Analytic continuation of nonholomorphic discrete series for classical groups

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In Honor of Jacques Carmona

ABSTRACT The question of unitarity of representations in the analytic continuation of discrete series from a Borel–de Siebenthal chamber is considered for those linear equal-rank classical simple Lie groups G that have not been treated fully before. Groups treated earlier by other authors include those for which G has real rank one or has a symmetric space with an invariant complex structure. Thus the groups in question are locally isomorphic to $SO(2m, n)_0$ with $m \geq 2$ and $n \geq 3$, or to $Sp(m, n)$ with $m \geq 2$ and $n \geq 2$.

The representations under study are obtained from cohomological induction. One starts from a finite-dimensional irreducible representation of a compact subgroup L of G associated to a Borel–de Siebenthal chamber, forms an upside-down generalized Verma module, applies a derived Bernstein functor, and passes to a specific irreducible quotient. Enright, Parthasarthy, Wallach, and Wolf had previously identified all cases where the representation of L is 1-dimensional and the generalized Verma-like module is irreducible; for these cases they proved that unitarity is automatic. B. Gross and Wallach had proved unitarity for additional cases for a restricted class of groups when the representation of L is 1-dimensional.

The present work gives results for all groups and allows higher-dimensional representations of L. In the case of 1-dimensional representations of L, the results address unitarity and nonunitarity and are conveniently summarized in a table that indicates how close the results are to being the best possible. In the case of higher-dimensional representations of L, the method addresses only unitarity and in effect proceeds by reducing matters to what happens for a 1-dimensional representation of L and a lower-dimensional group G.

Introduction

Let G be a connected simple Lie group with finite center, and let K be a maximal compact subgroup. Let $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the respective Lie algebras, and let $\mathfrak{g}$ and $\mathfrak{k}$ be their complexifications. If rank G = rank K, then G has discrete series representations [HC2], and they are parametrized roughly as follows. Fix a compact Cartan subalgebra

2000 Mathematics Subject Classification: Primary 22E46; Secondary 17B10, 20G20.
Key words: discrete series, analytic continuation, unitary representation.

$\mathfrak{b}_0$ inside $\mathfrak{k}_0$, and let $\mathfrak{b}^*$ be the dual of the complexification $\mathfrak{b}$ of $\mathfrak{b}_0$. To each system of positive roots corresponds a set of discrete series representations parametrized by the dominant nonsingular elements of a translate of the set of integral points in $\mathfrak{b}^*$. As the system of positive roots varies, the entire discrete series is obtained, each representation appearing a number of times equal to the order of the Weyl group of K.

In a suitable realization of the representations, once the positive system is fixed, the parameter for the discrete series can be moved a certain amount outside the region of dominance to give new representations. These new representations are obtained by a two-step process, the first step involving the moving of a continuous parameter and the second step involving an operation that makes sense only at a discrete set of parameter values. For this reason the two-step process has been called "analytic continuation of discrete series."

N. Wallach was the person to introduce this term, and in [Wa1] he made the first progress in deciding which of the continued representations had a natural infinitesimally unitary irreducible $(\mathfrak{g}, K)$ module associated to it. He worked with the situation that G/K is Hermitian symmetric and the discrete series are the holomorphic ones constructed by Harish-Chandra in [HC1]. Following the reformulation in [Kn3], we can describe matters this way: Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition corresponding to K. The condition that G/K is Hermitian symmetric is equivalent with the existence of a K stable splitting $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$, and each of $\mathfrak{p}^+$ and $\mathfrak{p}^-$ is an abelian subspace of $\mathfrak{p}$. If Z is an irreducible finite-dimensional representation of $\mathfrak{k}$, then the parameters of Z are rigid in most directions but can be moved continuously in a direction that corresponds to the 1-dimensional center of $\mathfrak{k}$. The two-step construction of the analytically continued representations is to form the upside down generalized Verma module $\mathrm{ind}_{\mathfrak{k}+\mathfrak{p}^-}^{\mathfrak{g}}(Z \otimes \bigwedge^{\mathrm{top}} \mathfrak{p}^+)$ and then to restrict attention to those parameters where this $\mathfrak{g}$ module makes sense as a representation of K. The resulting $(\mathfrak{g}, K)$ module need not be irreducible, but it has a unique irreducible quotient containing the K type $Z \otimes \bigwedge^{\mathrm{top}} \mathfrak{p}^+$. In [Wa1] Wallach exactly determined, under the hypothesis that Z is 1-dimensional, those parameters for which this irreducible quotient is infinitesimally unitary. Enright, Howe, and Wallach in [EHW] and Jakobsen [Ja] independently determined the parameters for which this irreducible quotient is infinitesimally unitary when Z is higher-dimensional. We shall refer to the cases with 1-dimensional Z as the "line-bundle cases" and to the higher-dimensional cases as the "vector-bundle cases."

In [EPWW], Enright, Parthasarathy, Wallach, and Wolf, as part of a study of conditions for cohomological induction to preserve unitarity, made a beginning at addressing the question of unitarity of the analytic continuation of nonholomorphic discrete series. A defining property of the positive system of roots for the holomorphic case is that there is just one noncompact simple root and that it occurs exactly once in the highest root. The authors of [EPWW] examined cases in which there is just one noncompact simple root and it occurs exactly twice in the highest root. According to a theorem of Borel and de Siebenthal [BoS], every G for which $\operatorname{rank} G = \operatorname{rank} K$ but G/K is not Hermitian symmetric has such a system of positive roots; see Theorem 6.96 of [Kn1] for a quick proof. In the notation of [Kn3], the situation may be described as follows. With $\mathfrak{b}$ and the positive system fixed, let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the parabolic subalgebra of $\mathfrak{g}$ such that the semisimple part of the Levi factor $\mathfrak{l}$ is generated by the root spaces for the roots in the span of the compact simple roots and such that the nilpotent radical $\mathfrak{u}$ is generated by the root spaces for the remaining positive roots. Let $\bar{\mathfrak{q}} = \mathfrak{l} \oplus \bar{\mathfrak{u}}$ be the opposite parabolic subalgebra to $\mathfrak{q}$, and let L be the analytic subgroup of G corresponding to the Lie subalgebra $\mathfrak{l}_0 = \mathfrak{g}_0 \cap \mathfrak{l}$ of $\mathfrak{g}_0$; the group L is a compact connected subgroup of K. If Z is an irreducible finite-dimensional representation of $\mathfrak{l}$, then the parameters of Z are again rigid in most directions but can be moved continuously in a direction that corresponds to the 1-dimensional center of $\mathfrak{l}$. The two-step construction of the analytically continued representations is to form the upside-down generalized Verma module $\operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}(Z \otimes \bigwedge^{\mathrm{top}}\mathfrak{u})$ and then to apply the S^{th} derived Bernstein functor Π_S, where $S = \dim(\mathfrak{u} \cap \mathfrak{k})$, when the parameters have the property that this $\mathfrak{g}$ module makes sense as a representation of L; see Section 1 below for more detail. The resulting $(\mathfrak{g}, K)$ module is said to be cohomologically induced. If in addition the highest weight Λ of the L representation $Z \otimes \bigwedge^{\mathrm{top}}(\mathfrak{u} \cap \mathfrak{p})$ is dominant for K, then the K representation with highest weight Λ occurs with multiplicity one in the cohomologically induced representation, and the cohomologically induced representation has a unique irreducible subquotient containing that K type. It is this subquotient whose unitarity we investigate.

When Z is 1-dimensional, we say that the cohomologically induced representation is a "line-bundle case"; otherwise it is a "vector-bundle case." The paper [EPWW] identified, more or less, those line-bundle cases where Λ is dominant for K and the generalized Verma-like module is irreducible for the given value of the central parameter for L and also for all larger values; the authors showed that the cohomologically induced representation is infinitesimally unitary in those cases.

In [GW1] and [GW2], Gross and Wallach dealt with additional line-bundle cases when G/K has a quaternionic structure. In the presence of such a structure, G has $\operatorname{rank} G = \operatorname{rank} K$ and, except when the Dynkin diagram is of type A_l, there is a canonically associated positive system of Borel–de Siebenthal type. The groups in question are locally isomorphic to any of $SO(4, n)_0$ with $n \geq 3$, $Sp(1, n)$, or five exceptional groups. Gross and Wallach were able to prove for a few line-bundle cases beyond those in [EPWW] that the unique irreducible subquotient containing the K type Λ is infinitesimally unitary. They were able to do so despite the complications introduced by reducibility of the generalized Verma-like modules.

In the present paper we work with analytic continuation of discrete series for arbitrary classical groups, starting from any positive system of Borel–de Siebenthal type. We address the unitarity question for line-bundle cases and vector-bundle cases alike, even when the generalized Verma-like modules are reducible. The main restriction is that we assume G to be *linear*; the need for G to be linear seems to be an essential feature of our method. For nearly all of the line-bundle cases where we do not prove unitarity, we prove a certain amount of nonunitarity, in order to give an indication that our result seems to be close to best possible. For the vector-bundle cases, we examine only parameters that the line-bundle case suggests might be unitary, and we offer only examples of nonunitarity results.

Let us be more precise about the groups in question. The group G may as well be taken to be a simple Lie group with a simply connected complexification. Theorems 6.74 and 6.88 of [Kn1] show that the Lie algebra of G is completely determined up to isomorphism by specifying a Dynkin diagram and identifying which simple root is to be the noncompact one. We omit the cases corresponding to G/K Hermitian symmetric and also those corresponding to G of real rank one. The analytic continuation of discrete series has been settled in the first case in [EHW] and in [Ja], and the unitary dual is completely known in the second case (e.g., [Hi], [Ot], [Kr], [Ba], and [BaB]; see [BaK] for a uniform description).

The Lie algebras in the remaining cases are as follows: For a Dynkin diagram of type B_l, neither of the end simple roots is to be the noncompact one; the Lie algebra is $\mathfrak{so}(2m, 2(l-m)+1)$ with $m \geq 2$ and $l-m \geq 1$. For a Dynkin diagram of type C_l, neither of the end simple roots is to be the noncompact one, and nor is the one next to the long simple root; the Lie algebra is $\mathfrak{sp}(m, l-m)$ with $m \geq 2$ and $l-m \geq 2$. For a Dynkin diagram of type D_l, none of the three end simple roots is to be the noncompact one; the Lie algebra is $\mathfrak{so}(2m, 2l-2m)$ with $m \geq 2$ and $l-m \geq 2$.

The proofs of unitarity involve three techniques. The first is the main theorem of [Kn3], which provides an intertwining operator between certain cohomologically induced representations, essentially converting the cohomologically induced representation under study into one that comes from a different parabolic subalgebra $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$ and a 1-dimensional representation of $\mathfrak{l}'$. The second is Vogan's Unitarizability Theorem ([Vo]; see also [Wa2]), which provides certain sufficient conditions for cohomologically induced representations to be unitary. The third is a combinatorial result that we return to in a moment.

Let us elaborate on the use of Vogan's theorem. Vogan actually gave at least three sufficient conditions for unitarity. The general-purpose condition is that the infinitesimal character of the inducing representation is in the "weakly-good" range. This result will be of relatively little use for our current purposes. A more sensitive condition, applicable when the inducing representation is 1-dimensional, is that the parameter is in the "weakly-fair" range. This result is easy to apply and, when applied to $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$, can handle all the cases that arise from line-bundle cases at the start of our construction. But it does so in an unnatural way and gives incomplete results when we have a vector-bundle case at the start. The third sufficient condition, applicable when the inducing representation is 1-dimensional, is that a certain equation in the dual of the Cartan subalgebra has no nontrivial solution. This result handles all the cases of interest for us, but it is the most difficult to apply. Our technique for applying it is to take advantage of a combinatorial result that is complicated to prove and is given below as Proposition 4.8 in Section 4.

This use of the same three techniques arose in [Kn3] earlier when we gave a relatively simple proof of unitarity for the relevant vector-bundle cases of analytic continuation of holomorphic discrete series when the root system is simply laced. For most groups the weakly-fair condition handled everything. But for groups with Lie algebra $\mathfrak{su}(m, l-m)$, it did not. We had to use the above third condition, the one involving an equation in the dual of the Cartan subalgebra, to handle matters. Unfortunately the argument we gave for $\mathfrak{su}(m, l-m)$ in [Kn3] was flawed, and we need a correct argument now. In fact, the combinatorial result needed for $\mathfrak{su}(m, l-m)$ turns out to be a preliminary step toward the result needed for B_l, C_l, and D_l, and we shall prove the preliminary step below as Proposition 4.3. In Section 6h we describe briefly how to use Proposition 4.3 to repair the argument in [Kn3].

In the line-bundle cases for B_l, C_l, and D_l, the Langlands parameters of the irreducible unitary representations that we obtain are known from [Kn2], which gives an algorithm for computing them. The verification in [Kn2] that the parameters are the correct ones is combinatorial in nature. P. Friedman ([Fr1], [Fr2]) interpreted the result in [Kn2] in terms of intertwining operators, and he generalized it. His work may have some bearing on the Langlands parameters in the vector-bundle cases for B_l, C_l, and D_l.

The present paper is organized as follows: Some notation is introduced in Section 1, and the two main theorems (Theorems 1.1 and 1.2) are stated there. Table 2 in that section shows the extent to which Theorems 1.1 and 1.2 are complementary for line-bundle cases. Section 2 gives the proof of the nonunitarity in line-bundle cases, using a technique in [GW2]. Section 3 reviews material from [Kn3], some describing the intertwining operators of interest between cohomologically induced representations and some summarizing aspects of Vogan's Unitarizability Theorem. Section 4 is the long one, establishing the combinatorial result (Proposition 4.8) that is to be used to check the hypotheses of Vogan's theorem. Section 5 gives the proof of the unitarity in line-bundle cases and vector-bundle cases alike, and Section 6 gives some examples and other remarks, including examples of nonunitarity in vector-bundle cases.

For a general exposition of cohomological induction and historical references concerning it, see [KnV]. For a brief summary of some useful properties of cohomological induction, see [Kn3].

1 Main theorems

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, let $\mathfrak{b}$ be a Cartan subalgebra, let $\Delta^+(\mathfrak{g})$ be a positive system of roots relative to $\mathfrak{b}$, and let α be a simple root. Define $\mathfrak{b}_0$ to be the real subspace of $\mathfrak{b}$ on which all roots are purely imaginary. By Theorems 6.74 and 6.88 of [Kn1], there exists a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ such that $\mathfrak{b}_0$ is a compact Cartan subalgebra of $\mathfrak{g}_0$, such that α is a noncompact root, and such that all other simple roots are compact. Moreover $\mathfrak{g}_0$ is unique up to isomorphism.

Let $\mathfrak{k}$ be the sum of $\mathfrak{b}$ and the root spaces for the compact roots, and let $\mathfrak{p}$ be the sum of the root spaces for the noncompact roots. Put $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0$ and $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0$. Then $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is a Cartan decomposition of $\mathfrak{g}_0$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is its complexification. Let θ be the Cartan involution.

Let $\Delta(\mathfrak{l})$ be the set of roots in the linear span of the compact simple roots, let $\mathfrak{l}$ be the sum of $\mathfrak{b}$ and the root spaces for the members of $\Delta(\mathfrak{l})$, and let $\mathfrak{u}$ be the sum of the root spaces for the positive roots that are not in $\Delta(\mathfrak{l})$. Then $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is a θ stable parabolic subalgebra of $\mathfrak{g}$. We shall be especially interested in the case that α occurs at most twice in the largest root (always the case if $\mathfrak{g}_0$ is classical), and then we say that $\Delta^+(\mathfrak{g})$ and $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ are of *Borel–de Siebenthal type*.

The center $Z_{\mathfrak{l}}$ of $\mathfrak{l}$ is the common kernel in $\mathfrak{b}$ of the members of $\Delta(\mathfrak{l})$, and this is 1-dimensional. It is the complexification of its intersection $Z_{\mathfrak{l}_0}$ with $\mathfrak{b}_0$. If a bar denotes the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$, then $\bar{\mathfrak{q}} = \mathfrak{l} \oplus \bar{\mathfrak{u}}$ is the opposite parabolic of $\mathfrak{q}$.

If E is a complex subspace of $\mathfrak{g}$ spanned by root spaces and a subspace of $\mathfrak{b}$, let $\Delta(E)$ be the set of roots contributing to E, and let $\Delta^+(E)$ be the set of positive roots contributing to E. We write $\delta(E)$ for half the sum of the members of $\Delta^+(E)$, and we abbreviate $\delta(\mathfrak{g})$ as δ. If $H_{\delta(\mathfrak{u})}$ denotes the member of $\mathfrak{b}$ paired with $\delta(\mathfrak{u})$ by the Killing form, then Corollary 4.69 of [KnV] shows that $H_{\delta(\mathfrak{u})}$ is in $iZ_{\mathfrak{l}_0}$, and every member β of $\Delta(\mathfrak{u})$ has $\beta(H_{\delta(\mathfrak{u})}) > 0$.

