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Limits of Holomorphic Discrete Series

A. W. KNAPP*

Department of Mathematics, Cornell University, Ithaca, New York 14850

AND

K. OKAMOTO†

Department of Mathematics, Hiroshima University, Hiroshima, Japan

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New irreducible unitary representations of the (semisimple) automorphism groups of Cartan domains are constructed. These representations are used to exhibit the reducibility of some of the continuous series representations occurring in the Plancherel formula. The imbedding that exhibits the reducibility is similar to the imbedding of the Hardy class H^2 of analytic functions in the disc into the space of all L^2 functions on the circle, given by passing from an analytic function to its boundary values.

1. INTRODUCTION

The irreducible unitary representations of the group $G = SU(1, 1)$ of all two-by-two complex matrices of the form

$$(\alpha, \beta; \bar{\beta}, \bar{\alpha}) = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$$

with $|\alpha|^2 - |\beta|^2 = 1$ were classified by Bargmann [1]. The ones that occur as direct summands of $L^2(G)$ comprise the "discrete series" and are of two types, holomorphic and antiholomorphic, each parametrized by the integers $n \geqslant 2$. For the n-th representation of the

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holomorphic discrete series ($n \geqslant 2$), the Hilbert space is the space of holomorphic functions $F(z)$ in the unit disc Ω with norm

$$\| F \|_n^2 = \int_\Omega | F(z)|^2 (1 - | z |^2)^{n-2} \, dx \, dy \tag{1.1}$$

and with group action

$$\mathscr{U}_n(g) F(z) = (\alpha + \bar{\beta} z)^{-n} F\left(\frac{\beta + \bar{\alpha} z}{\alpha + \bar{\beta} z}\right) \tag{1.2}$$

if $g = (\alpha, \beta; \bar{\beta}, \bar{\alpha})$.

It is possible to associate to the integer $n = 1$ a representation of G that is similar in appearance to those above but is not in the discrete series. To do so, one does not use the norm (1.1) with $n = 1$, which would result in a null Hilbert space, but instead uses the norm

$$\| F \|_1^2 = \lim_{n \downarrow 1} (n - 1) \| F \|_n^2 , \tag{1.3}$$

which apart from a constant factor is equal to

$$\lim_{r \uparrow 1} \int_0^{2\pi} | F(re^{i\theta})|^2 \, d\theta. \tag{1.4}$$

This Hilbert space is the space H^2 of Hardy and is nonvanishing. The group action is given by (1.2) with $n = 1$. In the sense given by Eq. (1.3), we can then regard $\mathscr{U}_1(g)$ as a limit of holomorphic discrete series.

The representation $\mathscr{U}_1(g)$ is of special interest, partly because of the well-known imbedding of H^2 in $L^2(\partial\Omega)$, given by associating a boundary function on the circle to each H^2 function on Ω. In fact, the representation $\mathscr{V}(g)$ in $L^2(\partial\Omega)$ given by

$$\mathscr{V}(g) F(e^{i\theta}) = (\alpha + \bar{\beta} e^{i\theta})^{-1} F\left(\frac{\beta + \bar{\alpha} e^{i\theta}}{\alpha + \bar{\beta} e^{i\theta}}\right)$$

is a member of what Bargmann called the "principal continuous series"; i.e., it is unitarily equivalent with a representation induced from an irreducible finite-dimensional representation of the subgroup

$$MA^+N = \pm \begin{pmatrix} \cosh t - ixe^t & \sinh t - ixe^t \\ \sinh t + ixe^t & \cosh t + ixe^t \end{pmatrix}, \qquad t \in \mathbf{R}, \quad x \in \mathbf{R}.$$

The boundary-value imbedding of H^2 in $L^2(\partial\Omega)$ clearly commutes with the action of G, and therefore the existence of $\mathscr{U}_1(g)$ exhibits the representation $\mathscr{V}(g)$ of the principal series as reducible. That this is an exceptional situation is indicated by the fact that all other members of the principal series of G are irreducible.

In this paper we propose to investigate this boundary-value imbedding and reducibility more generally; we shall allow G to be any simple Lie group that has a faithful matrix representation and whose associated symmetric space has an invariant complex structure. There is one previous result in this direction, other than for $SU(1, 1)$. Harish-Chandra in [5, p. 770], by an argument involving positive definite functions, obtained the existence of exceptional representations having "extreme vectors" in the sense of Lemma 6.2 below, and the representations that we shall here construct have this property. (The proof of this theorem of Harish-Chandra is omitted in [5] and later papers, but his result will not be used in our work.) In any case, Harish-Chandra's realization of exceptional representations is not constructive and therefore does not help in describing the imbedding of the exceptional representations in continuous series geometrically as a passage to boundary values.

The paper is arranged as follows. Starting from appropriate singular characters of a compact Cartan subgroup of G, we define a subgroup MA^+N in Section 2 and a representation $U(g)$ in Section 4. In Section 4, we prove that $U(g)$ is unitary and that its Hilbert space is nonvanishing. In Section 5 we imbed $U(g)$ in a continuous series representation $V(g)$ obtained from MA^+N, and in Section 7 we prove that $U(g)$ is irreducible and that the image in the representation space of $V(g)$ is proper.

The problems that are dealt with in this paper arose naturally from the work [10] of the first author with E. M. Stein, and the authors are grateful for Professor Stein's help at an early stage of the present work. The interplay of our results and those in [10] will be discussed in Section 8, where we consider in a special case the extent to which we have accounted for all reducible representations of the principal series.

2. Notation and Construction of MA^+N

This is the first of two sections in which we introduce notation. Let G be a connected semisimple Lie group with a faithful matrix representation, let K be a maximal compact subgroup, and suppose that G/K is hermitian symmetric. If $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K, there is a corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$;

let θ be the associated involution of $\mathfrak{g}$. For any subspace of $\mathfrak{g}$, we use a superscript $^{\mathbf{C}}$ in referring to its complexification. Thus we have defined $\mathfrak{g}^{\mathbf{C}}$, $\mathfrak{k}^{\mathbf{C}}$, and $\mathfrak{p}^{\mathbf{C}}$. Since G has a faithful matrix representation, we can regard G as a subgroup of a connected group $G^{\mathbf{C}}$ with Lie algebra $\mathfrak{g}^{\mathbf{C}}$. Let $K^{\mathbf{C}} \subseteq G^{\mathbf{C}}$ be the analytic subgroup with Lie algebra $\mathfrak{k}^{\mathbf{C}}$.

Let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{k}$, let T be the corresponding analytic subgroup of G, and let $T^{\mathbf{C}} \subseteq G^{\mathbf{C}}$ be the analytic subgroup with Lie algebra $\mathfrak{h}^{\mathbf{C}}$. It is known [9, p. 312] that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{k}$ and of $\mathfrak{g}$ and that T is a Cartan subgroup of K and of G. Let Σ be the set of nonzero roots of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{h}^{\mathbf{C}})$. If $\alpha \in \Sigma$ has E_α as root vector, then $E_\alpha \in \mathfrak{k}^{\mathbf{C}}$ or $E_\alpha \in \mathfrak{p}^{\mathbf{C}}$, and we call α compact or noncompact accordingly. Let Σ_k and Σ_n be the sets of compact and noncompact roots, respectively.

An ordering yielding a system of positive roots in Σ will be said to be compatible with the complex structure of G/K if each positive noncompact root is larger than all compact roots. Let P be such a system of positive roots, and let $P_k = P \cap \Sigma_k$ and $P_n = P \cap \Sigma_n$. One way of obtaining such an ordering is to order the center of $\mathfrak{k}$ before the rest of $\mathfrak{h}$; the resulting ordering has the required properties because the compact roots are exactly the roots that vanish on the center of $\mathfrak{k}$ (Corollary 7.3, p. 314 of [9]). With such an ordering, the sum of coefficients of the noncompact simple roots in the expansion of any positive root in terms of simple roots is either 0 or 1 [5, p. 761]; also if G is noncompact and simple, there is only one noncompact simple root.

Let $\mathfrak{p}^+$ and $\mathfrak{p}^-$ be the sum of all the root spaces for positive and, respectively, negative noncompact roots. Then $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are abelian subalgebras of $\mathfrak{g}^{\mathbf{C}}$ with sum $\mathfrak{p}^{\mathbf{C}}$, and each is stable under $\mathfrak{k}$; $\mathfrak{k}$ acts irreducibly on $\mathfrak{p}^+$ and $\mathfrak{p}^-$ if $\mathfrak{g}$ is simple. Let P^+ and P^- be the corresponding analytic subgroups of $G^{\mathbf{C}}$.

We recall the Harish-Chandra decomposition of G [6, p. 4; 7, p. 590]. Let $\mathfrak{b}$ be the Borel subalgebra

$$\mathfrak{b} = \mathfrak{h}^{\mathbf{C}} + \sum_{\alpha \in P} \mathfrak{g}_{-\alpha},$$

and let B be the corresponding analytic subgroup of $G^{\mathbf{C}}$. Then we have the inclusions

$$BG \subseteq P^- K^{\mathbf{C}} P^+ \subseteq G^{\mathbf{C}}.$$

Moreover, BG and $P^- K^{\mathbf{C}} P^+$ are open in $G^{\mathbf{C}}$, the complex structure that $P^- K^{\mathbf{C}} P^+$ inherits from $G^{\mathbf{C}}$ is the same as the product structure, and $BG = P^- K^{\mathbf{C}} \Omega$ for a bounded open subset $\Omega \subseteq P^+$.

Let Λ be an integral linear form on $\mathfrak{h}^{\mathbf{C}}$, dominant with respect to $\mathfrak{k}$. That is, we suppose that there is a character $\xi_\Lambda(h)$ defined on $T^{\mathbf{C}}$ such that $\xi_\Lambda(\exp H) = e^{\Lambda(H)}$ for $H \in \mathfrak{h}^{\mathbf{C}}$, and we suppose that $\langle \Lambda, \alpha \rangle \geqslant 0$ for all $\alpha \in P_k$. Then $2\langle \Lambda, \alpha \rangle / \langle \alpha, \alpha \rangle$ is an integer for all $\alpha \in \Sigma_k$. Let ρ be half the sum of the positive roots (compact and noncompact). Then $2\langle \rho, \alpha \rangle / \langle \alpha, \alpha \rangle = 1$ for α simple, and it follows that an integral form Λ is dominant with respect to $\mathfrak{k}$ if and only if $\langle \Lambda + \rho, \alpha \rangle > 0$ for all $\alpha \in P_k$. (This equivalence uses the fact that the compact roots that are simple in P generate P_k, which is a consequence of the $0 - 1$ property of coefficients of the noncompact simple roots.)

Thus $\langle \Lambda + \rho, \alpha \rangle > 0$ for $\alpha \in P_k$. We define q_Λ to be the number of $\alpha \in P_n$ such that $\langle \Lambda + \rho, \alpha \rangle > 0$. The condition $q_\Lambda = 0$ is a condition of holomorphicity. In fact, Harish-Chandra showed in [7], under the assumption that $q_\Lambda = 0$ and $\Lambda + \rho$ is nonsingular (i.e., $\langle \Lambda + \rho, \alpha \rangle \neq 0$ for all $\alpha \in \Sigma$), that one could associate to Λ an irreducible unitary representation in a space of holomorphic functions. In this paper, we shall be concerned partly with a similar problem for the case that $q_\Lambda = 0$ and $\Lambda + \rho$ is singular.

If G is compact, then $\Lambda + \rho$ is automatically nonsingular, and our theory will be empty for this case. Thus let G be noncompact. Let α_0 be the largest root.

LEMMA 2.1. *α_0 is in P_n. If G is simple, if $q_\Lambda = 0$, and if $\Lambda + \rho$ is singular, then $\langle \Lambda + \rho, \alpha_0 \rangle = 0$ and $\langle \Lambda + \rho, \alpha \rangle < 0$ for all other $\alpha \in P_n$.*

Proof. α_0 is noncompact by the compatibility of the ordering. Let $\alpha \in P_n$. If $\mathfrak{g}$ is simple, then $\mathfrak{k}^{\mathbf{C}}$ acts irreducibly on $\mathfrak{p}^+$, and α and α_0 are weights of this representation, with α_0 the highest weight. Then $\alpha = \alpha_0 - \Sigma n_i \alpha_i$, with $n_i \geqslant 0$ and with the α_i simple for P. By the $0 - 1$ property of the coefficients of noncompact roots, each of the α_i must be compact. Then

$$\langle \Lambda + \rho, \alpha \rangle = \langle \Lambda + \rho, \alpha_0 \rangle - \Sigma n_i \langle \Lambda + \rho, \alpha_i \rangle \leqslant \langle \Lambda + \rho, \alpha_0 \rangle$$

with equality if and only if all $n_i = 0$ (and $\alpha = \alpha_0$). Since $q_\Lambda = 0$, $\langle \Lambda + \rho, \alpha_0 \rangle \leqslant 0$, and since $\Lambda + \rho$ is singular, we conclude $\langle \Lambda + \rho, \alpha_0 \rangle = 0$ and $\langle \Lambda + \rho, \alpha \rangle < 0$ for all other $\alpha \in P_n$. The lemma is proved.

In the semisimple case with $q_\Lambda = 0$, $\Lambda + \rho$ is nonsingular in some simple components, and Lemma 2.1 applies in the other components. We could attempt to take these matters into account, but we prefer not to complicate the notation by doing so. *As a result, we shall assume in all our theorems that G is simple.* (However, to prove our theorems, we shall have to use some results for G not simple that were proved by Harish-Chandra.) We shall not emphasize this point in this section and the next, because the results of these two sections have obvious versions in the semisimple case.

