# IRREDUCIBLE UNITARY REPRESENTATIONS OF SOME GROUPS OF REAL RANK TWO

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We have been collecting data on the unitary duals of various linear connected semisimple Lie groups in an effort to find out whether it is reasonable to have a simply-stated explicit classification for all such groups. For groups of real rank one, Baldoni-Silva and Barbasch [2] obtained an explicit classification, and our paper [5], when specialized to these groups, shows how that classification can be stated simply.

Our concern here is with simple groups $G$ of real rank two. We prefer to think of these as divided into two classes, those with $\operatorname{rank} G = \operatorname{rank} K$ (for $K$ maximal compact) and those with $\operatorname{rank} G > \operatorname{rank} K$. Within each class, some of the groups appear to us as "regular cases," some are variants of regular cases, and some are exceptions.

For $\operatorname{rank} G = \operatorname{rank} K$, the regular cases are the "single-line" cases: $SU(n,2)$, $\widetilde{SO}(2n,2)$, $SO^*(10)$, and $E_{6(-14)}$. The groups $Sp(n,2)$ may be viewed as variants of $SU(n,2)$, while the various $\widetilde{SO}(2n+1,2)$ are variants of $\widetilde{SO}(2n,2)$. The group $G_2^{\mathbb{R}}$ is exceptional. We gave a classification for $SU(n,2)$ in [4]. Angelopoulos [1] announced a classification for $\widetilde{SO}(2n,2)$ and $\widetilde{SO}(2n+1,2)$, but we are unable to relate his results to Langlands parameters nor do we have enough details to check his results. Thus we have recently obtained our own classification (unpublished) for these groups. Our methods appear to handle $Sp(n,2)$ as well, but they are insufficient for

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$SO^*(10)$ and $E_{6(-14)}$. The classifications for all these groups have a qualitative similarity to them, but a detailed statement of the classification requires (at present) treatment of one class at a time. In short, we find these classifications discouraging.

The situation is nicer for rank $G >$ rank $K$. The regular cases are those with one conjugacy class of Cartan subgroups and with $A_2$ as restricted root diagram: $SL(3,\mathbb{C})$, $SL(3,\mathbb{H})$, and $E_{6(-26)}$ (which we regard as $SL(3,\mathbb{O})$ with $\mathbb{O} = \{\text{Cayley numbers}\}$). The group $SL(3,\mathbb{R})$ is a variant of $SL(3,\mathbb{C})$, while $Sp(2,\mathbb{C})$ and $G_2^{\mathbb{C}}$ are exceptional. Classifications were done by Tsuchikawa [18] for $SL(3,\mathbb{C})$, Vogan [23] for $SL(3,\mathbb{H})$, Vahutinskii [19] for $SL(3,\mathbb{R})$, and Duflo [6] for $Sp(2,\mathbb{C})$ and $G_2^{\mathbb{C}}$. Our objective in this paper will be to complete the classifications for the real rank two groups with rank $G >$ rank $K$ by doing $E_{6(-26)}$. In doing so, we shall work with an abstract group with one conjugacy class of Cartan subgroups and with $A_2$ as restricted root diagram. This refusal to use explicit knowledge of $E_{6(-26)}$ is in line with our desire to have a simple final classification.

Turning to the precise statement of our result, we begin with notation and background. We let $G$ be linear connected simple, $K$ be maximal compact, and $G = KAN$ be an Iwasawa decomposition. For any subgroup we denote the Lie algebra by the corresponding lower-case German letter. We assume that $G$ has just one conjugacy class of Cartan subgroups, that $\dim A = 2$, and that the restricted roots (the roots of $(\mathfrak{g},\mathfrak{a})$) form a root system of type $A_2$. Let $M = Z_K(A)$ be the (compact) centralizer of $A$ in $K$, so that $P = MAN$ is a minimal parabolic subgroup of $G$. For $\sigma$ an irreducible (finite-dimensional) representation of $M$ and $\nu$ in $(\mathfrak{a}')^{\mathbb{C}}$, the representation $U(P,\sigma,\nu)$ given by normalized induction as

$$U(P,\sigma,\nu) = \operatorname{ind}_P^G(\sigma \otimes e^{\nu} \otimes 1)$$

is a member of the <u>nonunitary principal series</u>. If $\nu$ has the additional property that $\mathrm{Re}\,\nu$ is in the closed positive Weyl chamber of $\mathfrak{a}'$ relative to $N$, then it follows from Langlands [16], Miličić [17], and Knapp [9] for this kind of $G$ that $U(P,\sigma,\nu)$ has a unique irreducible quotient $J(P,\sigma,\nu)$, which is known as the <u>Langlands quotient</u>.

The representations $J(P,\sigma,\nu)$ exhaust the candidates for irreducible unitary representations, and the classification problem is to decide which of them are infinitesimally unitary. From [14] and [12], it is known that $J(P,\sigma,\nu)$ admits a nonzero invariant Hermitian form (on its K-finite vectors) if and only if the following <u>formal symmetry</u> condition holds: there exists $w$ in the normalizer $N_K(A)$ such that $w^2$ is in $M$, $w\sigma \cong \sigma$, and $w\nu = -\bar{\nu}$. This form lifts to a form $\langle\cdot,\cdot\rangle$ on $U(P,\sigma,\nu)$ that is given by an explicit intertwining operator $\mathcal{A}$ on $L^2(K)$ : $\langle f,g\rangle = (\mathcal{A}f,g)_{L^2(K)}$. Moreover, $J$ is infinitesimally unitary if and only if $\langle\cdot,\cdot\rangle$ is semidefinite, if and only if $\mathcal{A}$ is semidefinite. By a theorem of Vogan (see Theorem 16.10 of [11]), it is enough to decide the unitarizability for $\nu$ real.

Let us denote the simple restricted roots by $\alpha_R' = e_1 - e_2$ and $\alpha_R'' = e_2 - e_3$. The Weyl group $W(A:G) = N_K(A)/M$ is the symmetric group on three letters, and the only Weyl group element of order two sending a real element $\nu \neq 0$ in the closed positive Weyl chamber into its negative is the reflection $s_{\alpha_R}$ in the sum $\alpha_R = \alpha_R' + \alpha_R'' = e_1 - e_3$. The set of $\nu$'s to study is therefore the one-dimensional set $\nu = c\alpha_R$ with $c \geq 0$. Let $w_{12}$, $w_{23}$, and $w_{13}$ be representatives in $K$ of the Weyl group elements $s_{\alpha_R'}$, $s_{\alpha_R''}$, and $s_{\alpha_R}$. The formal symmetry condition $w_{13}\sigma \cong \sigma$ imposes a certain nontrivial condition on $\sigma$ that we consider later.

