.2) of ν_0^+ or ν_0^- is not 0.

(I.1) Suppose $k = n-1$, $j = n-2$, $\mu = \frac{1}{2}\alpha$, and $e_{n-1} - e_n$ and e_n are both basic. Then $\nu_0^- = 2 < 3 \leqslant \nu_0^+$, and the conditions of (iii) in Theorem 1.1 are satisfied. Here ζ is $e_{n-1} - e_n$, which is noncompact. Thus exception (i) of Lemma 11.2 asserts irreducibility only for $0 \leqslant c < 1$, and there is nothing to prove.

(I.2) Suppose $k = n-1$, $j = n-1$, and e_n is as in (7.8).

(I.2a) Suppose $e_{n-2} - e_{n-1}$, if it exists, is not compact basic. If $\mu \neq \frac{1}{2}\alpha$, then $\nu_0^+ = 1 + \mu_\alpha < 2$, and there is nothing to prove. So suppose $\mu = \frac{1}{2}\alpha$. If $e_{n-2} - e_{n-1}$ does not exist or is not noncompact basic, then $\nu_0^- = 1$, and there is nothing to prove. If $e_{n-2} - e_{n-1}$ exists and is noncompact basic, then $\nu_0^+ = 2 < 3 \leqslant \nu_0^-$ and the conditions of (iii) in Theorem 1.1 are satisfied. Here ζ is 0, and exception (i) of Lemma 11.2 asserts irreducibility only for $0 \leqslant c < 1$. Thus there is nothing to prove.

(I.2b) Suppose $e_{n-2}-e_{n-1}$ exists and is compact basic. If $\mu = +\frac{1}{2}\alpha$, then $\nu_0^- = 2-\mu_\alpha = 1 < \nu_0^+$, and there is nothing to prove. If $\mu \neq +\frac{1}{2}\alpha$, we first let Δ_L be generated by $e_{n-2}-e_{n-1}$, α, and e_n. In Δ_L, part (z) of Lemma 8.6 gives irreducibility at $c=1$. In applying (SV), the worst root to check is $\beta = e_{n-3}-e_{n-1}$, and we have

$$\langle \lambda_0 + \tfrac{1}{2}\alpha, \beta \rangle = \langle \lambda_0, e_{n-3}-e_{n-2} \rangle + \tfrac{1}{2}(1+\mu_\alpha) - \tfrac{1}{2}. \tag{11.2}$$

If $\mu = 0$, then (11.2) is $\geqslant 0$; thus (SV) applies and the irreducibility at $c=1$ extends to Δ. Moreover, $\nu_0^- = 2-\mu_\alpha = 2$. Hence there is nothing further to prove when $\mu = 0$.

Suppose $\mu = -\frac{1}{2}\alpha$. If $e_{n-3}-e_{n-2}$ does not exist, there is nothing to prove. Otherwise first assume $e_{n-3}-e_{n-2}$ is not basic. Then (11.2) is $\geqslant 0$, (SV) applies, and the irreducibility at $c=1$ extends to Δ. Moreover, $\nu_0^+ = 2 < 3 = \nu_0^-$, and there is nothing further to prove.

Next assume that $\mu = -\frac{1}{2}\alpha$ and that $e_{n-3}-e_{n-2}$ is noncompact basic. Then $e_{n-3}-e_n$ is a $\Delta_{K,\perp}^+$ simple root of type (g) in Lemma 2.2, and $\nu_0^+ = 2 < 3 \leqslant \nu_0^-$. We have to prove irreducibility at $c=1$, and (11.2) is no help (being negative). Instead we let Δ_L be generated by all the long simple roots. Since $\mu = -\frac{1}{2}\alpha$ and Δ_L has only single lines in its Dynkin diagram, there is irreducibility in Δ_L at $c=1$. In applying (SV), the worst root to check is e_n. Since (7.8) says that e_n is one step removed from basic, we have

$$\frac{2\langle \lambda_0 + \frac{1}{2}\alpha, e_n \rangle}{|e_n|^2} = 2 + \frac{\langle \alpha, e_n \rangle}{|e_n|^2} = 1 > 0.$$

Thus (SV) does apply, and the irreducibility at $c=1$ extends to Δ.

Finally assume that $\mu = -\frac{1}{2}\alpha$ and that $e_{n-3}-e_{n-2}$ is compact basic. Then (11.2) is $\geqslant 0$, (SV) applies, and the irreducibility at $c=1$ extends to Δ. However, $\nu_0^- = 3 < \nu_0^+$ in this case, and we have to prove irreducibility at $c=2$ also. Thus we enlarge Δ_L so as to be generated by $e_{n-3}-e_{n-2}$, $e_{n-2}-e_{n-1}$, α, and e_n. Part (z) of Lemma 8.6 gives irreducibility in Δ_L at $c=2$. In applying (SV), the worst root to check is $\beta = e_{n-4}-e_{n-1}$, for which

$$\langle \lambda_0 + \alpha, \beta \rangle = \langle \lambda_0, e_{n-4}-e_{n-3} \rangle + 1 - 1 \geqslant 0.$$

Thus (SV) applies to show that the irreducibility at $c=2$ extends to Δ.

(I.3) Suppose $k=n$ and $j \leqslant n-2$.

(I.3a) Exceptional term of ν_0^+ nonzero. Then $\nu_0^- = 2 < \nu_0^+$. Let Δ_L be the subsystem with e_n deleted. Then $\nu_{0,L}^- = 2 \leqslant \nu_{0,L}^+$. By Corollary 8.3 we have irreducibility at $c=1$ in Δ_L. In applying (SV), we do not need to check roots $e_i - e_{i'}$ (since these are in Δ_L), and the worst root to check is therefore $\beta = e_{j+1}$. Since e_n is compact and orthogonal to α, we have

$$\langle \lambda_0 + \tfrac{1}{2}\alpha, e_{j+1} \rangle = \langle \lambda_0, e_{j+1} \rangle - \tfrac{1}{2} \geqslant \langle \lambda_0, e_n \rangle - \tfrac{1}{2} \geqslant 0.$$

Thus (SV) holds, and the irreducibility at $c = 1$ extends to Δ.

(I.3b) Exceptional term of ν_0^- nonzero. The first of two preliminary subcases is that $e_{j-2} - e_{j+1}$ is a $\Delta_{K,\perp}^+$ root of type (g). Then $\nu_0^+ = 2 < \nu_0^-$. With Δ_L defined as in (I.3a), we have $\nu_{0,L}^+ = 2 \leqslant \nu_{0,L}^-$. Corollary 8.3 gives us irreducibility at $c = 1$ in Δ_L, and the argument in (I.3a) shows that (SV) holds, so that the irreducibility extends to Δ.

The second preliminary subcase is that $e_{j-1} - e_j$, if it exists, is noncompact. Then $\nu_0^+ = 1 + \mu_\alpha < \nu_0^-$. There is nothing to prove unless $\mu_\alpha = +1$, in which case we can again proceed as in (I.3a) to obtain irreducibility at $c = 1$.

The main subcase is that the component of α in the special basic case is of real rank one. One node in this component is $e_{n-1} - e_n$; let the other one be $e_i - e_{i+1}$. Then we have

$$\nu_0^+ = 1 + \mu_\alpha + 2(j - i),$$
$$\nu_0^- = 2 - \mu_\alpha + 2(n - j - 1),$$

and these numbers are of opposite parity. First suppose $\nu_0^+ < \nu_0^-$. Then we let Δ_L be the subsystem of Δ with e_n deleted. Then $\nu_{0,L}^+ = \nu_0^+$ and $\nu_{0,L}^- = \nu_0^- - 1 \geqslant \nu_0^+$, so that Lemma 10.1 gives irreducibility in Δ_L for $c \leqslant \nu_0^+ - 1$. In applying (SV), the worst root to check is $\beta = e_{j+1}$. Then $c \leqslant \nu_0^+ - 1 \leqslant \nu_0^- - 2$ implies

$$\begin{aligned}
&\langle \lambda_0 + \tfrac{1}{2}c\alpha, e_{j+1} \rangle \\
&\quad \geqslant \langle \lambda_{0,b}, e_{j+1} - e_n \rangle + \langle \lambda_{0,b}, e_n \rangle - \tfrac{1}{2}(\nu_0^- - 2) \\
&\quad = (\tfrac{1}{2}(1 - \mu_\alpha) + n - j - 2) + \tfrac{1}{2} - \tfrac{1}{2}(2 - \mu_\alpha + 2(n - j - 2)) = 0.
\end{aligned}$$

and the irreducibility extends to Δ.

Otherwise suppose $\nu_0^- < \nu_0^+$. Then we let Δ_L be the result of adjoining e_n to the component of α in the special basic case. For this L, $\nu_{0,L}^+ = \nu_0^+$ and $\nu_{0,L}^- = \nu_0^-$, so that part (aa) of Lemma 8.6 gives irreducibility in Δ_L for $c \leqslant \nu_0^- - 1$. In applying (SV), the worst root to check is $\beta = e_{i-1} - e_j$. If $i < j$, then $c \leqslant \nu_0^- - 1 \leqslant \nu_0^+ - 2$ implies

$$\langle \lambda_0 + \tfrac{1}{2}c\alpha, e_{i-1} - e_j \rangle \geqslant \langle \lambda_{0,b}, e_i - e_j \rangle - \tfrac{1}{2}(\nu_0^+ - 2) = 0,$$

and the irreducibility extends to Δ. If $i = j$, then $\nu_0^+ \leqslant 2$ and hence $\nu_0^- \leqslant 1$; thus there is nothing to prove.

(II) Suppose that α or $-\alpha$ is conjugate by the Weyl group of $\Delta^+_{K,\perp}$ to $\beta_0 = e_j + e_{j+1}$.

(II.1) Suppose $j < n-1$, so that e_n is compact and orthogonal to Δ. If we let Δ_L be the subsystem of Δ with e_n deleted, then the worst case for applying (SV) is e_{j+1}, and we have

$$\langle \lambda_0 + \tfrac{1}{2}\alpha, e_{j+1} \rangle \geqslant \langle \lambda_0, e_n \rangle - \tfrac{1}{2} \geqslant 0.$$

Hence irreducibility at $c = 1$ in Δ_L will imply irreducibility in Δ.

If there is a $\Delta^+_{K,\perp}$ root of type (f) in Lemma 2.2, then $\nu_0^- = 2 \leqslant \nu_0^+$ in Δ_L and in Δ. If there is a $\Delta^+_{K,\perp}$ root of type (g) in Lemma 2.2, then $\nu_0^+ = 2 \leqslant \nu_0^-$ in Δ_L and in Δ. If $e_{j-1} - e_j$ exists and is noncompact basic, then $\nu_0^+ = 1 + \mu_\alpha \leqslant \nu_0^-$ in Δ_L and in Δ. Hence in all of these cases, the remarks in the previous paragraph show that we have nothing further to prove for the desired irreducibility.

