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A QUICK PROOF OF THE CLASSIFICATION OF SIMPLE REAL LIE ALGEBRAS

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ABSTRACT. Élie Cartan's classification of the simple Lie algebras over $\mathbb{R}$ is derived quickly from some structure theory over $\mathbb{R}$ and the classification over $\mathbb{C}$.

Élie Cartan classified the simple Lie algebras over $\mathbb{R}$ for the first time in 1914. There have been a number of simplifications in the proof since then, and these are described in [3, p. 537]. All proofs assume the classification over $\mathbb{C}$ and a certain amount of structure theory over $\mathbb{R}$. Recent proofs tend to run to 25 pages. Here is a shorter argument.

Theorem. *Up to isomorphism, the only simple Lie algebras over* $\mathbb{R}$ *that are neither complex nor compact are those in Cartan's list as organized in* [3, p. 518].

We use terminology as in [3]. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of a noncomplex simple Lie algebra over $\mathbb{R}$, and let θ be the Cartan involution. Choose a maximal abelian subspace $\mathfrak{t}_0$ of $\mathfrak{k}_0$ and extend to a maximally compact Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ of $\mathfrak{g}_0$. Removal of subscripts 0 will indicate complexifications. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the root system. Roots are imaginary on $\mathfrak{t}_0$ and real on $\mathfrak{a}_0$. All roots are imaginary-valued or complex on $\mathfrak{h}_0$; there are no real-valued roots. Introduce a positive system Δ^+ that takes $i\mathfrak{t}_0$ before $\mathfrak{a}_0$. The map θ carries roots to roots and permutes the simple roots. The complex simple roots move in two-element orbits, while the imaginary simple roots are fixed. By the **Diagram** of $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$, we mean the Dynkin diagram of Δ with the two-element orbits under θ so labeled and with the imaginary roots shaded or not, according as the simple root is noncompact (root vector in $\mathfrak{p}$) or compact (root vector in $\mathfrak{k}$).

Lemma 1. *If* $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$ *and* $(\mathfrak{g}_0', \mathfrak{h}_0', (\Delta')^+)$ *have the same Diagram, then* $\mathfrak{g}_0$ *and* $\mathfrak{g}_0'$ *are isomorphic.*

Proof. We may assume that the complexifications $(\mathfrak{g}, \mathfrak{h}, \Delta^+)$ are the same and that the associated compact forms are the same: $\mathfrak{u}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0 = \mathfrak{k}_0' \oplus i\mathfrak{p}_0'$. Using the conjugacy of compact forms, the conjugacy of maximal abelian subspaces within them, and the standard construction of a compact form from $\mathfrak{h}$, we see that we can normalize root vectors X_α, $\alpha \in \Delta$, as in Theorem 5.5 of [3, p. 176] and obtain $\mathfrak{u}_0$ from $\{X_\alpha\}$ as in Theorem 6.3 of [3, p. 181].

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First suppose $\mathfrak{a}_0 = 0$, so that all roots are imaginary. For α simple we have $\theta X_\alpha = \pm X_\alpha$, the sign being $+$ if α is compact and $-$ is α is noncompact. The same formula holds for θ'. Since $\mathfrak{h}$ and the X_α's for α simple generate $\mathfrak{g}$, it follows that $\theta = \theta'$, hence that $\mathfrak{k} = \mathfrak{k}'$ and $\mathfrak{p} = \mathfrak{p}'$. Then $\mathfrak{g}_0 = \mathfrak{g}_0'$ is recovered as $(\mathfrak{u}_0 \cap \mathfrak{k}) \oplus i(\mathfrak{u}_0 \cap \mathfrak{p})$.

If $\mathfrak{a}_0 \neq 0$, we may not have $\theta = \theta'$. For $\alpha \in \Delta$, write $\theta X_\alpha = a_\alpha X_{\theta\alpha}$. Then $a_\alpha a_{-\alpha} = 1$ and $a_\alpha a_{\theta\alpha} = 1$. Since θ maps $\mathfrak{u}_0 \cap \operatorname{span}\{X_\alpha, X_{-\alpha}\}$ to $\mathfrak{u}_0 \cap \operatorname{span}\{X_{\theta\alpha}, X_{-\theta\alpha}\}$, we see that $\bar{a}_\alpha = a_{-\alpha}$. Therefore $|a_\alpha| = 1$. For each pair of complex simple roots α and $\theta\alpha$, choose square roots $a_\alpha^{1/2}$ and $a_{\theta\alpha}^{1/2}$ whose product is 1. Similarly write $\theta' X_\alpha = b_\alpha X_{\theta\alpha}$ with $|b_\alpha| = 1$, and define $b_\alpha^{1/2}$ and $b_{\theta\alpha}^{1/2}$ for α and $\theta\alpha$ simple. Define H and H' in $\mathfrak{h} \cap \mathfrak{u}_0$ by the conditions that $\alpha(H) = \alpha(H') = 0$ for α simple imaginary and that $\exp(\frac{1}{2}\alpha(H)) = a_\alpha^{1/2}$, $\exp(\frac{1}{2}\theta\alpha(H)) = a_{\theta\alpha}^{1/2}$, $\exp(\frac{1}{2}\alpha(H')) = b_\alpha^{1/2}$, and $\exp(\frac{1}{2}\theta\alpha(H')) = b_{\theta\alpha}^{1/2}$ if α and $\theta\alpha$ are complex simple. A little computation shows that $\theta' \circ \operatorname{Ad}(\exp \frac{1}{2}(H - H')) = \operatorname{Ad}(\exp \frac{1}{2}(H - H')) \circ \theta$, from which it follows that $\mathfrak{k}' = \operatorname{Ad}(\exp \frac{1}{2}(H - H'))\mathfrak{k}$, $\mathfrak{p}' = \operatorname{Ad}(\exp \frac{1}{2}(H - H'))\mathfrak{p}$, and $\mathfrak{g}_0' = \operatorname{Ad}(\exp \frac{1}{2}(H - H'))\mathfrak{g}_0$. □