For each $\alpha \in \Sigma$, let H_α be the member of $\mathfrak{h}^{\mathbf{C}}$ such that $\alpha(H) = B(H_\alpha, H)$ for all $H \in \mathfrak{h}^{\mathbf{C}}$, where B is the Killing form. Let $\mathfrak{h}^+ = i\mathbf{R}H_{\alpha_0} \subseteq \mathfrak{h}$, and let $\mathfrak{h}^-$ be the orthogonal complement of $\mathfrak{h}^+$ in $\mathfrak{h}$. Let T^+ and T^- be the analytic subgroups of G with Lie algebras $\mathfrak{h}^+$ and $\mathfrak{h}^-$, so that $T = T^+T^-$.

LEMMA 2.2. *T^+ and T^- are closed, hence compact.*

Proof. Each root is an integral form on $\mathfrak{h}^{\mathbf{C}}$, being a weight of the adjoint representation. Then T^- is closed, because it is the identity component of $\ker(\xi_{\alpha_0}|_T)$. T^+ will be closed if we can show that $\mathrm{Ad}(T^+)$ is closed, which will be the case if we can show that $\mathrm{Ad}(\exp(icH_{\alpha_0}))$ is the identity for a suitable positive c. For any c, this transformation is the identity on $\mathfrak{h}^{\mathbf{C}}$, and on $\mathfrak{g}_\alpha$ it is the scalar $e^{ic\langle\alpha_0,\alpha\rangle}$. This scalar is 1 if c is chosen as $4\pi/\langle\alpha_0, \alpha_0\rangle$. Hence T^+ is closed.

For $\alpha \in \Sigma$, define

$$H_\alpha' = 2\langle\alpha, \alpha\rangle^{-1}H_\alpha .$$

Choose by Lemma 3.1 of p. 219 of [9], for each $\alpha \in P$, vectors E_α and $E_{-\alpha}$ in $\mathfrak{g}^{\mathbf{C}}$ such that

(i) $B(E_\alpha, E_{-\alpha}) = 2\langle\alpha, \alpha\rangle^{-1}$, and
(ii) $E_\alpha - E_{-\alpha}$ and $i(E_\alpha + E_{-\alpha})$ are in $\mathfrak{k} + i\mathfrak{p}$.

Then we have the bracket relations

$$[H_\alpha', E_\alpha] = 2E_\alpha, \qquad [H_\alpha', E_{-\alpha}] = -2E_{-\alpha}, \qquad [E_\alpha, E_{-\alpha}] = H_\alpha'.$$

Now $i(E_{\alpha_0} + E_{-\alpha_0})$ is in $\mathfrak{p}^{\mathbf{C}}$ since α_0 is noncompact, and $i(E_{\alpha_0} + E_{-\alpha_0})$ is in $\mathfrak{k} + i\mathfrak{p}$ by (ii); thus $E_{\alpha_0} + E_{-\alpha_0}$ is in $\mathfrak{p}$. Define

$$\begin{aligned} \mathfrak{a}^+ &= \mathbf{R}(E_{\alpha_0} + E_{-\alpha_0}) \subseteq \mathfrak{p} \\ \mathfrak{a}^- &= \mathfrak{h}^- \subseteq \mathfrak{h} \subseteq \mathfrak{k} \\ \mathfrak{a} &= \mathfrak{a}^+ + \mathfrak{a}^- \\ u_1 &= \exp\frac{\pi}{4}(E_{\alpha_0} - E_{-\alpha_0}) \in G^{\mathbf{C}}. \end{aligned}$$

The following lemma is a well-known simple computation.

LEMMA 2.3. $\mathrm{Ad}(u_1)$ *is the identity on* $\mathfrak{a}^-$ *and* $\mathrm{Ad}(u_1)\,\mathfrak{a}^+ = i\mathfrak{h}^+$. *Consequently,* $\mathrm{Ad}(u_1)\,\mathfrak{a}^{\mathbf{C}} = \mathfrak{h}^{\mathbf{C}}$ *and* $\mathfrak{a}$ *is a* θ*-stable Cartan subalgebra of* $\mathfrak{g}$.

Once we have a θ-stable Cartan subalgebra, the construction of a group MA^+N becomes a standard one, due originally to Harish-Chandra. We omit the well-known proofs in the construction, which occupies most of the rest of this section. (See, for example, [3] and [8, p. 212].) For the rest of the section, we assume that $G \subseteq G^{\mathbf{C}}$, that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$ with involution θ, that K is the analytic subgroup with algebra $\mathfrak{k}$, that $\mathfrak{a}$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$, and that $\mathfrak{a}^+ = \mathfrak{a} \cap \mathfrak{p}$ and $\mathfrak{a}^- = \mathfrak{a} \cap \mathfrak{k}$.

Let Σ' be the set of nonzero roots of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{a}^{\mathbf{C}})$, and let P' be the positive roots in some ordering. Let ρ' be half the sum of the positive roots. Let Σ_-' be the set of roots that vanish on $\mathfrak{a}^+$, and put $P_-' = \Sigma_-' \cap P'$. Let

$$Z_r(s) = \text{centralizer of } s \text{ in } r, \text{ for given } r \text{ and } s,$$

$$\mathfrak{m} = \text{orthocomplement of } \mathfrak{a}^+ \text{ in } Z_{\mathfrak{g}}(\mathfrak{a}^+),$$

$$M_0 = \text{analytic subgroup of } G \text{ with Lie algebra } \mathfrak{m},$$

$$M_0^{\mathbf{C}} = \text{analytic subgroup of } G^{\mathbf{C}} \text{ with Lie algebra } \mathfrak{m}^{\mathbf{C}},$$

$$A = Z_G(\mathfrak{a}),$$

$$A^{\mathbf{C}} = \text{analytic subgroup of } G^{\mathbf{C}} \text{ with Lie algebra } \mathfrak{a}^{\mathbf{C}}.$$

LEMMA 2.4. $\mathfrak{a}^-$ *is a compact Cartan subalgebra of* $\mathfrak{m}$, *and*

$$\mathfrak{m}^{\mathbf{C}} = (\mathfrak{a}^-)^{\mathbf{C}} + \sum_{\alpha \in \Sigma_-'} \mathfrak{g}_\alpha .$$

Also $Z_K(\mathfrak{a}^+)$ *normalizes* $\mathfrak{m}$.

The lemma shows that we obtain a group by defining

$$M = Z_K(\mathfrak{a}^+)\, M_0 .$$

Also let

$$A^+ = \text{analytic subgroup of } G \text{ with Lie algebra } \mathfrak{a}^+,$$

$$A^- = Z_M(\mathfrak{a}^-),$$

$$\mathfrak{n} = \Big(\sum_{\alpha \in P'} \mathfrak{g}_{-\alpha}\Big) \cap \mathfrak{g},$$

$$N = \text{analytic subgroup of } G \text{ with Lie algebra } \mathfrak{n}.$$

LEMMA 2.5. (a) $M = (M \cap K)(M \cap \exp \mathfrak{p})$, *and the latter factor is connected and is the exponential of the orthocomplement of* $\mathfrak{a}^+$ *in* $Z_{\mathfrak{p}}(\mathfrak{a}^+)$. *M is closed,*

(b) $MA^+ = Z_G(\mathfrak{a}^+)$, *and* MA^+ *is a direct product,*

(c) $A = A^C \cap G = A^+A^-$ *and* $A^- = A \cap K = A \cap M$,

(d) MA^+ *normalizes* N *and* MA^+N *is a closed subgroup,*

(e) $G = NA^+MK$ *with the* N, A^+, *and* MK *components unique,*

(f) *For* $m \in M$, $\det \operatorname{Ad}(m)|_{\mathfrak{a}^+ + \mathfrak{n}} = \pm 1$. *For* $a^+ \in A^+$,

$$\det \operatorname{Ad}(a^+)|_{\mathfrak{n}} = e^{-2\rho_+' \log a^+},$$

where $\rho' = \rho_+' + \rho_-'$ *is the decomposition of* ρ' *according to* $\mathfrak{a}^C = (\mathfrak{a}^+)^C + (\mathfrak{a}^-)^C$.

3. FURTHER PROPERTIES OF THE SUBGROUP M

We assume in Sections 3–7 that G is of the form described in Section 2. The purpose of this section is to develop further properties of the subgroup M. More specifically, first, we give the connection between the systems P and P' of positive roots; second, we prove in Proposition 3.1 the main structure theorem for M; third, we define $\mathfrak{h}_+^-$ and some subgroups and subalgebras bearing the subscript s; and last of all we prove an important simple identity for our linear form Λ.

First, we connect P and P'. We have chosen an ordering for $(\mathfrak{h}^C)'$, and P is therefore fixed. By Lemma 2.3, the mapping $\operatorname{Ad}(u_1) \in \operatorname{Ad}(G^C)$ sends $\mathfrak{a}^C$ into $\mathfrak{h}^C$. Defining ${}^t\operatorname{Ad}(u_1)\,\lambda(H') = \lambda(\operatorname{Ad}(u_1)\,H')$ for $\lambda \in (\mathfrak{h}^C)'$ and $H' \in \mathfrak{a}^C$, we see that ${}^t\operatorname{Ad}(u_1)\,\alpha$ is a root of $(\mathfrak{g}^C, \mathfrak{a}^C)$ whenever α is a root of $(\mathfrak{g}^C, \mathfrak{h}^C)$. If we use ${}^t\operatorname{Ad}(u_1)$ to transform the ordering from $(\mathfrak{h}^C)'$ to an ordering of $(\mathfrak{a}^C)'$, the result is that $P' = {}^t\operatorname{Ad}(u_1)\,P$. Let $\Sigma_- = {}^t\operatorname{Ad}(u_1)^{-1}\,\Sigma_-'$ and $P_- = {}^t\operatorname{Ad}(u_1)^{-1}\,P_-'$.

If $\alpha \in \Sigma$, we write α' for the corresponding member of Σ' given by

$$\alpha' = {}^t\operatorname{Ad}(u_1)\alpha. \tag{3.1}$$

It is easy to see that $\operatorname{Ad}(u_1)^{-1} E_\alpha$ is a root vector for the root α', and we therefore define

$$E_{\alpha'} = \operatorname{Ad}(u_1)^{-1} E_\alpha . \tag{3.2}$$

Since $\mathrm{Ad}(u_1)$ preserves the Killing form, we have the expected bracket relations

$$[H'_{\alpha'}, E_{\alpha'}] = 2E_{\alpha'}, \qquad [H'_{\alpha'}, E_{-\alpha'}] = -2E_{-\alpha'}, \qquad [E_{\alpha'}, E_{-\alpha'}] = H'_{\alpha'},$$

where

$$H'_{\alpha'} = 2\langle\alpha', \alpha'\rangle^{-1}H_{\alpha'} = 2\langle\alpha, \alpha\rangle^{-1}H_{\alpha'}.$$

LEMMA 3.1. *If* $X \in \mathfrak{g}^{\mathbf{C}}$ *and* $\exp X \in G$, *then* $\exp \bar{X} = \exp X$, *where bar denotes the conjugation of* $\mathfrak{g}^{\mathbf{C}}$ *with respect to* $\mathfrak{g}$.

LEMMA 3.2. $G \cap u_1^{-1}Bu_1 = AN.$

Proof. The left side contains the right side. In fact, by Lemma 2.5(c),

$$A = G \cap A^{\mathbf{C}} = G \cap u_1^{-1}T^{\mathbf{C}}u_1 \subseteq G \cap u_1^{-1}Bu_1.$$

Since N is connected, the inclusion will follow if we show $\mathfrak{g}_{\alpha'} \subseteq \mathrm{Ad}(u_1^{-1})\,\mathfrak{b}$ if $-\alpha' \in P'$. But this follows from (3.2), since $\mathfrak{b}$ is constructed from the negative roots for $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{h}^{\mathbf{C}})$.

For the reverse inclusion, we have

$$G \cap u_1^{-1}T^{\mathbf{C}}u_1 = G \cap A^{\mathbf{C}} = A \subseteq AN.$$

If we let N^- be the nilpotent part of B, the result will follow if we show that $G \cap u_1^{-1}N^-u_1 \subseteq AN$. This inclusion follows from the inclusion of the Lie algebras if we show that $G \cap u_1^{-1}N^-u_1$ is connected. Now $u_1^{-1}N^-u_1$ is simply connected, nilpotent, and complex. If $g \in G \cap u_1^{-1}N^-u_1$, then $g = \exp X$ for some X in the Lie algebra of $u_1^{-1}N^-u_1$. By Lemma 3.1, $g = \exp \bar{X}$ also. But $\bar{X}$ is in the Lie algebra also, since the algebra is complex. Since the exponential map is one-one on this algebra, $X = \bar{X}$. That is, X is in $\mathfrak{g}$. Then $\exp tX$, $0 \leqslant t \leqslant 1$, is a curve in $G \cap u_1^{-1}N^-u_1$ connecting the identity and g. Hence $G \cap u_1^{-1}N^-u_1$ is connected.

LEMMA 3.3. $B = P^-(B \cap K^{\mathbf{C}})$ *and* $B \cap K^{\mathbf{C}} = T^{\mathbf{C}}(N^- \cap K^{\mathbf{C}})$, *where* N^- *is the nilpotent part of* B.

