Investigations of Unitary Representations of Semisimple Lie Groups

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These notes give a survey of some current areas of research in the representation theory of semisimple Lie groups, all discussed in the context of examples. The first section reviews the unsolved problem of classifying the irreducible unitary representations for such groups. An extensive bibliography for the classification question appears in [8] and will not, for the most part, be repeated here. Later sections deal with three active areas of research——minimal K-types, Zuckerman's conjecture, and Flensted-Jensen representations——and relate them to each other and to the classification question.

1. The problem of classifying irreducible unitary representations

Let G be a linear connected semisimple Lie group, by which we mean any connected closed group of real or complex matrices that is closed under conjugate transpose and has finite center. The problem is to classify the irreducible unitary representations of G. These will normally be infinite-dimensional.

Examples of noncompact simple groups fall into two classes, as follows. In each case we assume that the indices are sufficiently large:

Complex groups:

$SL(n,\mathbb{C})$	special linear
$SO(n,\mathbb{C})$	orthogonal
$Sp(n,\mathbb{C})$	symplectic
5 exceptional groups	

Real groups:

$$\left.\begin{array}{l} SL(n,\mathbb{R}) \\ SL(n,\mathbb{H}) = SU^*(2n) \end{array}\right\} \text{special linear groups}$$

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$SO_e(m,n)$, $SU(m,n)$, $Sp(m,n)$ — isometry groups of Hermitian forms

$Sp(n,R)$, $SO^*(2n)$ — other automorphism groups of bounded symmetric domains

12 exceptional groups.

The status of the problem is that it is unsolved in general. However,

(1) it is known that it is sufficient to decide the unitarity of some specific representations,

(2) it is known which of these admit an invariant Hermitian form,

(3) it is known that the Hermitian form is given by a specific integral operator, which therefore needs to be semidefinite,

(4) it is known how to use a few techniques to decide in some cases whether the operator is semidefinite.

Simple noncompact groups for which the classification is complete are as follows:

SL(2,R)	Bargmann 1947
SL(2,C)	Bargmann 1947 and Gelfand, Naimark 1947
SL(3,R)	Vahutinskii 1968
SL(3,C)	Tsuchikawa 1968
$SO_e(4,1)$ double cover	Dixmier 1961
$SO_e(n,1)$ double cover	Hirai 1962
SU(n,1)	Ottoson 1968 and Kraljevic 1973

-----Langlands classification, 1973-----

Sp(2,C) and G_2^C	Duflo 1976
SL(4,R) (actually GL)	Speh 1977
Sp(n,1)	Baldoni Silva 1980
SL(4,C) and SL(5,C)	Duflo 1980
Sp(2,R)	several people, about 1981
SU(2,2)	Knapp; Speh 1981
$F_{4(-20)}$	Baldoni Silva, Barbasch 1981

This list is not intended to suggest that one should or does proceed case-by-case. Rather it serves to test the strength of new methods.

The two parts of the list are qualitatively different. The answer is relatively simple in the first cases, not so in the last ones. In the first ones, people spent most of their effort classifying representations of a broader class than the unitary ones. The Langlands classification and results shortly after it swept away that problem, in a way that we discuss shortly.

Partly for illustration and partly for use later, let us examine the case of SL(2,R). The irreducible unitary representations fall into four classes:

1) Discrete series $\mathcal{D}_n^+$ and $\mathcal{D}_n^-$, $n \geq 2$ an integer

$$\mathcal{D}_n^+ \text{ space} = \left\{ f \text{ analytic for } \operatorname{Im} z > 0 \;\middle|\; \|f\|^2 = \iint_{\operatorname{Im} z > 0} |f(z)|^2 y^{n-2}\, dx\, dy < \infty \right\}$$

$$\mathcal{D}_n^+ \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = (bz+d)^{-n} f\left(\frac{az+c}{bz+d}\right).$$

$\mathcal{D}_n^-$ is analogous with antianalytic functions. These representations are square integrable in the sense of having one (or equivalently every) nonzero matrix coefficient $(\mathcal{D}_n(g)u,v)$ in L^2.

2) Principal series $\mathcal{P}^{+,iv}$ and $\mathcal{P}^{-,iv}$, $v \in R$

Space = $L^2(R)$

$$\mathcal{P}^{\pm,iv} \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = \begin{cases} |bx+d|^{-1-iv} f\left(\frac{ax+c}{bx+d}\right) & \text{if } + \\ \operatorname{sgn}(bx+d)\, |bx+d|^{-1-iv} f\left(\frac{ax+c}{bx+d}\right) & \text{if } - \end{cases}$$

These representations can be viewed as induced, in a suitable sense, from the upper triangular subgroup. Parameters v and $-v$ give equivalent representations, and $\mathcal{P}^{-,0}$ must be excluded because it is reducible.

3) Complementary series $\mathcal{C}^u$, $0 < u < 1$

$$\text{Space} = \left\{ f : R \to C \;\middle|\; \|f\|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(x)\overline{f(y)}\,dx\,dy}{|x-y|^{1-u}} < \infty \right\}$$

$$\mathcal{C}^u \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = |bx+d|^{-1-u} f\left(\frac{ax+c}{bx+d}\right).$$

These representations can be viewed as follows: $\mathcal{P}^{\pm,w}$ makes sense for all complex w, acting in $L^2(R, (1+x^2)^{\mathrm{Re}\, w}\,dx)$, and $\mathcal{C}^u$ comes about by redefining the inner product for $\mathcal{P}^{+,w}$.

4) Trivial representation 1, and
Limits of discrete series $\mathcal{D}_1^+$ and $\mathcal{D}_1^-$ with a Hardy space norm.

In the general case, the breakthrough beginning in 1973 was control over candidates for irreducible unitary representations. The problem became one of checking whether certain representations were unitary. This reduction came about in several steps:

1) Parametrization of discrete series (Harish-Chandra, 1966). Discrete series are irreducible unitary representations whose matrix coefficients are square integrable; they are exactly the irreducible representations that are direct summands in L^2 of the group.
2) Classification of "irreducible admissible representations" (Langlands, 1973).
3) Construction of limits of discrete series (Schmid, Zuckerman, 1975). These are representations whose algebraic properties are similar to those of discrete series but whose analytic properties are different.
4) Classification of "irreducible tempered representations" (Knapp, Zuckerman, 1976).

The reduction, then, is to a consideration of some standard series of representations. We shall say what these standard series are for the group $SU(2,2)$.

The standard series are constructed from subgroups known as "cuspidal standard parabolic subgroups," and we begin by writing down these subgroups for SU(2,2), indicating where they come from. Let

$$G = SU(2,2) = \left\{ g \in SL(4,\mathbb{C}) \;;\; g^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} g = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \right\}$$

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} \;;\; a \text{ and } d \text{ skew Hermitian, } \operatorname{Tr} a + \operatorname{Tr} d = 0 \right\}.$$

The group G has an Iwasawa decomposition $G = KA_{min}N_{min}$, whose components are as follows: K is maximal compact and is given by

$$K = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \;;\; a \in U(2),\; d \in U(2),\; (\det a)(\det d) = 1 \right\}.$$

The Lie algebra of A_{min} is

$$\mathfrak{a}_{min} = \left\{ \begin{pmatrix} 0 & 0 & s & 0 \\ 0 & 0 & 0 & t \\ s & 0 & 0 & 0 \\ 0 & t & 0 & 0 \end{pmatrix} \;;\; s, t \in \mathbb{R} \right\},$$

and $A_{min} = \exp \mathfrak{a}_{min}$ is then isomorphic to $\mathbb{R}^2$. The group N_{min} is a certain 6-dimensional nilpotent subgroup of G whose exact nature is not very important.

