SOME NEW INTERTWINING OPERATORS FOR SEMISIMPLE GROUPS

A. W. Knapp* and E. M. Stein*

1. Background and problem

This paper gives a progress report on work not yet complete on the construction of further intertwining operators beyond those studied in [8]. The general problem we have in mind is to understand the reducibility of degenerate series of unitary representations of a semisimple Lie group G with finite center. The hope is that the information obtained will be a step toward classifying the irreducible unitary representations of G.

The setting consists of a <u>maximal</u> parabolic subgroup of G with Langlands decomposition MAN relative to a maximal compact subgroup K of G. We consider the series of induced representations

$$\operatorname{ind}_{MAN}^{G}(\xi \otimes e^{\lambda} \otimes 1), \tag{1.1}$$

where ξ is an irreducible unitary representation of M and e^{λ} is a character of A. Parameters are arranged so that the induced representation is unitary when e^{λ} is unitary, and we write the action of G on the left. Usually we abbreviate (1.1) as

$$\operatorname{ind}_{M}^{G}(\xi \otimes e^{\lambda}). \tag{1.2}$$

We study self-intertwining operators for (1.2) when e^{λ} is unitary. It is known (Harish-Chandra, unpublished) that if ξ has a real infinitesimal character and λ is nonzero imaginary, then (1.2) is irreducible. Thus we shall confine our investigation of intertwining operators to the case $\lambda = 0$.

---

* Supported by grants from the National Science Foundation.

The first systematic study of this problem was made by Bruhat [1], who associated distributions on double cosets of MAN\G/MAN to such operators. Some, but usually not all, of these double cosets are indexed by the Weyl group $W(\mathfrak{a})$ of the parabolic, defined as the normalizer modulo centralizer $N_K(A)/Z_K(A)$. If w represents a member of $W(\mathfrak{a})$, the associated MAN double coset is MANwMAN. For these nice double cosets, there is a preliminary algebraic obstruction to having any associated operator at all — namely that we must have $w\xi \cong \xi$ (and $w\lambda = \lambda$, which holds for $\lambda = 0$).

If this condition $w\xi \cong \xi$ is satisfied, then Kunze and Stein [9] discovered that the relevant operator is formally roughly[1]

$$A(w,\xi,\lambda)f(x) = \int_{V\cap w^{-1}Nw} f(xwv)\,dv, \tag{1.3}$$

where $V = \theta N$ and θ is the Cartan involution of G corresponding to K. We call (1.3) a standard intertwining operator. The integral (1.3) is divergent, but [8] shows how to define it by analytic continuation in λ. Normalization of (1.3) by dividing by a suitable scalar-valued meromorphic function of λ yields a unitary operator for e^{λ} unitary, and the dependence of the operator on λ is holomorphic for these values. When w is non-trivial and the normalizing factor is regular at $\lambda = 0$ (so that the operator is well-defined without normalization), the operator exhibits reducibility. If the normalizing factor is singular, the normalization effectively pushes the mass of the distribution off MANwMAN to lower-dimensional double cosets in the closure, defining an operator associated with these other double cosets. In a sense the operator should have been studied in a different setting.

In practice, the intertwining distributions of [1] are hard to

---

[1] The self-intertwining operator is $\xi(w)A(w,\xi,\lambda)$. See §7 of [8].

understand on G. But we can consider the induced representations (1.2) in the noncompact picture (by restriction of functions from G to V), and then the distributions can be regarded as on subsets of V that mirror the double coset structure of G. This approach will be useful in §2.

The standard self-intertwining operators, then, are well understood. The problem here is to study those self-intertwining operators whose distributions are attached to MAN double cosets not coming from $W(\mathfrak{a})$. We shall give in this paper a way of dealing with such operators sometimes (all the time?) by associating diamond-shaped diagrams to these double cosets. Understanding of the operators will come from going the long way around the diamond.

The plan of the paper is as follows. In §2, we discuss in detail the examples that led us to consider diamonds. The rigorous part of the paper begins in §§3-4 with results on double cosets, including in Theorem 4.1 a proof of a conjecture by Bruhat concerning their structure. In §5 we define diamonds and work with the algebraic formalism of them. The analytic problems of diamonds are listed in §6. We conclude in §7 by showing how diamonds can be used to account for a number of known phenomena.

Our discovery of diamonds was facilitated greatly by examples and suggestions provided to us by B. Speh, R. Strichartz, and M. Vergne. We thank these people for all their help.

2. Motivating examples

In this section we shall reinterpret known reducibility phenomena for two groups as motivation for using diamonds.

1. SU(2,2). The reducible continuous series representation in this example has been understood for some time and has been studied more recently by Jakobsen-Vergne [5], Kashiwara-Vergne [6], and Speh

[12]. The restricted roots form a system of type C_2, and we shall denote the simple restricted roots $e_1 - e_2$ and $2e_2$. If we build a parabolic subgroup MAN out of $e_1 - e_2$, then the M of the parabolic is isomorphic to $SL(2,\mathbb{C})$ with the scalar matrix iI adjoined. The representation we study is $U = \mathrm{ind}_{M_{e_1-e_2}}^{G} (1 \otimes e^0)$.

More concretely we can conjugate $SU(2,2)$ so that its Lie algebra takes the form

$$\mathfrak{su}(2,2) = \{\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^* \end{pmatrix}\} \tag{2.1}$$

with the 2-by-2 matrix $\alpha - \alpha^*$ of trace 0 and with β and γ Hermitian 2-by-2 matrices. The Lie algebra of the parabolic in question is all matrices (2.1) with $\gamma = 0$. The group V is abelian, isomorphic with the additive group of Hermitian 2-by-2 matrices, and the representation U takes on a nice form in the noncompact picture, namely

$$U(g)f(x) = \det(a + bx)^{-2} f((c + dx)(a + bx)^{-1}) \tag{2.2}$$

for $g^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, x Hermitian 2-by-2, and f in L^2. Holomorphic such f (i.e., boundary values of functions in the associated Hardy space) form an invariant subspace, as is apparent from (2.2), and antiholomorphic f also form an invariant subspace. But these two subspaces do not exhaust L^2. (On the Fourier transform side, these subspaces become the spaces of functions supported on the positive definite matrices and negative definite matrices, respectively, and the functions supported on the indefinite matrices are missing.) The third subspace is invariant and easily seen to be irreducible, and it follows that the space $\mathcal{C}$ of self-intertwining operators for U has dimension 3.

We look at the standard intertwining operators coming from the Weyl group of the parabolic. Here A is one-dimensional, and the group has two elements. The identity element leads to the identity operator, and the nontrivial element leads to a standard operator that does not require normalization (to become regular) and is therefore not scalar. We therefore obtain two independent members of C but still need a third operator.

To get a clue to the nature of a third operator, we note that the Cauchy kernel and the conjugate Cauchy kernel provide intertwining operators when we pass to boundary values. The limiting operators are convolutions with the distributions

$$\lim_{\epsilon \downarrow 0} \det(x + i\epsilon I)^{-2} \quad \text{and} \quad \lim_{\epsilon \downarrow 0} \det(x - i\epsilon I)^{-2} .$$

We form the difference of these distributions, which will clearly be supported on the set $\{\det x = 0\}$. A little computation shows that the difference (away from the origin) is not a signed measure on the set $\{\det x = 0\}$ but involves a first-order transverse derivative.

A second clue comes from consideration of MAN double cosets. Let $M_p A_p N_p$ be a minimal parabolic subgroup of G contained in MAN. The Weyl group $W(\mathfrak{a}_p)$ of the minimal parabolic has 8 elements. Bruhat [1] noted that we obtain a map of $W(\mathfrak{a}_p)$ onto $MAN\backslash G/MAN$ by passing from w in $W(\mathfrak{a}_p)$ to a representative (also denoted w) in G and then to the double coset MANwMAN. If W_M denotes the subgroup of $W(\mathfrak{a}_p)$ with representatives in M, then we can pass to the quotient and obtain a map

$$W_M \backslash W(\mathfrak{a}_p)/W_M \xrightarrow{\text{onto}} MAN\backslash G/MAN.$$

By Theorem 4.1 below (originally conjectured by Bruhat) this map is one-one. Thus we can determine the MAN double cosets by reference to $W(\mathfrak{a}_p)$. The group $W(\mathfrak{a}_p)$ consists of all permutations and sign changes on a two-element set, and the elements fall into double cosets of $W_M \backslash W(\mathfrak{a}_p)/W_M$ according as how many sign changes (0,1, or 2) they involve. Zero and two sign changes, respectively, correspond to the two elements of the Weyl group of the parabolic MAN, and it is the middle double coset (with 4 members) that will be of interest.