Proof. $B \subseteq K^{\mathbf{C}}P^-$ since B is connected and $\mathfrak{b} \subseteq \mathfrak{k}^{\mathbf{C}} + \mathfrak{p}^-$. Hence $B \subseteq P^-(B \cap K^{\mathbf{C}})$. Since $P^- \subseteq B$, $P^-(B \cap K^{\mathbf{C}}) \subseteq B$ also.

Next, $T^{\mathbf{C}} \subseteq B \cap K^{\mathbf{C}}$ and $N^- \cap K^{\mathbf{C}} \subseteq B \cap K^{\mathbf{C}}$. So $T^{\mathbf{C}}(N^- \cap K^{\mathbf{C}}) \subseteq B \cap K^{\mathbf{C}}$. Also $B = T^{\mathbf{C}}N^-$. If $b = tn \in K^{\mathbf{C}}$, then $n \in K^{\mathbf{C}}$ since $t \in K^{\mathbf{C}}$. Hence $B \cap K^{\mathbf{C}} \subseteq T^{\mathbf{C}}(N^- \cap K^{\mathbf{C}})$.

LEMMA 3.4. (a) E_{α_0} *and* $E_{-\alpha_0}$ *commute with all* E_α *for* $\alpha \in \Sigma_-$;

(b) u_1 *commutes with all* E_α *for* $\alpha \in \Sigma_-$;

(c) $E_{\alpha'} = E_\alpha$ *if* $\alpha \in \Sigma_-$;

(d) $\mathfrak{m}^{\mathbf{C}} = (\mathfrak{h}^-)^{\mathbf{C}} + \sum_{\alpha \in \Sigma_-} \mathfrak{g}_\alpha$;

(e) T^+, E_{α_0}, $E_{-\alpha_0}$, *and* u_1 *commute with* $\mathfrak{m}^{\mathbf{C}}$.

Proof. For (a), it is enough by symmetry to show $[E_{\alpha_0}, E_{-\alpha}] = 0$ for $\alpha \in \Sigma_-$. Since $\alpha \in \Sigma_-$, $\langle \alpha_0, \alpha \rangle = \alpha(H_{\alpha_0}) = 0$. Hence the α-string containing α_0 is $\alpha_0 - p\alpha, \ldots, \alpha_0 + q\alpha$ with $p - q = 0$. If $\alpha < 0$, then $p = 0$ since α_0 is the largest root, and $\alpha_0 - \alpha$ is not a root. If $\alpha > 0$, then $q = 0$ since α_0 is the largest root, and so $p = q = 0$. Again $\alpha_0 - \alpha$ is not a root. Hence $[E_{\alpha_0}, E_{-\alpha}] = 0$. Conclusion (b) follows from (a) and the definition of u_1, and (c) follows from (b) and Eq. (3.2). For (d), we have

$$\mathfrak{m}^{\mathbf{C}} = (\mathfrak{a}^-)^{\mathbf{C}} + \sum_{\alpha' \in \Sigma_-'} \mathfrak{g}_{\alpha'} .$$

But $\mathfrak{a}^- = \mathfrak{h}^-$, and $\mathfrak{g}_{\alpha'} = \mathfrak{g}_\alpha$ for $\alpha \in \Sigma_-$, by (c). Thus (d) follows. For (e), T^+ commutes with $\mathfrak{m}^{\mathbf{C}}$ by (d), since $\mathrm{ad}(H_{\alpha_0})$ operates as 0 on each factor of $\mathfrak{m}^{\mathbf{C}}$, and E_{α_0}, $E_{-\alpha_0}$, and u_1 commute with $\mathfrak{m}^{\mathbf{C}}$ by (a), (b), and (d).

LEMMA 3.5. *If* $\alpha' \in P'$, *then* $\alpha'(H_{\alpha_0'}) \geqslant 0$.

Proof. The root α_0 is the highest weight of the adjoint representation, is therefore dominant. Thus $\langle \alpha_0, \alpha \rangle \geqslant 0$. Since $\mathrm{Ad}(u_1)$ preserves the Killing form, Eq. (3.1) shows that $\alpha'(H_{\alpha_0'}) = \langle \alpha_0', \alpha' \rangle \geqslant 0$.

We turn to the main structure theorem for M, which is given as Proposition 3.1 below. The proof given here is due to the referee and is shorter than our own. Let Z_M be the center of M, and let

$$\eta = u_1^4 = \exp \pi(E_{\alpha_0} - E_{-\alpha_0}). \tag{3.3}$$

The main content of the result is that η is in the center of M and $M = M_0 \cup \eta M_0$. Consequently, $M = Z_M M_0$. Although the latter formulation of this result makes sense for the M constructed from any Cartan subalgebra A (in the manner described in Section 2), the result is not true in such generality. For example, with $SL(3, \mathbf{R})$ and with $Sp(2, \mathbf{R})$, it is possible to choose A so that A^+ is one-dimensional and M is the group of real 2-by-2 matrices of determinant ± 1. This group is not generated by its center and its identity component.

PROPOSITION 3.1. (a) *$M = M_0 \cup \eta M_0$, and $\eta \in Z_M \cap T^+$;*
(b) *$\mathrm{Ad}(M)$ is the identity on E_{α_0}, $E_{-\alpha_0}$, $E_{\alpha_0'}$, and $E_{-\alpha_0'}$;*
(c) *M commutes with u_1.*

The proof will be preceded by two lemmas. By [9, Proposition 7.4, pp. 314–315], there exists a subset $\gamma_1,..., \gamma_s$ of P_n such that the subspace $\mathfrak{a}_0 = \sum_{j=1}^{s} \mathbf{R}(E_{\gamma_j} + E_{-\gamma_j})$ is maximal abelian in $\mathfrak{p}$. According to Moore [14, p. 364], the roots γ_j can be chosen in such a way that

(i) γ_1 is the smallest noncompact positive root,

(ii) the γ_j have the same length,

(iii) the γ_j are strongly orthogonal in the sense that sums and differences of pairs of them are not roots,

(iv) the only restricted roots relative to $\mathfrak{a}_0$ are $d(\pm\frac{1}{2}\gamma_j \pm \frac{1}{2}\gamma_k)$ and possibly $d(\pm\frac{1}{2}\gamma_j)$, where $d = \mathrm{Ad}(\exp \pi/4 \sum_{j=1}^{s} (E_{-\gamma_j} - E_{\gamma_j}))$.

The construction at the end of Section 2 produced M from the largest noncompact positive root α_0, and we repeat this construction with the smallest noncompact positive root γ_1, writing $M_1,..., \eta_1$ in place of $M,..., \eta$. Lemma 3.6 is contained in Lemmas 1 and 3 of Moore [13].

LEMMA 3.6. $M_1 = (M_1)_0(\exp(i\mathfrak{a}_0) \cap K)$.

LEMMA 3.7. *If $G^{\mathbf{C}}$ is simply-connected, then the lattice*

$$\{H \in i\mathfrak{a}_0 \mid \exp H \in K\}$$

is generated by the vectors $2\pi i\langle\beta, \beta\rangle^{-1} H_\beta$, where β ranges over the restricted roots.

Proof. The vectors in question are in the lattice, by a computation in $SL(2, \mathbf{R})$. Conversely, if H is in the lattice, then it is well known that $\exp 2H = 1$. By Theorems 3.4(a) and 3.6 of [11, pp. 75–77], $2H$ is in the lattice generated by the vectors $4\pi i\langle\beta, \beta\rangle^{-1} H_\beta$.

Proof of Proposition 3.1. We may assume that $G^{\mathbf{C}}$ is simply-connected. We consider first the group M_1. Since the roots γ_j have the same length and are orthogonal, we have

$$\pi i H'_{\gamma_j/2 \pm \gamma_k/2} = \pi i H'_{\gamma_j} \pm \pi i H'_{\gamma_k} \qquad \text{and} \qquad \pi i H'_{\gamma_j/2} = 2(\pi i H'_{\gamma_j}).$$

The element d carries $H'_{\gamma_j} = 2\langle\gamma_j, \gamma_j\rangle^{-1} H_{\gamma_j}$ into

$$2\langle d(\gamma_j), d(\gamma_j)\rangle^{-1} H_{d(\gamma_j)} = E_{\gamma_j} + E_{-\gamma_j},$$

and Lemma 3.7 therefore shows that $\exp(i\mathfrak{a}_0) \cap K$ is generated by the elements

$$\eta_j = \exp \pi i(E_{\gamma_j} + E_{-\gamma_j}), \qquad 1 \leqslant j \leqslant s.$$

A computation in $SL(2, \mathbf{R})$ with the explicit form of d shows that

$$\eta_j = \exp \pi(E_{\gamma_j} - E_{-\gamma_j}) = \exp \pi i H'_{\gamma_j}$$

and $\eta_j^2 = 1$. Now $\mathfrak{a}_1^+$ is spanned by $E_{\gamma_1} + E_{-\gamma_1}$. Since γ_j is orthogonal to γ_1 for $j \neq 1$, iH'_{γ_j} is in $\mathfrak{m}_1$ for $j \neq 1$. Thus η_j is in $(M_1)_0$ for $j \neq 1$. By Lemma 3.6, M_1 is then generated by $(M_1)_0$ and η_1. Since η_1 is in $\exp i\mathfrak{a}_1^+$, η_1 is in the center of M_1. Since $\eta_1^2 = 1$, $M_1 = (M_1)_0 \cup \eta_1(M_1)_0$.

Next we pass from M_1 to M. It is easy to see that part (a) of the proposition follows if we can show that α_0 and γ_1 are conjugate under the Weyl group of $(\mathfrak{k}^{\mathbf{C}}, \mathfrak{h}^{\mathbf{C}})$. To see this conjugacy, let w be the element of the Weyl group of K that carries all the positive roots into negative roots. Regarded as an element of the Weyl group of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{h}^{\mathbf{C}})$, w carries all the positive compact roots into negative compact roots and it permutes the positive noncompact roots (since $\mathfrak{k}$ normalizes $\mathfrak{p}^+$). Since α_0 is the unique positive noncompact root whose sum with each positive compact root fails to be a root, w maps α_0 into the unique positive noncompact root whose sum with each negative compact root fails to be a root. Thus w maps α_0 into γ_1. This completes the proof of (a).

For (c), u_1 commutes with M_0 by Lemma 3.4(e) and with η because $\eta = u_1^4$. For (b), $\mathrm{Ad}(M_0)$ is the identity on E_{α_0} and $E_{-\alpha_0}$ by Lemma 3.4(e), and $\mathrm{Ad}(\eta)$ is the identity on E_{α_0}, $E_{-\alpha_0}$, $E_{\alpha_0'}$, and $E_{-\alpha_0'}$ because it is the identity on the entire span of H_{α_0}, E_{α_0}, and $E_{-\alpha_0}$. This result, together with the commutativity of u_1 with $\mathfrak{m}^{\mathbf{C}}$ (Lemma 3.4(e)), shows that $\mathrm{Ad}(M_0)$ is the identity on $E_{\alpha_0'}$ and $E_{-\alpha_0'}$. Thus (b) is proved, and the proof of Proposition 3.1 is complete.

The third part of Section 3 is a discussion of the semisimple part of M. The Lie algebra $\mathfrak{m}$ is stable under the Cartan involution and so is reductive, and we can therefore write

$$\mathfrak{m} = \mathfrak{h}_+^- + \mathfrak{m}_s,$$

where $\mathfrak{h}_+^-$ and $\mathfrak{m}_s$ are, by definition, the center and commutator subalgebra of $\mathfrak{m}$, respectively. Here $\mathfrak{m}_s = 0$ or $\mathfrak{m}_s$ is semisimple.

Clearly, $\mathfrak{h}_+^- \subseteq \mathfrak{h}^-$, and we let $\mathfrak{h}_s$ be the orthogonal complement of $\mathfrak{h}_+^-$ in $\mathfrak{h}^-$. Then

$$\mathfrak{m}_s^{\mathbf{C}} = \mathfrak{h}_s^{\mathbf{C}} + \sum_{\alpha \in \Sigma_-} \mathfrak{g}_\alpha , \tag{3.4}$$

by Lemma 3.4(d). The roots $\alpha \in \Sigma_-$ vanish on $\mathfrak{h}_+^-$ and thus cannot vanish identically on $\mathfrak{h}_s$. It follows from (3.4) that $\mathfrak{h}_s$ is a Cartan subalgebra of $\mathfrak{m}_s$ and that the roots of $(\mathfrak{m}_s^{\mathbf{C}}, \mathfrak{h}_s^{\mathbf{C}})$ are the restrictions to $\mathfrak{h}_s^{\mathbf{C}}$ of the members of Σ_-. Since $\mathfrak{m}_s$ is stable under θ, $\mathfrak{m}_s = (\mathfrak{k} \cap \mathfrak{m}_s) + (\mathfrak{p} \cap \mathfrak{m}_s)$ is a Cartan decomposition of $\mathfrak{m}_s$ and $\mathfrak{k} \cap \mathfrak{m}_s$ is a maximal compactly imbedded subalgebra. Let T_+^-, M_s, T_s, and K_s be the analytic subgroups with Lie algebras $\mathfrak{h}_+^-$, $\mathfrak{m}_s$, $\mathfrak{h}_s$, and $\mathfrak{k} \cap \mathfrak{m}_s$, respectively.

