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Langlands Classification and Unitary Dual of $SU(2,2)$

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This paper continues a theme addressed by some seminar participants: investigation of the irreducible unitary representations of semisimple Lie groups. For the group

$$G = SU(2,2) = \left\{ g \in SL(4,\mathbf{C}) \,\middle|\, g^* \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} g = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \right\},$$

which is locally isomorphic to the conformal group of space-time, the classification problem has been completely solved in joint work with B. Speh [**8**].

The detailed answer appears in [**8**], and some parts of that answer will be reproduced presently. Qualitatively there are no surprises, and moreover the argument shows that all the unitary representations are unitary for simple reasons. This is so even for the ladder representations that have been studied by a number of mathematical physicists!

Instead of concentrating here on the answer to the classification problem, we shall emphasize the approach to such a problem. In particular, the Langlands classification (of irreducible "admissible" representations) and its method of proof will be of special interest because the classification and proof may well have independent applications in mathematical physics and they are available in full only as unpublished

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[1] Supported by NSF Grant MCS 80-01854. The research on the unitary dual of $SU(2,2)$ was joint work with B. Speh [**8**].

notes [**6, 7, 9, 3, 14, 11**]. Partial proofs appear in published form in Warner [**15**], Borel and Wallach [**2**] and Miličić [**12**].

Historically, approaches to classification of the irreducible unitary representations of semisimple groups have always proceeded in two steps:

(1) Get control of some larger class of representations (such as the irreducible "admissible" representations defined below in §1).

(2) Decide which ones are unitary.

In fact, in some groups, seemingly only a little more complicated than $SU(2,2)$, the pattern of unitary parameters is much more complicated than in $SU(2,2)$. Thus either this two-step approach is forced on us as a nontrivial subdivision of the problem, or else people are completely off track in the current thinking of what kinds of variables to use as parameters.

Step (1) above is nowadays solved in considerable generality—completely in connected semisimple groups G having faithful matrix representations. The relevant theorem is the Langlands classification [**9**] obtained in 1973. This result is not simply a generalization to G of earlier techniques, and we shall contrast the earlier techniques with the Langlands approach in §1.

Step (2) is the relatively new part for $SU(2,2)$. For $SU(2,2)$ this step involves only a little more than an exercise with known techniques. The techniques succeed because of the small size of the group. Aspects of this step are the subject of §2.

1. Admissible representations of $SU(2,2)$. Let K be the maximal compact subgroup of $G = SU(2,2)$ given by

$$K = S(U(2)\times U(2)) = SU(2,2)\cap U(4) = \left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\}.$$

A (continuous) representation π of G on a (complex) Hilbert space V, say with K acting unitarily, is said to be *admissible* if each irreducible representation of K occurs only finitely often in the restriction $\pi|_K$. It is known [**5**] that irredicible unitary representations are admissible.

In this case we can pass to the Lie algebra $\mathfrak{g}$ of G, which acts on

$$V_K = \{v \in V \mid \pi(K)v \text{ spans a finite-dimensional space}\}.$$

The vector space V_K is called the space of *K-finite vectors* in V, and the representation of $\mathfrak{g}$ on V_K provides an instance of an *admissible representation of* $(\mathfrak{g}, K)$ (= a complex vector space on which $\mathfrak{g}$ and K both have representations, in compatible fashion, with every vector K-finite and with each irreducible representation of K occurring only finitely often).

A number of properties of admissible representations were established by Harish-Chandra [5]. The representation of G on V is irreducible if and only if the representation of $(\mathfrak{g}, K)$ on V_K is algebraically irreducible. An irreducible admissible representation of $(\mathfrak{g}, K)$ comes from a unitary representation of G if and only if the representation of $(\mathfrak{g}, K)$ admits an inner product such that $\mathfrak{g}$ acts by skew-Hermitian operators, and any two such irreducible unitary representations of G leading to the same irreducible representation of $(\mathfrak{g}, K)$ are unitarily equivalent. (We summarize this condition on the representation of $(\mathfrak{g}, K)$ by saying it is *infinitesimally unitary*.)

