BOUNDED SYMMETRIC DOMAINS AND HOLOMORPHIC DISCRETE SERIES

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In this paper we shall study some special features of those symmetric spaces of noncompact type that are complex manifolds in a natural way. After discussing the case of the disk, we give Harish-Chandra's realization of all such spaces as bounded symmetric domains in $\mathbf{C}^n$, homogeneous under a connected group G, and we reproduce Harish-Chandra's construction of some irreducible infinite-dimensional group representations of G by means of this realization. Finally we show how the Hardy H^2 theory of the disk generalizes to yield further irreducible representations of G and to show that certain standard induced representations of G are reducible. A list of the material in each section follows.

1. Discrete series for $SU(1,1)$. Bargmann [*1*] classified the infinite-dimensional representations of $SU(1,1)$. The theorems mentioned at the beginning of this chapter were proved by him and then put in a more general setting by Harish-Chandra [*2*].

2. Bounded symmetric domains: examples, role of the center of K, structure of roots, Harish-Chandra decomposition. Much of this material appears in [*3*, pp. 354, 304–322], where it is treated from the point of view of differential geometry. The structure of the roots and the Harish-Chandra decomposition appear originally in [*2*, pp. 750, 756–761, 4, 590].

3. Holomorphic discrete series: definition, the "constant function," nonvanishing theorem, square-integrability, examples. See [*2*, pp. 6–13, 598–612].

4. Limits of homomorphic discrete series: case of $SU(1,1)$, a more general setting, statements of theorems. See [*4*].

5. Notes and references.

1. Discrete Series for $SU(1,1)$

Let

$$G = SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| \, |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

The members of this group act as analytic automorphisms of the disk $\Omega = \{|z| < 1\}$ under $z \to zg = (\bar{\alpha}z + \beta)/(\bar{\beta}z + \alpha)$. [Here $(zg)h = z(gh)$.] The discrete series for G occurs in two parts:

$$\text{(1)} \qquad n \geqq 2,$$

$$\mathscr{H}_n = \{F(z) \text{ analytic in } \Omega,\ \|F\|^2 = \textstyle\int_\Omega |F(z)|^2\,(1 - |z|^2)^{n-2} dx\,dy < \infty\},$$

$$\mathscr{U}_n(g)F(z) = (\bar{\beta}z + \alpha)^{-n}\, F((\bar{\alpha}z + \beta)/(\bar{\beta}z + \alpha)).$$

This part will be called the holomorphic discrete series.

(2) The same, except that conjugates of the functions in $\mathscr{H}_n$ are used and $\mathscr{U}$ involves $(\overline{\beta z + \alpha})^{-n}$.

Some properties of $\mathscr{H}_n$ and $\mathscr{U}_n$ in the first case are the following:

(1) $\mathscr{H}_n$ is a Hilbert space, i.e., the analytic functions in

$$L^2(\Omega,\ (1 - |z|^2)^{n-2}\ dx\, dy)$$

form a closed subspace.

(2) $\mathscr{U}_n$ is a representation, i.e., $\mathscr{U}_n(gh) = \mathscr{U}_n(g)\mathscr{U}_n(h)$ and $\mathscr{U}_n(e) = I$.

(3) $\mathscr{U}_n$ is unitary, i.e., each $\mathscr{U}_n(g)$ is a linear isometry onto. (The proof consists of a change of variables, which uses the invariance of $(1 - |z|^2)^{-2}\, dx\, dy$ under $z \to zg$.)

(4) $\mathscr{U}_n$ is strongly continuous, i.e., $\lim_{g\to e}\|\mathscr{U}_n(g)f - f\| = 0$ for each f in $\mathscr{H}_n$.

(5) $\mathscr{U}_n$ is irreducible, i.e., there are no nontrivial closed invariant subspaces in $\mathscr{H}_n$.

(6) $\mathscr{U}_n$ is square integrable, i.e., $\int |(\mathscr{U}_n(g)\ \phi,\ \phi)|^2\ dg < \infty$ for some $\phi \in \mathscr{H}_n$. In the presence of (1)–(5), this implies that $\int |(\mathscr{U}_n(g)\,\phi, \psi)|^2\, dg < \infty$ for all $\phi, \psi \in \mathscr{H}_n$ and that $(\mathscr{H}_n, \mathscr{U}_n)$ is unitarily equivalent with a closed subspace of $L^2(G)$ with right translation as the operation.

We shall now describe the Harish-Chandra realization of the holomorphic discrete series for this G. Let $SL(2,\mathbf{C})$ be the group of 2-by-2 complex matrices of determinant 1 and let B be the lower triangular subgroup $\left\{\begin{pmatrix} a & 0 \\ c & a^{-1}\end{pmatrix}\right\}$.

Lemma. Every element of the set $BG \subseteq SL(2,\mathbf{C})$ has a unique decomposition as a product

$$\begin{pmatrix} 1 & 0 \\ \zeta & 1\end{pmatrix}\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1}\end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix}, \qquad \zeta \in \mathbf{C},\quad \gamma \in \mathbf{C} - \{0\},\quad |z| < 1, \quad (*)$$

and every matrix (*) is in BG.

Indication of Proof. Since $B = \left\{\begin{pmatrix} 1 & 0 \\ \zeta & 1\end{pmatrix}\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1}\end{pmatrix}\right\}$ is a group, existence of the decomposition follows from the identity

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \bar{\beta}/\alpha & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & \beta/\alpha \\ 0 & 1 \end{pmatrix}.$$

The lemma suggests that we give BG the structure of a complex manifold by giving it the product structure. Now $SL(2,\mathbf{C})$ has its own complex structure, and the next lemma gives the connection.

Lemma. *BG is an open subset of $SL(2,\mathbf{C})$, and its product complex structure obtained from (*) is the same as what it inherits from $SL(2,\mathbf{C})$. In particular, right translation by $g \in G$ is a holomorphic automorphism of BG.*

The second statement in the lemma can be verified directly, without reference to the complex structure on $SL(2,\mathbf{C})$.

To construct the representation, let $n \geqq 2$ be an integer and let ξ_n be the holomorphic character of B given by $\xi_n\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} = a^{-n}$. Let

$$H_n = \left\{ F \,\middle|\, \begin{array}{ll} \text{(i)} & F(x) \text{ is holomorphic on } BG \\ \text{(ii)} & F(bx) = \xi_n(b)F(x) \quad \text{for} \quad b \in B, \quad x \in BG\ , \\ \text{(iii)} & \|F\|^2 = \displaystyle\int_G |F(g)|^2\, dg < \infty \end{array} \right\},$$

$$U_n(g)F(x) = F(xg).$$

If F satisfies (i), so does $U_n(g)F$ by the lemma above; thus it is clear that $U_n(g)$ preserves H_n and is unitary.

We shall give a correspondence between $\mathscr{H}_n$ and H_n and show that it is a unitary equivalence if the Haar measure dg is normalized suitably. Given $F(\zeta, \gamma, z) \in H_n$, define $f(z)$ on the disk Ω by

$$f(z) = F(0, 1, z).$$

For the inverse, if $f(z) \in \mathscr{H}_n$ is given, put

$$F(\zeta, \gamma, z) = \gamma^{-n} f(z).$$

Then F satisfies (i) and (ii) at least. To see that the correspondence $F \leftrightarrow f$ is an isometry, we note that it is possible to normalize dg in such a way that for any reasonable $h_0(z)$ on Ω the function $h(g) = h_0(0g)$ satisfies

$$\int_G h(g)\, dg = \int_\Omega h_0(z)(1 - |z|^2)^{-2}\, dx\, dy. \qquad (**)$$

Let $F \in H_n$, $g = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}$. Then

$$F(g) = F(\bar\beta/\alpha, \alpha, \beta/\alpha) = \alpha^{-n}F(0, 1, \beta/\alpha) = \alpha^{-n}f(\beta/\alpha) = \alpha^{-n}f(0g)$$

since $\beta/\alpha = (\bar\alpha z + \beta)/(\bar\beta z + \alpha)$ for $z = 0$. Thus $|F(g)|^2 = |\alpha|^{-2n}|f(0g)|^2$. Since $|\alpha|^2 - |\beta|^2 = 1$, $1 - |0g|^2 = |\alpha|^{-2}$ and

$$|F(g)|^2 = (1 - |0g|^2)^n|f(0g)|^2.$$

If we put $h_0(z) = (1 - |z|^2)^n|f(z)|^2$ and $h(g) = |F(g)|^2$ in (**), we obtain

$$\|F\|^2 = \int_G |F(g)|^2\, dg = \int_\Omega |f(z)|^2(1 - |z|^2)^{n-2}\, dx\, dy = \|f\|^2,$$

and it follows that the correspondence is unitary.

To see that $F \leftrightarrow f$ commutes with the group actions, we compute that

$$U_n(g)F(\zeta,\gamma,z) = F\left(\zeta',\gamma(\bar\beta z + \alpha), \frac{\bar\alpha z + \beta}{\bar\beta z + \alpha}\right).$$

Thus

$$\begin{aligned} U_n(g)F(0,1,z) &= (\bar\beta z + \alpha)^{-n}F\left(0,1,\frac{\bar\alpha z + \beta}{\bar\beta z + \alpha}\right) \\ &= (\bar\beta z + \alpha)^{-n}f\left(\frac{\bar\alpha z + \beta}{\bar\beta z + \alpha}\right) = \mathcal{U}_n(g)f(z), \end{aligned}$$

and $F \leftrightarrow f$ is a unitary equivalence.

2. Bounded Symmetric Domains

2.1 Examples

These are examples of domains Ω in $\mathbf{C}^n$ of the form G/K, where G is non-compact semisimple and K is maximal compact in G, such that G operates holomorphically on Ω. These domains were classified by Cartan, and we use his numbering.

I. Let $m \leqq n$ and $\Omega = \{Z \in M_{nm}(\mathbf{C}) | I_m - Z^*Z > 0\}$.
Here M_{nm} refers to all n-by-m matrices, I_m is the identity of size m, and "> 0" means "is positive definite." Let

$$G = SU(n, m) = \left\{ g \in SL(n + m, \mathbf{C}) \middle| g^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} g = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \right\},$$

$$K = S(U(n) \times U(m)) = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \middle| A \in U(n), D \in U(m), \det = 1 \right\}.$$

The condition for g to be in G is that g preserve the Hermitian quadratic form $|z_1|^2 + \cdots + |z_n|^2 - |z_{n+1}|^2 - \cdots - |z_{n+m}|^2$. G operates holomorphically on Ω by

$$gZ = (AZ + B)(CZ + D)^{-1} \qquad \text{if } \; g = \begin{array}{cc} \begin{matrix} n & \; m \end{matrix} & \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \begin{matrix} n \\ m \end{matrix} \end{array}$$

[To see that $(CZ + D)^{-1}$ is defined and $g(\Omega) \subseteq \Omega$, write

$$\begin{aligned} (AZ + B)^*(AZ + B) - (CZ + D)^*(CZ + D) &= (Z^* \;\; I_m) g^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} g \begin{pmatrix} Z \\ I_m \end{pmatrix} \\ &= (Z^* \;\; I_m) \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} Z \\ I_m \end{pmatrix} \\ &= Z^*Z - I_m. \end{aligned}$$

Let $(CZ + D)v = 0$. Unless $v = 0$, we have

$$0 \leqq v^*(AZ + B)^*(AZ + B)v = v^*(Z^*Z - I_m)v < 0,$$

a contradiction. So $(CZ + D)^{-1}$ exists and

$$(gZ)^*(gZ) - I = (CZ + D)^{-1*}(Z^*Z - I_m)(CZ + D)^{-1} < 0.$$

Thus $g(\Omega) \subseteq \Omega$.

The isotropy subgroup at $Z = 0$ is K. One can show that G operates transitively on Ω, and thus $\Omega = G/K$. The Lie algebra of G is obtained from the definition of G and is

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ \bar{b}^t & d \end{pmatrix} \in \mathfrak{sl}(n + m, \mathbf{C}) \middle| a = -a^*, d = -d^* \right\}$$

It is possible to show that G is the identity component of the group of all holomorphic automorphisms of Ω. This fact holds much more generally than for this Ω, and its proof is concealed in the proof of the first lemma in Section 2.3.

