INTERTWINING OPERATORS INTO $L^2(G/H)$

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Let G/H be a semisimple symmetric space. Here we take G to be a linear connected semisimple group and H to be the identity component of the subgroup fixed by an involution τ. Such spaces were classified by Berger [2]. Examples include ordinary symmetric spaces, the groups themselves (the *group case*, from $(G \times G)$/diagonal), any simply connected complex group modulo a real form, and various spaces obtained from splittings of quadratic forms.

The space G/H admits an invariant measure, and the left regular representation on $L^2(G/H)$ sometimes has irreducible direct summands. Such an irreducible representation of G is said to be in the *discrete series* of $L^2(G/H)$. Under the equal-rank condition (1.1) below, Flensted-Jensen [3] gave a construction of many such discrete series. Oshima and Matsuki [13] obtained all remaining discrete series and also showed that Flensted-Jensen's equal-rank condition is necessary for the discrete series of $L^2(G/H)$ to be nonempty. Schlichtkrull [14] found the Langlands parameters [9, Theorem 14.92] of generic such representations ("generic" to be defined in (1.2)); that is, he found how such representations fit into the classification of all irreducible admissible representations of G.

Vogan [20] gave an argument suggesting that all discrete series occur with multiplicity one in $L^2(G/H)$. In this case there should be an essentially unique G-commuting operator carrying a particular copy of a discrete series representation of $L^2(G/H)$ into its realization in $L^2(G/H)$.

The particular copy that we have in mind is the Langlands realization [9, Theorem 14.92] used for classification of irreducible representations. Our objective

*Supported by National Science Foundation Grant DMS 87-23046.

therefore is to find an intertwining operator carrying such a discrete series representation in its Langlands realization into a space of functions on G/H. We restrict attention to the generic case, where the Langlands parameters are known, and Theorem 3.1 gives such an operator $\mathcal{E}$. Under an additional hypothesis (that a subgroup L defined in §1 has the same real rank as G), Theorem 3.2 shows that $\mathcal{E}$ has image in $L^2(G/H)$, and Theorem 3.3 shows that $\mathcal{E}$ is nonzero if some choices are made suitably. The detailed proofs of these theorems are carried out in §§4-7.

The opposite extreme to our additional hypothesis is that the subgroup L is compact (in which case the representation is in the discrete series of G). In this case we have a heuristic argument that $\mathcal{E}$ is nonzero in general. In two special cases with L compact (the group case and $SO(2,1)/SO(1,1)$), we give in §8 a rigorous argument that the image is in $L^2(G/H)$ and $\mathcal{E}$ is nonzero.

Our interest in the subject of this paper came from trying to understand from an analytic point of view the unitarity of some representations shown earlier by Vogan [19] to be unitary by algebraic means. We realized that the formula we sought could be derived for one particular example (the one in §2), and we were able to guess the rest. We are happy to acknowledge crucial assistance of D. A. Vogan, as well as suggestions of T. Oshima and W. Schmid, that helped us carry out this work.

Contents.

1. Parameters for generic discrete series

Our semisimple symmetric space will be G/H. Lie algebras of G, H, etc. will be denoted $\mathfrak{g}_0, \mathfrak{h}_0$, etc., and their complexifications will be denoted $\mathfrak{g}, \mathfrak{h}$, etc. Let $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{q}_0$ be the decomposition of $\mathfrak{g}_0$ relative to our given involution τ, and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a compatible Cartan decomposition of $\mathfrak{g}_0$. We write K for the maximal compact subgroup of G corresponding to $\mathfrak{k}_0$.

Let $\mathfrak{t}_0$ be a maximal abelian subspace of $\mathfrak{k}_0 \cap \mathfrak{q}_0$, and let $T = \exp \mathfrak{t}_0$. The space $\mathfrak{t}_0$ need not be maximal abelian in $\mathfrak{k}_0$, but any two choices of $\mathfrak{t}_0$ are conjugate via $H \cap K$, according to [9, Theorem 5.13] applied to the noncompact dual of $K/(H \cap K)$. Let Δ and Δ_c be the sets of roots of $\mathfrak{g}$ and $\mathfrak{k}$ with respect to $\mathfrak{t}$, let Δ^+ and Δ_c^+ be compatible positive systems, and let δ and δ_c be the corresponding half sums of positive roots with multiplicities counted.

In the notation of Schlichtkrull [14], Flensted-Jensen [3] considers parameters λ in the dual $\mathfrak{t}^*$, along with the corresponding parameters

$$\mu_\lambda = \lambda + \delta - 2\delta_c \quad \text{in } \mathfrak{t}^*,$$

such that

(i) $\langle \mu_\lambda, \alpha\rangle/|\alpha|^2$ is an integer ≥ 0 for all $\alpha \in \Delta_c^+$
(ii) $\mu_\lambda(\exp^{-1}\{1\} \cap \mathfrak{t}) \subseteq 2i\mathbb{Z}$
(iii) $\langle \lambda + \delta, \alpha\rangle \geq 0$ for all $\alpha \in \Delta^+$.

Here $\langle \cdot, \cdot \rangle$ and $|\cdot|$ are the inner product and norm induced by the Killing form for $\mathfrak{g}$.

Let $(\mathfrak{t}_1)_0$ be a Cartan subalgebra of $\mathfrak{k}_0$ containing $\mathfrak{t}_0$, and extend μ_λ to $\mathfrak{t}_1$ by making it be 0 on the orthogonal complement of $\mathfrak{t}$. Write μ_λ also for this extension. In terms of a compatible positive system of roots of $\mathfrak{k}$ with respect to $\mathfrak{t}_1$, conditions (i) and (ii) are equivalent with the requirement that μ_λ be the highest weight of a representation π_{μ_λ} of K with a $(K \cap H)$-fixed vector. (See [9, Theorem 9.14].)

For each such λ, Flensted-Jensen constructs a specific nonzero function ψ_λ in $C^\infty(G/H)$ of K type μ_λ. With two additional assumptions, he proves that ψ_λ is square integrable on G/H and that it generates an irreducible direct summand U^λ of $L^2(G/H)$. The assumptions are that an equal-rank condition holds, namely

$$\mathfrak{t} \text{ is maximal abelian in } \mathfrak{q}, \tag{1.1}$$

and that λ is *generic* in the sense that

$$|\langle \lambda, \alpha\rangle| \geq C \qquad \text{for all } \alpha \in \Delta, \tag{1.2}$$

where C is a constant depending only on G/H.

Schlichtkrull [14] gives the Langlands parameters of U^λ for generic λ. Let L be the centralizer of T in G, let $M_L A N_L$ be the Langlands decomposition of a minimal parabolic subgroup of L, let ρ_L be half the sum of the positive roots of $\mathfrak{l}$ with respect to $\mathfrak{a}$, with positivity determined by N_L, and let MAN be a cuspidal parabolic subgroup of G having the same split component A and having $N_L \subseteq N$. By choosing N suitably, we may assume that ρ_L is G-dominant (i.e., dominant relative to N). Then Vogan's Proposition 4.1 in [18] associates to the K type μ_λ a discrete series representation σ of M, and U^λ is infinitesimally equivalent with the unique irreducible quotient of the induced representation (with normalized induction)

$$U(MAN, \sigma, \rho_L) = \operatorname{ind}_{MAN}^G(\sigma \otimes \exp \rho_L \otimes 1) \tag{1.3}$$

that contains the K type μ_λ. This irreducible quotient will be denoted $J(MAN, \sigma, \rho_L, \mu_\lambda)$ and is called the *Langlands quotient* containing the minimal K type μ_λ.

Let ρ_G be half the sum of the positive roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. We shall use the fact that

$$\mathfrak{a} \subseteq \mathfrak{h}, \tag{1.4}$$

which allows us to define ρ_H as half the sum of the positive roots of $\mathfrak{h}$ with respect to $\mathfrak{a}$. To prove (1.4), we shall use the equal rank condition (1.1). First note that

$$\mathfrak{a} \oplus \mathfrak{t} \quad \text{is abelian.} \tag{1.5}$$

since $\mathfrak{a} \subseteq \mathfrak{l}$. Now let $X \in \mathfrak{a}$ and $Y \in \mathfrak{t}$. By (1.5), $[X, Y] = 0$. Since $\tau(Y) = -Y$, $[\tau(X), Y] = 0$. Thus $X - \tau(X)$ commutes with $\mathfrak{t}$ and, by (1.1), must be in $\mathfrak{t}$. But also $X - \tau(X)$ is in $\mathfrak{p}$, and hence $\tau(X) = X$. Then X is in $\mathfrak{h}$, and (1.4) follows.

2. Motivating example

Let $G = SO(4,1)$, so that $\mathfrak{g}$ consists of 5-by-5 complex matrices that are skew in the first four rows and columns, are symmetric in the last row and column, and are 0 on the diagonal. Let the automorphism τ be conjugation by $diag(-1,1,1,1,1)$, so that H equals $SO(3,1)$ imbedded in G as $SO(1) \times SO(3,1)$. Then the nonzero entries of members of $\mathfrak{h}$ are in the last four rows and columns.

