The Existence of Complementary Series

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1. Introduction

Let G be a semisimple Lie group. The principal series for G consists of unitary representations induced from finite-dimensional unitary representations of a certain subgroup of G. These representations are not all mutually inequivalent, and their study begins with a study of the operators that give the various equivalences—the so-called intertwining operators.

For $G = SL(2, R)$, these operators are classical transformations. The principal series can be viewed conveniently as representations on L^2 of the line or L^2 of the circle. In the first case, the operators are given formally by scalar multiples of

$$f(x) \to \int_{-\infty}^{\infty} f(x-y)|y|^{-1+it}\,dy \tag{1.1a}$$

and

$$f(x) \to \int_{-\infty}^{\infty} f(x-y)(\operatorname{sign} y)|y|^{-1+it}\,dy. \tag{1.1b}$$

The operator (1.1a) is fractional integration of the imaginary order it and is also known as a Riesz potential operator of imaginary order; for $t = 0$, the operator (1.1b) is the Hilbert transform. If the principal series instead is viewed on the circle, the operators are less familiar analogs of these, given formally in the case of (1.1a) by

$$f(\theta) \to \int_0^{2\pi} f(\theta-\varphi)(1-\cos\varphi)^{-(1-it)/2}\,d\varphi. \tag{1.2}$$

In [3] the authors investigated the operators that generalize (1.1) to an arbitrary group G of real-rank one in order to determine which representations of the principal series are irreducible. The idea was roughly that

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reducibility occurs exactly when the operator generalizing (1.1) can be interpreted as a bounded operator given as a principal-value integral.

Here we shall study these operators for the same groups G for a different purpose. We wish to determine what unitary representations of G can be obtained by inducing from *nonunitary* finite-dimensional representations of the special subgroup. In other words, we ask what the representations are of the complementary series of G.

We shall treat the problem of existence of complementary series by considering analytic continuations of the operators generalizing (1.1) and (1.2). The essential question will be to determine which of the continued operators are positive-definite in a suitable sense. [In (1.1) the operators (1.1a) are positive-definite when it is replaced by a *real* number between 0 and 1, and the operators (1.1b), with it replaced by a complex parameter, are never positive-definite.] The ideas used in answering this question will be given in Section 2, and a more precise exposition will follow in the later sections. Most of the arguments will involve operators $A(z)$ generalizing (1.2), rather than (1.1), but at one point indicated in Section 3 we shall pass to the operators generalizing (1.1). This passage back and forth between integration on a compact group and integration on a noncompact group appears to play an important role in our work.

Our results are special, in that we work only with semisimple groups of real rank 1. Among other results concerning existence of complementary series in special situations are those of Kostant [4] (for general G but only for "class 1" induced representations) and Kunze [5] (for complex semisimple groups G).

The sections of the paper are arranged as follows. The notation and motivation are in Section 2, the precise definition of complementary series and the main theorem (Theorem 3.3) are in Section 3, and a discussion of the applicability of the main theorem is in Section 4. Since it is our intention to present here only the main ideas, we defer most proofs until another time.

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2. Notation and heuristics

In what follows, G will denote a connected semisimple Lie group with finite center. Let $G = ANK$ be an Iwasawa decomposition of G, let θ be the Cartan involution of G corresponding to K, let M be the centralizer of A in K, let M' be the normalizer of A in K, let ρ be half the sum of the positive restricted roots, and let $\bar{N} = \theta N$. Then MAN is a closed subgroup whose finite-dimensional irreducible unitary representations are all of the form $man \to \lambda(a)\sigma(m)$, where λ is a unitary character of A and σ is an irreducible representation of M. The *principal series* of unitary representations of G is parametrized by (σ, λ) and is obtained by inducing these representations of MAN to G.

These representations may be viewed as operating on a space of functions on K by restriction. That is, let σ operate on the finite-dimensional space V_σ and let H^σ be the subspace of members f of $L^2(K) \otimes V_\sigma$ such that, for each m in M,

$$f(mk) = \sigma(m)f(k)$$

for almost all k in K. Define operators on H^σ by

$$(U^{\sigma,\lambda}(x)f)(k) = e^{\rho H(kx)}\lambda(\exp H(kx))f(\kappa(kx)) \qquad x \in G \tag{2.1}$$

where the notation on the right refers to the Iwasawa decomposition $kx = \exp H(kx)\cdot n \cdot \kappa(kx)$. The representation $U^{\sigma,\lambda}$ is unitarily equivalent with the member of the principal series corresponding to the pair (σ, λ).