II. $\Omega = \{Z \in M_{nn}(\mathbf{C}) \mid I_n - Z^*Z > 0, Z = -Z^t\}$. Let

$$G = SO^*(2n) = \left\{g \in SU(n,n) \,\middle|\, g^t\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}g = \begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\right\}.$$

$$K = \left\{\begin{pmatrix}A & 0\\ 0 & \bar{A}\end{pmatrix} \,\middle|\, A \in U(n)\right\}.$$

Then G acts in the same way as in type I, preserving Ω. The action is transitive with K the isotropy subgroup at 0. G has Lie algebra

$$\mathfrak{g} = \left\{\begin{pmatrix}a & b\\ -\bar{b} & \bar{a}\end{pmatrix} \,\middle|\, a = -a^*, b = -b^t\right\}.$$

III. $\Omega = \{z \in \mathbf{C}^n \mid |z^tz|^2 - 2z^*z + 1 < 0, |z^tz|^2 - 1 < 0\}$. The group $G = SO(n,2)$ has a transitive holomorphic action on Ω with isotropy subgroup $K = SO(n) \times SO(2)$.

IV. $\Omega = \{Z \in M_{nn}(\mathbf{C}) \mid I_n - Z^*Z > 0, Z = Z^t\}$. Let

$$G = Sp(n, \mathbf{R}) = \left\{g \in SU(n,n) \,\middle|\, g^t\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}g = \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\right\},$$

$$K = \left\{\begin{pmatrix}A & 0\\ 0 & \bar{A}\end{pmatrix} \,\middle|\, A \in U(n)\right\}.$$

(This is not the standard realization of $Sp(n, \mathbf{R})$ but is conjugate to it.) Then G acts in the same way as in type I, preserving Ω. The action is transitive with K the isotropy subgroup at 0. G has Lie algebra

$$\mathfrak{g} = \left\{\begin{pmatrix}a & b\\ \bar{b} & \bar{a}\end{pmatrix} \,\middle|\, a = -a^*, b = b^t\right\},$$

In addition to domains of types I to IV, there are two exceptional domains, and we can form products of any of these domains. In each case, we can verify that dim(center K) > 0. During the rest of this chapter, we shall be concerned with the following problems.

Problem 1. Characterize those pairs (G, K) such that G/K admits an invariant complex structure. The answer for G simple will be that the dimension of the center of K is positive.

Problem 2. When G/K has an invariant complex structure, exhibit G/K as a bounded domain in $\mathbf{C}^n$ in a group-theoretic way.

2.2 Almost Complex Manifolds

Let M be a C^∞-manifold, M_p the tangent space at p. A map $p \to J_p \in \mathrm{End}(M_p)$ is an *almost complex structure* on M if

(i) $J_p^2 = -I$ for all p,
(ii) JX is a smooth vector field whenever X is.

In this case, (M, J) is an *almost complex manifold.*

Example. $M =$ complex manifold of dimension m with charts (U_a, φ_a), where $\varphi_a : U_a \to \mathbf{C}^m$. M_p has a basis

$$\left(\frac{\partial}{\partial x_1}\right)_p, \left(\frac{\partial}{\partial y_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_m}\right)_p, \left(\frac{\partial}{\partial y_m}\right)_p,$$

defined relative to (U_a, φ_a). Put

$$J^\alpha\left(\frac{\partial}{\partial x_i}\right)_p = \left(\frac{\partial}{\partial y_i}\right)_p, \qquad J^\alpha\left(\frac{\partial}{\partial y_i}\right)_p = -\left(\frac{\partial}{\partial x_i}\right)_p.$$

Then $(J^\alpha)^2 = -I$. One can check that $J^\alpha = J^\beta$ if $p \in U_a \cap U_\beta$. Thus $J_p = J^\alpha = J^\beta$ is independent of the coordinates and the result J is an almost complex structure on M.

Let (M, J) and (M', J') be almost complex manifolds, $\Phi : M \to M'$ differentiable. We say Φ is *almost complex* if

$$d\Phi_p \circ J_p = J'_{\Phi(p)} \circ d\Phi_p \qquad \text{for} \quad p \in M, \qquad (*)$$

i.e., if Φ infinitesimally carries J into J'.

Example. Let M and M' be complex manifolds, $\Phi : M \to M'$. Then Φ is holomorphic if and only if Φ is almost complex. In fact, Φ is holomorphic if and only if Φ is holomorphic in each variable (see Bochner-Martin, p. 33), if and only if Φ satisfies the Cauchy-Riemann equations in each variable, if and only if Φ satisfies (*).

2.3 Cartan Decomposition

Let

$G =$ connected semisimple Lie group with finite center,
$K =$ maximal compact subgroup,
$\mathfrak{g}, \mathfrak{k} =$ corresponding Lie algebras,

$B(X, Y) = \mathrm{Tr}(\mathrm{ad}\, X\, \mathrm{ad}\, Y) =$ Killing form, and
$\mathfrak{p} = \mathfrak{k}^{\perp}$ relative to B.

Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$, in that

(1) B is negative definite on $\mathfrak{k}$, positive definite on $\mathfrak{p}$,

(2) $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$,

(3) the map $K \times \mathfrak{p} \to G$ given by $(k, X) \to k \exp X$ is a diffeomorphism onto G.

The tangent space $(G/K)_{eK}$ is naturally identified with $\mathfrak{p}$ in such a way that $dL_k = \mathrm{Ad}(k)|_{\mathfrak{p}}$, dL_k being the differential at eK of left translation by k. Suppose G/K is Hermitian, i.e., has an invariant complex structure. Form the corresponding almost complex structure, and let J be the structure on $(G/K)_{eK} \leftrightarrow \mathfrak{p}$. Since left translation by $k \in K$ is holomorphic, it is almost complex. That is,

$$dL_k \circ J = J \circ dL_k,$$

where $dL_k = \mathrm{Ad}(k)|_{\mathfrak{p}}$. Thus if G/K is Hermitian, then there is a $J \in \mathrm{Aut}(\mathfrak{p})$ with $J^2 = -I$ and with $J\,\mathrm{Ad}(k)|_{\mathfrak{p}} = \mathrm{Ad}(k)|_{\mathfrak{p}} J$ for $k \in K$.

Theorem. If G/K is Hermitian and if J is the associated automorphism of $\mathfrak{p}$, then

(i) there exists $V \in \mathrm{center}(\mathfrak{k})$ such that $\mathrm{ad}\, V|_{\mathfrak{p}} = J$,

(ii) if $V = V_1 + \cdots + V_n \in \mathfrak{g}_1 + \cdots + \mathfrak{g}_n$ is the decomposition according to the simple components of $\mathfrak{g}$, then $V_j \in \mathrm{center}(\mathfrak{k}_j)$ and $(\mathrm{ad}\, V_j)^2|_{\mathfrak{p}_j} = -I$. Consequently V_j defines a G_j-invariant almost complex structure on G_j/K_j.

Remarks. (ii) is immediate from (i) since the $\mathfrak{g}_j$ are ideals. (i) will be proved after the lemma below, which does not depend on Hermitian structures. The lemma contains the fact that G is the full connected component of the group of automorphisms of G/K. J-i. Hano remarked that if there were not independent interest in the lemma, then one could give a shorter proof of the theorem by using the fact that every derivation of $\mathfrak{g}$ is inner.

Lemma. Suppose $\mathfrak{g}$ has no compact factor. Let C be a K-invariant inner product on $\mathfrak{p}$ and let

$$\mathfrak{k}' = \left\{ M \in \mathrm{End}\, \mathfrak{p} \;\middle|\; \begin{array}{ll} \text{(i)} & C(MX, Y) + C(X, MY) = 0 \\ \text{(ii)} & [M, \mathrm{ad}[X,Y]|_{\mathfrak{p}}] = \mathrm{ad}[MX, Y]|_{\mathfrak{p}} \\ & \qquad + \mathrm{ad}[X, MY]|_{\mathfrak{p}};\ X, Y \in \mathfrak{p} \end{array} \right\}$$

Then $\mathfrak{k}'$ is a Lie subalgebra of End $\mathfrak{p}$ and the mapping $\mathfrak{k} \to \mathfrak{k}'$ given by $X \to$ ad $X|_\mathfrak{p}$ is an isomorphism onto.

Proof. Closure under brackets is straightforward. Also $\mathfrak{k}$ maps into $\mathfrak{k}'$ because (i) is just the $\mathfrak{k}$-invariance of C and (ii) is just the Jacobi identity for an element of $\mathfrak{k}$ and two elements of $\mathfrak{p}$. The map is one–one because $\{x \in \mathfrak{k} \mid \text{ad } X|_\mathfrak{p} = 0\}$ is easily seen to be an ideal in $\mathfrak{g}$, and $\mathfrak{g}$ was assumed to have no compact factor.

We are to show the map is onto. Since it is one–one, we can regard $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ as a subspace of the vector space $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}$. Make $\mathfrak{g}'$ into a Lie algebra with $\mathfrak{g}$ as a subalgebra by defining

$$\begin{aligned} &[X, Y]_{\text{new}} = \text{ad}[X, Y]|_\mathfrak{p} && \text{for } X, Y \in \mathfrak{p}, \\ &[T, X]_{\text{new}} = -[X, T]_{\text{new}} = T(X) && \text{for } T \in \mathfrak{k}', X \in \mathfrak{p}, \\ &[S, T]_{\text{new}} = ST - TS && \text{for } S, T \in \mathfrak{k}'. \end{aligned}$$

(Here $[\cdot\,,\,\cdot]_{\text{new}}$ is clearly skew, and the Jacobi identity is trivial for three vectors unless two of them are in $\mathfrak{p}$, the other in $\mathfrak{k}'$. In this case, the identity follows from (ii).) Since $\mathfrak{k} \supseteq [\mathfrak{p}, \mathfrak{p}]$, the proof will be complete if we show that $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}'$.

Let $SO(\mathfrak{p})$ be the rotation group on $\mathfrak{p}$ relative to C, $\mathfrak{so}(\mathfrak{p})$ the Lie algebra. Then $\mathfrak{k}' \subseteq \mathfrak{so}(\mathfrak{p})$. Choose an $SO(\mathfrak{p})$-invariant inner product on $\mathfrak{so}(\mathfrak{p})$, restrict it to $\mathfrak{k}'$, and extend it by means of C to be defined on all of $\mathfrak{g}'$. The result is that $\mathfrak{g}'$ possesses a $\mathfrak{k}'$-invariant inner product. If $\mathfrak{z}$ = center $\mathfrak{g}'$, then $\mathfrak{z} \cap \mathfrak{k}' = 0$ since $\mathfrak{k}' \subseteq$ End $\mathfrak{p}$. From these two facts, it follows that the Killing form B of $\mathfrak{g}'$ is negative definite on $\mathfrak{k}'$ (cf. [*3*, p. 123]).

Since $[\mathfrak{k}', \mathfrak{k}'] \subseteq \mathfrak{k}'$ and $[\mathfrak{k}', \mathfrak{p}] \subseteq \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}'$, $\mathfrak{k}'$ and $\mathfrak{p}$ are orthogonal with respect to B. From this fact and the fact that B is negative definite on $\mathfrak{k}'$, it follows that

$$\mathscr{I} = \{X \in \mathfrak{g}' \mid B(X, Y) = 0 \quad \text{for all} \quad Y \in \mathfrak{g}'\},$$

which is an ideal in $\mathfrak{g}'$, is contained in $\mathfrak{p}$. Thus $\mathscr{I}$ is an ideal in $\mathfrak{g}$ contained in $\mathfrak{p}$, and therefore $\mathscr{I} = 0$. Consequently B is nondegenerate on $\mathfrak{p}$.