[To define N_{min} precisely, we let f_1 and f_2 be the linear functionals on $\mathfrak{a}_{min}$ defined as s and t, respectively, on the above matrix in $\mathfrak{a}_{min}$. Then $\operatorname{ad}(\mathfrak{a}_{min})$ acts diagonably on $\mathfrak{g}$ with simultaneous eigenvalues $\pm f_1 \pm f_2$, $\pm 2f_1$, $\pm 2f_2$, 0, and we choose

$$N_{min} = \exp(\mathfrak{g}_{2f_1} \oplus \mathfrak{g}_{f_1+f_2} \oplus \mathfrak{g}_{f_1-f_2} \oplus \mathfrak{g}_{2f_2}).\;]$$

Let

$$T = \left\{ \begin{pmatrix} e^{i\theta} & & & \\ & e^{-i\theta} & & \\ & & e^{i\theta} & \\ & & & e^{-i\theta} \end{pmatrix} \right\}$$

$$M_{min} = Z_K(A_{min}) = \{i^n t \;\; ; \;\; 0 \le n \le 3 \;\; \text{and} \;\; t \in T\} \cong T \oplus Z_2$$

$$P_{min} = M_{min}A_{min}N_{min} = \text{"minimal parabolic subgroup."}$$

The group M_{min} is compact, and P_{min} is the analog of the upper triangular subgroup of $SL(2,R)$.

A standard parabolic subgroup is any closed subgroup containing P_{min}. Such a subgroup always has a suitable decomposition as MAN, and M is noncompact except in the case of P_{min}. The number of these subgroups is $2^{\dim(A_{min})}$ and is 4 in the case of SU(2,2). We retain only those standard parabolic subgroups for which M has a compact Cartan subgroup (equivalently, M has a compact torus whose Lie algebra is maximal abelian in $\mathfrak{m}$); such standard parabolic subgroups are called cuspidal. (The condition of having a compact Cartan subgroup is Harish-Chandra's famous necessary and sufficient condition for the existence of discrete series.)

In SU(2,2) there are three cuspidal standard parabolic subgroups, two of them being P_{min} and $G = G\{1\}\{1\}$. The other is $P_{2f_2} = M_{2f_2}A_{2f_2}N_{2f_2}$, where

$$\mathfrak{a}_{2f_2} = \{H \in \mathfrak{a}_{min} \mid 2f_2(H) = 0\}$$

$$A_{2f_2} = \exp \mathfrak{a}_{2f_2} \quad (\cong R^1)$$

$$M_{2f_2} = T \oplus \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & 1 & 0 \\ 0 & \bar{\beta} & 0 & \bar{\alpha} \end{pmatrix} \; ; \; \alpha, \beta \in C \;\; \text{and} \;\; |\alpha|^2 - |\beta|^2 = 1 \right\}$$

$$= T \oplus SL(2,R)$$

$$N_{2f_2} = \exp(\mathfrak{n}_{2f_1} \oplus \mathfrak{n}_{f_1+f_2} \oplus \mathfrak{n}_{f_1-f_2}) \;\; \text{with} \;\; \dim N_{2f_2} = 5.$$

Now we can describe the standard series of representations associated to a cuspidal standard parabolic subgroup $P = MAN$. The parameters are pairs (σ, ν), where

σ = discrete series or "nondegenerate" limit of discrete series on M

ν = member of $(\mathfrak{a}')^C$ with $\operatorname{Re}\nu$ in the closed positive Weyl chamber.

In the case of $SU(2,2)$, the condition on $\operatorname{Re}\nu$ is as follows:

$$\mathfrak{a}_{min}: \quad \operatorname{Re}\nu = af_1 + bf_2, \quad a \geq b \geq 0$$

$$\mathfrak{a}_{2f_2}: \quad \operatorname{Re}\nu = af_1, \quad a \geq 0 .$$

Let

$$U(P,\sigma,\nu) = \operatorname{ind}_P^G(\sigma \otimes e^{\nu} \otimes 1).$$

[To define the induced representation, we use "unitary induction," so that the induced representation is unitary if ν is imaginary. The representation is realized in a space of functions on K and obtains a norm from $L^2(K)$; however, it is not unitary as it stands if ν is not imaginary.] Under known conditions on ν, $U(P,\sigma,\nu)$ has a unique irreducible quotient $J(P,\sigma,\nu)$, the "Langlands quotient." A sufficient condition for J to be defined is that $\operatorname{Re}\nu$ be in the open positive Weyl chamber.

Reduced problem: Decide whether $J(P,\sigma,\nu)$ is unitary.

By way of explanation, the J's include all the representations that can be unitary, but there may be equivalences. However, these equivalences are understood, and so an answer to the reduced problem does yield a classification of irreducible unitary representations.

The meaning of the reduced problem in the case of $SL(2,R)$ is as follows: P is either G or the upper triangular group MAN. For G one is to test $\mathcal{D}_n^{\pm}$ for $n \geq 1$. For MAN one is to test the irreducible quotient of $\mathcal{P}^{\pm,w}$ for $\operatorname{Re} w \geq 0$, except that $\mathcal{P}^{-,0}$ need not be tested.

It is a general fact that the series attached to G itself always consists entirely of unitary representations. Thus in SU(2,2), where we initially have to deal with three standard series, only the series attached to P_{2f_2} and P_{min} really need attention.

There are four further reductions now, much smaller in scope:

1) $J(P,\sigma,\nu)$ has an invariant Hermitian form (necessarily unique up to a scalar) if and only if there is an element w in K normalizing A with $w\sigma \cong \sigma$, $w\nu = -\bar{\nu}$, and w^2 centralizing A. In the case of SL(2,R) this means that the complex parameter in the case of the MAN series has to be real or imaginary, or J cannot be unitary.

2) If the condition in (1) holds, then the Hermitian form is given by a specific integral operator on $L^2(K)$, and the representation is unitary if and only if the operator is semidefinite.

3) (Vogan) If one is willing to proceed by induction on dim G, then it is enough to take ν real. (Specifically, one needs to assume that the classification is known for all M's of proper standard parabolic subgroups, not necessarily cuspidal.) In the case of SL(2,R), this means that we need only consider a real parameter for the MAN series, and a similar simplification occurs for SU(2,2). The inductive assumption for SU(2,2) is that one can handle groups that are essentially a circle or SL(2,R) or SL(2,C).

4) A necessary condition (now that we are assuming ν is real) is that ν must lie in a certain bounded set. The reason is that ν gives the gross asymptotic behavior of matrix coefficients and the matrix coefficients must be bounded for a unitary representation.

We discuss results concerning techniques shortly. First let us look at the pattern of unitary representations in groups that have been settled. We consider a table of three indices of difficulty in handling a group, the first index being the most important. The idea is that each σ gives a picture of ν parameters, and the indices relate to the complexity of such pictures.

Group	(i) Max. dim. of -1 eigenspace in $\mathfrak{a}$ for w as in (1)	(ii) Real rank $= \dim \mathfrak{a}_{min}$	(iii) Complex rank $=$ rank of $\mathfrak{g}^C$
SL(2,R)	1	1	1
SL(2,C)	1	1	2
SL(3,R)	1	2	2
SL(3,C)	1	2	4
$SO_e(4,1)$	1	1	2
$SO_e(n,1)$	1	1	$\left[\frac{n+1}{2}\right]$
SU(n,1)	1	1	n
Sp(2,C) and G_2^C	2	2	4
SL(4,R)	2	3	3
Sp(n,1)	1	1	n+1
SL(4,C)	2	3	6
SL(5,C)	2	4	8
Sp(2,R)	2	2	2
SU(2,2)	2	2	3
$F_{4(-20)}$	1	1	4

Table. Indices of difficulty in known cases.

In no case has a simple group with index (i) greater than 2 been settled. Index (i) equal to 1 means that all pictures of interest are one-dimensional.

In the first set of groups when index (ii) is 1, the picture is an interval starting from the origin and going to a point depending on σ that is easily described: For SL(3,R) and SL(3,C), it is such an interval, plus a point for the trivial representation when σ is trivial. (In all these cases, only P_{min} needs attention. For SL(3,R) there are two intermediate P's to consider, but the formal symmetry condition (needed for the existence of the invariant Hermitian form) is never satisfied.