The definition (2.1) of a representation in the Hilbert space H^σ also makes sense when λ is a nonunitary character of A. In this case, $U^{\sigma,\lambda}(x)$ is a bounded operator with norm $\leqq \sup_{k \in K} |\lambda(\exp H(kx))|$, but it is not unitary. We call these representations the *nonunitary principal series*. Somewhat imprecisely, the complementary series consists of those representations of the nonunitary principal series that can be made unitary by redefining the inner product. (A precise definition will be given in Section 3.)

Temporarily we shall proceed only formally and see what has to happen for a representation to be in the complementary series. Suppose $\langle \cdot, \cdot \rangle$ is an inner product for which $U^{\sigma,\lambda}$ is unitary. This inner product will be given by an operator, possibly unbounded, say

$$\langle f, g \rangle = (Lf, g).$$

Here $(\cdot,\cdot)$ is the usual inner product on $L^2(K)$ given by integration. The condition that $\langle U^{\sigma,\lambda}(x)f, U^{\sigma,\lambda}(x)g\rangle = \langle f, g\rangle$ means that

$$LU^{\sigma,\lambda}(x) = U^{\sigma,\lambda}(x^{-1})^*L,$$

where the adjoint is defined relative to $(\cdot,\cdot)$. On the other hand, we have the lemma below, which follows from a change of variables.

LEMMA 2.1. $U^{\sigma,\lambda}(x^{-1})^* = U^{\sigma,\bar{\lambda}^{-1}}(x)$.

We conclude that

$$LU^{\sigma,\lambda}(x) = U^{\sigma,\bar{\lambda}^{-1}}(x)L \tag{2.2}$$

with $L \not\equiv 0$. There is a theorem of F. Bruhat [1], in the case that λ is unitary, that most $U^{\sigma,\lambda}$ are irreducible and that $U^{\sigma,\lambda}$ and $U^{\tau,\mu}$ are equivalent if and only if there is some member m' of M' such that $\sigma^{m'}$, defined by $\sigma^{m'}(m) = \sigma(m'mm'^{-1})$, is equivalent with τ and such that $\lambda^{m'} = \mu$. If we assume (slightly inaccurately) that these facts persist for nonunitary λ, then we expect that, for most λ, if (2.2) holds, then L is unique up to a scalar and there exists some m' in M' with $\sigma^{m'}$ equivalent with σ and $\lambda^{m'}$ equal to $\bar{\lambda}^{-1}$.

From now on, assume that $\dim A = 1$ (the *real-rank one* case). Then M'/M has order 2. Fix an m' in M' but not in M, and introduce a complex parameter z by the definition

$$\lambda(a) = e^{z\rho H(a)}.$$

If λ corresponds to z, then $\bar{\lambda}^{-1}$ corresponds to $-\bar{z}$ and $\lambda^{m'}$ corresponds to $-z$. From what we have just said, there are only two possibilities:

(i) $z = -\bar{z}$. That is, z is imaginary and λ is unitary; hence $U^{\sigma,\lambda}$ is in the principal series.

(ii) $\sigma^{m'}$ is equivalent with σ, and $-z = -\bar{z}$. That is, it is possible to define $\sigma(m')$, and z is real.

Thus we are looking for an operator L such that $LU^{\sigma,z} = U^{\sigma,-z}L$, and we expect it to be unique up to a scalar for most z. Such an operator was obtained by Kunze and Stein [6] for $\operatorname{Re} z > 0$. It is

$$A(z)f(k_0) = \int_K e^{(1-z)\rho \log a(km')}\sigma(m')\sigma^{-1}(m(km'))f(kk_0)\,dk, \tag{2.3}$$

where the notation on the right refers to the decomposition of G into $MAN\bar{N}$, namely

$$km' = m(km')\cdot a(km')\cdot n\cdot \bar{n};$$

this decomposition exists uniquely for all k not in M.

In short, for (σ, z) to give a representation of the complementary series, we expect that σ must be equivalent with $\sigma^{m'}$ and that z must be real. In this case if $z > 0$, the inner product should be a multiple of $(A(z)f, g)$. That is, a multiple of $A(z)$ must be a positive Hermitian operator. $A(z)$ is always Hermitian for z real, and it is positive if and only if its kernel is a positive-definite function.