Let $\mathfrak{v}$ be the orthogonal complement relative to B of $[\mathfrak{p}, \mathfrak{p}]$ in $\mathfrak{k}'$. Fix T in $\mathfrak{v}$ and let X and Y be arbitrary in $\mathfrak{p}$. Then

$$B([T, X], Y) = B(T, [X, Y]) = 0.$$

By the nondegeneracy on $\mathfrak{p}$, $T(X) = [T, X] = 0$ for all X. Thus $T = 0$, $\mathfrak{v} = 0$, and $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}'$. This proves the lemma.

Proof of (i) in theorem. Without loss of generality, we can assume $\mathfrak{g}$ has no compact factor. Put $C(X, Y) = B(X, Y) + B(JX, JY)$ for $X, Y \in \mathfrak{p}$. Then C is a K-invariant inner product and J satisfies (i) in the definition of $\mathfrak{k}'$ in the lemma. In view of the lemma, the proof will be complete if we show that J satisfies

$$[J, \operatorname{ad}[X, Y]|_{\mathfrak{p}}] = \operatorname{ad}[JX, Y]|_{\mathfrak{p}} + \operatorname{ad}[X, JY]|_{\mathfrak{p}}$$

for $X, Y \in \mathfrak{p}$. The left side is 0 since J commutes with $\operatorname{ad} \mathfrak{k}|_{\mathfrak{p}}$. Complexify $\mathfrak{k}$, $\mathfrak{p}$, and C. Then the right side is a restriction of

$$-\operatorname{Im} \operatorname{ad}[X - iJX, Y - iJY],$$

where $J(X - iJX) = i(X - iJX)$, $J(Y - iJY) = i(Y - iJY)$. Changing notation, we see that it is enough to prove that $\operatorname{ad}[X, Y]|_{\mathfrak{p}^c} = 0$ if $JX = iX$ and $JY = iY$. Now

$$C(X, Y) = C(JX, CY) = C(iX, iY) = -C(X, Y),$$

so that

$$C(X, Y) = 0.$$

Let $Z \in \mathfrak{p}^{\mathbf{C}}$ and $T \in \mathfrak{p}^{\mathbf{C}}$ be arbitrary. Then

$$C([[Z, T], X], Y) = 0$$

since $J([[Z,T],X]) = i[[Z,T],X]$. On the other hand,

$$D(X,Y,Z,T) = C([[X,Y],Z],T)$$

is a quadrilinear form on $\mathfrak{p}^{\mathbf{C}}$ that

(i) is skew in X and Y,

(ii) is skew in Z and T (a small verification required here), and

(iii) satisfies the Jacobi identity in X, Y, and Z.

By Lemma 12.4 in [*3*, pp. 68–69],

$$C([[X,Y],Z],T) = C([[Z,T],X],Y),$$

and we have seen that the right side is 0. Now C is nondegenerate, being the complexification of an inner product, and thus $[[X,Y],Z] = 0$ for all $Z \in \mathfrak{p}^{\mathbf{C}}$. That is, $\operatorname{ad}[X,Y]|_{\mathfrak{p}^c} = 0$. The proof of the theorem is complete.

At the moment we settle for a weak converse, which will be used in the next section.

Proposition. If G is simple and $\mathfrak{c} = \text{center } (\mathfrak{k}) \neq 0$, then G/K has an invariant almost complex structure.

Lemma 1. $\mathfrak{k}$ is a maximal subalgebra of $\mathfrak{g}$.

Proof. Since $\mathfrak{g}$ is simple, ad $\mathfrak{k}$ acts irreducibly on $\mathfrak{p}$. (Namely, if $[\mathfrak{k},\mathfrak{p}_1] \subseteq \mathfrak{p}_1$ and $\mathfrak{p}_1 \subseteq \mathfrak{p}$, then $[\mathfrak{p},\mathfrak{p}_1] + \mathfrak{p}_1$ is an ideal in $\mathfrak{g}$.) If $\mathfrak{s} \subseteq \mathfrak{k}$ is a subalgebra of $\mathfrak{g}$, then $\mathfrak{s} = \mathfrak{k} + (\mathfrak{s} \cap \mathfrak{p})$. If $X \in \mathfrak{s} \cap \mathfrak{p}$ with $X \neq 0$, then iterated brackets of members of $\mathfrak{k}$ with X span $\mathfrak{p}$. So $\mathfrak{s} \cap \mathfrak{p} = \mathfrak{p}$ and $\mathfrak{s} = \mathfrak{g}$.

Corollary. K is a maximal Lie subgroup of G.

Proof. It is known that no other Lie subgroup of G has $\mathfrak{k}$ as Lie algebra.

Lemma 2. If $X \in \mathfrak{c}$ with $X \neq 0$, then the centralizer $Z_{\mathfrak{g}}(X)$ equals $\mathfrak{k}$. If $x \in \mathrm{Ad}_G(\text{center } K)$ with $x \neq e$, then $Z_{\mathrm{Ad}\ G}(x) = \mathrm{Ad}_G K$.

Proof. $Z_{\mathfrak{g}}(X) \supseteq \mathfrak{k}$ and $Z_{\mathfrak{g}}(X)$ is a subalgebra. Since $\mathfrak{g}$ is semisimple, $Z_{\mathfrak{g}}(X) \neq \mathfrak{g}$. By Lemma 1, $Z_{\mathfrak{g}}(X) = \mathfrak{k}$. The second statement is similar.

Proof of Proposition. The identity component of $\mathrm{Ad}_G(\text{center } K)$ is a torus and thus contains an element $\mathrm{Ad}(j)$ of order 4. By Lemma 2, Ad K $= Z_{\mathrm{Ad}G}(\mathrm{Ad}\ j^2)$. $\mathrm{Ad}(j^2)$ has ± 1 as eigenvalues and is $+1$ on $\mathfrak{k}$. Since $Z_{\mathrm{Ad}G}(\mathrm{Ad}\ j^2) = \mathrm{Ad}\ K$, $\mathrm{Ad}(j^2) = -1$ on $\mathfrak{p}$. Thus $\mathrm{Ad}(j)|_{\mathfrak{p}}$ provides the required automorphism of $(G/K)_e$. Translation of this by G gives the almost complex structure at the other points of G/K.

2.4 Compact Cartan Subgroups and Roots

In this section and the next, we sharpen the proposition of Section 2.3 to produce a complex structure on G/K and to exhibit G/K as a bounded domain in $\mathbf{C}_n$. In order not to complicate the notation, we assume that G is simple, rather than just semisimple. We therefore assume

$\mathfrak{g}$ = simple Lie algebra,

$\mathfrak{k}$ = maximal compact subalgebra,

$G \supset K$, corresponding subgroups, G having finite center,

$\mathfrak{c} = \text{center } (\mathfrak{k}) \neq 0$,

$\mathfrak{h}$ = maximal Abelian subalgebra of $\mathfrak{k}$, and
T^- = exp $\mathfrak{h}$, a subgroup.

Proposition. $\mathfrak{h}$ is a maximal Abelian subalgebra in $\mathfrak{g}$, and ad $\mathfrak{h}$ is fully reducible on $\mathfrak{g}^{\mathbf{C}}$. T^- is a (compact) torus, and T^- is maximal Abelian in G.

Proof. By Lemma 2, $Z_{\mathfrak{g}}(\mathfrak{h}) \subseteq Z_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{k}$. So $Z_{\mathfrak{g}}(\mathfrak{h}) = Z_{\mathfrak{k}}(\mathfrak{h}) = \mathfrak{h}$, and $\mathfrak{h}$ is maximal Abelian in $\mathfrak{g}$. Next, ad $\mathfrak{h}$ is skew relative to the positive definite form $B_\theta(X, Y) = -B(X, \theta Y)$ (where $\theta = 1$ on $\mathfrak{k}$ and $\theta = -1$ on $\mathfrak{p}$) and is therefore fully reducible. Since $\mathfrak{h}$ is maximal Abelian in $\mathfrak{k}$, $\overline{T^-} = T^-$; thus T^- is compact and is a torus. By Corollary 2.7 of [*3*, p. 247], T^- is maximal Abelian in K.

To see that T^- is maximal Abelian in G, let $z = k \exp X \in Z_G(T^-)$ with $k \in K$, $X \in \mathfrak{p}$. We want $X = 0$. We have $k(\exp X)^{-1} = \theta z \in Z_G(T^-)$, and thus $\exp 2X = (\theta z^{-1})z \in Z_G(T^-)$. That is, $h(\exp 2X)h^{-1} = \exp 2X$ for $h \in T^-$. Hence $\exp(\mathrm{Ad}(h)2X) = \exp 2X$, and $\mathrm{Ad}(h)2X = 2X$ because exp is one–one on $\mathfrak{p}$. Consequently $[\mathfrak{h}, X] = 0$ and $X \in Z_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. Since $X \in \mathfrak{p}$, $X \in \mathfrak{h} \cap \mathfrak{p} = 0$. Thus T^- is maximal Abelian in G.

It follows from the proposition that $\mathfrak{h}^{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbf{C}}$. Form roots in the usual way: If $\alpha \in (\mathfrak{h}^{\mathbf{C}})'$, let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}^{\mathbf{C}} | [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}^{\mathbf{C}}\}$$

and call α a *root* if $\mathfrak{g}_\alpha \neq 0$. Let Σ be the set of nonzero roots. Then $\mathfrak{g}^{\mathbf{C}} = \mathfrak{h}^{\mathbf{C}} + \sum_{\alpha\in\Sigma} \mathfrak{g}_\alpha$. If $\alpha \in \Sigma$, it is known that $-\alpha \in \Sigma$, that $\dim_{\mathbf{C}} \mathfrak{g}_\alpha = 1$, and that $\frac{1}{2}$ is purely imaginary on $\mathfrak{h}$. Since $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, we have

$$[\mathfrak{h}^{\mathbf{C}}, \mathfrak{k}^{\mathbf{C}}] \subseteq \mathfrak{k}^{\mathbf{C}} \qquad \text{and} \qquad [\mathfrak{h}^{\mathbf{C}}, \mathfrak{p}^{\mathbf{C}}] \subseteq \mathfrak{p}^{\mathbf{C}}.$$

If $\alpha \in \Sigma$, it follows from the one-dimensionality of $\mathfrak{g}_\alpha$ that $\mathfrak{g}_\alpha \subseteq \mathfrak{k}^{\mathbf{C}}$ or $\mathfrak{g}_\alpha \subseteq \mathfrak{p}^{\mathbf{C}}$. We call α *compact* or *noncompact* accordingly. We have $\mathfrak{k}^{\mathbf{C}} = \mathfrak{h}^{\mathbf{C}} + \sum_{\alpha \text{ compact}} \mathfrak{g}_\alpha$ and $\mathfrak{p}^{\mathbf{C}} = \sum_{\alpha \text{ noncompact}} \mathfrak{g}_\alpha$.

Example. Let $\mathfrak{g} = \mathfrak{su}(m,n)$ as in 2.1. Then $\mathfrak{g}^{\mathbf{C}} = \mathfrak{sl}(m+n, \mathbf{C})$ and

$$\mathfrak{k}^{\mathbf{C}} = \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \text{trace } 0\right\} \quad \mathfrak{p}^{\mathbf{C}} = \left\{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\right\}.$$

If $\mathfrak{h}$ is chosen as the subalgebra of diagonal matrices in $\mathfrak{g}$, then $\mathfrak{h}^{\mathbf{C}}$ consists of all diagonal matrices of trace 0. If e_i denotes evaluation of the ith diagonal entry, the roots are $e_i - e_j$, $i \neq j$. With $\alpha = e_i - e_j$, $\mathfrak{g}_\alpha = \mathbf{C}E_{ij}$, where E_{ij} is 1 in the i-jth entry and 0 elsewhere. The compact roots $e_i - e_j$ are thus the ones for which both i and j are $\leq m$ or both i and j are $\geq m + 1$.

Proposition. If $\alpha \in \Sigma$, the following are equivalent:
(i) α is compact,
(ii) α vanishes on $\mathfrak{c}$,
(iii) α vanishes on some $H \neq 0$ in $\mathfrak{c}$.

