DETERMINATION OF INTERTWINING OPERATORS

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The subject is representations of the principal series and complementary series. Let $G=KAN$ be a connected semisimple Lie group of matrices, and let MAN be a minimal parabolic subgroup, where M is the centralizer of A in K. If σ is an irreducible unitary representation of M and λ is a unitary character of A, then

$$U(\sigma, \lambda)= \operatorname*{ind}_{MAN\uparrow G} \ (man \to \lambda(a)\,\sigma(m))$$

is a representation of the *principal series.* The principal series is one of the series contributing to the Plancherel formula and corresponds to a Cartan subgroup as noncompact as possible. The principal series with $\sigma=1$ was investigated by Kostant [**5**], who proved that $U(1, \lambda)$ is irreducible. However, $U(\sigma, \lambda)$ need not be irreducible in general.

It is still possible to define $U(\sigma, \lambda)$ as a nonunitary representation on a Hilbert space when λ is nonunitary. We say $U(\sigma, \lambda)$ is in the *complementary series* if there is an invariant inner product on the C^∞ vectors that is continuous in the C^∞ topology.

We consider two problems: (1) Find the dimension and algebra structure of the commuting ring $C(\sigma, \lambda)$ of $U(\sigma, \lambda)$ when λ is unitary; (2) produce complementary series. At this point we could state our solutions to these problems, but we prefer first to motivate the results by introducing the intertwining operators. The development of these operators was begun by Kunze and Stein [**6**], continued by Schiffmann [**8**] and to an extent by Helgason [**2**], and completed by Knapp and Stein [**3**].

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Let M' be the normalizer of A in K and let $W=M'/M$. If w is in M', we write $[w]$ for the coset wM in W. The group M' operates on λ and σ by conjugation of the A or M variable by w^{-1}. The class of $w\sigma$ depends only on $[w]$. Let

$$W_{\sigma,\lambda}=\{w\in W \mid w\sigma\sim\sigma \text{ and } w\lambda=\lambda\}.$$

Bruhat [1] obtained the results for the principal series that

(1) $\dim C(\sigma,\lambda)\leqq|W_{\sigma,\lambda}|$ (and so $=1$ for almost all λ),

(2) $U(\sigma,\lambda)$ is unitarily equivalent with $U(w\sigma, w\lambda)$.

The intertwining operator that implements (2) is, by (1), unique up to a scalar for almost all λ. By [3] such operators $\mathscr{A}(w,\sigma,\lambda)$ can be chosen so that the following hold:

(i) $U(w\sigma, w\lambda)\,\mathscr{A}(w,\sigma,\lambda)=\mathscr{A}(w,\sigma,\lambda)\,U(\sigma,\lambda)$.

(ii) $\mathscr{A}(w,\sigma,\lambda)$ is unitary and in its action on smooth functions varies real-analytically in λ.

(iii) $\mathscr{A}(w_1w_2,\sigma,\lambda)=\mathscr{A}(w_1, w_2\sigma, w_2\lambda)\,\mathscr{A}(w_2,\sigma,\lambda)$.

(iv) $\mathscr{A}(w, E\sigma E^{-1},\lambda)=E\mathscr{A}(w,\sigma,\lambda)\,E^{-1}$.

(v) If $[w]$ is the reflection relative to a simple restricted root α and if $\mathfrak{g}_\alpha$ is the real-rank-one algebra generated by $\mathfrak{n}_\alpha$ and $\theta\mathfrak{n}_\alpha$, then $\mathscr{A}(w,\sigma,\lambda)$ is essentially $\mathscr{A}_\alpha(w,\sigma|_{M_\alpha},\lambda|_{A_\alpha})$. [Here M_α and A_α denote the M and A subgroups for the group corresponding to $\mathfrak{g}_\alpha$. The representation $\sigma|_{M_\alpha}$ is a multiple of a single irreducible representation of M_α, and consequently there is no difficulty in defining the operator $\mathscr{A}_\alpha$.]

If $w\sigma=\sigma$ and $w\lambda=\lambda$, (i) says $\mathscr{A}(w,\sigma,\lambda)$ is in $C(\sigma,\lambda)$. More generally suppose $w\sigma\sim\sigma$ and $w\lambda=\lambda$. Then it is possible to extend σ to a representation of the group generated by M and w. So $\sigma(w)$ is defined; it is unique up to a root of unity. In this case, (i) and (iv) show that $\sigma(w)\,\mathscr{A}(w,\sigma,\lambda)$ is in $C(\sigma,\lambda)$. This operator depends only on $[w]$ and we may write $\sigma([w])\,\mathscr{A}([w],\sigma,\lambda)$ instead. Then

$$\operatorname{span}\{\sigma(p)\,\mathscr{A}(p,\sigma,\lambda)\mid p\in W_{\sigma,\lambda}\}\subseteq C(\sigma,\lambda).$$

The following unpublished theorem was proved in other notation by Harish-Chandra and translated into this notation by Wallach; it is given here with Harish-Chandra's permission.