Fix w in this middle double coset. General considerations in the spirit of [1] and [9] still indicate we should look at

$$\int_{V \cap w^{-1}MANw} Df(xwv \downarrow)\,dv\ , \tag{2.3}$$

where D is some differentiation of f transverse to the space of integration and where the arrow points to the place where group elements are to be inserted for the differentiation. Our previous clue above indicates that D should be first-order. The four choices for w from the middle double coset do not lead to obviously equal expressions. But let us take $w = p_{2e_2}$ (second sign change) anyway. Then (2.3) becomes

$$\int_{V_{e_1+e_2} V_{2e_2}} Df(xp_{2e_2}v \downarrow)\,dv\ .$$

We can hope that D will be a constant coefficient operator on V, and then the only possibility is $D = X_{-2e_1}$. Hence our guess at a third self-intertwining operator is

$$Lf(x) = \int_{V_{e_1+e_2} V_{2e_2}} X_{-2e_1} f(xp_{2e_2}v \downarrow)\,dv \tag{2.4}$$

in the induced picture.

One can check by a long calculation that (2.4) is formally an intertwining operator. However, the integral in (2.4) is not convergent, and we need to make analytic sense out of it. In concrete terms, with f identified with a function on Hermitian matrices, equation (2.4) at $x = \text{identity}$ becomes

$$Lf(0) = \int_{s\in\mathbb{C}, r\in\mathbb{R}} \frac{\partial}{\partial t} f\begin{pmatrix} t + \frac{r}{rs}|s|^2 & rs \\ & r \end{pmatrix}_{t=0} ds\, dr. \qquad (2.5)$$

Even if f has compact support, the integrand does not; this fact is the source of the analytic problem.

A first — and natural — attempt at regularization is to introduce a parameter λ corresponding to inducing from $1 \otimes e^{\lambda(e_1+e_2)}$ and write down the expression corresponding to (2.5), namely

$$\int_{s\in\mathbb{C}, r\in\mathbb{R}} \frac{\partial}{\partial t} f\begin{pmatrix} t + \frac{r}{rs}|s|^2 & rs \\ & r \end{pmatrix}_{t=0} |r|^{\lambda} ds\, dr. \qquad (2.6)$$

Then we let λ tend to 0 from the right half plane. It turns out that the limit of (2.6) exists and gives a distribution, but the result is not an intertwining operator for U.

Experimentation shows that the limit (as $\lambda \to 0$) of

$$L_\lambda f(0) = \int_{s\in\mathbb{C}, r\in\mathbb{R}} \frac{\partial}{\partial t} f\begin{pmatrix} t + \frac{r}{rs}|s|^2 & rs \\ & r \end{pmatrix}_{t=0} |r|^{\lambda}|s|^{\lambda} ds\, dr$$

also exists and does give an intertwining operator for U. A heuristic calculation indicates that this regularization corresponds to using $e^{\lambda e_2}$ in place of $e^{\lambda(e_1+e_2)}$. However, $2e_2$ is not a parameter in the A direction — there is not enough space — and it is not possible to use $e^{\lambda e_2}$ as part of the inducing parameter. We need to make it possible by providing enough space.

It is at this stage that we introduce a diamond to provide the extra A-parameter needed for regularizing L. We back away from our $M = M_{e_1-e_2}$ to the minimal parabolic, using the imbedding of the trivial representation of M in the nonunitary principal series of M at parameters $(1, -(e_1 - e_2))$ on $(M_p, \mathfrak{A}_p)$, and pulling back via a standard intertwining operator $\varphi = A(p_{e_1-e_2})$. Schematically the picture is that given in Figure 1.

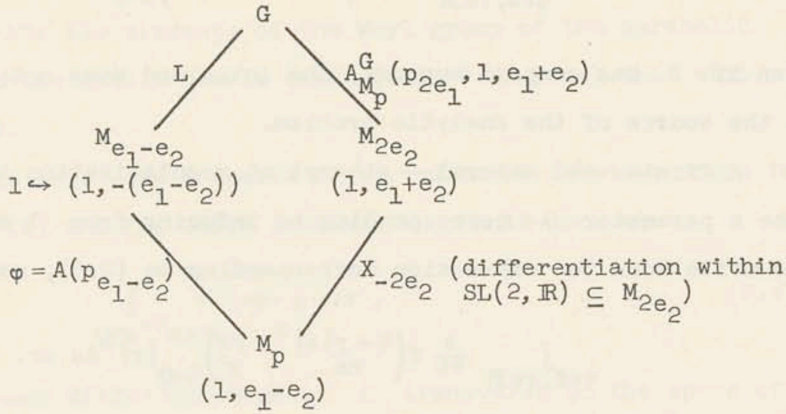


Figure 1. Diamond for SU(2,2).

With λ introduced, the diamond picture becomes that in Figure 2.

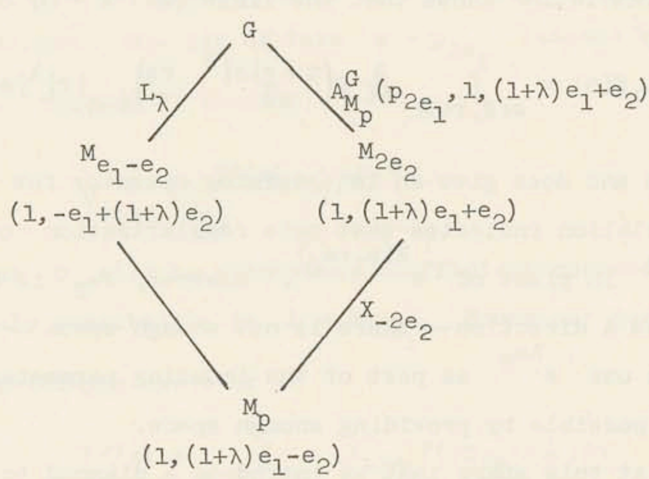
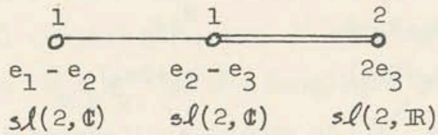


Figure 2. Diamond for SU(2,2) after λ is introduced.

There are several analytic facts that need proving, and they can all be carried out. See the beginning of §6 for a list.

2. SU(3,3). Kashiwara and Vergne [6] treated a reducibility

problem in $SU(n,n)$, and we specialize to $n = 3$ to translate matters into diamonds. The restricted roots form a system of type C_3:



We consider $U = \mathrm{ind}_{M_{e_1-e_2,\,e_2-e_3}}^{G} (\mathrm{sgn}\,\det \otimes e^0)$.

Again V can be identified with Hermitian matrices, this time 3-by-3. In the noncompact picture, the irreducible invariant subspaces of U are given on the Fourier transform side by support on matrices of a particular signature (4 such subspaces). The identity and the standard intertwining operator give two self-intertwining operators, and we need two more. There are two remaining MAN double cosets, and we can construct a diamond for each. We drop the parameter on M_p, which involves a signum, in the diamonds in Figures 3 and 4. The operator $(\det \partial)$ is a differential operator in $SU(2,2)$ studied in [4] and [5]; its role in a diamond for $SU(3,3)$ is deduced by a direct attempt at proving L is an intertwining operator.

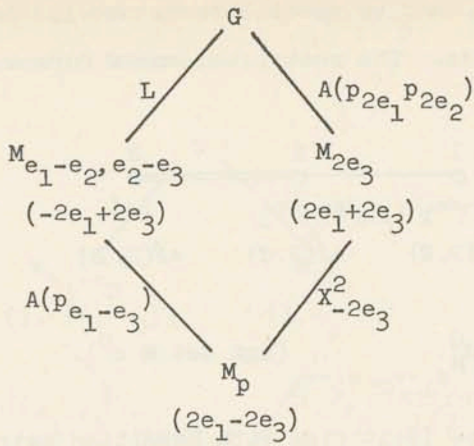


Figure 3: First diamond for SU(3,3).