It is convenient to define equivalence of admissible representations of G on the Lie algebra level, saying that two admissible representations of G are *infinitesimally equivalent* if the corresponding representations of $(\mathfrak{g}, K)$ are algebraically equivalent. Every irreducible admissible representation of $(\mathfrak{g}, K)$ arises from an admissible representation of G, according to a theorem of Lepowsky [10]. Let $\mathfrak{k}$ be the Lie algebra of K.

The traditional approach to Step (1) of the introduction would be as follows: Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the set of Hermitian members of $\mathfrak{g}$, and complexify to get $\mathfrak{g}^{\mathbf{C}} = \mathfrak{k}^{\mathbf{C}} \oplus \mathfrak{p}^{\mathbf{C}}$. We can assume that we understand the action of $\mathfrak{k}^{\mathbf{C}}$ in an irreducible representation, and we want to understand the action of $\mathfrak{p}^{\mathbf{C}}$. In the first place, this action has to be consistent with the bracket relations in $\mathfrak{g}^{\mathbf{C}}$. (Recall that $[\mathfrak{k}^{\mathbf{C}}, \mathfrak{p}^{\mathbf{C}}] \subseteq \mathfrak{p}^{\mathbf{C}}$ and $[\mathfrak{p}^{\mathbf{C}}, \mathfrak{p}^{\mathbf{C}}] \subseteq \mathfrak{k}^{\mathbf{C}}$.) But also it must satisfy some further relations because the center $Z(\mathfrak{g}^{\mathbf{C}})$ of the universal enveloping algebra (the generalized Casimir elements) must act as scalars. In the case of $SU(2,2)$, $Z(\mathfrak{g}^{\mathbf{C}})$ is a full polynomial algebra (with no relations other than commutativity) in three operators that one could write down explicitly if necessary; so we essentially have three additional relations beyond the bracket relations. The idea is to play these relations off against each other and see what happens.

This calculation seems to be just barely possible for $SU(2,2)$ (cf. Angelopoulos [1]), but the result that is obtained does not obviously fall into any general pattern.

The Langlands approach to Step (1) is quite different. One begins by constructing models for some irreducible admissible representations; these will be denoted $J(P, \sigma, \nu)$. To construct these representations, let $G = KA_{\min}N_{\min}$ be an Iwasawa decomposition of G. Here $A_{\min}$ and $N_{\min}$ have Lie algebras $\mathfrak{a}_{\min}$ and $\mathfrak{n}_{\min}$, respectively, where

$$\mathfrak{a}_{\min} = \left\{ \begin{pmatrix} 0 & 0 & s & 0 \\ 0 & 0 & 0 & t \\ s & 0 & 0 & 0 \\ 0 & t & 0 & 0 \end{pmatrix} \right\} \tag{1.1}$$

and where $\mathfrak{n}_{\min}$ is a certain nilpotent subalgebra of $\mathfrak{g}$ of dimension 6. We define $M_{\min}$ to be the centralizer of $A_{\min}$ in K; this is the group generated by the scalar fourth roots of unity and the circle group

$$T = \left\{ \begin{pmatrix} e^{i\theta} & & & \\ & e^{-i\theta} & & \\ & & e^{i\theta} & \\ & & & e^{-i\theta} \end{pmatrix} \right\}. \tag{1.2}$$

Then $P_{\min} = M_{\min} A_{\min} N_{\min}$ is a closed subgroup of G that plays the same role for $SU(2,2)$ that the upper triangular group plays for $SL(n,\mathbf{R})$ or $SL(n,\mathbf{C})$.