For $z > 1$, we can settle the question of positivity immediately. For such a value of z, the kernel vanishes at the identity and is continuous and bounded on K; since it is not identically 0, it is not positive-definite. Thus there should be no complementary series for $z > 1$.

Our approach to the question of positivity when $0 < z < 1$ involves complex methods. To begin with, $z \to A(z)f$ is analytic for $\operatorname{Re} z > 0$, and, if f is smooth, we show that this function of z extends to be meromorphic in the whole plane. Denoting the new operators, defined for $\operatorname{Re} z \leqq 0$, by $A(z)$ also, we shall see that $z \to A(-z)A(z)f$ is meromorphic in the whole plane. For z purely imaginary and not 0, $A(-z)A(z)$ is an intertwining operator for the unitary representation $U^{\sigma,z}$, which is irreducible by Bruhat's theorem. Thus $A(-z)A(z)f = c(z)f$ with $c(z)$ scalar for z imaginary. If we introduce a suitable normalization $B(z) = \gamma(z)^{-1}A(z)$, we shall obtain $B(-z)B(z)f = f$ for imaginary z, hence for all z by analytic continuation. Suppose $B(0)$ is the identity. Then for $B(z_0)$ to fail to be positive-definite for some positive z_0, the equality $B(-z)B(z)f = f$ says that either $B(z)$ or $B(-z)$ must have a singularity for some z with $0 < z < z_0$. Thus an investigation of the singularities of $B(z)$ will be the key to the whole problem of the existence of complementary series associated with σ.

3. The existence theorem

We continue to assume that G has real-rank one. The representations $U^{\sigma,z}(x)$ of the nonunitary principal series, which was defined in Section 2, are parametrized by the finite-dimensional irreducible unitary representations σ of M and by the complex number z, which corresponds to the character $a \to e^{z\rho H(a)}$ of A.

The space H^σ on which $U^{\sigma,z}$ operates is a subspace of $L^2(K) \otimes V_\sigma$ that depends on σ but not on z, and the action of K, by right translation, is independent of z. The space of C^∞ vectors for $U^{\sigma,z}$ is the subspace of C^∞ functions in H^σ; thus, it too is independent of z. We denote this subspace by $C^\infty(\sigma)$.

We shall say that $U^{\sigma,z}$ is a member of the *complementary series* if there exists a positive-definite continuous inner product $\langle \cdot,\cdot \rangle$ on $C^\infty(\sigma) \times C^\infty(\sigma)$ such that

$$\langle U^{\sigma,z}(x)f, U^{\sigma,z}(x)g \rangle = \langle f, g \rangle \tag{3.1}$$

for all x in G and all f and g in $C^\infty(\sigma)$. If there is a nontrivial positive-semidefinite continuous inner product on $C^\infty(\sigma) \times C^\infty(\sigma)$ such that (3.1) holds, we shall say that $U^{\sigma,z}$ is a member of the *quasi-complementary series*.

In either case, the continuity of the inner product, equation (3.1) for x in K, and the Schwartz Kernel Theorem together imply the existence of a continuous operator L mapping $C^\infty(\sigma)$ into itself such that

$$\langle f, g \rangle = (Lf, g)$$

for f and g in $C^\infty(\sigma)$. Here $(\cdot,\cdot)$ denotes the usual inner product on $L^2(K) \otimes V_\sigma$.

As in Section 2, equation (3.1) translates into the fact that L intertwines $U^{\sigma,z}$ and its contragredient. Applying Lemma 2.1, we see that

$$LU^{\sigma,z}(x) = U^{\sigma,-\bar{z}}(x)L \tag{3.2}$$

for all x in G.

Recall that m' is a fixed member of M' that is not in M. If σ is equivalent with $\sigma^{m'}$, then it is possible to extend σ to a representation of all of M' on V_σ; that is, we can regard $\sigma(m')$ as defined. In this case, we define the operator $A(z)$ for $\operatorname{Re} z > 0$ by equation (2.3).