On the basis of the first set of examples, one might expect an easy solution to the problem in general. The work by Duflo on $Sp(2,C)$ and G_2^C eliminated any such expectation. See [3] for pictures when σ is trivial; to have just the positive Weyl chamber, one should look at only 1/8 or 1/12 of the plane in the two cases. Moreover, in $Sp(2,C)$ for a certain nontrivial σ, there is an isolated representation (not the trivial representation of G), and Duflo points out that it is a constituent of the Weil representation of $Sp(2,C)$.

Later work showed that the situation is difficult but not hopeless. Baldoni Silva, partly with Barbasch, handled the remaining real-rank-one groups. The only P's are G and P_{min}, and each σ has a one-dimensional picture. The picture is an interval from the origin, together with possibly one additional point.

The groups $SL(4,R)$, $SL(4,C)$, $SL(5,C)$, $Sp(2,R)$, and $SU(2,2)$ required some new techniques, but the pictures were fairly simple--- triangles, squares, line segments, and the trivial representation (isolated in all these cases in the picture for σ trivial).

More of a sense of what to expect comes from looking at some pictures for a class of groups that cannot yet be handled completely--- $SU(N,2)$ for $N \geq 3$. The results noted here are joint work with B. Speh. Here the P's are

G :	all cases unitary, as usual
intermediate P :	$\dim = 1$
P_{min} :	$\dim A_{min} = 2$,

and all real ν satisfy the formal symmetry condition. The intermediate P leads, in the case of discrete series on M, to an interval from 0 that is understood. The interesting pictures come from P_{min}. We have

$$M_{min} = \left\{ \begin{pmatrix} e^{i\theta} & & & & \\ & e^{i\varphi} & & & \\ & & \omega & & \\ & & & e^{i\varphi} & \\ & & & & e^{i\theta} \end{pmatrix} ;\ \omega \in U(N-2),\ \text{total det} = 1 \right\}$$

Let

$$\sigma(\text{this}) = e^{i(m\theta+n\varphi)}\ \sigma_0(\omega)\ .$$

Three pictures are worth concentrating on. Let us take $N = 8$ and consider

(1) σ trivial

(2) σ_0 with highest weight $(k,0,0,0,0,-k)$ with $k \geq 2$

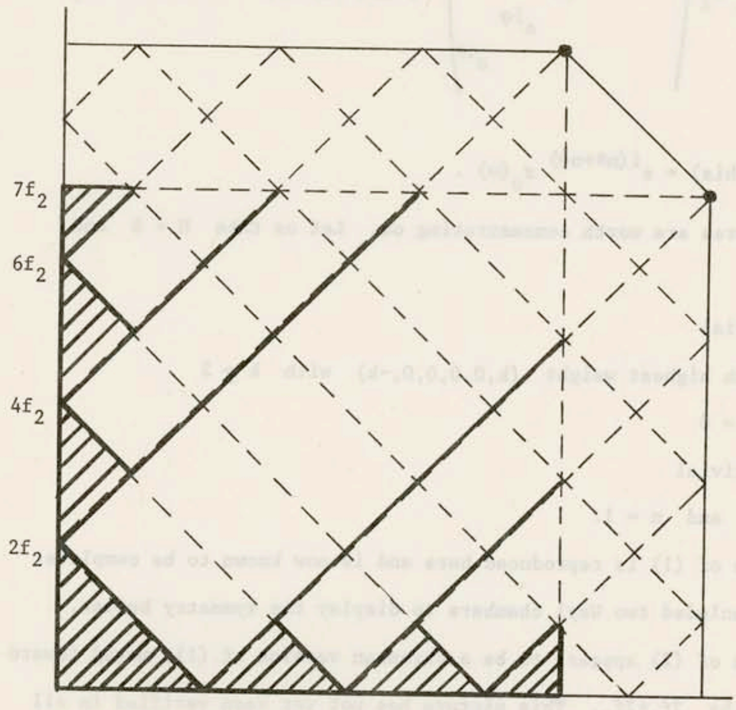
$m = n = 0$

(3) σ_0 trivial

$m = 0$ and $n = 1$.

The picture of (1) is reproduced here and is now known to be complete. We have included two Weyl chambers to display the symmetry better. The picture of (2) appears to be a shrunken version of (1), moved toward the origin by $2f_1+2f_2$. This picture has not yet been verified in all details: it is known to be correct within the small square around the origin, it is known that the point $7f_1+5f_2$ corresponds to a unitary representation (one of those obtained by Flensted-Jensen [4], in view of results of Schlichtkrull [12]), and it is known that there are no unitary points farther from the origin than this one. The picture of (3) was given in [8] and is now known to be complete; qualitatively it looks something like (1), but spread out farther from the diagonal, without any isolated point, and asymmetric with respect to the diagonal.

Other pictures for $SU(N,2)$ with σ such that $U(P_{min},\sigma,0)$ is irreducible do not seem to be substantially different from these three.



$\nu = af_1 + bf_2$, two chambers showing

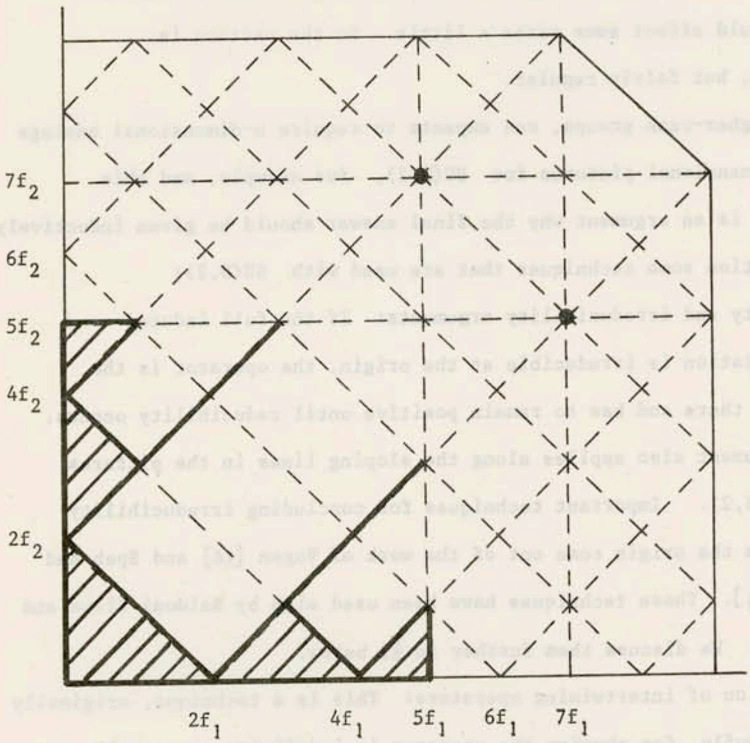
Positive chamber: $0 \leq b \leq a$

Key: unitary points

reducible points

edge of region of bounded matrix coefficients

Figure 1. Unitary $J(P_{min}, \sigma, \nu)$ in $SU(8,2)$ with ν real and σ trivial.



$\nu = af_1 + bf_2$, two chambers showing

Positive chamber: $0 \leq b \leq a$

Key: unitary points

reducible points

edge of region of bounded matrix coefficients

Figure 2. Unitary $J(P_{min}, \sigma, \nu)$ in SU(8,2) with ν real and σ such that $m = n = 0$ and $\sigma_0 \leftrightarrow (k,0,0,0,0,-k)$.

For other groups of real-rank two, one expects nothing very different from what is already seen in $SU(N,2)$ and the cases that have been solved completely, except that the Baldoni Silva isolated representations of $Sp(n,1)$ should affect some cases a little. So the pattern is complicated, but fairly regular.

For higher-rank groups, one expects to require n-dimensional analogs of the 2-dimensional pictures for $SU(N,2)$, for example, and this expectation is an argument why the final answer should be given inductively.