THEOREM. $\operatorname{span}\{\sigma(p)\,\mathscr{A}(p,\sigma,\lambda)\mid p\in W_{\sigma,\lambda}\}=C(\sigma,\lambda)$.

In view of the theorem it is of interest to determine a linear basis of the left side of the equality, in particular to determine which operators are scalar. Another reason for wanting this information is given by the next theorem [3].

THEOREM. *Let p be an element of order 2 in $W_{\sigma,1}$. If $\sigma(p)\,\mathscr{A}(p,\sigma,1)$ is scalar,* *then* $U(\sigma, \lambda)$ *is in the complementary series for all* λ *sufficiently close to* 1 *such that* $p\lambda = \bar{\lambda}^{-1}$. *"Sufficiently close" depends on G but not* σ *or* p.

We shall now describe $C(\sigma, \lambda)$. Let Δ be the set of restricted roots and let

$$\Delta' = \{\alpha \in \Delta \mid p_\alpha \in W_{\sigma, \lambda} \text{ and } \sigma(p_\alpha)\, \mathscr{A}(p_\alpha, \sigma, \lambda) = cI\}.$$

From [3] one knows that $\sigma(p_\alpha)\, \mathscr{A}(p_\alpha, \sigma, \lambda)$ is scalar if and only if the real-rank-one Plancherel density satisfies $p_{\sigma|M_\alpha}(\lambda|_{A_\alpha}) = 0$, and so it is an easy matter to determine the members of Δ'. Now Δ' is a root system, and we let $W'_{\sigma, \lambda} \subseteq W_{\sigma, \lambda}$ be its Weyl group. Let

$$R_{\sigma, \lambda} = \{p \in W_{\sigma, \lambda} \mid p\alpha > 0 \text{ for all } \alpha > 0 \text{ in } \Delta'\}.$$

THEOREM. (i) $W_{\sigma, \lambda}$ *is the semidirect product* $W_{\sigma, \lambda} = W'_{\sigma, \lambda} R_{\sigma, \lambda}$ *with* $W'_{\sigma, \lambda}$ *normal. The operators* $\sigma(w)\, \mathscr{A}(w, \sigma, \lambda)$ *are scalar exactly for* w *in* $W'_{\sigma, \lambda}$ *and they are linearly independent for* w *in* $R_{\sigma, \lambda}$. *Consequently, the operators for* $R_{\sigma, \lambda}$ *are a basis for* $C(\sigma, \lambda)$ *and*

$$\dim C(\sigma, \lambda) = |R_{\sigma, \lambda}|.$$

(ii) *For* w *in* W *let* $p_\sigma^w(\lambda)$ *be the product of* $p_{\sigma|M_\alpha}(\lambda|_{A_\alpha})$ *over all* $\alpha > 0$ *in* Δ *such that* $\alpha/2$ *is not in* Δ *and* $w\alpha$ *is* < 0. *Then*

$$\dim C(\sigma, \lambda) = |\{w \in W_{\sigma, \lambda} \mid p_\sigma^w(\lambda) \neq 0\}|.$$

(iii) $R_{\sigma, \lambda} = \sum Z_2$ *with the number of summands* $\leqq \dim A$.

In the theorem, part (i) is elementary and self-proving, and (ii) comes out of the proof of (i). Part (i) shows that the subgroup of $W_{\sigma, \lambda}$ corresponding to trivial operators is a Weyl group; consequently the elements p in the theorem about complementary series are all given by commuting products of reflections relative to Δ' and are easy to determine. Part (iii) is the part that is hard to prove, and it is the one that gives insight into the nature of $R_{\sigma, \lambda}$. Despite property (iii) of $\mathscr{A}(w, \sigma, \lambda)$, this result falls short of saying that $C(\sigma, \lambda)$ is commutative, saying only that is commutative modulo $\pm$ signs. It seems possible to analyze this matter further and use the methods of proof of (iii) to prove commutativity of $C(\sigma, \lambda)$, but such a proof has yet to be carried out.[1]

We shall discuss one aspect of the proof of (iii). First, to prove (iii) for $\lambda = \lambda_0$, it suffices to prove (iii) for $\lambda = 1$. Then the basic idea is that $R_{\sigma, 1}$ can be under-