Thus we may assume that the component of α in the special basic case Δ_S is of type $\mathfrak{so}(\text{odd}, 2)$. Let Δ_{L_1} be the system Δ_S with e_n deleted. Since the conditions of (v) in Theorem 1.1 are satisfied, Lemma 11.2 asserts irreducibility only for $c < \min(\nu_0^+, \nu_{0,L_1}^- + 1)$. In the notation of Section 7, ν_0^+ and $\nu_{0,L_1}^- + 1$ satisfy

$$\nu_0^+ = 1 + \mu_\alpha + 2(j - l),$$
$$\nu_{0,L_1}^- + 1 = 2 - \mu_\alpha + 2(n - j - 1),$$

and are of opposite parity. If $\nu_0^+ < \nu_{0,L_1}^- + 1$, then we let Δ_L be the subsystem of Δ with e_n deleted. Then $\nu_{0,L}^+ = \nu_0^+$ and $\nu_{0,L}^- = \nu_{0,L_1}^-$, so that Lemma 10.1 gives irreducibility in Δ_L for $c \leqslant \nu_0^+ - 1$. In applying (SV), the worst root to check is $\beta = e_{j+1}$. Then $c \leqslant \nu_0^+ - 1 \leqslant \nu_{0,L_1}^- - 1$ implies

$$\begin{aligned}
&\langle \lambda_0 + \tfrac{1}{2}c\alpha, e_{j+1} \rangle \\
&\quad \geqslant \langle \lambda_{0,b}, e_{j+1} - e_n \rangle + \langle \lambda_{0,b}, e_n \rangle - \tfrac{1}{2}(\nu_{0,L_1}^- - 1) \\
&\quad = (\tfrac{1}{2}(1 - \mu_\alpha) + n - j - 2) + \tfrac{1}{2} - \tfrac{1}{2}(1 - \mu_\alpha + 2(n - j - 1) - 1) \\
&\quad = 0,
\end{aligned}$$

and the irreducibility extends to Δ. (Here the equality $\langle \lambda_{0,b}, e_n \rangle = \tfrac{1}{2}$ used the compactness of e_n.)

If $\nu_{0,L_1}^- + 1 < \nu_0^+$, then we let $\Delta_L = \Delta_S$. Part (cc) of Lemma 8.6 gives irreducibility in Δ_L for $c \leqslant \nu_{0,L_1}^-$. In applying (SV), the worst root to check is $\beta = e_{l-1} - e_j$. If $l < j$, then $c \leqslant \nu_{0,L_1}^- \leqslant \nu_0^+ - 2$ implies

$$\langle \lambda_0 + \tfrac{1}{2}c\alpha, \beta \rangle \geqslant \langle \lambda_{0,b}, e_l - e_j \rangle - \tfrac{1}{2}(\nu_0^+ - 2) = 0,$$

and the irreducibility extends to Δ. If $l=j$, then $\nu_0^+ \leqslant 2$ and hence $\nu_{0,L_1}^- = 0$; thus there is nothing to prove.

(II.2) Suppose $j=n-1$. Then $\mu \neq +\frac{1}{2}\alpha$. If $e_{n-2}-e_{n-1}$ does not exist or is not compact basic, then $\nu_0^+ = 1+\mu_\alpha \leqslant 1 < \nu_0^-$, and there is nothing to prove. So suppose $e_{n-2}-e_{n-1}$ exists and is compact basic.

If $e_{n-3}-e_n$ is a $\Delta_{K,\perp}^+$ simple root of type (g) in Lemma 2.2, then $\mu = -\frac{1}{2}\alpha$ and $\nu_0^+ = 2 \leqslant \nu_0^-$. We have to prove irreducibility at $c=1$. Here α does not satisfy the parity condition. By Theorem 8.1, reducibility can occur only when there is a root $\beta \neq \pm\alpha$ with $\langle \lambda_0 + \frac{1}{2}\alpha, \beta\rangle > 0$ and $\langle \lambda_0 - \frac{1}{2}\alpha, \beta\rangle < 0$ such that $2\langle \lambda_0 + \frac{1}{2}\alpha, \beta\rangle/|\beta|^2$ is an integer. Since λ_0 is Δ^+ integral (this being a cotangent case), $\langle \alpha, \beta\rangle/|\beta|^2$ is an integer. Thus β is short, and we must have $\beta = \pm e_{n-1}$ or $\pm e_n$. Table 2.1 gives $\langle \lambda_0, e_n\rangle = \frac{1}{2} = \langle \lambda_0, e_{n-1}\rangle$, and thus we see that the condition of Theorem 8.1 is not met. Hence we have irreducibility at $c=1$.

Now suppose that no $\Delta_{K,\perp}^+$ simple root of type (g) in Lemma 2.2 is present. Then the conditions of (v) or (vi) in Theorem 1.1 are satisfied, and it is enough to prove irreducibility for $0 \leqslant c < \nu_{0,L}^- + 1$, where $\nu_{0,L}^- = 1 - \mu_\alpha$. Let Δ_L be the subsystem generated by $e_{n-2}-e_{n-1}$, $e_{n-1}-e_n$, and e_n. Part (bb) of Lemma 8.6 gives us irreducibility in Δ_L for $c \leqslant 1-\mu_\alpha$. In applying (SV), the worst root to check is $\beta = e_{n-3}-e_{n-1}$. Then $c \leqslant 1-\mu_\alpha$ implies

$$\begin{aligned} \langle \lambda_0 + \tfrac{1}{2}c\alpha, \beta\rangle &\geqslant \langle \lambda_0, e_{n-3}-e_{n-2}\rangle + \langle \lambda_{0,b}, e_{n-2}-e_{n-1}\rangle - \tfrac{1}{2}c \\ &\geqslant \langle \lambda_0, e_{n-3}-e_{n-2}\rangle + \tfrac{1}{2}(1+\mu_\alpha) - \tfrac{1}{2}(1-\mu_\alpha) \\ &= \langle \lambda_0, e_{n-3}-e_{n-2}\rangle + \mu_\alpha. \end{aligned} \tag{11.3}$$

Since $e_{n-3}-e_n$ is assumed not to be of type (g) for Lemma 2.2, we cannot have both $\mu_\alpha = -1$ and $\langle \lambda_0, e_{n-3}-e_{n-2}\rangle = 0$. Thus (11.3) is $\geqslant 0$, (SV) holds, and the irreducibility extends to Δ.

(III) Suppose that neither α nor $-\alpha$ is conjugate by the Weyl group of $\Delta_{K,\perp}^+$ to $\beta_0 = e_j + e_{j+1}$ and that the exceptional terms of ν_0^+ and ν_0^- are 0. We know from Section 7 that e_n is not in the component of α within the special basic case.

Let Δ_S be the special basic case, and let $\Delta_L \supseteq \Delta_S$ be the subsystem of Δ obtained by deleting e_n. First suppose $\min(\nu_0^+, \nu_0^-) = 2$. In Δ_L, Lemma 10.1 gives us irreducibility for $c=1$. In applying (SV), the worst root to check is e_{j+1}, and we have $\langle \lambda_0 + \frac{1}{2}\alpha, e_{j+1}\rangle = \langle \lambda_0, e_{j+1}\rangle - \frac{1}{2}$. Thus the irreducibility at $c=1$ extends to Δ unless $\langle \lambda_0, e_{j+1}\rangle = 0$, i.e., either

$j<n-1$, $e_{j+1}-e_{j+2}$ is (noncompact) basic, $\mu = +\frac{1}{2}\alpha$, and $e_{j+2}-e_{j+3}, \ldots, e_n$ are all noncompact basic

or

$j=n-1$, e_n is noncompact basic, and $\mu \neq -\frac{1}{2}\alpha$.

In the first case, nondegeneracy forces $j=n-2$; this is case (I.1), and we have already considered it. In the second case, $\mu=+\frac{1}{2}\alpha$ is case (I.2) and $\mu=0$ is case (II.2); we have already considered these cases.

Consequently we may assume that the component of α in Δ_S is of real rank one and that $\min(\nu_0^+, \nu_0^-)>2$. Let $e_i-e_{i+1},..., e_{l-1}-e_l$ be the simple roots of this component; we know $l>j+1$ since $\min(\nu_0^+, \nu_0^-)>2$. We saw in Section 10 that

$$\langle \lambda_0+\tfrac{1}{2}(\nu_0^- -2)\,\alpha, e_{j+1}-e_l\rangle=0. \tag{11.4}$$

Let Δ_L be the subsystem of Δ obtained by deleting e_n. Lemma 10.1 gives us irreducibility in Δ_L for $c\leqslant\min(\nu_0^+, \nu_0^-)-1$. In applying (SV), the worst root to check is e_{j+1}, and (11.4) gives

$$\begin{aligned}\langle \lambda_0+\tfrac{1}{2}(\nu_0^- -1)\,\alpha, e_{j+1}\rangle &= \langle \lambda_0+\tfrac{1}{2}(\nu_0^- -2)\,\alpha, e_{j+1}-e_l\rangle+\langle\lambda_0, e_l\rangle-\tfrac{1}{2}\\ &=\langle\lambda_0, e_l\rangle-\tfrac{1}{2}.\end{aligned}$$

Consequently the irreducibility in Δ_L extends to Δ unless $\nu_0^-\leqslant\nu_0^+$ and $\langle\lambda_0, e_l\rangle=0$. This condition forces all simple roots after $e_{l-1}-e_l$ to be noncompact basic; by nondegeneracy, we must have $l=n$ (and e_n noncompact basic).

Thus the conditions of (iii) in Theorem 1.1 are satisfied. The root ζ is $e_{j+1}-e_n$, which is noncompact. Hence Lemma 11.2 asks for irreducibility only when $c<\nu_0^- -1$. That much irreducibility follows from (11.4), and the proof of Lemma 11.2 is complete.

12. Isolated Representations

Situations (i), (ii), (iii), and (vi) in Theorem 1.1b indicate unitarity for some isolated representations. Situation (ii) requires no proof, and situation (i) is well known for nonsplit F_4. Thus it is enough to prove this unitarity for situations (i), (iii), and (vi), with (i) restricted to $\mathfrak{sp}(p, q)$.

Many of the ideas and results in this section are due to D. A. Vogan, partly in response to questions posed by the authors, and we are grateful for his help.

The chief idea to prove the unitarity is to use Zuckerman's derived functor modules $A_{\mathfrak{q}}(\lambda)$, as explained in Vogan and Zuckerman [26], but with the parameter λ outside the usual range. (See also Enright and Wallach [6].) Unitarity is proved for such representations under suitable conditions by Vogan [24]. In situations (i) and (iii), we shall identify the span of the minimal K-type as the desired Langlands quotient, while in situation (vi), we shall identify a different irreducible constituent of the $A_{\mathfrak{q}}(\lambda)$ as the desired Langlands quotient.

