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Multiplicity one fails for $\mathfrak{p}$-adic unitary principal series

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For a real semisimple group of matrices, every unitary principal series representation splits into inequivalent irreducible representations [5]. This multiplicity-one result was conjectured [7] in 1971 because one expected in general and knew in some special cases that all reducibility was accounted for by canonical geometric constructions, such as the imbedding of a space of analytic functions on the disc in the space of functions on the circle by passage to boundary values.

We shall give an example to show that the corresponding multiplicity-one statement is false for a semisimple group of matrices defined over a locally compact, totally disconnected, nondiscrete field of characteristic 0. This example is summarized in § 2, and its properties are verified in § 5. It is ultimately motivated by the work of Langlands [11] on classification of irreducible admissible representations. Langlands [12] was able to give a formulation of the results of [8] that suggests that straightforward generalization of the multiplicity-one theorem to other fields is not likely to succeed. An exposition of [12] is given in [9], and the way in which this work motivates our example is explained in § 6.

Verification of the properties of our example depends on a suitable development of intertwining operators for split groups. Most of such a development has been carried out by Sally [15] and Winarsky [18]. Some small modifications and elaborations of their work are the subject of § 3 and 4.

The work in this paper grew out of conversations at the American Mathematical Society Summer Institute in 1977 with Langlands, Lusztig, Schiffmann, Shelstad, and Wallach. The work by I. Muller [13] on intertwining operators was also of influence; Muller developed a variation on Winarsky's work and was able to push through analogs of the results of [8]. In addition, she came close to discovering the example of § 2. We thank all these people for their help.

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§1. Notation

Let $\mathbf{F}$ be a locally compact nondiscrete field of characteristic 0, and let G be a semisimple algebraic group defined over $\mathbf{F}$. We shall assume that G is split over $\mathbf{F}$ and that G is simply-connected in the algebraic sense. Let T be a maximal split torus in G, fix a positive system of roots, and let $P=TN$ be the corresponding minimal parabolic subgroup in G. Here N is the nilpotent radical of P.

Let μ be the positive character of T whose square is the module of the action of conjugation of T on N. In terms of Haar measures on N, $d(tnt^{-1})=\mu(t)^2dn$.

The *unitary principal series* for G is a series of unitary representations of G parametrized by the unitary characters of T. If $\xi\colon T\to\{z\in\mathbf{C}\mid |z|=1\}$ is such a character, the representation $U(\xi, g)$ is given in the dense subspace

$$\left\{ f\colon G\to\mathbf{C} \;\middle|\; \begin{array}{l} f \text{ continuous,} \\ f(xtn) = \mu(t)^{-1}\xi(t)^{-1}f(x) \text{ for } x\in G,\ t\in T,\ n\in N \end{array} \right\}$$

by the action

$$U(\xi, g)f(x) = f(g^{-1}x).$$

We can allow a generalization in which a character $\xi\colon T\to\mathbf{C}^\times$ is not necessarily unitary, and then we speak of the *nonunitary principal series.*

Since G is simply-connected, the torus T is isomorphic to a product of copies of $\mathbf{F}^\times$. We introduce notation that makes this isomorphism explicit. Let α be a root and form the image in G of the corresponding $SL(2, \mathbf{F})$. The image in G of $\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix}$, $x\in\mathbf{F}^\times$, is denoted $\psi_\alpha(x)$. Number the simple roots $\alpha_1,\dots,\alpha_\ell$, and define $\psi\colon(\mathbf{F}^\times)^\ell\to T$ by

$$\psi(x_1,\dots,x_\ell) = \prod_{j=1}^{\ell}\psi_{\alpha_j}(x_j).$$

Then ψ is the required isomorphism of $(\mathbf{F}^\times)^\ell$ onto T.

§2. Example

To describe our example, we specialize to the case that $\mathbf{F}$ is a finite extension of the field $\mathbf{Q}_p$ of p-adic numbers with p odd. Let $\mathcal{O}$ be the ring of integers, $\mathfrak{P}$ the maximal ideal, and π an element with $\mathfrak{P}=\pi\mathcal{O}$. Define q by $|\pi|=q^{-1}$. We shall take the group G to be (simply-connected) of type D_4. With standard notation we list the simple roots in the order e_1-e_2, e_2-e_3, e_3-e_4, e_3+e_4, and this list defines the function ψ that coordinatizes T.