There is a standard series of admissible representations $U(P_{\min}, \sigma, \nu)$ induced from $P_{\min}$ known as the *nonunitary principal series*. The parameter σ is a unitary character of the compact abelian group $M_{\min}$, acting in the one-dimensional complex vector space $V^{\sigma} \cong \mathbf{C}$. The parameter ν is a complex-valued real-linear functional on $\mathfrak{a}_{\min}$, which we may write in coordinates as

$$\nu = cf_1 + df_2, \tag{1.3}$$

where f_1 and f_2 on the matrix (1.1) are s and t, respectively. Then $\sigma \otimes e^{\nu} \otimes 1$ is a representation of $P_{\min}$ (nonunitary unless ν is imaginary), and we let

$$U(P_{\min}, \sigma, \nu) = \operatorname{ind}_{P_{\min}}^{G}(\sigma \otimes e^{\nu} \otimes 1).$$

The conventions in the definition of $U(P_{\min}, \sigma, \nu)$ are that G is to act on the left on the representation space in the form

$$\left\{ f \in L^2(K, V^{\sigma}) \mid f(km) = \sigma(m)^{-1} f(k) \text{ for } k \in K, m \in M_{\min} \right\}, \tag{1.4}$$

and the parameters are arranged so that unitary data lead to unitary representations U ("Mackey induction").

In the notation of (1.3), let $\rho = 3f_1 + f_2$. If ν in (1.3) satisfies $\operatorname{Re} c > \operatorname{Re} d > 0$ and f and g are continuous, then one has the limit formula

$$\lim_{a \to +\infty} e^{-(\nu-\rho)\log a}\big(U(P_{\min}, \sigma, \nu, ma) f, g\big) = \left(\int_{N^*_{\min}} \sigma(m) f(n^*)\, dn^* \right) \overline{g(1)} \tag{1.5}$$

for m in $M_{\min}$. Here "$*$" refers to adjoints, and $a \to +\infty$ means that $(f_1 - f_2)(\log a) \to +\infty$ and $f_2(\log a) \to +\infty$. Equation (1.5) is easy to see on a formal level by changing variables from K to $N^*_{\min}$ and passing to the limit, provided one is content to omit the justification of the interchange of limit and integral. We shall rewrite (1.5) a little. We define an operator by

$$A(P_{\min}^* : P_{\min} : \sigma : \nu)f(x) = \int_{N_{\min}^*} f(xn^*)\,dn^*. \tag{1.6}$$

Under our assumption on ν that $\operatorname{Re} c > \operatorname{Re} d > 0$, one can show that this is a bounded operator on the space (1.4) that satisfies

$$U(P_{\min}^*, \sigma, \nu)A(P_{\min}^* : P_{\min} : \sigma : \nu) = A(P_{\min}^* : P_{\min} : \sigma : \nu)U(P_{\min}, \sigma, \nu). \tag{1.7}$$

We can then rewrite (1.5) as

$$\begin{aligned}\lim_{a \to +\infty} e^{-(\nu-\rho)\log a}&(U(P_{\min}, \sigma, \nu, ma)f, g)_{L^2(K, V^\sigma)} \\ &= (\sigma(m)(A(P_{\min}^* : P_{\min} : \sigma : \nu)f)(1), g(1))_{V^\sigma}. \end{aligned} \tag{1.8}$$

It is easy to see from (1.8) that if f is not in the kernel of $A(P_{\min}^* : P_{\min} : \sigma : \nu)$, then f is cyclic for $U(P_{\min}, \sigma, \nu)$. It follows readily that $U(P_{\min}, \sigma, \nu)$ has a unique irreducible quotient $J(P_{\min}, \sigma, \nu)$, known as the *Langlands quotient*, and that $J(P_{\min}, \sigma, \nu)$ is isomorphic to the image of $A(P_{\min}^* : P_{\min} : \sigma : \nu)$. In view of (1.7), we can therefore regard $J(P_{\min}, \sigma, \nu)$ as operating in a subspace of $U(P_{\min}^*, \sigma, \nu)$.