The process of identifying the Langlands parameters is simplified by our assumption $\dim A = 1$, as we show in the following proposition.

PROPOSITION 12.1. *Suppose that $J(MAN, \sigma, \nu)$ and $J(M'A'N', \sigma', \nu')$ each have the same unique minimal K-type and the same infinitesimal character, and suppose that σ is a discrete series or nondegenerate limit of discrete series and that $\dim A = 1$. Then $J(MAN, \sigma, \nu)$ and $J(M'A'N', \sigma', \nu')$ are infinitesimally equivalent.*

Proof. Applying the theory of [20], we conclude from the presence of the same minimal K-type in each of the given representations that $J(MAN, \sigma, \nu)$ is an irreducible quotient of some $U(M_*A_*N_*, \sigma_*, \nu_*)$ while $J(M'A'N', \sigma', \nu')$ is an irreducible quotient of $U(M_*A_*N_*, \sigma_*, \nu'_*)$ with the same σ_*; here σ_* is a discrete series representation of M_*. The number of irreducible quotients of $U(M_*A_*N_*, \sigma_*, \nu_*)$ is $|R_{\sigma_*,\nu_*}|$, and the various irreducible quotients of $U(M_*A_*N_*, \sigma_*, \nu_*)$ all have the same number of minimal K-types, which must be one since $J(MAN, \sigma, \nu)$ has a unique minimal K-type. Therefore $U(M_*A_*N_*, \sigma_*, \nu_*)$ has $|R_{\sigma_*,\nu_*}|$ minimal K-types. Since $|R_{\sigma_*,\nu_*}| \leqslant |R_{\sigma_*,0}|$, we conclude $|R_{\sigma_*,\nu_*}| = |R_{\sigma_*,0}|$. Therefore ν_* satisfies $r\nu_* = \nu_*$ for every r in $R_{\sigma_*,0}$. Similarly $r\nu'_* = \nu'_*$ for every r in $R_{\sigma_*,0}$.

Since $J(MAN, \sigma, \nu)$ has just one minimal K-type, so does $U(MAN, \sigma, 0)$, and thus $U(MAN, \sigma, 0)$ is irreducible. We now bring in the theory of [17]. Since we have nondegeneracy, this theory tells us that $R_{\sigma_*,0}$ determines a superorthogonal set $\{\alpha_j\}$ of real roots such that

$$\mathfrak{a}_* = \mathfrak{a} \oplus \sum \mathbb{R} H_{\alpha_j}$$

and such that $r\alpha_j = -\alpha_j$ for each j. Thus the elements in $\mathfrak{a}'_*$ fixed by $R_{\sigma_*,0}$ are in $\mathfrak{a}'$.

Since $\mathfrak{a}$ is one dimensional, ν'_* must be a multiple of ν_*. Since the infinitesimal character is the same for our two given representations, $|\nu'_*| = |\nu_*|$. Therefore $\nu'_* = \nu_*$, and it follows that our given representations are infinitesimally equivalent.

Since the Langlands quotients under study have unique minimal K-types, Proposition 12.1 allows us to match them with representations $A_{\mathfrak{q}}(\lambda)$ by matching the minimal K-type and the infinitesimal character and by checking that the minimal K-type of $A_{\mathfrak{q}}(\lambda)$ is unique.

We shall not need the detailed construction of $A_{\mathfrak{q}}(\lambda)$. It is enough to have the following result.

THEOREM 12.2 (*Vogan* [24]). *Suppose* rank $G = $ rank K. *Let* Δ^+ *be a positive system for* $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$, *let* Δ_L *be a subset of* Δ *generated by* Δ^+ *simple roots, let* $\Delta(\mathfrak{u})$ *be the set of positive roots not in* Δ_L, *and define*

$$\mathfrak{l} = \mathfrak{g} \cap \left(\mathfrak{b}^{\mathbb{C}} \oplus \sum_{\beta \in \Delta_L} \mathfrak{g}_\beta \right), \qquad \mathfrak{u} = \sum_{\beta \in \Delta(\mathfrak{u})} \mathfrak{g}_\beta, \qquad \mathfrak{q} = \mathfrak{l}^{\mathbb{C}} \oplus \mathfrak{u}.$$

Let L be the analytic subgroup of G with Lie algebra $\mathfrak{l}$. *Denote by* $\delta(\cdot)$ *the half sum of the positive roots contributing to the specified vector space. If* λ *in* $i\mathfrak{b}'$ *is the differential of a unitary (one-dimensional) character of L such that*

$$\langle \lambda + \delta(\mathfrak{u}), \beta \rangle \geqslant 0 \qquad \textit{for all } \beta \textit{ in } \Delta(\mathfrak{u}), \tag{12.1}$$

and if

$$\Lambda = \lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}}),$$

then there exists an admissible representation $A_{\mathfrak{q}}(\lambda)$ *of* $\mathfrak{g}$ *with infinitesimal character* $\lambda + \delta$ *such that*

(a) *the K-types have multiplicities given by the following version of Blattner's formula*:

$$\text{mult } \tau_{\Lambda''} = \sum_{s \in W_K} (\det s)\, \mathscr{P}(s(\Lambda'' + \delta_K) - (\Lambda + \delta_K)),$$

where W_K *is the Weyl group of* Δ_K *and* $\mathscr{P}$ *is the partition function relative to expansions in terms of noncompact members of* $\Delta(\mathfrak{u})$, *and*

(b) *the representation* $A_{\mathfrak{q}}(\lambda)$ *admits a positive* definite *invariant inner product.*

Proof. Let Y be the one-dimensional character e^λ of L, and let $A_{\mathfrak{q}}(\lambda) = \mathscr{R}^S(Y)$ in the notation of Vogan [24]. We shall apply Theorem 7.1 of [24]. Hypothesis (a) of that theorem is just that Y is unitary, which we have assumed. Hypothesis (b) is satisfied by virtue of Vogan's Proposition 8.5 (in which Vogan's λ is to be our $\lambda + \delta$), since Y is one dimensional and (12.1) holds. The theorem says that $\mathscr{R}^S Y$ admits a (specific) positive definite inner product and that $\mathscr{R}^i Y = 0$ for $i \neq S$. Applying Theorem 6.3.12 of [23], we obtain the Blattner formula as stated.

PROPOSITION 12.3. *Under the assumptions of Theorem* 12.2, *suppose that* Λ *is* Δ_K^+ *dominant. Then* $A_{\mathfrak{q}}(\lambda)$ *is nonzero and the K-type* τ_Λ *occurs with multiplicity one. If in addition* $\langle \Lambda + 2\delta_K, \beta \rangle \geqslant 0$ *for all* β *in* $\Delta(\mathfrak{u})$, *then* τ_Λ *is the unique minimal K-type of* $A_{\mathfrak{q}}(\lambda)$.

Proof. For $\Lambda'' = \Lambda$, the term $s = 1$ gives a contribution of 1 to mult τ_Λ. Conversely suppose Λ'' and s give a nonzero term. With $\Delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}})$ denoting the set of noncompact members of $\Delta(\mathfrak{u})$, we have

$$\Lambda'' + \delta_K = s(\Lambda'' + \delta_K) + \sum_{\gamma \in \Delta_K^+} k_\gamma \gamma,$$

$$s(\Lambda'' + \delta_K) = \Lambda + \delta_K + \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}})} n_\beta \beta.$$

Hence

$$\Lambda'' = \Lambda + \sum_{\gamma \in \Delta_K^+} k_\gamma \gamma + \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}})} n_\beta \beta. \tag{12.2}$$

If $\Lambda'' = \Lambda$, then $\sum n_\beta \beta = 0$ and $s(\Lambda + \delta_K) = \Lambda + \delta_K$, so that $s = 1$. Thus τ_Λ has multiplicity one and $A_{\mathfrak{q}}(\lambda)$ is nonzero. For general Λ'', (12.2) gives

$$|\Lambda'' + 2\delta_K|^2 = |\Lambda + 2\delta_K|^2 + 2\left\langle \Lambda + 2\delta_K, \sum k_\gamma \gamma + \sum n_\beta \beta \right\rangle + \left| \sum k_\gamma \gamma + \sum n_\beta \beta \right|^2 \tag{12.3}$$

We have $\langle \Lambda + 2\delta_K, \gamma \rangle > 0$ for all γ in Δ_K^+. If $\langle \Lambda + 2\delta_K, \beta \rangle \geqslant 0$ for all β in $\Delta(\mathfrak{u})$, then the right side of (12.3) is $\geqslant |\Lambda + 2\delta_K|^2$, with equality only when $\sum k_\gamma \gamma = \sum n_\beta \beta = 0$. This proves the minimality under the additional hypothesis.

PROPOSITION 12.4. *In the setting of Section* 1 *with* rank $G =$ rank K, *suppose that* Δ_L *is a root subsystem of* Δ *generated by simple roots and containing* α, *and suppose* Δ_L *has real rank one. Let* Λ *be defined by* (1.3). *If the parameter*

$$\lambda = \Lambda - 2\delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}}) \tag{12.4}$$

is orthogonal to Δ_L, *then* $J(MAN, \sigma, \rho_L)$ *is infinitesimally unitary. Here* ρ_L *is the half sum of the positive roots of* $(\mathfrak{l}, \mathfrak{a})$ *computed just from roots that lie in* Δ_L.

Proof. We use the given L and the corresponding $\mathfrak{u}$ (from our usual Δ^+) as data for Theorem 12.2. First we exponentiate λ. Let $L^{\mathbb{C}}$ be the analytic subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{l}^{\mathbb{C}}$. The Lie algebra $\mathfrak{b}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{l}^{\mathbb{C}}$, and λ exponentiates to $\exp(\mathfrak{b}^{\mathbb{C}}) \subseteq L^{\mathbb{C}}$ since Λ and $2\delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}})$ do. Since λ is orthogonal to Δ_L, it is Δ_L^+ dominant. Therefore the Theorem of the Highest Weight supplies an irreducible representation of $L^{\mathbb{C}}$ with highest weight λ. Naturally this representation is one dimensional and its restriction e^{λ} to L is unitary.