Proof. Let $H \in \mathfrak{c} \subseteq \mathfrak{h}$ with $H \neq 0$. Then $\alpha(H) = 0 \Leftrightarrow [H, X_\alpha] = 0$ for $X_\alpha \in \mathfrak{g}_\alpha \Leftrightarrow X_\alpha \in Z_{\mathfrak{g}^c}(H) = \mathfrak{k}^{\mathbf{C}} \Leftrightarrow \alpha$ is compact. The right side of this chain is independent of H, and the proposition follows.

Corollary. $\dim \mathfrak{c} = 1$.

Proof. Let α be any noncompact root. Such exist since $\mathfrak{p} \neq 0$. By the proposition, $0 = \dim(\mathfrak{c} \cap \ker \alpha) \geq \dim \mathfrak{c} - 1$ since α is a linear functional.

Corollary. G/K admits only two invariant complex structures.

Proof. In view of the previous corollary and of the theorem in 2.3, there are at most two possibilities for J. J determines the complex structure since an almost complex mapping between complex manifolds is holomorphic.

All roots are real on the real vector space $i\mathfrak{h}$. By choosing an ordered basis for $i\mathfrak{h}$, we can order the roots by saying that α is positive if it is positive on the first basis vector on which it is nonzero. Choose a *good ordering*, in that every noncompact positive root is larger than every compact root. (For instance, take a vector in $\mathfrak{c}$ as the first basis vector and appeal to the proposition.) Let

$$\mathfrak{p}^+ = \sum_{\substack{\alpha>0 \\ \alpha \text{ noncompact}}} \mathfrak{g}_\alpha , \qquad \mathfrak{p}^- = \sum_{\substack{\alpha<0 \\ \alpha \text{ noncompact}}} \mathfrak{g}_\alpha .$$

In the example $\mathfrak{su}(m, n)$, it is possible to choose a good ordering in such a way that the positive roots are $e_i - e_j$ with $i < j$. Then

$$\mathfrak{p}^+ = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right\} \text{ and } \mathfrak{p}^- = \left\{\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\right\}.$$

Proposition. $\mathfrak{p}^{\mathbf{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$, $[\mathfrak{k}^{\mathbf{C}}, \mathfrak{p}^+] \subseteq \mathfrak{p}^+$, $[\mathfrak{k}^{\mathbf{C}}, \mathfrak{p}^-] \subseteq \mathfrak{p}^-$, and $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are Abelian subspaces.

Proof. Clearly $\mathfrak{p}^{\mathbf{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$. Let α, β, and $\alpha + \beta$ be in Σ with α compact and β noncompact. Then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ and β and $\alpha + \beta$ are both positive or both negative. Summing on α and β, we obtain $[\mathfrak{k}^{\mathbf{C}}, \mathfrak{p}^+] \subseteq \mathfrak{p}^+$ and $[\mathfrak{k}^{\mathbf{C}}, \mathfrak{p}^-] \subseteq \mathfrak{p}^-$.

The idea in the proof that $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are Abelian is that they are the eigenspaces of the automorphism J of $\mathfrak{p}^{\mathbf{C}}$ that was considered in 2.3. However, we are given the existence of the center $\mathfrak{c}$, not of J, and J must be constructed. Thus by the proposition in 2.3, choose $J \in \operatorname{Aut} \mathfrak{p}$ with $J^2 = -I$ so that J commutes with $\operatorname{Ad} K$. The proof of the theorem in 2.3 shows that $J = \operatorname{ad} V\big|_{\mathfrak{p}}$ for some $V \in \mathfrak{c}$. Suppose for definiteness that the positive noncompact roots are positive on $-iV$. If $\alpha > 0$ is noncompact and $X \in \mathfrak{g}_\alpha$, then $-i\,\alpha(V) > 0$ and

$$JX = (\operatorname{ad} V)X = \alpha(V)X = (-i\,\alpha(V))\,(iX).$$

Since $J^2 = -I$, J has eigenvalues $\pm i$, and it follows that $JX = iX$. If $\beta > 0$ is noncompact and $Y \in \mathfrak{g}_\beta$, then $JY = iY$ and the proof of the theorem in 2.3 shows that $\operatorname{ad} [X, Y]\big|_{\mathfrak{p}^c} = 0$. Since $\mathfrak{g}$ has no compact factor, $[X, Y] = 0$. Summing on α and β, we see that $[\mathfrak{p}^+, \mathfrak{p}^+] = 0$. Similarly $[\mathfrak{p}^-, \mathfrak{p}^-] = 0$.

Corollary (OF PROOF). All noncompact positive roots agree on $\mathfrak{c}$.

Proof. $-i\,\alpha(V) = 1$ for all positive noncompact roots.

2.5 Harish-Chandra Decomposition

We retain the notation of 2.4 but assume further that G is a matrix group. Define

$$\mathfrak{b} = \mathfrak{h}^{\mathbf{C}} + \sum_{\alpha < 0} \mathfrak{g}_\alpha,$$

$$\begin{aligned}
\mathfrak{g}^{\mathbf{C}} &= \text{complexified algebra of matrices } (\mathfrak{g} \text{ being an algebra} \\
&\quad \text{of matrices}), \\
G^{\mathbf{C}} &= \text{analytic subgroup of matrices with Lie algebra } \mathfrak{g}^{\mathbf{C}}, \text{ and} \\
P^-, K^{\mathbf{C}}, P^+, B &= \text{subgroups of } G^{\mathbf{C}} \text{ corresponding to } \mathfrak{p}^-, \mathfrak{k}^{\mathbf{C}}, \mathfrak{p}^+, \mathfrak{b}.
\end{aligned}$$

Each of these groups has a natural complex structure consistent with multiplication by i in $\mathfrak{g}^{\mathbf{C}}$. It is not hard to see that exp is a diffeomorphism on $\mathfrak{p}^+$ and $\mathfrak{p}^-$ and hence that P^+ and P^- are holomorphically isomorphic with $\mathbf{C}^n$ for a suitable n.

Theorem. Multiplication $P^- \times K^{\mathbf{C}} \times P^+ \to G^{\mathbf{C}}$ is one–one, holomorphic, and regular (with the image open in $G^{\mathbf{C}}$), BG is open in $G^{\mathbf{C}}$, and there exists a bounded open subset $\Omega \subseteq P^+$ such that

$$BG = P^- K^{\mathbf{C}} G = P^- K^{\mathbf{C}} \Omega.$$

Interpretation. G acts on Ω by holomorphic automorphisms under the definition $(w)g = p^+(wg)$, for $w \in \Omega$ and $g \in G$, where $p^+(\,\cdot\,)$ refers to the P^+ component of an element of $P^- K^{\mathbf{C}} P^+$. The formula $P^- K^{\mathbf{C}} G = P^- K^{\mathbf{C}} \Omega$ shows the action is transitive. For the isotropy subgroup at $1 \in \Omega$, we ask when $p^+(1g) = 1$ or $g \in P^- K^{\mathbf{C}}$. Thus the isotropy subgroup is $G \cap P^- K^{\mathbf{C}}$. On the Lie algebra level, one verifies that $\mathfrak{g} \cap (\mathfrak{p}^- + \mathfrak{k}^{\mathbf{C}}) = \mathfrak{k}$, and it follows that $G \cap P^- K^{\mathbf{C}} = K$ since no other Lie subgroup of G has $\mathfrak{k}$ as Lie algebra. Thus $\Omega = K \backslash G$, Ω is a bounded domain in $\mathbf{C}^n$, and G operates holomorphically. The theorem therefore completes the solution of the two problems at the end of 2.1.

For the proof of the theorem we shall use some further notation:

$$\mathfrak{n}_c^- = \sum_{\alpha < 0} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_c^+ = \sum_{\alpha > 0} \mathfrak{g}_\alpha, \qquad \mathfrak{h}^+ = i\mathfrak{h},$$

N_c^-, T^+ = analytic subgroups of $G^{\mathbf{C}}$ with Lie algebras $\mathfrak{n}_c^-$, $\mathfrak{h}^+$.

Then
$$B = N_c^- T^+ T^-.$$

Lemma 1. $P^- K^{\mathbf{C}} \cap P^+ = \{1\}$ and $K^{\mathbf{C}} P^+ \cap P^- = \{1\}$.

Proof. Let $y \in P^- K^{\mathbf{C}} \cap P^+$. Since $y \in P^- K^{\mathbf{C}}$, $\mathrm{Ad}(y)\mathfrak{p}^- \subseteq \mathfrak{p}^-$. Choose $Y \in \mathfrak{p}^+$ so that $y = \exp Y$. Write $Y = \sum c_\alpha X_\alpha$, $X_\alpha \neq 0$ in $\mathfrak{g}_\alpha$, where the sum is over positive noncompact α. If $Y \neq 0$, let α_0 be the smallest α so that $c_\alpha \neq 0$. Then $[Y, X_{-\alpha_0}] \equiv c_{\alpha_0}[X_{\alpha_0}, X_{-\alpha_0}] \bmod \mathfrak{n}_c^+$, and so

$$\mathrm{Ad}(y)X_{-\alpha_0} \equiv X_{-\alpha_0} + c_{\alpha_0}[X_{\alpha_0}, X_{-\alpha_0}] \bmod \mathfrak{n}_c^+,$$
$$\mathrm{Ad}(y)X_{-\alpha_0} \equiv c_{\alpha_0}[X_{\alpha_0}, X_{-\alpha_0}] \bmod (\mathfrak{n}_c^+ + \mathfrak{p}^-).$$

Thus $\mathrm{Ad}(y)X_{-\alpha_0} \notin \mathfrak{p}^-$, contradiction. So $Y = 0$ and $P^- K^{\mathbf{C}} \cap P^+ = \{1\}$. The other identity is proved similarly.

Lemma 2. $(\mathfrak{n}_c^- + \mathfrak{h}^+) \cap \mathfrak{g} = 0$.

Reference. Corollary, [*2*, p. 2].

Lemma 3. Let $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, and let G, H_1, and H_2 be corresponding analytic subgroups. Then multiplication $H_1 \times H_2 \to G$ is everywhere regular.

Reference. Lemma 5.2, [*3*, p. 235].

Lemma 4. *BG is open in $G^{\mathbf{C}}$.*

Proof. $BG = N_c^- T^+ T^- G = (N_c^- T^+)G$ and $(\mathfrak{n}_c^- + \mathfrak{h}^+) \cap \mathfrak{g} = 0$ by Lemma 2. Also $\dim_{\mathbf{R}} \mathfrak{g}^{\mathbf{C}} = \dim_{\mathbf{R}}(\mathfrak{n}_c^- + \mathfrak{h}^+) + \dim_{\mathbf{R}} \mathfrak{g}$ because

$$\begin{aligned}\dim_{\mathbf{R}} \mathfrak{g}^{\mathbf{C}} &= \dim_{\mathbf{R}} \mathfrak{g} + \dim_{\mathbf{C}} \mathfrak{g}^{\mathbf{C}} = \dim_{\mathbf{R}} \mathfrak{g} + \dim_{\mathbf{C}} \mathfrak{n}_c^- + \dim_{\mathbf{C}} \mathfrak{n}_c^+ + \dim_{\mathbf{C}} \mathfrak{h}^{\mathbf{C}} \\ &= \dim_{\mathbf{R}} \mathfrak{g} + \dim_{\mathbf{R}} \mathfrak{n}_c^- + \dim_{\mathbf{R}} \mathfrak{h}^+ = \dim_{\mathbf{R}} \mathfrak{g} + \dim_{\mathbf{R}}(\mathfrak{n}_c^- + \mathfrak{h}^+).\end{aligned}$$

So $\mathfrak{g}^{\mathbf{C}} = (\mathfrak{n}_c^- + \mathfrak{h}^+) \oplus \mathfrak{g}$, and the result follows from Lemma 3.

Lemma 5. Multiplication $P^- \times K^{\mathbf{C}} \times P^+ \to G^{\mathbf{C}}$ is one–one, holomorphic, and regular.

Proof. It is clearly holomorphic, and it is regular by two applications of Lemma 3. By Lemma 1, $(P^- K^{\mathbf{C}}) \cap P^+ = \{1\}$. Since $P^- \cap (K^{\mathbf{C}} P^+) = \{1\}$, $P^- \cap K^{\mathbf{C}} = \{1\}$. Thus 1 has only one preimage. But $p_1 k_1 q_1 = p_2 k_2 q_2$ implies $(p_2^{-1} p_1)(k_1 k_2^{-1})(k_2(q_1 q_2^{-1})k_2^{-1}) = 1$ and hence $p_1 = p_2$, $k_1 = k_2$, and $q_1 = q_2$.