The operator $A(z)$ (actually a slight variant of it) was considered in [6]. It was shown that the kernel

$$e^{(1-z)\rho \log a(km')}\sigma(m')\sigma^{-1}(m(km')) \tag{3.3}$$

is an integrable function of k for $\operatorname{Re} z > 0$ and hence that $A(z)$ is a bounded operator on H^σ. Moreover, $A(z)$ satisfies

$$A(z)U^{\sigma,z}(x) = U^{\sigma,-z}(x)A(z). \tag{3.4}$$

(For $\operatorname{Re} z < 0$, the expression (3.3) is not an integrable function, and we consequently shall not deal directly with this case. In any event, one expects that $U^{\sigma,z}$ is in the complementary series if and only if $U^{\sigma,-z}$ is and that, in this case, $U^{\sigma,z}$ and $U^{\sigma,-z}$ lead to the same unitary representation.)

LEMMA 3.1. *Fix σ and z, and suppose* $\operatorname{Re} z > 0$. *Unless σ is equivalent with $\sigma^{m'}$ and z is real, the only continuous linear operator L on $C^\infty(\sigma)$ satisfying* (3.2) *is* 0. *If σ is equivalent with $\sigma^{m'}$ and z is real, then the continuous operators on $C^\infty(\sigma)$ satisfying* (3.2) *are exactly the scalar multiples of* $A(z)$. *$A(z)$ is bounded and Hermitian.*

Before we pass to a study of the analyticity of $A(z)$, let us observe that the $A(z)$ have a common finite-dimensional resolution. Specifically, let H_D^σ be the subspace of H^σ of functions that transform under K according to the equivalence class D of irreducible representations of K. H_D^σ is finite-dimensional since $H^\sigma \subseteq L^2(K) \otimes V_\sigma$, and it is independent of z. Then each $A(z)$ maps each H_D^σ into itself, by equation (3.4) for x in K.

If f is in $C^\infty(\sigma)$, the mapping $z \to A(z)f$ is an analytic mapping of $\{\operatorname{Re} z > 0\}$ into $C^\infty(\sigma)$. We shall be concerned with extending this mapping to a meromorphic function defined in the whole complex plane. It will be enough to consider the simpler function $z \to A(z)f(1)$, where 1 is the identity of K, provided we prove joint continuity of this function in z and f. Since the singularities of the kernel (3.3) occur only for k in M, we can suppose that f is supported near M, particularly away from $M' - M$.

This turns out to mean that we can transform the whole problem to a problem about the simply connected nilpotent group $\bar{N}$. In fact, using the change-of-variables formula of [2, p. 287], we find that

$$(3.5)\quad A(z)f(1) = \int_{\bar{N}} e^{(1-z)\rho \log a(ym')}\sigma(m')\sigma^{-1}(m(ym'))\{e^{(1+z)\rho H(y)}f(\kappa(y))\}\,dy.$$

The notation here is the same as in formulas (2.1) and (2.3). The ingredients of this formula are technically much simpler than those of formula (2.3), and we consider them one at a time.

First we make some comments about $\bar{N}$. The restricted roots of the Lie algebra $\mathfrak{g}$ of G are either $2\alpha, \alpha, 0, -\alpha, -2\alpha$ or $\alpha, 0, -\alpha$. In this notation, $\bar{N} = \exp(\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha})$. Let $p = \dim \mathfrak{g}_{-\alpha}$ and $q = \dim \mathfrak{g}_{-2\alpha}$. The group A acts on $\bar{N}$ by conjugation; geometrically this action looks like dilations, except that the $\mathfrak{g}_{-2\alpha}$ directions are dilated twice as fast as the $\mathfrak{g}_{-\alpha}$ directions.

Now consider the first factor in the integrand of (3.5). Although it is not necessary to do so for the present problem, one can compute this factor explicitly. If $y = \exp(X + Y)$ with $X \in \mathfrak{g}_{-\alpha}$ and $Y \in \mathfrak{g}_{-2\alpha}$, then[3]

$$(3.6)\qquad e^{(1-z)\rho \log a(ym')} = (\tfrac{1}{4}c^2\|X\|^4 + 2c\|Y\|^2)^{-(p+2q)(1-z)/4},$$

where the norm is that induced by the Killing form of $\mathfrak{g}$ and where $c = (2p + 8q)^{-1}$. Put $|y| = e^{-\rho \log a(ym')}$. Then the function (3.6) has an important property of homogeneity relative to A: if b is in A, then

$$|byb^{-1}| = e^{-2\rho H(b)}|y|.$$

Next we consider $\sigma(m')\sigma^{-1}(m(ym'))$, which we shall denote $\sigma(y)$. This is a matrix-valued function defined everywhere but at the identity and satisfying the homogeneity property $\sigma(byb^{-1}) = \sigma(y)$ for all b in A.