Substituting from (1.3), we have

$$\begin{aligned}\lambda+\delta(\mathfrak{u}) &= \Lambda-2\delta(\mathfrak{u}\cap\mathfrak{p}^{\mathbb{C}})+\delta(\mathfrak{u})\\ &= \lambda_0+\delta-2\delta_K+\mu-\tfrac{1}{2}\alpha-2\delta(\mathfrak{u}\cap\mathfrak{p}^{\mathbb{C}})+\delta(\mathfrak{u})\\ &= \lambda_0-\delta+\delta(\mathfrak{u})+2\delta(\mathfrak{l}^{\mathbb{C}}\cap\mathfrak{p}^{\mathbb{C}})+\mu-\tfrac{1}{2}\alpha\\ &= \lambda_0-\delta(\mathfrak{l}^{\mathbb{C}})+2\delta(\mathfrak{l}^{\mathbb{C}}\cap\mathfrak{p}^{\mathbb{C}})+\mu-\tfrac{1}{2}\alpha.\end{aligned}\tag{12.5}$$

The right side is the sum of λ_0 and a real combination of members of Δ_L, and any member γ of Δ_L satisfies $\sum_{w\in W(\Delta_L)} w\gamma=0$, where $W(\Delta_L)$ is the Weyl group of the root system Δ_L. Since λ (by assumption) and $\delta(\mathfrak{u})$ are invariant under $W(\Delta_L)$, we obtain

$$\lambda+\delta(\mathfrak{u})=\sum_{w\in W(\Delta_L)} w\lambda_0.$$

If β is in $\Delta(\mathfrak{u})$, then

$$\langle\lambda+\delta(\mathfrak{u}),\beta\rangle=\sum_{w\in W(\Delta_L)}\langle\lambda_0,w^{-1}\beta\rangle\geqslant 0$$

since $w^{-1}\beta$ is in $\Delta(\mathfrak{u})$, hence is $\geqslant 0$.

Thus Theorem 12.2 applies. The form Λ in the theorem is the minimal K-type here, by (12.4). Thus $A_{\mathfrak{q}}(\lambda)$ has infinitesimal character $\lambda+\delta$ and is unitary. Since Λ is by assumption Δ_K^+ dominant, Proposition 12.3 says that $A_{\mathfrak{q}}(\lambda)$ is nonzero. Let us compute $\langle\Lambda+2\delta_K,\beta\rangle$ for β in $\Delta(\mathfrak{u})$. By (1.3),

$$\frac{2\langle\Lambda+2\delta_K,\beta\rangle}{|\beta|^2}=\frac{2\langle\lambda_0,\beta\rangle}{|\beta|^2}+\frac{2\langle\delta,\beta\rangle}{|\beta|^2}+\frac{2\langle c\alpha,\beta\rangle}{|\beta|^2},$$

where $c=0$, $-\frac{1}{2}$, or -1, depending on the value of μ. The first term on the right side is $\geqslant 0$, and the second term is $\geqslant 1$. The only way that the left side can be <0 is for c to be -1 and $2\langle\alpha,\beta\rangle/|\beta|^2$ to be $+2$. In this case $\beta-\alpha$ is a root. Hence β is not simple and $2\langle\delta,\beta\rangle/|\beta|^2\geqslant 2$. Thus $\langle\Lambda+2\delta_K,\beta\rangle\geqslant 0$ for β in $\Delta(\mathfrak{u})$.

By Proposition 12.3, $A_{\mathfrak{q}}(\lambda)$ has the unique minimal K-type Λ and infinitesimal character $\lambda+\delta$. Here

$$\lambda+\delta=\lambda+\delta(\mathfrak{u})+\delta(\mathfrak{l}^{\mathbb{C}}).$$

The first two terms on the right side are fixed by $W(\Delta_L)$. Applying a member of $W(\Delta_L)$ that results in a positive system for Δ_L that takes $\mathfrak{a}$ before $\mathfrak{m}$, we see that the infinitesimal character of $A_{\mathfrak{q}}(\lambda)$ is given also by

$$\lambda + \delta(\mathfrak{u}) + \delta_{-}(\mathfrak{l}^{\mathbb{C}}) + \rho_L.$$

The projection of this form on $\mathbb{R}\alpha$ is ρ_L; the proof will be complete if we show that the projection orthogonal to α is λ_0.

For this purpose we apply Lemma 3 of [9] to $\mathfrak{l}^{\mathbb{C}}$ to obtain

$$\delta(\mathfrak{l}^{\mathbb{C}}) - 2\delta_K(\mathfrak{l}^{\mathbb{C}}) = \delta_{-}(\mathfrak{l}^{\mathbb{C}}) - 2\delta_{-,c}(\mathfrak{l}^{\mathbb{C}}) + \tfrac{1}{2}\alpha - E(2\delta_K(\mathfrak{l}^{\mathbb{C}})),$$

where E is the orthogonal projection on $\mathbb{R}\alpha$. Since Δ_L has real rank one, $\delta_{-}(\mathfrak{l}^{\mathbb{C}}) = \delta_{-,c}(\mathfrak{l}^{\mathbb{C}})$. Thus we can rewrite this identity as

$$\delta(\mathfrak{l}^{\mathbb{C}}) + \delta_{-}(\mathfrak{l}^{\mathbb{C}}) = 2\delta_K(\mathfrak{l}^{\mathbb{C}}) + \tfrac{1}{2}\alpha - E(2\delta_K(\mathfrak{l}^{\mathbb{C}})). \tag{12.6}$$

We are to check the component orthogonal to α of

$$\begin{aligned}
&\lambda + \delta(\mathfrak{u}) + \delta_{-}(\mathfrak{l}^{\mathbb{C}}) + \rho_L \\
&\quad = \lambda_0 - \delta(\mathfrak{l}^{\mathbb{C}}) + 2\delta(\mathfrak{l}^{\mathbb{C}} \cap \mathfrak{p}^{\mathbb{C}}) + \mu - \tfrac{1}{2}\alpha + \delta_{-}(\mathfrak{l}^{\mathbb{C}}) + \rho_L \qquad \text{by } (12.5) \\
&\quad = \lambda_0 + \delta(\mathfrak{l}^{\mathbb{C}}) - 2\delta_K(\mathfrak{l}^{\mathbb{C}}) + \mu - \tfrac{1}{2}\alpha + \delta_{-}(\mathfrak{l}^{\mathbb{C}}) + \rho_L \\
&\quad = \lambda_0 + \rho_L + \mu - E(2\delta_K(\mathfrak{l}^{\mathbb{C}})) \qquad \text{by } (12.6)
\end{aligned}$$

and the component is clearly λ_0. Thus the infinitesimal character and minimal K-type of $A_{\mathfrak{q}}(\lambda)$ match those of $J(MAN, \sigma, \rho_L)$, and the result follows from Proposition 12.1.

Situation (i). We are assuming that the total group is $\mathrm{Sp}(p, q)$, that $\mu = 0$, and that α is adjacent to the long simple root. We take Δ_L to be the component of α in the special basic case, which we assume is of type $\mathfrak{sp}(n, 1)$ for some $n \geqslant 2$. We check directly that $\frac{1}{2}(v_0^+)\,\alpha = \frac{1}{2}(v_0^-)\,\alpha = \rho_L$. The members of $\Delta_{K,\perp}^+$ span Δ_L over $\mathbb{R}$ (see Sect. 6, item (II.3c)), and thus Λ is orthogonal to Δ_L. The roots of $\Delta(\mathfrak{u})$ are those involving an index less than $p+q-n-1$, and the noncompact ones come in pairs $e_i \pm e_j$. Thus $2\delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}})$ is orthogonal to Δ_L, and Proposition 12.4 says that $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary for $c = v_0^+ = v_0^-$.

Situation (iii). We are assuming that we have $\mathfrak{so}(2n, 3)$ imbedded in Δ, with the short simple root ε basic and with the remaining $\mathfrak{su}(n, 1)$ equal to the component of α in the special basic case. We saw in the detailed treatment of $\mathfrak{so}(\text{odd}, \text{even})$ that there is no loss of generality in assuming that α and ε are not adjacent and that the sum ζ of the simple roots strictly between α and ε is noncompact. In this case, ε is noncompact. Under the assumption that $\nu_0^+ \geqslant \nu_0^-$, we are to prove that $J(MAN, \sigma, \frac{1}{2}\nu_0^- \alpha)$ is infinitesimally unitary.

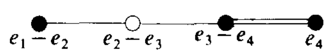
We have seen that the exceptional term of ν_0^- is zero in this circumstance. If Δ_S denotes the special basic case, we thus have $\nu_{0,S}^+ \leqslant \nu_0^+$ and $\nu_{0,S}^- = \nu_0^-$, with $\nu_0^+ - \nu_{0,S}^+ \leqslant 1$. Since $\nu_{0,S}^+$ and $\nu_{0,S}^-$ have the same parity, our assumption is that $\nu_{0,S}^+ \geqslant \nu_{0,S}^-$. Let Δ_L be the component of α in the special basic case, but with $\frac{1}{2}(\nu_{0,S}^+ - \nu_{0,S}^-)$ roots deleted from the end opposite to the one where ε is adjoined. Then we have $\nu_{0,L}^+ = \nu_{0,L}^- = \nu_{0,S}^- = \nu_0^-$, and an easy computation with the basic cases of $SU(N, 1)$ shows that $\frac{1}{2}(\nu_{0,L}^+)\,\alpha = \rho_L$. Thus we will be done (by Proposition 12.4) if we show that $\Lambda - 2\delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}})$ is orthogonal to Δ_L.

Now Δ_L is generated by $\Delta_{K,\perp}$ and α, and thus it is enough to show orthogonality with α. Let E be the orthogonal projection along $\mathbb{R}\alpha$. By Theorem 1 of [9], we have

$$\begin{aligned} E(\Lambda - 2\delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}})) &= -E(2\delta_K) + \mu - E(2\delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}})) \\ &= -E(2\delta) + E(2\delta(\mathfrak{l}^{\mathbb{C}} \cap \mathfrak{p}^{\mathbb{C}})) + \mu \\ &= -\alpha + E(2\delta(\mathfrak{l}^{\mathbb{C}})) - E(2\delta_K(\mathfrak{l}^{\mathbb{C}})) + \mu \\ &= -E(2\delta_K(\mathfrak{l}^{\mathbb{C}})) + \mu. \end{aligned} \tag{12.7}$$

A little check in $SU(N, 1)$ shows that Δ_L is the basic case for $\sigma = 1$ in $SU(N, 1)$ (since $\nu_{0,L}^+ = \nu_{0,L}^-$), and Theorem 1 of [9] therefore identifies (12.7) as the minimal K-type of the spherical principal series, namely 0. This proves the required orthogonality.

SITUATION (vi). First we work in $\mathfrak{so}(5, 4)$ with the basic case for



with $\mu = 0$ and $\alpha = e_3 - e_4$. Here $\lambda_0 = (\frac{1}{2}, \frac{1}{2}, 0, 0)$ and we are to prove unitarity at $\nu = \frac{3}{2}\alpha$. The minimal K-type is $\Lambda' = (2, 0, 0, 1)$.