We mention some techniques that are used with $SU(N,2)$:

1) Continuity and irreducibility arguments: If the full induced representation is irreducible at the origin, the operator is the identity there and has to remain positive until reducibility occurs. This argument also applies along the sloping lines in the pictures for $SU(N,2)$. Important techniques for concluding irreducibility away from the origin come out of the work of Vogan [16] and Speh and Vogan [14]. These techniques have been used also by Baldoni Silva and Barbasch. We discuss them further in §3 below.
2) Calculation of intertwining operators: This is a technique, originally used by Duflo, for showing the operator is indefinite on a small finite-dimensional space and the representation is therefore not unitary.
3) Dirac inequality (Schmid, Parthasarathy, Baldoni Silva, Enright): This rules out unitarity beyond a certain radius.
4) Movement of M parameter (Knapp and Speh): Under certain conditions, unitarity is preserved when the A parameter is fixed and the M parameter is moved. (The reason is that the one operator is related to the other in a way that maintains positivity.)

We conclude by mentioning **some other** aspects of the classification problem that will play a role in later sections of these notes. These include Vogan's theory of minimal K-types, a new formula for minimal K-types, the theory of discrete series of semisimple symmetric spaces, as developed by Flensted-Jensen, Oshima, and Schlichtkrull, and Zuckerman's conjecture concerning a family of algebraically defined representations that are thought to be all unitary. For other work on the classification problem, see the survey [8].

2. Background and notation

We shall now go into more detail about some areas of research that appear to bear on the problem of classifying irreducible unitary representations. We shall use extensively the Cartan-Weyl theory of roots and weights and also the Harish-Chandra homomorphism defined on the center of the universal enveloping algebra; see [15] for this material.

The context for our discussion will normally be $G = SU(N,2)$. We let

$$\mathfrak{g} = \mathfrak{su}(N,2)$$
$$\mathfrak{g}^C = \mathfrak{sl}(N+2,C)$$
$$K = S(U(N)\times U(2))$$
$$\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(N)\oplus\mathfrak{u}(2))$$
$$\mathfrak{h} = \text{diagonal subalgebra of } \mathfrak{g}$$
$$\mathfrak{h}^C = \text{diagonal subalgebra of } \mathfrak{g}^C.$$

Here $\mathfrak{h}^C$ is a Cartan subalgebra of both $\mathfrak{g}^C$ and $\mathfrak{k}^C$. We let Δ and Δ_K denote the sets of roots of $(\mathfrak{g}^C,\mathfrak{h}^C)$ and $(\mathfrak{k}^C,\mathfrak{h}^C)$, respectively, and we let W and W_K be the respective Weyl groups.

<u>Integral forms</u> on $\mathfrak{h}^C$ are complex-linear functionals $\sum_{j=1}^{N+2} c_j e_j$ with $c_i - c_j$ an integer for all i and j. Here e_j denotes evaluation of the j^{th} diagonal entry. Note that the same integral form may be

realized in different ways since $c(e_1 + \dots + e_{N+2}) = 0$. We sometimes write

$$\sum_{j=1}^{N+2} c_j e_j \leftrightarrow (c_1, c_2, \dots, c_{N+2})$$

to make the correspondence with classical notation. The Weyl groups W and W_K act by permutations, with W_K leaving stable the sets of indices $\{1,\dots,N\}$ and $\{N+1, N+2\}$.

A positive system $\Delta^+ \subseteq \Delta$ results from an ordering of the indices $1,\dots,N+2$ and induces a positive system $\Delta_K^+ \subseteq \Delta_K$. We shall have occasion to use several distinct positive systems. For any positive system Δ^+, we let ρ be half the sum of the positive roots, and we let ρ_K be half the sum of the members of Δ_K^+.

For $\mathfrak{g}^C = \mathfrak{sl}(N+2, C)$, let $U(\mathfrak{g}^C)$ be the universal enveloping algebra, let $Z(\mathfrak{g}^C)$ be its center, and let $U(\mathfrak{h}^C)$ be the universal enveloping algebra of $\mathfrak{h}^C$. The Harish-Chandra homomorphism γ is an algebra isomorphism of $Z(\mathfrak{g}^C)$ onto the set $U(\mathfrak{h}^C)^W$ of Weyl-group-invariant members of $U(\mathfrak{h}^C)$.

The Harish-Chandra homomorphism γ leads to a description of the algebra homomorphisms of $Z(\mathfrak{g}^C)$ into C. If Λ is in $(\mathfrak{h}^C)'$, then

$$z \to \chi_\Lambda(z) = \Lambda(\gamma(z))$$

is such a homomorphism. It is easy to see that $\chi_\Lambda = \chi_{\Lambda'}$ if and only if $\Lambda' = w\Lambda$ for some w in W. Moreover, one can show that every homomorphism of $Z(\mathfrak{g}^C)$ into C is of the form χ_Λ for some Λ.

For any reasonable irreducible (infinite-dimensional) representation π of G, $Z(\mathfrak{g}^C)$ acts as scalars. Then $z \to \pi(z)$ gives a homorphism of $Z(\mathfrak{g}^C)$ into C, and the above discussion means that

$$\pi(z) = \chi_\Lambda(z) I$$

for some Λ in $(\mathfrak{h}^{\mathbb{C}})'$. We say that π has <u>infinitesimal character</u> Λ. The infinitesimal character is determined up to a member of W.

An example of infinitesimal character in a classical context occurs with $SU(n)$: If a (finite-dimensional) irreducible π has highest weight λ, then the infinitesimal character of π is $\lambda + \rho$.

An irreducible unitary representation π of G is said to be in the <u>discrete series</u> if it is equivalent with a direct summand of $L^2(G)$. Equivalently its matrix coefficients $g \to (\pi(g)\varphi, \psi)$ are to be in $L^2(G)$. Harish-Chandra [5] proved that the discrete series for G is nonempty if and only if there is a maximal torus T in K that is maximal abelian in G. This condition is satisfied in $SU(N,2)$ with T equal to the diagonal subgroup. Harish-Chandra [5] proved also the following theorem parametrizing the discrete series.

<u>Theorem</u> (Harish-Chandra). Let $\mathcal{L}_0$ be the set of integral forms on $\mathfrak{h}^{\mathbb{C}}$. Then discrete series representations of G are parametrized (up to equivalence) in terms of global characters by nonsingular elements Λ of $\mathcal{L}_0 + \rho$ (i.e., $\langle\Lambda,\alpha\rangle \neq 0$ for all $\alpha \in \Delta$), modulo the action of W_K. Moreover, the discrete series representation ω_Λ has infinitesimal character Λ, i.e., $\omega_\Lambda(z) = \chi_\Lambda(z)I$ for z in $Z(\mathfrak{g}^{\mathbb{C}})$.

Meaning in special cases other than $SU(N,2)$:

1) $G = SU(n)$, compact. Every irreducible representation is in the discrete series. If Λ is given, introduce an ordering to make Λ dominant. Then ω_Λ is the irreducible representation with highest weight $\Lambda - \rho$.

2) $G = SU(1,1)$, conjugate to $SL(2,\mathbb{R})$. For $n \geq 2$ an integer, we have a discrete series $\mathcal{D}_n^+$ acting in a space of analytic functions in the disc by

$$\mathcal{D}_n^+\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} f(z) = (-\beta z + \bar{\alpha})^{-n} f\left(\frac{\alpha z - \bar{\beta}}{-\beta z + \bar{\alpha}}\right).$$

The corresponding Harish-Chandra parameter Λ has $\Lambda\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = n - 1$.

formula for the multiplicities of the K-types in $\omega_\Lambda|_K$. It has the following two important features:

1) If we introduce an ordering to make Λ dominant and put

$$\lambda = \Lambda - 2\rho_K + \rho ,$$

then the representation τ_λ of K with highest weight λ occurs exactly once in $\omega_\Lambda|_K$.