An Iwasawa decomposition $G = KA_0N$ is developed from a maximal Abelian subspace $\mathfrak{a}_0$ of $\mathfrak{p}$. Then $G = KAK$ also, though not uniquely. We shall construct an $\mathfrak{a}_0$ with special properties.

Lemma 6. There exist positive noncompact roots $\gamma_1, \ldots, \gamma_s$ such that $\sum_{i=1}^{s} \mathbf{C}(X_{\gamma_j} + X_{-\gamma_j})$ is a maximal Abelian subspace of $\mathfrak{p}^{\mathbf{C}}$.

Reference. [*3*, pp. 314–315].

The Killing form is nondegenerate on $\mathfrak{h}^{\mathbf{C}}$, and we can use this form by duality to define $\langle \alpha, \beta \rangle$ for $\alpha, \beta \in (\mathfrak{h}^{\mathbf{C}})'$. If α is a nonzero root, then one can show that $\langle \alpha, \alpha \rangle > 0$. According to [*3*, p. 220], it is possible to choose vectors $X_\alpha \in \mathfrak{g}_\alpha$ in such a way that

$$X_\alpha - X_{-\alpha} \in \mathfrak{k} + i\mathfrak{p}, \qquad i(X_\alpha + X_{-\alpha}) \in \mathfrak{k} + i\mathfrak{p},$$
$$B(X_\alpha, X_{-\alpha}) = 2/\langle \alpha, \alpha \rangle. \tag{*}$$

This normalization is arranged to connect $\mathfrak{g}^{\mathbf{C}}$ with $\mathfrak{sl}(2, \mathbf{C})$. Namely if $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, then $\mathfrak{sl}(2, \mathbf{C}) = \mathbf{C}H + \mathbf{C}X + \mathbf{C}Y$ and the bracket relations are $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$. The normalization (*) is arranged so that the mapping

$$H \to [X_\alpha, X_{-\alpha}], \qquad X \to X_\alpha, \qquad Y \to X_{-\alpha}$$

extends to an isomorphism of $\mathfrak{sl}(2, \mathbf{C})$ into $\mathfrak{g}^{\mathbf{C}}$ that carries $\mathfrak{su}(1,1)$ into $\mathfrak{g}$ and $\mathbf{R}H$ into $\mathfrak{h}^+$.

Corollary. Under the normalization (*), the positive noncompact roots $\gamma_1, \ldots, \gamma_s$ of Lemma 6 are such that

$$\mathfrak{a}_0 = \sum_{i=1}^{s} \mathbf{R}(X_{\gamma_i} + X_{-\gamma_i})$$

is a maximal Abelian subspace of $\mathfrak{p}$.

Proof. $X_{\gamma_i} + X_{-\gamma_i} \in \mathfrak{p}^{\mathbf{C}} \cap i^{-1}(\mathfrak{k} + i\mathfrak{p}) = \mathfrak{p}$. So $\mathfrak{a} \subset \mathfrak{p}$. The rest follows from Lemma 6.

Lemma 7. Let L be a Lie group with Lie algebra $\mathfrak{sl}(2, \mathbf{C}) = \mathbf{C}H + \mathbf{C}X + \mathbf{C}Y$ as above. Then

$$\exp t(X + Y) = \exp(\tanh t)Y \exp(\log(\cosh t))H \exp(\tanh t)X.$$

Proof. Since $SL(2, \mathbf{C})$ is simply connected, it is enough to verify this identity there, where it is the special case of the identity

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \bar{\beta}/\alpha & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & \beta/\alpha \\ 0 & 1 \end{pmatrix}$$

with $\alpha = \cosh t$ and $\beta = \sinh t$.

Corollary. If $Z = \sum_{i=1}^{s} t_i(X_{\gamma_i} + X_{-\gamma_i}) \in \mathfrak{a}_0$, then

$$\exp Z = \exp Y_0 \exp H_0 \exp X_0$$

with

$$Y_0 = \sum(\tanh t_i)X_{-\gamma_i}, H_0 = \sum(\log\cosh t_i)[X_{\gamma_i}, X_{-\gamma_i}], X_0 = \sum(\tanh t_i)X_{\gamma_i}.$$

Proof. Apply Lemma 7, using the commutativity of X_{γ_i} and $X_{\pm\gamma_j}$ for $i \neq j$.

In the corollary, we have $Y_0 \in \mathfrak{p}^-$, $H_0 \in \mathfrak{h}^+ \subseteq \mathfrak{k}^{\mathbf{C}}$, and $X_0 \in \mathfrak{p}^+$. The corollary therefore gives explicitly the imbedding of the group $A_0 = \exp \mathfrak{a}_0$ in $P^-K^{\mathbf{C}}P^+$.

Lemma 8. $BG \subseteq P^-K^{\mathbf{C}}G \subseteq P^-K^{\mathbf{C}}P^+$.

Proof. Since $\mathfrak{b} \subseteq \mathfrak{p}^- + \mathfrak{k}^{\mathbf{C}}$, we have $B \subseteq P^-K^{\mathbf{C}}$, and it is enough to prove $G \subseteq P^-K^{\mathbf{C}}P^+$. Since $G = KA_0K$ and K normalizes P^- and P^+, it is enough to show $A_0 \subseteq P^-K^{\mathbf{C}}P^+$. But this follows from the corollary above.

Proof of Theorem. Let $\mathfrak{b}_K = \mathfrak{h}^{\mathbf{C}} + \sum_{\substack{\alpha<0 \\ \alpha \text{ compact}}} \mathfrak{g}_\alpha$, B_K the corresponding analytic subgroup. B_K is known to be closed, and K is compact; thus $B_K K$ is closed. On the other hand, $B_K K$ is open in K^C by the same kind of argument as in Lemma 4. Thus $K^{\mathbf{C}} = B_K K$. Therefore

$$P^-K^{\mathbf{C}}G \subseteq BK^{\mathbf{C}}G = B\, B_K KG = BG.$$

That is, $BG = P^-K^{\mathbf{C}}G$ and so $BG = P^-K^{\mathbf{C}}\Omega$ with $\Omega \subseteq P^+$. Since BG is open in $G^{\mathbf{C}}$, Ω is open in P^+.

We are to show that Ω is bounded, i.e., that $p^+(G)$ is bounded. Choose a K-invariant inner product on $\mathfrak{p}^+$. Since $G = KA_0K$ with K normalizing P^- and P^+, we see that we are to prove that $\|\log p^+(A_0)\|$ is bounded. In view of the corollary to Lemma 7,

$$\|\log p^+(\exp \sum t_i(X_{\gamma_i} + X_{-\gamma_i})\| = \|\sum(\tanh t_i)X_{\gamma_i}\| \leqq \sum_{i=1}^{s} \|X_{\gamma_i}\|,$$

and the right side is bounded independently of the t_i. This proves the theorem.

3. Holomorphic Discrete Series

3.1 Construction of Representations

Throughout this chapter, we retain the notation of 2.5. We begin by summarizing some properties of irreducible unitary representations of the compact group K. Let τ be such a representation (necessarily finite-dimensional) on a complex vector space V. Since the subgroup T^- of K is Abelian, V is the orthogonal direct sum of subspaces on which each member of $\tau(T^-)$ operates as a scalar. If these subspaces are chosen as large as possible, they are called the *weight spaces* for τ. For the corresponding representation of $\mathfrak{k}$, each member of $\tau(\mathfrak{h})$ operates as a scalar on each weight space. If the weight space is held fixed and the member of $\mathfrak{h}$ is allowed to vary, the scalar value of the member of $\tau(\mathfrak{h})$ defines a complex-valued linear functional on $\mathfrak{h}$. The corresponding complex-linear functionals on $\mathfrak{h}^{\mathbf{C}}$ are called the *weights* of τ. Since τ is unitary, the weights are imaginary on $\mathfrak{h}$, and hence real on $\mathfrak{h}^+ = i\mathfrak{h}$. We have already defined an ordering on $(\mathfrak{h}^+)'$, and we can therefore speak of the *highest weight* of τ, which is the largest weight in this ordering.

Some known properties of the highest weight Λ of τ are as follows:

(1) Λ is integral, i.e., $\xi_\Lambda(h) = \exp \Lambda(\log h)$ is a well-defined character of $T^{\mathbf{C}} = \exp \mathfrak{h}^{\mathbf{C}}$.

(2) Λ is *dominant with respect to* $\mathfrak{k}$, i.e., $\langle \Lambda, \alpha \rangle \geqq 0$ for all positive compact roots α.

(3) The weight space for the weight Λ has complex dimension one.

(4) τ is determined up to unitary equivalence by Λ.

Conversely every integral linear form on $\mathfrak{h}^{\mathbf{C}}$, dominant with respect to $\mathfrak{k}$, is the highest weight of some irreducible unitary representation of K.

Let Λ be an integral form on $\mathfrak{h}^{\mathbf{C}}$ dominant with respect to $\mathfrak{k}$, and let ξ_Λ be the associated character of $T^{\mathbf{C}}$. We have $B = N_c^- T^{\mathbf{C}}$, and we extend ξ_Λ to a holomorphic character of B by setting $\xi_\Lambda(n) = 1$ for $n \in N_c^-$. Define

$$\Gamma(\Lambda) = \left\{ F : BG \to \mathbf{C} \;\middle|\; \begin{array}{ll} \text{(i)} & F \text{ holomorphic on } BG \\ \text{(ii)} & F(bx) = \xi(b)F(x) \text{ for } b \in B,\ x \in BG \end{array} \right. ,$$

$$H(\Lambda) = \left\{ F \in \Gamma(\Lambda) \middle| \; \|F\|^2 = \int_G |F(g)|\, dg < \infty \right\},$$

$$U_\Lambda(g)F(x) = F(xg) \qquad \text{for} \quad F \in \Gamma(\Lambda),\ g \in G,\ x \in BG.$$

Then $U_\Lambda(g)$ preserves both $\Gamma(\Lambda)$ and $H(\Lambda)$. These definitions are completely analogous to the ones discussed in Section 1 for $G = SU(1,1)$. The main result is as follows:

Theorem (Harish-Chandra). If Λ is an integral form on $\mathfrak{h}^{\mathbf{C}}$ dominant with respect to $\mathfrak{k}$, then $H(\Lambda)$ is a Hilbert space, and U is a continuous unitary representation on it. If furthermore

$$\langle \Lambda + \rho, \alpha \rangle < 0 \qquad \left(\rho = \tfrac{1}{2} \sum_{\beta > 0} \beta\right)$$

for every noncompact positive root α, then $H(\Lambda)$ is nonzero, U is irreducible, and the matrix coefficients of U are square integrable.

Example of $SU(1,1)$. Here

$$\mathfrak{h} = \begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix}, \qquad \mathfrak{h}^{\mathbf{C}} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

$$\Lambda \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} = -na, \quad \alpha \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} = 2a, \quad \rho = \tfrac{1}{2}\alpha, \quad \rho \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} = a.$$

Λ integral means n is an integer; Λ is automatically dominant with respect to $\mathfrak{k}$ since the only positive root α is noncompact. Let

$$\Lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = c_1, \qquad \Lambda_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = c_2.$$

Then $\langle \Lambda_1, \Lambda_2 \rangle = cc_1c_2$ for a certain constant $c > 0$. Thus

$$\langle \Lambda + \rho, \alpha \rangle = c(-n + 1)2,$$

and Harish-Chandra's condition $\langle \Lambda + \rho, \alpha \rangle < 0$ is the condition $n > 1$. Therefore

$$\xi_\Lambda \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a^{-n} \qquad \text{with} \quad n \geqq 2.$$

In this example, when the representations are viewed as operating on functions on the disk, the constant function plays a special role, in that the mean value property is expressed in terms of it. That is, we have

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{2i\theta}z)\,d\theta = f(0),$$

which we rewrite as

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\,U_n\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta}\end{pmatrix} f(z)\,d\theta = f(0),$$

and we can regard the right side as equal to $f(0)1(z)$, where $1(z)$ is the constant function. Passing to Harish-Chandra's realization by putting $F(\zeta, \gamma, z) = \gamma^{-n}f(z)$, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} F\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix} x \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta}\end{pmatrix}\right) d\theta = F(1)\psi_n(x), \qquad x \in BG,$$

where $\psi_n(\zeta, \gamma, z) = \gamma^{-n}$ is the extension of the constant function. We begin the discussion of general G by defining the analog of ψ_n and by giving the analog of this mean value property.