LEMMA 3.8. *M_s is closed, and T_+^-, T_s, and K_s are compact. K_s is a maximal compact subgroup of M_s, and $T_s \subseteq K_s$ is a compact Cartan subgroup of M_s. The groups K_s, T_+^-, and T^+ mutually commute.*

Proof. M_s is a semisimple group of matrices and is therefore closed. (See p. 128 of [9].) T_+^- is closed because it is the identity component of the center of M, which is closed by Lemma 2.2. Since $T_+^- \subseteq K$, T_+^- is compact. K_s is maximal compact since $\mathfrak{k} \cap \mathfrak{m}$ is maximal compactly imbedded and M_s has finite center. Since $\mathfrak{h}_s$ is a compact Cartan subalgebra of $\mathfrak{m}_s$, its centralizer is compact and connected and is therefore T_s. Finally, the only nontrivial relation of commutativity is between K_s and T^+, and this is given in Lemma 3.4(e). The lemma is proved.

Before passing to the next proposition, we remark that

$$G = NA^+M_sK \tag{3.5}$$

because, by Lemma 2.5(e), $G = NA^+MK = NA^+M_0Z_K(\mathfrak{a}^+)\, K = NA^+M_0K = NA^+M_sT_+^-K = NA^+M_sK$.

Now we must emphasize that G is assumed to be simple.

PROPOSITION 3.2. *If $\mathfrak{m}_s$ is noncompact, then $\mathfrak{m}_s$ is the direct sum of a compact semisimple subalgebra and a noncompact simple subalgebra. In any case, M_s/K_s is hermitian symmetric, and the restriction of the ordering on Σ to the roots of $(\mathfrak{m}_s^{\mathbf{C}}, \mathfrak{h}_s^{\mathbf{C}})$ is compatible with the complex structure.*

Proof. The simple roots (in P_-) for $\mathfrak{m}_s$ are the simple roots of P that are in P_-. In fact, let $\alpha \in P_-$ be simple for $\mathfrak{m}_s$, and suppose $\alpha = \beta + \gamma$ with β and γ in P but not P_-. Then Lemma 3.5 and the nonvanishing of $\beta(H_{\alpha_0})$ and $\gamma(H_{\alpha_0})$ imply that $\beta(H_{\alpha_0}) > 0$ and $\gamma(H_{\alpha_0}) > 0$. Hence $\alpha(H_{\alpha_0}) > 0$, and α cannot be in P_-, contradiction.

Divide the simple roots for $\mathfrak{m}_s$ into systems corresponding to each simple component of $\mathfrak{m}_s$. Since P has exactly one noncompact simple root, the result of the preceding paragraph shows that at most one of the components has a noncompact simple root. Thus at the most one component is noncompact.

It follows that M_s/K_s is hermitian if $\mathfrak{k}_s$ has a nonzero center. Let E^+, E_+^-, and E_s be the projections of $\mathfrak{h}$ corresponding to the decomposition $\mathfrak{h} = \mathfrak{h}^+ + \mathfrak{h}_+^- + \mathfrak{h}_s$, and let X be a nonzero vector in the center of $\mathfrak{k}$. If $\alpha \in \Sigma_-$, we have

$$\alpha(X) = \alpha(E^+X) + \alpha(E_+^-X) + \alpha(E_sX) = \alpha(E_sX).$$

If also $\alpha \in \Sigma_k$, then $\alpha(X) = 0$ and so $\alpha(E_sX) = 0$. That is, E_sX is in the center of $\mathfrak{k}_s$. If $\mathfrak{m}_s$ is noncompact, then there is a noncompact root α in Σ_-. For such an α, $\alpha(X) \neq 0$ since no noncompact root vanishes on the center. Then $\alpha(E_sX) \neq 0$, and $E_sX \neq 0$. Hence $\mathfrak{k}_s$ has a nonzero center.

The restricted ordering on Σ_- is compatible with the complex structure because every positive noncompact root is still larger than every compact root. This completes the proof.

The proposition shows that we can define groups P^-, P^+, and B for M_s. We denote these by P_s^-, P_s^+, and B_s. We have a corresponding Harish-Chandra decomposition

$$B_sM_s \subseteq P_s^-B_sP_s^+ \subseteq M_s^{\mathbf{C}},$$

and $P_s^- \subseteq P^-$, $B_s \subseteq B$, and $P_s^+ \subseteq P^+$. Let us remark that u_1 commutes with $M_0^{\mathbf{C}}$, by Lemma 3.4(e), and therefore u_1 commutes with the subgroups M_s, B_s, P_s^-, and P_s^+.

PROPOSITION 3.3. *$Z_M = (T_+^-Z_{M_s}) \cup \eta(T_+^-Z_{M_s})$, where Z_{M_s} is the center of M_s. Also $Z_M \subseteq T$, Z_M commutes with $M_0^{\mathbf{C}}$, and $M = Z_MM_s$.*

Proof. By Proposition 3.1(a), $Z_M = Z_{M_0} \cup \eta Z_{M_0}$. Since $M_0 = T_+^-M_s$ and $T_+^- \subseteq Z_{M_0}$, $Z_{M_0} = T_+^-Z_{M_s}$. To see that $Z_M \subseteq T$, we use that $Z_{M_s} \subseteq T_s$ since T_s is a Cartan subgroup of M_s (Lemma 3.8). By Proposition 3.1(a), $\eta \in T^+$, and therefore $Z_M \subseteq T$. Z_M commutes with $\mathfrak{m}$, and thus Z_M commutes with $M_0^{\mathbf{C}}$. Also, by Proposition 3.1(a), $M = Z_MM_0 = Z_MT_+^-M_s = Z_MM_s$.

Finally, we return to a consideration of the integral form Λ on $\mathfrak{h}^{\mathbf{C}}$, dominant with respect to $\mathfrak{k}$, such that $\langle \Lambda + \rho, \alpha_0 \rangle = 0$. We shall exhibit the simple key identity that will allow us in the next section to associate a unitary representation of G to Λ. We have a direct sum decomposition $\mathfrak{a}^{\mathbf{C}} = (\mathfrak{a}^+)^{\mathbf{C}} + (\mathfrak{a}^-)^{\mathbf{C}}$, and corresponding to this we decompose each $\lambda \in (\mathfrak{a}^{\mathbf{C}})'$ as $\lambda = \lambda_+ + \lambda_-$. Let us put $\Lambda' = {}^t\mathrm{Ad}(u_1)\,\Lambda$. As particular cases of our notation, we have $\rho' = \rho_+' + \rho_-'$ and $\Lambda' = \Lambda_+' + \Lambda_-'$. The key identity is the identification of Λ_+' (or of Λ_-'), which we can write in the equivalent forms

$$\Lambda_-' = \Lambda' + \rho' - \rho_-' \qquad \text{or} \quad \Lambda_+' = -\rho_+'. \tag{3.6}$$

To prove these identities, all we need do is verify that $\Lambda' + \rho' - \rho_-'$ vanishes on $\mathfrak{a}^+$. But ρ_-' vanishes on $\mathfrak{a}^+$ by definition, and

$$\begin{aligned}(\Lambda' + \rho')(H_{\alpha_0'}) &= \langle \Lambda' + \rho', \alpha_0' \rangle = \langle {}^t\mathrm{Ad}(u_1)(\Lambda + \rho), {}^t\mathrm{Ad}(u_1)\,\alpha_0 \rangle \\ &= \langle \Lambda + \rho, \alpha_0 \rangle = 0.\end{aligned}$$

This proves (3.6). The particular form in which we shall use (3.6) is given in the proposition below, which is an immediate consequence of the definition of Λ'.

PROPOSITION 3.4. *If $a^+ \in A^+$, then* $({}^t\mathrm{Ad}(u_1)\Lambda)\log a^+ = -\rho_+' \log a^+$.

4. Construction of Limits of Holomorphic Discrete Series

Let Λ be an integral form on $\mathfrak{h}^{\mathbf{C}}$ dominant with respect to $\mathfrak{k}$. We have seen that $\langle \Lambda + \rho, \alpha \rangle > 0$ for every compact positive root α, and we have defined q_Λ to be the number of noncompact positive roots α such that $\langle \Lambda + \rho, \alpha \rangle > 0$. We shall always assume that $q_\Lambda = 0$.

To each such Λ, it is possible to associate in a natural way an irreducible unitary representation $U_\Lambda(g)$ of G in a space of holomorphic functions. For the case that $\Lambda + \rho$ is nonsingular (i.e., $\langle \Lambda + \rho, \alpha \rangle \neq 0$ for all positive roots α), U_Λ was constructed by Harish-Chandra in [6, 7]. We shall be concerned here with the case that $\Lambda + \rho$ is singular. Our argument will require a precise statement of Harish-Chandra's theorem, which we give below, and we therefore begin without the assumption that $\Lambda + \rho$ is singular.

We know that BG and $P^-K^{\mathbf{C}}P^+$ are open subsets of $G^{\mathbf{C}}$ and that $BG \subseteq P^-K^{\mathbf{C}}P^+ \subseteq G^{\mathbf{C}}$. Then BG inherits a complex structure from $G^{\mathbf{C}}$. Since Λ is integral, there exists a character ξ_Λ of T such that

$\xi_\Lambda(\exp H) = \exp \Lambda(H)$ for all $H \in \mathfrak{h}$. Then ξ_Λ extends uniquely to a holomorphic character of B. Define

$$\Gamma(\Lambda) = \{f : BG \to \mathbf{C} \mid (1)\, f \text{ is holomorphic} \quad (2)\, f(bx) = \xi_\Lambda(b) f(x) \text{ for } b \in B,\ x \in BG\},$$

and let $U_\Lambda(g) f(x) = f(xg)$ for $f \in \Gamma(\Lambda)$, $x \in BG$, and $g \in G$.

Harish-Chandra proved in Lemma 6 of [6] that $\Gamma(\Lambda) \neq 0$, and he gave in Lemma 14 a formula for a distinguished member of $\Gamma(\Lambda)$. We shall give a slightly different formula for this member of $\Gamma(\Lambda)$. The equivalence of these formulas will be verified in Section 6 after Lemma 6.1.

Let τ_Λ be an irreducible unitary representation of K with highest weight Λ and a highest weight vector ϕ_Λ of norm one. Extend τ_Λ to a holomorphic representation of $K^{\mathbf{C}}$ on the same vector space. If $x \in P^- K^{\mathbf{C}} P^+$, we let $\mu(x)$ be its $K^{\mathbf{C}}$ component. Define, for $x \in P^- K^{\mathbf{C}} P^+$,

$$\psi_\Lambda(x) = (\tau_\Lambda(\mu(x))\, \phi_\Lambda\,, \phi_\Lambda).$$

The inner product is assumed linear in the first variable and conjugate-linear in the second. As a straightforward consequence of Lemma 3.3, one can prove

LEMMA 4.1. *ψ_Λ is in $\Gamma(\Lambda)$.*

Suppose now for the moment that $\Lambda + \rho$ is *nonsingular*. For $f \in \Gamma(\Lambda)$, define $\| f \|^2 = \int_G | f(x)|^2\, dx$. Let $H(\Lambda) \subseteq \Gamma(\Lambda)$ be the subspace of functions of finite norm. $H(\Lambda)$ can be shown to be a (complete) Hilbert space. It is clear that $U_\Lambda(g)$ is a unitary representation on this space. Remembering that G in this paper is assumed to have a faithful matrix representation, we can state Theorem 4 of [7] as follows. (See also p. 612 of [7].)

THEOREM (Harish-Chandra). *If $q_\Lambda = 0$ and if $\Lambda + \rho$ is nonsingular, then $\| \psi_\Lambda \| < \infty$. The representation $U_\Lambda(g)$ of G on $H(\Lambda)$ is nonzero, irreducible, and unitary, and its matrix coefficients are square-integrable.*

Now suppose that $\Lambda + \rho$ is *singular* (and $q_\Lambda = 0$). In case G is simple, we shall define a norm on the members of $\Gamma(\Lambda)$ by means of an integral of their boundary values. First we need to know that Bu_1G is contained in the closure of BG. Define

$$u_t = \exp \frac{\pi t}{4} (E_{\alpha_0} - E_{-\alpha_0})$$

for $0 \leqslant t < 1$, so that $u_1 = \lim_{t \to 1} u_t$.

LEMMA 4.2. *For* $0 \leqslant t < 1$, u_t *is in* BG. *The decomposition of* u_t *according to* $BG = P^- K^{\mathbf{C}} \Omega$ *is* $u_t = \zeta_t k_t z_t$, *for* $0 \leqslant t < 1$, *where*

$$\zeta_t = \exp(-(\tan \pi t/4)\, E_{-\alpha_0}),$$

$$k_t = \exp(\log(\cos \pi t/4)\, H'_{\alpha_0}),$$

$$z_t = \exp((\tan \pi t/4)\, E_{\alpha_0}).$$

Consequently, Bu_1G *is contained in the closure of* BG.

Proof. The identity $u_t = \zeta_t k_t z_t$ is a straightforward computation in $SL(2, \mathbf{C})$. To prove the lemma, we must show that z_t is in Ω. To do this, we observe that $v_s = \exp s(E_{\alpha_0} + E_{-\alpha_0})$ is in G. Therefore, in the decomposition $v_s = \zeta_s' k_s' z_s'$, we must have $z_s' \in \Omega$. By Lemma 9 of [7], $z_s' = \exp((\tanh s)\, E_{\alpha_0})$. If $0 \leqslant t < 1$, we can choose s so that $\tan \pi t/4 = \tanh s$. For such a choice of s, $z_s' = z_t$. Therefore z_t is in Ω.