[1] (Footnote added January, 1973.) The proof of the commutativity of $C(\sigma, \lambda)$ has now been carried out. The algebra structure of $C(\sigma, \lambda)$ is as follows: The ambiguous signs of $\sigma(w)$ for w in $R_{\sigma, \lambda}$ can be chosen so that the operators $\sigma(w)\, \mathscr{A}(w, \pi, \lambda)$ for w in $R_{\sigma, \lambda}$ form both a group isomorphic to $\sum Z_2$ and a linear basis of $C(\sigma, \lambda)$.

stood provided σ is moved to some "standard position." By such a device the proof for general G is reduced to the case that G is split over $\boldsymbol{R}$, and then this case is considered separately. To simplify the exposition we shall not deal with general G here but will content ourselves with two cases.

Case 1. We assume that, for each simple α in Δ, $\mathfrak{g}_\alpha$ is not isomorphic with $sl(2, \boldsymbol{R})$. This condition implies that M is connected. It is satisfied, for example, if G is complex semisimple or if G is simple and twice some restricted root is again a restricted root. For complex G, the whole principal series is irreducible, by [7] and [11], and there is a corresponding simple computation that one can do to show, without the irreducibility theorem, that all the $\sigma(w)\mathscr{A}(w, \sigma, \lambda)$ are scalar. The idea in Case 1 will be to imbed into G as much of the complex case as possible to show that most of the operators are scalar.

Let $\mathfrak{h} \subseteq \mathfrak{m}$ be a maximal abelian subspace, so that $\mathfrak{a}+\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{m}$. The roots of $(\mathfrak{g}^C, (\mathfrak{a}+\mathfrak{h})^C)$ are real on $\mathfrak{a}$ and imaginary on $\mathfrak{h}$. The restricted roots are the restrictions to $\mathfrak{a}$ of the roots, and the roots of $(\mathfrak{m}^C, \mathfrak{h}^C)$ are the restrictions to $\mathfrak{h}^C$ of the roots that vanish on $\mathfrak{a}$. We may assume that the ordering on the roots is chosen so that the $\mathfrak{a}$ part is more significant than the $\mathfrak{h}$ part.

We say that α in Δ is *essential* if neither α nor 2α is a root when extended to be 0 on $\mathfrak{h}$. Otherwise α is *inessential*. The name refers to the possibilities for how the Weyl group reflection p_α can be extended to $\mathfrak{h}'$; essential restricted roots cannot act trivially on $\mathfrak{h}'$. Specifically, if α is inessential, there is w in M' so that $[w]=p_\alpha$ and $\mathrm{Ad}(w)=1$ on $i\mathfrak{h}'$. If α is essential and $\alpha \pm \beta$ are roots (with β in $i\mathfrak{h}'$), there is w in M' so that $[w]=p_\alpha$ and $\mathrm{Ad}(w)=p_\beta$ on $i\mathfrak{h}'$.

In the complex case every restricted root is essential. Quite generally the idea is that essential restricted roots lead to trivial intertwining operators, and we intend to discard these by imbedding into $i\mathfrak{h}'$ the part of $\mathfrak{a}'$ corresponding to the essential restricted roots. Let

$$\pi_e = \{\text{essential simple restricted roots}\},$$

$$\mathfrak{a}'_e = \text{span of } \pi_e \text{ in } \mathfrak{a}',$$

$$W_e = \text{subgroup of } W \text{ generated by the } p_\alpha \text{ for } \alpha \text{ in } \pi_e.$$

IMBEDDING LEMMA. *It is possible to choose β in $i\mathfrak{h}'$ corresponding to each α in π_e so that $\alpha+\beta$ is a root, so that p_β preserves the set of positive roots of $\mathfrak{m}$, and so that the linear extension of the mapping given by $\alpha \to J(\alpha)=\beta$ is an isometry of $\mathfrak{a}'_e$ into $i\mathfrak{h}'$.*

Fix J as in the lemma and let W_π be the set of simple reflections in W. J defines a map of W_π into the orthogonal group $O(i\mathfrak{h}')$ as follows: If α is in π_e, map p_α into $p_{J\alpha}$. If α is in $\pi - \pi_e$, map p_α into the identity.

THEOREM. *The mapping of W_π into $O(i\mathfrak{h}')$ defined by J extends to a group homomorphism of W into $O(i\mathfrak{h}')$. The resulting action of W on $i\mathfrak{h}'$ has the properties that*

(a) *for w in W_e, $Jw = wJ$ on $\mathfrak{a}_e'$,*

(b) *for w in W, if γ is a positive root of $\mathfrak{m}$, so is $w\gamma$,*

(c) *for w in W, if σ has highest weight Λ, then $w\sigma$ has highest weight $w\Lambda$ (and so $\sigma \sim w\sigma$ if and only if $\Lambda = w\Lambda$).*

The map J is not unique, but (c) in the theorem shows that the action of W on $i\mathfrak{h}'$ is canonical.