Recall that $BG = P^-K^{\mathbf{C}}\Omega \subseteq P^+K^{\mathbf{C}}P^+$. Let $\mu(x)$ be the middle (or $K^{\mathbf{C}}$) component of the member x of $P^-K^{\mathbf{C}}P^+$. Let τ be an irreducible unitary representation of K, and extend τ holomorphically to $K^{\mathbf{C}}$ (by passing from K to $\mathfrak{k}$ to $\mathfrak{k}^{\mathbf{C}}$ to $K^{\mathbf{C}}$). Let ϕ_Λ be a highest weight vector for τ_Λ and suppose $\|\phi_\Lambda\| = 1$. Define

$$\psi_\Lambda(x) = (\tau_\Lambda(\mu(x))\phi_\Lambda, \phi_\Lambda) \qquad \text{for} \quad x \in P^-K^{\mathbf{C}}P^+.$$

Lemma. The restriction to BG of ψ_Λ is in $\Gamma(\Lambda)$.

Proof. ψ_Λ is holomorphic, being the composition of the holomorphic maps $x \in P^-K^{\mathbf{C}}P^+ \to \mu(x) \in K^{\mathbf{C}}$, $k \in K^{\mathbf{C}} \to \tau_\Lambda(k) \in GL(n, \mathbf{C})$, $g \in GL(n, \mathbf{C}) \to (g\phi_1, \phi_2)$. To see that it transforms appropriately under B on the left, let $b \in B$ and $x \in BG$. Write $b = p^-k^{\mathbf{C}}$ with $p^- \in P^-$ and $k^{\mathbf{C}} \in K^{\mathbf{C}} \cap B$; then $\mu(bx) = \mu(k^{\mathbf{C}}x) = k^{\mathbf{C}}\mu(x)$ since $K^{\mathbf{C}}$ normalizes P^-. Thus

$$\psi_\Lambda(bx) = (\tau_\Lambda(\mu(bx))\phi_\Lambda, \phi_\Lambda) = (\tau_\Lambda(k^{\mathbf{C}})\tau_\Lambda(\mu(x))\phi_\Lambda, \phi_\Lambda).$$

Let $\phi_1 = \phi_\Lambda, \phi_2, \ldots, \phi_d$ be an orthonormal system of weight vectors arranged in order so that the corresponding weights decrease, and write

$$\tau_\Lambda(\mu(x)) = \sum c_j \phi_j$$

and $k^{\mathbf{C}} = hn$ with $h \in T^{\mathbf{C}} = \exp \mathfrak{h}^{\mathbf{C}}$ and $n \in N_c^- \cap K^{\mathbf{C}}$. Then the matrix of $\tau_\Lambda(n)$ can be seen to be lower triangular in the basis $\{\phi_j\}$, with ones on the diagonal, and so

$$\begin{aligned}\tau_\Lambda(k^{\mathbf{C}})\tau_\Lambda(\mu(x))\phi_\Lambda = \tau_\Lambda(h)\tau_\Lambda(n)\sum c_j\phi_j &= \tau_\Lambda(h)[c_1\phi_\Lambda + \sum_{j\geqq 2} d_j\phi_j] \\ &= c_1\xi_\Lambda(h)\phi_\Lambda + \sum_{j\geqq 2} d_j\xi_{\Lambda_j}(h)\phi_j.\end{aligned}$$

Thus

$$\psi_\Lambda(bx) = c_1\xi_\Lambda(h) = \xi_\Lambda(h)\,(\tau_\Lambda(\mu(x))\phi_\Lambda,\phi_\Lambda) = \xi_\Lambda(h)\psi_\Lambda(x) = \xi_\Lambda(b)\psi_\Lambda(x)$$

since $\xi_\Lambda(p^-) = \xi_\Lambda(n) = 1$.

Main Lemma. For every F in $\Gamma(\Lambda)$,

$$\int_{T^-} F(hxh^{-1})\,dh = F(1)\psi_\Lambda(x).$$

The main lemma gives the generalization of the mean value property. We shall give the idea of the proof in 3.3.

Note that the members of $\Gamma(\Lambda)$ are determined by their values on G. Since the functions in $\Gamma(\Lambda)$ are continuous, $\|F\| = 0$ implies $F = 0$.

Proposition. $|F(1)| \leqq \|\psi_\Lambda\|^{-1}\,\|F\|$ for all $F \in H(\Lambda)$.

Proof. We can assume $F(1) \neq 0$. Applying the main lemma to F and recalling that $T^- \subseteq B$, we have

$$\int_{T^-} F(xh)\overline{\xi_\Lambda(h)}\,dh = \int_{T^-} F(h^{-1}xh)\,dh = F(1)\psi_\Lambda(x).$$

Thus

$$\|F\|^2 = \int_G |F(x)|^2 dx = \int_G |F(xh)\overline{\xi_\Lambda(h)}|^2 dx \qquad \text{for all} \quad h \in T^-$$

$$
\begin{aligned}
&= \int_G \int_{T^-} \left|F(xh)\overline{\xi_\Lambda(h)}\right|^2 \, dh \, dx \\
&\geq \int_G \left| \int_{T^-} F(xh)\overline{\xi_\Lambda(h)} \, dh \right|^2 dx \qquad \text{by the Schwarz inequality} \\
&= \int_G |F(1)|^2 \, |\psi_\Lambda(x)|^2 \, dx \\
&= |F(1)|^2 \, \|\psi_\Lambda\|^2.
\end{aligned}
$$

Corollary. $H(\Lambda) \neq 0$ if and only if $\|\psi_\Lambda\| < \infty$.

Proof. If $\|\psi_\Lambda\| < \infty$, then $\psi_\Lambda \in H(\Lambda)$. If $\|\psi_\Lambda\| = \infty$, then the proposition shows that $U_\Lambda(g)F(1) = 0$ for all $g \in G$ and $F \in H(\Lambda)$. Hence $F \equiv 0$.

Corollary. To each compact set $E \subseteq BG$ corresponds a constant $C_E < \infty$ such that

$$|F(x)| \leqq C_E \|F\|$$

for all $F \in H(\Lambda)$ and $x \in E$.

This is a simple consequence of the proposition since $\|U_\Lambda(g)F\| = \|F\|$ and since $|F(bx)| = |\xi_\Lambda(b)| \, |F(x)|$.

Corollary. $H(\Lambda)$ is complete.

Proof. Norm convergence in $H(\Lambda)$ implies uniform convergence on compact sets of BG, by the previous corollary. This shows that $H(\Lambda)$ is identified with a closed subspace of $L^2(G)$. Hence $H(\Lambda)$ is complete.

Clearly U_Λ is a unitary representation. It is continuous because $(H(\Lambda), U_\Lambda)$ is isomorphic with a subspace of $L^2(G)$, and right translation is continuous in $L^2(G)$.

3.2 Properties of the Representations

Proposition. If $H(\Lambda) \neq 0$, then U_Λ is irreducible.

Proof. Let $H \subseteq H(\Lambda)$ be a nonzero closed invariant subspace, and let

$F \neq 0$ be in H. After applying some $U_\Lambda(g)$, we can assume that $F(1) \neq 0$. The limit of a linear combination of translates of F given by

$$\int \overline{\xi_\Lambda(h)}\, U_\Lambda(h) F \, dh$$

is again in H. But the main lemma shows that this is $F(1)\psi_\Lambda$. Thus ψ_Λ is in H. If $H^\perp \neq 0$ also, then ψ_Λ is in $H^\perp$, which is a contradiction. Hence $H = 0$ or $H^\perp = 0$.

In the previous section we showed that $H(\Lambda) \neq 0$ if and only if $\|\psi_\Lambda\| < \infty$. We now turn to the evaluation of $\|\psi_\Lambda\|$. In $\mathfrak{a}_0$, it is possible to choose a closed cone $\mathfrak{a}_0^+$ that is a fundamental domain for the action of $\mathrm{Ad}(K)$ in the following sense: Any point of $\mathfrak{a}_0$ is conjugate by some member of K to a member of $\mathfrak{a}_0^+$, and no two points of the interior of $\mathfrak{a}_0^+$ are conjugate to each other. (In $SU(1,1)$, $\mathfrak{a}_0 = \left\{ \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, t \in \mathbf{R} \right\}$, and $\mathfrak{a}_0^+ = \left\{ \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, t \geq 0 \right\}$. Let $A_0^+ = \exp \mathfrak{a}_0^+$, da = Haar measure on A_0. Then $G = KA_0^+K$. By [3, pp. 381–382], there is a continuous function $D(a) \geqq 0$ on A_0^+ such that

$$dx = D(a)\, dk_1\, da\, dk_2 \qquad \text{under} \quad x = k_1\, a\, k_2.$$

Lemma. $\|\psi_\Lambda\|^2 = d_\Lambda^{-2} \int_{A_0^+} \mathrm{Tr}(\tau_\Lambda(\mu(a))^2 D(a)\, da$ where d_Λ is the degree of τ_Λ.

Proof. Again choose an orthonormal basis of weight vectors $\phi_1 = \phi_\Lambda$, $\phi_2, \ldots, \phi_d$. If v is in the representation space of τ_Λ, then

$$\int_K |(\tau_\Lambda(k)v, \phi)|^2\, dk = d_\Lambda^{-1} \|v\|^2. \qquad (*)$$

[This is a consequence of the Schur orthogonality relations

$$\int_K (\tau_\Lambda(k)\phi_i, \phi_j)\, \overline{(\tau_\Lambda(k)\phi_{i'}, \phi_{j'})}\, dk = \delta_{ii'}\, \delta_{jj'}\, d_\Lambda^{-1}. \;]$$

Since $\mu(k'\, a\, k) = k'\mu\,(a)k$, we have

$$\begin{aligned} \|\psi_\Lambda\|^2 = \int_G |\psi_\Lambda(x)|^2\, dx &= \int_{K \times A_0^+ \times K} |(\tau_\Lambda(k'\mu(a)k)\phi_\Lambda, \phi_\Lambda)|^2 D(a)\, dk'\, da\, dk \\ &= d_\Lambda^{-1} \int_{A_0^+ \times K} \|\tau_\Lambda(\mu(a)\tau_\Lambda(k)\phi)\|^2 D(a)\, dk\, da \quad \text{by } (*). \end{aligned}$$

Write $\tau_\Lambda(k)\phi_i = \sum_j \tau_{ij}(k)\phi_j$ and put $i = 1$. Then

$$\|\psi_\Lambda\|^2 = d_\Lambda^{-1}\int_{A_0^+\times K} \sum_{j,l}(\tau_\Lambda(\mu(a))\phi_j,\ \tau_\Lambda(\mu(a))\phi_l)\tau_{j1}(k)\overline{\tau_{l1}(k)}D(a)\,dk\,da$$

$$= d_\Lambda^{-2}\int_{A_0^+} \sum_j\|\tau_\Lambda(\mu(a)\,)\phi_j\|^2\ D(a)\,da.$$

In the corollary to Lemma 7 of Section 2.5, we saw that $\mu(a)\in T^+$. Therefore $\tau_\Lambda(\mu(a)\,)$ is diagonal and self-adjoint. The lemma follows.

The evaluation of the integral in the statement of the lemma is complicated and will be omitted. In Section 2.3, we shall look at the integral more closely in two special cases as illustrations. We shall need the condition $\langle\Lambda + \rho, \alpha\rangle < 0$ for α positive and noncompact, in order to get a finite answer.