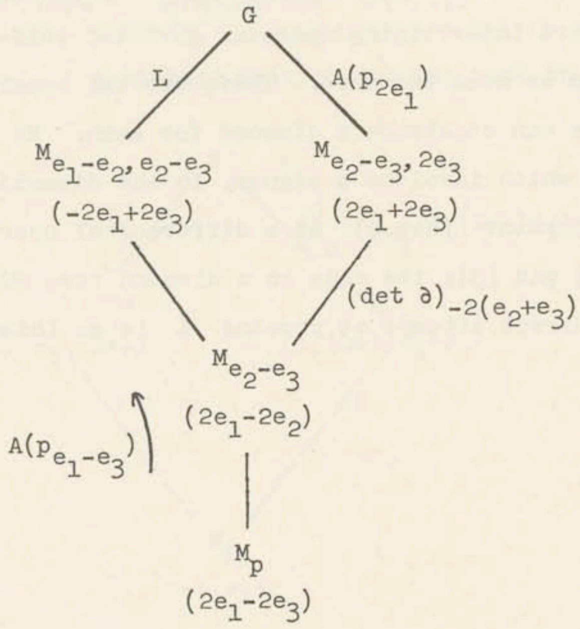


Figure 4: Second diamond for SU(3,3).

3. Generalities on double cosets

Let G be a connected semisimple Lie group with finite center, or more generally a reductive group satisfying the axioms (1.1) of [8]. Fix a maximal compact subgroup K and corresponding Cartan involution θ. Let $M_pA_pN_p$ be the Langlands decomposition (relative to θ) of a minimal parabolic subgroup of G. We work with the finite collection of standard parabolic subgroups MAN — those parabolic subgroups containing $M_pA_pN_p$. In addition, we let $V = \theta N$ and we let $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}$, and $\mathfrak{v}$ denote the Lie algebras of M, A, N, and V.

The Weyl group $W(\mathfrak{a}_p)$ is the quotient of the normalizer $N_K(\mathfrak{a}_p)$ by the centralizer $Z_K(\mathfrak{a}_p)$. The choice of N_p determines which $\mathfrak{a}_p$-roots are positive and imposes a notion of length $|w|$ on the members w of $W(\mathfrak{a}_p)$.

If MAN is given, we let $A_M = M \cap A_p$ and $N_M = M \cap N_p$. Then $M_pA_MN_M$ is the Langlands decomposition of a minimal parabolic subgroup of M. The Weyl group $W(\mathfrak{a}_M)$ is naturally identified with the subgroup W_M of $W(\mathfrak{a}_p)$ leaving each element of $\mathfrak{a}$ fixed.

For the purposes of the remainder of this section, fix a double coset of $W_M\backslash W(\mathfrak{a}_p)/W_M$. Let ℓ be the shortest length of all elements in the double coset. We consider two conditions on a member w of the double coset:

(I) If α is any $\mathfrak{a}_p$-root > 0 with $\alpha = 0$ on $\mathfrak{a}$, then either $w^{-1}\alpha < 0$ or $w^{-1}\alpha = 0$ on $\mathfrak{a}$ (or both).

(II) If β is any $\mathfrak{a}_p$-root > 0 with $\beta = 0$ on $\mathfrak{a}$, then $w\beta > 0$.

Lemma 3.1. If w_1 in the double coset has $|w_1| = \ell$, then w_1 and w_1^{-1} both satisfy (II).

Proof. If w_1 does not satisfy (II), then there is an $\mathfrak{a}_p$-root $\beta > 0$ with $\beta = 0$ on $\mathfrak{a}$ such that $w_1\beta < 0$. Without loss of generality we may take β to be simple. Then the reflection p_β is such that $w_1 p_\beta$ is not a minimal product, and p_β is in W_M. Hence $|w_1 p_\beta| = \ell - 1$ and $w_1 p_\beta$ is a shorter element of the double coset. This contradiction shows that w_1 satisfies (II). Similarly w_1^{-1} satisfies (II).

<u>Lemma 3.2</u>. If w_1 in the double coset has $|w_1| = \ell$ and if u is in W_M, then uw_1 is a minimal product.

Proof. Failure of the product to be minimal would mean there is an $\mathfrak{a}_p$-root $\alpha > 0$ such that $u\alpha < 0$ and $w_1^{-1}\alpha < 0$. Since $u\alpha < 0$, α is 0 on $\mathfrak{a}$. Then $w_1^{-1}\alpha < 0$ contradicts Lemma 3.1 for w_1^{-1}.

<u>Lemma 3.3</u>. If w and w' are in the double coset and satisfy (II), then $w' = uw$ for some u in W_M.

Proof. Write $w' = uwv$ with u and v in W_M, and among all such decompositions assume that v is as short as possible. We shall show $v = 1$. Thus assume (on the contrary) that γ is simple, vanishes on $\mathfrak{a}$, and has $v\gamma < 0$. We have $|vp_\gamma| < |v|$ and

$$uwv = uwvp_\gamma v^{-1}w^{-1}wvp_\gamma = (up_{wv\gamma})w(vp_\gamma),$$

and it is enough to show that $p_{wv\gamma}$ is in W_M, i.e., that $wv\gamma = 0$ on $\mathfrak{a}$. We know $v\gamma < 0$, and (II) for w gives $wv\gamma < 0$. If $wv\gamma \neq 0$ on $\mathfrak{a}$, then $0 > uwv\gamma = w'\gamma$ in contradiction to (II) for w'. Thus $wv\gamma = 0$ on $\mathfrak{a}$, and the lemma follows.

<u>Proposition 3.4</u>. The element w_1 of minimal length ℓ in the double coset is unique.

Proof. If w and w_1 in the double coset have length ℓ, they both satisfy (II), by Lemma 3.1, and so $w = uw_1$ with u in W_M, by Lemma 3.3. By Lemma 3.2, the product uw_1 is minimal. Hence $|w| = |w_1|$ implies $u = 1$ and $w = w_1$.

Lemma 3.5. If w_1 is the unique element of length ℓ in the double coset and w_M is the long element for W_M, then $w_M w_1$ satisfies (I).

Proof. By Lemma 3.2, $w_M w_1$ is a minimal product. If $\gamma > 0$ is an $\mathfrak{a}_p$-root vanishing on $\mathfrak{a}$, then $w_M^{-1}\gamma = w_M\gamma$ is < 0. Since $w_1^{-1}w_M^{-1}$ is minimal we have $w_1^{-1}w_M^{-1}\gamma < 0$. Hence (I) holds.

Proposition 3.6. Within the double coset, there exists one and only one element w that satisfies both (I) and (II). The element w is characterized as the unique shortest one satisfying (I).

Proof of existence. Let w be any element in the double coset that has the shortest possible length among elements satisfying (I); w exists by Lemma 3.5. We prove w satisfies (II). Thus suppose $w\beta < 0$ for some $\mathfrak{a}_p$-root $\beta > 0$ with $\beta = 0$ on $\mathfrak{a}$. Without loss of generality we may assume β is simple. Then $|wp_\beta| < |w|$, and the minimality of lengths implies that wp_β does not satisfy (I).

However, consider an $\mathfrak{a}_p$-root $\alpha > 0$ with $\alpha = 0$ on $\mathfrak{a}$ such that (I) fails for wp_β and α. Then both (i) $p_\beta w^{-1}\alpha > 0$ and (ii) $p_\beta w^{-1}\alpha \neq 0$ on $\mathfrak{a}$. By (ii) we can apply p_β to $p_\beta w^{-1}\alpha$ without changing the sign. Then (i) gives $w^{-1}\alpha > 0$. Since w satisfies (I), we conclude that $w^{-1}\alpha = 0$ on $\mathfrak{a}$. But then $p_\beta w^{-1}\alpha = 0$ on $\mathfrak{a}$, in contradiction to (ii). This contradiction means that β does not exist. Hence w satisfies (II), and existence is proved.

Proof of uniqueness. Let w and w' both satisfy (I) and (II). By Lemma 3.3 write $w' = uw$ with u in W_M. Then we have

$$w'w^{-1}N_p ww'^{-1} \cap V_p$$

$$= w'(w^{-1}N_p w \cap w'^{-1}V_p w')w'^{-1}$$

$$= w'(w^{-1}N_p w \cap V \cap w'^{-1}V_p w')(w^{-1}N_p w \cap N_p \cap w'^{-1}V_p w')w'^{-1}$$

by (II) for w

$$= w'(w^{-1}N_p w \cap V \cap w'^{-1}Vw')(w^{-1}N_p w \cap N_p \cap w'^{-1}V_p w')w'^{-1}$$

by (I) for w'

$$= w'(w^{-1}N_p w \cap V \cap w'^{-1}Vw')(w^{-1}N_p w \cap N \cap w'^{-1}V_p w')w'^{-1}$$

by (II) for w'

$$= w'(w^{-1}N_p w \cap V \cap w'^{-1}Vw')(w^{-1}Nw \cap N \cap w'^{-1}V_p w')w'^{-1}$$

by (I) for w

$$= w'(w^{-1}N_p w \cap V \cap w^{-1}Vw)(w'^{-1}Nw' \cap N \cap w'^{-1}V_p w')w'^{-1}$$

since $w' = uw$

$$= \{1\},$$

the last equality holding since $w^{-1}(N_p \cap V)w = \{1\}$ and $w'^{-1}(N \cap V_p)w' = \{1\}$. Hence $w'w^{-1} = 1$, and uniqueness is proved.