Let ε be a generator of the unique cyclic subgroup of $\mathbf{F}^\times$ of order $q-1$. Then each x has a unique decomposition [17, p. 11] as

$$x = \pi^n \varepsilon^\ell a$$

with $0 \le \ell \le q-2$ and a in $1+\mathfrak{P}$. We write $n=n(x)$ and $\ell=\ell(x)$.

Define a unitary character ξ_0 of T by

$$\xi_0(\psi(x_1, x_2, x_3, x_4)) = (-1)^{n(x_1x_2x_3x_4)+\ell(x_2)},$$

and form the corresponding unitary principal series representation $U(\xi_0, g)$.

THEOREM 2.1. *The commuting algebra of $U(\xi_0, g)$ has dimension* 32 *and is noncommutative. Therefore some irreducible constituent of ξ_0 occurs with multiplicity greater than one.*

More precise information about the reducibility will be given in Corollary 5.4. The noncommutativity of the commuting algebra depends upon elementary properties of intertwining operators that will be assembled in the next two sections. The fact that the dimension is at most 32, as well as the precise information about reducibility, requires in addition a theorem of Harish-Chandra that is the analog for $\mathbf{F}$ of Theorem 38.1 of [3] for the field $\mathbf{R}$. See Harish-Chandra's lecture notes for 1971–73; cf. Theorem 5.5.3.2 of [16].

§ 3. Intertwining operators for $SL(2, \mathbf{F})$

The theory of intertwining operators for $SL(2, \mathbf{F})$ was developed by Sally [15]. We shall briefly redevelop the theory here, in a different form that is more parallel to the development in [6].

For this section let $G=SL(2, \mathbf{F})$, let $K=SL(2, \mathcal{O})$, let T be the diagonal subgroup of G, let A be the subgroup of matrices $\operatorname{diag}(\pi^n, \pi^{-n})$, let $M=T\cap K$, and let

$$N=\left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right\}, \quad V=\left\{\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right\}, \quad w=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then we have $G=KAN$, with the A-component unique; we write $g=\kappa(g)h(g)n$. The formula for $h(g)$ is

$$h\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} \pi^{\min(n(a),n(c))} & 0 \\ 0 & \pi^{-\min(n(a),n(c))} \end{pmatrix},$$

with $n(x)$ as in § 2. For this element g, $\mu(h(g))=\max\{|a|, |c|\}=\max\{q^{-n(a)}, q^{-n(c)}\}$.

LEMMA 3.1. *For a suitable normalization of Haar measures, every continuous function f on K that is right-invariant under $K \cap N$ satisfies*

$$\int_K f(k)dk = \int_{V \times M} f(\kappa(v)m)\mu(h(v))^{-2}dvdm.$$

This is proved in the same manner as Lemma 18 of [10].

The formal intertwining operator $A(w, \xi)$, with ξ a character of T, is defined by

$$A(w, \xi)f(x) = \int_V f(xwv)dv$$

for f in the representation space for $U(\xi, g)$, and it satisfies formally

$$A(w, \xi)U(\xi, g) = U(\xi^{-1}, g)A(w, \xi) \tag{3.1}$$

for all g in G. Corresponding to ξ there are a complex number z_0 (determined modulo $2\pi i/\log q$) and a character ξ_* trivial on A such that

$$\xi(t) = \xi_*(t)\mu(t)^{z_0}, \quad t \in T.$$

Let ξ_z for z in $\mathbf{C}$ be defined by

$$\xi_z(t) = \xi(t)\mu(t)^z.$$

For $\mathrm{Re}(z_0+z)>0$, the integral defining $A(w, \xi_z)$ is convergent and is given by a simple formula, as we shall see below. To write down the formula, we note that $G \doteq VTN$ in the sense that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & ba^{-1} \\ 0 & 1 \end{pmatrix}$$

if $a \neq 0$. We extend μ and ξ from T to VTN by having them ignore the V and N factors. The formula for $A(w, \xi_z)$, proved in the same manner as Lemma 56 of [6], is