The next step is to identify some pairs of distinct Diagrams that correspond merely to changes of Δ^+. The argument is inspired by [2]. First let us assume that $\mathfrak{a}_0 = 0$, i.e., that the automorphism of Δ given by θ is the identity. Let Λ be the subset of $i\mathfrak{t}_0$ where all roots take integer values and where all noncompact roots take odd-integer values. If $\{\omega_j\}$ is the basis dual to the simple roots, then the sum of those ω_j corresponding to the noncompact simple roots is a member of Λ. The set Λ is discrete, and we let H_0 be a member of Λ as close to 0 as possible.

Lemma 2. *If $(\Delta^+)'$ is a positive system that makes H_0 dominant, then there is at most one noncompact simple root, say α_i. If the basis dual to the simple roots of $(\Delta^+)'$ is $\{\omega_j\}$, then there cannot exist i' such that $\langle \omega_i - \omega_{i'}, \omega_{i'} \rangle > 0$.*

Proof. Since H_0 is in Λ and is dominant, $H_0 = \sum n_j \omega_j$ with all n_j integers ≥ 0. If $n_i > 0$, then $H_0 - \omega_i$ is dominant and thus has $\langle H_0 - \omega_i, \omega_i \rangle \geq 0$ with equality if and only if $H_0 = \omega_i$. Then $|H_0 - 2\omega_i|^2 \leq |H_0|^2$ with equality only if $H_0 = \omega_i$, and minimality forces $H_0 = \omega_i$. Now let $H_0 = \omega_i$. If $\langle \omega_i - \omega_{i'}, \omega_{i'} \rangle > 0$, then $|H_0 - 2\omega_{i'}|^2 < |H_0|^2$, in contradiction to minimality. □

When $\mathfrak{a}_0 \neq 0$, Lemma 2 is to be applied to the part of $i\mathfrak{t}_0$ corresponding to the span of the imaginary simple roots. The result is that we can associate to any $\mathfrak{g}_0$ at least one Diagram in which at most one imaginary root is shaded.

Now we can read off the possibilities. First suppose that the automorphism of Δ is the identity. If all roots are unshaded, then $\mathfrak{g}_0$ is the compact form. Otherwise exactly one simple root is shaded. For the classical Dynkin diagrams, let the double line or triple point be at the right end, and let the i^{th} root be shaded. In A_n, we are led to $\mathfrak{su}(i, n+1-i)$. In B_n, we are led to $\mathfrak{so}(2i, 2n+1-2i)$. In C_n, we are led to $\mathfrak{sp}(i, n-i)$ if $i < n$ and to $\mathfrak{sp}(n, \mathbb{R})$ if $i = n$. In D_n, we are led to $\mathfrak{so}(2i, 2n-2i)$ if $i \leq n-2$ and to $\mathfrak{so}^*(2n)$ otherwise.

For the exceptional Dynkin diagrams, a little checking that compares the second conclusion of Lemma 2 with the fundamental weights (see [1, pp. 260-275]) shows that α_i in Lemma 2 has to be a node (endpoint vertex) of the Dynkin diagram. Moreover, in G_2, α_i has to be the long simple root, while in E_8, it cannot be the node on the short branch. In E_6 two nodes are equivalent by outer automorphism. Thus we obtain at most three Lie algebras for E_7; at most two for E_6, E_8, F_4; and at most one for G_2. These are E II, E III for E_6, E V, E VI, E VII for E_7; E VIII, E IX for E_8; F I, F II for F_4; and G for G_2.

When the automorphism of Δ is not the identity, the Dynkin diagram is A_n, D_n, or E_6. For A_n, there is no imaginary simple root if n is even, and there is one if n is odd. For n even we are led to $\mathfrak{sl}(n+1,\mathbb{R})$, while for n odd we are led to $\mathfrak{sl}(n+1,\mathbb{R})$ if the root is shaded and to $\mathfrak{su}^*(n+1)$ if the root is unshaded. For D_n, the first $n-2$ simple roots are imaginary. If all are unshaded, we are led to $\mathfrak{so}(1,2n-1)$. If the i^{th} simple root is shaded, $i \leq n-2$, we are led to $\mathfrak{so}(2i+1,2n-2i-1)$. For E_6, the triple point and the node on the short branch are imaginary. If neither is shaded, we are led to E IV, while if either one is shaded, we are led to E I.

Note added in proof. David Vogan has pointed out that any Dynkin diagram marked with an involution and having a subset of its one-element orbits shaded is a Diagram for some $\mathfrak{g}_0$. The proof is in the spirit of Lemma 1. Existence of the exceptional simple real Lie algebras follows.

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