Remarks. Apart from questions of connectedness, one can compare $w^{-1}MANw \cap V$ with $V_p \cap w^{-1}N_p w$ by examining their Lie algebras, and one sees readily that a necessary and sufficient condition to have an equality

$$w^{-1}MANw \cap V = V_p \cap w^{-1}N_p w \tag{3.1}$$

is that $V \cap w^{-1}V_M w = \{1\}$ and $V_M \cap w^{-1}N_p w = \{1\}$, i.e., that (I) and (II) hold. Hence Proposition 3.6 says that each double coset contains exactly one element w for which equation (3.1) holds.

4. A conjecture of Bruhat

In this section we allow ourselves to use the same symbol w to denote both a member of $N_K(\mathfrak{a}_p)$ and the corresponding member of $W(\mathfrak{a}_p)$. With this convention the map sending $W_M w W_M$ to MANwMAN is a well-defined function from the double coset space $W_M \backslash W(\mathfrak{a}_p)/W_M$ into the double coset space $MAN \backslash G/MAN$. This function is onto, by the Bruhat decomposition theorem for G. Bruhat conjectured in [1] that this function is one-one. We shall prove his conjecture in the theorem below. The theorem provides for us a convenient parametrization for the double cosets of $MAN \backslash G/MAN$.

Theorem 4.1. Let w_1 and w_2 in $W(\mathfrak{a}_p)$ be such that $MANw_1MAN = MANw_2MAN$. Then w_1 and w_2 are in the same double coset of $W_M \backslash W(\mathfrak{a}_p)/W_M$.

Proof. Without loss of generality we may assume w_1 and w_2 are as short as possible within their respective double cosets in $W_M \backslash W(\mathfrak{a}_p)/W_M$. Then Lemma 3.1 implies that

$$w_1 N_M w_1^{-1} \subseteq N_p \tag{4.1}$$

and

$$w_2^{-1} N_M w_2 \subseteq N_p \ . \tag{4.2}$$

Using the Bruhat decomposition of M, we see that w_2 is in

$$\begin{aligned} MANw_1MAN &= MANw_1\Big(\bigcup_{s \in W_M} N_M s M_p A_M N_M\Big) AN \\ &= \bigcup_{s \in W_M} MAN(w_1 N_M w_1^{-1}) w_1 s M_p A_p N_p \\ &\subseteq \bigcup_{s \in W_M} MANN_p w_1 s M_p A_p N_p \quad \text{by (4.1)} \\ &= \bigcup_{s \in W_M} MANw_1 s M_p A_p N_p \ . \end{aligned}$$

Thus we can choose s in W_M, man in MAN, and $m_p a_p n_p$ in $M_p A_p N_p$ such that

$$w_2 = (man)^{-1} w_1 s m_p a_p n_p ,$$

i.e.,

$$an'mw_2 = w_1 s m_p a_p n_p \ . \qquad (4.3)$$

Applying the Bruhat decomposition theorem to m and using (4.2), we have

$$m = m_p' a_p' n_p' t n_p'' = m_p' a_p' n_p' t w_2 (w_2^{-1} n_p'' w_2) w_2^{-1}$$

for some t in W_M. Hence

$$an'mw_2 = an'm_p' a_p' n_p' t w_2 n_p''' \ . \qquad (4.4)$$

By the uniqueness part of the Bruhat decomposition theorem for G, we conclude from (4.3) and (4.4) that tw_2 and $w_1 s$ give the same element of $W(\mathfrak{a}_p)$, and the theorem follows.

5. Algebraic framework of diamonds

The Weyl group associated with the parabolic MAN is $W(\mathfrak{a}) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. Any element in $W(\mathfrak{a}_p)$ that normalizes $\mathfrak{a}$ yields an element of $W(\mathfrak{a})$ by restriction. Conversely every element of $W(\mathfrak{a})$ arises this way (see Lemma 8 of [7]). Among all members of $W(\mathfrak{a}_p)$ yielding a particular element $\bar{w}$ of $W(\mathfrak{a})$, there is a unique one w of shortest length, and it is characterized by the property that $w\gamma > 0$ for every $\mathfrak{a}_p$-root $\gamma > 0$ vanishing on $\mathfrak{a}$.

Let w_M and w_G denote the long elements of W_M and $W(\mathfrak{a}_p)$, respectively. There is an element $\bar{w}$ of $W(\mathfrak{a})$ carrying all the positive $\mathfrak{a}$-roots to negative roots if and only if w_G normalizes $\mathfrak{a}$, and in this case $w_G w_M$ is the shortest representative of $\bar{w}$ in $W(\mathfrak{a}_p)$. The element $\bar{w}$ has order 2, and hence so does $w_G w_M$. It follows that w_G commutes with w_M when $\bar{w}$ exists.

A standard parabolic is completely determined by its M component, and we shall often drop the AN in referring to it. For discussion of diamonds, fix M corresponding to a standard parabolic. We shall assume that MAN is maximal parabolic, i.e., that A has dimension one. A diamond is a diagram of parabolic subgroups and elements of $W(\mathfrak{a}_p)$ of the form in Figure 5 that respects inclusions and has $M_* = M \cap M^*$. We assume that M_* is a proper subgroup of M^*.

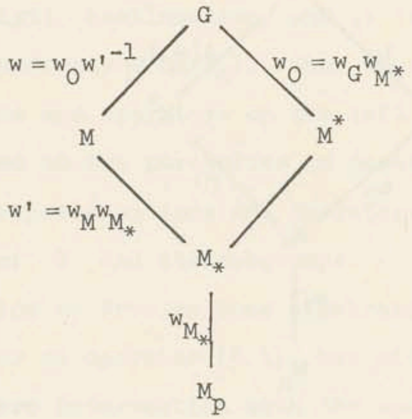


Figure 5: General diamond.

The diamond in Figure 5 is an allowable diamond if

(i) w_0 represents an element of the Weyl group of the parabolic corresponding to M^*, and

(ii) w_0 commutes with w_{M_*}.

Our intention is to use allowable diamonds to construct self-intertwining operators L for representations induced from the standard parabolic corresponding to M. A formal (divergent) expression for L is

$$Lf(x) = \int_{V \cap w^{-1}MANw} Df(xwv \downarrow)\,dv\,, \tag{5.1}$$

where D is a suitable left-invariant differential operator for which the differentiations occur in the position marked by the arrow and are transverse to the space of integration. We shall make sense out of L by introducing operators on the sides of the diamond corresponding to the equation

$$L\varphi_2 = A(w_0)D' \tag{5.2}$$

as in Figure 6.

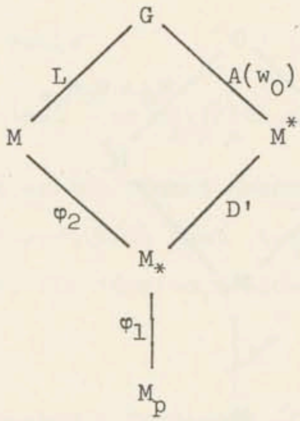


Figure 6: Operators associated with a diamond.

The operators in Figure 6 require some explanation. The diagram uses the same notation for an intertwining operator B within a subgroup M_1 and the induced operator $\widetilde{B}$ for representations $\mathrm{ind}_{M_1}^{G}(-)$ given by

$$(\widetilde{B}F)(x) = B(F(x)).$$

The operator $\varphi = \varphi_2\varphi_1$ is assumed to be the minimal decomposition of a standard unnormalized intertwining operator within M whose image is exactly the given representation ξ of M; to interpret $L\varphi_2$, we must induce φ_2 to G. The operator φ_1 occurs within M_*, and D' is an M^*-intertwining differential operator between two representations of the continuous series

$$\mathrm{ind}_{M_*}^{M^*}\,(\text{image } \varphi_1 \otimes A_* \text{ parameter}).$$

The operators D and D' will be related by the formula

$$D = \mathrm{Ad}(w')D' . \tag{5.3}$$

Finally the operator $A(w_0)$ is a standard unnormalized intertwining operator between two representations of the continuous series $\mathrm{ind}_{M^*}^{G}$ (image $D' \otimes A^*$ parameter).