$$\begin{aligned} A(w, \xi_z)f(k_0) &= \int_V \mu(h(v))^{-(1+z_0+z)}\xi_*(h(v))^{-1}f(k_0 w\kappa(v))dv \\ &= \int_K \mu(w^{-1}k)^{-(1-z_0-z)}\xi_*(w^{-1}k)f(k_0 k)dk \\ &= \int_V \mu(w^{-1}v)^{-(1-z_0-z)}\xi_*(w^{-1}v)\mu(h(v))^{-(1+z_0+z)}f(k_0\kappa(v))dv, \end{aligned} \tag{3.2}$$

and the integrals are convergent for $\mathrm{Re}(z_0+z)>0$ (cf. [17]) because

$$\mu(w^{-1}v)^{-(1-z_0-z)}\xi_*(w^{-1}v)dv = |x|^{-(1-z_0-z)}\xi_*(-x)dx$$

if $v=\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}$. Consequently we can obtain an analytic continuation by decomposing f into two pieces as in the proof of Theorem 3 of [6]. The result is

LEMMA 3.2. *Let f be locally constant on K and right-invariant under $K\cap N$. If the character ξ is ramified (i.e., ξ_* is nontrivial), then*

$$A(w,\ \xi_z)f(k_0)$$

extends to be entire in z and continuous in the pair (z, k_0). Otherwise, $\xi(t) =\mu(t)^{z_0}$, and $A(w, \xi_z)f(k_0)$ extends to be meromorphic in z with poles, at most simple, only at $-z_0$ and points congruent modulo $2\pi i/\log q$; moreover, $A(w, \xi_z)f$ is continuous in (z, k_0) away from the poles.

In terms of the analytically continued operators, equation (3.1) extends to an identity of meromorphic functions. Also the operators satisfy the adjoint relation, K-space by K-space, given in the following lemma. The proof is the same as for Lemma 24 of [10].

LEMMA 3.3. *In terms of the character ξ_* trivial on A,*

$$A(w,\ \xi_*,\ \mu^z)^* = A(w^{-1},\ \xi_*^{-1}\mu^{\bar{z}}).$$

If the unitary character ξ does not have order exactly two, $U(\xi, g)$ is irreducible, by [2, p. 164]. Consequently the proof of Proposition 27 of [6] can be repeated in the context of the field $\mathbf{F}$.

LEMMA 3.4. *For each character ξ on T there exists a meromorphic complex-valued function $\eta_\xi(z)$ of one complex variable such that*

$$A(w^{-1},\ (\xi_z)^{-1})A(w,\ \xi_z) = \eta_\xi(z)I.$$

The function $\eta_\xi(z)$ has the further properties that $\eta_{\xi_{z_0}}(z)=\eta_\xi(z_0+z)$ and

(i) *$\eta_{\xi_*}(z)$ is unchanged if w is replaced by wt with t in T,*
(ii) *$\eta_{\xi_*}(z)$ is ≥ 0 on the imaginary axis,*
(iii) *$\eta_{\xi_*^{-1}}(z)=\eta_{\xi_*}(-z)$,*
(iv) *$\eta_{\xi_*}(z)=\overline{\eta_{\xi_*}(-\bar{z})}$,*
(v) *$\eta_{\xi_*^\varphi}(z)=\eta_{\xi_*}(z)$ if φ is an automorphism of G leaving A fixed and K stable.*

By Lemma 36 of [6] there exists a meromorphic function $\gamma_{\xi_*}(z)$ in the plane such that

$$\eta_{\xi_*}(z) = \overline{\gamma_{\xi_*}(-\bar{z})}\gamma_{\xi_*}(z).$$

If ξ_* has order two, then (iii) above, together with the same Lemma 36, shows that $\gamma_{\xi_*}(z)$ can be taken to be real for real z. In any case, we define

$$\mathscr{A}(w, \xi_*\mu^{z_0}) = \gamma_{\xi_*}(z_0 + z)^{-1}A(w, \xi_*\mu^{z_0}).$$

Then we have, for all ξ,

$$\mathscr{A}(w^{-1}, \xi^{-1})\mathscr{A}(w, \xi) = I$$

and

$$\mathscr{A}(w, \xi_*\mu^z)^* = \mathscr{A}(w^{-1}, \xi_*^{-1}\mu^{\bar{z}}).$$

Hence $\mathscr{A}(w, \xi)$ is unitary for ξ unitary.