A version of this construction works when $P_{\min}$ is replaced by any larger closed subgroup of G. There are four such subgroups P (including $P_{\min}$ and G) in the case of $SU(2,2)$, and they are listed explicitly in [**8**, p. 44]. Each can be written as $P = MAN$ with $M \supseteq M_{\min}$, $A \subseteq A_{\min}$ and $N \subseteq N_{\min}$. Except in the case of $P_{\min}$, M will be noncompact. We form

$$U(P, \sigma, \nu) = \operatorname{ind}_P^G(\sigma \otimes e^\nu \otimes 1),$$

where σ is an irreducible unitary representation of M whose $(K \cap M)$-finite matrix coefficients are in $L^{2+\varepsilon}(M)$ for every $\varepsilon > 0$ (i.e., σ is *irreducible tempered*) and where ν is a complex-valued real-linear functional on the Lie algebra of A such that $\operatorname{Re}\nu$ satisfies a suitable positivity condition. Then (1.6), with the subscripts "min" erased, is a convergent integral for K-finite f, (1.7) holds at least on the Lie algebra level when the subscripts "min" are erased, and (1.8) is valid for f and g K-finite when the subscripts "min" are erased. In the same way it then follows that $U(P, \sigma, \nu)$ has a unique irreducible quotient $J(P, \sigma, \nu)$, the "Langlands quotient," and $J(P, \sigma, \nu)$ is isomorphic to the image of $A(P^* : P : \sigma : \nu)$.

Langlands showed conversely that all irreducible admissible representations of G are obtained this way, and each arises from this construction only once. A precise statement follows. We shall refer to the relevant triple (P, σ, ν) as the "Langlands parameters" of an irreducible admissible representation.

THEOREM (LANGLANDS CLASSIFICATION FOR $SU(2,2)$). *The equivalence classes of irreducible admissible representations of $SU(2,2)$, under infinitesimal equivalence, stand in one-one correspondence with triples (P, σ, ν), where*

$P = MAN$ is a closed subgroup of $SU(2,2)$ containing $P_{\min}$,

σ is an irreducible tempered representation of M, two such representations being regarded as the same if they are unitarily equivalent,

ν is a complex-valued real-linear functional on the Lie algebra of A with $\operatorname{Re}\nu$ *in the open positive Weyl chamber.*

The correspondence is that (P, σ, ν) corresponds to the class of $J(P, \sigma, \nu)$.

We shall give the idea behind the proof of completeness. Let π be irreducible admissible on a Hilbert space V, and consider a finite K-stable block of matrix coefficients, which we write as $E_2\pi(x)E_1$, where E_1 and E_2 are orthogonal projections. This function has known behavior on the left and right under K. Also, if we regard each member X of the universal enveloping algebra of $\mathfrak{g}^{\mathbf{C}}$ as a left-invariant differential operator on G, then we have

$$X(E_1\pi(x)E_2) = E_1\pi(x)\pi(X)E_2.$$

For X in the center $Z(\mathfrak{g}^{\mathbf{C}})$, $\pi(X)$ is a scalar, and it follows that

$$Z(E_1\pi(x)E_2) = c(Z)E_1\pi(x)E_2 \quad \text{for } Z \in Z(\mathfrak{g}^{\mathbf{C}}). \tag{1.9}$$

Now $G = K\overline{A^+_{\min}}K$, and it turns out that one can use the transformation laws under K to rewrite the system (1.9) as a system of differential equations on $A^+_{\min}$ with variable coefficients. Solutions of the rewritten system will be functions on $A^+_{\min}$ whose value at each point is in the space W of linear maps from image(E_1) to image(E_2) commuting with the action of $M_{\min}$. One can arrange that the coefficient functions in the rewritten system have values in the space of linear maps from W into W.