The group $M$ is compact and connected, and thus $\sigma$ is given by the theory of the highest weight. Let $\mathfrak{b} \subseteq \mathfrak{m}$ be a maximal abelian subspace, so that $\mathfrak{b} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta_M^+$

be a positive system for the root system $\Delta_M = \Delta(\mathfrak{m}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$, let $\lambda$ be the highest weight of $\sigma$, and let $\delta_M$ be half the sum of the members of $\Delta_M^+$, so that $\lambda_0 = \lambda + \delta_M$ is the infinitesimal character of $\sigma$.

Let $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, (\mathfrak{b} \oplus \mathfrak{a})^{\mathbb{C}})$ be the root system of $\mathfrak{g}$; we can regard $\Delta_M$ as the set of members of $\Delta$ that vanish on $\mathfrak{a}$. We introduce a positive system $\Delta^+$ for $\Delta$ containing $\Delta_M^+$ so that $\lambda$ is $\Delta^+$ dominant and $i\mathfrak{b}$ comes before $\mathfrak{a}$. (For example, we can use the lexicographic order obtained by adjoining an orthogonal basis of $\mathfrak{a}'$ at the end of an orthogonal basis of $i\mathfrak{b}'$ that starts with $\lambda$.)

Let $L = L(\sigma)$ be the analytic subgroup of $G$ containing $\mathfrak{b} \oplus \mathfrak{a}$ and corresponding to the set $\Delta^L$ of all roots $\beta$ in $\Delta$ with $\langle \lambda, \beta \rangle = 0$. Since $\lambda$ is dominant, $\Delta^L$ is generated by $\Delta^+$ simple roots. The group $L$ is another real rank two group, though not necessarily simple, and it has an Iwasawa decomposition $L = (K \cap L)A(N \cap L)$. Let $\rho_L$ be half the sum of its positive restricted roots, counting multiplicities; $\rho_L$ is a positive multiple of $\alpha_R = e_1 - e_3$.

Main Theorem. Let $G$ be linear connected simple of real rank two with just one conjugacy class of Cartan subgroups and with restricted root diagram of type $A_2$. Let $\sigma$ be an irreducible representation of $M$ such that $w_{13}\sigma \cong \sigma$, let $L = L(\sigma)$, and let $L_{ss}$ be the semisimple part of $L$. Then $L_{ss}$ has real rank one or two. Moreover, for $\nu$ real in the closed positive Weyl chamber, $J(P,\sigma,\nu)$ is infinitesimally unitary if and only if $\nu = c\rho_L$ with

$$\begin{cases} 0 \leq c \leq 1 & \text{if } L_{ss} \text{ has real rank one} \\ 0 \leq c \leq \frac{1}{2} \text{ or } c = 1 & \text{if } L_{ss} \text{ has real rank two.} \end{cases} \tag{0.1}$$

Remarks. The classification of real groups shows that $L_{ss}$ is locally $SL(2)$ over $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ if $L_{ss}$ has real rank one; alternatively $L_{ss}$ is locally $SO(n,1)$ for $n = 3$, $5$, or $9$ if $L_{ss}$ has real rank one. If $L_{ss}$ has real rank two, $L_{ss}$ is locally

$SL(3)$ over $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. The statement of the theorem is that the unitarity for the series for $\sigma$ in $G$ is the same as the unitarity for the series for $1$ in $L_{ss}$, which is given by the simple formula (0.1).

## 1. Structure of L and the roots

We proceed with $L$ as in the introduction but temporarily do not assume $w_{13}\sigma \cong \sigma$. The positive restricted roots are $\alpha_R' = e_1 - e_2$, $\alpha_R'' = e_2 - e_3$, and $\alpha_R = \alpha_R' + \alpha_R'' = e_1 - e_3$.

Let us bring to bear some results from [9]. Since there is just one conjugacy class of Cartan subgroups, there are no real roots in $\Delta$. Then Lemma 2.2 of [9] says that all restricted roots are "even," in the sense of that paper. Moreover, all the restricted roots are "useful," as one sees from §4 of [8]. For any root $\beta$ in $\Delta$, we decompose $\beta$ as $\beta = \beta_I + \beta_R$, its parts on $i\mathfrak{b}$ and $\mathfrak{a}$, respectively. If $\beta$ is complex, Lemma 2.5 of [9] says that

$$|\beta|^2 = 2|\beta_I|^2 = 2|\beta_R|^2. \tag{1.1}$$

Therefore all complex roots have the same length.

For each of the simple restricted roots $\beta_R$ ($= \alpha_R'$ or $\alpha_R''$), it is possible by Proposition 3.1 of [9] to choose $\beta_I$ in $i\mathfrak{b}'$ so that $\beta_I + \beta_R$ is in $\Delta$, so that the reflection $s_{\beta_I}$ preserves $\Delta_M^+$, and so that the linear extension of the map

$$\alpha_R' \to \alpha_I', \quad \alpha_R'' \to \alpha_I''$$

to $\mathfrak{a}'$ is an isometry of $\mathfrak{a}'$ into $i\mathfrak{b}'$. Put $\alpha_I = \alpha_I' + \alpha_I''$. Then it follows that $\alpha_I + \alpha_R$ is in $\Delta$. Theorem 3.7 of [9] says that we obtain an action of $W(A:G)$ on $i\mathfrak{b}'$ from this correspondence——with $s_{\alpha_R'}$ acting by $s_{\alpha_I'}$ and so forth.

The group $M$ is connected. In fact, $M = M^{\#}$ in the notation of

[9] (see §1 of that paper), and Lemma 2.1 of [15] shows that $M^{\#} = M_0$. Therefore $\sigma$ is determined by its highest weight. If the highest weight of $\sigma$ is $\lambda$ and if $w$ is a representative in $K$ of a member $s$ of $W(A:G)$, then Proposition 4.7 of [9] says that $w\sigma$ has highest weight $s\lambda$, with $s\lambda$ defined from the previous paragraph. Therefore

$$w_{12}\sigma \cong \sigma \quad \text{if and only if} \quad \langle \lambda, \alpha_I' \rangle = 0 \tag{1.2a}$$

$$w_{23}\sigma \cong \sigma \quad \text{if and only if} \quad \langle \lambda, \alpha_I'' \rangle = 0 \tag{1.2b}$$

$$w_{13}\sigma \cong \sigma \quad \text{if and only if} \quad \langle \lambda, \alpha_I \rangle = 0 . \tag{1.2c}$$

Lemma 1.1. Suppose $\beta = \beta_I + \alpha_R$ is in $\Delta$ and $\langle \lambda, \beta \rangle = 0$. Then $\langle \lambda, \alpha_I \rangle = 0$. Similar results hold for $\alpha_R'$ and $\alpha_R''$.