With E_{ij} denoting the matrix that is 1 in the $(i,j)^{\text{th}}$ entry and 0 elsewhere, let

$$\mathfrak{t}_0 = \mathbb{R}(E_{12} - E_{21}).$$

Then

$$\mathfrak{l}_0 = \mathfrak{t}_0 \oplus \mathfrak{so}(2,1)$$

with $\mathfrak{so}(2,1)$ imbedded in $\mathfrak{g}_0$ in the last three rows and columns. If we take

$$\mathfrak{a}_0 = \mathbb{R}(E_{45} + E_{54}),$$

then M is the upper left copy of $SO(3)$ in G.

For G, there is one positive restricted root, say α, and it has multiplicity 3. The half sums of positive restricted roots are then

$$\rho_L = \frac{1}{2}\alpha, \qquad \rho_H = \alpha, \qquad \rho_G = \frac{3}{2}\alpha.$$

According to Schlichtkrull, $\mu_\lambda = 0$ is an allowable parameter leading to a discrete series for $L^2(G/H)$. The trivial K type μ_λ leads by the Vogan construction [18] to the trivial M type $\sigma = 1$, and thus the irreducible quotient of

$$\operatorname{ind}_{MAN}^{G}(1 \otimes e^{\frac{1}{2}\alpha} \otimes 1) \tag{2.1}$$

containing the trivial K type occurs in $L^2(G/H)$. (As it happens, (2.1) is irreducible, but this fact is not important for us.)

Functions f in the space for (2.1) transform according to the law

$$f(xman) = e^{-(\frac{1}{2}\alpha + \frac{3}{2}\alpha)\log a} f(x) \qquad \text{for } man \in MAN.$$

Here the term $\frac{3}{2}\alpha = \rho_G$ in the exponent is required for normalized induction. For $x \in G$ and $h \in H$, put $f_x(h) = f(xh)$. Then

$$f_x(hman) = e^{-(\alpha+\alpha)\log a} f_x(h) \qquad \text{for } man \in H \cap MAN.$$

Since the second α in the exponent is just ρ_H, f_x is in the space for

$$\operatorname{ind}_{H \cap MAN}^{H}(1 \otimes e^{\alpha} \otimes 1).$$

This is exactly the induced representation of H for which the trivial representation of H occurs as Langlands quotient. According to the Langlands theory, the quotient is picked out by a certain intertwining operator. Briefly

$$Ef_x(h) = \int_{H\cap \bar{N}} f_x(h\bar{n})\, d\bar{n}, \qquad \text{with } x \in G,\ h \in H,$$

is constant in h. (Here $\bar{N}$ is the nilpotent group opposite to N.) Thus the operator

$$\mathcal{E}f(x) = \int_{H\cap \bar{N}} f(x\bar{n})\, d\bar{n}, \qquad \text{with } x \in G, \tag{2.2}$$

carries f to a function right invariant under H.

The key property in the above analysis was the identity

$$\frac{1}{2}\alpha + \frac{3}{2}\alpha = \alpha + \alpha$$

in the form

$$\rho_L + \rho_G = 2\rho_H. \tag{2.3}$$

This identity turns out to be completely general.

The one way in which formula (2.2) is not completely general is that the function f in the general case has values in the space V^σ on which σ operates, while the image functions on G/H are to be scalar-valued. The passage from V^σ to scalars will be accomplished by a linear functional denoted l below.

3. Outline of argument

The development of our intertwining operator takes the following five steps. Details of these steps will be provided in subsequent sections. Notation is as in §1.

1. *Formula for operator.* Let l be a continuous linear functional on the space of analytic vectors of V^σ that intertwines $\sigma|_{H\cap M}$ with the trivial representation:

$$l\,\sigma(m) = l \qquad \text{for } m \in H \cap M. \tag{3.1}$$

Construction of such an l will be addressed in Step 3 below, under the assumption that λ is generic. For K-finite functions f in the space of the induced representation (1.3), define

$$\mathcal{E}f(x) = \int_{H\cap K} l\,(f(xk))\, dk \qquad \text{for } x \in G. \tag{3.2}$$

Since the image of K under f consists of $(K \cap M)$-finite vectors in V^σ, which are necessarily analytic, it follows readily that $\mathcal{E}f(x)$ is a well defined function on G. See Proposition 4.1.

2. *Intertwining property.*

a. The formula

$$\rho_L + \rho_G = 2\rho_H \tag{3.3}$$

is valid in complete generality. See Proposition 4.2.

b. On any K-finite f, the operator $\mathcal{E}$ of (3.2) is given also by

$$\mathcal{E}f(x) = \int_{H\cap\bar{N}} l\,(f(x\bar{n}))\,d\bar{n}, \tag{3.4}$$

as a consequence of (3.3) and the definition of l. See Corollary 4.3. Thus the formula (3.2) for our intertwining operator is consistent with the formula (2.2) in the special case $SO(4,1)/SO(3,1)$.

c. For K-finite f, $\mathcal{E}f$ is right H-invariant. See Proposition 4.4. Thus $\mathcal{E}$ gives a well defined $(\mathfrak{g}, K)$ map from the Harish-Chandra module of $\operatorname{ind}_{MAN}^G(\sigma \otimes e^{\rho_L} \otimes 1)$ into $C^\infty(G/H)$.

3. *Construction of l when λ is generic.* If λ is generic, then σ is an integrable discrete series representation of M. This result, given as Proposition 5.1, uses the necessary and sufficient condition for integrability of σ due to Trombi-Varadarajan [17] and Hecht-Schmid [5].

Now fix a $(K\cap M)$-finite vector v_0 in V^σ, and let

$$\mathcal{D}(\sigma) = \{\sigma(f)v_0 \mid f \in C^\infty_{\text{com}}(M)\}. \tag{3.5}$$

By a result of Schmid [15, Corollary 1], $\mathcal{D}(\sigma)$ coincides with the space of analytic vectors for V^σ.

Under the assumption that σ is integrable (valid, as we have seen, if λ is generic), fix u in $\mathcal{D}(\sigma)$. Then Proposition 5.5 shows that the integral in the definition

$$l\,(v) = l_u(v) = \int_{H\cap M} \langle \sigma(m)v,\, u\rangle\, dm \qquad \text{for } v \in \mathcal{D}(\sigma) \tag{3.6}$$

is convergent, and it follows that

$$l\,(\sigma(m)v) = l(v) \qquad \text{for } v \in \mathcal{D}(\sigma),\ m \in H\cap M.$$

Thus l satisfies (3.1). We can summarize the above steps as follows.

Theorem 3.1. Let λ be the Flensted-Jensen parameter of a generic discrete series representation U^λ of $L^2(G/H)$, and let $(MAN, \sigma, \rho_L, \mu_\lambda)$ be its Langlands parameters. Also let l be a linear functional as in (3.1), such as the one in (3.6). If $\mathcal{E}f$ is given by (3.2) for K-finite functions f in $U(MAN, \sigma, \rho_L)$, then $\mathcal{E}$ defines a $(\mathfrak{g}, K)$ intertwining operator from $U(MAN, \sigma, \rho_L)$ into $C^\infty(G/H)$.

4. *Image in* $L^2(G/H)$. Since the K type μ_λ occurs with multiplicity one in $U(MAN, \sigma, \rho_L)$, there exists a nonzero operator P in $\mathrm{Hom}_{K\cap M}(\mu_\lambda, \sigma)$ unique up to a scalar. For v in the space for π_{μ_λ}, let f_v be the K-finite function on K given by

$$f_v(k) = P(\pi_{\mu_\lambda}(k)^{-1}v). \tag{3.7}$$

Then f_v extends to a member of the space for $U(MAN, \sigma, \rho_L)$ with K type μ_λ. Adapting an argument of Flensted-Jensen [3], we show in Lemma 6.1 that $\mathcal{E}f_v(x)$ is bounded for x in G, under the assumption that L and G have the same real rank. Whenever $\mathcal{E}f_v(x)$ is bounded for x in G, Proposition 6.2 derives as a consequence that $\mathcal{E}f_v$ is in $L^2(G/H)$ provided λ is generic. Thus we obtain the following supplement to Theorem 3.1.

Theorem 3.2. With notation as in Theorem 3.1, suppose that $\mathcal{E}f_v$ is bounded, as is the case when L and G have the same real rank. If $\mathcal{E}f$ is given by (3.2) just for K-finite functions f in the cyclic span of the μ_λ K type in $U(MAN, \sigma, \rho_L)$, then $\mathcal{E}$ defines a $(\mathfrak{g}, K)$ intertwining operator from $J(MAN, \sigma, \rho_L, \mu_\lambda)$ into an irreducible subspace of $L^2(G/H)$ of type U^λ.