Let F be an irreducible finite-dimensional $\mathfrak{l}$ module. If ν is its highest weight, we write $F = F_\nu$. Define a 1-dimensional $\mathfrak{l}$ module ξ by $\xi([\mathfrak{l}, \mathfrak{l}]) = 0$ and $\xi(H_{\delta(\mathfrak{u})}) = 1$. For $z \in \mathbb{C}$, $z\xi$ is another 1-dimensional $\mathfrak{l}$ module, and we write $\mathbb{C}_{z\xi}$ rather than $F_{z\xi}$ for its space. Then $F_\nu \otimes \mathbb{C}_{z\xi}$ is a family of $\mathfrak{l}$ modules parametrized by $\mathbb{C}$.

We regard $\bigwedge^{\text{top}}\mathfrak{u}$ as a 1-dimensional $\mathfrak{l}$ module with unique weight $2\delta(\mathfrak{u})$. We define $F_\nu^\#$ to be the $\mathfrak{l}$ module $F_\nu \otimes \bigwedge^{\text{top}}\mathfrak{u}$, and we convert it into a $\bar{\mathfrak{q}}$ module by having $\bar{\mathfrak{u}}$ act by 0. With $U(\mathfrak{g})$ denoting the universal enveloping algebra of $\mathfrak{g}$, we define a $\mathfrak{g}$ module $N(\nu + 2\delta(\mathfrak{u}))$ by

$$N(\nu + 2\delta(\mathfrak{u})) = \operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}} F_\nu^\# = U(\mathfrak{g}) \otimes_{\bar{\mathfrak{q}}} F_\nu^\#.$$

Replacing F_ν by $F_\nu \otimes \mathbb{C}_{z\xi}$, we can form $N(\nu + z\xi + 2\delta(\mathfrak{u}))$. We shall be interested in properties of this family as z varies.

To define the Bernstein functor Π, we need to fix a group with Lie algebra $\mathfrak{g}_0$. Let G be such a group, and assume that G has finite center. In our theorems we shall make the stronger assumption that G has a simply connected complexification; but we do not make the stronger assumption yet. Let K and L be the analytic subgroups of G with respective Lie algebras $\mathfrak{k}_0$ and $\mathfrak{l}_0$. The group L is a compact connected subgroup of the maximal compact subgroup K.

We shall be interested only in values of ν and z such that F_ν and $\mathbb{C}_{z\xi}$ are actually L modules. Since L is compact, this condition forces z to be real. When F_ν and $\mathbb{C}_{z\xi}$ are L modules, the extension of the action to $\bar{\mathfrak{q}}$ makes them and $\bigwedge^{\text{top}}\mathfrak{u}$ into $(\bar{\mathfrak{q}}, L)$ modules, and $N(\nu + z\xi + 2\delta(\mathfrak{u}))$ becomes a $(\mathfrak{g}, L)$ module. Chapter I of [KnV] introduces rings $R(\mathfrak{g}, K)$ and $R(\mathfrak{g}, L)$ that play the same role for $(\mathfrak{g}, K)$ and $(\mathfrak{g}, L)$ modules that $U(\mathfrak{g})$ plays for $\mathfrak{g}$ modules. The Bernstein functor Π for this situation carries $(\mathfrak{g}, L)$ modules to $(\mathfrak{g}, K)$ modules and is given by $\Pi(X) = R(\mathfrak{g}, K) \otimes_{R(\mathfrak{g},L)} X$; Π is a covariant right exact functor, and the interest is in its S^{th} derived functor Π_S, where $S = \dim(\mathfrak{u} \cap \mathfrak{k})$.

The $(\mathfrak{g}, K)$ modules that we study in the first instance are

$$\pi(\nu + z\xi) = \Pi_S(N(\nu + z\xi + 2\delta(\mathfrak{u})))$$

under the assumption that F_ν and $\mathbb{C}_{z\xi}$ are actually representations of L, not merely of $\mathfrak{l}$. Again, z must be real for this condition to be satisfied. The $(\mathfrak{g}, K)$ module $\pi(\nu + z\xi)$ is known to have a composition series. If $\nu + z\xi + \delta$ is strictly dominant, as happens when z is sufficiently large positively, $\pi(\nu + z\xi)$ is known to be irreducible, to be in the discrete series, and to be the underlying $(\mathfrak{g}, K)$ module of the representation that Harish-Chandra called $\pi_{\nu+z\xi+\delta}$ in [HC2]. A relatively direct proof of the square integrability of $\pi(\nu + z\xi)$ may be found in [Wa1]; all other facts needed for this identification may be found in [KnV]. We are interested in what happens when z moves out of the interval $(z_0, +\infty)$ where we have a discrete series representation.

We need to impose one more condition to get a reasonable $(\mathfrak{g}, K)$ module to study, namely that the parameter $\nu + z\xi$ is such that $\nu + z\xi + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is $\Delta^+(\mathfrak{k})$ dominant. In this case, according to Corollary 5.85 of [KnV], the K type with highest weight $\nu + z\xi + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ occurs with multiplicity one in $\pi(\nu + z\xi)$. It is said to be in the *bottom layer*.

When this K type occurs in $\pi(\nu + z\xi)$, the fact that $\pi(\nu + z\xi)$ admits a composition series implies that it makes sense to speak of the irreducible subquotient $\bar{\pi}(\nu + z\xi)$ of $\pi(\nu + z\xi)$ containing this K type. The $(\mathfrak{g}, K)$ module $\bar{\pi}(\nu + z\xi)$ is the one that we study for unitarity in Theorem 1.1 below.

There is a related $(\mathfrak{g}, K)$ module that arises naturally. If $N'(\nu + z\xi + 2\delta(\mathfrak{u}))$ is the unique irreducible quotient of $N(\nu + z\xi + 2\delta(\mathfrak{u}))$, this related $(\mathfrak{g}, K)$ module is $\pi'(\nu + z\xi) = \Pi_S(N'(\nu + z\xi + 2\delta(\mathfrak{u})))$. For $\nu = 0$, we study $\pi'(z\xi)$ for nonunitarity in Theorem 1.2 below. Specifically $\pi'(z\xi)$ carries a natural nondegenerate invariant Hermitian form known as the *Shapovalov form*, and we study whether this form is definite or indefinite.

It is currently not known whether $\pi'(z\xi)$ is irreducible.* If it is, then it is isomorphic to $\bar{\pi}(z\xi)$, as we shall see in Section 2, and it follows that the Shapovalov form is, up to a scalar, the unique candidate for an invariant Hermitian form on $\bar{\pi}(z\xi)$. We discuss the relationship between $\bar{\pi}(z\xi)$ and $\pi'(z\xi)$ briefly in Section 2 and again in Section 6b.

Now let us specialize to $\mathfrak{g}$ of type B_l, C_l, or D_l, with α given in standard notation as $e_m - e_{m+1}$. The integers m and $l-m$ are assumed to be ≥ 2, except that $l-m$ is allowed to be 1 in the case of B_l. The positive system of Borel–de Siebenthal type makes $\mathfrak{g}_0$ equal to $\mathfrak{so}(2m, 2(l-m)+1)$, $\mathfrak{sp}(m, l-m)$, or $\mathfrak{so}(2m, 2(l-m))$ in the respective cases. Let G be a connected Lie group with Lie algebra $\mathfrak{g}_0$ and with a simply connected complexification.

We use l tuples to denote members of $\mathfrak{b}^*$, with $(a_1, \ldots, a_l)$ standing for $a_1e_1 + \cdots + a_le_l$. Normally we use a semicolon to separate the first m entries from the last $l-m$. We begin by listing the values of various half-sums of positive roots. In order to treat B_l, C_l, and D_l simultaneously, we introduce h and h' as in Table 1.

$$h = \tfrac{1}{2} \text{ and } h' = 0 \text{ for } B_l,$$
$$h = 1 \text{ and } h' = 1 \text{ for } C_l,$$
$$h = 0 \text{ and } h' = 0 \text{ for } D_l.$$

TABLE 1. Definitions of h and h'

**Added in proof*: Since the writing of this paper, Peter Trapa has shown for some groups of types B_l and D_l that $\pi'(z\xi)$ is irreducible when the Shapovalov form is definite.

Then

$$\begin{aligned}
\delta &= (l-1+h, l-2+h, \ldots, l-m+h;\ l-m-1+h, \ldots, h),\\
\delta(\mathfrak{k}) &= (m-1+h', m-2+h', \ldots, h';\ l-m-1+h, \ldots, h),\\
\delta(\mathfrak{l}) &= \left(\tfrac{m-1}{2}, \ldots, -\tfrac{m-1}{2};\ l-m-1+h, \ldots, h\right),\\
\delta(\mathfrak{u}) &= \left(l-\tfrac{m+1}{2}+h, \ldots, l-\tfrac{m+1}{2}+h;\ 0, \ldots, 0\right),\\
\delta(\mathfrak{u}\cap\mathfrak{k}) &= \left(\tfrac{m-1}{2}+h', \ldots, \tfrac{m-1}{2}+h';\ 0, \ldots, 0\right),\\
\delta(\mathfrak{u}\cap\mathfrak{p}) &= (l-m+h-h', \ldots, l-m+h-h';\ 0, \ldots, 0).
\end{aligned}$$

The use $\nu + z\xi$ as parameter of the representation of L is inconvenient because the 1-dimensional part has to be extracted from the sum. Instead we shall make the 1-dimensional part be paramount. To do so, we introduce the 1-dimensional representation of L with weight

$$\lambda = (-l+t, \ldots, -l+t;\ 0, \ldots, 0) \qquad \text{with } t \in \mathbb{Z}.$$

What λ does is to detect the jump between the m^{th} and $(m+1)^{\text{st}}$ entries of a highest weight for L. The $-l$ may be regarded as an additive normalization. Let $\omega = (\omega_1, \ldots, \omega_l)$ be $\Delta^+(\mathfrak{g})$ dominant integral with $\omega_m = \omega_{m+1}$, i.e., with 0 jump between the m^{th} and $(m+1)^{\text{st}}$ entries. It will be convenient to assume that $\omega_l \geq 0$. This condition is automatic for B_l and C_l; for D_l the operation of negating the l^{th} entry of ω can be achieved by an outer automorphism of G, and hence theorems when ω_l is < 0 can be derived from what happens when ω_l is > 0.

The representation of L that we use is the one with highest weight $\lambda+\omega$. The related parameters are

$$\begin{aligned}
\lambda+\omega+\delta &= (\omega_1+t-1+h,\ \omega_2+t-2+h, \ldots,\\
&\qquad \omega_m+t-m+h;\ \omega_{m+1}+l-m-1+h, \ldots, \omega_l+h),\\
\lambda+\omega+2\delta(\mathfrak{u}\cap\mathfrak{p}) &= (\omega_1+t+l-2m+2h-2h', \ldots,\\
&\qquad \omega_m+t+l-2m+2h-2h';\ \omega_{m+1}, \ldots, \omega_l).
\end{aligned}$$

Strict dominance of $\lambda+\omega+\delta$ relative to $\Delta^+(\mathfrak{g})$ is the condition for discrete series, and the only thing to check is that the m^{th} entry exceeds the $(m+1)^{\text{st}}$. Since $\omega_m = \omega_{m+1}$, the condition for discrete series is that $t \geq l$. When $t = l-1$, $\pi(\lambda+\omega)$ is a limit of discrete series. Our interest is in $t \leq l-2$.

The condition for the existence of the bottom layer K type is the $\Delta^+(\mathfrak{k})$ dominance of $\lambda+\omega+2\delta(\mathfrak{u}\cap\mathfrak{p})$, and the only thing to check is that the difference of the m^{th} and $(m+1)^{\text{st}}$ entries is ≥ 0. Since $\omega_m = \omega_{m+1}$, the condition for the existence of the bottom-layer K type is that $t \geq 2m-l+2h'-2h$.

Theorem 1.1. *Let $\mathfrak{g}_0$ be any one of the following:*

$$\begin{array}{lll}
\mathfrak{so}(2m, 2(l-m)+1) & \textit{of type } B_l & \textit{with } m \geq 2 \textit{ and } l-m \geq 1,\\
\mathfrak{sp}(m, l-m) & \textit{of type } C_l & \textit{with } m \geq 2 \textit{ and } l-m \geq 2,\\
\mathfrak{so}(2m, 2(l-m)) & \textit{of type } D_l & \textit{with } m \geq 2 \textit{ and } l-m \geq 2,
\end{array}$$

and let G be a connected Lie group with Lie algebra $\mathfrak{g}_0$ and with a simply connected complexification. Define h and h' in the respective cases B_l, C_l, and D_l as in Table 1. Let $\mathfrak{b}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$ lying in $\mathfrak{k}_0$, and introduce the positive system $\Delta^+(\mathfrak{g})$ of roots of Borel–de Siebenthal type such that $e_m - e_{m+1}$ is the unique noncompact simple root. Let t be an integer, define $\lambda \in \mathfrak{b}^*$ by $\lambda_i = -l + t$ for $i \leq m$ and $\lambda_i = 0$ for $i \geq m + 1$, and let $\omega \in \mathfrak{b}^*$ be a $\Delta^+(\mathfrak{g})$ dominant integral form with $\omega_m = \omega_{m+1}$ and $\omega_l \geq 0$. The expression $\lambda + \omega + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is $\Delta^+(\mathfrak{k})$ dominant for

$$t \geq 2m - l + 2h' - 2h,$$

and hence $\bar{\pi}(\lambda + \omega)$ is well defined in these cases. Let i_0 be the smallest index i with $1 \leq i \leq m$ such that $\omega_i = \omega_m$, let j_0 be the largest index j with $m \leq j \leq l - 1$ such that $\omega_{j+1} = \omega_{m+1}$, and put $t_0 = l - (j_0 - i_0 + 2)$, so that $0 \leq t_0 \leq l - 2$. If $t \geq 2m - l + 2h' - 2h$ and, in addition,

$$t \geq \max\{t_0, 2m - l + 1 - 2h\},$$

then $\bar{\pi}(\lambda + \omega)$ is infinitesimally unitary.

The line-bundle case is $\omega = 0$, and then $i_0 = 1$, $j_0 = l - 1$, and $t_0 = 0$. Theorem 1.2 is an indication of parameters for this situation that might correspond to nonunitarity.

Theorem 1.2. *Let $\mathfrak{g}_0$, h, h', $\mathfrak{b}_0$, and $\Delta^+(\mathfrak{g})$ be as in Theorem 1.1. Let t be an integer, define $\lambda \in \mathfrak{b}^*$ by $\lambda_i = -l + t$ for $i \leq m$ and $\lambda_i = 0$ for $i \geq m + 1$, and let $t \geq 2m - l + 2h' - 2h$, so that the expression $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is $\Delta^+(\mathfrak{k})$ dominant. Then the Shapovalov form on $\pi'(\lambda)$ is indefinite when t satisfies the additional condition $t < 0$.*

If $\pi'(\lambda)$ is irreducible, then, as we mentioned, it equals $\bar{\pi}(\lambda)$, and Theorem 1.2 gives a result that for $\omega = 0$ is close to being complementary to Theorem 1.1. A precise comparison of the results for $\omega = 0$ when $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is $\Delta^+(\mathfrak{k})$ dominant, putting together the results from Theorems 1.1 and 1.2, is given in Table 2. Let us emphasize that to say a representation with parameter t is "decided" means either that $\bar{\pi}(\lambda)$ has been shown to be infinitesimlly unitary or that the Shapovalov form on $\pi'(\lambda)$ has been shown to be indefinite. This is not quite the same as saying that either $\bar{\pi}(\lambda)$ has been shown to be infinitesimally unitary or it has been shown to fail to be infinitesimally unitary.

Curiously the line-bundle cases listed in Table 2 as not "decided" are all spherical. At least some of them have been shown by other authors to be unitary. Kostant [Kos1] and [Kos2] extensively investigated the $t = 0$ representation of the group $SO(4, 4)_0$, which is of type D_4 with $m = 2$. This representation is unitary. It has been investigated also in [KaS], [BrK], and [GW1]. As we shall observe in Section 6a, some of the other representations listed as not "decided" lie in the range treated in [EPWW] and are therefore unitary.

The unitarity in the line-bundle cases of B_l and D_l when $m = 2$ was already known from [GW2]. Some vector-bundle cases for B_4 with $m = 2$ were shown to be unitary in [Kn3].

B_l		$\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ dominant for K for $2m - l - 1 \le t$.
	$2m \le l$	$\bar{\pi}(\lambda)$ defined and unitary for $0 \le t$, Shapovalov form on $\pi'(\lambda)$ indefinite for $t < 0$, All points t decided.
	$2m > l$	$\bar{\pi}(\lambda)$ defined and unitary for $2m - l \le t$, All points but $t = 2m - l - 1$ decided.
C_l		$\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ dominant for K for $2m - l \le t$.
	$2m \le l$	$\bar{\pi}(\lambda)$ defined and unitary for $0 \le t$, Shapovalov form on $\pi'(\lambda)$ indefinite for $t < 0$, All points t decided.
	$2m > l$	$\bar{\pi}(\lambda)$ defined and unitary for $2m - l \le t$, All points t decided.
D_l		$\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ dominant for K for $2m - l \le t$.
	$2m < l$	$\bar{\pi}(\lambda)$ defined and unitary for $0 \le t$, Shapovalov form on $\pi'(\lambda)$ indefinite for $t < 0$, All points t decided.
	$2m \ge l$	$\bar{\pi}(\lambda)$ defined and unitary for $2m - l + 1 \le t$, All points but $t = 2m - l$ decided.