Let $\Gamma_0(\Lambda) \subseteq \Gamma(\Lambda)$ be the subspace of functions having a holomorphic extension to a neighborhood of the closure of BG in $G^{\mathbf{C}}$. Suppose G is simple. Since $\Lambda + \rho$ is singular, we can define M and M_s as in Sections 2–3. For f in $\Gamma_0(\Lambda)$, let

$$\| f \|^2 = \int_{M_s \times K} | f(u_1 mk)|^2\, dm\, dk, \tag{4.1}$$

and let $\Gamma_2(\Lambda)$ be the subspace of $\Gamma_0(\Lambda)$ of functions of finite norm. In Section 7, we shall see that each nonzero member of $\Gamma_2(\Lambda)$ has nonzero norm, but we ignore this fact for the present. Factor $\Gamma_2(\Lambda)$ by the subspace of functions of zero norm, and let $H(\Lambda)$ be the completion. Then $H(\Lambda)$ is a Hilbert space. We recall the action of G on $\Gamma(\Lambda)$ by $U_\Lambda(g) f(x) = f(xg)$.

THEOREM 4.1. *If* G *is simple, if* $q_\Lambda = 0$, *and if* $\Lambda + \rho$ *is singular, then* $\Gamma_2(\Lambda)$ *is stable under* $U_\Lambda(g)$, *and* $U_\Lambda(g)$ *acts by unitary transformations on it.* $U_\Lambda(g)$ *extends to a continuous unitary representation of* G *on* $H(\Lambda)$.

It is worth noticing that the proof of Theorem 4.1 uses the full strength of the holomorphicity condition $q_\Lambda = 0$, whereas the proof of the nonvanishing of $H(\Lambda)$, given as Theorem 4.2, depends upon only a weaker condition. This fact suggests generalizations to nonunitary representations that we have not pursued.

Before proving the theorem, we make two remarks. The first is a transformation law for $\Gamma_0(\Lambda)$. Let $f \in \Gamma_0(\Lambda)$. Define a function F on G by $F(x) = f(u_1 x)$. Then F satisfies

$$F(ax) = \xi_\Lambda(u_1 a u_1^{-1}) F(x) \tag{4.2}$$

for all $a \in AN$. In fact, Lemma 3.2 shows that $G \cap u_1^{-1} B u_1 = AN$. Therefore $u_1 a u_1^{-1} \in B$ and

$$F(ax) = f(u_1 ax) = f(u_1 a u_1^{-1} u_1 x) = \xi_\Lambda(u_1 a u_1^{-1}) f(u_1 x) = \xi_\Lambda(u_1 a u_1^{-1}) F(x).$$

The second observation is a simple integration formula, which is a special case of a result on p. 66 of [2]. We state the formula as a lemma.

LEMMA 4.3. *Let W be a unimodular Lie group, and suppose X and Y are closed subgroups such that $W = XY$ and $X \cap Y$ is compact. Then the Haar measures of W, X, and Y may be normalized in such a way that, for any nonnegative or integrable Borel function f on W,*

$$\int_W f(w)\, dw = \int_{X \times Y} f(xy)\, d_l x\, d_r y.$$

Here $d_l x$ is a left Haar measure on X, and $d_r y$ is a right Haar measure on Y.

Proof of Theorem 4.1. For $a \in AN$, let $\sigma(a) = |\xi_\Lambda(u_1 a u_1^{-1})|^{-2}$. Letting $d_l(na^+)$ and $d_r(na^+)$ stand for left and right Haar measures on NA^+, we claim that

$$d_l(na^+) = \sigma(na^+)\, d_r(na^+). \tag{4.3}$$

In fact, by Lemma 1.2 on p. 365 of [9],

$$\begin{aligned} d_r(na^+) &= \det \mathrm{Ad}(na^+)\,|_{\mathfrak{a}^+ + \mathfrak{n}}\, d_l(na^+) \\ &= \det \mathrm{Ad}(a^+)\,|_{\mathfrak{n}}\, d_l(na^+) \\ &= \exp(-2\rho_+' \log a^+)\, d_l(na^+), \end{aligned} \tag{4.4}$$

by Lemma 2.5(f). Also

$$\sigma(na^+) = \sigma(a^+) = |\xi_\Lambda(u_1 a^+ u_1^{-1})|^{-2} \tag{4.5}$$

and

$$\begin{aligned} \xi_\Lambda(u_1 a^+ u_1^{-1}) &= \xi_\Lambda(\exp \mathrm{Ad}(u_1) \log a^+) = \exp(\Lambda(\mathrm{Ad}(u_1) \log a^+)) \\ &= \exp(({}^t\mathrm{Ad}(u_1)\Lambda) \log a^+) = \exp(-\rho_+' \log a^+) \end{aligned} \tag{4.6}$$

by Proposition 3.4. Equations (4.4–4.6) together prove (4.3).

Next, we observe that left Haar measure $d_l(na^+m)$ on NA^+M_s satisfies

$$d_l(na^+m) = d_l(na^+)\, dm \tag{4.7}$$

because M_s normalizes NA^+ and $\det \operatorname{Ad}(m)|_{\mathfrak{a}^+ + \mathfrak{n}} = 1$ for $m \in M_s$.

The rest of the proof is not very different from a standard argument that certain induced representations are unitary. Choose by Lemma 2.5(e) a continuous function $\varphi \geqslant 0$ on G such that

$$\int_{NA^+} \varphi(na^+x)\, d_r(na^+) = 1 \tag{4.8}$$

for all x in G. Let f be a member of $\Gamma_0(\Lambda)$ and define $F(x) = f(u_1x)$ for x in G. Then

$$\int_{M_s \times K} |F(mk)|^2\, dm\, dk = \int_G |F(x)|^2 \varphi(x)\, dx \tag{4.9}$$

by successive application of the formulas (4.8), (4.2), (4.3), (4.7) and then application of Lemma 4.3 with $G = (NA^+M)K$. To prove that $U_\Lambda(g)$ preserves $\Gamma_2(\Lambda)$ and acts by unitary transformations, we are to prove that the integral $\int_G |F(xg)|^2 \varphi(x)\, dx$ is independent of G. When we replace xg by x in the integral and unwind the proof of (4.9), using the identity of Lemma 4.3 and formulas (4.3) and (4.8), we obtain $\int_{M_s \times K} |F(mk)|^2\, dm\, dk$ as the value of the integral. This is independent of g, and so $U_\Lambda(g)$ is unitary.

This completes the proof of Theorem 4.1, except for the proof that $U_\Lambda(g)$ is strongly continuous. This fact will follow from the imbedding of $U_\Lambda(g)$ as a subrepresentation of an induced representation. Since the strong continuity will not be needed until after the imbedding is proved, we postpone the proof of the strong continuity to Section 5.

THEOREM 4.2. *If G is simple, if $q_\Lambda = 0$, and if $\Lambda + \rho$ is singular, then* $0 < \| \psi_\Lambda \| < \infty$. *Consequently, $H(\Lambda)$ is not* 0.

We shall reduce this theorem to Harish-Chandra's theorem stated earlier in this section. To do so, we require some intermediate steps in the proof of his theorem, which we collect as the following lemma, valid without the assumption that G is simple. The proof is contained in [7, pp. 598–599].

Let $\mathfrak{a}_0$ be the maximal abelian subspace of $\mathfrak{p}$ described in Section 3, let A_0 be the analytic subgroup corresponding to $\mathfrak{a}_0$, and let A_0^+ be the closed positive Weyl chamber in A_0. Then $G = KA_0^+K$, and there is a corresponding decomposition of Haar measure as $dx = D(a)\, dk'\, da\, dk$ for a function $D(a)$ on A_0^+.

LEMMA 4.4. (Harish-Chandra). *Let Φ be an integral form on $\mathfrak{h}^{\mathbf{C}}$, dominant with respect to $\mathfrak{k}$. Suppose that $q_\Phi = 0$ and $\Phi + \rho$ is nonsingular. Let τ_Φ be an irreducible unitary representation of K with highest weight Φ and with degree d_Φ. Then*

$$\| \psi_\Phi \|^2 = d_\Phi^{-2} \int_{A_0+} \mathrm{Tr}(\tau_\Phi(\mu(a))^2)\, D(a)\, da.$$

In Lemma 3.8 we proved that K_s is a maximal compact subgroup of M_s and that the group T_s, whose Lie algebra is $\mathfrak{h}_s$, is a Cartan subgroup of K_s and M_s. If λ is a linear form on $\mathfrak{h}^{\mathbf{C}}$, we define $\bar{\lambda}$ to be the restriction of λ to $\mathfrak{h}_s^{\mathbf{C}}$. With this notation we have the following lemma.

LEMMA 4.5. *The span of $\tau_\Lambda(K_s)\, \phi_\Lambda$ is irreducible under K_s, and the highest weight of this representation relative to $\mathfrak{h}_s$ is $\bar{\Lambda}$.*

Proof. It follows from Lemma 3.8 that $K_s T_+^- T^+$ is a connected compact subgroup of K with $T_+^- T^+$ contained in the center. We have $T \subseteq K_s T_+^- T^+$ because $\mathfrak{h} = \mathfrak{h}_s + \mathfrak{h}_+^- + \mathfrak{h}^+$ and because T is connected. Decompose $V_1 = \mathrm{span}\{\tau_\Lambda(K_s T_+^- T^+)\, \phi_\Lambda\}$ into irreducible components under $K_s T_+^- T^+$, and choose weight vectors $\phi_1, \ldots, \phi_u$ relative to T in the irreducible components. Write $\phi_\Lambda = \Sigma c_l \phi_l$ and apply T. Each ϕ_l with nonzero coefficients belongs to Λ under T. Since the space belonging to Λ is one-dimensional, $\phi_\Lambda = c_l \phi_l$ for some l. That is, ϕ_Λ lies in an irreducible subspace under $K_s T_+^- T^+$, and so V_1 is irreducible under $K_s T_+^- T^+$. Since $T_+^- T^+ \subseteq \mathrm{center}(K_s T_+^- T^+)$, K_s acts irreducibly on V_1. Since $\mathrm{span}\{\tau_\Lambda(K_s)\, \phi_\Lambda\} \subseteq V_1$, K_s acts irreducibly on $\mathrm{span}\{\tau_\Lambda(K_s)\, \phi_\Lambda\}$. For the statement about the highest weight, we need observe only that $E_\alpha \phi_\Lambda = 0$ for all $\alpha \in P_- \cap P_k$ to conclude that $\bar{\Lambda}$ is the highest weight. The lemma follows.

COROLLARY. *$\bar{\Lambda}$ is an integral form on $\mathfrak{h}_s^{\mathbf{C}}$, dominant with respect to $\mathfrak{k}_s$.*

The group M_s is semisimple, M_s/K_s is hermitian, and the restriction of a compatible ordering of the roots for $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{h})$ is compatible with the complex structure of M_s/K_s. (See Proposition 3.2.) We shall apply Harish-Chandra's results to M_s. Let ρ_s be half the sum of the positive roots of $(\mathfrak{m}_s^{\mathbf{C}}, \mathfrak{h}_s^{\mathbf{C}})$.

LEMMA 4.6. $\rho_s = \bar{\rho}$.

Proof. Let w be the Weyl group reflection corresponding to α_0. If $\alpha \in P - P_-$, then $\langle \alpha, \alpha_0 \rangle > 0$. Since $\langle -\alpha^w, \alpha_0 \rangle = \langle \alpha, \alpha_0 \rangle$ and since α_0 is the largest root, it follows that $-\alpha^w \in P$. Hence $-\alpha^w \in P - P_-$. Moreover, $(\alpha + (-\alpha^w))|_{\mathfrak{h}^-} = 0$ by definition of w. Thus by grouping the summands in pairs, we see that

$$\sum_{\alpha \in P - P_-} \alpha \,|_{\mathfrak{h}^-} = 0$$

and hence that

$$\sum_{\alpha \in P_-} \alpha \,|_{\mathfrak{h}^-} = \sum_{\alpha \in P} \alpha \,|_{\mathfrak{h}^-} .$$

Consequently, $\rho_s = \bar{\rho}$.

LEMMA 4.7. *$q_{\bar{\Lambda}} = 0$ and $\bar{\Lambda} + \rho_s$ is nonsingular with respect to M_s.*

Proof. We first prove: If $\alpha \in P_-$, then there exists $c_\alpha > 0$ such that $\langle \bar{\lambda}, \bar{\alpha} \rangle_s = c_\alpha \langle \lambda, \alpha \rangle$ for all linear forms λ on $\mathfrak{h}^{\mathbf{C}}$. In fact, let $H_{\bar{\alpha}} \in \mathfrak{h}_s{}^{\mathbf{C}}$ and $H_\alpha \in \mathfrak{h}^{\mathbf{C}}$ be defined relative to the Killing forms of $\mathfrak{m}_s{}^{\mathbf{C}}$ and $\mathfrak{g}^{\mathbf{C}}$, respectively. We have

$$[E_\alpha , E_{-\alpha}] = c_1 H_\alpha$$

for a constant $c_1 \neq 0$. Since $\alpha \in P_-$, $\bar{\alpha}$ is a root of $(\mathfrak{m}_s{}^{\mathbf{C}}, \mathfrak{h}_s{}^{\mathbf{C}})$ with $E_\alpha \in \mathfrak{m}_s{}^{\mathbf{C}}$ as root vector. Hence

$$[E_\alpha , E_{-\alpha}] = c_2 H_{\bar{\alpha}}$$

with $c_2 \neq 0$. Then

$$\begin{aligned} \langle \bar{\lambda}, \bar{\alpha} \rangle_s &= \bar{\lambda}(H_{\bar{\alpha}}) = c_2^{-1} \bar{\lambda}([E_\alpha , E_{-\alpha}]) \\ &= c_2^{-1} \lambda([E_\alpha , E_{-\alpha}]) = c_2^{-1} c_1 \lambda(H_\alpha) = c_2^{-1} c_1 \langle \lambda, \alpha \rangle. \end{aligned}$$

Thus $\langle \bar{\lambda}, \bar{\alpha} \rangle_s = c_\alpha \langle \lambda, \alpha \rangle$ with $c_\alpha = c_2^{-1} c_1$ independent of λ. Choose $\lambda = \alpha$, and it follows that $c_\alpha > 0$.