The one conclusion in Theorem 3.2 that needs further explanation is that $\mathcal{E}$ actually descends to $J(MAN, \sigma, \rho_L, \mu_\lambda)$. In fact, let U' be the cyclic span of the μ_λ K type in $U(MAN, \sigma, \rho_L)$. Since $\mathcal{E}(U')$ is unitary and has a finite composition series, it is fully reducible. Assume it is not 0. The cyclic K type μ_λ has multiplicity one in U' (being of multiplicity one in all of $U(MAN, \sigma, \rho_L)$). Since it has to be cyclic for $\mathcal{E}(U')$, it has multiplicity one in $\mathcal{E}(U')$. Therefore it has to occur in some irreducible constituent of $\mathcal{E}(U')$, and $\mathcal{E}(U')$ must be irreducible. In the case that $\mathcal{E}(U')$ is nonzero, $\mathcal{E}(U')$ is thus identified as an irreducible quotient of $U(MAN, \sigma, \rho_L)$ containing the K type μ_λ. Hence $\mathcal{E}(U')$ is infinitesimally equivalent with $J(MAN, \sigma, \rho_L, \mu_\lambda)$, and $\mathcal{E}$ must descend in the required fashion.

5. *Operator nonzero.* We are at present able to show that $\mathcal{E}$ is nonzero only in certain circumstances, as mentioned in the introduction. For now we shall give a qualitative version of the main result that we have in this direction. The proof will be given in §7.

Theorem 3.3. With notation as in Theorem 3.1, suppose that L and G have the same real rank. Then with suitable choices of the subgroup A and the element u in (3.6), the function $\mathcal{E}f_v$ is not identically 0. In particular $\mathcal{E}$ is not the 0 operator on $J(MAN, \sigma, \rho_L, \mu_\lambda)$.

4. Intertwining operator

Sections 4-7 carry out the detailed steps announced in §3. The notation for these sections is as in §1. In the present section, we shall prove the results in Subsections 1 and 2 of §3. Let the $G = K(M \cap \exp \mathfrak{p})AN$ decomposition of an element g of G be $g = \kappa(g)\mu(g)e^{H(g)}n$.

Proposition 4.1. If f is a K-finite function in the space of the induced representation (1.3), if l is a continuous linear functional on the space of analytic vectors of V^σ, and if x is in G, then the integral

$$\int_{H\cap K} l(f(xk))\,dk \tag{4.1}$$

is well defined and is a continuous function of x.

Proof. We have

$$f(g) = e^{-(\rho_L+\rho_G)H(g)}\sigma(\mu(g))^{-1}f(\kappa(g)). \tag{4.2}$$

The K-finiteness of f implies that each member of $f(K)$ is an analytic vector. Since the space of analytic vectors is stable under M, (4.2) is an analytic vector. Thus the integrand of (4.1) is well defined. As g varies in G, we see from (4.2) that $f(g)$ varies continuously in the space of analytic vectors. Since l is continuous, the integrand of (4.1) is continuous in the pair (x, k). Therefore (4.1) is well defined and is continuous in x.

Proposition 4.2. $\rho_L + \rho_G = 2\rho_H$.

Proof. By (1.5), $\mathfrak{a} \oplus \mathfrak{t}$ is abelian, and (1.4) shows that it is stable under both τ and the Cartan involution. By (1.1) we can therefore find a subspace $\mathfrak{s} \subseteq \mathfrak{h}$ such that $\mathfrak{a} \oplus \mathfrak{t} \oplus \mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{g}$. We introduce a lexicographic ordering that takes $\mathfrak{a}$ first and that is compatible with our choice of positive restricted roots in §1.

For any root α with $\alpha|_\mathfrak{a} > 0$, let X_α be a corresponding root vector. Then we have

$$\begin{aligned}\mathfrak{n} &= \sum_{\substack{\alpha|\mathfrak{a}>0\\ \alpha|\mathfrak{t}=0}} \mathbb{C}X_\alpha \oplus \sum_{\substack{\alpha|\mathfrak{a}>0\\ \alpha|\mathfrak{t}>0}} \mathbb{C}(X_\alpha + \tau X_\alpha) \oplus \sum_{\substack{\alpha|\mathfrak{a}>0\\ \alpha|\mathfrak{t}>0}} \mathbb{C}(X_\alpha - \tau X_\alpha)\\ &= (\mathfrak{n}\cap\mathfrak{l}) \oplus (\mathfrak{n}\cap(\mathfrak{h}\ominus\mathfrak{l})) \oplus (\mathfrak{n}\cap\mathfrak{q}).\end{aligned} \tag{4.3}$$

If $2\rho(\cdot)$ denotes the sum of the positive restricted roots contributing to $(\cdot)$ with multiplicities counted, then (4.3) shows that

$$2\rho_G = 2\rho(\mathfrak{n}\cap\mathfrak{l}) + 2\rho(\mathfrak{n}\cap(\mathfrak{h}\ominus\mathfrak{l})) + 2\rho(\mathfrak{n}\cap\mathfrak{q}) \tag{4.4}$$

with

$$2\rho(\mathfrak{n}\cap\mathfrak{l}) = 2\rho_L \tag{4.5a}$$

$$2\rho(\mathfrak{n}\cap(\mathfrak{h}\ominus\mathfrak{l})) = 2\rho(\mathfrak{n}\cap\mathfrak{q}) \tag{4.5b}$$

$$2\rho(\mathfrak{n}\cap\mathfrak{l}) + 2\rho(\mathfrak{n}\cap(\mathfrak{h}\ominus\mathfrak{l})) = 2\rho_H. \tag{4.5c}$$

Substituting (4.5) into (4.4), we obtain

$$2\rho_G = 2\rho_H + (2\rho_H - 2\rho_L),$$

and the proposition follows.

Corollary 4.3. The operator $\mathcal{E}$ on K-finite functions f in the space of the induced representation (1.3), given by

$$\mathcal{E}f(x) = \int_{H\cap K} l(f(xk))\,dk, \tag{4.6}$$

is given also by

$$\mathcal{E}f(x) = \int_{H\cap\bar{N}} l(f(x\bar{n}))\,d\bar{n}. \tag{4.7}$$

Proof. The expression (4.6) is well defined, according to Proposition 4.1. If m is in $H\cap K\cap M$, then

$$l(f(xkm)) = l(\sigma(m)^{-1}f(xk)) = l(f(xk)).$$

Applying the change of variables formula [9, (5.25)] to the group H, we see that

$$\begin{aligned}\int_{H\cap K} l(f(xk))\,dk &= \int_{H\cap\bar{N}} l(f(x\kappa(\bar{n})))e^{-2\rho_H H(\bar{n})}d\bar{n}\\ &= \int_{H\cap\bar{N}} l(f(x\kappa(\bar{n})\mu(\bar{n})))e^{-2\rho_H H(\bar{n})}d\bar{n} \quad \text{by (3.1)}\\ &= \int_{H\cap\bar{N}} l(f(x\kappa(\bar{n})\mu(\bar{n})e^{H(\bar{n})}))\,d\bar{n} \quad \text{by Proposition 4.2}\\ &= \int_{H\cap\bar{N}} l(f(x\bar{n}))\,d\bar{n}.\end{aligned}$$

Proposition 4.3. For any K-finite function f in the space of the induced representation (1.3),

$$\mathcal{E}f(xh) = \mathcal{E}f(x) \qquad \text{for all } x\in G,\ h\in H. \tag{4.8}$$

Proof. Since $H = (H\cap K)(H\cap M)A(H\cap\bar{N})$, it is enough to handle h in each factor separately. For $h\in H\cap K$, (4.8) follows from (4.6). For $h\in H\cap\bar{N}$, (4.8) follows from (4.7).

Now let $h \in H \cap M$. Then (4.7) gives

$$\begin{aligned}
\mathcal{E}f(xh) &= \int_{H\cap\bar{N}} l(f(x(h\bar{n}h^{-1})h))\,d\bar{n} \\
&= \int_{H\cap\bar{N}} l(f(x(h\bar{n}h^{-1})))\,d\bar{n} \qquad \text{by (3.1)} \\
&= \int_{H\cap\bar{N}} l(f(x\bar{n}))\,d\bar{n} \qquad \text{by change of variables} \\
&= \mathcal{E}f(x).
\end{aligned}$$

The argument for $h \in A$ is similar, and (4.8) follows.

5. Construction of linear functional l

In this section we shall prove the results in Subsection 3 of §3. An observation by W. Schmid helped simplify the proofs.

Proposition 5.1. If the Flensted-Jensen parameter λ is generic, then the Langlands M parameter σ is an integrable discrete series of M.

Proof. We shall make use of some qualitative features of the Vogan construction [18] for passing from μ_λ to M and σ. First we extend $\mathfrak{t}_0$ to a Cartan subalgebra $(\mathfrak{t}_1)_0$ of $\mathfrak{k}_0$ as in §1, and then we extend $(\mathfrak{t}_1)_0$ to a Cartan subalgebra $(\mathfrak{t}_2)_0$ of $\mathfrak{g}_0$. Let

$$\Delta_1 = \Delta(\mathfrak{g}, \mathfrak{t}_1), \quad \Delta_2 = \Delta(\mathfrak{g}, \mathfrak{t}_2), \quad \Delta_{1,c} = \Delta(\mathfrak{k}, \mathfrak{t}_1).$$

We can extend Δ_c^+ to a compatible positive system $\Delta_{1,c}^+$ for $\Delta_{1,c}$, and we can use the result to define a compatible positive system Δ_2^+ for Δ_2 as in [18, p. 19] such that $\mu_\lambda + 2\delta_{1,c}$ is Δ_2^+ dominant and such that the positivity for members of Δ_2 consistently defines Δ_1^+ by restriction. Since λ is generic, Δ_1^+ will be consistent with Δ^+ under restriction.