TABLE 2. Unitarity and nonunitarity for line-bundle cases

Authors of some papers have constructed similar-appearing finite or infinite sequences of small unitary representations of the groups under study. It seems that these sequences often have some representations in common with the ones obtained from Theorem 1.1 but are basically just different sequences of representations. Among papers of this kind are [BiZ], [Kob], and [ZhH]. Sections 8 and 9 of [EPWW], which are sections addressing examples other than analytic continuation of discrete series, construct more sequences of this kind. The papers [Li1] and [Li2] classify a certain kind of small representation for classical groups, and one may expect that many of the representations shown to be unitary by Theorem 1.1 are small in the sense of those papers.

2 Nonunitarity

In this section we prove Theorem 1.2. The method is largely from [EPWW] and [GW2], except that the notation is different and the applicability of that method to our situation may not be immediately apparent. We work under the assumption that the θ stable parabolic algebra is of Borel–de Siebenthal type and the parameter of the

1-dimensional $\mathfrak{l}$ module $\mathbb{C}_\lambda$ leads to the existence of a bottom-layer K type, i.e., that

$$\lambda + 2\delta(\mathfrak{u}\cap\mathfrak{p}) \qquad \text{is } \Delta^+(\mathfrak{k}) \text{ dominant.}$$

Section 3 of [EPWW] indicates a number of special facts about $N(\lambda+2\delta(\mathfrak{u}))$. One of these is a very complicated isomorphism

$$N(\lambda+2\delta(\mathfrak{u})) \cong U(\mathfrak{k}) \otimes_{U(\bar{\mathfrak{q}}\cap\mathfrak{k})} (S(\mathfrak{u}\cap\mathfrak{p}) \otimes \mathbb{C}_{\lambda+2\delta(\mathfrak{u})}) \qquad \text{as } U(\mathfrak{k}) \text{ modules,} \tag{2.1}$$

where $S(\cdot)$ denotes a symmetric algebra and the action by $\mathfrak{l}$ on $S(\mathfrak{u}\cap\mathfrak{p})$ has been extended to $\bar{\mathfrak{q}}\cap\mathfrak{k}$ by having $\bar{\mathfrak{u}}\cap\mathfrak{k}$ act by 0. In (2.1), $S(\mathfrak{u}\cap\mathfrak{p})\otimes\mathbb{C}_{\lambda+2\delta(\mathfrak{u})}$ is fully reducible as an $\mathfrak{l}$ module and even as a $\bar{\mathfrak{q}}\cap\mathfrak{k}$ module. Hence so is the left side of (2.1). The highest weights of the constituents are of the form $\sigma'+\lambda+2\delta(\mathfrak{u})$, where σ' is the highest weight of an irreducible summand of $S(\mathfrak{u}\cap\mathfrak{p})$, hence a sum of members of $\Delta(\mathfrak{u}\cap\mathfrak{p})$.

Following Section 3 of [EPWW], but not in its logical order, let us examine such a constituent. Put $\tau=\sigma'+\lambda+2\delta(\mathfrak{u})$. The $\mathfrak{k}$ infinitesimal character of $U(\mathfrak{k})\otimes_{U(\bar{\mathfrak{q}}\cap\mathfrak{k})} F_\tau$ is $\tau+\delta(\mathfrak{l})-\delta(\mathfrak{u}\cap\mathfrak{k})$, by Theorem 5.24 of [KnV], for example. If β is in $\Delta(\mathfrak{u}\cap\mathfrak{k})$, then this τ satisfies

$$\begin{aligned}\langle\tau-2\delta(\mathfrak{u}\cap\mathfrak{k}),\,\beta\rangle &= \langle\sigma'+\lambda+2\delta(\mathfrak{u})-2\delta(\mathfrak{u}\cap\mathfrak{k}),\,\beta\rangle\\ &= \langle\sigma'+(\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p})),\,\beta\rangle.\end{aligned}$$

On the right side, by the hypotheses, $\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p})$ is $\Delta^+(\mathfrak{k})$ dominant and σ' is a sum of members of $\Delta(\cap\mathfrak{p})$. Thus $\langle\tau-2\delta(\mathfrak{u}\cap\mathfrak{k}),\,\beta\rangle\geq 0$. The $\mathfrak{k}$ infinitesimal character in question is the sum of $\tau-2\delta(\mathfrak{u}\cap\mathfrak{k})$ and $\delta(l)+\delta(\mathfrak{u}\cap\mathfrak{k})=\delta(k)$. Hence its inner product with any $\beta\in\Delta(\mathfrak{u}\cap\mathfrak{k})$ is ≥ 0, and it follows from Corollary 5.105 of [KnV] that each $U(\mathfrak{k})\otimes_{U(\bar{\mathfrak{q}}\cap\mathfrak{k})} F_\tau$ is irreducible as a $U(\mathfrak{k})$ module.

Making a change of variables in the τ variable, we summarize as follows:

> $N(\lambda+2\delta(\mathfrak{u}))$, which is defined to be $U(\mathfrak{g})\otimes_{U(\bar{\mathfrak{q}})}\mathbb{C}_{\lambda+2\delta(\mathfrak{u})}$, is semisimple as a $U(\mathfrak{k})$ module, and the irreducible summands are all of the form $N^K(\tau+2\delta(\mathfrak{u}\cap\mathfrak{k})) = U(\mathfrak{k})\otimes_{U(\bar{\mathfrak{q}}\cap\mathfrak{k})}(F_\tau\otimes\mathbb{C}_{2\delta(\mathfrak{u}\cap\mathfrak{k})})$, where τ is an irreducible $\mathfrak{l}$ module whose highest weight is of the form
>
> $$\tau=\sigma'+\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p}), \tag{2.2}$$
>
> σ' being an $\mathfrak{l}$ dominant sum of members of $\Delta(\mathfrak{u}\cap\mathfrak{p})$. Each such τ is $\Delta^+(\mathfrak{k})$ dominant.

The Bernstein functor Π defined above carries $(\mathfrak{g}, L)$ modules to $(\mathfrak{g}, K)$ modules. There is also a Bernstein functor Π^K carrying $(\mathfrak{k}, L)$ modules to $(\mathfrak{k}, K)$ modules; the formula is $\Pi^K(\cdot)=R(\mathfrak{k},K)\otimes_{R(\mathfrak{k},L)}(\cdot)$, and it too is covariant and right exact. The derived functors Π^K_j of Π^K have two properties that are relevant for us. One, by (4.170) of [KnV], is that any integral parameter τ that is $\Delta^+(\mathfrak{k})$ dominant has $\Pi^K_j(N^K(\tau+2\delta(\mathfrak{u}\cap\mathfrak{k})))=0$ if $j\neq S$ and has $\Pi^K_S(N^K(\tau+2\delta(\mathfrak{u}\cap\mathfrak{k})))$ equal to an irreducible K module with highest weight τ; this is an algebraic version of the Borel–Weil–Bott Theorem. The other property, by Proposition 2.69b of [KnV], is that $\mathcal{F} \circ \Pi_j \cong \Pi_j^K \circ \mathcal{F}'$ for all j, where $\mathcal{F}$ is the forgetful functor from $(\mathfrak{g}, K)$ modules to $(\mathfrak{k}, K)$ modules and $\mathcal{F}'$ is the forgetful functor from $(\mathfrak{g}, L)$ modules to $(\mathfrak{l}, L)$ modules.

In conjunction with (2.2), these properties allow us to draw conclusions about the K decomposition of $\pi(\lambda) = \Pi_S(N(\lambda + 2\delta(\mathfrak{u})))$, namely that the multiplicity of the K type with highest weight τ in $\pi(\lambda)$ equals the multiplicity of the L type with highest weight $\tau - 2\delta(\mathfrak{u} \cap \mathfrak{p})$ in $S(\mathfrak{u} \cap \mathfrak{p}) \otimes \mathbb{C}_{\lambda+2\delta(\mathfrak{u})}$.

The $\mathfrak{g}$ module $N(\lambda + 2\delta(\mathfrak{u}))$ carries a natural invariant Hermitian form called the *Shapovalov form* and defined, apart from an adjustment of notation, in (3.2) of [EPWW]. This descends to a nondegenerate invariant Hermitian form on the quotient $\mathfrak{g}$ module $N'(\lambda + 2\delta(\mathfrak{u}))$.

For any $\mathfrak{k}$ module V, let $V^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$ be the $\mathfrak{l}$ module of invariants under $\bar{\mathfrak{u}} \cap \mathfrak{k}$. Because of the irreducibility of each $N^K(\tau + 2\delta(\mathfrak{u} \cap \mathfrak{k}))$ in (2.2), it follows that

$$N^K(\tau + 2\delta(\mathfrak{u} \cap \mathfrak{k}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}} = 1 \otimes (F_\tau \otimes \mathbb{C}_{2\delta(\mathfrak{u}\cap\mathfrak{k})}).$$

The Shapovalov form can be carried from $N(\lambda+2\delta(\mathfrak{u}))$ to $\pi(\lambda) = \Pi_S(N(\lambda+2\delta(\mathfrak{u})))$, and a key result (Proposition 6.6) of [EPWW] (cf. Proposition 6.50 of [KnV]) is that the signature on each $1 \otimes (F_\tau \otimes \mathbb{C}_{2\delta(\mathfrak{u}\cap\mathfrak{k})})$ as an L multiplicity matches the signature on the corresponding subspace of type τ in $\pi(\lambda)$ as a K multiplicity. Roughly speaking, the unitarity or nonunitarity of $\pi(\lambda)$ can be detected from $N(\lambda + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$.

However, it is not $\pi(\lambda) = \Pi_S(N(\lambda + 2\delta(\mathfrak{u})))$ that is of ultimate interest to us. Instead it is the irreducible subquotient $\bar{\pi}(\lambda)$. How to detect exact signatures on this from signatures on $N(\lambda + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$ is unclear.

There is a substitute. The Shapovalov form makes sense on $N'(\lambda + 2\delta(\mathfrak{u}))$, and Π_S carries the form from $N'(\lambda+2\delta(\mathfrak{u}))$ to $\pi'(\lambda) = \Pi_S(N'(\lambda+2\delta(\mathfrak{u})))$. Since the $\mathfrak{k}$ module $N(\lambda + 2\delta(\mathfrak{u}))$ is semisimple, so is the $\mathfrak{k}$ module $N'(\lambda + 2\delta(\mathfrak{u}))$. Thus the signatures of the form on K types of $\pi'(\lambda)$ can be related to the signatures of the form on L types of $N'(\lambda + 2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$. But what is the relationship between $\pi'(\lambda)$ and $\bar{\pi}(\lambda)$?

Here we use facts about Π_{S-1}. We have seen from the algebraic version of the Borel–Weil–Bott theorem that Π_{S-1}^K is 0 on each $N^K(\tau + 2\delta(\mathfrak{u} \cap \mathfrak{k}))$ in (2.2), and it follows from (2.2) that $\Pi_{S-1}^K(\ker(N(\lambda + 2\delta(\mathfrak{u})) \to N'(\lambda + 2\delta(\mathfrak{u})))) = 0$. Therefore $\Pi_{S-1}(\ker(N(\lambda + 2\delta(\mathfrak{u})) \to N'(\lambda + 2\delta(\mathfrak{u})))) = 0$. By the long exact sequence of the derived functors of Π, $\Pi_S(N'(\lambda + 2\delta(\mathfrak{u})))$ is a quotient of $\Pi_S(N(\lambda + 2\delta(\mathfrak{u})))$. Also $\Pi_S(N'(\lambda + 2\delta(\mathfrak{u})))$ contains with multiplicity one the K type with highest weight $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$. Therefore one of the irreducible subquotients of $\Pi_S(N'(\lambda + 2\delta(\mathfrak{u})))$ is $\bar{\pi}(\lambda)$. In other words, $\bar{\pi}(\lambda)$ is the unique irreducible subquotient of $\pi'(\lambda) = \Pi_S(N'(\lambda + 2\delta(\mathfrak{u})))$ containing the K type with highest weight $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$. If it should happen that $\pi'(\lambda)$ is irreducible, then $\pi'(\lambda)$ is isomorphic with $\bar{\pi}(\lambda)$.

With this preparation let us turn to a consideration of the Shapovalov form on $\pi'(\lambda)$, following the lines of [GW2]. It is enough to consider the form on $N'(\lambda+2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$ and therefore the form on $N(\lambda+2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$. Write $(\cdot\,,\,\cdot)$ for the form on $N(\lambda+2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$. The authors of [GW2] make a computation of the Casimir operator Ω of $\mathfrak{g}$ in terms of the Casimir operator $\Omega_{\mathfrak{l}}$ of $\mathfrak{l}$. They make a certain normalization of the root vectors X_γ of noncompact roots γ such that $\overline{X_\gamma} = X_{-\gamma}$. Taking into account the differences between their notation and ours, we obtain

$$\Omega = \Omega_{\mathfrak{l}} - 2H_{\delta(\mathfrak{u})} + 2\sum_{\gamma\in\Delta(\mathfrak{u}\cap\mathfrak{k})} X_\gamma X_{-\gamma} + 2\sum_{\gamma\in\Delta(\mathfrak{u}\cap\mathfrak{p})} X_\gamma X_{-\gamma}.$$

The two sides of this formula are to be applied to a member v of $N(\lambda+2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$ that lies in the space of $1\otimes(F_\tau\otimes\mathbb{C}_{2\delta(\mathfrak{u}\cap\mathfrak{k})})$, where $\tau = \sigma'+\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p})$. The infinitesimal character of $N(\lambda+2\delta(\mathfrak{u}))$ is $\lambda+\delta$, and the $\mathfrak{l}$ infinitesimal character of $F_\tau\otimes\mathbb{C}_{2\delta(\mathfrak{u}\cap\mathfrak{k})}$ is $\tau+2\delta(\mathfrak{u}\cap\mathfrak{k})+\delta(\mathfrak{l})$. The element $\delta(\mathfrak{u})$ has the same inner product with all weights of $F_\tau\otimes\mathbb{C}_{2\delta(\mathfrak{u}\cap\mathfrak{k})}$, namely whatever the highest weight gives. Therefore

$$\begin{aligned}
\Omega v &= (\|\lambda+\delta\|^2 - \|\delta\|^2)v,\\
\Omega_{\mathfrak{l}} v &= (\|\sigma'+\lambda+2\delta(\mathfrak{u})+\delta(\mathfrak{l})\|^2 - \|\delta(\mathfrak{l})\|^2)v,\\
H_{\delta(\mathfrak{u})}v &= \langle\delta(\mathfrak{u}),\, \sigma'+\lambda+2\delta(\mathfrak{u})\rangle v,\\
X_{-\gamma}v &= 0 \qquad \text{for } \gamma\in\Delta(\mathfrak{u}\cap\mathfrak{k}),
\end{aligned}$$

and the orthogonality of $\delta(\mathfrak{l})$ and $\delta(\mathfrak{u})$ gives

$$\begin{aligned}
-2\sum_{\gamma\in\Delta(\mathfrak{u}\cap\mathfrak{p})} X_\gamma X_{-\gamma} v &= \{-\|\lambda+\delta\|^2+\|\delta\|^2+\|\sigma'+\lambda+2\delta(\mathfrak{u})+\delta(\mathfrak{l})\|^2-\|\delta(\mathfrak{l})\|^2\\
&\qquad -2\langle\delta(\mathfrak{u}),\, \sigma'+\lambda+2\delta(\mathfrak{u})\rangle\}v\\
&= \{-\|\lambda+\delta\|^2+\|\delta(\mathfrak{u})\|^2+\|\sigma'+\lambda+\delta+\delta(\mathfrak{u})\|^2\\
&\qquad -2\langle\sigma'+\lambda,\, \delta(\mathfrak{u})\rangle - 4\|\delta(\mathfrak{u})\|^2\}v\\
&= \{-\|\lambda+\delta\|^2+\|\sigma'+\lambda+\delta\|^2+2\langle\sigma'+\lambda+\delta,\, \delta(\mathfrak{u})\rangle\\
&\qquad -2\langle\sigma'+\lambda,\, \delta(\mathfrak{u})\rangle - 2\|\delta(\mathfrak{u})\|^2\}v\\
&= \{\|\sigma'+\lambda+\delta\|^2 - \|\lambda+\delta\|^2\}v.
\end{aligned}$$

We apply $(\,\cdot\,, v)$ to both sides, and the resulting formula

$$2\sum_{\gamma\in\Delta(\mathfrak{u}\cap\mathfrak{p})} (X_{-\gamma}v, X_{-\gamma}v) = \{\|\sigma'+\lambda+\delta\|^2 - \|\lambda+\delta\|^2\}(v,v) \tag{2.3}$$

is to be regarded as an inductive formula for (v,v).

In fact, if v is in $N(\lambda+2\delta(\mathfrak{u}))^{\bar{\mathfrak{u}}\cap\mathfrak{k}}$, then so is each $X_{-\gamma}v$ for $\gamma\in\mathfrak{u}\cap\mathfrak{p}$, since $[\mathfrak{u}\cap\mathfrak{k}, \mathfrak{u}\cap\mathfrak{p}] = 0$. We can regard v as in $1\otimes(S(\mathfrak{u}\cap\mathfrak{p})\otimes\mathbb{C}_{\lambda+2\delta(\mathfrak{u})})$ under the isomorphism (2.1), and the same thing is then true of each $X_{-\gamma}v$. The index for the induction becomes the degree in $S(\mathfrak{u}\cap\mathfrak{p})$, which shows up in $\sigma'+\lambda+\delta$ as the number of times that the expansion of σ' in terms of simple roots involves the noncompact simple root α.