2) If $\tau_{\lambda'}$ occurs in $\omega_\Lambda|_K$, then λ' is necessarily of the form

$$\lambda' = \lambda + \sum_{\alpha_i \in \Delta^+} k_i \alpha_i$$

with the k_i integers ≥ 0.

3. Minimal K-types

Let π be a representation of G, and let

$$\pi|_K = \sum_{\tau \in \hat{K}} n_\tau \tau$$

be the decomposition of its restriction to K into irreducible representations, lumped by equivalence class (or "K-type"). We say that π is <u>admissible</u> if each n_τ is finite.

In this context Vogan has introduced a useful notion of minimal (or "lowest") K-type. The <u>minimal K-types</u> τ_λ of π are those K-types occurring in $\pi|_K$ for which $|\lambda + 2\rho_K|^2$ is a minimum. It is clear that π has a minimal K-type; it may have several.

Example. For the discrete series representation ω_Λ the Blattner weight $\lambda = \Lambda - 2\rho_K + \rho$ gives a minimal K-type, and it is the only one.

Some results obtained by Vogan [16] about minimal K-types are the following:

1) If π is irreducible and τ_λ is a minimal K-type, then τ_λ occurs with multiplicity one.

2) For the standard series of representations $U(P,\sigma,\nu) = \mathrm{ind}_P^G(\sigma \otimes e^{\nu} \otimes 1)$ with

$P = MAN$

σ = discrete series on M

$\nu \in (\mathfrak{a}')^{\mathbb{C}}$

(in which the K-types that occur do not depend on ν),

a) if $\operatorname{Re} \nu \in (\mathfrak{a}')^{+}$, so that the Langlands quotient $J(P,\sigma,\nu)$ is defined, then all the minimal K-types of U occur in J.

b) if $\nu = 0$, so that $U(P,\sigma,\nu)$ is unitary, then each irreducible summand of U contains exactly one of the minimal K-types of U.

c) if $\operatorname{Re} \nu$ is in the closure of $(\mathfrak{a}')^{+}$, then there is a result incorporating both (a) and (b) above.

In this work of Vogan's and in some later work, the role of the minimal K-types seems to be as an anchor for beginning a study of the representation, using information about its K-types.

We have a new formula [7] for the minimal K-types of Langlands quotients $J(P,\sigma,\nu)$ and give it here for $G = SU(N,2)$, $P = P_{min}$, and certain representations σ (approximately 1/4 of them). We write $P_{min} = M_{min}A_{min}N_{min}$ and use the formulas for M_{min} and σ given in §1. In particular, σ is given by three parameters: m, n, and σ_0. Our assumption on σ will be that $m \equiv n \equiv N \mod 2$.

The key thing in the formula is to use the correct ordering. Thus let the highest weight of σ be

$$\lambda_{\sigma} = \tfrac{1}{2}m(e_1 + e_{N+2}) + \tfrac{1}{2}n(e_2 + e_{N+1}) + \sum_{j=3}^{N} c_j e_j$$

with all c_j in Z and with $c_3 \geq \cdots \geq c_N$.

The half sum of the positive roots of M is

$$\rho_M = \sum_{j=3}^{N} \tfrac{1}{2}(N - 2j + 3)e_j .$$

We introduce an ordering to make $\lambda_\sigma + \rho_M$ dominant for Δ^+ and to make e_1-e_{N+2} and e_2-e_{N+1} be simple roots, i.e., to keep the pairs $1,N+2$ and $2,N+1$ intact in the ordering. (There is an additional condition in general, but it is automatically satisfied here.) Schematically this means that we take the string of integers

$$3,\ 4,\ 5,\ \ldots,\ N-2,\ N-1,\ N$$

and insert the pair $1,N+2$ somewhere and the pair $2,N+1$ somewhere so that the corresponding coefficients for $\lambda_\sigma + \rho_M$ are nonincreasing. This ordering defines Δ^+ and ρ. We let $\Delta_K^+ = \Delta_K \cap \Delta^+$ and form ρ_K. Then the formula for the highest weight of the minimal K-type, which in this case will be unique, is

$$\lambda = \lambda_\sigma - \begin{pmatrix} \text{orthogonal projection of } 2\rho_K \text{ on} \\ \text{span of } e_1 - e_{N+2} \text{ and } e_2 - e_{N+1} \end{pmatrix}$$

The significance of the roots e_1-e_{N+2} and e_2-e_{N+1} in the formula is that they are the roots that determine A_{min}.

Example. Let us take the situation in Figure 2. Thus $N = 8$, and σ is given by $m = n = 0$ and $\sigma_0 \leftrightarrow (k,0,0,0,0,-k)$. The highest weight of σ is $\lambda_\sigma = k(e_3-e_8)$, and thus

$$\begin{array}{rcccccc} & 3 & 4 & 5 & 6 & 7 & 8 \\ \lambda_\sigma + \rho_M \leftrightarrow (& \frac{5}{2}+k, & \frac{3}{2}, & \frac{1}{2}, & -\frac{1}{2}, & -\frac{3}{2}, & -\frac{5}{2}-k) \end{array}$$

The coefficients for indices $1,\ 10,\ 2,\ 9$ are all 0, and these indices can be placed in this order between 5 and 6. We compute ρ_K and find that it is orthogonal to e_1-e_{10} and e_2-e_9, and thus the result is that

$$\lambda = \lambda_\sigma = k(e_3 - e_8)\ .$$

The current area of research with minimal K-types is to establish relationships between representations $U^G(P,\sigma,\nu)$ and $U^L(P_{L,min},\sigma_L,\nu)$, where L is a subgroup of G, using minimal K-types as anchor. We give three examples of this kind of research:

1) The theory [14] of Speh and Vogan gives conditions under which U^L irreducible implies U^G irreducible. In our situation with $G = SU(N,2)$ and with $m \equiv n \equiv N \mod 2$, the allowable groups L are obtained as follows: We list the indices in the order described above. Then L can be any block diagonal subgroup containing the diagonal and the (1,10) block and the (2,9) block. The formula for the highest weight of σ_L is

$$\lambda_{\sigma_L} = \lambda_\sigma + (\rho_M - \rho_M^L) - (\rho - \rho^L) .$$

The allowable ν's are those with $\langle \lambda_\sigma + \rho_M + \nu, \alpha \rangle > 0$ for all α in Δ^+ that do not contribute to L.

For an illustration let us go back to the situation in Figure 2. If we abuse notation by thinking of $\mathfrak{a}$ as contained in $\mathfrak{h}^C$ when it is actually only conjugate to a subalgebra of it, we can write

$$\nu = \tfrac{1}{2}a(e_1 - e_{10}) + \tfrac{1}{2}b(e_2 - e_9)$$

with $0 \leq b \leq a$. (The correspondence with the figure has $f_1 \leftrightarrow \frac{1}{2}(e_1 - e_{10})$ and $f_2 \leftrightarrow \frac{1}{2}(e_2 - e_9)$.) We give three examples of L by indicating how the indices are arranged in blocks in each one:

$$\underbrace{3},\ \underbrace{4},\ \underbrace{5},\ \underbrace{1,\ 10},\ \underbrace{2,\ 9},\ \underbrace{6},\ \underbrace{7},\ \underbrace{8} \qquad \text{(i)}$$

$$\underbrace{3},\ \underbrace{4,\ 5,\ 1,\ 10,\ 2,\ 9,\ 6,\ 7},\ \underbrace{8} \qquad \text{(ii)}$$

$$\underbrace{3},\ \underbrace{4,\ 5,\ 1,\ 10,\ 2,\ 9,\ 6,\ 7,\ 8} \qquad \text{(iii)}$$

Let us examine the three examples separately:

(i) Here L = torus $\times$ $SU(1,1) \times SU(1,1)$. A little computation shows that σ_L is trivial on $SU(1,1) \times SU(1,1)$ and nontrivial on the torus and that the allowable ν's have $0 \leq b \leq a < 1$.