Proposition 4.1 of [18] instructs us to work with $\mu_\lambda + 2\delta_{1,c} - \delta_2$. Since λ is generic, this and all of its bounded translates have large positive inner products with any member of Δ_2^+ that is nonzero on $\mathfrak{t}$. Consequently the imaginary roots $\beta_1, \ldots, \beta_r$ constructed in [18, Proposition 4.1] all vanish on $\mathfrak{t}$. As a result, the $\mathfrak{a}$ that is constructed is always in $\mathfrak{l}$, no matter how the β_i are chosen. And the Harish-Chandra parameter of σ, which we shall call λ_0, differs from $\mu_\lambda + 2\delta_{1,c} - \delta_2$ by a linear functional vanishing on $\mathfrak{t}$. Thus $\langle \lambda_0, \alpha \rangle > C$ for every $\alpha \in \Delta_2^+$ that is nonzero on $\mathfrak{t}$.

Let Δ_-^+ be the set of positive roots for M (i.e., the members of Δ_2^+ orthogonal to $\beta_1, \ldots, \beta_r$ and carried on $\mathfrak{t}_1$). The condition for integrability of σ is that

$$\langle \lambda_0, \beta \rangle > \frac{1}{2} \sum_{\substack{\gamma \in \Delta_-^+ \\ \langle \gamma, \beta \rangle > 0}} \langle \gamma, \beta \rangle$$

for all $\beta \in \Delta_-^+$ that are M-noncompact, according to [17] and [5]. For this condition to fail for some β, β would have to vanish on $\mathfrak{t}$, according to the previous paragraph. But then the root vector E_β would centralize $\mathfrak{t}$ and so be in $\mathfrak{l}$. But Schlichtkrull [14] says that A corresponds to a minimal parabolic subgroup of L, and thus L has no M-noncompact roots. The result follows.

We turn to the integrability of matrix coefficients of σ over $H \cap M$. The first few steps are general facts about integrable discrete series of a linear reductive group M whose identity component has compact center; these steps are carried out in the lemmas below.

Let A_M be an Iwasawa A of M, and let φ_0^M be the 0^{th} spherical function, as in the notation of [9, Chapter VII]. As in §3, let $\mathcal{D}(\sigma)$ be the space of analytic vectors of σ, given by (3.5) according to [15].

Lemma 5.2. For f and g in $C_{\text{com}}^\infty(M)$, let $E_1 = \text{support}(f)$ and $E_2 = \text{support}(g)$, and put

$$C = \big(\sup |f|\big)\big(\sup |g^*|\big),$$

where $g^*(x) = \overline{g(x^{-1})}$. For $u_0 \in \mathcal{D}(\sigma)$,

$$|\langle \sigma(x)\sigma(f)u_0,\, \sigma(g)u_0 \rangle| \le C|E_1||E_2| \sup_{\substack{y \in E_1 \\ z \in (E_2)^{-1}}} |\langle \sigma(zxy)u_0,\, u_0 \rangle|,$$

where $|E_1|$ and $|E_2|$ denote the measures of E_1 and E_2.

Proof.

$$\begin{aligned} \langle \sigma(x)\sigma(f)u_0,\, \sigma(g)u_0 \rangle &= \langle \sigma(g^*)\sigma(x)\sigma(f)u_0,\, u_0 \rangle \\ &= \int_M \int_M \langle \sigma(z)\sigma(x)\sigma(y)u_0,\, u_0 \rangle g^*(z) f(y)\, dz\, dy, \end{aligned}$$

and the result follows directly.

Lemma 5.3. If u_0 is $(K \cap M)$-finite and if $\epsilon > 0$ is sufficiently small, then there exists C' such that

$$|\langle \sigma(x)u_0, u_0\rangle| \leq C' \varphi_0^M(x)^{2+\epsilon} \tag{5.1}$$

for all $x \in M$.

Proof. We refer to the analysis of asymptotics of matrix coefficients of σ, as in [9, Chapter VIII]. Let A_M^+ be an exponentiated positive Weyl chamber for A, let ρ_M be half the sum of the corresponding positive restricted roots, and let $\{\omega_j\}$ be the dual basis to the basis of simple restricted roots. By [9, Theorem 8.48], integrability of σ implies that every leading exponent $\nu - \rho_M$ of σ has

$$\langle \nu - \rho_M, \omega_j\rangle < -2\langle \rho_M, \omega_j\rangle,$$

i.e.,

$$\langle \nu, \omega_j\rangle < -\langle \rho_M, \omega_j\rangle.$$

Thus

$$\langle \nu, \omega_j\rangle \leq -\langle (1+2\epsilon)\rho_M, \omega_j\rangle \tag{5.2}$$

if $\epsilon > 0$ is sufficiently small. By [9, Theorem 8.47], (5.2) implies that there is a $q \geq 0$ such that each $(K \cap M)$-finite matrix coeffcient of σ is dominated on A_M^+ by a multiple of $e^{-(2+2\epsilon)\rho_M \log a}(1 + \|a\|)^q$, hence by a multiple of $e^{-(2+\epsilon)\rho_M \log a}$. Since $ce^{-\rho_M \log a} \leq \varphi_0^M(a)$, each $(K \cap M)$-finite matrix coefficient is dominated on A_M^+ by a multiple of $\varphi_0^M(a)^{2+\epsilon}$. Then (5.1) follows readily.

Lemma 5.4. Let ϵ be as in Lemma 5.3. If u and v are in $\mathcal{D}(\sigma)$, then there exists C'' (depending on u and v) such that

$$|\langle \sigma(x)v, u\rangle| \leq C'' \varphi_0^M(x)^{2+\epsilon}$$

for all $x \in M$.

Proof. Write $v = \sigma(f)u_0$ and $u = \sigma(g)u_0$. Then Lemmas 5.2 and 5.3 give

$$\begin{aligned} |\langle \sigma(x)v, u\rangle| &\leq C|E_1||E_2| \sup_{\substack{y \in E_1 \\ z \in (E_2)^{-1}}} |\langle \sigma(zxy)u_0, u_0\rangle| \\ &\leq C|E_1||E_2| \sup_{\substack{y \in E_1 \\ z \in (E_2)^{-1}}} \varphi_0^M(zxy)^{2+\epsilon}. \end{aligned}$$

Now [9, p. 429] shows that

$$\varphi_0^M(zxy) \leq \Big(\sup_{k \in K \cap M} e^{-\rho_M H(y^{-1}k)}\Big)\Big(\sup_{k \in K \cap M} e^{-\rho_M H(zk)}\Big)\varphi_0^M(x),$$

and hence the lemma follows.

Proposition 5.5. If σ is an integrable discrete series on M and if u and v are in $\mathcal{D}(\sigma)$, then the integral

$$\int_{H\cap M} \langle \sigma(m)v, u\rangle \, dm$$

is convergent.

Proof. Let Iwasawa A groups for M and $H \cap M$ be denoted A_M and $A_{H\cap M}$, respectively. We may suppose that $A_{H\cap M} \subseteq A_M$. We denote typical open Weyl chambers in $(\mathfrak{a}_{H\cap M})_0$ and $(\mathfrak{a}_M)_0$ by $\mathcal{C}_{H\cap M}$ and $\mathcal{C}_M$.

Let $\mathcal{C}^+_{H\cap M}$ be a fixed positive Weyl chamber for $(\mathfrak{a}_{H\cap M})_0$, and consider all $\mathcal{C}_M$ with the property that $\overline{\mathcal{C}_M} \cap \mathcal{C}^+_{H\cap M}$ has nonempty interior in $\mathcal{C}^+_{H\cap M}$. There are finitely many such, and they cover $\mathcal{C}^+_{H\cap M}$. Since φ_0^M is bi-$(K\cap M)$-invariant, we are to show that

$$\int_{X\in\mathcal{C}^+_{H\cap M}} \varphi_0^M(\exp X)^{2+\epsilon} e^{2\rho_{H\cap M}(X)}\, dX < \infty,$$

in view of Lemma 5.4. It is enough to show that

$$\int_{X\in\overline{\mathcal{C}_M}\cap\mathcal{C}^+_{H\cap M}} \varphi_0^M(\exp X)^{2+\epsilon} e^{2\rho_{H\cap M}(X)}\, dX < \infty$$

for each $\mathcal{C}_M$ as above. Define ρ_M relative to $\overline{\mathcal{C}_M}$. Proposition 7.15c of [9] gives

$$\varphi_0^M(\exp X) \le C_0 e^{-\rho_M(X)}(1+\|X\|)^r$$

for a suitable integer r. Then

$$\varphi_0^M(\exp X)^{2+\epsilon} \le C_0' e^{-(2+\epsilon)\rho_M(X)}(1+\|X\|)^{r(2+\epsilon)} \le C_0'' e^{-(2+\frac{1}{2}\epsilon)\rho_M(X)},$$

and

$$\varphi_0^M(\exp X)^{2+\epsilon} e^{2\rho_{H\cap M}(X)} \le C_0'' e^{2[\rho_{H\cap M}(X)-\rho_M(X)]-\frac{1}{2}\epsilon\rho_M(X)}.$$