The rewritten system behaves as if it has a regular singular point at $+\infty$ in $A_{\min}$, except that the domain is two-dimensional. All solutions have series expansions (about $+\infty$) with coefficients in W, and the space of solutions is finite-dimensional.

Any leading term of a solution leads in a natural way to an imbedding of the K-finite vectors of π as a subrepresentation in some $U(P^*, \sigma, \nu)$, hence to a realization as $J(P, \sigma, \nu)$. In more detail, there are only finitely many candidates for growth/decay rates on $A_{\min}$ of leading terms, independently of E_1 and E_2. Thus fix v' in V_K. For any v in V_K choose E_1

and E_2 with $E_1 v = v$ and $E_2 v' = v'$. Then the analysis above enables us to expand $(\pi(x)v, v')$ in series and to pick off the coefficient of the leading term we are studying. The result is a linear functional l on V_K with good behavior relative to $\mathfrak{a}_{\min}$ and with $l(\pi(\mathfrak{n}^*_{\min})V_K) = 0$. Formally we complete the argument by mapping v in V_K to the function f_v on G given by $f_v(x) = l(\pi(x)^{-1}v)$ to obtain an imbedding of V_K in an induced representation from $A_{\min}N^*_{\min}$ to G. Bringing in the $M_{\min}$ behavior, we extract a $\mathfrak{g}$-commuting mapping of V_K into the space of some $U(P^*_{\min}, \sigma_0, \nu_0)$ with $\operatorname{Re}\nu_0$ in the closed positive Weyl chamber. Separating out directions in which $\operatorname{Re}\nu_0 = 0$ and building a bigger M subgroup from them, we obtain a $\mathfrak{g}$-commuting mapping of V_K into the space of some $U(P^*, \sigma, \nu)$ with $\operatorname{Re}\nu$ in the open positive Weyl chamber and with σ irreducible tempered, provided the leading term that we use is suitably extremal.

2. Irreducible unitary representations of $SU(2,2)$. In view of the results mentioned in §1, the problem of classifying the irreducible unitary representations of $SU(2,2)$ comes down to deciding which Langlands quotients $J(P, \sigma, \nu)$ are infinitesimally unitary. This decision is simplified by the following general facts:

(1) If $J(P, \sigma, \nu)$ is infinitesimally unitary, then *a fortiori* $J(P, \sigma, \nu)$ has a nonzero invariant Hermitian form. The existence of such a form is equivalent with the condition that $J(P, \sigma, \nu)$ be infinitesimally equivalent with its complex contragredient, which can be realized as $J(P^*, \sigma, -\bar{\nu})$. This equivalence forces a conjugacy of (P, σ, ν) with $(P^*, \sigma, -\bar{\nu})$, and we conclude that if $J(P, \sigma, \nu)$ is infinitesimally unitary, then there exists w in the normalizer of A in K with $wPw^{-1} = P^*$, $w\sigma \cong \sigma$ and $w\nu = -\bar{\nu}$.

(2) If $J(P, \sigma, \nu)$ has a nonzero invariant Hermitian form, then the form is unique up to a scalar and is given by a simple modification of $(A(P^* : P : \sigma : \nu)f, g)$. The question of unitarity is then whether this specific form is semidefinite.

(3) There are some standard techniques to make the decision in many cases whether the form in (2) is semidefinite. We list two of them.

(a) Continuity: If the operator appearing in the form is definite for one value of ν (e.g., if it is the identity), then it must remain definite on any connected set where the symmetry conditions of (1) hold as long as the operator remains invertible on the K-finite vectors. And the operator is invertible until some reducibility occurs for $U(P, \sigma, \nu)$.

(b) Boundedness of matrix coefficients: The K-finite matrix coefficients are unaffected by changing the inner product for an admissible representation, and they must be bounded for a unitary representation. Hence the K-finite matrix coefficients of $J(P, \sigma, \nu)$ are bounded if $J(P, \sigma, \nu)$ is infinitesimally unitary. The limit relation (1.5), rewritten with P in place of $P_{\min}$, enables us to conclude that $J(P, \sigma, \nu)$ cannot be infinitesimally unitary if $\operatorname{Re}\nu$ lies outside a certain bounded region.