If $\alpha \in P_-$, then the above result and Lemma 4.6 give

$$\langle \bar{\Lambda} + \rho_s , \bar{\alpha} \rangle_s = c_\alpha \langle \Lambda + \rho, \alpha \rangle$$

with $c_\alpha > 0$. Moreover, α is compact if and only if $\bar{\alpha}$ is compact, since the Cartan decompositions of $\mathfrak{g}$ and $\mathfrak{m}_s$ are compatible. Hence

$\langle \bar{\Lambda} + \rho_s, \bar{\alpha} \rangle_s \leqslant 0$ for all noncompact $\bar{\alpha}$, and $q_{\bar{\Lambda}} = 0$. Finally, $\langle \Lambda + \rho, \alpha \rangle$ vanishes only for $\alpha = \pm\alpha_0$, by Lemma 2.1 since G is simple, and $\alpha_0 \notin P_-$ by construction. Hence $\bar{\Lambda} + \rho_s$ is nonsingular.

Proof of Theorem 4.2. Let $u_1 = \zeta_1 k_1 z_1$, $m \in M_s$, and $k \in K$. By Proposition 3.2, we can write $m = p\mu(m)\, q \in P_s^- K_s^{\mathbf{C}} P_s^+ \subseteq P^- K^{\mathbf{C}} P^+$. Now $P_s^- \subseteq M_s^{\mathbf{C}} \subseteq M_0^{\mathbf{C}}$, and so u_1 commutes with p by Lemma 3.4(e). Then

$$\begin{aligned} u_1 mk &= u_1\, p\mu(m)\, qk = pu_1\mu(m)\, qk = p\zeta_1 k_1 z_1 \mu(m)\, qk \\ &= [\,p\zeta_1][k_1\mu(m)k][k^{-1}(\mu(m)^{-1}z_1\mu(m))\, qk] \in P^- K^{\mathbf{C}} P^+. \end{aligned}$$

Thus

$$\begin{aligned} \| \psi_\Lambda \|^2 &= \int_{M_s \times K} |(\tau_\Lambda(\mu(u_1 mk))\, \phi_\Lambda, \phi_\Lambda)|^2\, dm\, dk \\ &= \int_{M_s \times K} |(\tau_\Lambda(k_1\mu(m)k)\, \phi_\Lambda, \phi_\Lambda)|^2\, dm\, dk \\ &= \int_{M_s \times K} |(\tau_\Lambda(k)\, \phi_\Lambda, \tau_\Lambda(\mu(m))^* \tau_\Lambda(k_1)^* \phi_\Lambda)|^2\, dm\, dk. \end{aligned}$$

By Lemma 4.2, $k_1 \in \exp i\mathfrak{h}$. Thus $\tau_\Lambda(k_1)^*\, \phi_\Lambda = \xi_\Lambda(k_1)\, \phi_\Lambda$ with $\xi_\Lambda(k_1)$ real. Substituting and applying Schur's Lemma, we obtain

$$\| \psi_\Lambda \|^2 = d_\Lambda^{-1} \xi_\Lambda(k_1)^2 \int_{M_s} \| \tau_\Lambda(\mu(m))^* \phi_\Lambda \|^2\, dm.$$

Construct an Iwasawa decomposition of M_s in the manner described before Lemma 4.4 and call the abelian factor A_s and the corresponding $D(a)$ function $D_s(a)$. Since $dm = D_s(a)\, dk\, da\, dk'$ for the decomposition $M_s = K_s A_s^+ K_s$, we have

$$\| \psi_\Lambda \|^2 = d_\Lambda^{-1} \xi_\Lambda(k_1)^2 \int_{K_s \times A_s^+ \times K_s} \| \tau_\Lambda(\mu(kak'))^* \phi_\Lambda \|^2\, D_s(a)\, dk\, da\, dk'.$$

Again K_s normalizes P^+ and P^-, and therefore $\mu(kak') = k\mu(a)\, k'$. Since $\tau_\Lambda(k')^*$ is unitary,

$$\| \psi_\Lambda \|^2 = d_\Lambda^{-1} \xi_\Lambda(k_1)^2 \int_{K_s \times A_s^+} \| \tau_\Lambda(\mu(a))^* \tau_\Lambda(k)^* \phi_\Lambda \|^2\, D_s(a)\, dk\, da.$$

Choose an orthonormal basis $\phi_1 = \phi_\Lambda, \ldots, \phi_{d_\Lambda}$ of weight vectors for τ_Λ in such a way that the first $d_{\bar{\Lambda}}$ vectors are in the K_s-irreducible subspace $\operatorname{span}\{\tau_\Lambda(K_s)\, \phi_\Lambda\}$. (Recall Lemma 4.5.) Write, for $k \in K_s$, $\tau_\Lambda(k)\, \phi_i =$

$\sum_j \tau_{ji}(k)\,\phi_j$. Then $\tau_\Lambda(k)^*\phi_\Lambda = \sum_{j\leqslant d_{\bar\Lambda}} \overline{\tau_{1j}(k)}\,\phi_j$. Substituting this expression, we obtain

$$\|\psi_\Lambda\|^2 = d_\Lambda^{-1}\xi_\Lambda(k_1)^2 \int_{K_s\times A_s^+} \sum_{j,l\leqslant d_{\bar\Lambda}} (\tau_\Lambda(\mu(a))^*\phi_j\,,\tau_\Lambda(\mu(a))^*\phi_l)$$

$$\times\ \overline{\tau_{1j}(k)}\,\tau_{1l}(k)\,D_s(a)\,dk\,da$$

$$= d_\Lambda^{-1}d_{\bar\Lambda}^{-1}\xi_\Lambda(k_1)^2 \int_{A_s^+} \sum_{j\leqslant d_{\bar\Lambda}} \|\tau_\Lambda(\mu(a))^*\phi_j\|^2\,D_s(a)\,da$$

by the Schur orthogonality relations for K_s, which apply by Lemma 4.5. As in the proof of Lemma 4.4, a is in $\exp i\mathfrak{h}$, and therefore $\tau_\Lambda(\mu(a))$ is self-adjoint and diagonal relative to the basis ϕ_j. Since the sum in the last integral extends only over $j \leqslant d_{\bar\Lambda}$, we conclude

$$\|\psi_\Lambda\|^2 = d_\Lambda^{-1}d_{\bar\Lambda}^{-1}\xi_\Lambda(k_1)^2 \int_{A_s^+} \operatorname{Tr}(\tau_{\bar\Lambda}(\mu(a))^2)\,D_s(a)\,da. \tag{4.10}$$

On the other hand, we can apply Lemma 4.4 to the group M_s. Lemma 4.7 and the corollary to Lemma 4.5 show that $\bar\Lambda$ satisfies the hypotheses of Lemma 4.4. Therefore

$$\|\psi_{\bar\Lambda}\|^2 = d_{\bar\Lambda}^{-2} \int_{A_s^+} \operatorname{Tr}(\tau_{\bar\Lambda}(\mu(a))^2)\,D_s(a)\,da.$$

Combining this expression with (4.10), we find that

$$\|\psi_\Lambda\|^2 = d_\Lambda^{-1}d_{\bar\Lambda}\xi_\Lambda(k_1)^2\,\|\psi_{\bar\Lambda}\|^2. \tag{4.11}$$

Again Lemma 4.7 says that $q_{\bar\Lambda} = 0$ and $\bar\Lambda + \rho_s$ is nonsingular. The assumptions of Harish-Chandra's theorem at the beginning of this section are satisfied, and therefore $\|\psi_{\bar\Lambda}\| < \infty$. By (4.11), we conclude that $\|\psi_\Lambda\| < \infty$.

This completes the proof of the finiteness of norm in Theorem 4.2, except for one remark. Our definition of $\psi_{\bar\Lambda}$ differs in form from Harish-Chandra's, and we should not apply his theorem until we have checked that the two definitions are equivalent. This equivalence will be verified after Lemma 6.1 and will not depend on any results of this section or the next, except for Lemma 4.1.

Finally, we observe that $\|\psi_\Lambda\| \neq 0$. In fact, define a continuous function ψ on $M_s \times K$ by $\psi(m, k) = |\psi_\Lambda(u_1mk)|^2$. Since ψ is continuous, it will follow that $\|\psi_\Lambda\| \neq 0$ if we show that $\psi(1, 1) \neq 0$. We have

$$\psi(1, 1) = |\psi_\Lambda(u_1)|^2 = |(\tau_\Lambda(k_1)\,\phi_\Lambda\,,\phi_\Lambda)|^2 = \xi_\Lambda(k_1)^2 \neq 0,$$

and the proof is complete.

5. Imbedding in Continuous Series

Suppose G is simple. Let Λ be an integral form on $\mathfrak{h}^{\mathbf{C}}$, dominant with respect to $\mathfrak{k}$, such that $q_\Lambda = 0$ and $\Lambda + \rho$ is singular. In this section we construct a discrete series representation ω_Λ of M and exhibit the representation $U_\Lambda(g)$, which was defined in Section 4, as a direct summand of the induced representation

$$V_\Lambda = \operatorname*{ind}_{MA^+N\uparrow G} (\omega_\Lambda \otimes 1 \otimes 1).$$

In Section 7 we shall see that the image of $U_\Lambda(g)$ is proper and hence that the induced representation $V_\Lambda(g)$ is reducible.

We begin by defining ω_Λ. By Lemma 4.7 the restriction $\bar{\Lambda}$ of Λ to $\mathfrak{h}_s$ has $q_{\bar{\Lambda}} = 0$, and $\bar{\Lambda} + \rho_s$ is nonsingular. We have defined $\Gamma(\bar{\Lambda})$ to be the space of holomorphic functions φ on $B_s M_s$ such that $\varphi(bx) = \xi_{\bar{\Lambda}}(b)\,\varphi(x)$ for $b \in B_s$ and $x \in B_s M_s$. Also $H(\bar{\Lambda})$ is the subspace of $\Gamma(\bar{\Lambda})$ of functions of finite norm, where

$$\|\varphi\|^2 = \int_{M_s} |\varphi(x)|^2\, dx < \infty.$$

By Proposition 3.3, we have $Z_M \subseteq T \subseteq B$, and hence $\xi_\Lambda(z)$ is defined if $z \in Z_M$. Moreover, $M = Z_M M_s$ by Proposition 3.3. We therefore define ω_Λ on $\Gamma(\bar{\Lambda})$ by

$$\omega_\Lambda(zm)\,\varphi(x) = \xi_\Lambda(z)\,\varphi(xm)$$

for $\varphi \in \Gamma(\bar{\Lambda})$, $z \in Z_M$, $m \in M_s$, and $x \in B_s M_s$.

Lemma 5.1. *The representation ω_Λ of M is unambiguously defined on $\Gamma(\bar{\Lambda})$ and is unitary, strongly continuous, and irreducible on $H(\bar{\Lambda})$.*

Proof. To see that ω_Λ is well-defined, we are to show that if $z \in Z_M \cap M_s$ and $\varphi \in \Gamma(\bar{\Lambda})$, then $\varphi(xz) = \xi_\Lambda(z)\,\varphi(x)$ for $x \in B_s M_s$. (Here $xz \in B_s M_s$ since $z \in M_s$.) Now z commutes with M_s because $z \in Z_M$, and z commutes with B_s by Proposition 3.3. Thus z commutes with x. Since $z \in Z_M \subseteq T \subseteq B$ (Proposition 3.3), we have $\varphi(xz) = \varphi(zx) = \xi_\Lambda(z)\,\varphi(x)$, as required.

Then ω_Λ is clearly a representation and is unitary and strongly continuous. Its restriction to M_s is irreducible by Harish-Chandra's theorem, and hence ω_Λ is irreducible as a representation of M. The lemma is proved.

We remark that ω_Λ has square-integrable matrix coefficients.

In fact, by the corollary to Theorem 2 of [6], we have, for $x = mz \in M_s Z_M = M$,

$$(\omega_\Lambda(x)\, \psi_{\bar{\Lambda}}, \psi_{\bar{\Lambda}}) = \xi_\Lambda(z)(U_{\bar{\Lambda}}(m)\, \psi_{\bar{\Lambda}}, \psi_{\bar{\Lambda}}) = \xi_\Lambda(z)\, \| \psi_{\bar{\Lambda}} \|^2\, \psi_{\bar{\Lambda}}(m).$$

The square-integrability of the left side (and hence of all matrix coefficients) follows by applying Lemma 4.3 with $W = M$, $X = M_s$, and $Y = Z_M$.