If α is a restricted root for $H\cap M$ that is positive on $\mathcal{C}^+_{H\cap M}$, then α is positive on $\overline{\mathcal{C}_M}\cap\mathcal{C}^+_{H\cap M}$. Let $\tilde\alpha$ be an extension of α from $\mathfrak{a}_{H\cap M}$ to $\mathfrak{a}_M$ as a restricted root for M. Then $\tilde\alpha > 0$ on $\overline{\mathcal{C}_M}\cap\mathcal{C}^+_{H\cap M}$, $\tilde\alpha > 0$ on some of $\mathcal{C}_M$, and $\tilde\alpha > 0$ on all of $\mathcal{C}_M$. Thus every α contributing to $\rho_{H\cap M}$ contributes to ρ_M. The roots β contributing to ρ_M that are not counted in this way at least have $\beta(X)\ge 0$ for $X\in\overline{\mathcal{C}_M}\cap\mathcal{C}^+_{H\cap M}$. Thus

$$e^{2[\rho_{H\cap M}(X)-\rho_M(X)]} \le 1 \qquad \text{for } X\in\overline{\mathcal{C}_M}\cap\mathcal{C}^+_{H\cap M}.$$

Hence

$$\varphi_0^M(\exp X)^{2+\epsilon} e^{2\rho_{H\cap M}(X)} \le C_0'' e^{-\frac{1}{2}\epsilon\rho_M(X)}.$$

For $X\in\overline{\mathcal{C}_M}$, $\rho_M(X) > 0$ (since $\langle\rho_M,\omega_j\rangle > 0$ for all j and since every element of $\overline{\mathcal{C}_M}$ is of the form $\sum c_j\omega_j$ with $c_j\ge 0$). Hence our integrand is dominated by an exponential whose integral is convergent, and the proposition follows.

6. Image in $L^2(G/H)$

In this section we shall prove the results in Subsection 4 of §3. T. Oshima suggested to us that the square integrability of $\mathcal{E}f_v$, where f_v is as in (3.7), should follow from his adaptation [11, 12] of the theory of asymptotics of matrix coefficients. D. A. Vogan showed us how to carry out such a program under the assumption that $\mathcal{E}f_v$ is bounded. A similar result was proved by Tong and Wang [16, §2], using a somewhat longer argument to obtain a more precise theorem.

We are able at present to show that $\mathcal{E}f_v$ is bounded only under the additional assumption that L and G have the same real rank. The argument is a variant of the proof of Theorem 4.8(i) of Flensted-Jensen [3]. We let $\mathfrak{b}_0$ be a maximal abelian subspace of $\mathfrak{p}_0 \cap \mathfrak{q}_0$, and we define $B = \exp \mathfrak{b}_0$.

Lemma 6.1. Suppose that L and G have the same real rank. For v in the space for π_{μ_λ}, let f_v be the K-finite function on K given by (3.7). Then $\mathcal{E}f_v$ is a bounded function.

Proof. According to [10, p. 161], we have $G = KBH$, and it follows that it is enough to prove that $\mathcal{E}f_v$ is bounded on B.

Let π be any finite-dimensional representation of G, and introduce a Hermitian inner product so that the action of the compact form is unitary. Since $\{\pi(b) \mid b \in B\}$ is a commuting family of self-adjoint operators, it has an orthonormal basis of simultaneous eigenvectors v_i, say with eigenvalues $\lambda_i(b) \in \mathbb{R}$. Let ω be the highest $\mathfrak{a}$-weight of π, and let ξ be a unit vector of $\mathfrak{a}$-weight ω. Since M is compact (because L and G have the same real rank),

$$\|\pi(x)\xi\|^2 = e^{2\omega H(x)} \qquad \text{for } x \in G. \tag{6.1}$$

We shall use this identity to prove that

$$e^{2\omega H(bk)} \geq 1 \qquad \text{for } b \in B,\ k \in H \cap K. \tag{6.2}$$

In fact, the involution τ leaves K and N stable and fixes A. Thus $H(x) = H(\tau x)$. Since $\tau(bk) = b^{-1}k$ for $b \in B$ and $k \in H \cap K$, (6.1) gives

$$e^{2\omega H(bk)} = \|\pi(bk)\xi\|^2 = \|\pi(b^{-1}k)\xi\|^2. \tag{6.3}$$

Let us write $\pi(k)\xi = \sum c_i(k)v_i$ for constants $c_i(k)$ with $\sum |c_i(k)|^2 = 1$. Then

$$\|\pi(bk)\xi\|^2 = \|\sum c_i(k)\lambda_i(b)v_i\|^2 = \sum |c_i(k)|^2 \lambda_i(b)^2$$

$$\|\pi(b^{-1}k)\xi\|^2 = \|\sum c_i(k)\lambda_i(b)^{-1}v_i\|^2 = \sum |c_i(k)|^2 \lambda_i(b)^{-2},$$

so that (6.3) gives

$$e^{2\omega H(bk)} = \sum |c_i(k)|^2 \frac{1}{2}(\lambda_i(b)^2 + \lambda_i(b)^{-2}) \geq \sum |c_i(k)|^2 = 1.$$

This proves (6.2).

Since N was chosen to make ρ_L dominant for G, and since ρ_G is automatically dominant, $(\rho_L + \rho_G)H(x)$ is a nonnegative combination of expressions $2\omega H(x)$ with ω as above. Then it follows from (6.2) that

$$e^{-(\rho_L+\rho_G)H(bk)} \leq 1 \qquad \text{for } b \in B,\ k \in H \cap K. \tag{6.4}$$

Since M is compact, we have

$$\mathcal{E} f_v(b) = \int_{H\cap K} l\,(f_v(bk))\,dk = \int_{H\cap K} e^{-(\rho_L+\rho_G)H(bk)} l\,(f_v(\kappa(bk)))\,dk. \tag{6.5}$$

Since f_v is bounded on K and since V^σ is finite-dimensional, $l\,(f_v(\kappa(bk)))$ is bounded as a function of b. Thus (6.4) and (6.5) allow us to conclude that $\mathcal{E} f_v(b)$ is bounded as a function of b.

Proposition 6.2. Suppose λ is generic. For v in the space for $\pi_{\mu\lambda}$, let f_v be the K-finite function on K given by (3.7). If $\mathcal{E} f_v$ is bounded, as is the case when L and G have the same real rank, then $\mathcal{E} f_v$ is square integrable on G/H.

T. Oshima [11, 12] developed a theory of asymptotic expansions on B of K-finite functions on G/H that are eigenfunctions of $D(G/H)$, the algebra of invariant linear differential operators on G/H. D. A. Vogan showed us how to apply this theory to $\mathcal{E} f_v$ and deduce Proposition 6.2. The remainder of this section gives the details of the proof. See also [16, §2].

The function $\mathcal{E} f_v$ is right H-invariant on G and is an eigenfunction of the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, since $\mathcal{E}$ is equivariant and $U(MAN, \sigma, \rho_L)$ has an infinitesimal character. The relationship between $Z(\mathfrak{g})$ and $D(G/H)$ is subtle, however. Restriction to right H-invariant functions yields a map of $Z(\mathfrak{g})$ into $D(G/H)$, and Helgason [6, Theorem 4] proved that this map is onto for classical G but not for certain exceptional G. Consequently $\mathcal{E} f_v$ is an eigenfunction of $D(G/H)$ at least for G classical. In the general case, Lemma 6.6 gives a substitute.

For purposes of Lemmas 6.3 through 6.5, let $G = KAN$ be the Iwasawa decomposition of a linear connected semisimple group, let M be the centralizer of A in K, and let $\mathfrak{b}_0$ be a Cartan subalgebra of $\mathfrak{m}_0$. Then $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{g}$. Let Δ be the set of roots of $(\mathfrak{g}, \mathfrak{h})$. We shall study $D(G/K)$. Let $W = W(\mathfrak{h})$ and $W_p = W(\mathfrak{a})$ be the usual Weyl groups, and let $I(\mathfrak{h})$ and $I(\mathfrak{a})$ be the respective subalgebras of Weyl-group invariants in $U(\mathfrak{h})$ and $U(\mathfrak{a})$. There are Harish-Chandra isomorphisms

$$\gamma_{\mathfrak{h}} : Z(\mathfrak{g}) \to I(\mathfrak{h}) \qquad \text{and} \qquad \gamma_{\mathfrak{a}} : D(G/K) \to I(\mathfrak{a}),$$

and any (simultaneous) eigenvalue for the operation of $Z(\mathfrak{g})$ or of $D(G/K)$ is given by a member of $\mathfrak{h}^*$ or $\mathfrak{a}^*$, respectively. Lemma 6.3 sorts out the effect on eigenvalues of the map of $Z(\mathfrak{g})$ into $D(G/K)$ given by restriction to right K-invariant functions.