For Theorem 1.2 we compute the signature of $(\,\cdot\,,\,\cdot\,)$ on $1\otimes(S(\mathfrak{u}\cap\mathfrak{p})\otimes\mathbb{C}_{\lambda+2\delta(\mathfrak{u})})$ one degree at a time. The expression σ' is a highest weight occurring in $S(\mathfrak{u}\cap\mathfrak{p})$, according to (2.2). The base stage of an induction is that the degree of the subspace of type σ' within $S(\mathfrak{u} \cap \mathfrak{p})$ is 0. Then $\sigma' = 0$, and the form is positive definite for degree 0. For degree 1, σ' has to be $\sigma' = e_1 + e_{m+1}$ since L acts irreducibly on $\mathfrak{u} \cap \mathfrak{p}$, and this highest weight occurs in $S(\mathfrak{u} \cap \mathfrak{p})$ with multiplicity one. If we take v in (2.3) to be a vector of this type, the formula tells us that the signature at degree 1 is therefore the sign of

$$\begin{aligned}
&\|\sigma' + \lambda + \delta\|^2 - \|\lambda + \delta\|^2 \\
&\quad = \|(t-1+h+1, t-2+h, \ldots, t-m+h;\ l-m-1+h+1, \ldots, h)\|^2 \\
&\qquad - \|(t-1+h, t-2+h, \ldots, t-m+h;\ l-m-1+h, \ldots, h)\|^2 \\
&\quad = 2(t-1+l-m+2h) \\
&\quad \geq 2((2m-l+2h'-2h)-1+l-m+2h) \\
&\qquad\qquad \text{from Section 1 since } \lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p}) \text{ is } \Delta^+(\mathfrak{k}) \text{ dominant} \\
&\quad = 2(m-1+2h').
\end{aligned}$$

This is positive, and the form is therefore positive definite for degree 1.

We show for some σ' of degree 2 that $\|\sigma' + \lambda + \delta\|^2 - \|\lambda + \delta\|^2$ is < 0. Taking v to be in the corresponding space of degree 2 and applying (2.3), we conclude that the Shapovalov form is not semidefinite on the subspace of interest, and then the proof of Theorem 1.2 will be complete.

For B_l and D_l, L is locally $U(m) \times SO(n)$ with n equal to $2(l-m)+1$ and $2(l-m)$ in the respective cases. The action on $\mathfrak{u} \cap \mathfrak{p}$ is isomorphic to the action with $U(m)$ on the left and $SO(n)$ on the right of the matrix space $M_{mn}(\mathbb{C})$. Then we take symmetric tensors. It is classical (see, e.g., [GoW], p. 256) that when $U(m) \times U(n)$ acts on $M_{mn}(\mathbb{C})$, the action on $S(M_{mn}(\mathbb{C}))$ decomposes with multiplicity 1 into the sum of all outer tensor products $\tau_1 \widehat{\otimes} \tau_2$, where τ_1 has an arbitrary nonnegative highest weight of depth at most $\min(m, n)$ and τ_2 has the highest weight of the contragredient of τ_1 except that its number of variables is changed from m to n. We take τ_1 to be the representation of $U(m)$ in holomorphic polynomials in m-variables homogeneous of degree 2. Its highest weight is $2e_1$. The restriction of the corresponding τ_2 to $SO(n)$ contains the trivial representation of $SO(n)$. Thus the representation of L with highest weight $\sigma' = (2, 0, \ldots, 0; 0, \ldots, 0)$ occurs in $S(\mathfrak{u} \cap \mathfrak{p})$, and we have

$$\begin{aligned}
&\|\sigma' + \lambda + \delta\|^2 - \|\lambda + \delta\|^2 \\
&\qquad = \|(t-1+h+2, t-2+h, \ldots, t-m+h;\ l-m-1+h, \ldots, h)\|^2 \\
&\qquad\quad - \|(t-1+h, t-2+h, \ldots, t-m+h;\ l-m-1+h, \ldots, h)\|^2 \\
&\qquad = 4(t+h).
\end{aligned}$$

This is negative for $t < 0$ since h is $\frac{1}{2}$ or 0, and hence the form is not semidefinite for B_l or D_l when $t < 0$.

For C_l, L is locally $U(m) \times Sp(l-m)$. The action on $\mathfrak{u} \cap \mathfrak{p}$ is isomorphic to the action with $U(m)$ on the left and $Sp(l-m)$ on the right of the matrix space $M_{m,2(l-m)}(\mathbb{C})$. We view $Sp(l-m)$ as a subgroup of $U(2(l-m))$. Then we form the representation

$\tau_1 \widehat{\otimes} \tau_2$ of $U(m) \times U(2(l-m))$ as above with τ_1 the representation of $U(m)$ on $\bigwedge^2 \mathbb{C}^m$. Its highest weight is $e_1 + e_2$. The restriction of the corresponding τ_2 to $Sp(l-m)$ contains the trivial representation of $Sp(l-m)$. Thus the representation of L with highest weight $\sigma' = (1, 1, 0, \ldots, 0; 0, \ldots, 0)$ occurs in $S(\mathfrak{u} \cap \mathfrak{p})$, and, since $h = 1$,

$$\begin{aligned}\|\sigma' + \lambda + \delta\|^2 - \|\lambda + \delta\|^2 &= \|(t+1, t, t-2, \ldots, t-m+1;\ l-m, \ldots, 1)\|^2 \\ &\quad - \|(t, t-1, t-2, \ldots, t-m+1;\ l-m, \ldots, 1)\|^2 \\ &= 4t.\end{aligned}$$

This is negative for $t < 0$, and hence the form is not semidefinite for C_l when $t < 0$.

3 Tools for unitarity

Cohomological induction is defined relative to a θ stable parabolic subalgebra $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$ in $\mathfrak{g}$, $\mathfrak{u}'$ being the nilpotent radical. We use primes here to distinguish $\mathfrak{q}'$ from the subalgebra $\mathfrak{q}$ in Section 1. For a general development of cohomological induction, see [KnV]. For a summary of some important properties, with references, see Section 3 of [Kn3].

Part of the definition of θ stable parabolic is that $\mathfrak{q}'$ has a Levi factor $\mathfrak{l}'$ that is the complexification of $\mathfrak{l}'_0 = \mathfrak{l}' \cap \mathfrak{g}_0$. For our G, if L' denotes the corresponding analytic subgroup, then L' is closed connected reductive, and $L' \cap K$ is maximal compact in it. Let $\bar{\mathfrak{q}}' = \mathfrak{l}' \oplus \bar{\mathfrak{u}}'$ be the opposite parabolic, which is the complex conjugate of $\mathfrak{q}'$.

Let $S' = \dim(\mathfrak{u}' \cap \mathfrak{k})$. Cohomological induction carries $(\mathfrak{l}', L' \cap K)$-modules into $(\mathfrak{g}, K)$-modules. If Z is an $(\mathfrak{l}', L' \cap K)$-module, define

$$(\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l}',\mathfrak{u}'})_j(Z) = (\Pi^{\mathfrak{g},K}_{\mathfrak{g},L'\cap K})_j(\operatorname{ind}^{\mathfrak{g},L'\cap K}_{\bar{\mathfrak{q}}',L'\cap K}(Z \otimes \textstyle\bigwedge^{\mathrm{top}} \mathfrak{u}')).$$

Here $\Pi^{\mathfrak{g},K}_{\mathfrak{g},L'\cap K}$ is the Bernstein functor given by tensoring over $R(\mathfrak{g}, L' \cap K)$ with $R(\mathfrak{g}, K)$, the index j refers to the j^{th} derived functor, and ind refers to the tensor product over $U(\bar{\mathfrak{q}}')$ with $U(\mathfrak{g})$.

In our situation, $\mathfrak{l}'_0$ will contain the compact Cartan subalgebra $\mathfrak{b}_0$ that we started with, and the positive system $\Delta^+(\mathfrak{g})$ of roots will not change. Since $\mathfrak{b} \subseteq \mathfrak{l}'$, the sets $\Delta^+(\mathfrak{l}')$ and $\Delta(\mathfrak{u}')$ are well defined.

The Vogan Unitarizability Theorem consists of several results that appeared originally in [Vo]. Wallach [Wa2] obtained a simplified proof, which is what appears in [KnV]. The theorem is applicable to infinite-dimensional $(\mathfrak{l}', L' \cap K)$ modules, but we abstract from the exposition [Kn3] only the results that we need; these concern only finite-dimensional $(\mathfrak{l}', L' \cap K)$-modules.

Theorem 3.1 (Vogan Unitarizability Theorem). *Let F_ν be an infinitesimally unitary irreducible finite-dimensional $(\mathfrak{l}', L' \cap K)$-module with highest weight ν.*

(a) *If ν is in the weakly-good range, i.e., $\langle \nu + \delta, \gamma \rangle \geq 0$ for all $\gamma \in \Delta(\mathfrak{u}')$, then $(\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l}',\mathfrak{u}'})_{S'}(F_\nu)$ is infinitesimally unitary.*

(b) *If F_ν is 1-dimensional and ν is in the weakly-fair range, i.e., if*

$$\langle \nu + \delta(\mathfrak{u}'), \gamma \rangle \geq 0$$

for all $\gamma \in \Delta(\mathfrak{u}')$, then $(\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l}',\mathfrak{u}'})_{S'}(F_\nu)$ is infinitesimally unitary.

(c) *Suppose that F_ν is 1-dimensional and that, for each number $c \geq 0$, the only $\Delta^+(\mathfrak{l}')$ dominant sum μ of members of $\Delta(\mathfrak{u}')$ such that*

$$\nu + \mu + \delta + c\,\delta(\mathfrak{u}') = w(\nu + \delta + c\,\delta(\mathfrak{u}'))$$

for some w in the Weyl group $W(\mathfrak{g}, \mathfrak{b})$ is $\mu = 0$. Then $(\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l}',\mathfrak{u}'})_{S'}(F_\nu)$ is infinitesimally unitary.

The interest is in $\nu = \lambda + \omega$ with λ and ω as in the paragraphs before Theorem 1.1. When $\mathfrak{q}'$ equals the Borel–de Siebenthal parabolic subalgebra $\mathfrak{q}$ of Section 1, conclusion (a) gives the unitarity of $\pi(\lambda + \omega)$ when $\lambda + \omega + \delta$ is dominant. Dominance in this situation implies that $\pi(\lambda + \omega)$ is a discrete series or limit of discrete series representation, and the conclusion of unitarity tells us nothing new. Still with $\mathfrak{q}' = \mathfrak{q}$, conclusion (b) applies only when $\omega = 0$. Conclusion (b) does say in this special case that unitarity persists while $\lambda + \delta(\mathfrak{u})$ is dominant. This amount of unitarity is nearly what [EPWW] detects, as will be pointed out in Section 6a, but it is not close to what Theorem 1.1 asserts.

The approach in the present paper is to map $(\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l},\mathfrak{u}})_S(F_\omega \otimes \mathbb{C}_\lambda)$ into a certain $(\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l}',\mathfrak{u}'})_{S'}(F_{\nu'})$ with $F_{\nu'}$ 1-dimensional, so that conclusions (b) and (c) are available, but with ν and $\mathfrak{q}$ replaced by ν' and $\mathfrak{q}'$. Conclusion (b) will be enough to prove Theorem 1.1 when $\omega = 0$, but we need conclusion (c) for general ω. Unfortunately conclusion (c) is difficult to verify. The theorem that produces the desired mapping is Theorem 5.1 of [Kn3], whose notation we change so that the theorem reads as below. In the statement, $\mathfrak{q}$ will be allowed to be fairly arbitrary, but in our application it will be the Borel–de Siebenthal parabolic of Theorem 1.1.

Theorem 3.2. *Fix a compact Cartan subalgebra $\mathfrak{b}_0$ of $\mathfrak{g}_0$ and a system $\Delta^+(\mathfrak{g})$ of positive roots relative to $\mathfrak{b}$. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ and $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$ be two θ stable parabolic subalgebras of $\mathfrak{g}$ containing $\mathfrak{b}$ and compatible with $\Delta^+(\mathfrak{g})$. Suppose that L is compact, $\mathfrak{l}' \cap \mathfrak{k}$ is contained in $\mathfrak{l}$, and $L'/(L' \cap K)$ is Hermitian symmetric in a fashion compatible with $\Delta^+(\mathfrak{l}')$. Let $\nu \in \mathfrak{b}^*$ be the highest weight of an irreducible finite-dimensional representation F_ν of L such that $\Lambda = \nu + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is $\Delta^+(\mathfrak{k})$ dominant, and put*

$$\nu' = \nu + 2\delta(\mathfrak{l}' \cap \mathfrak{p}).$$

Suppose that ν' is orthogonal to the roots of $\mathfrak{l}'$. Then

(a) *there exists a 1-dimensional $(\mathfrak{l}', L' \cap K)$ module $\mathbb{C}_{\nu'}$ with unique weight ν',*

(b) *the K type with highest weight Λ has multiplicity 1 in both $(\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l},\mathfrak{u}})_S(F_\nu)$ and $(\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l}',\mathfrak{u}'})_S(\mathbb{C}_{\nu'})$, and*

(c) *there exists a* $(\mathfrak{g}, K)$ *map*

$$(\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l},\mathfrak{u}})_S(F_\nu) \to (\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l}',\mathfrak{u}'})_{S'}(\mathbb{C}_{\nu'})$$

that is one-one on the K type with highest weight Λ.

Consequently if $(\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l}',\mathfrak{u}'})_{S'}(\mathbb{C}_{\nu'})$ *is infinitesimally unitary, then the unique irreducible subquotient of* $(\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l},\mathfrak{u}})_S(F_\nu)$ *having the K type with highest weight* Λ *is infinitesimally unitary.*

4 Combinatorial tools

In this section we prove three propositions that allow one sometimes to verify the combinatorial hypothesis in Theorem 3.1c. The main result is Proposition 4.8, which we shall use in Section 5 to prove Theorem 1.1. Proposition 4.8 ultimately reduces to a special case that is proved separately as Proposition 4.6. In turn the proof of Proposition 4.6 reduces to a version of Proposition 4.8 given in Proposition 4.3 that can be used to prove unitarity for the analytic continuation of holomorphic discrete series in $\mathfrak{su}(m, l-m)$. We shall use Proposition 4.3 in Section 6h to correct a proof in [Kn3].

Throughout this section, r, s, and l denote integers with $1 \le r < s \le l$. We work in $\mathbb{R}^l$, and $e_1, \ldots, e_l$ denotes the standard basis. The i^{th} entry of a member $\nu = (\nu_1, \ldots, \nu_l)$ of $\mathbb{R}^l$ may be written as $\nu_i = \langle \nu, e_i \rangle$. A permutation p on $\{1, \ldots, l\}$ acts on the basis vectors e_i by $pe_i = e_{p(i)}$, and this action extends linearly to members ν of $\mathbb{R}^l$ in such a way that $\langle p\nu, e_i\rangle = \langle \nu, p^{-1}e_i\rangle = \langle \nu, e_{p^{-1}i}\rangle$.

Lemma 4.1. *If* $\mu = (\mu_1, \ldots, \mu_l)$ *is a sum of expressions* $e_i - e_j$ *with* $i < j$ *and if* $\mu_k = 0$ *for some* k, *then* μ *is a sum of expressions* $e_i - e_j$ *in which* $i < j$ *and neither* i *nor* j *is equal to* k.

Proof. Let the expressions $e_i - e_j$ with $i < j$ that contribute to μ and involve entry k, including repetitions, be the following: $e_{i_1} - e_k, \ldots, e_{i_u} - e_k$ with $i_1, \ldots, i_u$ less than k, and $e_k - e_{j_1}, \ldots, e_k - e_{j_v}$ with $j_1, \ldots, j_v$ greater than k. The entry μ_k is then $-u + v$, and we have assumed μ_k to be 0. Thus $u = v$. We can therefore rewrite the contribution to μ from these expressions as coming from $e_{i_1} - e_{j_1}, \ldots, e_{i_u} - e_{j_u}$, and index k has been eliminated.

Lemma 4.2. *Let* $\sigma = \{a_1, \ldots, a_l\}$ *be a member of* $\mathbb{R}^l$ *such that* $i \mapsto a_i$ *is at most two-to-one. Fix a permutation* p, *and have* p *act on* $\mathbb{R}^l$. *Then there exists a permutation* τ *of order* 2 *such that* $\tau\sigma = \sigma$ *and such that* $p' = p\tau$ *has the following property: whenever* $\langle \sigma, e_i\rangle = \langle \sigma, p'^{-1}e_i\rangle$, *then* p' *fixes* i.