Let Δ_L be spanned by $e_1 - e_2$, $e_2 - e_3$, $e_3 - e_4$, and put $\lambda = -\delta(\mathfrak{u}) = (-2, -2, -2, -2)$. Then (12.1) is trivial, and Λ, defined by

$$\Lambda = \lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}}) = (1, 0, 0, 1)$$

is dominant for $\Delta_K^+ = \{e_1 \pm e_4,\ e_2 \pm e_3,\ e_2,\ e_3\}$. Moreover,

$$\Lambda + 2\delta_K = (3, 3, 1, 1)$$

is Δ^+ dominant. Thus Theorem 12.2 and Proposition 12.3 say that $A_{\mathfrak{q}}(\lambda)$

has infinitesimal character $\lambda+\delta$, that every irreducible subquotient of $A_{\mathfrak{q}}(\lambda)$ is unitary, and that Λ is the unique minimal K-type. Here

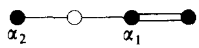
$$\lambda+\delta=(-2,\ -2,\ -2,\ -2)+(\tfrac{7}{2},\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2})=(\tfrac{3}{2},\tfrac{1}{2},\ -\tfrac{1}{2},\ -\tfrac{3}{2})$$

is conjugate by the Weyl group of Δ to

$$\lambda_0+\nu=(\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2},\ -\tfrac{3}{2}).$$

Hence $A_{\mathfrak{q}}(\lambda)$ has the same infinitesimal character as $J(MAN, \sigma, \tfrac{3}{2}\alpha)$.

To complete the proof, we shall show that $A_{\mathfrak{q}}(\lambda)$ is reducible, having an irreducible constituent with Λ' as minimal K-type. We begin by finding the Langlands parameters of the (irreducible) cyclic subspace of $A_{\mathfrak{q}}(\lambda)$ generated by the K-type τ_Λ. In fact, take as parameters $\tilde{M}\tilde{A}\tilde{N}$, $\tilde{\nu}=\tfrac{3}{2}\alpha_1+\tfrac{1}{2}\alpha_2$, and $\tilde{\lambda}_0=0$ for $\tilde{\Delta}^+$ given by



Theorem 1 of [9] shows that the minimal K-type of the corresponding induced series is indeed $\Lambda=(1, 0, 0, 1)$. Since we know that Λ determines $\tilde{\lambda}_0$ and that $\tilde{\lambda}_0$ is 0, $\tilde{\nu}$ must be the full infinitesimal character (put in the positive Weyl chamber).

Let us see that $\tau_{\Lambda'}$ does not occur in the induced series $U(\tilde{M}\tilde{A}\tilde{N}, \tilde{\sigma}, \cdot)$. Let $\tilde{\lambda}$ be the minimal $(K\cap\tilde{M})$-type of $\tilde{\sigma}$, namely $\tilde{\lambda}=(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})$. For Λ' to occur, we must have

$$\Lambda=\tilde{\lambda}+d_1\alpha_1+d_2\alpha_2,$$

$$\lambda'=\tilde{\lambda}+\sum_{\beta\in\Delta^+_{-,n}} n_\beta\beta,$$

$$\Lambda'-\sum_{\gamma\in\Delta_K^+} k_\gamma\gamma=\lambda'+c_1\alpha_1+c_2\alpha_2=\text{weight of } \tau_{\Lambda'}.$$

for integers $n_\beta\geqslant 0$ and $k_\gamma\geqslant 0$ and for real numbers d_1, d_2, c_1, c_2. Then it follows that

$$(1, 0, 0, 0)=\Lambda'-\Lambda=\sum_{\beta\in\Delta^+_{-,n}} n_\beta\beta+\sum_{\gamma\in\Delta_K^+} k_\gamma\gamma+(x_1\alpha_1+x_2\alpha_2).$$

The only possible β's are $\beta_1=e_1+e_2$ and $\beta_2=e_3+e_4$. Taking the inner product with $\beta_1+\beta_2$, we obtain

$$1=2(n_1+n_2)+\sum k_\gamma\langle\gamma, e_1+e_2+e_3+e_4\rangle.$$

Since $e_1+e_2+e_3+e_4$ is Δ^+ dominant, it follows that $n_1=n_2=0$, that the

Δ_K^+ simple root e_1+e_4 has coefficient 0, and that the Δ_K^+ simple root e_3 has coefficient 1. Thus

$$\begin{aligned}(1,0,0,0) &= \sum_{\gamma\in\Delta_K^+} k_\gamma\gamma+(x_1\alpha_1+x_2\alpha_2)\\ &= a(e_1-e_4)+b(e_2-e_3)+e_3+(x_1\alpha_1+x_2\alpha_2).\end{aligned}$$

The inner product with e_1+e_2 shows $a+b=1$. Thus the only possible solutions have $a=1$, $b=0$ and $a=0$, $b=1$. In these respective cases, $\sum k_\gamma\gamma$ is $(1,0,1,-1)$ or $(0,1,0,0)$, and $\Lambda'-\sum k_\gamma\gamma$ is $(1,0,-1,2)$ or $(2,-1,0,1)$, both of which are too large to be weights. Hence $\tau_{\Lambda'}$ does not occur in the induced series.

Next we consider the occurrence of a general $\tau_{\Lambda''}$ in $A_\mathfrak{q}(\lambda)$. If the s term is nonzero, (12.2) says that

$$\Lambda''=\Lambda+\sum_{\gamma\in\Delta_K^+} k_\gamma\gamma+\sum_{\beta\in\Delta(\mathfrak{u}\cap\mathfrak{p}^{\mathbb{C}})} n_\beta\beta,\quad k_\gamma\geqslant 0\quad\text{and}\quad n_\beta\geqslant 0. \tag{12.9}$$

Here we can restrict γ in Δ_K^+ to be Δ_K^+ simple, thus a member of

$$\{e_1+e_4,\ e_1-e_4,\ e_2-e_3,\ e_3\},$$

while β is a member of

$$\{e_1,\ e_4,\ e_1+e_2,\ e_1+e_3,\ e_2+e_4,\ e_3+e_4\}.$$

Moreover, at least one n_β is >0 if $\Lambda''\neq\Lambda$.

For $\Lambda''=\Lambda'$, we take the inner product of (12.9) with $e_1+e_2+e_3+e_4$ and conclude from the remarks above that

$$e_1=\Lambda'-\Lambda=a(e_1-e_4)+b(e_2-e_3)+ce_1+de_4+he_3$$

with a, b, c, d, h nonnegative integers and with $c+d+h=1$. Solving, we find the two solutions

$$\begin{aligned} c=1 \quad&\text{and}\quad a=b=d=h=0,\\ a=d=1 \quad&\text{and}\quad b=c=h=0.\end{aligned}$$

These translate into

$$\sum k_\gamma\gamma=0 \qquad\text{and}\qquad \sum n_\beta\beta=e_1,$$

$$\sum k_\gamma\gamma=e_1-e_4 \qquad\text{and}\qquad \sum n_\beta\beta=e_4.$$

Hence they correspond to $s=1$ and $s=s_{e_1-e_4}$. But we check directly that

$$s_{e_1-e_4}(\Lambda'+\delta_K)\neq\Lambda'+\delta_K-(e_1-e_4),$$

and thus only $s=1$ is possible. On the other hand, the $s=1$ term for $\text{mult}(\tau_{\Lambda'})$ does equal one, and thus we conclude that $\tau_{\Lambda'}$ occurs in $A_{\mathfrak{q}}(\lambda)$ with multiplicity one.

Finally we show that $|\Lambda''+2\delta_K|^2$ is minimized among all K-types other than τ_Λ in $A_{\mathfrak{q}}(\lambda)$ uniquely by $\Lambda''=\Lambda'$, so that the cyclic span of the $\tau_{\Lambda'}$ subspace is an irreducible unitary representation with the same minimal K-type and infinitesimal character as $J(MAN,\sigma,\frac{3}{2}\alpha)$. We continue with the normalization of inner products that makes $|e_1|^2=1$. First let us note that

$$|\Lambda'+2\delta_K|^2-|\Lambda+2\delta_K|^2=7.$$

Suppose Λ'' gives a smaller difference. We refer to (12.3) and note that

$$2\left\langle\Lambda+2\delta_K,\sum k_\gamma\gamma+\sum n_\beta\beta\right\rangle$$

gets bigger when more γ's and β's are used, while the term

$$\left|\sum k_\gamma\gamma+\sum n_\beta\,\beta\right|^2 \tag{12.10}$$

is always at least one (except when $\Lambda''=\Lambda$). For β in $\Delta(\mathfrak{u}\cap\mathfrak{p}^{\mathbb{C}})$, we find that $2\langle\Lambda+2\delta_K,\beta\rangle\leqslant 6$ (as required) only for $\beta=e_1$ and $\beta=e_4$. With $\beta=e_1$, we must have $\sum k_\gamma\gamma+\sum n_\beta\beta=e_1$, and thus $\Lambda''=\Lambda'$. With $\beta=e_4$, we must have $2\langle\Lambda+2\delta_K,\sum k_\gamma\gamma\rangle\leqslant 6-2n_{e_4}$. This equation implies $k_{e_1+e_4}=0$, and (12.10) forces also $k_{e_2-e_3}=0$. If $k_{e_1-e_4}>0$, we are led to $\sum k_\gamma\gamma=e_1-e_4$ and $\sum n_\beta\beta=e_4$, from which we obtain $\Lambda''=\Lambda'$. The only remaining possibility is that $k_{e_3}>0$. Then we must have $\Lambda''=\Lambda+ke_3+ne_4$. But this Λ'' is not Δ_K^+ dominant if $k>0$ or $n>0$. This proves the minimizing property of $|\Lambda''+2\delta_K|^2$ and completes the proof of unitarity of $J(MAN,\sigma,\frac{3}{2}\alpha)$ in $\mathfrak{so}(5,4)$.

Now let us pass to the general case in situation (vi). Possibly after reflecting in α, we can arrange that the $\mathfrak{so}(5,4)$ case is imbedded in the general case, and we take Δ_L to correspond to the $\mathfrak{so}(5,4)$. In standard notation, the last entries of λ_0 and $\lambda_0+\frac{3}{2}\alpha$ are

$$\lambda_0=(...,\tfrac{1}{2},\tfrac{1}{2},0,0),$$

$$\lambda_0+\tfrac{3}{2}\alpha=(...,\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2},-\tfrac{3}{2}),$$

and the last entry of the "..." within λ_0 and $\lambda_0+\frac{3}{2}\alpha$ is at least $\frac{3}{2}$. Then it follows that $\langle\lambda_0+\frac{3}{2}\alpha,\beta\rangle\geqslant 0$ for all β in $\Delta(\mathfrak{u})$, and $J(MAN,\sigma,\frac{3}{2}\alpha)$ is infinitesimally unitary by Theorem 1.3a of Vogan [24].