LEMMA 3.5. *If ξ is a unitary character on T, then $\mathscr{A}(w, \xi)$ is nonscalar except for ξ trivial.*

PROOF. Write $\xi = \xi_*\mu^z$. Except when ξ_* is trivial and $z \equiv 0 \bmod (2\pi i/\log q)$, $A(w, \xi)$ and $A(w^{-1}, \xi^{-1})$ do not have a pole and are not identically 0, by Lemma 3.2. By Lemmas 3.3 and 3.4, $\eta_{\xi_*}(z)$ is regular and not 0. Hence $\gamma_\xi(z)$ is regular and not 0. In addition, (3.2) shows that $A(w, \xi)$ cannot be scalar (cf. p. 575 of [6]). Hence $\mathscr{A}(w, \xi)$ is not scalar.

§4. Intertwining operators for G split over F

For general G split over F, the principal series is still defined as in §1. If ξ is a character of T and w is in the normalizer $N_G(T)$, the intertwining operator $A(w, \xi)$ is defined on the representation space for $U(\xi, g)$ and is given formally by

$$A(w, \xi)f(x) = \int_{V \cap w^{-1}Nw} f(xwv)dv. \tag{4.1}$$

The operator, except for a scalar, depends only on the coset of w in the Weyl group $W_G(A) = N_G(T)/T$. The formal intertwining relation is

$$A(w, \xi)U(\xi, g) = U(w\xi, g)A(w, \xi). \tag{4.2}$$

The details of how these operators are defined rigorously by analytic continuation are not very different from the real case and were carried out by Winarsky [18]. Winarsky used an integration over a quotient space instead of the one in (4.1), and Muller [13] began her work by redeveloping matters from (4.1) as definition. After either development, (4.2) results.

Some features of this analytic continuation are relevant for us. Let K be a "good" maximal compact subgroup (relative to T), and let $M = K \cap T$. With

ψ as in § 1, let

$$A = \{\psi(\pi^{n_1},\dots,\pi^{n_\ell}) \mid n_1,\dots,n_\ell \in \mathbf{Z}\}.$$

Then $T=MA$, direct product. In corresponding fashion, we can decompose ξ as

$$\xi = \xi_* \lambda,$$

where ξ_* is trivial on A and λ is trivial on M. Then λ is a character, not necessarily unitary, on $A \cong \mathbf{Z}^\ell$. That is, λ can be regarded as a parameter on $(\mathbf{C}^\times)^\ell$ or as a periodic parameter on $\mathbf{C}^\ell$. If functions in the representation space for $U(\xi, g)$ are restricted to K, the space of restrictions is independent of λ, and the analyticity property of $A(w, \xi)$ is that $A(w, \xi)f$ is meromorphic in λ (when λ is regarded as a parameter on $\mathbf{C}^\ell$) for each locally constant f on K. Moreover, if $w=w_1\cdots w_n$ corresponds to a minimal decomposition into simple reflections in the Weyl group, then

$$A(w, \xi) = A(w_1, w_2\cdots w_n\xi)A(w_2, w_3\cdots w_n\xi)\cdots A(w_n, \xi). \tag{4.3}$$

To obtain normalized intertwining operators $\mathscr{A}(w, \xi)$, we normalize each factor in (4.3) by means of the normalizing factors in § 3. Thus, changing notation, suppose $w \bmod T$ is a simple reflection in the Weyl group. If $w \bmod T$ is the reflection with respect to the simple root α and if α is as in § 1, we define $\xi_\alpha = \xi \cdot \psi_\alpha$. Then ξ_α is a character of the torus in a subgroup $SL(2, \mathrm{F})$ and decomposes as $\xi_\alpha = (\xi_\alpha)_* \mu_\alpha^z$. In this situation we define

$$\mathscr{A}(w, \xi) = \gamma_{(\xi_\alpha)_*}(z)^{-1} A(w, \xi).$$

This defines normalized intertwining operators for the case that w is a simple reflection. Changing notation back again, we can then normalize each operator on the right side of (4.3), and we take the normalized $\mathscr{A}(w, \xi)$ to be the product of the corresponding normalized operators. This procedure is independent of the decomposition of w that led to (4.3). One can then proceed as in the real case [6] to prove the cocycle relation

$$\mathscr{A}(w_1 w_2, \xi) = \mathscr{A}(w_1, w_2\xi)\mathscr{A}(w_2, \xi).$$

See [13] for details.