Proposition. If $\|\psi_\Lambda\| < \infty$, then $(U_\Lambda(x)\psi_\Lambda, \psi_\Lambda) = \psi_\Lambda(x)\|\psi_\Lambda\|^2$.

Remark. Thus when $\|\psi_\Lambda\| < \infty$, U_Λ has a square integrable matrix coefficient and is therefore square integrable.

Proof. Apply the main lemma to $U_\Lambda(y)\psi_\Lambda$ and get

$$\int_{T^-}\psi_\Lambda(hxh^{-1}y)\,dh = \psi_\Lambda(y)\psi_\Lambda(x).$$

Then

$$\psi_\Lambda(y)\|\psi_\Lambda\|^2 = \int_G \psi_\Lambda(y)\psi_\Lambda(x)\overline{\psi_\Lambda(x)}dx = \int_G\int_{T^-}\overline{\psi_\Lambda(x)}\psi_\Lambda(hxh^{-1}y)\,dh\,dx$$

$$= \int_{T^-}\int_G \overline{\psi_\Lambda(x)}\psi_\Lambda(hxh^{-1}y)\,dx\,dh = \int_{T^-}\int_G\overline{\psi_\Lambda(x)}\psi_\Lambda(xy)\,dx\,dh$$

$$= (U_\Lambda(y)\psi_\Lambda, \psi_\Lambda),$$

the next-to-last equality holding after the change of variables $x \to h^{-1}xh$ since $\psi_\Lambda(h^{-1}xh) = \psi_\Lambda(x)$.

3.3 Comments about the Main Lemma and $\|\psi_\Lambda\|$

Main Lemma. For every F in $\Gamma(\Lambda)$,

$$\int_{T^-} F(hxh^{-1})\, dh = F(1)\psi_\Lambda(x).$$

Idea of Proof. The idea is to show near $x = 1$ that the left side is $F(1)$ times a power series in x that is independent of F. The power series is evaluated as the series for $\psi_\Lambda(x)$ by putting $F = \psi_\Lambda$. Then since both sides are holomorphic on BG and equal in a neighborhood of 1, they are equal everywhere. Thus for $X \in \mathfrak{g}^{\mathbf{C}}$ and $g \in BG$, let

$$\tilde{X}f(g) = d/dt\, f(g \exp tX)\big|_{t=0}.$$

Since F is real analytic on BG, we can expand F in a power series about $1 \in G^{\mathbf{C}}$ (cf. [*3*, p. 95]):

$$F(\exp X) = \sum_{n\geq 0} \frac{1}{n!} (\tilde{X}^n F)(1).$$

Conjugate by $h \in T^-$ and integrate, letting L–S denote the left hand side:

$$L-S = \sum_{n\geq 0} \frac{1}{n!} (\{ \int_{T^-} Ad(h)\, \tilde{X}^n\, dh\} F)(1).$$

Choose a basis $\{X_1, \ldots, X_s\}$ of $\mathfrak{g}^{\mathbf{C}}$ over $\mathbf{C}$, write $X = z_1 X_1 + \cdots + z_s X_s$, and expand the integral into a sum of integrals of monomials. In this expansion, the various z_j's factor out because $\tilde{X}$ is complex-linear when applied to a holomorphic function. The integral of each monomial is T^--invariant. If $\{X_1, \ldots, X_s\}$ is chosen with the first vectors in $\mathfrak{n}_c^+$, the next ones in $\mathfrak{h}^{\mathbf{C}}$, and the last ones in $\mathfrak{n}_c^-$, then the integral of a monomial is 0 unless the monomial itself was T^--invariant. In the monomial, the $\mathfrak{n}_c^-$ vectors can be put at the right end, and any monomial that does have vectors in $\mathfrak{n}_c^-$ present contributes 0 to L-S since F is constant on N_c^-. It follows that after the integration, the only nonzero terms come from monomials with all vectors in $\mathfrak{h}^{\mathbf{C}}$. Such monomials have F as eigenfunction since $T^{\mathbf{C}} \subseteq B$, and consequently the value of these terms is $F(1)$ times an expression independent of F. The lemma follows.

In computing $\|\psi_\Lambda\|^2$, we are to look at the integral

$$\int_{A_0^+} \mathrm{Tr}\, (\tau_\Lambda(\mu(a))^2) D(a)\, da.$$

First consider $SU(1,1)$. Here we know from Lemma 7 in 2.5 that $a = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ and $\mu(a) = \begin{pmatrix} \cosh t & 0 \\ 0 & (\cosh t)^{-1} \end{pmatrix}$. From 3.1, we have $\tau_\Lambda \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \alpha^{-n}$, and one can show that $D(a) = \sinh 2t$, up to a constant factor. The integral is

$$\int_0^\infty (\cosh t)^{-2n} \sinh 2t \, dt = 2\int \frac{\sinh t \, dt}{(\cosh t)^{2n-1}} = \frac{1}{n-1} \quad \text{if} \quad n > 1.$$

The condition $n > 1$ agrees with Harish-Chandra's condition

$$\langle \Lambda + \rho, \alpha \rangle < 0.$$

Next consider $Sp(2, \mathbf{R})$. As in 2.1, we have

$$\mathfrak{sp}(2,\mathbf{R}) = \begin{array}{c} \begin{array}{cc} 2 & 2 \end{array} \\ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \end{array} \begin{array}{l} 2 \\ 2 \end{array}, \quad a = -a^*, \quad b = b^t$$

and

$$\mathfrak{k} = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}, \quad a = -a^*; \qquad \mathfrak{k}^{\mathbf{C}} = \begin{pmatrix} a & 0 \\ 0 & -a^t \end{pmatrix}, \quad a \in \mathfrak{gl}(2,\mathbf{C}).$$

Choose

$$\mathfrak{h} = \begin{pmatrix} i\theta_1 & & & \\ & i\theta_2 & & \\ & & -i\theta_1 & \\ & & & -i\theta_2 \end{pmatrix}, \quad \mathfrak{h}^{\mathbf{C}} = \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & -c_1 & \\ & & & -c_2 \end{pmatrix}.$$

If e_i denotes evaluation of the ith diagonal entry of a member of $\mathfrak{h}^{\mathbf{C}}$, then the roots are

$$e_i - e_j, \; i \neq j, \qquad \text{compact},$$
$$\pm(e_i + e_j), \; i < j, \qquad \text{noncompact}.$$

The positive roots can be taken as

$$e_1 - e_2; \qquad 2e_1, \, e_1 + e_2, \, 2e_2.$$

Choose $\mathfrak{g}_0$ to be the real matrices

$$\begin{pmatrix} 0 & \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \\ \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} & 0 \end{pmatrix} = t_1 H_1 + t_2 H_2.$$

By the Corollary to Lemma 7 in 2.5,

$$\mu(\exp(t_1 H_1 + t_2 H_2)) = \begin{pmatrix} \cosh t_1 & & & \\ & \cosh t_2 & & \\ & & (\cosh t_1)^{-1} & \\ & & & (\cosh t_2)^{-1} \end{pmatrix}.$$

Let $\Lambda = -me_1 - ne_2$. Here $\rho = 2e_1 + e_2$ and so

$$\Lambda + \rho = -(m-2)e_1 - (n-1)e_2.$$

Λ integral says m and n are integers. Λ dominant with respect to $\mathfrak{k}$ says $\langle \Lambda, e_1 - e_2 \rangle \geq 0$ or $n \geq m$. $\langle \Lambda + \rho, \alpha \rangle < 0$ for α positive noncompact says

$$\langle \Lambda + \rho, 2e_1 \rangle < 0 \quad \text{and} \quad \langle \Lambda + \rho, 2e_2 \rangle < 0$$

or

$$m > 2 \quad \text{and} \quad n > 1.$$

Since $n \geq m$, Harish-Chandra's condition is $m > 2$. K is the 2-by-2 unitary group, and one computes easily that if $a = \exp(t_1 H_1 + t_2 H_2)$, then $\mathrm{Tr}(\tau_\Lambda(\mu(a))^2)$ equals

$$(\cosh t_1 \cosh t_2)^{-m-n} \left[\left(\frac{\cosh t_1}{\cosh t_2} \right)^{-m+n} + \cdots + \left(\frac{\cosh t_1}{\cosh t_2} \right)^{-(-m+n)} \right].$$

with the exponents decreasing one at a time. Also

$$D(a) = \sinh 2t_1 \sinh 2t_2 \sinh(t_1 + t_2) \sinh(t_1 - t_2),$$

and the integral in question is

$$\int_{t_1=0}^{\infty} \int_{t_2=0}^{t_1} \mathrm{Tr}(\tau_\Lambda(\mu(a))^2)\, D(a)\, dt_2\, dt_1.$$

When $n \geq m$, it is easy to see that this is finite if and only if $m > 2$, but it is not so easy to evaluate the integral. In [*2*, pp. 598–612], Harish-Chandra does the evaluation in the general case we have been considering.

4. Limits of Holomorphic Discrete Series

4.1 Case of $SU(1,1)$

Recall from Section 1 the two formulations of holomorphic discrete series for $G = SU(1,1)$. Let $n \geqq 2$ be an integer. In Bargmann's formulation

$$\mathscr{H}_n = \left\{ f \,\middle|\, \begin{array}{l} \text{(i)} \quad f \quad \text{analytic in} \quad \Omega = \{|z| < 1\} \\ \text{(ii)} \quad \|f\|_n^2 = \int_\Omega |f(z)|^2 (1 - |z|^2)^{n-2}\, dx\, dy < \infty \end{array} \right\},$$

$$\mathscr{U}_n(g) f(z) = (\bar{\beta} z + \alpha)^{-n} f\left(\frac{\bar{\alpha} z + \beta}{\bar{\beta} z + \alpha}\right), \qquad g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

In Harish-Chandra's formulation, put $\xi_n \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} = a^{-n}$. Then

$$H_n = \left\{ F \,\middle|\, \begin{array}{ll} \text{(i)} & F \quad \text{holomorphic on} \quad BG = P^- K^{\mathbf{C}} \Omega \\ \text{(ii)} & F(bx) = \xi_n(b) F(x),\ b \in B,\ x \in BG \\ \text{(iii)} & \|F\|^2 = \int_G |F(g)|^2\, dg < \infty \end{array} \right\}$$

$$U_n(g)\, F(x) = F(xg).$$

The unitary equivalance is given by

$$f(z) = F(0,1,z),\ F(\zeta,\gamma,z) = \gamma^{-n} f(z).$$

We shall now consider the limiting case $n = 1$. Without some adjustment, the norm for $n = 1$ will lead to a null Hilbert space. But with the norm

$$\|f\|_1^2 = \lim_{n \downarrow 1} (n-1) \int_\Omega |f(z)|^2 (1 - |z|^2)^{n-2}\, dx\, dy,$$

one finds that

$$\|f\|_1^2 = \lim_{r \uparrow 1} \tfrac{1}{2} \int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta,$$

and the space $\mathscr{H}_1$ becomes the Hardy space $H^2(\Omega)$ defined by

$$H^2(\Omega) = \{ f \mid \text{analytic in } \Omega,\ \sup_{r<1} \int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta < \infty \}.$$

This space is not 0, since all polynomials are in it. To each function f in $\mathscr{H}_1$, one can associate a boundary function $f(e^{i\theta}) \in L^2$(circle) and the formula for the norm becomes

$$\|f\|_1^2 = \tfrac{1}{2}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta.$$

This norm on $\mathscr{H}_1$, together with the group action $\mathscr{U}_1(g)$, gives an irreducible unitary representation of G.