13. The Final Gap in Unitarity

To complete the proof of Theorem 1.1 when rank $G = \text{rank } K$, we have still to show in situation (vi) that $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is not infinitesimally unitary for $2 < c < 3$. This result does not seem to lend itself to the kind of analysis in Sections 3–7, and we shall use the theory of intertwining operators instead. The idea is that $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is reducible at $c = 2$ and the intertwining operator that defines the invariant Hermitian form has a simple zero on its (nontrivial) kernel at $c = 2$; consequently the signature of the form on any K-type that meets the kernel changes at $c = 2$, while the signature on the minimal K-type remains positive.

In the proof we shall treat just $\mathfrak{so}(5, 4)$, to keep the notation simple. It is an easy matter to revise the proof to apply to all cases of situation (vi), and we shall make some comments on this point at the end of the section. Possibly by reflecting in α, we may assume that the Dynkin diagram is as in (12.8).

First we prove reducibility at $\nu = \alpha$. We have $\mu = 0$ and

$$\lambda_0 = (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0),$$
$$\lambda_0 + \alpha = (\tfrac{1}{2}, \tfrac{1}{2}, 1, -1).$$

It follows from [5] that the reducibility question is the same as for the $SO(3, 2)$ subgroup (corresponding to integral infinitesimal character) with

$$\tilde{\lambda}_0 = (0, 0), \qquad \tilde{\lambda}_0 + \alpha = (1, -1), \qquad \mu = 0.$$

The root α does not satisfy the parity condition. The analysis of reducibility is carried out as in Section 4 of [4], but with $\mathrm{Sp}(3, \mathbb{R})$ cut down to $\mathrm{Sp}(2, \mathbb{R})$: The tool is Vogan's composition series algorithm, and one wall crossing is needed. The result is that we have reducibility into two pieces.

Now we bring in the intertwining operators of [15]. We shall use the notation of that paper without redefining it; alternatively the reader may consult [11], where the same notation is used. According to [16], the operator that defines the Hermitian form at ν is

$$\sigma(w)\, A_P(w, \sigma, \nu), \tag{13.1}$$

apart from normalization. Here w is a representative in K of the nontrivial element of $W(A:G)$, as in Section 1, and we may assume that this operator is positive definite (on each K-type) relative to $L^2(K, V^\sigma)$ for ν small and positive.

Let E be a finite-dimensional subspace of the domain of (13.1) equal to the sum of a number of K-types, and let $T(z): E \to E$ be the restriction to E

of $\sigma(w)\, A_P(w, \sigma, \frac{1}{2}(2-z)\,\alpha)$, for complex z with $|z|<1$. We can regard $T(z)$ as an analytic $n \times n$ matrix-valued function of z, by [15]. Following Jantzen [8] and Vogan [24], we define

$$E_k = \{v \in \mathbb{C}^n \mid \text{there exists } f\colon \{|z|<1\} \to \mathbb{C}^n \text{ analytic such that } f(0)=V \quad \text{and} \quad z^{-k}T(z)f(z) \text{ is analytic at } 0\}. \tag{13.2}$$

LEMMA 13.1. *In the above notation for* $\mathfrak{so}(5,4)$,

$$E_0 = E, \qquad E_1 = E \cap \ker T(0), \qquad \text{and} \qquad E_2 = 0.$$

We say that $T(z)$ has *only a simple zero* at $z=0$. We postpone the proof for a moment, first showing how the desired nonunitarity of $J(MAN, \sigma, \frac{1}{2}c\alpha)$ follows for $2<c<3$.

The operator $T(z)$ is Hermitian for real z, and we can use it as in Section 3 of [24] to define a nondegenerate Hermitian form on E_k/E_{k+1}, say with signature (p_k, q_k). According to Proposition 3.3 of [24], the signature of $T(z)$ for small positive z is $(\sum p_k, \sum q_k)$, while the signature for small negative z is

$$\Bigl(\sum_{k \text{ even}} p_k + \sum_{k \text{ odd}} q_k,\ \sum_{k \text{ odd}} p_k + \sum_{k \text{ even}} q_k\Bigr).$$

Lemma 13.1 says that $p_k = q_k = 0$ for $k \geqslant 2$, and the positivity of $T(z)$ for $z>0$ says that $q_0 = q_1 = 0$. Thus the signature on E of $T(z)$ for small negative z is (p_0, p_1). Here $p_0 = \dim E_0/E_1$ and $p_1 = \dim E_1$. Thus our operator is indefinite on any E large enough to contain the minimal K-type and a K-type that meets the (nontrivial) kernel of (13.1) at $v=\alpha$.

Thus the problem for $\mathfrak{so}(5,4)$ comes down to proving Lemma 13.1. We use the following two lemmas in its proof.

LEMMA 13.2. *Let* $A(z)$, $B(z)$, *and* $M(z)$ *be* $n \times n$ *matrix-valued analytic functions for* $|z|<1$ *with* $A(0)$ *and* $B(0)$ *nonsingular. If* $M(z)$ *has only a simple zero at* $z=0$, *then so does* $A(z)\,M(z)\,B(z)$.

Proof. Let us define $E_k(M)$ as in (13.2), with M replacing T. If v in $E_k(M)$ is represented by $f(z)$, then $B(z)^{-1} f(z)$ has $B(0)^{-1} f(0) = B(0)^{-1}\, v$ with

$$\lim_{z \to 0} z^{-k}[A(z)\,M(z)\,B(z)]\; B(z)^{-1} f(z) = A(0) \lim_{z \to 0} z^{-k} M(z) f(z)$$

existing. Hence $B(0)^{-1}\, v$ is in $E_k(AMB)$. Thus $B(0)^{-1}\, E_k(M) \subseteq E_k(AMB)$. Applying $B(0)$ and arguing similarly, we see that equality holds. The lemma follows.

LEMMA 13.3. *Let $M(z)$ be an $n \times n$ matrix-valued analytic function for $|z| < 1$ such that $M(z)$ is diagonal for all z. If each diagonal entry of $M(z)$ has at most a simple zero at $z = 0$, then $M(z)$ has at most a simple zero at $z = 0$.*

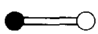
Proof. Elementary.

Lemma 13.2 allows us to strip off invertible factors from either side of $T(z)$. In particular, we can strip away invertible factors that do not depend on z. Thus we can identify our operators with matrices, and it does not matter what bases we use for the identification. First we discard $\sigma(w)$ from (13.1) because its action is invertible. Next we shall enlarge the domain of $A_P(w, \sigma, \nu)$ and then factor the resulting operator as a product of simpler operators.

To do so, we note that the roots of $\mathfrak{m}$ are given by the diagram

(13.3)

and λ_0 is 0 on the $\mathfrak{sl}(2, \mathbb{R})$ corresponding to $e_3 + e_4$. We imbed this limit of discrete series of $SL(2, \mathbb{R})$ (crossed with the rest of M) in a reducible unitary principal series of $SL(2, \mathbb{R})$ (crossed with the rest of M), and then we induce everything to G, using the double induction principle. The result is that $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is a direct summand of an induced representation $U(M_* A_* N_*, \sigma_*, \frac{1}{2}c\alpha)$ obtained from the rank two parabolic subgroup $M_* A_* N_*$ with A_* built from $\alpha = e_3 - e_4$ and $\alpha' = e_3 + e_4$. The Dynkin diagram of $\mathfrak{m}_*$ is simply



and the parameter $\tilde{\lambda}_0$ of σ_* is just the restriction of λ_0. For restricted roots relative to this parabolic subgroup, we can use a system of type B_2 with $f_1 + f_2 = \text{Cayley}(\alpha)$ and $f_1 - f_2 = \text{Cayley}(\alpha')$. To specify σ_* completely, we give only $\sigma_*(\gamma_{f_1+f_2})$ and $\sigma_*(\gamma_{f_1-f_2})$, since $\gamma_{f_1} = \gamma_{f_2} = \gamma_{f_1-f_2}\gamma_{f_1+f_2}$. We take σ_* to agree with σ on $\gamma_{f_1+f_2} = \gamma_\alpha$. Since $\mu = 0$, this means $\sigma_*(\gamma_{f_1+f_2}) = -1$. The value of σ_* on $\gamma_{f_1-f_2}$ is determined by the value of σ on the central element of the $SL(2, \mathbb{R})$ subgroup of M; thus $\sigma_*(\gamma_{f_1-f_2}) = -1$.

We can choose w in (13.1) to be a representative in K of the reflection $s_{f_1+f_2}$ in $W(A_*: G)$, and then the techniques of [15] show that

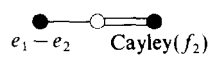
$$A_P(w, \sigma, \tfrac{1}{2}c\alpha) \subseteq A_{P_*}(w, \sigma_*, \tfrac{1}{2}c(f_1 + f_2)). \tag{13.4}$$

Actually since we can discard invertible operators in our analysis, we can simply write $s_{f_1+f_2}$ directly in place of w, and then Proposition 7.8 of [15] allows us to factor the right side of (13.4) according to a cocycle relation as

$$A_{P_*}(s_{f_2}, s_{f_1-f_2}s_{f_2}\sigma_*, -\tfrac{1}{2}c(f_1-f_2))\, A_{P_*}(s_{f_1-f_2}, s_{f_2}\sigma_*, \tfrac{1}{2}c(f_1-f_2)) \times A_{P_*}(s_{f_2}, \sigma_*, \tfrac{1}{2}c(f_1+f_2)). \tag{13.5}$$

Let us examine the third factor here more closely. This operator depends only on data in the subgroup of G given as the centralizer $Z = Z_G(\ker(f_2))$, and by means of the kind of identification in Proposition 7.5 of [15], it can be identified with a standard inertwining operator for Z. In more detail, we can see from Section 5 of [12] that the operator in G on a single K-type is the tensor product of a block diagonal operator with an identity operator, while the operator in Z on a $(K \cap Z)$-type contained in that K-type is the tensor product of one of the blocks with a different identity operator. At any rate, nonsingularity of the operator for Z implies nonsingularity of the operator for G.

The subgroup Z is essentially $SO(4, 3)$, and its $\mathfrak{m}$ is just $\mathfrak{m}_*$. (There is an additional abelian factor to Z, and there is some disconnectedness, but these features do not affect the intertwining operators in any essential way.) The intersection of $\mathfrak{a}_*$ with $\mathfrak{so}(4, 3)$ is one-dimensional, and we can write the Dynkin diagram of the Lie algebra of Z as



(13.6)

in order to fulfill the conditions of Section 1. (It is not important here to see that the middle simple root is compact in Z, even though that is the case.) Relative to this system, we can write $\tilde{\lambda}_0$ in coordinates as $(\frac{1}{2}, \frac{1}{2}, 0)$. This parameter is not integral, and we must have $\mu = 0$. Since Cayley(f_2) is short, Corollary 8.3 says that the induced representation of Z is irreducible at integral multiples of the root defining the $\mathfrak{a}$ of Z, hence in particular at $\frac{1}{2}cf_2$ for $c = 2$.