(ii) Here L = torus $\times$ $SU(6,2)$. We find that σ_L is trivial on $SU(6,2)$ and nontrivial on the torus and that the allowable ν's have $0 \leq b \leq a < 5 + 2k$.

(iii) Here L = torus $\times SU(7,2)$. On $SU(7,2)$, σ_L is given by $m = n = 1$ and $(\sigma_0)_L \leftrightarrow (0,0,0,0,-k)$. The allowable ν's have $0 \leq b \leq a < 5 + 2k$.

2) The Zuckerman conjecture (to be considered in detail in §4) expects $J^G(P,\sigma,\rho_L)$ to be unitary when $\sigma_L = 1$ on the semisimple part of L and the infinitesimal character of J is nonsingular. (Here ρ_L denotes the half sum of the positive restricted roots for L.) For an illustration, let us continue with the situation in Figure 2, going over the examples of L in (i) and (ii).

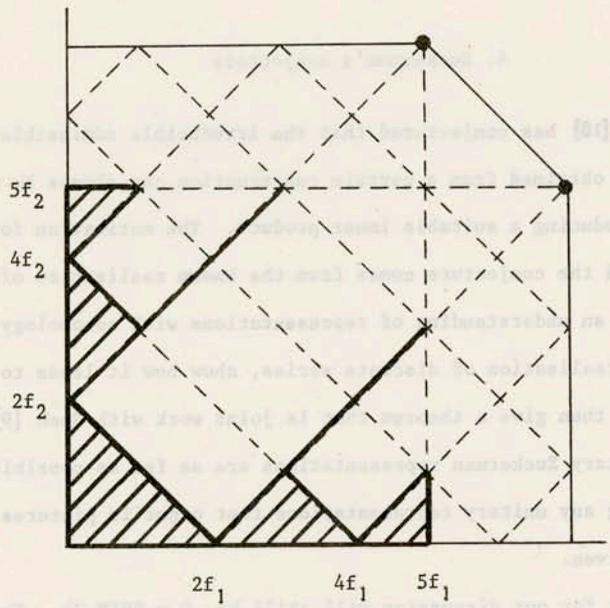
(ii) This L has semisimple part $SU(6,2)$, and σ_L is trivial there. The functional ρ_L corresponds to $a = 7$, $b = 5$. The infinitesimal character of J is

$$\begin{array}{cccccccccc} 3 & 4 & 5 & 6 & 7 & 8 & 1 & 10 & 2 & 9 \end{array}$$
$$(\tfrac{5}{2} + k, \tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2}, -\tfrac{5}{2} - k) \cup (\tfrac{7}{2}, -\tfrac{7}{2}, \tfrac{5}{2}, -\tfrac{5}{2}),$$

which is nonsingular (i.e., has no repeated entries) for $k \geq 2$. The conjecture expects J to be unitary in this case. And indeed J is unitary here. The point in question is the isolated unitary dot in Figure 2 and is a unitary point because J is a Flensted-Jensen representation, as we shall see in §5.

(i) This L has semisimple part $SU(1,1) \times SU(1,1)$, and σ_L is trivial there. The functional ρ_L corresponds to $a = b = 1$. The infinitesimal character is singular this time, so that the conjecture does not apply. But we can look at Figure 2 anyway and see that J is unitary.

3) A broader conjecture is possible. A picture of the unitary points for $\sigma = 1$ in $SU(6,2)$ is shown in Figure 3 and is known to be complete. A glance at Figure 2 shows that the unitary points in Figure 3 coincide with the points in Figure 2 that are known to be unitary. Thus we raise the



$\nu = af_1 + bf_2$, two chambers showing

Positive chamber: $0 \leq b \leq a$

Key: unitary points

reducible points

edge of region of bounded matrix coefficients

Figure 3. Unitary $J(P_{min}, \sigma, \nu)$ in $SU(6,2)$ with ν real and σ trivial.

question: In the setting of (2), can we expect that $J^G(P,\sigma,\nu)$ will be unitary if and only if $J^L(P_{L,min},1,\nu)$ is unitary?

4. Zuckerman's conjecture

Zuckerman [18] has conjectured that the irreducible admissible representations obtained from a certain construction can always be made unitary by introducing a suitable inner product. The motivation for the construction and the conjecture comes from the known realization of discrete series and from an understanding of representations with cohomology. We shall give the realization of discrete series, show how it leads to the conjecture, and then give a theorem that is joint work with Speh [9], saying that unitary Zuckerman representations are as far as possible from the origin among any unitary representations that occur in pictures like those we have given.

The context for our discussion will still be $G = SU(N,2)$. The generality of the theorem in [9] is that G is any linear connected semisimple Lie group with rank G = rank K, and Zuckerman's conjecture can be made in even greater generality.

A setting for realizing discrete series was proposed by Langlands [10]. We fix an ordering and hence Δ^+. Let B be the <u>lower</u> triangular subgroup of G^C relative to Δ^+. Then the orbit of 1 under G in the complex manifold G^C/B is open, and the isotropy subgroup is T. Hence G/T imbeds as an open subset of G^C/B and thereby becomes a complex manifold. Let χ be an integral form on $\mathfrak{h}^C$, and extend χ to the Lie algebra $\mathfrak{b}$ of B by defining it as 0 on the strictly lower triangular matrices. Then e^χ is a holomorphic character of B whose restriction to T is unitary. We form the associated holomorphic line bundle $\mathcal{L}_\chi$ over G/T :

$$\begin{tikzcd}[row sep=small] G \arrow[dd, dash] & & & \mathcal{L}_\chi \\ & T \arrow[r, "e^{\chi}"] & C & \\ G/T \arrow[urrr, dash] & & & \end{tikzcd}$$

Then we construct the space $\mathcal{H}^q(\mathcal{L}_\chi)$ of L^2 harmonic $(0,q)$ forms. [In more detail we let

$C^{0,q}(\mathcal{L}_\chi)$ = space of smooth $\mathcal{L}_\chi$-valued differential forms of type $(0,q)$ on G/T

$C^{0,q}_{com}(\mathcal{L}_\chi)$ = subspace of members of $C^{0,q}(\mathcal{L}_\chi)$ of compact support

$\bar\partial : C^{0,q}(\mathcal{L}_\chi) \to C^{0,q+1}(\mathcal{L}_\chi)$ the usual $\bar\partial$ operator

$\bar\partial^*$ = formal adjoint of $\bar\partial$ defined relative to $C^{0,q}_{com}(\mathcal{L}_\chi)$

$\square = \bar\partial\,\bar\partial^* + \bar\partial^*\bar\partial$

$\mathcal{H}^q(\mathcal{L}_\chi) = L^2$ subspace of $C^{0,q}(\mathcal{L}_\chi)$ in the kernel of $\square$.]

In any event, G acts and gives a unitary representation.