We have also the adjoint relation

$$\mathscr{A}(w, \xi)^* = \mathscr{A}(w^{-1}, \overline{w\xi^{-1}}),$$

and it follows that $\mathscr{A}(w, \xi)$ is unitary for ξ unitary.

From this point one can prove the theorem describing a basis for the commuting algebra of $U(\xi, g)$ when ξ is unitary. The following ingredients allow one to imitate the proof in the real case; Muller [13] has carried out the details.

(1) The fact that the definition of $A(w, \xi)$ can be written as an integral over $V \cap w^{-1}Nw$, not just as an integral over a quotient.
(2) The Bruhat double-coset decomposition of G.
(3) The description (due to Borel-Tits) of the closure of a Bruhat double coset.
(4) Harish-Chandra's completeness theorem referred to at the end of § 2.

The result is as follows. For ξ unitary, let

$$W(\xi) = \{w \in W_G(A) \mid w\xi = \xi\}.$$

If $w \bmod T$ is in $W(\xi)$, (4.2) shows that $\mathscr{A}(w, \xi)$ commutes with $U(\xi, g)$. Let

$$\Delta'(\xi) = \{\alpha \mid \mathscr{A}(w_\alpha, \xi) \text{ is scalar}\} = \{\alpha \mid \xi \circ \psi_\alpha = 1\}.$$

The equality of the two formulas for $\Delta'(\xi)$ follows from Lemma 3.5. The system $\Delta'(\xi)$ is a root system if it is not empty, and we let $W'(\xi)$ be its Weyl group. The R-group for ξ is defined as

$$R(\xi) = \{w \in W(\xi) \mid w\alpha > 0 \text{ for every } \alpha > 0 \text{ in } \Delta'(\xi)\}.$$

THEOREM 4.1. *$W(\xi)$ is the semidirect product $W(\xi) = W'(\xi)R(\xi)$, and $W'(\xi)$ is normal. The operators $\mathscr{A}(w, \xi)$ for $w \bmod T$ in $W'(\xi)$ are scalar, and the operators $\mathscr{A}(w, \xi)$ for $w \bmod T$ running through $R(\xi)$ are linearly independent. Consequently (by Harish-Chandra's completeness theorem), the operators $\mathscr{A}(w, \xi)$ for $w \bmod T$ running through $R(\xi)$ form a linear basis for the commuting algebra of $U(\xi, g)$.*

§ 5. The example of § 2

For the example of § 2, let w be in the Weyl group. We have

$$\psi(x_1, \ldots, x_4) = \textstyle\prod_{j=1}^4 \psi_{\alpha_j}(x_j) \tag{5.1}$$

for a particular listing of the simple roots. Then

$$w\xi_0(\psi(x_1, \ldots, x_4)) = \xi_0(\textstyle\prod_{j=1}^4 w^{-1}\psi_{\alpha_j}(x_j)) = \xi_0(\prod_{j=1}^4 \psi_{w\alpha_j}(x_j)). \tag{5.2}$$

We compute the groups W, W', and R. For W, the condition is that $w\xi_0 = \xi_0$, and in view of (5.1) and (5.2) it is necessary and sufficient that

$$\xi_0(\psi_\alpha(x)) = \xi_0(\psi_{w\alpha}(x)), \quad x \in \mathbf{F}^\times, \tag{5.3}$$

hold for each simple root α. If α, β, and $\alpha + \beta$ are roots, we have

$$\psi_{\alpha+\beta}(x) = \psi_\alpha(x)\psi_\beta(x); \tag{5.4}$$

thus (5.3) holds for all simple α if and only if it holds for all α.