Let Δ^+ be a positive system for Δ compatible with the positive restricted roots. Let $\Delta_M \subseteq \Delta$ be the set of roots of $(\mathfrak{m}, \mathfrak{b})$, let $\Delta_M^+ = \Delta^+ \cap \Delta_M$, and let δ_M be half the sum of the members of Δ_M^+.

Lemma 6.3. Let $\varphi \in C^\infty(G/K)$ be an eigenfunction of $D(G/K)$ with eigenvalue $\nu \in \mathfrak{a}^*$. If z is in $Z(\mathfrak{g})$, then

$$z\varphi = (\nu + \delta_M)(\gamma_{\mathfrak{h}}(z))\varphi.$$

Proof. Define the spherical function φ_ν^G as in [9] by

$$\varphi_\nu^G(x) = \int_K e^{-(\nu+\rho)H(x^{-1}k)}\,dk = \int_K e^{(\nu-\rho)H(xk)}\,dk.$$

Then [7, p. 431] shows that

$$D\varphi_\nu^G = \nu(\gamma_{\mathfrak{a}}(D))\varphi_\nu^G \qquad \text{for } D \in D(G/K). \tag{6.6}$$

In other words, φ_ν^G has eigenvalue ν. Now φ_ν^G is a matrix coefficient of $U(MAN,1,\nu)$, which has infinitesimal character $\nu + \delta_M$, by [9, Proposition 8.22]. Therefore

$$z\varphi_\nu^G = (\nu + \delta_M)(\gamma_{\mathfrak{h}}(z))\varphi_\nu^G \qquad \text{for } z \in Z(\mathfrak{g}). \tag{6.7}$$

If z_0 denotes the image of z in $D(G/K)$, then (6.6) and (6.7) together show that

$$\nu(\gamma_{\mathfrak{a}}(z_0)) = (\nu + \delta_M)(\gamma_{\mathfrak{h}}(z)). \tag{6.8}$$

By assumption, $D(G/K)$ acts on φ with eigenvalue ν, and (6.8) therefore proves the lemma.

Lemma 6.4. There exists $C > 0$ with the following property: Whenever ν and λ_A in $\mathfrak{a}^*$ and λ_M in $\mathfrak{b}^*$ have

$$\langle \lambda_A + \lambda_M, \alpha\rangle \geq C \qquad \text{for all } \alpha \in \Delta^+ \text{ with } \alpha \notin \Delta_M$$

and

$$\nu + \delta_M = w(\lambda_A + \lambda_M) \qquad \text{for some } w \in W,$$

then $\nu = w_p\lambda_A$ for some $w_p \in W_p$.

Proof. We fix C as any number greater than the maximum value of $2|\langle\delta_M,\alpha\rangle|$ for $\alpha\in\Delta$. Then

$$|\langle\nu+\delta_M,\alpha\rangle|<C \tag{6.9a}$$

for at least the roots α of Δ_M, while

$$|\langle\lambda_A+\lambda_M,\beta\rangle|<C \tag{6.9b}$$

for at most the roots β of Δ_M. Since $\nu+\delta_M$ and $\lambda_A+\lambda_M$ are conjugate by $w\in W$, (6.9a) and (6.9b) must hold for exactly the members of Δ_M.

It follows that $\nu+\delta_M$ is nonsingular. In fact, $\langle\nu+\delta_M,\alpha\rangle=0$ says that α satisfies (6.9a), so that α is in Δ_M. But then

$$0=\langle\nu+\delta_M,\alpha\rangle=\langle\delta_M,\alpha\rangle$$

is a contradiction.

By [9, Lemma 5.16], we may assume without loss of generality that ν is dominant with respect to the positive restricted roots. Also we may assume that λ_M is Δ_M^+ dominant. Then $\lambda_A+\lambda_M$ is Δ^+ dominant. If we can show that $\nu+\delta_M$ is Δ^+ dominant, then $w=1$ and $\nu=\lambda_A$.

Thus let α be in Δ^+. If α is in Δ_M^+, then

$$\langle\nu+\delta_M,\alpha\rangle=\langle\delta_M,\alpha\rangle>0.$$

For other α in Δ^+, we let $\bar\alpha$ be the member of Δ^+ with $\bar\alpha|_{\mathfrak{a}}=\alpha|_{\mathfrak{a}}$ and $\bar\alpha|_{\mathfrak{b}}=-\alpha|_{\mathfrak{b}}$. Suppose $\langle\nu+\delta_M,\alpha\rangle<0$. Then

$$0<2\langle\nu,\alpha\rangle=\langle\nu+\delta_M,\alpha\rangle+\langle\nu+\delta_M,\bar\alpha\rangle<\langle\nu+\delta_M,\bar\alpha\rangle,$$

so that (6.9a) gives $\langle\nu+\delta_M,\bar\alpha\rangle\geq C$. Hence

$$\langle\nu+\delta_M,\alpha\rangle=\langle\nu+\delta_M,\bar\alpha\rangle-2\langle\delta_M,\bar\alpha\rangle\geq C-2\langle\delta_M,\bar\alpha\rangle>0,$$

contradiction. We conclude that $\nu+\delta_M$ is Δ^+ dominant, and the lemma is proved.

Lemma 6.5. $D(G/K)$ is a finitely generated $Z(\mathfrak{g})$ module.

This is a special case of [4, Theorem 1]. Now let us return to the notation of Proposition 6.2. Let $\mathfrak{c}_0$ be a maximal abelian subspace of $\mathfrak{q}_0$ containing $\mathfrak{b}_0$, and let $\mathfrak{c}_0\oplus\mathfrak{d}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{c}_0$. Let G^0 be the Flensted-Jensen dual [3] of G, with a maximal compact subgroup H^0 and with other notation as in [3]. Under the duality we have

$$D(G/H)\cong D(G^0/H^0)\cong I(\mathfrak{c}) \tag{6.10a}$$

$$Z(\mathfrak{g})\cong Z(\mathfrak{g}^0)\cong I(\mathfrak{c}\oplus\mathfrak{d}). \tag{6.10b}$$

The Cartan subalgebra $\mathfrak{t}\oplus(\mathfrak{a}\oplus\mathfrak{s})$ of $\mathfrak{g}$ in Proposition 4.2 is conjugate to $\mathfrak{c}\oplus\mathfrak{d}$ under an element $w_0\in\mathrm{Ad}(G^{\mathbb{C}})$ that carries $\mathfrak{c}$ to $\mathfrak{t}$ and $\mathfrak{d}$ to $\mathfrak{a}\oplus\mathfrak{s}$. Define w_0^{tr} to be the transpose map from $\mathfrak{t}^*\oplus(\mathfrak{a}\oplus\mathfrak{s})^*$ to $\mathfrak{c}^*\oplus\mathfrak{d}^*$.

Lemma 6.6 Let λ be generic. Then the vector space of functions

$$V = \{D(\mathcal{E}f_v) \mid D \text{ is in } D(G/H),\ v \text{ is in space of } \pi_{\mu_\lambda}\}$$

is finite-dimensional, and $D(G/H)$ acts on it with a single generalized weight. Under the isomorphism (6.10a), the weight is $w_0^{\rm tr}(\lambda+\eta)$, where η is a member of $\mathfrak{t}^*$ independent of λ.

Proof. The representation $U(MAN, \sigma, \rho_L)$ has an infinitesimal character of the form $\lambda_0 + \rho_L$, where λ_0 is as in the proof of Proposition 5.1. We can rewrite this as $(\lambda+\eta)+\eta'$ with η and η' independent of λ, with η in $\mathfrak{t}^*$, and with η' in $(\mathfrak{a}\oplus\mathfrak{s})^*$. Then $\mathcal{E}f_v$ is a left K-finite eigenfunction of $Z(\mathfrak{g})$ with eigenvalue $w_0^{\rm tr}((\lambda+\eta)+\eta')$ relative to $\mathfrak{c}^*\oplus\mathfrak{d}^*$.

The finite-dimensionality of V now follows from Lemma 6.5 and Flensted-Jensen duality. By [8, p. 43], V is the direct sum of generalized weight spaces. Via (6.10a) we can regard each weight as a member of $\mathfrak{c}^*$. Fix such a weight ν, and let φ be a member of V that is an eigenfunction of $D(G/H)$ with weight ν.

Since φ is an eigenfunction of $Z(\mathfrak{g})$ with weight $w_0^{\rm tr}(\lambda+\eta)+w_0^{\rm tr}\eta'$, Lemma 6.3 shows that $\nu+\delta_M$ is in the same orbit as $w_0^{\rm tr}(\lambda+\eta)+w_0^{\rm tr}\eta'$ under the Weyl group of $\mathfrak{c}\oplus\mathfrak{d}$ for a certain δ_M in $\mathfrak{d}^*$. Since λ is generic, Lemma 6.4 says that ν is in the same orbit as $w_0^{\rm tr}(\lambda+\eta)$ under the Weyl group of $\mathfrak{c}$. Lemma 6.6 follows.