Proof. If i and j both map to x in $\mathbb{R}$ under $i \mapsto a_i$ and if $i \ne j$, we introduce the transposition $\tau_{i,j}$ of i and j. The various transpositions defined in this way commute, and each has $\tau_{i,j}\sigma = \sigma$. Now suppose that $\langle \sigma, e_i\rangle = \langle \sigma, p^{-1}e_i\rangle$. Write $p^{-1}e_i = e_j$. If

$i \neq j$, we shall adjust p by $\tau_{i,j}$. Namely let τ be the product, without repetitions, of all the transpositions $\tau_{i,j}$ such that $\langle\sigma, e_i\rangle = \langle\sigma, e_j\rangle$ and $p^{-1}e_i = e_j$. Then $\tau\sigma = \sigma$, and we define $p' = p\tau$. Suppose that $\langle\sigma, e_i\rangle = \langle\sigma, p'^{-1}e_i\rangle$. Then $\langle\sigma, e_i\rangle = \langle\sigma, p^{-1}e_i\rangle$ because $\langle\sigma, p'^{-1}e_i\rangle = \langle p\tau\sigma, e_i\rangle = \langle p\sigma, e_i\rangle = \langle\sigma, p^{-1}e_i\rangle$. Therefore $\tau_{i,p^{-1}j}$ is defined, and there are two cases. One is that $p^{-1}e_i = e_i$, and then τ fixes e_i; in this case, $p'^{-1}e_i = \tau^{-1}p^{-1}e_i = \tau^{-1}e_i = e_i$. The other is that $p^{-1}e_i$ equals some e_j not equal to e_i, and then $\tau e_i = e_j$; in this case $p'^{-1}e_i = \tau^{-1}p^{-1}e_i = \tau^{-1}e_j = e_i$. This proves the lemma.

Proposition 4.3. *Let $\sigma = (a_1, \ldots, a_{r-1}, a_r, \ldots, a_s, a_{s+1}, \ldots, a_l)$ be a tuple of real numbers such that, to the extent the terms are defined, the following conditions hold:*

(A1) $a_1 > \cdots > a_{r-1} > a_{s+1} > \cdots > a_l$,

(A2) $a_r > \cdots > a_s$,

(A3) $a_{r-1} > a_s$,

(A4) $a_r > a_{s+1}$.

Suppose that the equation

$$\sigma + \mu = w\sigma \tag{4.4}$$

is satisfied in such a way that

(B1) *$\mu = (\mu_1, \ldots, \mu_l)$ is in $\mathbb{R}^l$ with entries satisfying $\mu_r \geq \cdots \geq \mu_s$,*

(B2) *μ is a sum of expressions $e_i - e_j$ with $i < j$,*

(B3) *w is a permutation on $\{1, \ldots, l\}$.*

Then $\mu = 0$.

Proof. It follows from (A1) and (A2) that the map $i \mapsto a_i$ is at most two-to-one. Let us apply Lemma 4.2 to the permutation w. The lemma produces a permutation w' such that $w'\sigma = w\sigma$ and such that whenever σ and $w'\sigma$ agree in the i^{th} entry, then $w'e_i = e_i$. Changing notation, we may assume from the outset that whenever σ and $w\sigma$ agree in the i^{th} entry, then $we_i = e_i$.

Let us prove that $\mu_i = 0$ for all indices i for which $a_i > a_r$. By (A1), (A2), and (A4), all such indices i have $i \leq r - 1$. If there are any such indices, let u be the largest one; otherwise let $u = 0$. We are to show that $\mu_1 = \cdots = \mu_u = 0$. Assuming the contrary, let $i \leq u$ be the smallest index with $\mu_i \neq 0$. By (B2) this smallest index has $\mu_i > 0$ if $i \neq 0$. By (A1), (A2), and (A4), the largest i entries of σ are $a_1, \ldots, a_i$. Then the largest sum of i entries of σ is $a_1 + \cdots + a_i$, and the sum $a_1 + \cdots + a_{i-1} + (a_i + \mu_i)$ exceeds the largest sum of i entries of σ. But (4.4) and (B3) say that $a_1 + \cdots + a_{i-1} + (a_i + \mu_i)$ must match a certain sum of i entries of σ, and we have a contradiction. We conclude that there is no such i and that therefore $\mu_1 = \cdots = \mu_u = 0$.

Similarly, use of (A1), (A2), and (A3) shows that $\mu_i = 0$ for all indices i for which $a_s > a_i$. All of these indices have $i \geq s + 1$. If there are any such indices, let v be the largest one; otherwise let $v = l+1$. Then the same kind of argument as in the previous paragraph yields $\mu_v = \cdots = \mu_l = 0$.

Thus (4.4) shows that $\langle\sigma, e_i\rangle = \langle w\sigma, e_i\rangle = \langle\sigma, w^{-1}e_i\rangle$ for $i \le u$ and $i \ge v$. Since the fibers of $i \mapsto a_i$ over values $> a_r$ and $< a_s$ have just one element, $we_i = e_i$ for $i \le u$ and $i \ge v$. It therefore makes sense to regard w as a permutation of $\{u+1, \ldots, v-1\}$ and to restrict (4.4) to the corresponding entries. Inductive use of Lemma 4.1 shows that μ can be regarded as a sum of expressions $e_i - e_j$ in which $u+1 \le i < j \le v-1$.

Changing notation, we may assume from the outset that (A1) through (A4) hold and also that the inequalities $a_i \le a_r$ and $a_i \ge a_s$ hold for all i. From (4.4) and (B3), we must have $\mu_r \le 0$ and $\mu_s \ge 0$. By (B1), we conclude that $\mu_r = \cdots = \mu_s = 0$.

Then (4.4) shows that $\langle\sigma, e_i\rangle = \langle w\sigma, e_i\rangle = \langle\sigma, w^{-1}e_i\rangle$ for $r \le i \le s$. By the normalization of w in the first paragraph of the proof, $we_i = e_i$ for $r \le i \le s$. It therefore makes sense to regard w as a permutation of $\{1, \ldots, r-1, s+1, \ldots, l\}$ and to restrict (4.4) to the corresponding indices. Inductive use of Lemma 4.1 shows that μ can be regarded as a sum of expressions $e_i - e_j$ in which $i < j$ and the indices i and j are in $\{1, \ldots, r-1, s+1, \ldots, l\}$.

On this set of indices, σ is dominant in the traditional sense, and hence $w\sigma = \sigma - \nu$, where ν is a nonnegative linear combination of positive roots. We therefore obtain $\sigma + \mu = w\sigma = \sigma - \nu$, from which we see that $\mu + \nu = 0$. Since μ and ν are each equal to nonnegative linear combinations of positive roots, μ and ν are 0. This completes the proof.

Lemma 4.5. *If $\mu = (\mu_1, \ldots, \mu_l)$ is a sum of expressions $e_i - e_j$ with $i < j$ and expressions e_t with $1 \le t \le l$, and if $\mu_k = 0$ for some k, then μ is a sum of expressions $e_i - e_j$ and e_t in which $i < j$ and none of i, j, and t is equal to k.*

Proof. Let the expressions $e_i - e_j$ and e_t that contribute to μ and involve entry k, including repetitions, be the following: $e_{i_1} - e_k, \ldots, e_{i_u} - e_k$ with $i_1, \ldots, i_u$ less than k; $e_k - e_{j_1}, \ldots, e_k - e_{j_v}$ with $j_1, \ldots, j_v$ greater than k; and $e_{t_1}, \ldots, e_{t_n}$ with $t_1, \ldots, t_n$ all equal to k. The entry μ_k is then $-u + v + n$, and we have assumed μ_k to be 0. Thus $u = v + n$. We can therefore rewrite the contribution to μ from these expressions as coming from $e_{i_1} - e_{j_1}, \ldots, e_{i_v} - e_{j_v}$ and $e_{i_{v+1}}, \ldots, e_{i_{v+n}}$, and index k has been eliminated.

Proposition 4.6. *Let $\sigma = (a_1, \ldots, a_{r-1}, a_r, \ldots, a_s, a_{s+1}, \ldots, a_l)$ be a tuple of real numbers such that, to the extent the terms are defined, the following conditions hold: (A1), (A2), (A3), (A4),*

(A5) $a_l \ge 0$ *if* $l > s$, *or* $a_{r-1} \ge 0$ *if* $l = s$,

(A6) $a_s \ge 0$.

Suppose that the equation

$$\sigma + \mu = w\sigma \tag{4.7}$$

is satisfied in such a way that (B1) holds and also

(B2′) *μ is a sum of expressions $e_i - e_j$ with $i < j$ and expressions e_k,*

(B3′) *w is in the group generated by permutations and sign changes of the standard basis $e_1, \dots, e_l$ (the Weyl group of the system B_l).*

Then $\mu = 0$.

Proof. Write $w = qp$ with q a sign-change element and p a permutation. Summing the entries of (4.7) and writing $q(e_i)$ for the effect ± 1 of q on e_i, we have

$$\begin{aligned}\sum \langle \sigma, e_i \rangle + \sum \langle \mu, e_i \rangle &= \sum \langle w\sigma, e_i \rangle = \sum \langle qp\sigma, e_i \rangle = \sum \langle p\sigma, qe_i \rangle \\ &= \sum q(e_i)\langle p\sigma, e_i \rangle \le \sum \langle p\sigma, e_i \rangle = \sum \langle \sigma, e_i \rangle,\end{aligned}$$

with the inequality holding since $q(e_i) = \pm 1$ and $\langle p\sigma, e_i \rangle \ge 0$. Thus $\sum \langle \mu, e_i \rangle \le 0$. Since each term of μ, by (B2′), has sum of entries ≥ 0, we obtain $\sum \langle \mu, e_i \rangle = 0$. Then it follows from (B2′) that all the terms contributing to μ are of the form $e_i - e_j$.

Returning to the displayed inequality and substituting $\sum \langle \mu, e_i \rangle = 0$, we see that the end terms are equal. Since $q(e_i)\langle p\sigma, e_i \rangle \le \langle p\sigma, e_i \rangle$ for all i, we must have equality for all i. Consequently $\langle qp\sigma, e_i \rangle = \langle p\sigma, e_i \rangle$ for all i, and we have $qp\sigma = p\sigma$.

In other words, q plays no role in (4.7), and we may replace w by the permutation p. Taking into account Lemma 4.5, we are now reduced to the situation in Proposition 4.3, and thus Proposition 4.6 follows from Proposition 4.3.

Proposition 4.8. *Let $\sigma = (a_1, \dots, a_{r-1}, a_r, \dots, a_s, a_{s+1}, \dots, a_l)$ be a tuple of real numbers such that, to the extent the terms are defined, the following conditions hold: (A1), (A2), (A3), (A4), (A5), and*

(A6′) $a_r \ge |a_s|$.

Suppose that the equation

$$\sigma + \mu = w\sigma \tag{4.9}$$

is satisfied in such a way that (B1), (B2′), and (B3′) are valid. Then $\mu = 0$.

Remark. Hypotheses (A2) and (A6) together imply (A6′), and therefore Proposition 4.6 is a special case of Proposition 4.8.

Preliminary reduction. We can always discard indices i for which $|a_i| > a_r$ and then renumber accordingly. Let us see this. Write $w = qp$ with q a sign-change element and p a permutation, and write $q(e_i)$ for the effect ± 1 of q on e_i. The hypotheses force all indices i with $|a_i| > a_r$ to have $i \le r - 1$. If there are in fact any such indices, let u be the largest one; otherwise let $u = 0$.

We show first that $\mu_1 = \dots \mu_u = 0$. Assuming the contrary, let $i \le u$ be the smallest index with $\mu_i \ne 0$. This smallest index has $\mu_i > 0$. By the hypotheses the largest i entries of σ *in absolute value* are $a_1, \dots, a_i$. Then the largest sum of absolute values of i entries of σ is $a_1 + \dots + a_i$, and the sum $a_1 + \dots + a_i + (a_i + \mu_i)$ exceeds this. But (4.9) and (B3′) say that $a_1 + \dots + a_i + (a_i + \mu_i)$ must match some sum of

i entries of σ, each adjusted by a sign, and we have a contradiction. We conclude that $\mu_1 = \cdots = \mu_u = 0$.

For $1 \le i \le u$, (4.9) then gives $\langle w\sigma, e_i\rangle = \langle \sigma, e_i\rangle$, so that $q(e_i)\langle \sigma, e_{p^{-1}i}\rangle = \langle \sigma, e_i\rangle$. Since any entry of σ larger in magnitude than a_r is positive, we see that $q(e_i) = +1$ and hence that $\langle \sigma, e_{p^{-1}i}\rangle = \langle \sigma, e_i\rangle$. Since the fibers of $i \mapsto a_i$ over values $> a_r$ have just one element, $p^{-1}i = i$. Thus w fixes indices $1, \ldots, u$. Taking Lemma 4.5 into account, we can discard indices $1, \ldots, u$ from the given data, and we can renumber accordingly.

Proof of Proposition 4.8 in the special case $\mu_r = \cdots = \mu_s = 0$. We shall do an induction on the number of indices i with $r \le i \le s$ such that $a_i < 0$. The base case for the induction is that the number of such indices is 0, and this case follows from Proposition 4.6.

Reduction of the number of indices from 1 to 0 involves a problem with hypothesis (A3), which says that $a_{r-1} > a_s$, since a_s will increase to something ≥ 0 at this stage. From (A1) and (A5) we know that $a_{r-1} \ge 0$, but this inequality does not ensure that $a_{r-1} > a_s$ once a_s gets to be ≥ 0. Here is one way that this problem can be handled. First suppose that $a_{r-1} > 0$. If there is already an index i_0 between r and s with $a_{i_0} = 0$, then (A2) shows that a_s will be 0 once it gets to be ≥ 0, and hence (A3) presents no problem when a_s gets to be ≥ 0. If there is no such index i_0, we insert one in its proper position, defining $a_{i_0} = 0$, $\mu_{i_0} = 0$, and $we_{i_0} = e_{i_0}$. Again (A3) presents no problem when a_s gets to be ≥ 0. Now suppose that $a_{r-1} = 0$. In this case, (A1) and (A5) force $s = l$. The thing to do in this case is to move index $r - 1$ into position $s + 1 = l + 1$, afterward renumbering the indices. This move is valid when $\mu_r = \cdots = \mu_s = 0$, though not in general, and the effect is to reduce matters to the situation where $a_{r-1} > 0$. Verifying that this move is valid involves verifying that (A3) plays no role in the proof below for the special case $\mu_r = \cdots = \mu_s = 0$.

We turn to the general case of our induction. Let w be given, and write $w = qp$, with q a sign-change element and p a permutation. Write $q(e_i)$ for the effect ± 1 of q on e_i. Applying Lemma 4.2 to p, we may assume that

$$\langle \sigma, e_i\rangle = \langle \sigma, e_{p^{-1}i}\rangle \quad \text{implies} \quad pi = i. \tag{1}$$

For each i with $\mu_i = 0$, including those i's with $r \le i \le s$, we have

$$\langle \sigma, e_i\rangle = \langle \sigma, e_i\rangle + \langle \mu_i, e_i\rangle = \langle w\sigma, e_i\rangle = \langle qp\sigma, e_i\rangle = q(e_i)\langle \sigma, e_{p^{-1}i}\rangle. \tag{2}$$

If $q(e_i) = +1$, then (1) shows that $pi = i$. Then $we_i = e_i$. Since $\mu_i = 0$, we can drop index i if we want to, in view of Lemma 4.5. We choose to do so for all the cases where $q(e_i) = +1$ and $\langle \sigma, e_i\rangle < 0$, and we adjust notation accordingly. The hypotheses of Proposition 4.8 and of our special case remain valid, and therefore any such dropping of indices completes our induction. We may therefore assume that $q(e_i) = -1$ for every index i with $\langle \sigma, e_i\rangle < 0$ and that there is such an index.

Let s_i be the sign change transformation in the i^{th} entry, defined by $s_i(e_i) = -e_i$ and $s_i(e_j) = e_j$ for $j \ne i$. Suppose i is one of the indices with $r \le i \le s$ and $\langle \sigma, e_i\rangle < 0$, so that $q(e_i) = -1$ and $p^{-1}i \neq i$. Define $\tau = s_i s_{p^{-1}i} \tau_{i,p^{-1}i}$, where $\tau_{i,p^{-1}i}$ is the transposition of i and $p^{-1}i$, and consider τw. This satisfies

$$\tau w\sigma - \sigma = (\tau w\sigma - w\sigma) + (w\sigma - \sigma) = (\tau w\sigma - w\sigma) + \mu.$$

Since $\tau^{-1}e_j = e_j$ for all j other than i and $p^{-1}i$, $\langle \tau w\sigma - w\sigma, e_j\rangle = 0$ for all j except possibly i and $p^{-1}i$. For these indices we have

$$\begin{aligned}\langle \tau w\sigma - w\sigma, e_i\rangle &= \langle w\sigma, -e_{p^{-1}i}\rangle - \langle w\sigma, e_i\rangle \\ &= -(\langle\sigma, e_{p^{-1}i}\rangle + \langle\mu, e_{p^{-1}i}\rangle) + \langle\sigma, e_{p^{-1}i}\rangle = -\langle\mu, e_{p^{-1}i}\rangle\end{aligned}$$

and, in similar fashion,

$$\langle \tau w\sigma - w\sigma, e_{p^{-1}i}\rangle = \langle w\sigma, -e_i\rangle - \langle w\sigma, e_{p^{-1}i}\rangle = -\langle\mu, e_{p^{-1}i}\rangle.$$

This proves:

If i has $r \le i \le s$ and $\langle\sigma, e_i\rangle < 0$, and if $\langle\mu, e_{p^{-1}i}\rangle = 0$, then $\sigma + \mu = \tau w\sigma$. (3)

In other words, (4.9) remains valid when w is replaced by τw. The expressions σ and μ have not changed, and the hypotheses of Proposition 4.8 and the special case are still satisfied.