According to the definition of ξ_0, we have the following formulas for $\xi_0(\psi_\alpha(x))$ when α is simple:

$$\xi_0(\psi_\alpha(x)) = \begin{cases} (-1)^{n(x)} & \text{for} \quad \alpha = e_1 - e_2,\ e_3 - e_4, \text{ or } e_3 + e_4 \\ (-1)^{n(x)+\ell(x)} & \text{for} \quad \alpha = e_2 - e_3. \end{cases}$$

Then by (5.4) and a little computation

$$\xi_0(\psi_\alpha(x)) = \begin{cases} (-1)^{n(x)} & \text{for} \quad \alpha = \pm e_1 \pm e_2,\ \pm e_3 \pm e_4 & (5.5\text{a}) \\ (-1)^{\ell(x)} & \text{for} \quad \alpha = \pm e_1 \pm e_3,\ \pm e_2 \pm e_4 & (5.5\text{b}) \\ (-1)^{n(x)+\ell(x)} & \text{for} \quad \alpha = \pm e_1 \pm e_4,\ \pm e_2 \pm e_3 & (5.5\text{c}) \\ +1 & \text{for no} \quad \alpha. & (5.5\text{d}) \end{cases}$$

The whole Weyl group is the semidirect product of the even sign changes and the permutations on e_1, e_2, e_3, e_4. The condition for an element w of the Weyl group to be in W is that w leaves each line of roots stable in (5.5). The even sign changes have this property, and thus W is the semidirect product of the even sign changes and a group of permutations. Clearly the permutation group is the four-element group:

$$\{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}.$$

Thus W is a nonabelian group of order 32.

The roots in Δ' are those that appear in (5.5d); thus Δ' is empty and W' is trivial. It follows that $R = W$ and that R is a nonabelian group of order 32. Theorem 2.1 follows from this fact and from Theorem 4.1.

Actually we can describe precisely the number of irreducible constituents and their multiplicities for $U(\xi_0, g)$, and the remainder of this section will be devoted to deriving such a result. For this purpose we do need to use the result of Harish-Chandra's cited after Theorem 2.1.

LEMMA 5.1. *The R-group for ξ_0 can be written with generators and relations in such a way that every relation is of one of the forms*
(i) *the square of a generator, or*
(ii) *the commutator of two words, each of which is of order 2 when realized in R.*

PROOF. If α is a root, let p_α be the corresponding reflection in the Weyl group. Let s_i be the i-th sign change. Define

$$w_1 = p_{e_1-e_2}p_{e_1+e_2} = s_1 s_2,$$

$$w_2 = p_{e_1-e_3}p_{e_1+e_3} = s_1 s_3,$$

$$w_3 = p_{e_1-e_2}p_{e_3-e_4} = (1\ 2)(3\ 4),$$

$$w_4 = p_{e_1-e_3}p_{e_2-e_4} = (1\ 3)(2\ 4).$$

Since $s_1s_4 = w_3w_1w_2w_3^{-1}$, it is apparent that these 4 elements generate R. Also $w_i^2 = 1$; and

w_3 commutes with w_4;

w_1 commutes with w_3, with w_2, with $w_4w_1w_4^{-1}$, and with $w_3w_2w_3^{-1}$;

w_2 commutes with w_4, with w_1, with $w_4w_1w_4^{-1}$, and with $w_3w_2w_3^{-1}$;

w_3w_4 commutes with w_1w_3.

Consequently if we let R' be an abstract group with generators w_1, w_2, w_3, w_4 and with relations as in the previous sentence, then R is a homomorphic image of R'. The lemma will be proved if we show that $|R'| \leq 32$.

Consider the subgroup S_0 of R' generated by w_1, w_2 and $w_3w_1w_2w_3^{-1}$. These three elements commute and are of order ≤ 2. Thus $|S_0| \leq 8$. Also w_3 normalizes S_0 since

$$\begin{aligned} w_3w_1w_3^{-1} &= w_1, \\ w_3w_2w_3^{-1} &= (w_1^{-1})(w_3w_1w_2w_3^{-1}), \\ w_3(w_3w_1w_2w_3^{-1})w_3^{-1} &= (w_1)(w_2). \end{aligned}$$

Then w_3 and S_0 generate a subgroup S_1 of R' of order ≤ 16. We claim that this larger subgroup S_1 is normalized by w_4; if so, then the fact that w_4 is of order ≤ 2 implies that $|R'| \leq 32$. Now w_4 commutes with w_2 and w_3. Also

$$\begin{aligned} (w_4w_1w_4^{-1})(w_3w_1w_2w_3^{-1}) &= w_4w_1w_3w_4w_1w_2w_3^{-1} \\ &= w_4w_1w_1w_2w_3w_4w_3^{-1} \\ &= w_4w_2w_4 = w_2, \end{aligned}$$

so that

$$w_4w_1w_4^{-1} = w_2(w_3w_1w_2w_3^{-1}).$$

Finally

$$w_4(w_3w_1w_2w_3^{-1})w_4^{-1} = (w_4w_2w_4^{-1})w_4(w_4w_1w_4^{-1})w_4^{-1} = w_2w_1.$$

Thus S_1 is normalized by w_4, and the proof of the lemma is complete.