Now we can prove Proposition 6.2. The idea is to apply Oshima's theory [11, 12] to $\mathcal{E}f_v$. Lemma 6.6 says that $\mathcal{E}f_v$ belongs to a generalized eigenspace under $D(G/H)$, and Oshima's theory is directly applicable only to eigenfunctions. But the theory is easily modified to handle generalized eigenspaces, as long as one is not too fussy about what powers of logarithms are involved, and we shall take this extended theory as known.

The proof consists in playing off Theorem 4.1 and Corollary 4.3 of [11]. Oshima's $\mathfrak{a}$ is our $\mathfrak{b}_0$, his $\mathfrak{a}_\mathfrak{p}^d$ is our $\mathfrak{b}_0\oplus i(\mathfrak{c}_0\ominus\mathfrak{b}_0)$, and his λ is our $\lambda' = w_0^{\rm tr}(\lambda+\eta)$ in Lemma 6.6. For other notation, we refer the reader to Oshima's paper. Suppose $\mathcal{E}f_v$ is not square integrable. Let $\omega_1,\ldots,\omega_l$ be dual to the simple roots relative to $\mathfrak{b}$. Then Corollary 4.3 of [11] says that $FBI_{\lambda'}(\mathcal{E}f_v)\overline{P^d w^{-1} P^d}$ has an interior point for some $w\in W(\mathfrak{c})$ satisfying

$$\langle w\lambda', \omega_j\rangle \geq 0 \quad \text{for some } j = j_0. \tag{6.11}$$

Since there are only finitely many orbits in Oshima's G^d/P^d under his H^d, $FBI_{\lambda'}(\mathcal{E}f_v)\overline{P^d w^{-1} P^d}$ contains an open orbit in G^d/P^d for some w as above. By Remark 4.2i of [11], $FBI_{\lambda'}(\mathcal{E}f_v)\overline{P^d w^{-1} P^d} \supseteq w_0 P^d$ for some $w_0\in W(\mathfrak{b})$. In the notation of [11, p. 591], $\bar{W}(FBI_{\lambda'}(\mathcal{E}f_v), w_0P^d)$ contains some $w\in W(\mathfrak{c})$ satisfying (6.11).

Since $\langle w\lambda', \omega_{j_0}\rangle \geq 0$ and since λ is generic, $\langle w\lambda', \omega_{j_0}\rangle$ is large and positive. Taking $I=\{1,\ldots,l\}$ in [11, p. 591], we see that

$$\nu_I(w\lambda') = \big(\langle \rho-\lambda', \omega_1\rangle, \ldots, \langle \rho-\lambda', \omega_l\rangle\big)$$

has $\nu_I(w\lambda')_{j_0} < 0$. For some such w (one with an extremal property), the set

$$\Lambda = \Xi_I(FBI_{\lambda'}(\mathcal{E}f_v) : w_0P^d; \lambda')$$

contains $\nu_I(w\lambda')$, by comparison of the definitions at the top of [11, p. 592]. Theorem 4.1 of [11] says that

$$\mathcal{E}f_v(gw_0a_I(y)w_0^{-1}H) = \sum_{\nu\in\Lambda} c_{\nu,k}(g)y^\nu + r(g,y) \sum_{\substack{\nu\in\Lambda \\ 1\leq i\leq l}} y^\nu y_i^\epsilon \qquad (6.12)$$

with $c_{\nu,k}$ continuous on G and not identically 0 for ν in a subset Λ' of Λ, and with $r(g,y)$ continuous on $G\times[0,\infty)^l$. The relevant behavior is as $y \to 0$, and the second term represents error terms that are small relative to the first term when the first term is not 0.

By Remark 4.2iii of [11], $\Lambda' = \Lambda$. Thus Λ' contains the element $\nu = \nu_I(w\lambda')$ with $\nu_I(w\lambda')_{j_0} < 0$. For this ν, y^ν blows up as $y \to 0$ suitably. If we choose g_0 so that $c_{\nu,k}(g_0) \neq 0$ for this ν, then (6.12) shows that $\mathcal{E}f_v(g_0w_0a_I(y)w_0^{-1}H)$ blows up as $y \to 0$. But this behavior contradicts Lemma 6.1.

7. Operator nonzero

In this section we shall prove that $\mathcal{E}$ is not the 0 operator, provided L and G have the same real rank and our parameters are lined up suitably. Recall the definition of $l = l_u$ in (3.6) and the definitions of P and f_v in (3.7). Recall also from §1 that π_{μ_λ} has a nonzero $(H\cap K)$-fixed vector, say v_0.

Lemma 7.1. $\mathcal{E}f_{v_0}(1) = \langle Pv_0, u\rangle$, so that a suitable choice of u makes $\mathcal{E} \neq 0$ if $Pv_0 \neq 0$.

Proof. We have

$$\begin{aligned}
\mathcal{E}f_{v_0}(1) &= \int_{H\cap K} l\,(f_{v_0}(k))\,dk \\
&= \int_{H\cap K}\int_{H\cap M} \langle\sigma(m)P(\pi_{\mu_\lambda}(k)^{-1}v_0), u\rangle\,dm\,dk \\
&= \int_{H\cap K}\int_{H\cap M} \langle P(\pi_{\mu_\lambda}(mk^{-1})v_0), u\rangle\,dm\,dk \\
&= \langle Pv_0, u\rangle,
\end{aligned}$$

the last equality holding since v_0 is fixed by $H\cap K$ and the subgroup $H\cap M$.

The operator P may be regarded as a certain orthogonal projection within the space for π_{μ_λ}, namely the projection to the subspace of M type σ. Let v_h be a nonzero highest weight vector for π_{μ_λ}.

Lemma 7.2. $Pv_0 \neq 0$ if v_h lies in the image of P.

Proof. If v_h lies in the image of P, then $\langle Pv_0, v_h\rangle = \langle v_0, Pv_h\rangle = \langle v_0, v_h\rangle$. Thus the lemma follows from [9, (9.36)].

Recall the abelian subalgebras and root systems used internally in the proof of Proposition 5.1. We have $\mathfrak{t} \subseteq \mathfrak{t}_1 \subseteq \mathfrak{t}_2$; $\mathfrak{t}_1$ is a Cartan subalgebra of $\mathfrak{k}$, and $\mathfrak{t}_2$ is a Cartan subalgebra of $\mathfrak{g}$. We used the notation $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$, $\Delta_1 = \Delta(\mathfrak{g}, \mathfrak{t}_1)$, $\Delta_2 = \Delta(\mathfrak{g}, \mathfrak{t}_2)$. The positive system $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{t})$ is fixed, and the Flensted-Jensen parameter λ is very dominant.

We have at our disposal the choices of compatible positive systems Δ_1^+ and Δ_2^+, which determine v_h, and the choice of A, which determines the group M and ultimately the projection P.

The subalgebra $\mathfrak{t}_2$ is a Cartan subalgebra of $\mathfrak{l}$, and the members of $\Delta(\mathfrak{l}, \mathfrak{t}_2)$ are the roots in Δ_2 vanishing on $\mathfrak{t}$. In [1, §1 and §11], it is shown how to construct a positive system $\Delta^+(\mathfrak{l}, \mathfrak{t}_2)$ and an ordered sequence $\alpha_1, \dots, \alpha_m$ of noncompact imaginary roots in $\Delta(\mathfrak{l}, \mathfrak{t}_2)$ such that

(i) the α_i are strongly orthogonal

(ii) $\mathfrak{a}_0'' = \sum_{j=1}^m \mathbb{R}(E_{\alpha_j} + E_{-\alpha_j})$ has, for suitable normalization of root vectors, $\mathfrak{a}_0 = \mathfrak{a}_0'' \oplus (\mathfrak{t}_2 \ominus \mathfrak{t}_1)$ maximal abelian in $\mathfrak{l}_0 \cap \mathfrak{p}_0$

(iii) each α_j is (positive and) simple in the subsystem of roots of $\Delta(\mathfrak{l}, \mathfrak{t}_2)$ that are imaginary and orthogonal to $\alpha_1, \dots, \alpha_{j-1}$.

Together $\Delta^+(\mathfrak{l}, \mathfrak{t}_2)$ and Δ^+ determine Δ_2^+ (and therefore also Δ_1^+). The above construction defines $\mathfrak{a}$ as a Cayley transform, and M and $\mathfrak{m}$ are defined as usual. The proof of Proposition 3.1 of [1] shows that the $\mathfrak{m}$-cyclic span V' of v_h is irreducible and has v_h as a highest weight vector. The same proof shows that the one-dimensional space $\mathbb{C}v_h$ is stable under $L \cap K$, and it follows that V' is irreducible under M, say of M type σ'. The construction has the property that π_{μ_λ} is a minimal K type of the induced series $U(MAN, \sigma', \nu)$. By the uniqueness in [17, Proposition 4.1], it follows that σ' is equivalent with σ. In the above notation the image of P is then the M-cyclic span V' of v_h. In particular, v_h lies in the image of P. Application of Lemmas 7.1 and 7.2 then completes the proof of Theorem 3.3.