We say Λ in $i\mathfrak{h}'$ is *dominant* if $\langle \Lambda, J\alpha\rangle \geqq 0$ for all α in π_e. It is a simple exercise with the theorem to show that σ is conjugate under W to a representation of M whose highest weight is dominant. Now if σ is replaced by $p\sigma$ for some p in M', the whole situation for σ, intertwining operators and all, is conjugated to the situation for $p\sigma$. Thus it is enough to prove that $R_{\sigma,1} = \sum Z_2$ under the assumption that the highest weight Λ of σ is dominant. With this observation, we proceed as follows: Let

$$S = \{w \in W \mid w = 1 \text{ on } i\mathfrak{h}'\}.$$

Then S is normal in W and $W = W_e S$, as a semidirect product. Moreover, $W_{\sigma,1} = (W_{\sigma,1} \cap W_e)\, S$. If Λ is dominant, $W_{\sigma,1} \cap W_e$ is generated by the simple reflections that it contains. [In fact, an easy induction reduces this statement to showing that if w is in $W_{\sigma,1} \cap W_e$ and $w\alpha < 0$ for some α in π_e, then $p_\alpha \Lambda = \Lambda$. This last equality follows from the chain $0 \leqq \langle \Lambda, \alpha\rangle = \langle w^{-1}\Lambda, \alpha\rangle = \langle \Lambda, w\alpha\rangle \leqq 0$, which uses the dominance of Λ twice.] By means of properties (iii) and (v) of the $\mathscr{A}$ operators, we can therefore reduce the operators for $W_{\sigma,1} \cap W_e$ to operators for a real-rank-one group whose simple restricted root is essential. Such a group is a cover of the Lorentz group $SO(\text{odd}, 1)$, and the whole principal series for such a group is irreducible, by [**3**]. Consequently all the operators corresponding to $W_{\sigma,1} \cap W_e$ are scalar. Thus all the operators for $R_{\sigma,1}$ are already represented by members of S, except for scalar factors. Under our assumption on G, S is $\sum Z_2$, and it follows easily that $R_{\sigma,1} = \sum Z_2$.

An interesting special case occurs when every α in Δ is essential. Wallach has pointed out that this is exactly the case that G has only one conjugacy class of Cartan subgroups. For such a group, $W = W_e$ and so $S = \{1\}$ and $R_{\sigma,1} = \{1\}$. The principal series is therefore irreducible. This result was obtained earlier by Wallach in an unpublished work (cf. [**10**]).

Case 2. We assume that G is simple and is split over $\boldsymbol{R}$. Then $\mathfrak{a}$ is a Cartan subalgebra and M is a finite abelian group. Each $\mathfrak{g}_\alpha$ is isomorphic with $sl(2, \boldsymbol{R})$, and we let γ_α be the image of $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ under the corresponding map of $SL(2, \boldsymbol{R})$ into G. Then γ_α is in M and $\gamma_\alpha^2 = 1$. The γ_ε's for ε simple generate M.

Thus σ is determined by its values on the γ_ε's, and σ assumes only the values ± 1. It is easy to check that

$$\Delta' = \{\beta \in \Delta \mid \sigma(\gamma_\beta) = +1\}.$$

We assume that σ is not identically 1. In this case we say σ is *dominant* if $\sigma(\gamma_\varepsilon) = -1$ for exactly one simple ε, say $\varepsilon = \varepsilon_k$. Every $\sigma \not\equiv 1$ can be conjugated by W so as to be dominant, and we shall assume that σ is dominant from now on.

It is an easy matter to see that the simple roots of Δ' consist of the ε_i (for $i \neq k$) and at most one other root, say α. The root α exists if and only if there is a root $\beta > 0$ such that $\sigma(\gamma_\beta) = +1$ and $\langle \beta, \varepsilon_i \rangle \leqq 0$ for $i \neq k$. In this case α is the least such β. This fact makes it easy to determine Δ' explicitly in examples.

If α does not exist, a short argument shows that $R_{\sigma,1}$ is $\{1\}$ or Z_2. If α does exist, $W'_{\sigma,1}$ is a Weyl group of the same rank as W, and it follows that $R_{\sigma,1}$ must be small. In fact, it need not have 1 or 2 elements, but it is always $\sum Z_2$. Of the two proofs of this statement at present, one is by classification and one is not. The one by classification is shorter.

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