Form the continuous series representation

$$V_\Lambda = \operatorname*{ind}_{MA^+N \uparrow G} (\omega_\Lambda \otimes 1 \otimes 1).$$

In order to describe this representation explicitly, we need to know the modular factor for MA^+N. By Eqs. (4.3) and (4.6), the Haar measures for A^+N satisfy

$$d_r(a^+n) = \exp(-2\rho_+' \log a^+)\, d_l(a^+n).$$

Since M normalizes A^+N and $\det \mathrm{Ad}(m)|_{\mathfrak{a}^+ + \mathfrak{n}} = \pm 1$ for $m \in M$ (Lemma 2.5(f)), the Haar measures of MA^+N satisfy

$$d_r(ma^+n) = \exp(-2\rho_+' \log a^+)\, d_l(ma^+n).$$

Therefore we can regard $V_\Lambda(g)$ as operating in the space $\mathscr{H}(\Lambda)$ of almost-everywhere-defined functions $f : G \to H(\bar{\Lambda})$ such that

(i) for each $\varphi \in H(\bar{\Lambda})$, the function $x \to (f(x), \varphi)$ is measurable,

(ii) if $m \in M$, $a^+ \in A^+$, and $n \in N$, then

$$f(ma^+nx) = \exp(-\rho_+' \log a^+)\, \omega_\Lambda(m) f(x)$$

for almost every $x \in G$, and

(iii) $\int_K \| f(k) \|^2\, dk < \infty$.

The norm squared of f is given by the expression of (iii), and $V_\Lambda(g)$ operates by right translation: $V_\Lambda(g) f(x) = f(xg)$. It is well known that $V_\Lambda(g)$ is strongly continuous and unitary.

THEOREM 5.1. *If G is simple, if $q_\Lambda = 0$, and if $\Lambda + \rho$ is singular, then the mapping L defined on $\Gamma_2(\Lambda)$ by*

$$(LF(g))(x) = F(u_1 x g)$$

for $F \in \Gamma_2(\Lambda)$, $x \in B_s M_s$, and $g \in G$ is a linear isometry into $\mathscr{H}(\Lambda)$, equivariant with respect to $G : LU_\Lambda(g) = V_\Lambda(g) L$. Consequently, $U_\Lambda(g)$ is unitarily equivalent with a subrepresentation of $V_\Lambda(g)$.

Sketch of proof. The proof is completely elementary, and we shall list just the main steps. For $F \in \Gamma_0(\Lambda)$, put $f(g)(x) = F(u_1xg)$ if $x \in B_sM_s$ and $g \in G$. Then one shows:

(1) If $F \in \Gamma_0(\Lambda)$, then $f(g) \in \Gamma(\bar{\Lambda})$ for each $g \in G$.

(2) If $F \in \Gamma_2(\Lambda)$, then $\|f(g)\| < \infty$ almost everywhere, and the mapping $g \rightarrow (f(g), \varphi)$, for $\varphi \in H(\bar{\Lambda})$, is measurable.

(3) If $F \in \Gamma_0(\Lambda)$, $a^+nm \in A^+NM$, $g \in G$, and $x \in B_sM_s$, then

$$f(a^+nmg)(x) = \exp(-\rho_+' \log a^+)(\omega_\Lambda(m) f(g))(x).$$

(This is condition (ii) for f to be in $\mathscr{H}(\Lambda)$. The verification uses Lemma 3.2, Propositions 3.3 and 3.1(c), and Eq. (4.6).)

(4) If $F \in \Gamma_2(\Lambda)$, then $f \in \mathscr{H}(\Lambda)$ and $\|F\| = \|f\|$.

(5) $LU_\Lambda(g) = V_\Lambda(g)L$.

These five steps prove the theorem.

We now know that $U_\Lambda(g)$ is unitarily equivalent with a subrepresentation of $V_\Lambda(g)$ and that $V_\Lambda(g)$ is strongly continuous. It follows that $U_\Lambda(g)$ is strongly continuous. This is the conclusion of Theorem 4.1 that we had left unproved until now.

6. Two Lemmas of Harish-Chandra

To proceed further, we need to use two properties of ψ_Λ proved by Harish-Chandra in [6]. In this section, we assume that Λ is an integral form on $\mathfrak{h}^C$, dominant with respect to $\mathfrak{k}$, such that $q_\Lambda = 0$. The first result that we need is Lemma 6 of [6].

Lemma 6.1. *If $q_\Lambda = 0$, there exists a unique function $\psi \in \Gamma(\Lambda)$ such that $\psi(1) = 1$ and such that*

$$\int_T \varphi(hxh^{-1}g)\, dh = \varphi(g)\, \psi(x) \tag{6.1}$$

for all $\varphi \in \Gamma(\Lambda)$, $g \in G$, and $x \in BG$.

Harish-Chandra takes Lemma 6.1 as a definition of ψ_Λ and then derives a formula for ψ_Λ somewhat different from the one in our definition. In order to show the equivalence of his definition and ours (and thereby complete the proof of Theorem 4.2), we apply Eq. (6.1) to the function $\varphi = \psi_\Lambda$ and to the group element $g = 1$. Then

$$\psi_\Lambda(1)\, \psi(x) = \int_T \psi_\Lambda(hxh^{-1})\, dh. \tag{6.2}$$

Recall that $\psi_\Lambda(x) = (\tau_\Lambda(\mu(x))\,\phi_\Lambda\,,\phi_\Lambda)$. Then

$$\psi_\Lambda(1) = (\tau_\Lambda(1)\,\phi_\Lambda\,,\phi_\Lambda) = \|\phi_\Lambda\|^2 = 1.$$

Since $T \subseteq K$ and K normalizes P^- and P^+, $\mu(hxh^{-1}) = h\mu(x)\,h^{-1}$. It follows easily that

$$\psi_\Lambda(hxh^{-1}) = \psi_\Lambda(x), \tag{6.3}$$

Substituting in (6.2), we obtain $\psi(x) = \psi_\Lambda(x)$. Thus Harish-Chandra's definition and ours are equivalent.

Lemma 6.1 applies to all of $\Gamma(\Lambda)$. In Section 7 we shall see that $H(\Lambda) \subseteq \Gamma(\Lambda)$. On the subspace $H(\Lambda)$, Lemma 6.1 has the following interpretation: The multiplicity of the character ξ_Λ of T in the restriction of $U_\Lambda(g)$ to T is exactly one, and the space of functions in $H(\Lambda)$ transforming under T on the right according to ξ_Λ consists of the multiples of ψ_Λ.

The second result that we need is Lemma 8 of [6]. The notation for the lemma is as follows. Let W be an open set in $G^{\mathbf{C}}$ and let $Z = X + iY$ be in $\mathfrak{g}^{\mathbf{C}}$. Let f be a holomorphic function on W, and regard X and Y as operating as left-invariant vector fields. Define $Zf = Xf + iYf$. Then it follows from the fact that f is holomorphic that

$$Zf(w) = \frac{d}{dt} f(w \exp tZ)\Big|_{t=0}$$

for $w \in W$, where t can be taken to be complex in the differentiation.

LEMMA 6.2. *If $q_\Lambda = 0$, then the function ψ_Λ defined on the open subset $P^-K^{\mathbf{C}}P^+$ of $G^{\mathbf{C}}$ satisfies $H\psi_\Lambda = \Lambda(H)\,\psi_\Lambda$ for $H \in \mathfrak{h}$ and $E_\alpha\psi_\Lambda = 0$ for every positive root α.*

7. PROPERTIES OF THE CONSTRUCTION

We continue to assume that Λ is an integral form on $\mathfrak{h}^{\mathbf{C}}$, dominant with respect to $\mathfrak{k}$, such that $q_\Lambda = 0$. Once again we assume that G is simple and that $\Lambda + \rho$ is singular. In this section we shall use the lemmas of Section 6 to obtain some properties of the space $H(\Lambda)$ and of the representation $U_\Lambda(G)$. The first of these is a maximum principle.

LEMMA 7.1. *If G is simple, if $q_\Lambda = 0$, and if $\Lambda + \rho$ is singular, then there exists a constant $C < \infty$ such that*

$$|f(1)| \leqslant C \|f\| \tag{7.1}$$

for all f in $\Gamma_2(\Lambda)$.

Proof. For $f \in \Gamma_2(\Lambda)$, define $E_\Lambda f(x) = \int_T f(xh)\,\overline{\xi_\Lambda(h)}\,dh$. Then $E_\Lambda f$ is in $\Gamma(\Lambda)$, and $E_\Lambda f(1) = f(1)$. Moreover,

$$E_\Lambda f(xh_0) = \int_T f(xh_0h)\,\overline{\xi_\Lambda(h)}\,dh = \int_T f(xh)\,\overline{\xi_\Lambda(h_0^{-1}h)}\,dh = \xi_\Lambda(h_0)\,E_\Lambda f(x). \tag{7.2}$$

These equations, together with Lemma 6.1, imply that

$$E_\Lambda f(x) = E_\Lambda f(1)\,\psi_\Lambda(x) = f(1)\,\psi_\Lambda(x). \tag{7.3}$$

Then

$$\|f\|^2 = \int_{M_s \times K} \int_T |f(u_1mkh)\,\overline{\xi_\Lambda(h)}|^2\,dh\,dm\,dk \qquad \text{since } T \subseteq K$$

$$\geqslant \int_{M_s \times K} \left| \int_T f(u_1mkh)\,\overline{\xi_\Lambda(h)}\,dh \right|^2 dm\,dk \qquad \text{by Schwarz s inequality}$$

$$= |f(1)|^2\,\|\psi_\Lambda\|^2$$

by (7.3). This proves (7.1) with $C = \|\psi_\Lambda\|^{-1}$, which is finite by Theorem 4.2.

COROLLARY 1. *If G is simple, if $q_\Lambda = 0$, and if $\Lambda + \rho$ is singular, then to each compact set $E \subseteq BG$ corresponds a constant C_E such that*

$$|f(x)| \leqslant C_E \|f\|$$

for all $f \in \Gamma_2(\Lambda)$ and $x \in E$.

Proof. There exists a bounded nonempty set $S \subseteq B$ such that SG is open in $G^{\mathbf{C}}$. In fact, let S_n be an increasing sequence of compact sets in B with union B. Then $\bigcup S_nG = BG$, which is open, and each S_nG is closed. By the Baire Category Theorem, some S_nG has nonempty interior V. Since G acts on $G^{\mathbf{C}}$ by homeomorphisms, V is right G-invariant. Thus $V = (S_n \cap V)\,G$, and we can take $S = S_n \cap V$.

The translates bSG, for $b \in B$, form an open cover of E. Let $b_1SG, \ldots, b_nSG$ be a finite subcover, and put

$$C_E = C \max_{j \leqslant n, u \in S} |\xi_\Lambda(b_ju)|,$$

where C is the constant of Lemma 6.1. If $x \in E$, then $x = b_j ug$ for some $u \in S$ and $g \in G$ and for some j, and we have

$$|f(x)| = |\xi_\Lambda(b_j u) f(g)| = |\xi_\Lambda(b_j u)| \, |U_\Lambda(g) f(1)|$$
$$\leqslant |\xi_\Lambda(b_j u)| \, C \| U_\Lambda(g) f \| = |\xi_\Lambda(b_j u)| \, C \| f \| \leqslant C_E \| f \|,$$

by Lemma 6.1 and Theorem 4.1. This proves the corollary.

COROLLARY 2. *If G is simple, if $q_\Lambda = 0$, and if $\Lambda + \rho$ is singular, then the only member f of $\Gamma_2(\Lambda)$ with $\|f\| = 0$ is $f \equiv 0$.*

The second corollary follows from Corollary 1.

We can regard the normed space $\Gamma_2(\Lambda)$ as a space of holomorphic functions on BG and as a space of functions on the closed set $u_1 M_s K$ that are square-integrable with respect to a certain measure. Now let $f_n \in \Gamma_2(\Lambda)$ be a Cauchy sequence with limit $f \in H(\Lambda)$. Since $L^2(u_1 M_s K)$ is complete, we can regard f as a square-integrable function on $u_1 M_s K$. On the other hand, f_n is a Cauchy sequence in $L^2(u_1 M_s K)$, and it follows from Corollary 1 to Lemma 6.1 that the restrictions of the functions f_n to BG are uniformly Cauchy on compact subsets of BG. Consequently, $f_n(x)$ converges to a function, which we can denote $f(x)$, uniformly on compact subsets of BG. Then $f(x)$ is holomorphic on BG and is clearly a member of $\Gamma(\Lambda)$. The action by $U_\Lambda(g)$ on f goes into right translation of $f(x)$. Thus we can regard any member of $H(\Lambda)$ as the union of a function in $L^2(u_1 M_s K)$ and an associated function in $\Gamma(\Lambda)$. For members of $\Gamma_2(\Lambda)$, these two functions are related in that one is given as boundary values of the other, but the connection for the other members of $H(\Lambda)$ is less obvious.

Using this identification of members of $H(\Lambda)$ with functions both on $u_1 M_s K$ and on BG, we can pass to the limit in Lemma 7.1 and obtain a result valid for all f in $H(\Lambda)$.

LEMMA 7.1′. *If G is simple, if $q_\Lambda = 0$, and if $\Lambda + \rho$ is singular, then there exists a constant $C < \infty$ such that $|f(1)| \leqslant C \|f\|$ for all f in $H(\Lambda)$.*

Similarly we obtain the obvious extension of Corollary 1.

Now that we have realized all of $H(\Lambda)$ as a subspace of $\Gamma(\Lambda)$, the proof of irreducibility of $U_\Lambda(g)$ becomes a completely standard consequence of Lemma 6.1. (See Lemma 12 of [6].)

THEOREM 7.1. *If G is simple, if $q_\Lambda = 0$, and if $\Lambda + \rho$ is singular, then $U_\Lambda(g)$, as a representation on $H(\Lambda)$, is irreducible.*

Recall that in Section 5 we constructed a continuous series representation $V_\Lambda(g)$ on a space $\mathscr{H}(\Lambda)$ and exhibited $U_\Lambda(g)$ as a subrepresentation of $V_\Lambda(g)$ under a map L of $H(\Lambda)$ into $\mathscr{H}(\Lambda)$.