Proof. Without loss of generality, we may take $\beta$ to be positive. By (1.1) we may assume $\beta_I$ is not a multiple of $\alpha_I$. Then we have

$$\langle \beta , \alpha_I + \alpha_R \rangle = \langle \beta_I , \alpha_I \rangle + |\alpha_R|^2 > 0 \tag{1.3}$$

by (1.1) and the converse of the Schwarz inequality, and similarly

$$\langle \beta , \alpha_I - \alpha_R \rangle = \langle \beta_I , \alpha_I \rangle - |\alpha_R|^2 < 0 .$$

Therefore $\beta_I + \alpha_I$ and $\beta_I - \alpha_I$ are both roots. From (1.3) it follows that $2\langle \beta , \alpha_I + \alpha_R \rangle / |\alpha_I + \alpha_R|^2 = 1$, and from (1.1) we can then conclude that $\langle \beta_I , \alpha_I \rangle = 0$. Hence $s_{\alpha_I}(\beta_I + \alpha_I) = \beta_I - \alpha_I$, and the defining property of $\alpha_I$ forces $\beta_I + \alpha_I$ and $\beta_I - \alpha_I$ to have the same sign.

Meanwhile, $\beta = \beta_I + \alpha_R$ is positive, and our choice of $\Delta^+$ makes $\beta_I - \alpha_R$ positive. Thus $2\beta_I$ is the sum of positive roots, and it follows that $\beta_I + \alpha_I$ and $\beta_I - \alpha_I$ are both positive. Finally $\beta$ is equal to

$$(\beta_I - \alpha_I) + (\alpha_I + \alpha_R) = (\beta_I + \alpha_I) + (-\alpha_I + \alpha_R) , \tag{1.4}$$

and either $\alpha_I + \alpha_R$ or $-\alpha_I + \alpha_R$ will be positive, by our choice of $\Delta^+$. Hence one of the expressions in (1.4) exhibits $\beta$ as the sum of positive roots. Since $\langle\lambda,\beta\rangle = 0$ and $\lambda$ is $\Delta^+$ dominant, we conclude $\langle\lambda,\alpha_I\rangle = 0$.

Proposition 1.2. Under the assumption $w_{13}\sigma \cong \sigma$, $L_{ss}$ has real rank 1 or 2 and contains the roots $\pm\alpha_I \pm \alpha_R$. Moreover, the following conditions are equivalent:

(a) $L_{ss}$ has real rank 2

(b) $L_{ss}$ contains the roots $\pm\alpha_I' \pm \alpha_R'$ and $\pm\alpha_I'' \pm \alpha_R''$

(c) The whole Weyl group $W(A:G)$ fixes the class of $\sigma$.

Proof. In any case, the subspace $\mathfrak{a}$ is an Iwasawa $\mathfrak{a}$ for $L$. Thus the real rank of $L_{ss}$ is equal to the dimension of the span of the restricted roots that contribute to $L$. Since we are assuming $w_{13}\sigma \cong \sigma$, (1.2c) shows that $\pm\alpha_R \pm \alpha_I$ contribute to $L$; hence $\pm\alpha_R$ are restricted roots for $L$. Thus the real rank of $L_{ss}$ is 1 or 2.

With these considerations in mind, we see from (1.2) that (b) and (c) are equivalent and imply (a). On the other hand, (a) implies (b) by Lemma 1.1. This proves the proposition.

We need to relate our positive system $\Delta^+$ to the positive systems in various other papers, seeing that they are the same. What we need to see is that the infinitesimal character $\lambda_0 = \lambda + \delta_M$ of $\sigma$ is $\Delta^+$ dominant and that any positive system for $\Delta$ that takes $i\mathfrak{b}$ before $\mathfrak{a}$ and makes $\lambda + \delta_M$ dominant automatically makes $\lambda$ dominant.

Lemma 1.3. Let $(\Delta^+)'$ be any positive system for $\Delta$ that takes $i\mathfrak{b}$ before $\mathfrak{a}$ and contains $\Delta_M^+$. If $\beta$ is a $(\Delta^+)'$ simple root that is complex, then $\langle\delta_M,\beta\rangle = 0$.

Proof. We shall pair the members $\epsilon$ of $\Delta_M^+$ having $\langle\epsilon,\beta\rangle < 0$ with the members $\epsilon'$ of $\Delta_M^+$ having $\langle\epsilon',\beta\rangle > 0$, the pairing being $\epsilon' = \epsilon + 2\beta_I$ (where $\beta = \beta_I + \beta_R$) and satisfying $\langle\epsilon+\epsilon',\beta\rangle = 0$.

Notice from (1.1) that $\langle\beta,\theta\beta\rangle = 0$. Let $\langle\epsilon,\beta\rangle < 0$, and put

$\gamma = \varepsilon + \beta$. Then $\theta\gamma$ and $\beta$ are complex $(\Delta^+)'$ positive roots (necessarily of the same length) with

$$\langle \theta\gamma, \beta \rangle = \langle \varepsilon + \theta\beta, \beta \rangle = \langle \varepsilon, \beta \rangle < 0 .$$

Hence $\theta\gamma + \beta = \varepsilon + \theta\beta + \beta = \varepsilon + 2\beta_I$ is a positive root in $\Delta_M$ of the same length as $\beta$.

Conversely if $\varepsilon'$ in $\Delta_M^+$ has $\langle \varepsilon', \beta \rangle > 0$, then $\gamma = \varepsilon' - \beta$ is in $(\Delta^+)'$ (since $\beta$ is simple). Our choice of $i\mathfrak{b}$ before $\mathfrak{a}$ makes $\theta\gamma$ be positive, too, and

$$\langle \theta\gamma, \beta \rangle = \langle \varepsilon' - \theta\beta, \beta \rangle = \langle \varepsilon', \beta \rangle > 0 .$$

Since $\beta$ is simple, $\theta\gamma - \beta = \varepsilon' - \theta\beta - \beta = \varepsilon' - 2\beta_I$ is a positive root in $\Delta_M$. The roots $\theta\gamma$ and $\beta$ are complex and must be of the same length; hence $\varepsilon' - 2\beta_I$ has that same length.