Let us see in the situation of (3) that

$$q(e_{p^{-1}i}) = -1 \quad\text{and}\quad p^{-1}p^{-1}i = i. \tag{4}$$

In fact, in the situation of (3), we have

$$\begin{aligned}\langle\sigma, e_i\rangle &= \langle\sigma, e_i\rangle + \langle\mu, e_i\rangle = \langle\tau w\sigma, e_i\rangle \\ &= \langle w\sigma, \tau^{-1}e_i\rangle = -\langle w\sigma, e_{p^{-1}i}\rangle = -q(e_{p^{-1}i})\langle\sigma, e_{p^{-1}p^{-1}i}\rangle.\end{aligned} \tag{5}$$

If, instead of (4), we have $q(e_{p^{-1}i}) = +1$, then we have

$$\begin{aligned}\langle\sigma, e_{p^{-1}i}\rangle &= \langle\sigma, e_{p^{-1}i}\rangle + \langle\mu, e_{p^{-1}i}\rangle = \langle w\sigma, e_{p^{-1}i}\rangle \\ &= q(e_{p^{-1}i})\langle\sigma, e_{p^{-1}p^{-1}i}\rangle = \langle\sigma, e_{p^{-1}p^{-1}i}\rangle.\end{aligned}$$

This says that $\langle\sigma, e_j\rangle = \langle\sigma, e_{p^{-1}j}\rangle$ for $j = p^{-1}i$, and (1) allows us to conclude that $p^{-1}p^{-1}i = p^{-1}i$, hence that $p^{-1}i = i$. But (2) shows that $pi = i$ is impossible when $\langle\sigma, e_i\rangle < 0$. We have arrived at a contradiction, and hence $q(e_{p^{-1}i}) = -1$. Substituting into (5), we see that $\langle\sigma, e_i\rangle = \langle\sigma, e_{p^{-1}p^{-1}i}\rangle$. Since $\langle\sigma, e_i\rangle < 0$ and since $j \mapsto \langle\sigma, e_j\rangle$ can take on a given negative value only once, $p^{-1}p^{-1}i = i$. This proves (4).

Continuing with the situation in (3), write $\tau w = q'p'$ with q' a sign-change element and p' a permutation. Since the map of {sign changes}{permutations} to the second factor is a homomorphism, $p' = \tau_{i,p^{-1}i}\, p$. Therefore (4) yields

$$\langle\sigma, e_{p'^{-1}i}\rangle = \langle\sigma, e_{p^{-1}\tau_{i,p^{-1}i}(i)}\rangle = \langle\sigma, e_{p^{-1}p^{-1}i}\rangle = \langle\sigma, e_i\rangle$$

Since $\langle \sigma, e_i \rangle < 0$ and since $j \mapsto \langle \sigma, e_j \rangle$ takes on a given negative value only once, $p'^{-1}i = i$. Thus if we use τw in place of w, we can eliminate index i and reduce to a more primitive case in the induction. The upshot is as follows:

> If i has $r \leq i \leq s$ and $\langle \sigma, e_i \rangle < 0$, and if $\langle \mu, e_{p^{-1}i} \rangle = 0$, then use of τw in place of w allows index i to be eliminated, and the case is therefore handled by induction. (6)

Observe that the interpretation of (1) will be different at the next stage of the induction since w gets replaced by $w\tau$, but this change is not a problem since the above proof arranges for (1) to hold only after the stage of the induction is fixed.

Now, with a given positive number of indices i between r and s such that $\langle \sigma, e_i \rangle < 0$ and $q(e_i) = -1$, we do a further induction. Namely we induct on the number of indices outside the interval from r to s. If there are no such indices, then $\mu = 0$ and we are done. If there are some, then the smallest such index will be 1 or $s+1$; we write 1 for it in any event, to simplify the notation, and its position relative to r and s will make no difference.

Since 1 is the first index for which μ_i could conceivably be nonzero, hypothesis (B2′) shows that

$$\mu_1 \geq 0. \tag{7}$$

Formula (4.9) gives

$$\mu_1 = \langle w\sigma, e_1 \rangle - \langle \sigma, e_1 \rangle = q(e_1)\langle \sigma, e_{p^{-1}1} \rangle - \langle \sigma, e_1 \rangle. \tag{8}$$

Here $\langle \sigma, e_1 \rangle \geq 0$ by (A1) and (A5). In order to carry out the inductive step, we distinguish two cases.

The first case is that $p^{-1}1$ is outside the interval from r to s. If $p^{-1}1 > 1$, then hypotheses (A1) and (A5) show that $0 \leq \langle \sigma, e_{p^{-1}1} \rangle < \langle \sigma, e_1 \rangle$. It follows from (8) that $\mu_1 < 0$, in contradiction to (7). We conclude that $p^{-1}1$ can be outside the interval from r to s only if $p^{-1}1 = 1$. In this case,

$$\mu_1 = (q(e_1) - 1)\langle \sigma, e_1 \rangle. \tag{9}$$

If $\langle \sigma, e_1 \rangle > 0$, then $q(e_1) = -1$ would force $\mu_1 < 0$, in contradiction to (7). Thus $\langle \sigma, e_1 \rangle > 0$ implies $q(e_1) = +1$, and we obtain $w(e_1) = e_1$. Since (9) yields $\mu_1 = 0$, index 1 can be dropped, in view of Lemma 4.5. In other words, we are done by induction when $p^{-1}1 = 1$ and $\langle \sigma, e_1 \rangle > 0$. If, on the other hand, $\langle \sigma, e_1 \rangle$ is 0, then (4.9) remains valid when we adjust w to make $q(e_1) = -1$. Because of (9) this adjustment makes $w(e_1) = e_1$ and $\mu_1 = 0$, and index 1 can be dropped. In other words, we are done by induction when $p^{-1}1 = 1$ and $\langle \sigma, e_1 \rangle = 0$. This completes our discussion of the case that $p^{-1}1$ is outside the interval from r to s.

The remaining case is that $p^{-1}1$ is inside the interval from r to s. To have $\mu_1 \geq 0$, we must be in one of the following situations, since (8) holds and we know that

$\langle\sigma, e_1\rangle \geq 0$:

$$\langle\sigma, e_1\rangle = \langle\sigma, e_{p^{-1}1}\rangle = 0, \tag{10}$$

$$\langle\sigma, e_1\rangle \geq 0, \quad \langle\sigma, e_{p^{-1}1}\rangle > 0, \quad \text{and} \quad q(e_1) = +1, \tag{11}$$

$$\langle\sigma, e_1\rangle \geq 0, \quad \langle\sigma, e_{p^{-1}1}\rangle < 0, \quad \text{and} \quad q(e_1) = -1. \tag{12}$$

Situation (10), by (1), forces $p^{-1}1 = 1$, a contradiction to the fact that $p^{-1}1$ is inside the interval from r to s.

For situations (11) and (12), application of (2) with $i = p^{-1}1$ gives

$$\langle\sigma, e_{p^{-1}1}\rangle = q(e_{p^{-1}1})\langle\sigma, e_{p^{-1}p^{-1}1}\rangle. \tag{13}$$

If $q(e_{p^{-1}1}) = +1$, then (1) allows us to conclude that $p^{-1}1 = p^{-1}p^{-1}1$ and hence $p^{-1}1 = 1$, contradiction. So $q(e_{p^{-1}1}) = -1$ in both (11) and (12).

Let us now specialize to situation (11), with $\langle\sigma, e_{p^{-1}1}\rangle > 0$ and $q(e_1) = +1$. Then (13) gives $\langle\sigma, e_{p^{-1}p^{-1}1}\rangle < 0$, whence $p^{-1}p^{-1}1$ is between r and s. Application of (2) with $i = p^{-1}p^{-1}1$ gives

$$\langle\sigma, e_{p^{-1}p^{-1}1}\rangle = q(e_{p^{-1}p^{-1}1})\langle\sigma, e_{p^{-1}p^{-1}p^{-1}1}\rangle. \tag{14}$$

Referring to (6) with $i = p^{-1}p^{-1}1$, we see that if $p^{-1}p^{-1}p^{-1}1$ is between r and s, we can reduce to an earlier case of the outer induction, and we are done. So we may assume that $p^{-1}p^{-1}p^{-1}1$ is not between r and s. By (A1) and (A5) we have

$$0 \leq \langle\sigma, e_{p^{-1}p^{-1}p^{-1}1}\rangle \leq \langle\sigma, e_1\rangle.$$

But then (13) and (14) give

$$|\langle\sigma, e_{p^{-1}1}\rangle| = |\langle\sigma, e_{p^{-1}p^{-1}1}\rangle| = |\langle\sigma, e_{p^{-1}p^{-1}p^{-1}1}\rangle| \leq \langle\sigma, e_1\rangle.$$

If the inequality is strict here, then (8) yields $\mu_1 < 0$, contradiction. So equality must hold, and $\langle\sigma, e_{p^{-1}1}\rangle$, being > 0, must equal $\langle\sigma, e_1\rangle$. Then (1) yields $p^{-1}1 = 1$, a contradiction since $p^{-1}1$ is between r and s.

Thus either we have reduced matters to a previous stage of the induction, or we are in situation (12), with $\langle\sigma, e_{p^{-1}1}\rangle < 0$ and $q(e_1) = -1$. From $\langle\sigma, e_{p^{-1}1}\rangle < 0$, we know that $q(e_{p^{-1}1}) = -1$. Formula (13) is still valid, and we see that $\langle\sigma, e_{p^{-1}p^{-1}1}\rangle > 0$. Using (6) for index $p^{-1}1$, we see that if $p^{-1}p^{-1}1$ is between r and s, we can reduce to an earlier case of the outer induction, and we are done. So we may assume that $p^{-1}p^{-1}1$ is not between r and s. By (A1) and (A5) we have

$$0 \leq \langle\sigma, e_{p^{-1}p^{-1}1}\rangle \leq \langle\sigma, e_1\rangle.$$

But then (13) gives

$$|\langle\sigma, e_{p^{-1}1}\rangle| = |\langle\sigma, e_{p^{-1}p^{-1}1}\rangle| \leq \langle\sigma, e_1\rangle.$$

If the inequality is strict here, then (8) yields $\mu_1 < 0$, contradiction. So equality must hold, and $\langle \sigma, e_{p^{-1}p^{-1}1} \rangle$, being > 0, must equal $\langle \sigma, e_1 \rangle$. Since $j \mapsto \langle \sigma, e_j \rangle$ is one-one on the indices not between r and s, we obtain $p^{-1}p^{-1}1 = 1$. Also the equality in (8) forces $q(e_1) = -1$, since $\langle \sigma, e_{p^{-1}1} \rangle < 0$ for situation (12). Since $\mu_1 = 0$ and $p1 = p^{-1}1$, we can apply (6) with $i = p1$ to reduce to an earlier case of the outer induction.

Thus in all situations we can reduce to an earlier case of the induction. This completes the proof of the special case of Proposition 4.8 in which $\mu_r = \cdots = \mu_s = 0$.

Proof of Proposition 8 *in the general case.* Write $w = qp$ with q a sign-change element and p a permutation, and abbreviate the effect $q(e_i)$ of q on e_i as q_i. Applying Lemma 4.2 to p and changing notation, we can arrange that

$$\langle \sigma, e_i \rangle = \langle \sigma, e_{p^{-1}i} \rangle \quad \text{implies} \quad pi = i. \tag{15}$$

Next apply the "Preliminary reduction" that follows the statement of the proposition. This step allows us to eliminate certain indices and afterward to have

$$|\langle \sigma, e_i \rangle| \leq \langle \sigma, e_r \rangle \tag{16}$$

for all i.

Because of (16) the identity $\mu_r = \langle w\sigma, e_r \rangle - \langle \sigma, e_r \rangle$ obtained from (4.9) implies that

$$\mu_r \leq |\langle w\sigma, e_r \rangle| - \langle \sigma, e_r \rangle = |\langle \sigma, e_{p^{-1}r} \rangle| - \langle \sigma, e_r \rangle \leq 0.$$

If we can show that $\mu_s \geq 0$, then hypothesis (B1) implies that $\mu_r = \cdots = \mu_s = 0$, and we are reduced to the special case that has just been proved.

Arguing by contradiction, we suppose that $\mu_s < 0$. Then

$$\begin{aligned} \sum \mu_i &= \sum \langle w\sigma, e_i \rangle - \sum \langle \sigma, e_i \rangle = \sum_i q_i \langle \sigma, e_{p^{-1}i} \rangle - \sum \langle \sigma, e_i \rangle \\ &= \sum q_{p(i)} \langle \sigma, e_i \rangle - \sum \langle \sigma, e_i \rangle = \sum (q_{p(i)} - 1) \langle \sigma, e_i \rangle. \end{aligned} \tag{17}$$

The individual terms of the right side of (17) need not all be ≤ 0. However, we shall show that it is possible to group the terms into disjoint blocks in such a way that the sum of the terms in each block is ≤ 0 and that some block, because of the assumption $\mu_s < 0$, has sum < 0. This finding will give us a contradiction to the inequality $\sum \mu_i \geq 0$ that we have seen follows from (B2′), and the proof will be complete.

Write the permutation p as a product of disjoint cycles. Each block within the right side of (17) will consist of a subset of terms corresponding to consecutive elements of a single cycle. Every term for which $\langle \sigma, e_i \rangle \geq 0$ will be the start of a block, and the remaining terms of the block will have $\langle \sigma, e_i \rangle < 0$. In the case of a cycle that has no term with $\langle \sigma, e_i \rangle \geq 0$, the entire cycle is to form a block, and any term can be used to start the block; in this case we shall see that the sum of the terms for the block is 0.

Let us prove, by induction on n, the following formula, in which p carries i_0 to i_1, i_1 to $i_2, \ldots, i_{n-1}$ to i_n, and i_n to i_{n+1}. It is assumed that $i_0, \ldots, i_n$ are distinct, but i_{n+1} is allowed to equal i_0:

$$\begin{aligned}(q_{i_1}-1)\langle\sigma, e_{i_0}\rangle + (q_{i_2}-1)\langle\sigma, e_{i_1}\rangle + \cdots + (q_{i_{n+1}}-1)\langle\sigma, e_{i_n}\rangle \\ = \langle\sigma, e_{i_0}\rangle(q_{i_1}\cdots q_{i_{n+1}}-1) + (1-q_{i_2}\cdots q_{i_{n+1}})\langle\mu, e_{i_1}\rangle \\ +(1-q_{i_3}\cdots q_{i_{n+1}})\langle\mu, e_{i_2}\rangle + \cdots + (1-q_{i_{n+1}})\langle\mu, e_{i_n}\rangle.\end{aligned} \tag{18}$$

The base case of the induction is the case $n = 0$. In this case the formula in question is $(q_{i_1}-1)\langle\sigma, e_{i_0}\rangle = \langle\sigma, e_{i_0}\rangle(q_{i_1}-1)$ and is trivial. Assume the formula with n replaced by $k-1$, namely

$$\begin{aligned}(q_{i_1}-1)\langle\sigma, e_{i_0}\rangle + (q_{i_2}-1)\langle\sigma, e_{i_1}\rangle + \cdots + (q_{i_k}-1)\langle\sigma, e_{i_{k-1}}\rangle \\ = \langle\sigma, e_{i_0}\rangle(q_{i_1}\cdots q_{i_k}-1) + (1-q_{i_2}\cdots q_{i_k})\langle\mu, e_{i_1}\rangle \\ + (1-q_{i_3}\cdots q_{i_k})\langle\mu, e_{i_2}\rangle + \cdots + (1-q_{i_k})\langle\mu, e_{i_{k-1}}\rangle.\end{aligned}$$

We add $(q_{i_{k+1}}-1)\langle\sigma, e_{i_k}\rangle$ to both sides, and we see that we are show that

$$\begin{aligned}(q_{i_{k+1}}-1)\langle\sigma, e_{i_k}\rangle = \langle\sigma, e_{i_0}\rangle q_{i_1}\cdots q_{i_k}(q_{i_{k+1}}-1) + q_{i_2}\cdots q_{i_k}(1-q_{i_{k+1}})\langle\mu, e_{i_1}\rangle \\ + q_{i_3}\cdots q_{i_k}(1-q_{i_{k+1}})\langle\mu, e_{i_2}\rangle + \cdots + q_{i_k}(1-q_{i_{k+1}})\langle\mu, e_{i_{k-1}}\rangle \\ + (1-q_{i_{k+1}})\langle\mu, e_{i_k}\rangle.\end{aligned}$$

Thus we are to show that

$$\begin{aligned}\langle\sigma, e_{i_k}\rangle = q_{i_1}\cdots q_{i_k}\langle\sigma, e_{i_0}\rangle - q_{i_2}\cdots q_{i_k}\langle\mu, e_{i_1}\rangle \\ - q_{i_3}\cdots q_{i_k}\langle\mu, e_{i_2}\rangle - \cdots - q_{i_k}\langle\mu, e_{i_{k-1}}\rangle - \langle\mu, e_{i_k}\rangle.\end{aligned} \tag{19}$$

We do so by induction on k, the base case being $k = 0$. Assume (19) for $k = m-1$. We begin from

$$\langle\sigma, e_{i_m}\rangle + \langle\mu, e_{i_m}\rangle = \langle w\sigma, e_{i_m}\rangle = q_{i_m}\langle\sigma, e_{p^{-1}i_m}\rangle = q_{i_m}\langle\sigma, e_{i_{m-1}}\rangle.$$

We substitute for $\langle\sigma, e_{i_{m-1}}\rangle$ using (19) with $k = m-1$, and (19) follows for $k = m$. This completes the induction for (19), and the proof by induction of (18) is therefore complete.