Unfortunately φ_2 is usually not invertible, and the equation (5.2) does not obviously define L consistently. Moreover, $A(w_0)$ is defined by analytic continuation, and it is necessary to vary the A^* parameter to understand $A(w_0)$. When we vary this parameter, the representations and operators on the left side of the diamond do not remain attached to the parabolics in question but need to be reinterpreted as representations and operators for the nonunitary principal series of G and its subgroups.

In this section we develop some algebraic background and prove a formula (5.2) for an operator (5.1), but with domain $C^{\infty}_{com}(G)$. This domain has zero intersection with the spaces of induced representations, and the problem of altering the domain will be addressed in the next section.

Lemma 5.1. In any diamond,

(a) $w_0 = ww'$ is a minimal product, and

(b) $w_0 w_{M_*}$ is a minimal product.

Proof of (a). Assuming the contrary, suppose $\gamma > 0$ is an $\mathfrak{a}_p$-root such that $w_0 w'^{-1}\gamma < 0$ and $w'^{-1}\gamma < 0$. From $w'^{-1}\gamma < 0$, we conclude that γ is a restricted root of M. Let $\alpha = -w'^{-1}\gamma$, so that α is a positive restricted root of M. We have $w_0\alpha > 0$, and consequently α is a restricted root of M^*. Hence α is a restricted root of $M \cap M^* = M_*$, and $w_{M_*}^{-1}\alpha < 0$. Then

$$0 > w_{M_*}^{-1}\alpha = -w_{M_*}^{-1} w'^{-1}\gamma = -w_M^{-1}\gamma = -w_M\gamma ,$$

and hence $w_M\gamma > 0$. Since γ is a restricted root of M, this is a contradiction.

Proof of (b). If $\gamma > 0$ is a root vanishing on $\mathfrak{a}^*$, then $w_0\gamma > 0$ since $w_{M^*}\gamma < 0$. Hence $\gamma > 0$, $w_0\gamma < 0$, and $w_{M_*}\gamma < 0$ are impossible together, and $w_0 w_{M_*}$ is a minimal product.

Lemma 5.2. In any allowable diamond, the element w satisfies conditions (I) and (II) of §3.

Proof of (I). Suppose on the contrary that $\alpha > 0$ is an $\mathfrak{a}_p$-root with $\alpha = 0$ on $\mathfrak{a}$, $w^{-1}\alpha > 0$, and $w^{-1}\alpha \neq 0$ on $\mathfrak{a}$. Then w'^{-1} does not change the sign of $w^{-1}\alpha$, and we have $w_0^{-1}\alpha = w'^{-1}w^{-1}\alpha > 0$. Axiom (i) for allowable diamonds implies that $w_0^2 = 1$, and thus $w_0\,\alpha > 0$. Since Lemma 5.1a shows that $w_0 = ww'$ is a minimal product, we deduce that $w'\alpha > 0$. Consequently

$$w_M(-w_{M_*}\alpha) = w_M(-w_{M_*}^{-1}\alpha) = -w'\alpha < 0.$$

Since α vanishes on $\mathfrak{a}$, we obtain $-w_{M_*}\alpha > 0$, i.e., $w_{M_*}\alpha < 0$.

On the other hand, $w^{-1}\alpha > 0$ and $w^{-1}\alpha \neq 0$ on $\mathfrak{a}$ imply that $w_M^{-1}w^{-1}\alpha > 0$. By axioms (ii) and (i) for allowable diamonds,

$$w_0 w_{M_*} = w_{M_*}w_0 = w_{M_*}^{-1}w_0^{-1} = w_{M_*}^{-1}w'^{-1}w^{-1} = w_M^{-1}w^{-1}.$$

Therefore $w_0 w_{M_*}\alpha > 0$. But then we have $\alpha > 0$, $w_{M_*}\alpha < 0$, and $w_0 w_{M_*}\alpha > 0$, in contradiction to Lemma 5.1b.

Proof of (II). Suppose on the contrary that $\beta > 0$ is an $\mathfrak{a}_p$-root with $\beta = 0$ on $\mathfrak{a}$ and $w\beta < 0$. Since $w_0 = ww'$ is a minimal product, we must have $w'^{-1}\beta > 0$. On the other hand, $\beta = 0$ on $\mathfrak{a}$ implies $w_M^{-1}\beta < 0$, hence $w_{M_*}^{-1}(w'^{-1}\beta) < 0$. By Lemma 5.1b, $w_0 w_{M_*}$ is a minimal product. Hence we can conclude from $w'^{-1}\beta > 0$ and $w_{M_*}^{-1}(w'^{-1}\beta) < 0$ that $w_0 w'^{-1}\beta > 0$. But $w_0 w'^{-1} = w$, and we arrive at the contradiction $w\beta > 0$.

Lemma 5.3. In any diamond suppose β and γ are positive $\mathfrak{a}_p$-roots with the properties that $\beta \neq 0$ on $\mathfrak{a}$, $w'^{-1}\gamma < 0$, and $\beta - \gamma$ is a positive $\mathfrak{a}_p$-root. Then $w(\beta - \gamma) < 0$.

Proof. The facts that $\gamma > 0$, $w'^{-1}\gamma < 0$, and $w_0 = ww'$ is a minimal product together imply that $w\gamma > 0$. Thus $w\beta < 0$ would imply $w(\beta - \gamma) < 0$. So we may assume in the proof that $w\beta > 0$.

Under the correspondence $\alpha = w_M\epsilon$, we observe that

$$\alpha > 0,\ \alpha \neq 0 \text{ on } \mathfrak{a},\ w\alpha > 0$$

$$\text{if and only if } \epsilon > 0,\ w_M\epsilon > 0,\ ww_M\epsilon > 0,$$

$$\text{if and only if } \epsilon > 0,\ w_M\epsilon > 0,\ w_{M^*}w_{M_*}\epsilon < 0. \tag{5.4}$$

In fact, the first of these equivalences is clear. To see the second equivalence, we write

$$ww_M = ww'w_{M_*} = w_0w_{M_*} = w_Gw_{M^*}w_{M_*} \tag{5.5}$$

and use the fact that $-w_G$ preserves positivity of roots.

If the lemma fails, then β and $\beta - \gamma$ are both positive roots nonvanishing on $\mathfrak{a}$ such that $w\beta > 0$ and $w(\beta - \gamma) > 0$. We can apply the above observation. Putting $\gamma' = -w_M^{-1}\gamma$, we obtain $\epsilon = w_M^{-1}\beta$ and $\epsilon' = w_M^{-1}(\beta - \gamma) = \epsilon + \gamma'$ satisfying

$$\epsilon > 0,\ w_M\epsilon > 0,\ w_{M^*}w_{M_*}\epsilon < 0 \tag{5.6}$$

$$\epsilon' > 0,\ w_M\epsilon' > 0,\ w_{M^*}w_{M_*}\epsilon' < 0. \tag{5.7}$$

In (5.6), $\epsilon > 0$ and $w_M\epsilon > 0$ imply $w_{M_*}\epsilon > 0$. Hence $\epsilon'' = w_{M_*}\epsilon$ is a positive $\mathfrak{a}_p$-root such that $w_{M^*}\epsilon'' < 0$.

Let us study $\gamma' = -w_M^{-1}\gamma$. Since $\gamma > 0$ and $w'^{-1}\gamma < 0$, γ is 0 on $\mathfrak{a}$. Then $\gamma > 0$ and $\gamma = 0$ on $\mathfrak{a}$ imply γ' is > 0. From the first paragraph of the proof and from (5.5), we have

$$0 < w\gamma = -ww_M\gamma' = -w_G w_{M^*}(w_{M_*}\gamma')$$

and hence $w_{M^*}(w_{M_*}\gamma') > 0$. Also by hypothesis in the lemma,

$$w_{M_*}\gamma = -w_{M_*}w_M^{-1}\gamma = -w'^{-1}\gamma > 0 .$$

Therefore $\gamma'' = w_{M_*}\gamma'$ is a positive $\mathfrak{a}_p$-root such that $w_{M^*}\gamma'' > 0$.