<u>Theorem</u> (Schmid [13]). If $\chi+\rho$ is nonsingular, then $\mathcal{H}^{q(\chi+\rho)}(\mathcal{L}_\chi) = \omega_{\chi+\rho}$, where

$$q(\chi+\rho) = |\{\alpha > 0 \mid \alpha \text{ compact and } \langle \chi+\rho, \alpha\rangle < 0\}|$$
$$+ |\{\alpha > 0 \mid \alpha \text{ noncompact and } \langle \chi+\rho, \alpha\rangle > 0\}| .$$

Now let us track down a change of notation in Schmid's theorem. Let us suppose that $\chi - \rho$ is dominant for Δ^+ and is nonsingular. Also let us replace B by the <u>upper</u> triangular group. Following through the statment of the theorem, we find that $\mathcal{H}^q(\mathcal{L}_\chi) = \omega_{\chi-\rho}$ when

$$q = |\{\alpha > 0 \mid \alpha \text{ compact}\}| = \dim_{\mathbb{C}}(K/T).$$

Zuckerman considered a generalization of this setting in order to motivate his conjecture. For data he uses $(L, \text{ordering}, \chi)$, with L, the ordering, and χ all to be explained. The case $L = T$ will reduce to

Schmid's theorem in the changed notation. Thus let

L = any centralizer of a toral subgroup of T, Lie algebra $\mathfrak{l}$

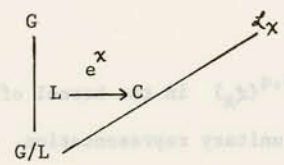
$\Delta^{(L)}$ = relative roots of $(\mathfrak{g}^C, Z_{\mathfrak{l}}^C)$

$(\Delta^{(L)})^+$ = positive system defined by "ordering"

$Q \subseteq G^C$ the block upper triangular group relative to L^C and $(\Delta^{(L)})^+$, so that G/L imbeds as an open subset of G^C/Q

χ = integral form on $\mathfrak{h}^C$ vanishing on $(Z_{\mathfrak{l}}^C)^{\perp} = [\mathfrak{l}^C, \mathfrak{l}^C]$, extended to $\mathfrak{q}$ as 0 on block strictly upper tiangular matrices.

Then e^{χ} is a holomorphic charcater of Q, and its restriction to L is unitary. We form the associated holomorphic line bundle $\mathcal{L}_{\chi}$ over G/L:



Let $H^{0,*}(\mathcal{L}_{\chi}) = \ker \bar{\partial}/\text{image } \bar{\partial}$ be the Dolbeault cohomology. Then the representation of interest is expected to be the representation of the Lie algebra $\mathfrak{g}$ in the K-finite vectors of $H^{0,d}$, where $d = \dim_C K/(K \cap L)$, provided $\chi - \rho_{(\Delta^{(L)})^+}$ is dominant for $(\Delta^{(L)})^+$ and nonsingular.

We say "is expected to be" because the actual construction that Zuckerman used is different. The analytic problems in dealing with the above setting are formidable. Instead of working directly with the above setting, he attempted to imitate it algebraically, by means of derived functors on injective resolutions. The details of the algebraic definition may be found in the book by Vogan [17]. The constructed representation is nonzero irreducible admissible and has nonsingular infinitesimal character.

Zuckerman's conjecture. Under the assumptions that L is the centralizer of a toral subgroup of T and that the integral form χ on $\mathfrak{h}^C$ vanishing on $[\mathfrak{l}^C,\mathfrak{l}^C]$ is such that $\chi - \rho_{(\Delta^{(L)})^+}$ is dominant for $(\Delta^{(L)})^+$ and nonsingular, the algebraically constructed representation admits an inner product with respect to which it is unitary.

We turn to the statement of a theorem about unitary parameters that is motivated by these considerations. Let Δ be the set of roots of $(\mathfrak{g}^C,\mathfrak{h}^C)$, and let Σ be the set of roots of $(\mathfrak{l}^C,\mathfrak{h}^C)$. Then we can choose positive systems Δ^+ and Σ^+ so that $\Sigma^+ \subseteq \Delta^+$ and the restrictions of members of Δ^+ to $Z_{\mathfrak{l}^C} \subseteq \mathfrak{h}^C$ are in $(\Delta^{(L)})^+$. The above considerations lead us to focus attention on representations with the following properties:

1) $\chi - \rho_\Delta + 2\rho_\Sigma$ = infinitesimal character. (This expression is Δ^+ dominant and nonsingular under the assumptions in the conjecture.)

2) $\chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K}$ = minimal K-type. (This expression is Δ_K^+ dominant under the assumptions in the conjecture.)

3) The Langlands parameter parabolic subgroup is one whose A is the Iwasawa A group of L, and the Langlands $\mathfrak{a}$ parameter is the $\rho_{L,\min}$ for L.

(These parameters do not coincide with those in Vogan [17] because Vogan uses a different ordering.)

Accordingly we make the following definition. We say that (Δ^+,Σ,χ) is a Zuckerman triple if

Δ^+ = positive system for $(\mathfrak{g}^C,\mathfrak{h}^C)$

Σ = root system generated by a set of simple roots in Δ^+ (i.e., blocks relative to the Δ^+ ordering)

χ = integral form on $\mathfrak{h}^C$ orthogonal to Σ with $\chi - \rho_\Delta + 2\rho_\Sigma$ dominant for Δ^+.

<u>Theorem</u> (with Speh [9]). Suppose that a Langlands quotient $J(P,\omega_\Lambda,\nu_0)$ is such that there is a Zuckerman triple (Δ^+,Σ,χ) for which $J(P,\omega_\Lambda,\nu_0)$ has infinitesimal character $\chi - \rho_\Delta + 2\rho_\Sigma$ and a minimal K-type $\chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K}$. If the infinitesimal character of $J(P,\omega_\Lambda,\nu_0)$ is nonsingular (or satisfies a slightly more general condition), then $J(P,\omega_\Lambda,\nu)$ cannot be made unitary for any real ν with $|\nu_0| < |\nu|$.

We shall obtain in the next section a wide class of unitary examples of representations $J(P,\omega_\Lambda,\nu_0)$ to which this theorem applies.

5. Flensted-Jensen representations

Flensted-Jensen representations are representations occurring discretely in $L^2(G/H)$, where H is the set of fixed points of an involution of G. Flensted-Jensen [4] has obtained sufficient conditions for the occurrence of such representations, Oshima [11] has refined Flensted-Jensen's results, and Schlichtkrull [12] has obtained minimal K-types and Langlands parameters for these representations. We shall discuss the results of these authors and relate them to the theorem at the end of the previous section.

The context for our discussion will still be $G = SU(N,2)$, and we take $H = S(U(r)\times U(N-r,2))$. Again the discussion could be carried out more generally. The Lie algebras of G, K, and H are

$$\mathfrak{g} = \mathfrak{su}(N,2)$$

$$\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(N)\oplus\mathfrak{u}(2))$$

$$\mathfrak{h}_1 = \mathfrak{s}(\mathfrak{u}(r)\oplus\mathfrak{u}(N-r,2)) = \begin{array}{c} \begin{array}{ccc} r & N-r & 2 \end{array} \\ \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \end{array}$$

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{q} \quad \text{with} \quad \mathfrak{q} = \begin{array}{c} \begin{array}{ccc} r & N-r & 2 \end{array} \\ \begin{pmatrix} 0 & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \end{array} .$$

Flensted-Jensen [4] obtained a sufficient condition for the existence of representations occurring discretely in $L^2(G/H)$:

$$\operatorname{rank}(G/H) = \operatorname{rank}(K/(K \cap H)),$$

i.e., there is a maximal abelian subspace of $\mathfrak{k} \cap \mathfrak{q}$ that is maximal abelian in $\mathfrak{q}$. For our particular G and H, this means that

$$r \leq N/2,$$

and then we can take

$$\mathfrak{t} = \left(\begin{array}{c|c|c} 0 & \begin{matrix} 0 & & \theta_1 \\ & \cdot^{\cdot^{\cdot}} & \\ \theta_r & & \end{matrix} & 0 \\ \hline \begin{matrix} & & 0 \\ & & -\theta_r \\ & \cdot^{\cdot^{\cdot}} & \\ -\theta_1 & & \end{matrix} & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

as the required maximal abelian subspace of $\mathfrak{k} \cap \mathfrak{q}$. The Lie algebra $\mathfrak{t}$ is conjugate via $\operatorname{Ad}(K)$ to a subalgebra of the Cartan subalgebra $\mathfrak{h}$, and we shall, by abuse of notation, often regard $\mathfrak{t}$ as a subalgebra of $\mathfrak{h}$. (If we wanted to study only Flensted-Jensen representations, we could wait until this point to choose our Cartan subalgebra and then define it in such a way that it contained $\mathfrak{t}$.)