Consequently we have a pairing $\varepsilon \longleftrightarrow \varepsilon'$ by addition or subtraction of $2\beta_I$, and what we have just seen implies $|\varepsilon| = |\beta| = |\varepsilon'|$. Thus $2\langle \varepsilon, \beta \rangle / |\beta|^2 = -1$ and $2\langle \varepsilon', \beta \rangle / |\beta|^2 = +1$, and it follows that $\varepsilon + \varepsilon'$ is orthogonal to $\beta$. Summing on $\varepsilon$ completes the proof of the lemma.

<u>Proposition 1.4</u>. The infinitesimal character $\lambda_0 = \lambda + \delta_M$ is $\Delta^+$ dominant. Conversely if $(\Delta^+)'$ is any positive system for $\Delta$ containing $\Delta_M^+$, taking $i\mathfrak{b}$ before $\mathfrak{a}$, and making $\lambda_0 = \lambda + \delta_M$ dominant, then $\lambda$ is $(\Delta^+)'$ dominant.

<u>Proof</u>. A $\Delta^+$ simple root $\beta$ is either imaginary or complex. If it is imaginary, then $2\langle \delta_M, \beta \rangle / |\beta|^2 = 1$, while if it is complex, Lemma 1.3 gives $2\langle \delta_M, \beta \rangle / |\beta|^2 = 0$. Thus $\delta_M$ is $\Delta^+$ dominant, and hence so is $\lambda + \delta_M$.

Conversely let $\beta$ be $(\Delta^+)'$ simple. If $\beta$ is imaginary, then $2\langle \lambda + \delta_M, \beta \rangle / |\beta|^2 \geq 1$ and hence $2\langle \lambda, \beta \rangle / |\beta|^2 \geq 0$. If $\beta$ is complex, then $2\langle \lambda + \delta_M, \beta \rangle / |\beta|^2 = 2\langle \lambda, \beta \rangle / |\beta|^2$ by Lemma 1.3. Hence $\lambda$ is $(\Delta^+)'$ dominant.

Let $G^{13}$ be the centralizer in $G$ of the subspace $\ker \alpha_R$ of $\mathfrak{a}$, and define $G^{12}$ and $G^{23}$ from $\alpha_R'$ and $\alpha_R''$ similarly. These groups are mutually conjugate by representatives of members of $W(A:G)$. The Lie algebra $(\mathfrak{g}^{13})^{\mathbb{C}}$ is the sum of $(\mathfrak{a} \oplus \mathfrak{b})^{\mathbb{C}}$ and all root spaces for roots whose $\mathfrak{a}$ part is a multiple of $\alpha_R$. Let $\Delta^{13}$ denote this subsystem of roots.

The semisimple part $G^{13}_{ss}$ of $G^{13}$ is then a group of real rank one with no real roots, and it is consequently locally isomorphic to the product of some $SO(2n+1,1)$ with a compact group. Similar remarks apply to $G^{12}$ and $G^{23}$.

Suppose henceforth that $w_{13}\sigma \cong \sigma$. Let $L^{13}$ be the centralizer in $L$ of the subspace $\ker \alpha_R$ of $\mathfrak{a}$. If $\Delta^L$ denotes the root system of $L$, then the root system of $L^{13}$ is $\Delta^L \cap \Delta^{13}$, the set of roots in $\Delta^{13}$ orthogonal to $\lambda$. From Theorem 1.1a of [5] and the identification of our positive system $\Delta^+$ in Proposition 1.4, we obtain the following result.

<u>Proposition 1.5</u>. The standard intertwining operator of $G^{13}$ for the nonunitary principal series of $G^{13}$ with $M$ parameter $\sigma$ and $A$ parameter $c\rho_{L^{13}}$ (i.e., the operator that defines the invariant Hermitian form on the Langlands quotient) is semidefinite and nonsingular for $0 \leq c < 1$ and is not semidefinite for $c > 1$.

If $L_{ss}$ has real rank one, then $\pm\alpha_R$ are its restricted roots, by Proposition 1.2. Therefore

$$\rho_L = \rho_{L^{13}} \quad \text{if} \quad \text{real rank}(L_{ss}) = 1. \tag{1.5a}$$

Suppose $L_{ss}$ has real rank two. Then the restricted root system of $L$ is of type $A_2$. In the inclusion $W(A:L) \subseteq W(A:G)$ we thus have equality, and we can therefore take our representatives $w_{12}$, $w_{23}$, and $w_{13}$ of Weyl group elements to be in $L$, hence in $L_{ss}$. Then we can define $L^{12}$ and $L^{23}$ to be the centralizers in $L$ of the subspaces $\ker \alpha_R'$ and $\ker \alpha_R''$ of $\mathfrak{a}$, and the three subgroups $L^{13}$, $L^{12}$, and $L^{23}$ are conjugate via representatives in $L$ of members of $W(A:G)$. So $\alpha_R$, $\alpha_R'$, and $\alpha_R''$ all have the same multiplicity as restricted roots in $L$, and it follows that

$$\rho_L = 2\rho_{L^{13}} \quad \text{if} \quad \text{real rank}(L_{ss}) = 2. \tag{1.5b}$$

## 2. Use of intertwining operators

We continue with the notation of the introduction, now assuming $w_{13}\sigma \cong \sigma$. In this section we shall prove the part of the Main Theorem that deals with $c\rho_L$ for $0 \leq c < 1$ and also the part for $c = 1$ when $L_{ss}$ has real rank one.

The tool will be the intertwining operators of [13], except that we write them consistently with an action by $G$ on the left in the induced space. For $w$ in $N_K(A)$, the operator $A(w,\sigma,\nu)$ is given initially by

$$A(w,\sigma,\nu)f(x) = \int_{\theta N \cap w^{-1}Nw} f(xw\bar{n})\,d\bar{n}.$$

It is continued meromorphically and then normalized suitably. We require that the normalizing factor have no poles or zeros for $\operatorname{Re}\nu$ in the open positive Weyl chamber, and we let $\mathcal{Q}(w,\sigma,\nu)$ be the normalized operator. The operator that defines the Hermitian form of interest is

$$\sigma(w_{13})\mathcal{Q}(w_{13},\sigma,\nu). \tag{2.1}$$

Here $\sigma(w_{13})$ is defined by means of Lemma 18 of [13].