Recall that $\epsilon'' = w_{M_*}\epsilon$ is a positive $\mathfrak{a}_p$-root such that $w_{M^*}\epsilon'' < 0$. Since $\mathfrak{a}^*$ is ordered before $\mathfrak{a}_{M^*}$, we conclude $w_{M^*}(\epsilon'' + \gamma'') > 0$. But

$$\epsilon'' + \gamma'' = w_{M_*}(\epsilon + \gamma') = w_{M_*}\epsilon' ,$$

and $w_{M^*}(\epsilon'' + \gamma'') > 0$ therefore contradicts (5.7). This contradiction finishes the proof.

<u>Lemma 5.4</u>. In any diamond suppose α and β are positive $\mathfrak{a}_p$-roots such that α and β are $\neq 0$ on $\mathfrak{a}$, $\alpha + \beta$ is a positive $\mathfrak{a}_p$-root, $w\alpha$ is > 0, and $w\beta$ is < 0. Then $w(\alpha+\beta) < 0$.

Proof. Define $\epsilon = w_M^{-1}\alpha$ and $\epsilon' = w_M^{-1}\beta$, and run through the proof of (5.4) to conclude

$$\epsilon > 0,\ w_M\epsilon > 0,\ w_{M^*}w_{M_*}\epsilon < 0$$

$$\epsilon' > 0,\ w_M\epsilon' > 0,\ w_{M^*}w_{M_*}\epsilon' > 0$$

$$\epsilon + \epsilon' \text{ is a positive } \mathfrak{a}_p\text{-root.}$$

From $w_M\epsilon > 0$ we obtain $w_{M_*}\epsilon > 0$. Similarly $w_{M_*}\epsilon' > 0$. Thus $w_{M_*}\epsilon$ and $w_{M_*}\epsilon'$ are positive, and $w_{M^*}(w_{M_*}\epsilon)$ is < 0 while $w_{M^*}(w_{M_*}\epsilon)$ is > 0. Since $\mathfrak{a}^*$ is ordered before $\mathfrak{a}_{M^*}$, we conclude

$$w_{M^*}(w_{M_*}\epsilon + w_{M_*}\epsilon') > 0.$$

That is, $\epsilon + \epsilon'$ satisfies

$$\epsilon + \epsilon' > 0,\ w_M(\epsilon + \epsilon') > 0,\ w_{M^*}w_{M_*}(\epsilon + \epsilon') > 0 .$$

Running through the proof of (5.4) again, we find that $\alpha + \beta = w_M(\epsilon + \epsilon')$ satisfies

$$\alpha + \beta > 0,\ \alpha + \beta \neq 0 \text{ on } \mathfrak{a},\ w(\alpha + \beta) < 0.$$

This conclusion proves the lemma.

Lemma 5.5. In any allowable diamond, let $\mathfrak{s}_1$ be the Lie algebra generated by $\mathfrak{a}_p$-root vectors $X_{-\alpha}$ with $\alpha > 0$, $\alpha \neq 0$ on $\mathfrak{a}$, and $w\alpha > 0$, and let $\mathfrak{s}_2$ be the Lie algebra generated by $\mathfrak{a}_p$-root vectors $X_{-\alpha}$ with $\alpha > 0$ and $w\alpha < 0$. Let D be in the universal enveloping algebra $\mathfrak{U}(\mathfrak{s}_1)$.

(a) If u is in the analytic subgroup whose Lie algebra is spanned by $\mathfrak{a}_p$-root vectors $Y_{-\beta}$ with $\beta > 0$ and $w'\beta < 0$, then

$$Ad(w'uw'^{-1})D \in D + \mathfrak{s}_2\,\mathfrak{U}(\mathfrak{v}).$$

(b) If v is in the analytic subgroup with Lie algebra $\mathfrak{s}_2$, then

$$Ad(v)D \in D + \mathfrak{U}(\mathfrak{v})\,\mathfrak{s}_2 .$$

Remarks. Since the diamond is allowable, the space of integration of L is

$$V \cap w^{-1}MANw = V_p \cap w^{-1}N_pw$$

by (3.1). One assumption is that the differential operator D is built from root vectors for V that are transverse to the space of integration. Conclusion (a) is that $Ad(w'uw'^{-1})D$ is the sum of D

and terms having at least one derivative parallel to the space of integration.

Proof of (a). The element $w'uw'^{-1}$ lies in the analytic subgroup with Lie algebra spanned by $\mathfrak{a}_p$-root vectors X_γ with $\gamma > 0$ and $w'^{-1}\gamma < 0$. We first observe that such an X_γ satisfies

$$[X_\gamma, \mathfrak{s}_1] \subseteq \mathfrak{s}_2$$
$$[X_\gamma, \mathfrak{s}_2] \subseteq \mathfrak{s}_2 . \qquad (5.8)$$

In fact, choose a generating vector $X_{-\alpha}$ for $\mathfrak{s}_1$ or $\mathfrak{s}_2$. Then $\alpha > 0$ and $\alpha \neq 0$ on $\mathfrak{a}$, since w satisfies (II) by Lemma 5.2. If $\alpha - \gamma$ is an $\mathfrak{a}_p$-root, it has to be positive since γ must vanish on $\mathfrak{a}$. Hence Lemma 5.3 says $w(\alpha - \gamma) < 0$, and $[X_\gamma, X_{-\alpha}]$ is in $\mathfrak{s}_2$.

Now expand $\mathrm{Ad}(w'uw'^{-1})$ in exponential series and apply each term to D. The zeroth order term gives D. The other terms, in view of (5.8), give a sum of monomials, all of whose factors are from $\mathfrak{s}_1$ or $\mathfrak{s}_2$ and one or more of whose factors are from $\mathfrak{s}_2$. We can commute an $\mathfrak{s}_2$ factor to the left end of each monomial if $[\mathfrak{s}_1, \mathfrak{s}_2] \subseteq \mathfrak{s}_2$, and this is the case by Lemma 5.4. Also $\mathfrak{s}_1 + \mathfrak{s}_2 = \mathfrak{v}$ since w satisfies (II).

Proof of (b). Expand $\mathrm{Ad}(v)$ in exponential series and apply each term to D. The argument in the previous paragraph can be repeated to yield (b) since $[\mathfrak{s}_1, \mathfrak{s}_2] \subseteq \mathfrak{s}_2$.

<u>Proposition 5.6</u>. In an allowable diamond, if φ_2 and $A(w_0)$ are defined on $C^\infty_{com}(G)$ by

$$\varphi_2 h(x) = \int_{V_p \cap w'^{-1} N p w'} h(xw'u)\,du$$

and

$$A(w_0)F(x) = \int_{V_p \cap w_0^{-1} N_p w_0} F(xw_0\tilde{v})\,d\tilde{v} ,$$

then L is well-defined on the image of φ_2 by (5.1), and $L\varphi_2 = A(w_0)D'$ holds if D' is defined by $D = \mathrm{Ad}(w')D'$.

Proof. We compute

$$A(w_0)D'h(x) = \int_{V_p \cap w_0^{-1} N_p w_0} D'h(xw_0\tilde{v}\downarrow)\,d\tilde{v}$$

$$= \int_{u \in V_p \cap w'^{-1} N_p w'} \left\{ \int_{v' \in V_p \cap w_0^{-1} N_p w_0 \cap w'^{-1} V_p w'} D'h(xw_0 v' u\downarrow)\,dv' \right\} du$$

since $w_0 = ww'$ is minimal. Under the change of variables $v' = w'^{-1} v w'$ the expression in braces is

$$= \int_{V_p \cap w^{-1} N_p w} D'h(xwvw'u\downarrow)\,dv$$

$$= \int_{V_p \cap w^{-1} N_p w} \mathrm{Ad}(w'uw'^{-1})\,\mathrm{Ad}(w')D'h(xwv\downarrow w'u)\,dv$$

$$= \int_{V_p \cap w^{-1} N_p w} \mathrm{Ad}(w'uw'^{-1})\,Dh(xwv\downarrow w'u)\,dv\,. \qquad (5.9)$$

We apply Lemma 5.5a. In the error terms the $\mathcal{S}_2$ contribution can be absorbed into the v, by a change of variables; hence the error terms contribute nothing to the integral. Thus (5.9) reduces to

$$= \int_{V_p \cap w^{-1} N_p w} Dh(xwv\downarrow w'u)\,dv\,,$$

and the space of integration can be rewritten as $V \cap w^{-1}MANw$ by (3.1). This completes the proof.