We form roots of $(\mathfrak{g}^C, \mathfrak{t}^C)$. If we identify $\mathfrak{t}^C$ with a subalgebra of $\mathfrak{h}^C$, then the roots of $(\mathfrak{g}^C, \mathfrak{t}^C)$ are the nonzero functionals among the various $(e_i - e_j)|_{\mathfrak{t}^C}$. Define

$$(e_1 - e_N)|_{\mathfrak{t}^C} = 2f_1$$

$$(e_2 - e_{N-1})|_{\mathfrak{t}^C} = 2f_2$$

$$(e_r - e_{N-r+1})|_{\mathfrak{t}^C} = 2f_r .$$

The multiplicities of these roots are given as follows:

Root	$\Delta_K^{(\mathfrak{t})}$	$\Delta^{(\mathfrak{t})}$
$\pm 2f_j$	1	1
$\pm f_j$	2(N-2r)	2(N-2r+2)
$\pm f_i \pm f_j$	2	2

If we take $2f_r$ and all $f_i - f_j$ with $i < j$ to be positive, then we find

$$\rho_K^{(\mathfrak{t})} = \sum_{j=1}^{r} (N-2j+1) f_j$$

$$\rho^{(\mathfrak{t})} = \sum_{j=1}^{r} (N-2j+3) f_j \ .$$

Apart from the fixed-point group H, Flensted-Jensen's construction uses one other parameter, a linear form λ on $\mathfrak{t}^C$, which we may write as

$$\lambda = \sum_{j=1}^{r} c_j f_j \ .$$

Let

$$\mu_\lambda = \lambda + \rho^{(\mathfrak{t})} - 2\rho_K^{(\mathfrak{t})} = \sum_{j=1}^{r} (c_j - N + 2j + 1) f_j \ .$$

We have already arranged the orderings so that a suitable compatibility condition holds for the inclusion of $\mathfrak{t}$ in $\mathfrak{h}$. The parameter λ is required to satisfy

1) $\langle \mu_\lambda, \beta \rangle / |\beta|^2$ is an integer ≥ 0 for each β in $(\Delta_K^{(\mathfrak{t})})^+$
2) μ_λ (extended to $\mathfrak{h}^C$ by 0 on the orthocomplement of $\mathfrak{t}^C$) is integral on $\mathfrak{h}^C$
3) $\langle \lambda + \rho^{(\mathfrak{t})}, \beta \rangle \geq 0$ for all $\beta \in (\Delta^{(\mathfrak{t})})^+$.

For such λ, Flensted-Jensen constructs ψ_λ in $C^\infty(G/H)$ generating (under K) the K-type μ_λ and (under G) an indecomposable representation π_λ in $C^\infty(G/H)$.

With an additional condition on λ, Flensted-Jensen was able to prove that π_λ was an irreducible representation in $L^2(G/H)$. Oshima [11] improved Flensted-Jensen's condition to

L^2) $\langle\lambda,\beta\rangle > 0$ for all $\beta\epsilon(\Delta^{(t)})^+$.

In the case that G is $SU(N,2)$ and λ is $\sum c_j f_j$, conditions (1) to (3) are together equivalent with

$c_j \epsilon Z$ and $c_j = N - 1 \mod 2$ for $1 \leq j \leq r$

$c_j \geq c_j + 2$

$c_r \geq N - 2r - 1$.

Condition (L^2) imposes the additional requirement that

$c_r > 0$.

This is a new condition if $r = \frac{N}{2}$ or $r = \frac{N-1}{2}$, but it is redundant otherwise.

Schlichtkrull [12] proved that μ_λ is a minimal K-type of π_λ and that it is the only one.

Schlichtkrull also obtained information about the Langlands parameters of π_λ. Namely let $\mathfrak{l} = Z_{\mathfrak{g}}(\mathfrak{t})$ be the 0 root space of $(\mathfrak{g}^C, \mathfrak{t}^C)$ intersected with $\mathfrak{g}$, and let $L = Z_G(\mathfrak{t})$ be the corresponding analytic subgroup. In our notation for $SU(N,2)$,

$L = \text{torus} \times SU(N-2r,2)$

and, under the identification of $\mathfrak{t}$ with a subalgebra of the Cartan subalgebra $\mathfrak{h}$, L is the block diagonal subgroup built from the diagonal and a block with indices $r+1,\ldots,N-r,N+1,N+2$. Schlichtkrull's result about Langlands parameters is that

a) the $\mathfrak{a}$ of the parabolic subalgebra can be taken as an Iwasawa $\mathfrak{a}$ of $\mathfrak{l}$.

b) the $\mathfrak{a}$ parameter of π_λ is $\rho_{L,\min}$.

c) the M parameter is such that the minimal K-type of π_λ is μ_λ.

Let us interpret Schlichtkrull's results under the assumption that $r < \frac{N-1}{2}$. (This assumption makes $\dim \mathfrak{a}$ equal to 2.) We can choose the roots from which $\mathfrak{a}$ is constructed as $e_{r+1}-e_{N+2}$ and $e_{r+2}-e_{N+1}$. Let

$$b_j = \tfrac{1}{2}(c_j - N + 2j + 1) \quad \text{for} \quad 1 \leq j \leq r\,.$$

Then μ_λ is given by

$$\begin{array}{ccccccccccc} & 1 & & r & r+3 & & N-r & N-r+1 & & N & \\ \mu_\lambda = (& b_1, & \ldots, & b_r, & 0, & \ldots, & 0, & -b_r, & \ldots, & -b_1 &) \end{array}$$

and is subject to the conditions that b_j is in $\mathbb{Z}$ and $b_1 \geq \ldots \geq b_r \geq 0$. The minimal K-type formula of §3 then shows that σ has $m = n = 0$ and

$$\sigma_0 \leftrightarrow (b_1, \ldots, b_r, 0, \ldots, 0, -b_r, \ldots, -b_1)\,.$$

Now we shall identify the parameters with those of a Zuckerman triple (Δ^+,Σ,χ). Let Δ^+ be defined by the ordering

$$\underbrace{1}_{},\ \underbrace{2}_{},\ \ldots,\ \underbrace{r}_{},\ \underbrace{r+1,\ r+2,\ r+3,\ \ldots,\ N-r}_{},\ \underbrace{N+1,\ N+2}_{},\ N-r+1,\ \ldots,\ \underbrace{N}_{}$$

and let Σ be the subsystem of Δ corresponding to blocks as indicated. Define

$$\chi = \sum_{j=1}^{r} (b_j + N - 2j + 1)(e_j - e_{N-j+1})\,.$$

Then it is easy to check that (Δ^+,Σ,χ) is a Zuckerman triple and that $J(P_{min},\sigma,\rho_{L,min})$ has infinitesimal character $\chi - \rho_\Delta + 2\rho_\Sigma$ and a minimal K-type $\chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K}$. The infinitesimal character is nonsingular if

$$b_r \geq 2\,,$$

and, under this condition, the theorem at the end of §4 applies.

We conclude the following: In $SU(N,2)$, let $H = S(U(r)\times U(N-r,2))$ with $r < \frac{N-1}{2}$. Then nonsingular Flensted-Jensen representations occur in the nonunitary principal series with parameters σ and ν_0, where

σ has $m = n = 0$ and $\sigma_0 \longleftrightarrow (b_1, \ldots, b_r, 0, \ldots, 0, -b_r, \ldots, -b_1)$ with $b_r \geq 2$

ν_0 has $a = N - 2r + 1$ and $b = N - 2r - 1$.

For that σ, no real ν with $|\nu| > |\nu_0|$ yields a unitary representation.

Example. If we choose $N = 8$ and $r = 1$, then $G = SU(8,2)$ and the above parameters are

σ with $m = n = 0$ and $\sigma_0 \longleftrightarrow (k, 0, 0, 0, 0, -k)$ with $k \geq 2$

ν_0 with $a = 7$ and $b = 5$.

These are the parameters of the isolated unitary dot in Figure 2, and the theorem at the end of §4 is saying that there are no unitary points in that picture at any position farther from the origin.

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