There is another representation of interest here, with Hilbert space all of L^2(circle) and with action given by

$$\mathscr{V}(g)f(e^{i\theta}) = (\bar{\beta}e^{i\theta} + \alpha)^{-1} f\left(\frac{\bar{\alpha}e^{i\theta} + \beta}{\bar{\beta}e^{i\theta} + \alpha}\right).$$

This representation is of interest because it is unitarily equivalent to a member of what Bargmann called the principal continuous series. The principal series of $SU(1,1)$ consists of representations of $SU(1,1)$ induced from the one-dimensional unitary representations of the closed subgroup

$$MA^+N = \pm\begin{pmatrix} \cosh t - ixe^t & \sinh t - ixe^t \\ \sinh t + ixe^t & \cosh t + ixe^t \end{pmatrix}.$$

(This group is conjugate within $SL(2, \mathbf{C})$ to the group of real upper triangular matrices. A precise definition of induced representation will not be needed here; it suffices to know at present that there is a construction that associates a unitary representation of G in a natural way to a unitary representation of a closed subgroup.) The one-dimensional representations in question are of two kinds:

$$\pm(\text{as above}) \to e^{i\lambda t}, \qquad \lambda \in \mathbf{R} \text{ fixed},$$

$$\pm(\text{as above}) \to \pm e^{i\lambda t}, \qquad \lambda \in \mathbf{R} \text{ fixed}.$$

$\mathscr{V}(g)$ is equivalent with the induced representation of the second kind (in which the sign $\pm$ is retained) with $\lambda = 0$. The imbedding $H^2 \subseteq L^2$ commutes with the action of $SU(1,1)$ and exhibits $\mathscr{V}(g)$ as reducible. That this is an exceptional situation is indicated by the fact that all other members of the principal series of $SU(1,1)$ are irreducible.

We shall consider the representation $(\mathscr{H}_1, \mathscr{U}_1)$ in the context of the Harish-Chandra decomposition. (The imbedding $H^2 \subseteq L^2$ can also be treated in this context.) We take F holomorphic on BG, transforming under B according to ξ_1. The norm of $F(\zeta, \gamma, z)$ is to be roughly the norm of

$F(0, 1, e^{i\theta})$, and so we suppose that F is holomorphic in a neighborhood of the closure of BG. Put

$$u_1 = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}, \qquad K = \left\{ k = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \right\}, \qquad f(e^{i\theta}) = F(0,1,e^{i\theta}).$$

Then

$$u_1 k = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} e^{-i\theta} & \frac{\sqrt{2}}{2} e^{i\theta} \\ -\frac{\sqrt{2}}{2} e^{-i\theta} & \frac{\sqrt{2}}{2} e^{i\theta} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} e^{-i\theta} & 0 \\ 0 & (-)^{-1} \end{pmatrix} \begin{pmatrix} 1 & e^{2i\theta} \\ 0 & 1 \end{pmatrix}.$$

Thus $u_1 k \in P^- K^{\mathbf{C}} \overline{\Omega} = \overline{BG}$, and we have

$$F(u_1 k) = \left(\frac{\sqrt{2}}{2} e^{-i\theta} \right)^{-1} F \begin{pmatrix} 1 & e^{2i\theta} \\ 0 & 1 \end{pmatrix} = \sqrt{2} e^{i\theta} f(e^{2i\theta}).$$

Since

$$\int_K |F(u_1 k)|^2 \, dk = \frac{1}{\pi} \int_0^{2\pi} |f(e^{2i\theta})|^2 \, d\theta = \frac{1}{\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta,$$

we can define

$$\|F\|^2 = \int_K |F(u_1 k)|^2 \, dk$$

and take the completion of the space of functions of finite norm as H_1. $U_1(g)$ is a right translation by g, and $(\mathscr{H}_1, \mathscr{U}_1)$ is unitarily equivalent with (H_1, U_1). Our intention in the next section is to generalize this construction and the imbedding $H^2 \subseteq L^2$.

4.2 General Case

We continue to use the notation of Section 2.5. In particular, G is a simple Lie group with $G \subseteq G^{\mathbf{C}}$, and center $\mathfrak{k}$ is not 0.

Let Λ be an integral form on $\mathfrak{h}^{\mathbf{C}}$ dominant with respect to $\mathfrak{k}$. Harish-Chandra's condition for nonvanishing $H(\Lambda)$ was $\langle \Lambda + \rho, \alpha \rangle < 0$ for all positive noncompact α, and we shall consider the limiting case of this condition in this section. One can show that the following three conditions are equivalent:

(i) $\langle \Lambda + \rho, \alpha \rangle \leq 0$ for all positive noncompact α with equality for some α,
(ii) $\langle \Lambda + \rho, \alpha_0 \rangle = 0$ for the largest root α_0, and $\langle \Lambda + \rho, \alpha \rangle < 0$ for all other positive noncompact α, and
(iii) $\langle \Lambda + \rho, \alpha_0 \rangle = 0$ for the largest root α_0.

Suppose that Λ satisfies these equivalent conditions.

We shall define shortly an element $u_1 \in G^{\mathbf{C}}$ and a closed subgroup $M_s \subseteq G$. Let

$$\Gamma_2(\Lambda) = \left\{ F \,\middle|\, \begin{array}{ll} \text{(i)} & F \text{ holomorphic in a neighborhood of } BG \\ \text{(ii)} & F(bx) = \xi_\Lambda(b)F(x) \text{ for } b \in B,\ x \in BG \\ \text{(iii)} & \|F\|^2 = \int_{M_s \times K} |F(u_1 mk)^2\, dm\, dk \end{array} \right\}$$

$$U_\Lambda(g)F(x) = F(xg) \qquad \text{for} \qquad F \in \Gamma_2(\Lambda), \quad g \in G, \quad x \in BG.$$

Here $\|F\|$ will make sense because we shall have $Bu_1G \subseteq \overline{BG}$. Also $\|F\| = 0$ will imply $F = 0$. Thus we define

$$H(\Lambda) = \text{completion of} \quad \Gamma_2(\Lambda).$$

Theorem 1. If Λ is an integral form on $\mathfrak{h}^{\mathbf{C}}$, dominant with respect to $\mathfrak{k}$, and if $\langle \Lambda + \rho, \alpha_0 \rangle = 0$ for the largest root α_0, then U_Λ extends to a continuous unitary representation on $H(\Lambda)$, $H(\Lambda)$ is not 0, and U_Λ is irreducible.

We shall define u_1 and M_s while we establish notation for Theorem 2, which concerns the reducibility of certain induced representations. Though we shall give most of the definitions for general G, we shall concentrate on the examples $SU(1,1)$ and $Sp(2, \mathbf{R})$. Recall the earlier discussion of $Sp(2, \mathbf{R})$ in Section 3.3. Our definitions are set forth in the following table.

Item	*General*	$SU(1,1)$	$Sp(2,\mathbf{R})$
$\mathfrak{h}^{\mathbf{C}}$	(*see* 2.4)	$\begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}$	$\begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & -c_1 & \\ & & & -c_2 \end{pmatrix}$

Compact roots > 0 (see 2.4)		none	$e_1 - e_2$
Noncompact roots > 0 (see 2.4)		$2e_1$	$2e_1, e_1 + e_2, 2e_2$
ρ	$\frac{1}{2} \sum_{\beta > 0} \beta$	e_1	$2e_1 + e_2$
α_0	largest root	$2e_1$	$2e_1$
Λ	–	$-ne_1$	$-me_1 - ne_2,\ n \geq m$
$\langle \Lambda + \rho, \alpha_0 \rangle = 0$	–	$n = 1$	$m = 2,\ n \geq m$
u_1	$\exp\frac{\pi}{4}(X_{\alpha_0} - X_{-\alpha_0})$ (see after 2.5, Lemma 6)	$\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$	$\begin{pmatrix} \begin{pmatrix} \sqrt{2}/2 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} \sqrt{2}/2 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -\sqrt{2}/2 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \sqrt{2}/2 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$
$\mathfrak{a}^+$	$\mathbf{R}(X_{\alpha_0} + X_{-\alpha_0})$	$\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} & 0 \end{pmatrix}$
$\mathfrak{m}$	$(\mathfrak{a}^+)^\perp \cap Z_{\mathfrak{g}}(\mathfrak{a}^+)$	0	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & \bar{b} & 0 & -a \end{pmatrix}$, a imaginary
M_0	corresponding group	1	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & 1 & 0 \\ 0 & \bar{\beta} & 0 & \bar{\alpha} \end{pmatrix}$, $\det = 1$
M_s	commutator subgroup of M_0	1	$M_s = M_0 \cong SU(1,1)$
M	$Z_K(\mathfrak{a}^+) \cdot M_0$	$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$	$\begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & \pm 1 & 0 \\ 0 & \bar{\beta} & 0 & \bar{\alpha} \end{pmatrix}$
A^+	$\exp \mathfrak{a}^+$	$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$	$\begin{pmatrix} \cosh t & 0 & \sinh t & 0 \\ 0 & 1 & 0 & 0 \\ \sinh t & 0 & \cosh t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

N	something	$\begin{pmatrix} 1-ix & -ix \\ ix & 1+ix \end{pmatrix}$	a 3-dimensional nilpotent group
$\mathfrak{a}^-$	$Z_{\mathfrak{h}}(\mathfrak{a}^+)$	0	$\begin{pmatrix} 0 & & & \\ & c & & \\ & & 0 & \\ & & & -c \end{pmatrix}$, c imaginary

Consider $Sp(2, \mathbf{R})$. Then $\mathfrak{a}^-$ is a compact Cartan subalgebra in $\mathfrak{m}$, and $\Lambda\big|_{\mathfrak{a}^-} = -ne_2$, $n \geq 2$. Consequently $\Lambda\big|_{\mathfrak{a}^-}$ gives rise to a holomorphic discrete series representation ω_Λ of $SU(1,1)$. M is generated by $\mathfrak{m}_0$ and -1. Extend ω_Λ to M by putting $\omega_\Lambda(-1) = \xi_\Lambda(-1)I = (-1)^n I$. Then ω_Λ is a square-integrable representation of M.

In the general case, it is possible to proceed similarly. M_s is semisimple and turns out to have holomorphic discrete series. From one of these, we can construct a special square-integrable representation ω_Λ of M.

Now let

$$V_\Lambda = \underset{MA^+N\uparrow G}{\text{induced rep.}} (\omega_\Lambda \otimes 1 \otimes 1).$$

Harish-Chandra has proved that most representations of the form

induced rep. $(\omega \otimes 1 \otimes 1)$, ω square-integrable, χ unitary character of A^+, $MA^+ N \uparrow G$

are irreducible. However we have

Theorem 2. Under the above assumptions, U_Λ is unitarily equivalent with a proper subrepresentation of V_Λ, and consequently V_Λ is reducible.

5. Notes

1. The representations $(\mathscr{H}_n, \mathscr{U}_n)$ were constructed by Bargmann [*1*], and the six listed properties were proved by him.

2.1. The bounded symmetric domains were classified by É. Cartan. See [*3*] for a bibliography of the relevant papers, and see [*3*, pp. 339–355], for the classification of noncompact symmetric spaces. Bounded symmetric domains are discussed in detail in the book by L. K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains.*

2.2. This summary is taken from [*3*, pp. 281–285]. See those pages for more details.

2.3. This material is in [3] in a geometric form. For the lemma, see [3, pp. 183–186, 194–197]. For the theorem and the proposition, see [3, pp. 305, 311].

2.4. Most of the results of this section are contained in [3, pp. 312–314]. See also [2, pp. 750, 756–761]; our ordering on the roots is such that the totally positive roots in Harish-Chandra's sense are exactly the noncompact positive roots.

2.5. Harish-Chandra's realization of Ω is given in [2, pp. 2–5, 589–591] and is reproduced in [3, pp. 314–321].

3.1. The idea of defining representation in spaces of holomorphic functions on BG has its origin in the Borel-Weil Theorem, which gives such a realization for all irreducible representations of a compact group; in this case $BG = G^{\mathbf{C}}$, and there are no serious technical problems. Harish-Chandra defined corresponding representations in [4, pp. 6, 15], and he developed the basic properties in [2, pp. 6–19]. The main lemma appears in Lemmas 6 and 14.

3.2–3. Irreducibility is proved in [2, pp. 19–20], the computation of $\|\psi_\Lambda\|^2$ appears in [2, pp. 598–609], and the square-integrability is given in [2, pp. 30, 612].

4. Bargmann [1] observed the identity between the limiting norm of the discrete series for $SU(1,1)$ and the norm of the Hardy class in the disk. The results of this chapter appear in [4].

References

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