LEMMA 5.2. *If the R-group for a character ξ of T can be given by gener-*

ators and relations as in Lemma 5.1, *then ξ extends to a character of the smallest subgroup $\bar{R}$ of $N_G(T)$ containing T and coset representatives of each element of R.*

PROOF. Let the generators be $r_1,\dots, r_m$, and let the relations be $r_1^2,\dots, r_m^2$ and $p_i q_i p_i^{-1} q_1^{-i}$ with each p_i and q_i a word in $r_1,\dots, r_m$. Choose representatives $w_1,\dots, w_m$ of $r_1,\dots, r_m$ in $N_G(T)$, and let $p_i(w)$ and $q_i(w)$ be the results of substituting the w_j's for the r_j's in p_i and q_i. Define elements of T by

$$h_i = w_i^2, \quad t_i = p_i(w)q_i(w)p_i(w)^{-1}q_i(w)^{-1}.$$

According to Lemma 59 of [6], $\bar{R}$ is given by generators $\bar{w}_1,\dots, \bar{w}_n$ and $\bar{t}$ ($t \in T$), with relations

(i) $\bar{1}$,

(ii) $(\overline{tt'})^{-1}\bar{t}\,\bar{t'}$,

(iii) $\bar{w}_i \bar{t} \bar{w}_i^{-1}(\overline{w_i t w_i^{-1}})^{-1}$,

(iv) $\bar{h}_i^{-1}\bar{w}_i^2$,

(v) $\bar{t}_i^{-1}p_i(\bar{w})q_i(\bar{w})p_i(\bar{w})^{-1}q_i(\bar{w})^{-1}$.

We define ξ on the free group on the above generators in such a way that

$$\xi(\bar{t}) = \xi(t),$$

$$\xi(\bar{w}_i)^2 = \xi(h_i).$$

To complete the proof, it is enough to show that, in the free group, ξ annihilates each relation (i),..., (v).

Our definitions make (i), (ii), and (iv) immediate. For (iii) the question is whether

$$\xi(\bar{t}) = \xi(\overline{w_i t w_i^{-1}}),$$

and hence whether

$$\xi(t) = \xi(w_i t w_i^{-1}).$$

But this holds since $r_i^{-1}\xi = \xi$ on T.

Thus we are left with (v). Applying ξ, we see that the question is whether $\xi(\bar{t}_i) = 1$, and hence whether

$$\xi(p_i(w)q_i(w)p_i(w)^{-1}q_i(w)^{-1}) = 1. \tag{5.6}$$

Let us consider the effect on (5.6) of replacing $p_i(w)$ by a different representative

$p_i(w)t$. Then the left side of (5.6) becomes

$$\xi((t(t^{-1})^{q_i})^{p_i})\xi(p_i(w)q_i(w)p_i(w)^{-1}q_i(w)^{-1}).$$

The first factor is 1 since $p_i\xi=\xi=q_i\xi$. Thus the validity of (5.6) is unaffected by using a representative $p_i(w)t$ for p_i in place of $p_i(w)$. Similarly the validity of (5.6) is unaffected by changing the representative of q_i. Since p_i and q_i are commuting elements of order 2, Lemma 3 of [5] says they have commuting representatives.* Then the left side of (5.6) collapses to 1, and the lemma is proved.

THEOREM 5.3. *If the R-group for a character ξ of T can be given by generators and relations as in Lemma* 5.1, *then the commuting algebra of $U(\xi, g)$ is isomorphic with the group algebra of R over* **C**.

PROOF. Apply Lemma 5.2 to obtain an extension of ξ to the group $\bar{R}$ of that lemma. Then the maps

$$w \longrightarrow \xi(w) \quad \text{and} \quad w \longrightarrow \mathscr{A}(w, \xi)$$

are homomorphisms on $\bar{R}$ and commute. Hence the map $w \to \xi(w)\mathscr{A}(w, \xi)$ is a homomorphism of $\bar{R}$ into unitary operators. The operator $\xi(w)\mathscr{A}(w, \xi)$ depends only on the coset of $w \bmod T$, and thus we have a homomorphism of R into unitary operators. In view of Theorem 4.1, this map is an isomorphism onto a basis of the commuting algebra of $U(\xi, g)$. By the universal mapping property of the group algebra, the map factors through the group algebra and provides the required isomorphism of the group algebra with the commuting algebra.