We conclude this section with some remarks about the role of double cosets in this construction. Any diamond determines an MAN double coset in G, namely the double coset to which w_0 and w

belong. If the diamond is allowable, then w has a characterization independent of the diamond by Lemma 5.2, Proposition 3.6, and Theorem 4.1. In this case we can define L and relate it to the double coset; Proposition 5.6 is predicting that when L is rigorously defined later, the distribution associated with L will be attached to the double coset of w and will involve transverse derivatives determined by D.

Examples show that the map of allowable diamonds to double cosets need not be one-one, but the fact that w is canonical should be expected to mean that allowable diamonds corresponding to the same double coset should yield the same operators L. Other examples show that the map of diamonds to double cosets is not onto, even if all diamonds are used.

6. Analytic framework of diamonds

Fix a maximal standard parabolic subgroup MAN, an irreducible unitary representation ξ of M, and an allowable diamond with notation as in §5. In giving a rigorous construction of a self-intertwining operator L for $\mathrm{ind}_M^G(\xi \otimes e^0)$ compatible with the algebraic framework of §5, one encounters the analytic problems listed below. Some of these problems are only partially solved, and we shall omit our partial solutions.

1. Introduce a parameter λ in the $\mathfrak{a}^*$ direction, with $\lambda = 0$ to correspond to the situation of interest. Construct for each λ with $\mathrm{Re}\,\lambda$ sufficiently large a holomorphic equation of G-intertwining operators

$$L_\lambda \varphi_{2,\lambda} = A_\lambda(w_0)D'$$

for representations induced from the minimal parabolic at the corresponding parameter values. Continue the operators and equation meromorphically to all λ. This problem is solved under general hypotheses, and the solution is given in Theorem 6.1.

2. Find a criterion for deciding when $A_\lambda(w_0)D'$ is holomorphic at $\lambda = 0$. This problem seems to be the heart of the matter.

3. When $A_\lambda(w_0)D'$ is holomorphic at $\lambda = 0$, show that L_0 can be defined on the image of $\varphi_{2,0}$.

4. When $A_\lambda(w_0)D'$ is holomorphic at $\lambda = 0$ and L_0 is defined, show that the expression for L_0 is given by (5.1). (This equation captures the support of the distribution corresponding to L_0 and is a step toward handling linear independence of intertwining operators.) This problem is solved at the same time as the first problem.

5. When $A_\lambda(w_0)D'$ is holomorphic at $\lambda = 0$ and L_0 is defined, show that the image space of L_0 transforms appropriately under the group MAN.

The solution to Problems 1 and 4 is contained in the following theorem.

Theorem 6.1. In the above context, suppose that

(i) ξ imbeds as a quotient at parameters (σ, Λ_0) of the nonunitary principal series of M, with Re Λ_0 in the open positive Weyl chamber of M,

(ii) $w_{M^*}w_{M_*}$ represents an element (necessarily -1) of the Weyl group of the parabolic corresponding to M_* in M^*,

(iii) if $\Lambda_0 = \mu_0 + \mu_1$ is the orthogonal decomposition according to $\mathfrak{m} = \mathfrak{m}_* \oplus \mathfrak{m}_*^\perp$, if $\mu_1 = -\mu + \mu^*$ is the orthogonal decomposition according to $\mathfrak{a}_{M^*} \oplus \mathfrak{a}^*$, and if ξ_0 denotes the Langlands quotient

$$\xi_0 = A_{M_p}^{M_*}(w_{M_*}, \sigma, \mu_0)[\mathrm{ind}_{M_p}^{M_*}(\sigma \otimes e^{\mu_0})],$$

then D' is an intertwining differential operator between representations of M^*, namely

$$D' : \mathrm{ind}_{M_*}^{M^*}(\xi_0 \otimes e^{-\mu}) \to \mathrm{ind}_{M_*}^{M^*}(\xi_0 \otimes e^{\mu}) .$$

Reinterpret D' as an intertwining operator on the G level by the formula $(D'f)(g) = D'(f(g))$. The following things then happen:

(a) any function f in the domain of the reinterpreted D', i.e., in

$$\mathrm{ind}_{M^*}^{G}(\mathrm{ind}_{M_*}^{M^*}(\xi_0 \otimes e^{-\mu}) \otimes e^{\mu^*+\lambda}), \tag{6.1}$$

can be identified with a member of

$$\mathrm{ind}_{M_p}^{G}(\sigma \otimes \exp w_{M_*}(\Lambda_0 + \lambda)) . \tag{6.2}$$

(b) under this identification if f is a K-finite member of the space (6.1) and if $\mathrm{Re}\,\lambda$ is sufficiently far out in the positive Weyl chamber of $\mathfrak{a}^*$, then

$$(\mathrm{Ad}(w_0)D')A_{M_p}^{G}(w, w_M\sigma, w_M(\Lambda_0 + \lambda))A_{M_p}^{G}(w', w_{M_*}\sigma, w_{M_*}(\Lambda_0 + \lambda))f$$

$$= A_{M^*}^{G}(w_0, \text{image } D', \mu^* + \lambda)D'f$$

with each factor a G-intertwining operator holomorphic in λ.

(c) if

$$L_\lambda = (\mathrm{Ad}(w_0)D')A_{M_p}^{G}(w, w_M\sigma, w_M(\Lambda_0 + \lambda))$$

$$\varphi_{2,\lambda} = A_{M_p}^{G}(w', w_{M_*}\sigma, w_{M_*}(\Lambda_0 + \lambda))$$

$$A_\lambda(w_0) = A_{M^*}^{G}(w_0, \text{image } D', \mu^* + \lambda) ,$$

then L_λ is given for large $\mathrm{Re}\,\lambda$ by the convergent expression (5.1) with $D = \mathrm{Ad}(w')D'$, and the equation

$$L_\lambda \varphi_{2,\lambda} = A_\lambda(w_0)D'$$

extends to a meromorphic identity for all λ.

Remark. The reader is invited to identify all these parameters in the case of $SU(3,3)$ with those displayed in Figures 3 and 4.

Sketch of proof. Conclusion (a) follows by tracking down the double inductions and imbeddings as subrepresentations that are involved. For (b), the two standard intertwining operators on the left are a minimal product and collapse to a single operator $A^G_{M_p}(w_0, w_{M_*}\sigma, w_{M_*}(\Lambda_0+\lambda))$. One checks that the domains and ranges of the various operators are compatible, and then matters reduce to the integral formula

$$\int_{V_p \cap w_0^{-1}N_p w_0} (\mathrm{Ad}(w_0)D')f(x \downarrow w_0\tilde{v})\,d\tilde{v} = \int_{V_p \cap w_0^{-1}N_p w_0} D'f(xw_0\tilde{v}\downarrow)\,d\tilde{v}$$

with both sides known to be convergent. The proof of the integral formula is similar to the proof of Proposition 5.6 and relies on Lemma 5.5. In (c), each factor extends to be meromorphic by [8], and the formula (5.1) for L_λ follows by use of Lemma 5.5b.

7. Further examples

We shall give six examples in which our philosophy of the diamond is partly or wholly applicable to explain known phenomena.

1. $Sp(n,\mathbb{R})$, n odd. The reducibility in this example was studied by Kashiwara and Vergne [6]. The $\mathfrak{a}_p$-roots form a system of

type C_n, and standard notation for the simple roots is

$$e_1 - e_2,\ e_2 - e_3,\ \ldots,\ e_{n-1} - e_n,\ 2e_n\ .$$

Let

$$M = M_{e_1-e_2,\, e_2-e_3,\, \ldots,\, e_{n-1}-e_n}\ . \tag{7.1}$$

The group M has two components, and we take ξ to be 1 on the identity component, $(-1)^{\frac{1}{2}(n+1)}$ on the other component. We study $\mathrm{ind}_M^G(\xi \otimes e^0)$. By [6], the commuting algebra has dimension $\frac{1}{2}(n+3)$.

The standard intertwining operator has a regular normalizing factor, and so the standard operator and the identity give two independent members of the commuting algebra. We can form n-1 allowable diamonds as follows: Fix k with $1 \leq k \leq n-1$ and let

$$M^* = M_{e_{k+1}-e_{k+2},\, \ldots,\, e_{n-1}-e_n,\, 2e_n} \tag{7.2}$$

$$w_0 = p_{2e_1} p_{2e_2} \cdots p_{2e_k} \tag{7.3}$$

$$M_* = M_{e_{k+1}-e_{k+2},\, \ldots,\, e_{n-1}-e_n}\ .$$

It is clear that the resulting diamonds are allowable.

At first it seems that we will obtain one new operator for each diamond and end up with $(n+1)$-fold reducibility, but something goes wrong when we search for D'. To see what happens, we need the parameters that are involved.