Now let us return to $G = SU(2,2)$. We defined a circle subgroup T in (1.2), and we let γ be the diagonal matrix

$$\gamma = \operatorname{diag}(1, -1, 1, -1).$$

Then

$$M_{\min} = T \oplus \{1, \gamma\}, \tag{2.1}$$

and the dual group is correspondingly parametrized by

$$\hat{M}_{\min} \leftrightarrow \{(n, \pm)\}. \tag{2.2}$$

There are four closed subgroups P containing $P_{\min}$, and we shall discuss unitarity of some of the representations attached to two of them. (For a full discussion of unitarity in all cases, see [**8**, Main Theorem and §5].)

For one of them, $P = MAN$ has dimension 10 with

$$M = SL(2, \mathbf{R}) \oplus T,$$

with A one-dimensional, and with N equal to a Heisenberg group of dimension 5. We discuss only the representations σ of M given by

$$\sigma \leftrightarrow \left(D_k^{\pm}, n\right),$$

where $D_k^{\pm}$ is a discrete series ($k \geqslant 2$) or limit of discrete series ($k = 1$) of $SL(2, \mathbf{R})$ and where n refers to a character of the circle group T. Here $\dim A = 1$, and general fact (1) at the start of this section says we may take ν to be real-valued (and equal to a positive multiple of f_1). The unitary points among such ν are given by $\nu = cf_1$ for the following values of c:

$$\begin{array}{ll} 0 < c \leqslant 1 & \text{if } k \equiv n \bmod 2, \\ 0 < c \leqslant 2 & \text{if } |n| = k - 1, \\ \text{no } c & \text{in the remaining cases.} \end{array}$$

The Langlands quotients that occur at the endpoints are of special interest. For $c = 2$ and $|n| = k - 1$, $J(P, \sigma, \nu)$ is a ladder representation, and all ladder representations (except the trivial representation) are of this form. The usual proofs (e.g., [4]) that the ladder representations are unitary use complex variable theory and/or the Fourier transform. Here we obtain the unitarity as a consequence of general fact (3a) after seeing that the induced representation at $\nu = 0$ is irreducible.

For $c = 1$ and $k \equiv n \bmod 2$, some of the representations $J(P, \sigma, \nu)$ are highest weight representations (but not of ladder type), and others are not.

Now let us consider $P_{\min}$. The group $M_{\min}$ and its characters are given in (2.1) and (2.2). Let us suppose $\nu = cf_1 + df_2$ is real with $c > d > 0$. For the character $\sigma \leftrightarrow (0, +)$, the unitary points are $(c, d) = (3, 1)$, where the trivial representation of G occurs, and the points with $c \leqslant 1$; those with $c = 1$ arise from degenerate complementary series induced from the 10-dimensional MAN with the trivial representation on M. For the character $\sigma \leftrightarrow (0, -)$, the unitary points are the ones with $c + d \leqslant 2$; those with $c + d = 2$ arise from degenerate complementary series induced from the 11-dimensional MAN with the signum character on the two-component group M.

For the character $\sigma \leftrightarrow (2n, +)$ with $n \neq 0$, the unitary points are the points with $c \leqslant 1$; those with $c = 1$ arise from degenerate complementary series induced from the 10-dimensional MAN with a unitary character on M. In the limiting case $d = 1$ along the line $c = 1$, the Langlands quotient is of the type constructed explicitly by Strichartz [**13**] for the analysis of the discrete spectrum of $L^2(SO(4,2)/SO(3,2))$.

Note added in proof. Since this paper was written, some of the material in [**3**] and [**11**] has been published as [**16**].

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