We can take $w_{13}$ to be in $L_{ss}$, and we can take $w_{12}$ to be in $L_{ss}$ if $L_{ss}$ has real rank two (see the end of §1). Then

$$w_{23} = w_{12}^{-1}w_{13}w_{12}$$

is a representative of $s_{\alpha_R''}$.

Lemma 2.1. For $w_{13}\nu = -\nu$ and $\nu$ real, the operator (2.1) satisfies

$$\sigma(w_{13})\mathcal{Q}(w_{13},\sigma,\nu) = \mathcal{Q}(w_{12}^{-1},\sigma,\nu)^*[(\sigma(w_{13})\mathcal{Q}(w_{23},w_{12}^{-1}\sigma,w_{12}^{-1}\nu)]\mathcal{Q}(w_{12}^{-1},\sigma,\nu)\ . \tag{2.3}$$

Moreover, the operator in brackets on the right side may be regarded as

$$w_{12}^{-1}\sigma(w_{23})\mathcal{Q}(w_{23},w_{12}^{-1}\sigma,w_{12}^{-1}\nu)\ . \tag{2.4}$$

Proof. From properties in [13], the operator (2.1) equals

$$\{\sigma(w_{13})\mathcal{Q}(w_{12},w_{23}w_{12}^{-1}\sigma,w_{23}w_{12}^{-1}\nu)\}\mathcal{Q}(w_{23},w_{12}^{-1}\sigma,w_{12}^{-1}\nu)\mathcal{Q}(w_{12}^{-1},\sigma,\nu)\ ,$$

and the operator in braces equals

$$\mathcal{Q}(w_{12},\sigma(w_{13})(w_{23}w_{12}^{-1}\sigma)\sigma(w_{13})^{-1},w_{23}w_{12}^{-1}\nu)\sigma(w_{13})\ . \tag{2.5}$$

Here

$$\begin{aligned}
\sigma(w_{13})(w_{23}w_{12}^{-1}\sigma)(m)\sigma(w_{13})^{-1} &= \sigma(w_{13})\sigma(w_{12}w_{23}^{-1}mw_{23}w_{12}^{-1})\sigma(w_{13})^{-1} \\
&= \sigma(w_{13}w_{12}w_{23}^{-1}mw_{23}w_{12}^{-1}w_{13}^{-1}) \\
&= \sigma(w_{12}mw_{12}^{-1}) \qquad \text{by (2.2)} \\
&= w_{12}^{-1}\sigma(m)\ .
\end{aligned}$$

Moreover $w_{13}\nu = -\nu$ implies $w_{12}w_{23}w_{12}^{-1}\nu = -\nu$ , and thus $w_{23}w_{12}^{-1}\nu = -w_{12}^{-1}\nu$ . Thus (2.5) is

$$\begin{aligned}
&= \mathcal{Q}(w_{12},w_{12}^{-1}\sigma,-w_{12}^{-1}\nu)\sigma(w_{13}) \\
&= \mathcal{Q}(w_{12}^{-1},\sigma,\nu)^*\sigma(w_{13}) \qquad \text{by [13].}
\end{aligned}$$

This proves (2.3). To prove that the operator in brackets may be regarded as (2.4), we must verify that $w_{12}^{-1}\sigma(w_{23})$ intertwines $w_{12}^{-1}\sigma$ and $w_{23}^{-1}w_{12}^{-1}\sigma$ and that its square is equal to $w_{12}^{-1}\sigma(w_{23}^2)$ , according to Lemma 18 of [13]. We have

$$w_{12}^{-1}\sigma(w_{23})w_{12}^{-1}\sigma(m)w_{12}^{-1}\sigma(w_{23}^{-1}) = \sigma(w_{13})\sigma(w_{12}mw_{12}^{-1})\sigma(w_{13})^{-1} \qquad \text{by (2.2)}$$

$$= \sigma(w_{13}w_{12}mw_{12}^{-1}w_{13}^{-1}) = \sigma(w_{12}w_{23}mw_{23}^{-1}w_{12}^{-1}) \qquad \text{by (2.2)}$$

$$= w_{23}^{-1}w_{12}^{-1}\sigma(m)$$

and

$$[w_{12}^{-1}\sigma(w_{23})]^2 = \sigma(w_{13})^2 = \sigma(w_{13}^2) = \sigma(w_{12}w_{23}^2w_{12}^{-1}) = w_{12}^{-1}\sigma(w_{23}^2)\,,$$

and thus the lemma follows.

Now let us return to the Main Theorem. The operator in brackets in (2.3) is an operator for a simple reflection in $G$, and [13] and §5 of [12] show that this operator is semidefinite if and only if the same operator for $G^{23}$ is semidefinite. Let us write the operator for $G^{23}$ as in (2.4) and use the conjugation $G^{13} = w_{12}G^{23}w_{12}^{-1}$. This conjugation carries the induced space for $G^{23}$, $w_{12}^{-1}\sigma$, and $w_{12}^{-1}\nu$ to the induced space for $G^{13}$, $\sigma$, and $\nu$, and the operator (2.4) is carried to the operator $\sigma(w_{13})G(w_{13},\sigma,\nu)$ defined in $G^{13}$. By Proposition 1.5 this operator is semidefinite for $\nu = c\rho_{L^{13}}$ with $0 \leq c \leq 1$. Thus the operator in brackets in (2.3) is semidefinite for $\nu = c\rho_{L^{13}}$ with $0 \leq c \leq 1$. Since (2.3) shows that (2.1) is just $B^*CB$, with $C$ the operator in brackets, the operator (2.1) is semidefinite for $\nu = c\rho_{L^{13}}$ with $0 \leq c \leq 1$. Taking into account (1.5), we see that $J(P,\sigma,\nu)$ is infinitesimally unitary for $\nu = c\rho_L$ with

$$\begin{cases} 0 \leq c \leq 1 & \text{if } L_{ss} \text{ has real rank one} \\ 0 \leq c \leq \tfrac{1}{2} & \text{if } L_{ss} \text{ has real rank two.} \end{cases}$$

Now suppose $L_{ss}$ has real rank 2. Then $W(A:G)$ has representatives in $L$, by Proposition 1.2, and it fixes the class of $\sigma$. From [13] and §5 of [12], we know that the nonsingularity of the right-hand operator $G(w_{12}^{-1},\sigma,\nu)$ in (2.3) is the same as for that operator in $G^{12}$, which depends only on the projection of $\nu$ in the direction of $\alpha'_R$. Write $\nu = c\rho_L = 2c\rho_{L^{13}}$. Since $\nu$ is a multiple of $\alpha_R$ and

$$\frac{\langle \alpha_R, \alpha'_R\rangle}{|\alpha'_R|^2} = \frac{1}{2},$$

we can replace $\nu$ in our operator for $G^{12}$ by $c\rho_{L^{12}}$. Proposition 1.5 says that our operator in $G^{12}$ is nonsingular out to $\rho_{L^{12}}$, hence for $0 \leq c < 1$.