Thus Lemma 13.2 allows us to discard the third factor on the right side of (13.5) from our analysis, and in similar fashion we can discard the first factor.

Let us examine more closely the second factor on the right side of (13.5). This operator depends only on data in the subgroup $Z' = Z_G(\ker(f_1 - f_2))$ and again can be identified with a standard intertwining operator for Z'. Here the relevant fact about the identification is that if the operator for Z' is diagonal with diagonal entries having at most a simple zero at $\frac{1}{2}(2-z)\,\alpha$ for $z = 0$, then the same thing is true of the operator in G.

In view of Lemma 13.3, we will therefore have proved Lemma 13.1 if we show that the operator for Z' is diagonal with diagonal entries having at most a simple zero at $\frac{1}{2}(2-z)\,\alpha$ for $z = 0$. The point now is that Z' is essentially a product of $SL(2, \mathbb{R})$, an abelian factor, and the identity component of M_*; moreover, only the $SL(2, \mathbb{R})$ is important to the operator. Thus we can regard the operator (on a $(K \cap Z')$-type) as the tensor product of an identity operator by the restriction to a K-type of $SL(2, \mathbb{R})$ of a standard intertwining operator for $SL(2, \mathbb{R})$.

The K-types for $SL(2, \mathbb{R})$ (and indeed any $\widetilde{SO}_0(n, 1)$) have multiplicity one, and thus any standard intertwining operator is scalar for a given K-type and given ν. This sclar function of ν is the subject of the lemma below, which completes the proof of Lemma 13.1 for $\mathfrak{g} = \mathfrak{so}(5, 4)$.

LEMMA 13.4. *Let $\gamma_\sigma(z)$ be the scalar value of a standard intertwining operator $A(w, \sigma, z\rho)$ for $\widetilde{SO}_0(n, 1)$ on some K-type. Then for z positive, any zero of $\gamma_\sigma(z)$ is simple.*

Proof. If $\gamma_\sigma(z_0) = 0$, then the induced representation is reducible at z_0. Hence the infinitesimal character is integral and, in the case of $SL(2, \mathbb{R}) \cong \widetilde{SO}_0(2, 1)$, α satisfies a parity condition at z_0. Then it follows from [14] that $A(w^{-1}, w\sigma, -z\rho)$ has no pole at $-z_0$, so that $\gamma_{w\sigma}(-z)\,\gamma_\sigma(z)$ is analytic at $z = z_0$ and vanishes there at least to the order that $\gamma_\sigma(z)$ vanishes.

But [14] shows that $\gamma_{w\sigma}(-z)\,\gamma_\sigma(z)$ is a nonzero multiple of the reciprocal of the Plancherel factor $p_\sigma(z)$. This factor is the product of a polynomial by a possible tangent or cotangent and has at most simple poles. Thus $\gamma_{w\sigma}(-z)\,\gamma_\sigma(z)$ vanishes at most to order one, and so does $\gamma_\sigma(z)$. This proves the lemma.

Let us briefly indicate how to revise the proof of Lemma 13.1 to apply to all cases of situation (vi). The reducibility at $\nu = \alpha$ is established by the same computation, still in $\mathfrak{so}(3, 2) \cong \mathfrak{sp}(2, \mathbb{R})$. After we form the intertwining operator, we embed our induced representations in representations induced from a rank-two parabolic subgroup, one built from $\alpha = e_{N-1} - e_N$ and $\alpha' = e_{N-1} + e_N$. Then we still have a factorization (13.5), and we can go through the same identification procedure. For the first and third factors, the diagram (13.6) is enlarged by more simple roots to the left of $e_1 - e_2$, but the irreducibility at integral multiples of f_2 is unaffected. For the second factor, we still have essentially an operator for $SL(2, \mathbb{R})$, and thus the argument goes through without essential change.

This completes the proof of Theorem 1.1 in the case that $\operatorname{rank} G = \operatorname{rank} K$.

14. CONSIDERATION OF $\mathfrak{so}$(ODD, ODD)

In this section we prove Theorem 1.1 when $\operatorname{rank} G > \operatorname{rank} K$. We use the general notation established in Section 1. We have already observed that we may take $\mathfrak{g} = \mathfrak{so}(\text{odd}, \text{odd})$, and we accordingly introduce further notation to reflect properties of that Lie algebra.

The root system $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}})$ is of type D_N, and we take $\alpha_R = e_N$. The root system $\Delta_- = \Delta(\mathfrak{m}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$ is of type D_{N-1} within Δ. Once (λ_0, Δ_-^+) is fixed, we can name the roots of Δ_- in such a way that the simple roots of Δ_-^+ are the standard ones in D_{N-1}: $e_1 - e_2, ..., e_{N-2} - e_{N-1}$, $e_{N-2} + e_{N-1}$. Then we can write

$$\lambda_0 = (n_1, ..., n_{N-2}, n_{N-1}, 0)$$

with

$$n_1 \geqslant \cdots \geqslant n_{N-2} \geqslant |n_{N-1}|$$

and with all n_j in $\mathbb{Z}$ or all n_j in $\mathbb{Z} + \frac{1}{2}$. The linear functional α_I in Section 1 is just e_{N-1}, and the condition $w[\sigma] = [\sigma]$ means exactly that $n_{N-1} = 0$. Since there is no unitarity without this condition, we assume it now. In particular, the n_j's are then integers.

Thus we write

$$\lambda_0 = (n_1, ..., n_{N-2}, 0, 0), \qquad \text{all} \quad n_j \in \mathbb{Z}. \tag{14.1}$$

Taking Δ^+ to be the system with simple roots $e_1 - e_2, ..., e_{N-1} - e_N$, $e_{N-1} + e_N$, we see that Δ^+ meets the conditions of Section 1. Here $e_1, ..., e_{N-1}$ span the dual of $i\mathfrak{b}$, and e_N spans the dual of $\mathfrak{a}$. We define $\langle \cdot, \cdot \rangle$ by $\langle e_i, e_j \rangle = \delta_{ij}$, so that $|e_i|^2 = 1$.

According to [9], $\Delta_K = \Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$ consists exactly of the restrictions to $\mathfrak{b}^{\mathbb{C}}$ of the members of $\Delta - \Delta_{-,n}$, and we may take Δ_K^+ to be the restrictions of the positive such elements. Then we see that

$$\Delta_K^+ = \{e_i \pm e_j \mid i < j \leqslant N-1 \text{ and } e_i \pm e_j \notin \Delta_{-,n}\} \cup \{e_i \mid i \leqslant N-1\},$$

with e_i the restriction to $\mathfrak{b}^{\mathbb{C}}$ of $e_i \pm e_N$. Theorem 1 of [9] says that the minimal K-type of the induced representations is

$$\Lambda = \lambda_0 + \delta - 2\delta_K. \tag{14.2}$$

LEMMA 14.1. *The Δ_K^+ simple roots γ such that $\langle \Lambda, \gamma \rangle = 0$ are e_{N-1} and all members $e_i - e_{i+1}$ of $\Delta_{-,c}^+$ such that $\langle \lambda_0, e_i - e_{i+1} \rangle = 1$.*

Proof. First suppose that $\gamma = e_i \pm e_j$ with $i < j \leqslant N-1$ and γ compact for Δ_-. If $\langle \Lambda, \gamma \rangle = 0$, then (14.2) and our normalization of the inner product give

$$0 = \langle \lambda_0, \gamma \rangle + \langle \delta, \gamma \rangle - 2. \tag{14.3}$$

The first term on the right side is $\geqslant 1$ by nondegeneracy, and thus it must be 1 and $\langle\delta, \gamma\rangle$ must be 1, i.e., γ must be Δ^+ simple. Hence $\gamma = e_i - e_{i+1}$. Conversely if $\gamma = e_i - e_{i+1}$ and $\langle\lambda_0, e_i - e_{i+1}\rangle = 1$, then (14.3) holds and γ satisfies $\langle\Lambda, \gamma\rangle = 0$.

Otherwise suppose $\gamma = e_i$ with $i \leqslant N-1$. If $\langle\Lambda, \gamma\rangle = 0$, then (14.2) gives

$$\begin{aligned} 0 &= 2\langle\lambda_0, \gamma\rangle + 2\langle\delta, \gamma\rangle - 2 \\ &= 2\langle\lambda_0, \gamma\rangle + \langle\delta, e_i + e_N\rangle + \langle\delta, e_i - e_N\rangle - 2 \\ &\geqslant \langle\delta, e_i + e_N\rangle + \langle\delta, e_i - e_N\rangle - 2. \end{aligned}$$

We conclude that $\langle\delta, e_i + e_N\rangle = \langle\delta, e_i - e_N\rangle = 1$, from which it follows that $e_i - e_N$ is simple. Thus $i = N-1$. Conversely if $i = N-1$, we know $\langle\lambda_0, e_{N-1}\rangle = 0$ and thus

$$\begin{aligned} \langle\Lambda, e_{N-1}\rangle &= 0 + \langle\delta, e_{N-1}\rangle - \langle 2\delta_K, e_{N-1}\rangle = \langle\delta, e_{N-1}\rangle - \frac{2\langle\delta_K, e_{N-1}\rangle}{|e_{N-1}|^2} \\ &= \langle\delta, e_{N-1}\rangle - 1 = 0. \end{aligned}$$

We recall from Section 1 the definition

$$\nu_0 = 2\#\{\beta \in \Delta^+ \mid \beta|_{\mathfrak{a}} > 0 \quad \text{and} \quad \langle\Lambda, \beta\rangle = 0\}.$$

Let i_0 be the smallest index i such that $e_i - e_{i+1}, \ldots, e_{N-2} - e_{N-1}$ are all compact and have $\langle\lambda_0, e_j - e_{j+1}\rangle = 1$ for $i \leqslant j \leqslant N-2$.

LEMMA 14.2. (a) *If $j < N$, then $\langle\Lambda, e_j\rangle = 0$ if and only if $i_0 \leqslant j \leqslant N-1$.*

(b) $\nu_0 = 2(N - i_0)$.