COROLLARY 5.4. *The principal series representation $U(\xi_0, g)$ of* § 2 *splits completely into* 16 *irreducible representations of multiplicity one and one irreducible representation of multiplicity four.*

PROOF. Lemma 5.1 and Theorem 5.3 show that the problem is to decompose the regular representation of the 32-element group R. In R, the center is the 2-element group consisting of the identity and the product of all four sign changes, and the quotient is abelian of order 16. Hence R has 16 characters. Also the standard 4-dimensional representation of the Weyl group on $\mathbf{R}^4$ or $\mathbf{C}^4$ is irreducible and remains irreducible when restricted to R. The corollary follows.

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* This lemma is stated over **R**, but the matrices in question are integer matrices, and the result is valid over **Q**.

§6. Motivation: Connection with Langlands theory of Weil group

By way of preliminaries, let G be a connected real semisimple Lie group with a simply-connected complexification $G^{\mathbf{C}}$. Langlands [11] developed a theory connected with a "Weil group" in the course of classifying irreducible admissible representations of G. See Borel [1]. The relevant objects for classifying representations (up to "L-indistinguishability") are homomorphisms φ of the Weil group into ${}^LG$, which is a complex centerless semisimple Lie group whose Dynkin diagram is dual to the one for $G^{\mathbf{C}}$. In the case of standard unitary continuous series representations of G, the image of φ is bounded, and Langlands shows that the reducibility group R of §4 for such a representation is essentially the component group of the centralizer of image φ; see Theorem 3.4 of [9] for a precise statement.

In the case that G is split over $\mathbf{R}$ and the representation is one from the unitary principal series (i.e., induced from a minimal parabolic subgroup), the theory simplifies as follows. In place of the homomorphism of the Weil group as above, we may use a homomorphism φ of the multiplicative group $\mathbf{R}^\times$ into ${}^LG$, we may assume that the image of φ is contained in a fixed Cartan subgroup of ${}^LG$, and then the R-group of §4 is *exactly* the component group of the centralizer of image φ. See Theorem 3.4 of [9].

We can speculate that the same thing will happen when $\mathbf{R}$ is replaced by our field $\mathbf{F}$, as long as G is split over $\mathbf{F}$ and is simply-connected in the algebraic sense. (Indeed, this turns out to be the case, but we omit the proof.)

We search for a two-element subset of a torus in a centerless compact semisimple group whose centralizer has a nonabelian component group. It is easy to see that in $SO(8)/\mathbf{Z}_2$, the component group of the centralizer of the pair of elements $\{x_1, x_2\}$ with

$$x_1 = \operatorname{diag}(-1, -1, 1, 1, -1, -1, 1, 1) \mod \mathbf{Z}_2,$$

$$x_2 = \operatorname{diag}(-1, -1, -1, -1, 1, 1, 1, 1) \mod \mathbf{Z}_2$$

is isomorphic to the 32-element group in §5. We therefore seek a homomorphism φ of $\mathbf{F}^\times$ into $SO(8)/\mathbf{Z}_2$ with image $\{1, x_1, x_2, x_1x_2\}$. As in §2, put $x = \pi^{n(x)}\varepsilon^{\ell(x)}a$ for x in $\mathbf{F}^\times$. If we define

$$\varphi(x) = x_1^{n(x)}x_2^{\ell(x)},$$

then φ is the required homomorphism.

To obtain a character ξ_0 of T from the homomorphism φ, we recall that ψ_α is a map of $\mathbf{F}^\times$ into G, and we let $\xi_{\alpha^\vee}$, where $\alpha^\vee$ is the root of ${}^LG$ dual to α for G, be the character corresponding to the root $\alpha^\vee$ of ${}^LG$. Then we set

$$\xi_0(\psi_\alpha(x)) = \xi_{\alpha^\vee}(\varphi(x))$$

and find easily that ξ_0 is the character given in § 2.

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