Thus for $\nu = c\rho_L$ with $0 \leq c < 1$, the operator (2.1) is of the form $B^*CB$ with $B$ nonsingular. Hence (2.1) is semidefinite only if $C$ is semidefinite. But $C$ is not semidefinite beyond $\rho_{L^{13}} = \frac{1}{2}\rho_L$, i.e., for $c > \frac{1}{2}$. Therefore $J(P,\sigma,\nu)$ is not infinitesimally unitary for $\frac{1}{2} < c < 1$.

## 3. Use of derived functor modules

We continue with the notation of the introduction, and we assume $w_{13}\sigma \cong \sigma$. Our goal in this section is to prove that $J(P,\sigma,\rho_L)$ is infinitesimally unitary if $L_{ss}$ has real rank two; this result will be stated as Proposition 3.4. (We know already that $J(P,\sigma,\rho_L)$ is infinitesimally unitary if $L_{ss}$ has real rank one, and that case will not concern us in this section.)

The proof is rather similar to the proof of unitarity of some isolated representations that occur in [5]. It uses Zuckerman's derived functor modules $A_q(\mu)$, as explained in Vogan and Zuckerman [24], but with the parameter $\mu$ outside the usual range. (See also Enright and Wallach [7].) A big theorem due to Vogan [22] establishes unitarity for such representations under suitable conditions. The supplementary arguments in §12 of [5] are due in part to Vogan as well, and we give the proofs only when they differ from those in §12 of [5].

Our subspace $\mathfrak{b}$ (see §1) is maximal abelian in $\mathfrak{k}$, and we let $\Delta_K$ be the set of roots $\Delta_K = \Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$. One knows (see, e.g., [10]) that the members of $\Delta_K$ are the restrictions to $\mathfrak{b}^{\mathbb{C}}$ of all members of $\Delta$ except the noncompact roots of $\Delta_M$ (of which there are none in our case). We take $\Delta_K^+$ to be the restrictions to $\mathfrak{b}^{\mathbb{C}}$ of the positive members of $\Delta$; this is a positive system since we have taken $i\mathfrak{b}$ before $\mathfrak{a}$ in defining $\Delta^+$. Let $\delta_K$ be half the sum of the members of $\Delta_K^+$.

For $\nu$ in the closed positive Weyl chamber, $U(P,\sigma,\nu)$ in our situation has a unique minimal K-type $\Lambda$ in the sense of Vogan [20], i.e., a subrepresentation $\tau_\Lambda$ of $U(P,\sigma,\nu)\big|_K$ with highest weight $\Lambda$ for which $|\Lambda + 2\delta_K|^2$ is a minimum, and Theorem 1 of [10] and the present Proposition 1.4 together say that the minimal K-type is given by

$$\Lambda = \lambda = \lambda_0 + \delta - 2\delta_K . \tag{3.1}$$

Moreover, $\tau_\Lambda$ occurs with multiplicity one in $U(P,\sigma,\nu)$ and lies in $J(P,\sigma,\nu)$.

Proposition 3.1. In the situation of §1, suppose that $J(P,\sigma,\nu)$ and $J(P,\sigma',\nu')$ each admit a nonzero Hermitian form and that they have the same real infinitesimal character and same minimal K-type. Then $J(P,\sigma,\nu)$ and $J(P,\sigma',\nu')$ are infinitesimally equivalent.

Proof. The theory of [20] shows that the assumption of a minimal K-type in common implies we may take $\sigma = \sigma'$. (This is clear from (3.1) only after a little work, since one must first adjust the positive systems suitably.) Since our representations admit invariant Hermitian forms, $\nu$ and $\nu'$ must both be mapped into their negatives by $w_{13}$, hence must both be multiples of $\alpha_R = e_1 - e_3$. Since the infinitesimal characters are equal, we conclude $|\nu| = |\nu'|$ and then $\nu = \nu'$. This proves the proposition.

We shall use Proposition 3.1 to match $J(P,\sigma,\rho_L)$ with a suitable $A_{\mathfrak{q}}(\mu)$. We shall not need the detailed construction of $A_{\mathfrak{q}}(\mu)$, only its existence and properties in Theorem 3.2 below. We have already defined $\Delta^L$ and we let $\Delta(\mathfrak{u})$ be the set of positive roots (of $\Delta^+$) outside $\Delta^L$. Let $\delta(\mathfrak{u})$ be half their sum. In the Vogan theory, the symbol stands for $\mathfrak{l}^{\mathbb{C}} \oplus \mathfrak{u}$.

There being no noncompact roots in $\Delta_M$, we let $\Delta(\mathfrak{p}^{\mathbb{C}})$ be the set of restrictions to $\mathfrak{b}^{\mathbb{C}}$ of the complex roots of $\Delta$, and we let $\Delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}})$ be the set of restrictions to $\mathfrak{b}^{\mathbb{C}}$ of the complex roots of $\Delta(\mathfrak{u})$. Let $\delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}})$ be half the sum of the members of $\Delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}})$. If $\delta_K(\mathfrak{u})$ denotes half the sum of the members of $\Delta_K^+$ that arise by restriction from $\Delta(\mathfrak{u})$, then we have

$$\delta(\mathfrak{u}) = \delta_K(\mathfrak{u}) + \delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}}) . \tag{3.2}$$

<u>Theorem 3.2</u> (Vogan [22]). Suppose $G$, $\Delta^L$, and $L$ are as in the introduction; here $L$ is the analytic subgroup of $G$ with Lie algebra

$$\mathfrak{l} = \mathfrak{g} \cap (\mathfrak{b} + \mathfrak{a} + \sum_{\beta \in \Delta^L} \mathfrak{g}_\beta) .$$

If $\mu$ in $i\mathfrak{b}'$ is the differential of a unitary (one-dimensional) character of $L$ such that

$$\langle \mu + \delta(\mathfrak{u}), \beta \rangle \geq 0 \quad \text{for all } \beta \text{ in } \Delta(\mathfrak{u}) , \tag{3.3}$$

and if

$$\Lambda' = \mu + 2\delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}}) ,$$

then there exists an admissible representation $A_{\mathfrak{q}}(\mu)$ of $\mathfrak{g}$ with infinitesimal character $\mu + \delta$ such that

(a) the K-types have multiplicities given by the following version of Blattner's formula:

$$\text{mult } \tau_{\Lambda''} = \sum_{s \in W_K} (\det s) \mathcal{P}(s(\Lambda'' + \delta_K) - (\Lambda' + \delta_K)) ,$$

where $W_K$ is the Weyl group of $\Delta_K$ and $\mathcal{P}$ is the partition function relative to $\Delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}})$, and

(b) the representation $A_{\mathfrak{q}}(\mu)$ admits a positive definite invariant inner product.