The group M^* is essentially $Sp(n-k, \mathbb{R})$, and M_* has two components. The representation ξ imbeds in the nonunitary principal series with $\mathfrak{a}_p$ parameter $-\rho_M$, and the $\mathfrak{a}_p$ parameter at the M_p position of the diamond is therefore

$$\rho_M = \tfrac{1}{2}(n-1) + \tfrac{1}{2}(n-3)e_2 + \ldots + \tfrac{1}{2}(1-n)e_n. \tag{7.4}$$

The standard intertwining operator for w_{M_*} has image the representation

$$\sigma = \begin{cases} +1 & \text{on identity component of } M_* \\ (-1)^{\frac{1}{2}(n+1)} & \text{on other component of } M_*, \end{cases}$$

and the continuous series representation upon which D' is to act is

$$\mathrm{ind}_{M_*}^{M^*}(\sigma \otimes e^{-\mu}),$$

where

$$-\mu = -\tfrac{1}{2}k(e_{k+1} + \ldots + e_n). \tag{7.5}$$

The operator D' must be fixed by $\mathrm{Ad}(M_*)$, and its homogeneity is dictated by μ. If k is odd, no D' of the appropriate homogeneity can be fixed even by $\mathrm{Ad}(M_p)$. On the other hand, if k is even, a determinant-like differential intertwining operator is available through the work of Jakobsen [4]. We take the operator D' to be

$$(\det \partial)^{\frac{1}{2}k} : \mathrm{ind}_{M_*}^{M^*}(\sigma \otimes e^{-\mu}) \to \mathrm{ind}_{M_*}^{M^*}(\sigma \otimes e^{\mu}).$$

Thus allowable diamonds indicate the existence of $\frac{1}{2}(n-1)$ operators L, and we should expect the commuting algebra to have dimension $\frac{1}{2}(n+3)$, in agreement with the result of [6].

2. <u>SU(n,n)</u>. The reducibility in this example was studied by Kashiwara and Vergne [6], and our treatment of it is a direct generalization of Figures 3 and 4. The $\mathfrak{a}_p$-roots form a system of type C_n, as with $Sp(n,\mathbb{R})$, and we take M to be as in (7.1). The group M has two components, and we take ξ to be 1 on the identity component, $(-1)^n$ on the other component.

Then M^* and w_0 are as in (7.2) and (7.3), but this time ρ_M is twice what it is in (7.4). Hence μ is twice what it is in (7.5), and the result is that we can find an operator D' for each of the $n-1$ allowable diamonds.[3] The standard intertwining operator has a regular normalizing factor, and thus we should expect the commuting algebra to have dimension $n+1$, in agreement with the result of [6].

3. $\widetilde{SO}_e(n,2)$, n even. The operators in this case were constructed by R. Strichartz, in answer to a question from us before we introduced diamonds. The $\mathfrak{a}_p$-roots form a system of type B_2. Let $M = M_{e_2}$ and let ξ be 1 on the identity component, $(-1)^{n/2}$ on the other component. There is just one choice for M^*, namely $M_{e_1-e_2}$, which is essentially $SL(2,\mathbb{R})$. The operator D' can be taken as $(X_{-(e_1-e_2)})^{\frac{1}{2}(n-2)}$. The analysis proceeds as with $SU(2,2)$. The resulting operator L, the standard intertwining operator, and the identity span a space of dimension 3 in the commuting algebra, and this space of operators is complete by an argument using the non-compact picture and Fourier transforms.

4. $SU(3,2)$. Pichet [11] studied certain analytic continuations of discrete series of G when G/K is Hermitian, and produced a number of examples of reducibility as a result. We shall interpret the reducibility for $SU(3,2)$ by means of diamonds. (This example has the property that D' is defined for an induced series in which the N group is nonabelian.) The $\mathfrak{a}_p$-roots form a system of type $(BC)_2$, and we let $M = M_{e_1-e_2}$. The group M is then the product $SL(2,\mathbb{C}) \times T$, where T is a circle, and we take $\xi = 1 \otimes e^{i\theta}$. Then $M^* = M_{e_2, 2e_2}$ is the product of a circle and $SU(2,1)$, and the

---

[3] For parameter k, $1 \leq k \leq n-1$, D' is $(\det \partial)^k$.

differential operator D' within $SU(2,1)$ is to carry the nonunitary principal series with parameters $(\theta, -e_2)$ to the one with parameters (θ, e_2). Within $SU(2,1)$ let $\{X_{-e_2}, Y_{-e_2}\}$ be an orthonormal basis of the $\mathfrak{a}_p$-root space for $-e_2$, and let X_{-2e_2} be in the space for $-2e_2$. Then

$$D' = (X^2_{-e_2} + Y^2_{-e_2}) + cX_{-2e_2}$$

has the required properties for a suitable choice of the real number c, and $A_\lambda(w_0)D'$ is holomorphic at $\lambda = 0$. See [13] for generalization of this D'.

5. $\underline{Sp(2,\mathbb{C})}$. The reducibility in this example was discovered by K. Gross [3] and was studied later by Duflo [2]. The $\mathfrak{a}_p$-roots form a system of type C_2, and we let ξ be the trivial representation of $M = M_{2e_2}$. The standard intertwining operator has a pole but does not become the identity upon being normalized; hence it exhibits reducibility. But the normalized operator is difficult to understand directly. By means of a diamond, we can construct a self-intertwining operator L that requires no normalization. Namely, we use $M^* = M_{e_1-e_2} \cong SL(2,\mathbb{C})$, and we take $D' = X^2_{-(e_1-e_2)} + Y^2_{-(e_1-e_2)}$ in the obvious notation. One sees that $A_\lambda(w_0)$ is convergent, hence holomorphic at $\lambda = 0$, by using the fact that the image of D' is tempered.

6. $\underline{SL(4,\mathbb{R})}$. This example was brought to our attention by B. Speh. Let $M = M_{e_1-e_2, e_3-e_4}$ consist of two 2-by-2 blocks, and let ξ be 1 on the identity component, -1 on the other component. The standard intertwining operator has a pole but does not become the identity upon being normalized. Reducibility is into two pieces, and a diamond gives a realization of a nonscalar intertwining operator that does not require normalization. The group M^* is $M_{e_2-e_3}$, and D' is $X_{-(e_2-e_3)}$.

References

[1] F. Bruhat, Sur les représentations induites des groupes de Lie, Bull. Soc. Math. France 84 (1956), 97-205.

[2] M. Duflo, Représentations unitaires irréductibles des groupes simples complexes de rang deux, Bull. Soc. Math. France 107 (1979), 55-96.

[3] K. Gross, Restriction to a parabolic subgroup and irreducibility of degenerate principal series of Sp(2,C), Bull. Amer. Math. Soc. 76 (1970), 1281-1285.

[4] H. P. Jakobsen, Intertwining differential operators for Mp(n,R) and SU(n,n), Trans. Amer. Math. Soc. 246 (1978), 311-337.

[5] H. P. Jakobsen and M. Vergne, Wave and Dirac operators, and representations of the conformal group, J. Func. Anal. 24 (1977) 52-106.

[6] M. Kashiwara and M. Vergne, Functions on the Shilov boundary of the generalized half plane, Non-Commutative Harmonic Analysis, Springer-Verlag Lecture Notes in Math. 728 (1979), 136-176.

[7] A. W. Knapp, Weyl group of a cuspidal parabolic, Ann. Sci. École Norm. Sup. 8 (1975), 275-294.

[8] A. W. Knapp and E. M. Stein, Intertwining operators for semisimple groups II, Inventiones Math. (1980).

[9] R. Kunze and E. M. Stein, Uniformly bounded representations III, Amer. J. Math. 89 (1967), 385-442.

[10] R. P. Langlands, On the classification of irreducible representations of real algebraic groups, mimeographed notes, Institute for Advanced Study, Princeton, New Jersey, 1973.

[11] C. Pichet, Unitary representations of simple Lie groups associated with the holomorphic discrete series, thesis, Rutgers University, 1978.

[12] B. Speh, Degenerate series representations of the universal covering group of SU(2,2), J. Func. Anal. 33 (1979), 95-118.

[13] N. R. Wallach, The analytic continuation of the discrete series II, Trans. Amer. Math. Soc. 251 (1979), 19-37.

Department of Mathematics
Cornell University
Ithaca, New York 14853

Department of Mathematics
Princeton University
Princeton, New Jersey 08544