8. Some cases with L compact

In §§6-7 we addressed the square integrability and nontriviality of the image of $\mathcal{E}$ under the assumption that L and G have the same real rank. The opposite extreme is that L is compact, in which case $A = \{1\}$ and $M = G$. The representation in question is just σ. Since $\bar{N} = \{1\}$, (3.4) shows that the formula for $\mathcal{E}$ reduces to

$$\mathcal{E}f(x) = l(f(x)). \tag{8.1}$$

There is no difficulty in showing that l exists as in (3.1) making $\mathcal{E}$ nonzero with image$(\mathcal{E}) \subseteq L^2(G/H)$. This existence follows from the existence and identification of generic discrete series for G/H proved in [3] and [14]; we have only to set up an abstract intertwining operator from the discrete series σ into $L^2(G/H)$ and take l to be evaluation at the identity coset of G/H.

The question is whether l is given by integration, as in (3.6). We suspect that l is indeed always given by integration in the generic case, and we give some evidence in this section for such a conjecture. The cases that we can handle are the group case and $G/H = SO(2,1)/SO(1,1)$.

In the group case the total group is $G \times G$, and the subgroup H fixed by τ is $\{(y,y) \mid y \in G\} \cong G$. An exposition of the Flensted-Jensen construction for the group case appears in [9, Chapter IX]. The parameter λ is essentially the Harish-Chandra parameter of a discrete series π_λ of G, and σ is $\pi_\lambda^* \otimes \pi_\lambda$. If π_λ acts on V, then π_λ^* acts on a space $\bar{V}$ whose elements are the same as those of V but whose complex structure is opposite to that of V. When we want to emphasize the distinction, we shall write members of $\bar{V}$ with overbars. The mapping $v \to \bar{v}$ is conjugate linear. Let d_λ be the formal degree of π_λ.

Proposition 8.1. In the group case with $\sigma = \pi_\lambda^* \otimes \pi_\lambda$, the linear functional l of (3.6) is given by

$$l_{\bar{u}_1 \otimes u_2}(\bar{v}_1 \otimes v_2) = \frac{1}{d_\lambda}\langle v_2, v_1\rangle\overline{\langle u_2, u_1\rangle}. \tag{8.2}$$

For the element

$$f_v(x_1,x_2) = \sigma(x_1,x_2)^{-1}(\bar{v}_1 \otimes v_2) = \pi_\lambda^*(x_1)^{-1}\bar{v}_1 \otimes \pi_\lambda(x_2)^{-1}v_2,$$

$\mathcal{E}$ is given by

$$\mathcal{E}f_v(x_1,x_2) = \frac{1}{d_\lambda}\langle \pi_\lambda(x_1x_2^{-1})v_2, v_1\rangle\overline{\langle u_2, u_1\rangle}. \tag{8.3}$$

Therefore the image of $\mathcal{E}$ is nonzero and is contained in L^2.

Proof. We use Schur orthogonality [9, Proposition 9.6] twice. For (8.2) we have

$$\begin{aligned} l_{\bar{u}_1\otimes u_2}(\bar{v}_1 \otimes v_2) &= \int_G \langle \pi_\lambda^* \otimes \pi_\lambda(y,y)(\bar{v}_1 \otimes v_2), \bar{u}_1 \otimes u_2 \rangle \, dy \\ &= \int_G \langle \pi_\lambda^*(y)\bar{v}_1, \bar{u}_1 \rangle \langle \pi_\lambda(y) v_2, u_2 \rangle \, dy \\ &= \int_G \overline{\langle \pi_\lambda(y) v_1, u_1 \rangle} \langle \pi_\lambda(y) v_2, u_2 \rangle \, dy \\ &= \frac{1}{d_\lambda} \langle v_2, v_1 \rangle \overline{\langle u_2, u_1 \rangle}, \end{aligned}$$

by a first application of Schur orthogonality.

Using (8.1) and substituting $f_v(x_1, x_2)$ as the argument of l, we obtain (8.3). When $x_1 = x_2$, $u_1 = u_2 \neq 0$, and $v_1 = v_2 \neq 0$, this expression is nonzero. Note that the value of (8.3) is unchanged upon replacing (x_1, x_2) by $(x_1 y, x_2 y)$. Integration over the quotient $(G \times G)/(\text{diagonal})$ is achieved by replacing $x_1 x_2^{-1}$ by a single variable and integrating over G. Consequently the square integrability of (8.3) over the quotient follows by a second application of Schur orthogonality.

The other case we can handle is $SO(2,1)/SO(1,1)$. It is a little simpler to consider $G = SU(1,1)$ with $H = \{h_t\} = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \right\}$. The holomorphic discrete series $\mathcal{D}_n^-$ of G is given in the space of analytic functions for $|z| < 1$ with $\|f\|^2 = \iint_{|z|<1} |f(z)|^2 (1 - |z|^2)^{n-2} \, dx \, dy$ by

$$\mathcal{D}_n^- \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} f(z) = (-\bar{\beta} z + \alpha)^{-n} f\left(\frac{\bar{\alpha} z - \beta}{-\bar{\beta} z + \alpha} \right).$$

Here n is an integer ≥ 2, and n is to be even if the discrete series is to descend to $SO(2,1)$. The only other discrete series of G are the antiholomorphic ones, whose formulas are similar.

Proposition 8.2. For $SU(1,1)/H$ and the discrete series $\mathcal{D}_n^-$ with $n \geq 2$, if u is taken to be the function 1 on the disc, then the linear functional $l = l_u$ in (3.6) is not 0, and the corresponding $\mathcal{E}$ is nonzero. Moreover, if in addition n is sufficiently large, then $\mathcal{E}$ has image in $L^2(SU(1,1)/H)$.

Proof. The subgroup B at the start of §6 is

$$B = \{b_s\} = \left\{ \begin{pmatrix} \cosh s & i \sinh s \\ -i \sinh s & \cosh s \end{pmatrix} \right\}.$$

The K type π_{μ_λ} is the minimal K type of $\mathcal{D}_n^-$, which corresponds to the constant functions on the disc. We have

$$l_1(\mathcal{D}_n^-(b_s)1) = \int_{-\infty}^{\infty} \langle \mathcal{D}_n^-(h_t)\mathcal{D}_n^-(b_s)1, 1\rangle \, dt$$

$$= \int_{-\infty}^{\infty} \langle \mathcal{D}_n^-(b_s)1, \mathcal{D}_n^-(h_{-t})1\rangle \, dt$$

$$= \int_{-\infty}^{\infty} \iint_{|z|<1} (\cosh s + iz \sinh s)^{-n} (\cosh t + \bar{z} \sinh t)^{-n} (1-|z|^2)^{n-2} \, dx \, dy \, dt$$

$$= (\cosh s)^{-n} \int_{-\infty}^{\infty} (\cosh t)^{-n} \left[\iint_{|z|<1} (1 + iz \tanh s)^{-n} (1 + \bar{z} \tanh t)^{-n} \cdot (1-|z|^2)^{n-2} \, dx \, dy \right] dt.$$

For $s = 0$, the expression in brackets is the nonzero constant

$$\iint_{|z|<1} (1-|z|^2)^{n-2} \, dx \, dy,$$

and thus $l_1(\mathcal{D}_n^-(b_0)1) \neq 0$. For general s, let $z = re^{i\theta}$. Then the expression in brackets is

$$= 2\pi \int_0^1 \left[\sum_{k=0}^{\infty} \binom{-n}{k}^2 (ir \tanh s)^k (r \tanh t)^k \right] (1-r^2)^{n-2} r \, dr$$

$$\leq 2\pi \int_0^1 \left[\sum_{k=0}^{\infty} \binom{-n}{k}^2 (r \tanh s)^k (r \tanh t)^k (\operatorname{sgn} st)^k \right] (1-r^2)^{n-2} r \, dr \quad \text{in absolute value}$$

$$= \iint_{|z|<1} (1 + z \tanh(s \operatorname{sgn}(st)))^{-n} (1 + \bar{z} \tanh t)^{-n} (1-|z|^2)^{n-2} \, dx \, dy,$$

which is the inner integral that arises with $h_{-s\operatorname{sgn}(st)}$ and h_{-t}. Hence

$$|l_1(\mathcal{D}_n^-(b_s)1)| \leq \int_{-\infty}^{\infty} \langle \mathcal{D}_n^-(h_{-s\operatorname{sgn}(st)})1, \mathcal{D}_n^-(h_{-t})1\rangle \, dt$$

$$= \int_{-\infty}^{\infty} \langle \mathcal{D}_n^-(h_{t-s\operatorname{sgn}(st)})1, 1\rangle \, dt$$

$$= \int_{-\infty}^{\infty} [\cosh(t - s\operatorname{sgn}(st))]^{-n} \, dt$$

$$\leq 2 \int_{-\infty}^{\infty} (\cosh t)^{-n} \, dt \; < \; \infty.$$

In other words, $\mathcal{E}(1)$ is bounded on B and hence is bounded everywhere. By Proposition 6.2, $\mathcal{E}(1)$ is square integrable on $SU(1,1)/H$ if n is generic.

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