Finally we consider the factor $e^{(1+z)\rho H(y)}f(\kappa(y))$. This function is a smooth function of compact support in $\bar{N}$ because f is assumed to be supported away from $M' - M$. The function depends on the complex parameter z but is entire in the variable z since $\rho H(y)$ has no singularities.

To see that (3.5) extends to be meromorphic in the whole z-plane, we choose a continuous function $\varphi(r)$ of compact support on $[0, \infty)$ so that $\varphi(|y|)f(\kappa(y)) = f(\kappa(y))$, we expand $e^{(1+z)\rho H(y)}f(\kappa(y))$ about $y =$ identity in a finite Taylor series with remainder term, we collect the polynomial terms of the same homogeneity relative to A, we multiply both sides of the expansion by $\varphi(|y|)$, and we substitute into (3.5). The terms of the expansion can be computed well enough to conclude the following: each term but the remainder has a meromorphic extension with at most one pole, that one simple and occurring at an integral multiple of $z = -(p + 2q)^{-1}$, and the

[3] That this explicit formula holds might be guessed from an earlier formula that S. Helgason had derived for $\exp\{-2\rho H(y)\}$.

remainder term gives a contribution analytic in a large right-half plane. Collecting these results, we have the following theorem:

THEOREM 3.2. *Let f be in $C^\infty(\sigma)$. As a mapping into $C^\infty(\sigma)$, the function $z \to A(z)f$ has a meromorphic extension to the whole complex plane with singularities only at the non-negative integral multiples of $-(p + 2q)^{-1}$. The singularities at these points are at most simple poles. The poles can occur only at integral multiples of $-2(p + 2q)^{-1}$ if $\sigma(y) = \sigma(m')\sigma^{-1}(m(ym'))$ satisfies*

$$\sigma(\exp(-X + Y)) = \sigma(\exp(X + Y)) \tag{3.7}$$

for $X \in \mathfrak{g}_{-\alpha}$ and $Y \in \mathfrak{g}_{-2\alpha}$. Moreover, the mapping $(z, f) \to A(z)f$ for z in the regular set and f in $C^\infty(\sigma)$ is a continuous mapping to $C^\infty(\sigma)$.

REMARKS. Condition (3.7) holds for all σ for the Lorentz groups $SO_e(n, 1)$, the Hermitian Lorentz groups $SU(n, 1)$, and the symplectic Lorentz groups $Sp(n, 1)$, but it fails for the spin groups $\mathrm{Spin}(n, 1)$. In any case, the parameter z is normalized so that $z = 1$ corresponds to ρ; therefore, $z = 2(p + 2q)^{-1}$ corresponds to the restricted root α. The result for $SO_e(n, 1)$ that the only poles of $A(z)f$ are simple and are at multiples of $-\alpha$ was obtained by Schiffmann [7].

Using Theorem 3.2, we can now define $A(z)$ for all z. To proceed further, however, we need more information about σ. It is possible to show, under the additional assumptions on G that G is simple and has a faithful matrix representation, that *some representation D of K, when restricted to M, contains σ exactly once.*[4] By the reciprocity theorem, this means that K acts irreducibly on some $H_D^\sigma \neq 0$. Fix such a $D = D_0$, and let $v(k)$ be a nonzero member of $H_{D_0}^\sigma$. Since $A(z)$ commutes with K, we obtain

$$A(z)v = \gamma(z)v \tag{3.8}$$

for a complex-valued meromorphic function $\gamma(z)$. Define

$$B(z) = \gamma(z)^{-1}A(z).$$

As we shall see in Section 4, there is no complementary series associated with σ near $z = 0$ unless the unitary representation $U^{\sigma,0}$ is irreducible. [And for $G = \mathrm{Spin}(n, 1)$ or $SU(n, 1)$ there is no complementary series for any z unless this condition is satisfied.] We therefore assume now that $U^{\sigma,0}$ is irreducible. A necessary and sufficient condition for this irreducibility is given as Theorem 3 of [3]. The condition implies that $\gamma(z)$ does have a pole at $z = 0$, which implies that the operators $B(z)$ are uniformly bounded on compact subsets of $0 \leqq \mathrm{Re}\, z \leqq c$ if c is sufficiently small. The irreducibility and equation (3.4) then imply that $B(0) = I$.