Proof. (a) It is immediate from Lemma 14.1 that $\langle\Lambda, e_j\rangle = 0$ for $i_0 \leqslant j \leqslant N-1$. Conversely let $\langle\Lambda, e_j\rangle = 0$. If we expand e_j in terms of Δ_K^+ simple roots, we obtain

$$e_j = (e_j - e_{j_1}) + (e_{j_1} - e_{j_2}) + \cdots + (e_{j_{l-1}} - e_{j_l}) + e_{j_l}.$$

Each term must be orthogonal to Λ, and then Lemma 14.1 shows that $j_l = N-1$ and the various pairs of indices j_k, j_{k+1} are consecutive with $\langle\lambda_0, e_{j_k} - e_{j_{k+1}}\rangle = 1$. Hence $i_0 \leqslant j \leqslant N-1$.

(b) The roots β in Δ^+ with $\beta|_{\mathfrak{a}} > 0$ are $\beta = e_j + e_N$, and the condition $\langle\Lambda, \beta\rangle = 0$ then means $\langle\Lambda, e_j\rangle = 0$. Thus $\nu_0 = 2(N - i_0)$ by (a).

Let Δ_L be the subsystem of Δ with simple roots

$$e_{i_0} - e_{i_0+1}, \ldots, e_{N-2} - e_{N-1}, e_{N-1} - e_N, e_{N-1} + e_N.$$

Since rank $L > \operatorname{rank}(L \cap K)$ and since every root of $\Delta_- \cap \Delta_L$ is compact, Δ_L is of the form $\mathfrak{so}(2(N-i_0)+1, 1)$. Therefore all $(K \cap L)$-types of any standard induced representation have multiplicity one. Now the linear form $\Lambda + e_{N-1}$, when made dominant for $\Delta_K^+ \cap \Delta_L$, becomes $\Lambda + e_{i_0}$, and Lemma 14.2a shows that this form is dominant for Δ_K^+. Therefore $\Lambda' = (\Lambda + e_{N-1})^\vee$ is a K-type in the bottom layer of $U(MAN, \sigma, \nu)$, in the sense of Speh and Vogan [20], and we can conclude from that paper that $\tau_{\Lambda'}$ has multiplicity one in $U(MAN, \sigma, \nu)$.

From the conclusion that $\tau_{\Lambda'}$ has multiplicity one and from a result like those in Section 3 (which we defer to another paper), we obtain

LEMMA 14.3. *With notation as above, put* $\Lambda' = (\Lambda + e_{N-1})^\vee$. *Normalize the standard Hermitian form for* $U(MAN, \sigma, \frac{1}{2}c\alpha_R)$ *so that it is positive on* τ_Λ. *Then* $\tau_{\Lambda'}$ *has multiplicity one and the signature of the standard form on* $\tau_{\Lambda'}$ *is* $\operatorname{sgn}(\nu_0 - c)$.

Consequently $J(MAN, \sigma, \frac{1}{2}c\alpha_R)$ is not infinitesimally unitary for $c > \nu_0$. To see that it is unitary for $c \leqslant \nu_0$, we prove irreducibility for $c < \nu_0$. Within Δ_L, the series of representations in question is the spherical principal series, which is irreducible out to $\rho_L = (N-i_0)\,\alpha_R = \frac{1}{2}\nu_0 \alpha_R$. We shall apply the results of Speh and Vogan [20] that are applicable here and are analogous to those quoted in Section 8. For one thing, irreducibility will follow in G at $\nu = \frac{1}{2}c\alpha_R$ if $c < \nu_0$ and ν_0 satisfies $\langle \lambda_0 + \nu, \beta \rangle \geqslant 0$ for all β in $\Delta^+ - \Delta_L$.

We can handle $c \leqslant 2(N - i_0 - 1)$ by showing that

$$\langle \lambda_0 + (N - i_0 - 1)\,\alpha_R, \beta \rangle \geqslant 0 \qquad \text{for all } \beta \text{ in } \Delta^+ - \Delta_L.$$

The worst β is evidently $\beta = e_{i_0 - 1} - e_N$, and we have

$$\begin{aligned} \langle \lambda_0 + (N - i_0 - 1)\,\alpha_R, e_{i_0-1} - e_N \rangle &= \langle \lambda_0, e_{i_0-1} - e_N \rangle - (N - i_0 - 1) \\ &\geqslant \langle \lambda_0, e_{i_0} - e_N \rangle - (N - i_0 - 1) = 0. \end{aligned}$$

For $2(N - i_0 - 1) < c < 2(N - i_0)$, we appeal to the following Lemma, based on [20].

LEMMA 14.4. *With* λ_0 *as in* (14.1) *and* ν *positive,* $U(MAN, \sigma, \nu)$ *can be reducible only when* ν *is an integral multiple of* α_R.

Proof. Since there are no real roots, Theorem 6.19 of Speh and Vogan [20] says that there can be reducibility only at points ν for which there is a complex root β such that $2\langle \lambda_0 + \nu, \beta \rangle / |\beta|^2$ is an integer and β

satisfies certain other properties. Possibly replacing β by $-\beta$, we can take β to be $e_j \pm e_N$ for some $j < N$. Then

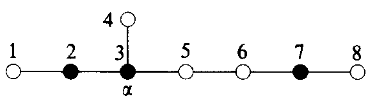
$$\frac{2\langle \lambda_0 + \nu, \beta \rangle}{|\beta|^2} = \langle \lambda_0, e_j \rangle \pm \langle \nu, e_N \rangle$$

has to be an integer. Since (14.1) says that $\langle \lambda_0, e_j \rangle$ is an integer, $\langle \nu, e_N \rangle$ has to be an integer. This proves the lemma and completes the proof of Theorem 1.1 when rank $G >$ rank K.

15. Remarks about Unitarity

1. *Calculations in Single-Line Diagrams*

In a single-line diagram with rank $G =$ rank K, Theorem 1.1 implies that the unitary points form an interval. This example will illustrate how to determine the endpoint of the interval easily. Actually no computation is needed after Δ_{-}^{+} has been imbedded in Δ^+. Let us suppose that Δ^+ is



and that $\lambda_0 = \left(\begin{smallmatrix} & & 1 & & & & \\ 1 & 0 & 0 & 1 & 1 & 0 & 3 \end{smallmatrix}\right)$, with the integers representing the numbers $2\langle \lambda_0, \beta \rangle / |\beta|^2$ for β simple.

Since λ_0 is Δ^+ integral and $\mathfrak{g} \not\cong \mathfrak{sp}(n, \mathbb{R})$, this is a cotangent case. We must determine whether $\mu = +\frac{1}{2}\alpha$ or $\mu = -\frac{1}{2}\alpha$. Using Table 2.1, we write out $\lambda_{0,b}$ for each choice of μ:

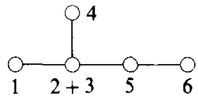
$$\lambda_{0,b}^{+} = \begin{pmatrix} & & 1 & & & & \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\lambda_{0,b}^{-} = \begin{pmatrix} & & 0 & & & & \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

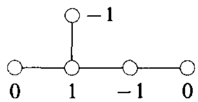
The infinitesimal character λ_0 must dominate one of these term-by-term; if it dominates both, then there are two minimal K-types and there is no unitarity. We see that λ_0 dominates $\lambda_{0,b}^{+}$ and not $\lambda_{0,b}^{-}$; therefore $\mu = +\frac{1}{2}\alpha$.

Comparing λ_0 and $\lambda_{0,b}^{+}$, we see that there is agreement on the E_7 subdiagram (consisting of all simple roots but the one marked "8"). Therefore the basic case is E_7, and we can discard root 8. Referring to Lemma 2.2, we see that the roots of the E_6 subdiagram (all the remaining ones but 7) are needed for $\Delta_{L,\perp}^{+}$, but root 7 is not. Thus the special basic case is E_6. Since it is not $\mathfrak{so}$(even, 2), the cut-off for unitarity is $\min(\nu_0^+, \nu_0^-)$.

Next we form the Dynkin diagram of $\Delta^+_{K,\perp}$. Lemma 2.2 shows that only the obvious simple roots of Δ^+_K within the basic case are eligible to be in $\Delta^+_{K,\perp}$. So $\Delta^+_{K,\perp}$ can be written down by inspection as



We attach to each simple root γ of $\Delta^+_{K,\perp}$ the number $2\langle\gamma, \alpha\rangle/|\alpha|^2$, obtaining the diagram



Positive roots in this system with positive total contribute to ν_0^-, while those with negative total contribute to ν_0^+. A little thought shows that only two have positive total, while at least three have negative total. Therefore $\nu_0^- = 1 - \mu_\alpha + 2(2) \leqslant 2(3) \leqslant \nu_0^+$. Therefore the cut-off for unitarity is at $\nu_0^- = 4$, i.e., at $\nu = 2\alpha$.

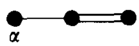
2. *A Conjecture of Knapp and Speh*

The above kind of computation is more complicated in a double-line diagram with α long because of the presence of the exceptional term that can contribute to ν_0^+ or ν_0^-. This exceptional term also provides a counterexample to a conjecture of Knapp and Speh [13] that the unitary points ν in the basic case are the same as the unitary points in G.

For a specific example, we take configuration (z) in Table 8.1 with $\mu = -\frac{1}{2}\alpha$ and $n = 2$. The total diagram is B_4 and has $\nu_0^+ = 4$ and $\nu_0^- = 3$. The basic case Δ_L is A_3 and has $\nu_0^+ = 4$ and $\nu_0^- = 2$. In both instances Theorem 1.1 says that the unitary points form an interval ending at $\min(\nu_0^+, \nu_0^-)$. Thus the unitary points in the basic case L are a proper subset of those in G.

3. *A Conjecture of Vogan*

Vogan [23, p. 408] conjectured that the parameter mapping from L to G described in Section 8 would carry unitary representations to unitary representations under certain circumstances. One set of circumstances is that (SV) holds as in Section 8, and Vogan proved this conjecture in [24]. Another set of circumstances is that $\mathfrak{l}^{\mathbb{C}} \oplus \mathfrak{u}$ contains the "classification parabolic" used in [23]; this form of the conjecture is still not settled. It seems to be only a slight change to insist only that (SV) hold at $\nu = 0$, which is what our conditions in Section 8 force. But with this change, situation (iii) in Theorem 1.1 provides a counterexample to the preservation of unitarity. In fact, let us take the basic case with $\mu = +\frac{1}{2}\alpha$ for the B_3 diagram



The group in question is $SO(4, 3)$ locally, and $\lambda_0 = 0$. Therem 1.1 says that the unitary points extend from 0 to $\frac{1}{2}\alpha$, with α as an isolated unitary point. Now we can take Δ_L to be generated by the two long simple roots, so that L is $SU(2, 1)$ locally (on the semisimple part). The corresponding parameters for L are those of the spherical principal series, where we have unitarity from 0 to α, with no break. Thus the open interval of ν from $\frac{1}{2}\alpha$ to α gives unitary Langlands quotients in L but not in G.

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