Now let us consider an entire block of terms on the right side of (17), expanded as in (18). First assume that $\langle\sigma, e_{i_0}\rangle \geq 0$ and that $\langle\sigma, e_{i_j}\rangle < 0$ for $1 \leq j \leq n$. The indices i_j must have $r \leq i_j \leq s$ for $1 \leq j \leq n$, and therefore $\langle\mu, e_{i_j}\rangle \leq \langle\mu, e_r\rangle \leq 0$ by (B1). Consequently every term of the right side of (18) is ≤ 0, and (18) itself is ≤ 0.

This proves that every cycle containing some index i with $\langle\sigma, e_i\rangle \geq 0$ makes a total contribution ≤ 0 to $\sum \mu_j$. Let us see what must happen for a cycle to fail to have some index i with $\langle\sigma, e_i\rangle \geq 0$. Suppose that $(j_1 \quad j_2 \quad \cdots \quad j_n)$ is a cycle of p with $n \geq 1$, $p(j_n) = j_1$, and $\langle\sigma, e_{j_k}\rangle < 0$ for all k. In notation that takes k modulo n, we have

$$0 > \langle\sigma, e_{j_k}\rangle = \langle w\sigma, e_{j_k}\rangle - \langle\mu, e_{j_k}\rangle \geq \langle w\sigma, e_{j_k}\rangle = q_{j_k}\langle\sigma, e_{j_{k-1}}\rangle.$$

Since $\langle \sigma, e_{j_{k-1}} \rangle < 0$, we must have $q_{j_k} = +1$. Thus all signs q_{j_k} for the cycle must be $+1$. Expanding as in (18) the sum of terms of the right side of (17) for the entire cycle and letting any index in the cycle be i_0, we see that every term of the right side of (18) is 0. Thus a cycle in which every index i has $\langle \sigma, e_i \rangle < 0$ contributes 0 to $\sum \mu_j$.

This completes the proof of Proposition 4.8 except for the demonstration, under the assumption that $\langle \mu, e_s \rangle < 0$, that some block has a strictly negative sum. This will be the block that contains index s. Since $\langle \mu, e_s \rangle < 0$, we have

$$q_s \langle \sigma, e_{p^{-1}s} \rangle = \langle w\sigma, e_s \rangle = \langle \sigma, e_s \rangle + \langle \mu, e_s \rangle < \langle \sigma, e_s \rangle.$$

If $q_s = +1$, then $\langle \sigma, e_{p^{-1}s} \rangle < \langle \sigma, e_s \rangle$, in contradiction to the fact that the s^{th} entry of σ is the unique smallest if $a_s < 0$; if $a_s \geq 0$, Proposition 4.8 already follows from Proposition 4.6. Thus we may assume that $q_s = -1$, and we see as well that $\langle \sigma, e_{p^{-1}s} \rangle > 0$. Hence $p^{-1}s$ begins a block. Let us write out the terms of this block as on the right side of (18), abbreviating the product of signs $q_{i_2} \cdots q_{i_{n+1}}$ as $q_0 = \pm 1$. The sum for the block is

$$= \langle \sigma, e_{p^{-1}s} \rangle (q_s q_0 - 1) + (1 - q_0) \langle \mu, e_s \rangle + (\text{terms} \leq 0)$$

with each term ≤ 0. If $q_0 = -1$, the term $(1 - q_0)\langle \mu, e_s \rangle$ is < 0. If $q_0 = +1$, the term $\langle \sigma, e_{p^{-1}s} \rangle (q_s q_0 - 1)$ is < 0. In either case the sum for the block is < 0. This completes the proof of Proposition 4.8.

5 Unitarity

In this section we shall combine Proposition 4.8 with the theorems of Section 3 to prove Theorem 1.1. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the Borel–de Siebenthal parabolic subalgebra of $\mathfrak{g}$ defined in Section 1, let λ and ω be as in the theorem, and define $\nu = \lambda + \omega$. By hypothesis $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is $\Delta^+(\mathfrak{k})$ dominant and ω is $\Delta^+(\mathfrak{g})$ dominant. Therefore $\nu + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is $\Delta^+(\mathfrak{k})$ dominant. The first step is to define another parabolic subalgebra $\mathfrak{q}'$ of $\mathfrak{g}$ so that we can apply Theorem 3.2.

We have defined i_0 and j_0 so that i_0 is as small as possible, j_0 is as large as possible, and $\omega_{i_0} = \omega_{i_0+1} = \cdots = \omega_{j_0+1}$. Then we defined $t_0 = l - (j_0 - i_0 + 2)$, so that $0 \leq t_0 \leq l - 2$. Let t be an integer satisfying

$$t \geq \max\{t_0,\ 2m - l + 1 - 2h,\ 2m - l + 2h' - 2h\}. \tag{5.1}$$

We are to prove unitarity of $(\mathcal{L}_{\mathfrak{l},\mathfrak{u}})_S(F_\nu)$. If $t \geq l - 1$, then $\nu + \delta$ is dominant, and unitarity follows from Theorem 3.1a.

Thus we may assume that $t \leq l - 2$. Since $t_0 \leq t \leq l - 2$, we can find integers i and j with $i_0 \leq i \leq m$, $m \leq j \leq j_0$, and

$$t = l - (j - i + 2). \tag{5.2}$$

Then we have $1 \le i \le m \le j < l$ and $\omega_i = \omega_{i+1} = \cdots = \omega_{j+1}$. Define a θ stable parabolic subalgebra $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$ so that $\mathfrak{b} \subseteq \mathfrak{l}'$, $\mathfrak{q}'$ is compatible with $\Delta^+(\mathfrak{g})$, and the simple roots contributing to $\mathfrak{l}'$ are $e_i - e_{i+1}, e_{i+1} - e_{i+2}, \ldots, e_j - e_{j+1}$. The Lie algebra $\mathfrak{l}' \cap \mathfrak{k}$ is spanned by $\mathfrak{b}$, the root vectors for the roots $\pm(e_k - e_{k'})$ with $k < k' \le m$, and the root vectors for the roots $\pm(e_{k+1} - e_{k'+1})$ with $m \le k < k'$. Thus $\mathfrak{l}' \cap \mathfrak{k} \subseteq \mathfrak{l}$. In addition, the semisimple part of $\mathfrak{l}'$ is simple of type A_{j-i+1}, and $e_m - e_{m+1}$ is the only noncompact simple root of $\mathfrak{l}'$. Therefore $L'/(L' \cap K)$ is Hermitian symmetric.

Following the prescription of Theorem 3.2, we define $\nu' = \nu + 2\delta(\mathfrak{l}' \cap \mathfrak{p})$. The noncompact positive roots of $\mathfrak{l}'$ are $e_k - e_{k'+1}$ with $i \le k \le m$ and $m \le k' \le j$. Thus

$$2\delta(\mathfrak{l}' \cap \mathfrak{p}) = (j - m + 1)(e_i + \cdots + e_m) - (m - i + 1)(e_{m+1} + \cdots + e_{j+1}).$$

To be able to apply Theorem 3.2, we are to check that ν' is orthogonal to the members of $\Delta(\mathfrak{l}')$. Since λ, ω, and $\delta(\mathfrak{l}' \cap \mathfrak{p})$ are all orthogonal to $e_i - e_{i+1}, \ldots, e_{m-1} - e_m$ and to $e_{m+1} - e_{m+2}, \ldots, e_j - e_{j+1}$, we have only to check orthogonality with $e_m - e_{m+1}$. We compute that

$$\begin{aligned}\langle \nu', e_m - e_{m+1} \rangle &= \langle \lambda, e_m - e_{m+1} \rangle + \langle \omega, e_m - e_{m+1} \rangle + \langle 2\delta(\mathfrak{l}' \cap \mathfrak{p}), e_m - e_{m+1} \rangle \\ &= (-l + t) + 0 + ((j - m + 1) + (m - i + 1)),\end{aligned}$$

and this is 0 by (5.2). Thus ν' is orthogonal to $\Delta(\mathfrak{l}')$.

Then $\mathbb{C}_{\nu'}$ is well defined as a 1-dimensional $(\mathfrak{l}', L' \cap K)$ module, Theorem 3.2 applies, and the result is that $(\mathcal{L}_{\mathfrak{l},\mathfrak{u}})_S(F_\omega \otimes \mathbb{C}_\lambda)$ will have been proved to be infinitesimally unitary for $\lambda = (-l+t)(e_1 + \cdots + e_m)$ if we show that $(\mathcal{L}_{\mathfrak{l}',\mathfrak{u}'})_{S'}(\mathbb{C}_{\nu'})$ is infinitesimally unitary. To show this latter unitarity, we shall apply Theorem 3.1c.

Put $\lambda' = \lambda + 2\delta(\mathfrak{l}' \cap \mathfrak{p})$ and $\sigma = \nu' + \delta + c\delta(\mathfrak{u}') = \lambda' + \delta + (\omega + c\delta(\mathfrak{u}'))$. We are going to check that $\sigma = (a_1, \ldots, a_l)$ satisfies the hypotheses (A1) through (A6′) of Proposition 4.8 with $r = i$ and $s = j + 1$.

Let us begin by considering these hypotheses when $\omega = 0$ and $c = 0$, and afterward we shall restore the general values of ω and c. For the moment, then, we have $\sigma = \lambda' + \delta = (a_1, \ldots, a_l)$, and the entries are given by

$$a_k = \begin{cases} t - k + h & \text{for } k \le i - 1 \\ t - k + j - m + 1 + h & \text{for } i \le k \le m \\ l - k + i - m - 1 + h & \text{for } m + 1 \le k \le j + 1 \\ l - k + h & \text{for } j + 2 \le k \le l. \end{cases} \tag{5.3}$$

Let us check that σ satisfies the hypotheses in this special case.

(A1) $a_1 > \cdots > a_{r-1} > a_{s+1} > \cdots > a_l$. This is obvious except for $a_{r-1} > a_{s+1}$, i.e., $a_{i-1} > a_{j+2}$, which follows from (5.3) and (5.2):

$$a_{i-1} - a_{j+2} = (t - i + 1 + h) - (l - j - 2 + h) = t - (l - (j - i + 2)) + 1 = 1 > 0.$$

(A2) $a_r > \cdots > a_s$, i.e., $a_i > \cdots > a_{j+1}$. This is immediate since λ' is orthogonal to $\Delta(\mathfrak{l}')$ and δ is dominant.

(A3) and (A4) $a_{r-1} > a_s$ and $a_r > a_{s+1}$, i.e., $a_{i-1} > a_{j+1}$ and $a_i > a_{j+2}$. These follow from (5.3) and (5.2):

$$a_{i-1} - a_{j+1} = (t-i+1+h) - (l-j-1+i-m-1+h) = m-i+1 > 0,$$
$$a_i - a_{j+2} = (t-i+j-m+1+h) - (l-j-2+h) = j-m+1 > 0.$$

(A5) $a_l \geq 0$ if $l > s$, or $a_{r-1} \geq 0$ if $l = s$. That is, $a_l \geq 0$ if $l > j+1$, or $a_{i-1} \geq 0$ if $l = j+1$. If $l > j+1$, (5.3) gives $a_l = l - l + h \geq 0$. If $l = j+1$, (5.3) gives

$$a_{i-1} = t-i+1+h = l-(j-i+2)-i+1+h = l-j-1+h \geq 0.$$

(A6′) $a_r \geq |a_s|$, i.e., $a_i \geq a_{j+1}$ and $a_i \geq -a_{j+1}$. In fact, we have $a_i = t-i+j-m+1+h = l-m-1+h$ and $a_{j+1} = t-m+h$. The inequality $a_i \geq a_{j+1}$ says $l-1 \geq t$, and this holds since we have arranged that $t \leq l-2$. The inequality $a_i \geq -a_{j+1}$ says $t \geq 2m-l+1-2h$, and this we have assumed as part of (5.1).

Thus the hypotheses of Proposition 4.8 are satisfied in the special case. Let us consider them when the general values of ω and c are restored. The table in (5.3) needs to be adjusted by adding $(\omega + c\delta(\mathfrak{u}'))_k$ to the value of a_k. Since $\omega + c\delta(\mathfrak{u}')$ is dominant and nonnegative, each of (A1) through (A5) remains true when $(\omega+c\delta(\mathfrak{u}'))_k$ is added to a_k. Thus we have only to verify (A6′). With ω and c in place, we have $a_i = l-m-1+h+(\omega+c\delta(\mathfrak{u}'))_i$ and $a_{j+1} = t-m+h+(\omega+c\delta(\mathfrak{u}'))_{j+1}$. We have seen that $l-m-1+h \geq |t-m+h|$, and we have also $(\omega+c\delta(\mathfrak{u}'))_i \geq |(\omega+c\delta(\mathfrak{u}'))_{j+1}|$. Adding these inequalities gives $a_i \geq |a_{j+1}|$.

Thus the hypotheses of Proposition 4.8 are satisfied for $\sigma = \nu' + \delta + c\delta(\mathfrak{u}')$. The proposition tells us that the equation $\nu' + \delta + c\delta(\mathfrak{u}') + \mu = w(\nu' + \delta + c\delta(\mathfrak{u}'))$ has no nonzero solutions μ of the type mentioned in Theorem 3.1c, and Theorem 3.1c therefore says that $(\mathcal{L}_{\mathfrak{l}',\mathfrak{u}'})_{S'}(\mathbb{C}_{\nu'})$ is infinitesimally unitary.

6 Complements

a. Scope of unitarity in [EPWW] for line-bundle cases. In both Section 13 of [EPWW] and the Vogan Unitarizability Theorem as stated in Theorem 3.1, it is proved that $\pi(\lambda)$ is infinitesimally unitary if $N(\lambda + c\xi + 2\delta(\mathfrak{u}))$ is irreducible for all real $c \geq 0$. All three conditions in Theorem 3.1 are designed to check this. The test that [EPWW] uses for this irreducibility is slightly more sensitive for this purpose than the weakly-fair test of Theorem 3.1b. In fact, the [EPWW] test is exact, and a table in [EPWW] identifies the smallest number $a \geq 0$ so that, in our notation, $N(\lambda + 2\delta(\mathfrak{u}))$ is reducible when $\lambda = (-l+t)(e_1 + \cdots + e_m)$ and $t = l-1-a$. The number a can be an integer or half integer and differs from $l - [\frac{m}{2}]$ by an amount that is bounded independently of l and m. The weakly-fair test of Theorem 3.1b says that the required irreducibility occurs if $\lambda + \delta(\mathfrak{u})$ is $\Delta^+(\mathfrak{g})$ dominant, and we check from the formulas in Section 1 that this means that $t \geq \frac{m+1}{2} - h$. Thus Vogan's test proves irreducibility for t from $l-1$ down to some number $l-1-a'$, where a' differs from $l - [\frac{m}{2}]$ by an amount that is bounded independently of l and m. Consequently Vogan's test misses being best possible by an amount that is bounded independently of l and m.

Our development for the line-bundle cases assumed that $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is $\Delta^+(\mathfrak{k})$ dominant in order to be able to isolate one particular irreducible subquotient of $\pi(\lambda)$ to study for unitarity. But there is no need to isolate one particular subquotient if all of $\pi(\lambda)$ is infinitesimally unitary, and indeed this can happen sometimes even when $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ fails to be $\Delta^+(\mathfrak{k})$ dominant.

There is no assumption in the above-mentioned tests of [EPWW] and the Vogan Unitarizability Theorem that $\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p})$ be $\Delta^+(\mathfrak{k})$ dominant. For an example of what can happen, consider $\lambda = (-l + t)(e_1 + \cdots + e_m)$ in D_l. The weakly-fair condition of Theorem 3.2b is that $\lambda + \delta(\mathfrak{u})$ is dominant, and this means that $t \geq \frac{m+1}{2}$. On the other hand, $\Delta^+(\mathfrak{k})$ dominance of $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is the condition that $t \geq 2m - l$. It is possible for the first of these to succeed and the second to fail. A particular example occurs with $m = 7$ and $l = 9$ and $t = 4$, in which case $\frac{m+1}{2} = 4$ and $2m - l = 5$. Thus $t \geq \frac{m+1}{2}$ but $t < 2m - l$.