[4] Independently J. Lepowsky has obtained this result and a generalization in his thesis at the Massachusetts Institute of Technology.

The definition of $\gamma(z)$ is arranged so that $B(-z)B(z)f = f$ for all f in $C^\infty(\sigma)$ and for all z. In fact, Theorem 3.3 implies that $B(-z)B(z)f$ is meromorphic in the whole plane. But $B(-z)B(z)$ for purely imaginary $z \neq 0$ intertwines $U^{\sigma,z}$ with itself. By Bruhat's irreducibility theorem in [1], $B(-z)B(z) = c(z)I$ with $c(z)$ scalar for z imaginary. Applying both sides to v, we see that $c(z) = 1$ for z imaginary. That is, $B(-z)B(z)f = f$ for z imaginary. By analytic continuation, $B(-z)B(z)f = f$ for all z.

$B(z)$ preserves each H_D^σ, and $B(0) = I$. Fix D, and suppose $B(z_0)|_D$ is not positive-definite for some $z_0 > 0$. Then either $B(z)$ has a pole nearer 0, or some $B(z)f$ has a 0, in which case $B(-z)$ has a pole, because $B(-z)B(z) = I$. The poles of $B(z)$ (and similarly for $B(-z)$) arise when $A(z)f$ has, for some f, a pole of higher order (possibly negative) than does $\gamma(z)$. These are the ideas behind the main theorem:

THEOREM 3.3. *Suppose that G is simple, that G has a faithful matrix representation, and that* $\dim A = 1$. *Let σ be an irreducible finite-dimensional unitary representation of M satisfying the necessary conditions above, namely, that*

(i) *σ is equivalent with $\sigma^{m'}$, where m' is a member of M that is not in M, and*
(ii) *the unitary representation $U^{\sigma,0}$ is irreducible.*

Define $A(z)$ by (2.3) *and $\gamma(z)$ by* (3.8). *Let z_0 be the least number $\geqq 0$ such that, for some $f \in C^\infty(\sigma)$, $z \to A(z)f$ has a pole at $-z_0$ and $\gamma(z)$ does not or such that γ has a zero at z_0 or at $-z_0$. Then $z_0 > 0$ and the parameters (σ, z) give rise to representations of the complementary series for* $0 < z < z_0$ *with inner product*

$$\langle f, g\rangle = \gamma(z)^{-1} \int_K (A(z)f, g)_{V_\sigma}\, dk$$

for f and g in $C^\infty(\sigma)$.

It is a simple matter to see also that the parameters (σ, z_0) give rise to a representation of the quasi-complementary series. It can happen that this representation is the trivial representation of G.

We should emphasize why the number z_0 in the theorem is strictly positive. The set whose least member is z_0 consists at most of the positive nonzero integral multiples of $(p + 2q)^{-1}$ and the non-negative values z such that one of the meromorphic functions $\gamma(z)$ and $\gamma(-z)$ vanishes. This set is discrete, and it does not contain $z = 0$ because $\gamma(z)$ has a pole at $z = 0$.

4. Further investigation of the singularities of $B(z)$

For the case that σ is the trivial representation of M, we can choose the eigenvector v of the $A(z)$ to be a constant function. The associated function $\gamma(z)$ is closely related to Harish–Chandra's $\mathbf{c}$ function (see [2]), and we can obtain very explicit results as a consequence. For G of general real rank, the $\mathbf{c}$ function can be defined as the analytic continuation in μ to the whole complexified dual of the Lie algebra of A of the function

$$\mathbf{c}(\mu) = \int_{\bar{N}} e^{(i\mu+\rho)H(x)}\,dx.$$

Let us return to the real-rank one case.

PROPOSITION 4.1. *Let σ be the trivial representation, and choose v to be a constant function. Then the function $\gamma(z)$ is given by $\gamma(z) = \mathbf{c}(-iz\rho)$. Therefore*

$$\gamma(z) = 2^{(p+2q)(1-z)/2}\,\frac{\Gamma(\frac{1}{2}(p+q+1))\Gamma(\frac{1}{2}(p+2q)z)}{\Gamma(\frac{1}{4}(p+2q)(1+z))\Gamma(\frac{1}{4}(p+2+(p+2q)z))},$$

where $p = \dim \mathfrak{g}_{-\alpha}$ and $q = \dim \mathfrak{g}_{-2\alpha}$.