Proof. This is derived from [22] and [21] in the same way that Theorem 12.2 is proved in [5].

Proposition 3.3. Under the assumptions of Theorem 3.2, suppose that $\Lambda'$ is $\Delta_K^+$ dominant. Then $A_{\mathfrak{q}}(\mu)$ is nonzero and the K-type $\tau_{\Lambda'}$ occurs with multiplicity one. If in addition $\langle \Lambda' + 2\delta_K, \beta \rangle \geq 0$ for all $\beta$ in $\Delta(\mathfrak{u})$, then $\tau_{\Lambda'}$ is the unique minimal K-type of $A_{\mathfrak{q}}(\mu)$.

Proof. The argument is the same as for Proposition 12.3 of [5].

Proposition 3.4. In the setting of the introduction, let $\Lambda$ be defined by (3.1). If $L_{ss}$ has real rank two, then $J(P, \sigma, \rho_L)$ is infinitesimally unitary.

Proof. We shall use $\Delta^L$ from the introduction, together with the corresponding $\mathfrak{u}$ built from $\Delta^+$, as data for Theorem 3.2. Put

$$\mu = \Lambda - 2\delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}}) . \tag{3.4}$$

First we exponentiate $\mu$; this is a little tricky.

There is no loss in generality in assuming that $G^{\mathbb{C}}$ is simply connected. Vogan and Zuckerman [24] show that $2\delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}})$ is integral, i.e., exponentiates to $(\mathfrak{b} \oplus \mathfrak{a})^{\mathbb{C}}$. To see that $\Lambda = \lambda$ is integral, let us note first that $2\langle \lambda, \beta \rangle / |\beta|^2$ is an integer if $\beta$ is in $\Delta_M^+$, since $\lambda$ is the highest weight of $\sigma$. Suppose $\beta$ is a complex root: $\beta = \beta_I + \beta_R$. The $\mathfrak{a}$ part $\beta_R$ is one of $\pm\alpha_R$, $\pm\alpha_R'$, and $\pm\alpha_R''$. Since the proof will be the same in all cases, let us suppose $\beta_R = \alpha_R'$. Then it follows from the first part of the proof of Lemma 1.1 that $\beta_I + \alpha_I'$ and $\beta_I - \alpha_I'$ are both roots. Hence

$$\beta = (\beta_I - \alpha_I') + (\alpha_I' + \alpha_R')$$

and

$$\frac{2\langle \lambda, \beta \rangle}{|\beta|^2} = \frac{2\langle \lambda, \beta_I - \alpha_I' \rangle}{|\beta_I - \alpha_I'|^2} \tag{3.5}$$

by Proposition 1.2. (Here we use that $L_{ss}$ has real rank two.) Since $\lambda$ is integral for members of $\Delta_M$, (3.5) shows $\lambda$ is integral for $\Delta^+$. Since $G^{\mathbb{C}}$ is simply connected, we conclude $e^{\mu}$ is a well defined one-dimensional quasicharacter of $(\mathfrak{b} \oplus \mathfrak{a})^{\mathbb{C}}$.

Next we show that $\mu$ is orthogonal to the members of $\Delta^L$. We know that $\Lambda = \lambda$ is orthogonal to $\Delta^L$, by definition of $\Delta^L$, and we have to check the orthogonality of $2\delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}})$ with $\Delta^L$. It is clear that $2\delta(\mathfrak{u})$ is orthogonal to $\Delta^L$, and (3.2) says it is enough to prove that $2\delta_K(\mathfrak{u})$ is orthogonal to $\Delta^L$. Now $2\delta_K(\mathfrak{u})$ vanishes (term-by-term) on $\mathfrak{a}$ and therefore sees only the restrictions of members of $\Delta^L$ to $\mathfrak{b}$. We have observed that these restrictions are exactly the members of $\Delta_K^L$ (since $\mathfrak{m}$ has no noncompact roots), and thus we are to prove that $2\delta_K(\mathfrak{u})$ is orthogonal to $\Delta_K^L$. But this is clear since the roots contributing to $2\delta_K(\mathfrak{u})$ are permuted by reflections in members of $\Delta_K^L$.

Therefore the Theorem of the Highest Weight supplies an irreducible finite-dimensional representation of $L^{\mathbb{C}}$ with highest weight $\mu$. Naturally this representation is one-dimensional. Since $\mu$ vanishes on $\mathfrak{a}$, the restriction of $e^{\mu}$ of this representation to $L$ is unitary.

To apply Theorem 3.2, we need to verify (3.3). We have

$$\begin{aligned}
\mu + \delta(\mathfrak{u}) &= \Lambda - 2\delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}}) + \delta(\mathfrak{u}) \\
&= \lambda_0 + \delta - 2\delta_K - 2\delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}}) + \delta(\mathfrak{u}) && \text{by (3.1)} \\
&= \lambda_0 + \delta - 2\delta_K + 2\delta_K(\mathfrak{u}) - \delta(\mathfrak{u}) && \text{by (3.2)} \\
&= \lambda_0 + \delta(\mathfrak{l}^{\mathbb{C}}) - 2\delta_K(\mathfrak{l}^{\mathbb{C}}) && \text{in obvious notation.}
\end{aligned}$$

The right side is the sum of $\lambda_0$ and a real combination of members of

$\Delta^L$, and any member $\gamma$ of $\Delta^L$ satisfies $\sum_{w\in W(\Delta^L)} w\gamma = 0$, where $W(\Delta^L)$ is the Weyl group of the root system $\Delta^L$. Since $\mu$ and $\delta(u)$ are invariant under $W(\Delta^L)$, we obtain

$$\mu + \delta(u) = \sum_{w\in W(\Delta^L)} w\lambda_0 .$$

Thus $\beta$ in $\Delta(u)$ implies

$$\langle \mu + \delta(u), \beta\rangle = \sum_{w\in W(\Delta^L)} \langle \lambda_0, w^{-1}\beta\rangle \geq 0 ,$$

since $w^{-1}\beta$ is in $\Delta(u)$ and $\langle \lambda_0, w^{-1}\beta\rangle$ is thus $\geq 0$ (Proposition 1.4).