When [EPWW] yields, for a particular group, a number a such that $l - 1 - a$ is $\leq$ the smallest t in Table 2 for which $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is $\Delta^+(\mathfrak{k})$ dominant, [EPWW] is saying that unitarity persists for all cases in the present paper for that particular group. In particular unitarity holds for all cases under study in D_9 when $m = 7$. Thus the work of [EPWW] settles some of the cases that Table 2 has marked as undecided.

b. Condition for nonunitarity of $\bar{\pi}(\lambda)$. We mentioned in Section 2 that $\bar{\pi}(\lambda)$ is an irreducible subquotient of $\pi'(\lambda)$ and hence is isomorphic to $\pi'(\lambda)$ if $\pi'(\lambda)$ is irreducible. When this happens, the nonunitarity that is asserted in Table 2 becomes the desired nonunitarity of $\bar{\pi}(\lambda)$.

If we look at the calculations in Section 2, we can see that the desired nonunitarity might be deducible in an easier way—without addressing irreducibility of $\pi'(\lambda)$. For the groups of type B_l and D_l, the conclusion of Theorem 1.2 follows because the Shapovalov form has opposite signature on the K types of $\pi'(\lambda)$ with highest weights $\Lambda = \lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ and $\Lambda + 2e_1$. Each of these K types occurs in $\pi'(\lambda)$ with multiplicity 1, and the first of them occurs by definition in $\bar{\pi}(\lambda)$. It therefore will follow that $\bar{\pi}(\lambda)$ is not infinitesimally unitary if it is shown that $\bar{\pi}(\lambda)$ contains the K type with highest weight $\Lambda + 2e_1$.

Similar remarks apply to the groups C_l, except that Λ and $\Lambda + 2e_1$ are to be replaced by Λ and $\Lambda + e_1 + e_2$.

c. Nonunitarity results for vector-bundle cases. In giving nonunitarity results, Theorem 1.2 sticks to line-bundle cases. But nonunitarity results for vector-bundle cases can be obtained in the same way. For an example consider D_l with $m = 3$. Take $\omega = (\omega_1, \omega_2, 0; 0, \ldots, 0)$ with $\omega_1 \geq \omega_2 > 0$. Following the prescription in Theorem 1.1 to see what unitarity is assured, we find that $i_0 = 3$, $j_0 = l - 1$, and $t_0 = 2$.

We can hope for nonunitarity when $t < 2$. Thus take $t = 1$ and consider $\pi(\lambda + \omega)$ with $\lambda = (-l + t)(e_1 + e_2 + e_3) = (-l + 1)(e_1 + e_2 + e_3)$. The key thing to compute in Section 2 is $\|\sigma' + \omega + \lambda + \delta\|^2 - \|\omega + \lambda + \delta\|^2$, where σ' is a weight of some representation F_σ of L that occurs in $S(\mathfrak{u} \cap \mathfrak{p})$ such that $\sigma' + \omega$ is dominant and $F_{\sigma'+\omega}$

occurs in $F_\sigma \otimes F_\omega$. We want this difference to be < 0.

We use $\sigma = (2, 0, 0; 0, \ldots, 0)$ just as in Section 2. If $\omega_2 \geq 2$, we can take $\sigma' = (0, 0, 2; 0, \ldots, 0)$, and a little computation shows that the difference is indeed < 0. If $\omega_2 = 1$ and $\omega_1 \geq 2$, we can take $\sigma' = (0, 1, 1; 0, \ldots, 0)$, and again the difference is < 0. If $\omega_1 = \omega_2 = 1$, we can take $\sigma' = (1, 0, 1; 0, \ldots, 0)$, but this time we find that the difference is 0. Thus we find nonunitarity for $t = 1$ for all cases except $\omega_1 = \omega_2 = 1$. For the case that $\omega_1 = \omega_2 = 1$, one could perhaps succeed with another highest weight in place of $(2, 0, 0; 0, \ldots, 0)$, but we have not tried to do so.

d. Insufficiency of "weakly-fair" condition in Section 5. In D_8 with $m = 2$, consider $\lambda = (-8+t,\ -8+t;\ 0, \ldots, 0)$ and $\omega = (b, 0; 0, \ldots, 0)$. First suppose that $b > 0$. In the notation of Theorem 1.1, we find that $i_0 = 2$ and $j_0 = 7$, so that $t_0 = 1$. Thus the theorem shows that $\bar{\pi}(\lambda + \omega)$ is infinitesimally unitary for $t = t_0 = 1$, and it does so by making use of $\Delta(\mathfrak{l}')$ built from $\{e_2 - e_3, \ldots, e_7 - e_8\}$. But the weakly-fair condition of Theorem 3.1b does not apply. In fact, $\nu' = (-7 + b, -1; -1, \ldots, -1)$. Thus $\nu' + \delta(\mathfrak{u}') = (b, 2; 2, \ldots, 2)$, and this is not dominant for $b = 1$. This example shows why we were forced to work with the more difficult condition (c) in Theorem 3.1.

This argument shows also that had b been 0, we could not have handled unitarity of $\bar{\pi}(\lambda)$ at $t = 1$ by using $i = 2$ and $j = 7$ and by applying the weakly-fair condition. However, a little calculation shows that the weakly-fair condition does show unitarity of $\bar{\pi}(\lambda)$ at $t = 1$ if we use $i = 1$ and $j = 6$. In this same way, as we shall see in the next subsection, one can handle all the line-bundle cases of Theorem 1.1 with the weakly-fair test. Such a proof, however, seems unnatural since the choices of i and j cannot be arbitrary.

e. Sufficiency of "weakly-fair" condition in line-bundle cases. The unitarity of all the line-bundle cases in Theorem 1.1, i.e., the unitarity of $\bar{\pi}(\lambda)$ as in Table 2, can be proved using the "weakly-fair" test in Theorem 3.1b. The combinatorial tools in Section 4 are therefore not needed for the line-bundle cases, and the proof of unitarity is considerably shorter for them. The drawback of this approach is that a certain aspect of a proof via Theorem 3.1b is unnatural, in a way that we explain in a moment.

Let us sketch the argument. We begin as in Section 5 except that $\omega = 0$ and hence $\nu = \lambda = (-l + t, \ldots, -l + t; 0, \ldots, 0)$. Since $\omega = 0$, the definitions of Section 5 yield $i_0 = 1$, $j_0 = l - 1$, and $t_0 = 0$. When the integer t defining λ is as in (5.1), we are to prove that $(\mathcal{L}_{\mathfrak{l},\mathfrak{u}})_S(\mathbb{C}_\lambda)$ is infinitesimally unitary. As in Section 5, Theorem 3.1a is applicable if $t \geq l - 1$, and thus we may assume that $t \leq l - 2$.

Let i and j be integers with $1 \leq i \leq m \leq j < l$ such that (5.2) holds, and define $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$ as in Section 5. The argument in Section 5 shows that it is enough to prove that $(\mathcal{L}_{\mathfrak{l}',\mathfrak{u}'})_{S'}(\mathbb{C}_{\lambda'})$ is infinitesimally unitary when λ' is defined by

$$\lambda' = \lambda + 2\delta(\mathfrak{l}' \cap \mathfrak{p}).$$

Section 5 used Theorem 3.1c to prove this unitarity for any pair (i, j) satisfying the above conditions. Theorem 3.1b can be used to show this unitarity, but only under additional odd-looking conditions on (i, j). It happens that there are enough pairs (i, j) satisfying these additional conditions that all cases in Table 2 are handled by

Theorem 3.1b. The need for *ad hoc* choices of pairs (i, j) is the sense in the which the proof using Theorem 3.1b is unnatural.

The condition for unitarity from Theorem 3.1b is that $\lambda' + \delta(\mathfrak{u}')$ is $\Delta^+(\mathfrak{g})$ dominant, hence that

$$\langle \lambda' + \delta(\mathfrak{u}'), \alpha \rangle \geq 0 \tag{6.1}$$

for every simple root α. Here

$$\lambda' + \delta(\mathfrak{u}') = \lambda' + \delta - \delta(\mathfrak{l}') = \lambda + \delta + 2\delta(\mathfrak{l}' \cap \mathfrak{p}) - \delta(\mathfrak{l}')$$

with

$$\delta(\mathfrak{l}') = (\tfrac{j+1-i}{2})e_i + (\tfrac{j-1-i}{2})e_{i+1} + \cdots + (-\tfrac{j+1-i}{2})e_{j+1}$$

and

$$2\delta(\mathfrak{l}' \cap \mathfrak{p}) = (j - m + 1)(e_i + \cdots + e_m) - (m - i + 1)(e_{m+1} + \cdots + e_{j+1}).$$

We need to check (6.1) for α equal to $e_{i-1} - e_i$ if $i > 1$, $e_{j+1} - e_{j+2}$ if $j \leq l - 2$, $e_m - e_{m+1}$ always, and all simple roots involving e_l. For the remaining simple roots α, we readily check that (6.1) equals 1. Computation gives

$$\begin{aligned}
\langle \lambda' + \delta(\mathfrak{u}'),\ e_{i-1} - e_i \rangle &= m - \tfrac{1}{2}(i + j) + \tfrac{1}{2} && \text{if } i > 1\\
\langle \lambda' + \delta(\mathfrak{u}'),\ e_{j+1} - e_{j+2} \rangle &= -m + \tfrac{1}{2}(i + j) + \tfrac{1}{2} && \text{if } j < l - 1\\
\langle \lambda' + \delta(\mathfrak{u}'),\ e_m - e_{m+1} \rangle &= 0 && \text{always}\\
\langle \lambda' + \delta(\mathfrak{u}'),\ e_{l-1} - e_l \rangle &= 0 && \text{if } j = l - 1\\
\langle \lambda' + \delta(\mathfrak{u}'),\ e_{l-1} - e_l \rangle &= -m + \tfrac{1}{2}(i + j) + \tfrac{1}{2} && \text{if } j = l - 2\\
\langle \lambda' + \delta(\mathfrak{u}'),\ e_{l-1} - e_l \rangle &= 1 && \text{if } j < l - 2\\
\langle \lambda' + \delta(\mathfrak{u}'),\ e_l \rangle &= -m + \tfrac{1}{2}(i + j) + h - \tfrac{1}{2} && \text{if } j = l - 1\\
\langle \lambda' + \delta(\mathfrak{u}'),\ e_l \rangle &= h && \text{if } j < l - 1.
\end{aligned}$$

Therefore (6.1) holds for all simple α if and only if all three of the following conditions hold:

$$\begin{aligned}
m - \tfrac{1}{2}(i + j) + \tfrac{1}{2} &\geq 0 \quad \text{if } i > 1\\
-m + \tfrac{1}{2}(i + j) + \tfrac{1}{2} &\geq 0 \quad \text{if } j < l - 1\\
-m + \tfrac{1}{2}(i + j) + h - \tfrac{1}{2} &\geq 0 \quad \text{if } j = l - 1.
\end{aligned}$$

The first two of these inequalities are together equivalent with

$$|m - \tfrac{1}{2}(i + j)| \leq \tfrac{1}{2}.$$

In other words, if $i > 1$ and $j < l - 1$, then Theorem 3.1b applies to the pair (i, j) if and only if $e_m - e_{m+1}$ is centered, as much as parity will allow, between the end simple roots $e_i - e_{i+1}$ and $e_j - e_{j+1}$ of $\Delta^+(\mathfrak{l}')$.

Briefly let us indicate that each case of unitarity in Table 2 has some pair (i, j) satisfying the above inequalities. First choose $i = m - r$ and $j = m + r$ with $0 \leq r \leq \min(m-2,\ l-m-2)$, so that $i > 1$ and $j < l - 1$. The above inequalities are satisfied, and $t = l - (j - i + 2) = l - 2r - 2$. Thus Theorem 3.1b handles t if $t \equiv l \bmod 2$ and $|l - 2m| + 2 \leq t \leq l - 2$.

Next choose $i = m - r - 1$ and $j = m + r$ with $0 \leq r \leq \min(m-2,\ l-m-2)$, so that $i \geq 1$ and $j < l - 1$. The above inequalities are satisfied, and $t = l - (j - i + 2) = l - 2r - 3$. Thus Theorem 3.1b handles t if $t \not\equiv l \bmod 2$ and $|l - 2m| + 1 \leq t \leq l - 3$.

So far, all cases with $|l - 2m| + 1 \leq t \leq l - 2$ have been handled. For the remaining cases we treat $2m \leq l$ and $2m > l$ separately. The case that $2m \leq l$ and $1 \leq t \leq l - 2m + 1$ is handled by taking $i = 1$ and $j \leq l - 2$, so that $t = l - (j + 1)$. The case that $2m \leq l$ and $t = 0$ is handled by taking $i = 1$ and $j = l - 1$; in this case the condition for Theorem 3.1b to apply is that $(h - \frac{1}{2}) + \frac{1}{2}(l - 2m) \geq 0$, which holds except for D_l when $l = 2m$. For the case that $2m > l$, Table 2 asks only that we consider t's in B_l and C_l with $t \geq 2m - l$ and t's in D_l with $t > 2m - l$. The only one that we have not considered yet is $t = 2m - l$ for B_l and C_l, and this is handled by taking $i = 2m - l + 1$ and $j = l - 1$.

f. Further vector-bundle cases—the example of B_4. Theorem 1.1 of the present paper is a systematic extension of the techniques used in the last section of [Kn3] to study the example of B_4 with $m = 2$. In terms of our present notation, [Kn3] worked with $\lambda + \omega = (-4+t+b, -4+t; 0, 0)$ when b is an integer ≥ 0. The value $t = 3$ yields a limit of discrete series representation, and the values $t = 2$ and $t = 1$ correspond to building $\Delta(\mathfrak{l}')$ in Section 5 from $\{e_2 - e_3\}$ and $\{e_2 - e_3,\ e_3 - e_4\}$, respectively, and then using Theorem 1.1.

For $t = 4$ the method of the present paper is applicable only when $b = 0$, and it gives unitarity. For $b > 0$, [Kn3] quotes from [BaK] to show that there is unitarity for $b = 1$ and nonunitarity for $b \geq 2$. This is a completely different pattern of unitarity from what comes out of Theorems 1.1 and 1.2, and we have no tools for addressing it in any generality.

For $t = 5$, Theorem 1.2 is applicable when $b = 0$ and gives nonunitarity. For $b > 0$, the method of proof of Theorem 1.1 can be tried with $\Delta(\mathfrak{l}')$ built from $\{e_2 - e_3,\ e_3 - e_4,\ e_4\}$. One finds that the method succeeds and yields unitarity for $b \geq 2$. Whatever the outcome for $b = 1$, the result is yet another completely different pattern of unitarity from what comes out of Theorems 1.1 and 1.2.

The extra unitarity obtained for $t = 4$ and $t = 5$ raises the question of where to stop in considering vector-bundle candidates for the analytic continuation of discrete series. We do not have a definitive answer for that question but have chosen to consider only those cases that point to a related line-bundle case for a lower-dimensional group G.

g. Examples in connection with Section 4. These examples show the need for some of the conditions in Propositions 4.3 and 4.8. In all of them, we take $l = 4$.

1) With $r = 2$ and $s = 3$,

$$(2, 3, 0, 1) + (1, -1, 1, -1) = (3, 2, 1, 0) = w(2, 3, 0, 1)$$

for a permutation w. For this example condition (B1) does not hold in Proposition 4.3.

2) With $r = 1$ and $s = 3$,

$$(2, 1, 0, 3) + (1, 0, 0, -1) = (3, 1, 0, 2) = w(2, 1, 0, 3)$$

for a permutation w. For this example condition (A4) does not hold in Proposition 4.3.

3) With $r = 1$ and $s = 4$,

$$(1, 0, -1, -2) + (1, 1, 1, 1) = (2, 1, 0, -1) = w(1, 0, -1, -2)$$

for w equal to the product of an even sign change and a permutation. For this example condition (A6′) does not hold in Proposition 4.8. In the notation of Section 5, $(1, 0, -1, -2)$ is $\lambda' + \delta$ for $\mathfrak{so}(4, 4)$ when $a = -4$, i.e., $t = 0$.

h. Correction to a proof in [Kn3]**.**

The paper [Kn3] gives prototypes for some of the arguments in the present paper. In particular Theorems 6.1e and 6.2e of [Kn3] reprove in the present style the assertions from [Wa1] about the analytic continuation of holomorphic discrete series, but just for groups G that are linear and have simply laced Dynkin diagrams. The arguments for all $\mathfrak{g}_0$ except $\mathfrak{su}(m, n)$ use the weakly-fair condition of the Vogan Unitarizability Theorem to address unitarity, and they are fine. For $\mathfrak{g}_0 = \mathfrak{su}(m, n)$, the argument extends from the bottom of page 425 to the bottom of page 427; it is meant to invoke the present Theorem 3.1c, but it does so incorrectly. Specifically, four lines after (6.9), it gives the wrong value for $\langle \lambda + \delta,\ e_{r-1} - e_r \rangle$.

To correct the argument, one applies Proposition 4.3 of the present paper. The expression for σ to which the proposition is to be applied is $\sigma = \lambda + \delta + t\delta(\mathfrak{u})$ in the notation of [Kn3] or $\sigma = \nu' + \delta + c\delta(\mathfrak{u}') = \lambda' + \delta + (\omega + c\delta(\mathfrak{u}'))$ in the notation of the present paper. The hypotheses of Proposition 4.3 are readily verified, the argument being completely similar to the one given in the present Section 5, and then the proof goes through.

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