PROOF.

$$\begin{aligned}\gamma(z) &= \int_K e^{(1-z)\rho\log a(km')}\,dk = \int_K e^{(1-z)\rho\log a(k)}\,dk\\ &= \int_{K/M} e^{(1-z)\rho\log a(k)}\,dk = \int_{\bar{N}} e^{(1-z)\rho\log a(\kappa(x))}e^{2\rho H(x)}\,dx,\end{aligned}$$

the last equality following from [2] (see p. 287). If $x = ank \in ANK$, then $k = a^{-1}(an^{-1}a^{-1})x \in MAN\bar{N}$. Hence $a(\kappa(x)) = \exp(-H(x))$, and we obtain $\gamma(z) = \int_{\bar{N}} e^{(1+z)\rho H(x)}\,dx$, as required. The formula for $\gamma(z)$ then follows from [2] (see p. 303).

In [4], B. Kostant obtained the existence of complementary series for σ trivial and G of any real rank. For a first application of Proposition 4.1, we shall compare his results in the rank-one case with what we can prove from Theorem 3.3 and Proposition 4.1 when σ is trivial.

Using the explicit value of $\gamma(z)$ in the proposition, we see that $\gamma(z)$ is nonvanishing for $-1 < z < 1$ and that the only poles of $\gamma(z)$ for $-1 < z \leqq 0$ occur at the non-negative integral multiples of $-2(p+2q)^{-1}$. If $q = 0$, there is a pole at every multiple of $-2p^{-1}$ less than 1. If $q > 0$, then p is even and a pole occurs at every multiple satisfying $-(p+2)(p+2q)^{-1} < z \leqq 0$. By Theorem 3.2 the poles of $A(z)f$ occur only at multiples of $-2(p+2q)^{-1}$. Thus by Theorem 3.3 there is a complementary series for

$$0 < z < z_0 = \begin{cases} 1 & \text{if } q = 0 \\ \dfrac{p+2}{p+2q} & \text{if } q > 0. \end{cases} \tag{4.1}$$

Kostant has this estimate (in [4], see Section 3.1 and Theorem 10). Kostant shows further that there is no (positive-definite) complementary series to the right of the point z_0 in this inequality. We can obtain this result by our method if $q = 0$, 1, or 3. But if $q = 7$, we obtain only the weaker result that there is no complementary series immediately to the right of z_0. This weaker result comes from comparing the signs near z_0 of $\gamma(z)$ and $(A(z)f, f)$ for an f such that $A(-z_0)f$ has a pole. (Such an f exists.)

We turn to other applications of Proposition 4.1. When $q = 0$ or 1, we have $z_0 = 1$ in (4.1). If $\exp\{(1 - z_1)\rho \log a(km')\}\sigma(m')\sigma^{-1}(m(km^{-1}))$ is a positive-definite function and if $0 < z < z_1 \leqq 1$, then the product with $\exp[\{1 - (1 - z_1 + z)\}\rho \log a(km')]$ is also positive-definite. We obtain the following corollary:

COROLLARY 4.2. *Let $G = SO_e(n, 1)$, Spin$(n, 1)$, or $SU(n, 1)$. Let σ be an irreducible unitary representation of M such that σ is equivalent with $\sigma^{m'}$. Then the positive z such that $U^{\sigma, z}$ is in the quasi-complementary series form an interval with* 0 *as left endpoint.*

COROLLARY 4.3. *Let $G = SO_e(n, 1)$, Spin$(n, 1)$, or $SU(n, 1)$. Let σ be an irreducible unitary representation of M such that σ is equivalent with $\sigma^{m'}$ and such that $U^{\sigma, 0}$ is reducible. Then there is no positive z such that $U^{\sigma, z}$ is in the quasi-complementary series.*

For the proof of the second corollary, it is possible to use Theorems 1 and 3 of [3] to show from the reducibility of $U^{\sigma, 0}$ that $B(0)$ is unitary and not scalar. But if U^{σ, z_0} is in the quasi-complementary series, $A(z)$ is semidefinite for $0 < z < z_0$, by Corollary 4.2; this fact implies that $B(0)$ is semidefinite, which is a contradiction.

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