Thus Theorem 3.2 applies. The form $\Lambda'$ in the theorem is our $\Lambda$, by (3.4). The theorem says that $A_q(\mu)$ has infinitesimal character $\mu + \delta$ and is unitary. By (3.1), $\Lambda + 2\delta_K$ is $\lambda_0 + \delta$, which is $\Delta^+$ dominant by Proposition 1.4. Thus Proposition 3.3 applies, showing that $A_q(\mu)$ is nonzero and that $\Lambda$ is the unique minimal K-type of $A_q(\mu)$.

Now $J(P,\sigma,\rho_L)$ has minimal K-type $\Lambda$ and infinitesimal character $\lambda_0 + \rho_L$. By Proposition 3.1 the proof of Proposition 3.4 will be complete if we show that $\lambda_0 + \rho_L$ is conjugate to $\mu + \delta$ by the Weyl group of $\Delta$. Here

$$\mu + \delta = \mu + \delta(u) + \delta(\mathfrak{l}^{\mathbb{C}}) .$$

The first two terms on the right side are fixed by $W(\Delta^L)$. Applying a member of $W(\Delta^L)$ that results in a positive system for $\Delta^L$ that takes $\mathfrak{a}$ before $i\mathfrak{b}$, we see from (3.1) and (3.4) that $\mu + \delta$ is conjugate to

$$\mu + \delta(u) + \delta_M(\mathfrak{l}^{\mathbb{C}}) + \rho_L = \lambda_0 + \delta - 2\delta_K - 2\delta(u \cap \mathfrak{p}^{\mathbb{C}}) + \delta(u) + \delta_M(\mathfrak{l}^{\mathbb{C}}) + \rho_L . \qquad (3.6)$$

Now Lemma 3 of [10], applied to $\mathfrak{l}^{\mathbb{C}}$, says that

$$\delta(\mathfrak{l}^{\mathbb{C}}) - 2\delta_K(\mathfrak{l}^{\mathbb{C}}) = \delta_M(\mathfrak{l}^{\mathbb{C}}) - 2\delta_{M\cap K}(\mathfrak{l}^{\mathbb{C}}) .$$

Since $M \cap L$ is compact, the right side is just $-\delta_M(\mathfrak{l}^{\mathbb{C}})$. Thus

$$\delta_M(\mathfrak{l}^{\mathbb{C}}) = 2\delta_K(\mathfrak{l}^{\mathbb{C}}) - \delta(\mathfrak{l}^{\mathbb{C}}) ,$$

and (3.6) becomes

$$= \lambda_0 + \delta - 2\delta_K - 2\delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}}) + \delta(\mathfrak{u}) + 2\delta_K(\mathfrak{l}^{\mathbb{C}}) - \delta(\mathfrak{l}^{\mathbb{C}}) + \rho_L$$

$$= \lambda_0 + \delta - 2\delta_K(\mathfrak{u}) - 2\delta(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}}) + \delta(\mathfrak{u}) - \delta(\mathfrak{l}^{\mathbb{C}}) + \rho_L$$

$$= \lambda_0 + \delta - 2\delta(\mathfrak{u}) + \delta(\mathfrak{u}) - \delta(\mathfrak{l}^{\mathbb{C}}) + \rho_L$$

$$= \lambda_0 + \rho_L .$$

This proves the required conjugacy and completes the proof of the proposition.

## 4. Cut-off for unitarity

To complete the proof of the Main Theorem, we are to show that there is no unitarity beyond $\rho_L$. Our argument uses the techniques of [3], as amplified in §§1-2 of [4] and §3 of [5]: We produce a K-type on which the signature of the invariant Hermitian form rules out unitarity. This kind of argument involves a certain calculation that is briefly indicated in [3] and [4] and will be written in more detail later. Accordingly, in the present paper, we merely identify what is to be calculated, what the result is, and what its effect is on unitarity.

In the setting of the introduction, we assume that $w_{13}\sigma \cong \sigma$. Thus (1.2) gives $\langle \lambda, \alpha_I \rangle = 0$, and hence (3.1) gives $\langle \Lambda, \alpha_I \rangle = 0$ for the minimal K-type $\Lambda$. The representation $\sigma$ of $M$ occurs in $\tau_\Lambda|_M$, and we let $B$ be a nonzero $M$ map from the space on which $\tau_\Lambda$ operates to the space on which $\sigma$ operates. If $v_\Lambda$ is a nonzero highest weight vector for $\tau_\Lambda$, then

$$f_0(k) = B(\tau_\Lambda(k)^{-1} v_\Lambda)$$

is a member of the induced space (in the compact picture). We normalize our invariant Hermitian form $\langle\cdot,\cdot\rangle$ so that $\langle f_0, f_0\rangle = 1$.

Let $\Lambda'$ be the result of making $\Lambda + \alpha_I$ dominant for $\Delta_K^+$ by applying a member of the Weyl group of $\Delta_K$, and let $P_{\Lambda'}$ be the projection of the induced space to the $\tau_{\Lambda'}$ subspace. Let $\alpha$ be the root $\alpha_I + \alpha_R$, and let $E_\alpha$ be a root vector in $\mathfrak{g}^{\mathbb{C}}$ for $\alpha$. Then we have the following result.

Proposition 4.1. In the setting of the introduction when $w_{13}\sigma \cong \sigma$, the function

$$f_1 = P_{\Lambda'} U(P,\sigma, c\rho_L, E_\alpha - \theta E_\alpha) f_0$$

is a nonzero member of the induced space, and $\langle f_1, f_1\rangle$ is a positive multiple of $1 - c^2$.

Then it follows that the form $\langle\cdot,\cdot\rangle$ is not semidefinite on the sum of its $\tau_\Lambda$ and $\tau_{\Lambda'}$ subspaces, and hence $U(P,\sigma,c\rho_L)$ is not infinitesimally unitary for $c > 1$.

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