THEOREM 7.2. *If G is simple, if $q_\Lambda = 0$, and if $\Lambda + \rho$ is singular, then the image of $H(\Lambda)$ in the space $\mathscr{H}(\Lambda)$ of the associated continuous series representation $V_\Lambda(g)$ is proper. Consequently, $V_\Lambda(g)$ is reducible.*

The proof will consist of examining the restrictions to T^+ of $U_\Lambda(g)$ and $V_\Lambda(g)$ to see that they are different. The group T^+ is isomorphic with a circle group, and we can think of its character group as the integers. In an obvious sense, the integers extend in two directions from 0, and Theorem 7.2 will therefore follow if we prove the two lemmas below.

LEMMA 7.2. *The restriction $V_\Lambda|_{T^+}$ contains infinitely many characters of T^+ in both directions with positive multiplicity.*

LEMMA 7.3. *In one direction, the restriction $U_\Lambda|_{T^+}$ contains only finitely many characters of T^+ with positive multiplicity.*

The proof of Lemma 7.2 will make use twice of the following very special case of Mackey's Theorem 12.1 on p. 127 of [11].

LEMMA 7.4 (Mackey). *Let H be a separable locally compact group, let H_1 and H_2 be closed subgroups with $H = H_1H_2$, and let R be a continuous unitary representation of H_1. Then, up to unitary equivalence,*

$$(\operatorname*{ind}_{H_1\uparrow H} R)|_{H_2} = \operatorname*{ind}_{H_1\cap H_2\uparrow H_2} (R|_{H_1\cap H_2}).$$

Proof of Lemma 7.2. Let σ be the representation $\omega_\Lambda \otimes 1 \otimes 1$ of MA^+N, so that

$$V_\Lambda = \operatorname*{ind}_{MA^+N\uparrow G} \sigma.$$

Let $K_M = M \cap K$. By Lemmas 2.5(e) and 7.4,

$$V_\Lambda|_K = \operatorname*{ind}_{K_M\uparrow K} (\sigma|_{K_M}).$$

Hence

$$V_\Lambda|_{T^+} = \{\operatorname*{ind}_{K_M\uparrow K} (\sigma|_{K_M})\}|_{T^+}. \tag{7.4}$$

By Proposition 3.1(a) and Lemma 3.4(e), T^+ commutes with M, so that K_MT^+ is a compact group. Let τ be any irreducible representation of K_M that occurs in $\sigma\,|_{K_M}$, and let $\supseteq$ mean "contains as a subrepresentation." Then (7.4) gives

$$V_\Lambda\,|_{T^+} \supseteq \{\operatorname*{ind}_{K_M\uparrow K} \tau\}\,|_{T^+} = \{\operatorname*{ind}_{K_MT^+\uparrow K}(\operatorname*{ind}_{K_M\uparrow K_MT^+}\tau)\}\,|_{T^+}\,.$$

We shall prove first that

$$(\operatorname*{ind}_{K_M\uparrow K_MT^+}\tau)\,|_{T^+} \tag{7.5}$$

contains infinitely many characters of T^+ in both directions. In fact, by Lemma 7.4,

$$(\operatorname*{ind}_{K_M\uparrow K_MT^+}\tau)\,|_{T^+} = \operatorname*{ind}_{K_M\cap T^+\uparrow T^+}(\tau\,|_{K_M\cap T^+}). \tag{7.6}$$

The Lie algebra of $K_M \cap T^+$ is 0 because $\mathfrak{h}^+$ does not commute with $\mathfrak{a}^+$. Thus $K_M \cap T^+$ is a finite cyclic group Z_n. Let ξ be any character of $K_M \cap T^+$ that occurs in $\tau\,|_{K_M\cap T^+}$. Then the right side of (7.6) contains

$$\operatorname*{ind}_{Z_n\uparrow T^+}\xi,$$

which is well known to contain infinitely many characters of T^+ in both directions. Hence the same thing is true of (7.5).

Now let χ be any character of T^+ occurring in (7.5) Choose an irreducible representation ω of K_MT^+ that occurs in

$$\operatorname*{ind}_{K_M\uparrow K_MT^+}\tau$$

and is such that the multiplicity $(\omega\,|_{T^+} : \chi)$ is positive. In view of our result about (7.5), the proof will be complete if we show that

$$((\operatorname*{ind}_{K_MT^+\uparrow K}\omega)\,|_{T^+} : \chi) > 0. \tag{7.7}$$

To prove (7.7), we write

$$(\operatorname*{ind}_{K_MT^+\uparrow K}\omega)\,|_{T^+} = \sum_\lambda (\tau_\lambda\,|_{K_MT^+} : \omega)\,\tau_\lambda\,|_{T^+}$$

by the Frobenius Reciprocity Theorem. Choose λ so that $(\tau_\lambda\,|_{K_MT^+} : \omega) > 0$, and write $\tau_\lambda\,|_{K_MT^+} = \omega \oplus \omega^\perp$ and $\omega\,|_{T^+} = \chi \oplus \chi^\perp$. Then (7.8) contains

$$\tau_\lambda\,|_{T^+} = \omega\,|_{T^+} \oplus \omega^\perp\,|_{1^+} = \chi \oplus \chi^\perp \oplus \omega^\perp\,|_{T^+}\,.$$

This proves (7.7) and the lemma.

Proof of Lemma 7.3. If $Z = X + iY \in \mathfrak{g}^{\mathbf{C}}$ and if f is a holomorphic function defined on an open subset of $G^{\mathbf{C}}$, we used in Section 6 the definition $Zf = Xf + iYf$, where X and Y operate as left-invariant vector fields. With this definition if $f \in H(\Lambda)$, then $Zf = U_\Lambda(Z)f$, because U_Λ operates as right translation.

By Lemma 6.2, ψ_Λ is K-finite under U_Λ, and by Theorem 7.1, U_Λ is irreducible. It therefore follows from the results of [4] that the space $U_\Lambda(\mathscr{U})\,\psi_\Lambda$, where $\mathscr{U}$ is the complexified universal enveloping algebra of $\mathfrak{g}$, is dense in $H(\Lambda)$. Since $Zf = U_\Lambda(Z)f$ for $Z \in \mathfrak{g}^{\mathbf{C}}$ and $f \in H(\Lambda)$, we have $W\psi_\Lambda = U_\Lambda(W)\,\psi_\Lambda$ for $W \in \mathscr{U}$. Applying Lemma 6.2 and the Birkhoff–Witt Theorem, we see that a dense subspace of $H(\Lambda)$ is spanned by all vectors

$$v = E_{\alpha_1} \cdots E_{\alpha_r}\psi_\Lambda ,$$

where $r \geqslant 0$ and where the α_j are negative roots, possibly with repetitions. Since $H_{\alpha_0}\psi_\Lambda = \Lambda(H_{\alpha_0})\,\psi_\Lambda$, it is apparent that such a vector v is an eigenvector for $U_\Lambda(H_{\alpha_0})$ with eigenvalue

$$\begin{aligned}\Lambda(H_{\alpha_0}) &+ \alpha_1(H_{\alpha_0}) + \cdots + \alpha_r(H_{\alpha_0})\\ &= \Lambda(H_{\alpha_0}) + \langle \alpha_1 , \alpha_0 \rangle + \cdots + \langle \alpha_r , \alpha_0 \rangle \leqslant \Lambda(H_{\alpha_0}),\end{aligned}$$

the inequality holding by Lemma 3.5, since all the α_j are negative. Thus the eigenvalues of $U_\Lambda(H_{\alpha_0})$ are bounded above on a dense subspace. Since $T^+ = \exp(i\mathbf{R}H_{\alpha_0})$, this is enough to guarantee that only finitely many characters of T^+ in one direction can occur in $U_\Lambda|_{T^+}$. The proofs of Lemma 7.3 and Theorem 7.2 are complete.

8. Discussion of $SU(n, 1)$

Let $G = SU(n, 1)$, $n \geqslant 2$. This is the group automorphisms of $\mathbf{C}^{n+1}$ preserving the hermitian quadratic form $|z_1|^2 + \cdots + |z_n|^2 - |z_{n+1}|^2$ and having determinant 1. In $\mathfrak{g}$, negative conjugate transpose is a Cartan involution, the diagonal elements of $\mathfrak{k}$ form a Cartan subalgebra $\mathfrak{h}$, and $\mathfrak{h}^{\mathbf{C}}$ is the set of diagonal matrices in $\mathfrak{g}^{\mathbf{C}} = \mathfrak{sl}(n + 1, \mathbf{C})$.

Let e_j be the linear functional on $\mathfrak{h}^{\mathbf{C}}$ whose value on a diagonal matrix is the $(n + 2 - j)$-th diagonal entry. Then the compact roots of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{h}^{\mathbf{C}})$ are the differences $e_i - e_j$ with i and j greater than 1, and the noncompact roots are the differences $\pm(e_1 - e_j)$. Choose an ordering so that the positive roots are $e_i - e_j$, $i < j$. Such an ordering is compatible with the complex structure in that every noncompact positive root is larger than every compact root. The largest root is $\alpha_0 = e_1 - e_{n+1}$.

Form MA^+N, etc., as in Section 2. Then $\mathfrak{h}^+$ consists of the diagonal matrices $(i\theta, 0,..., 0, -i\theta)$, and $\mathfrak{h}^-$ consists of the diagonal matrices $(i\theta, i\varphi_n, ..., i\varphi_2, i\theta)$ of trace 0. M is compact and connected, and $\mathfrak{h}^-$ is a Cartan subalgebra for it. Let σ be an irreducible unitary representation of M, and let μ be the highest weight of σ. Then μ is of the form

$$\mu(\theta, \varphi_n, ..., \varphi_2, \theta) = k\theta + \sum_{j=2}^{n} c_j \varphi_j,$$

where the only restrictions on the c_j are that the c_j are integers with $c_2 \geqslant c_3 \geqslant \cdots \geqslant c_n$. Since $\Sigma\varphi_j = -2\theta$, we can rewrite μ as

$$\mu = \sum_{j=2}^{n} \left(c_j - \frac{k}{2}\right) e_j.$$

(Notice that the parameters k, $c_2, ..., c_n$ do not lead independently to distinct μ's in this formula.) Form the representation of G induced from the representation $\sigma \otimes 1 \otimes 1$ of MA^+N. It is announced in [10] that this induced representation is reducible if and only if

(i) $k \equiv n \bmod 2$ and

(ii) $\langle \alpha, \mu + \rho^- \rangle \neq 0$ for every positive root α other than α_0,

where

$$\rho^- = \sum_{j=2}^{n} \left(\frac{n}{2} - j + 1\right) e_j.$$

Thus suppose $k \equiv n \bmod 2$. Changing notation by adding the same integer to each c_j, we can then write μ in the form

$$\mu = \sum_{j=2}^{n} \left(c_j + \frac{n}{2}\right) e_j. \tag{8.1}$$

Condition (ii) is the statement that $c_j \neq -(n + 1 - j)$ for $2 \leqslant j \leqslant n$. Because the integers c_j are decreasing, this condition divides the space of integer tuples $\{c_2, ..., c_n\}$ corresponding to reducibility into n components: $\{c_n \geqslant 0\}$, $\{c_{n-1} \geqslant -1, c_n \leqslant -2\}$, $\{c_{n-2} \geqslant -2, c_{n-1} \leqslant -3\}$,... . We shall see that Theorem 7.2 accounts exactly for the first component $\{c_n \geqslant 0\}$.

The fundamental weights on $\mathfrak{h}^{\mathbb{C}}$ are $\Lambda_j = e_1 + e_2 + \cdots + e_j$, $1 \leqslant j \leqslant n$. Thus the most general integral linear form is $\Lambda = \sum_{j=1}^{n} l_j \Lambda_j$

with $l_1, \ldots, l_n$ all integers. The form Λ is dominant with respect to $\mathfrak{k}$ if and only if $l_j \geqslant 0$, $2 \leqslant j \leqslant n$. The condition that $\langle \Lambda + \rho, \alpha_0 \rangle = 0$ is readily seen to be the condition that

$$l_1 + \cdots + l_n + n = 0. \tag{8.2}$$

Such a Λ decomposes as $-(n/2)(e_1 - e_{n+1}) + \Lambda_-$, where $-(n/2)(e_1 - e_{n+1})$ vanishes on $\mathfrak{h}^-$ and where

$$\Lambda_- = \sum_{j=2}^{n} \left(l_j + \cdots + l_n + \frac{n}{2}\right) e_j \tag{8.3}$$

vanishes on $\mathfrak{h}^+$. The restriction of Λ_- to $\mathfrak{h}^-$ is the highest weight of the representation ω_Λ of M constructed in Section 5. Comparing (8.1) and (8.3), we see that (i) is satisfied and that $c_j = l_j + \cdots + l_n$ for $2 \leqslant j \leqslant n$. The inequalities $l_j \geqslant 0$ for $2 \leqslant j \leqslant n - 1$ are equivalent with the known inequalities $c_2 \geqslant c_3 \geqslant \cdots \geqslant c_n$, and (8.2) can be regarded simply as the definition of l_1. The additional inequality $l_n \geqslant 0$ is equivalent with the condition $c_n \geqslant 0$. Thus Theorem 7.2 accounts exactly for the component $\{c_n \geqslant 0\}$.

This result is to be expected and corresponds to the fact that the holomorphic discrete series is only a part of the discrete series. In order to account for the other reducible representations induced from MA^+N, one expects to need a device like the cohomology spaces of [15] that were used to describe further discrete series.

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