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Intertwining Operators into Dolbeault Cohomology Representations

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Let G/L be the quotient of a semisimple Lie group G by the centralizer L of a torus. The space of Dolbeault cohomology sections of a holomorphic line bundle over G/L is a natural place to realize interesting irreducible unitary representations of G and was first studied for this purpose by Bott and Schmid. Zuckerman and Vogan later introduced derived functor modules to provide an algebraic analog of these representations. The authors give a nonzero integral intertwining operator from derived functor modules, realized in the Langlands classification, to the Dolbeault cohomology representations, under the assumption that L and G have the same real rank. © 1992 Academic Press, Inc.

Contents.

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0. Introduction

In the representation theory of a semisimple Lie group G, both the Bott–Borel–Weil Theorem [3] and Schmid's proof [25, 26; see also 1] of the Kostant–Langlands conjecture [17, 18] realize important classes of unitary representations as Dolbeault cohomology spaces of bundle-valued differential forms over a quotient of G. Because of formidable analytic problems, initial efforts to generalize this construction were largely unsuccessful. To get around the analytic difficulties, Zuckerman [37] introduced an algebraic *analog* of such representations based on a construction using derived functors. This construction has come to be known as cohomological induction and was developed more fully by Vogan [32]. There is by now a reasonable theory of cohomological induction. But even though some of the analytic problems have been solved by Schmid [27] and by Hecht and Taylor [8], the corresponding Dolbeault cohomology representations remain poorly understood.

In particular it is still only a conjecture [33, Conjecture 6.11] that the analytic and algebraic representations coincide. Beyond the work mentioned above, a paper of Rawnsley, Schmid, and Wolf [23] implicitly handles some highest weight representations, and the papers [28] and [29] of Schmid and Wolf address a different special case of a natural generalization of the conjecture. But substantially no other cases have been settled.

The present paper suggests a two-step approach to this conjecture when the inducing representation is one-dimensional. Under a dominance condition the algebraic representation is irreducible and its parameters in the Langlands classification [13, Theorem 14.92] are known. The first step is to map the Langlands representation into cocycles for the Dolbeault cohomology spaces, thereby exhibiting the algebraic representation as a subquotient of the analytic representation. The second step is to give an upper bound for the multiplicities of the K-types (K being a maximal compact subgroup of G) of the Dolbeault cohomology representation by those of the algebraic representation, which are known explicitly.

Arguments of the kind suggested for the second step are known for the case of discrete series and may be found in Schmid [25] and Hotta–Parthasarathy [11].

In this paper we address the first step, under an additional hypothesis given in the next paragraph. What we do is map certain Langlands quotient representations into spaces of cocycles for Dolbeault cohomology in such a way that the map into cohomology is nonzero. For this formulation of our results, it is not necessary to refer to cohomological induction at all.

To describe our main results more precisely, we introduce some notation. Let G be linear connected semisimple with complexification $G^{\mathbb{C}}$, let K be maximal compact with Cartan involution θ, let T be a torus in K, and let L be the centralizer of T in G. Our special additional hypothesis is that G and L have the same real rank. (This is the opposite extreme case from what happens for discrete series [25, 26], where L has real rank 0.) We denote Lie algebras of Lie groups by $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{t}_0$, $\mathfrak{l}_0$, etc., and their complexifications by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{t}$, $\mathfrak{l}$, etc.

The quotient G/L has a number of invariant complex structures, and we fix one obtained in the following way. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{l}$ [32, p. 226]. If Q denotes the analytic subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{q}$, then G/L imbeds as an open subset of the complex manifold $G^{\mathbb{C}}/Q$ and inherits an invariant complex structure in which $\mathfrak{q}/\mathfrak{l} \cong \mathfrak{u}$ is the antiholomorphic tangent space at the identity coset. A similar construction with $\mathfrak{q} \cap \mathfrak{k}$ makes the quotient $K/(L \cap K)$ into a compact complex submanifold of G/L.

Let ξ be a one-dimensional representation of L, and let $\xi^{\#} = \xi \otimes \bigwedge^{\text{top}} \mathfrak{u}$. The complex line bundle $G \times_L \mathbb{C} \to G/L$, with L acting on $\mathbb{C}$ via $\xi^{\#}$, canonically becomes a holomorphic line bundle [30], and we let $C^{0,m}(G/L, \xi^{\#})$ be its space of smooth $(0, m)$-form sections, i.e., the space of smooth sections of $G \times_L (\mathbb{C} \otimes (\bigwedge^m \mathfrak{u})^*)$. Relative to the standard

$$\bar{\partial}\colon C^{0,m}(G/L, \xi^{\#}) \to C^{0,m+1}(G/L, \xi^{\#}), \tag{0.1}$$

the space of Dolbeault cohomology sections is

$$H^{0,m}(G/L, \xi^{\#}) = \ker \bar{\partial}/\text{image } \bar{\partial}.$$

The group G acts on everything on the left, and we obtain an untopologized group representation.

Under a dominance condition on ξ, one expects interesting cohomology to occur in degree $s = \dim_{\mathbb{C}} K/(L \cap K)$. The dominance condition can be described invariantly by requiring that

$$H^{0,s}(K/(L \cap K), \xi^{\#}|_{L \cap K}) \neq 0. \tag{0.2}$$

(See (1.5) for a description in terms of dominance of weights.)

With the dominance condition in place, our main results are as follows: Theorem 6.1 associates to ξ a nonunitary principal series representation of G and an equivariant mapping $\mathscr{S}$ of it into $C^{0,s}(G/L, \xi^{\#})$. We prove that the image of $\mathscr{S}$ lies in the kernel of $\bar{\partial}$ (Theorem 8.4) and, parenthetically, also in the kernel of a naturally defined $\bar{\partial}^*$ operator (Theorem 9.4). By composing $\mathscr{S}$ with a kind of nonholomorphic Penrose transform $\mathscr{P}$ (see [25], [35], and [2]) from $H^{0,s}(G/L, \xi^{\#})$ to sections of a vector bundle over G/K, we prove that the image of $\mathscr{S}$, when viewed in $H^{0,s}(G/L, \xi^{\#})$, is not 0 (Corollary 10.4).

We shall see in Theorem 10.3 that the composition $\mathscr{P} \circ \mathscr{S}$ is the Szegö operator that has been studied in special cases in [5] and [21] and was introduced earlier in a different context [15] for the realization of discrete series.

The detailed proofs of our results are made more complicated by their generality. The reader may be helped by first understanding the extreme cases that rank G = rank K and that G is complex semisimple.

In originally carrying out this research, Barchini and Knapp worked together, and Zierau worked independently. We arrived at substantially the same theorem at the same time and decided to extend it a little and publish it jointly.

We are all indebted to D. A. Vogan for advice and assistance with this project. Our work has been assisted also by conversations with a number of other people, and we are happy to acknowledge their help: M. G. Eastwood, S. G. Gindikin, P. Lima Filho, M. K. Murray, and J. W. Rice.

1. Roots and Orderings

In this section we shall introduce notation that will allow us to work with $H^{0,s}(G/L, \xi^{\#})$. Our underlying group G is assumed to be linear connected semisimple, with a complexification $G^{\mathbb{C}}$. Our standing assumption on L is that G and L have the same real rank. The linearity of G simplifies the notation but is not essential; we show in §12 how to dispense with it.

We defined K, θ, T, L, Q, and various Lie algebras in the introduction. We write the Cartan decomposition of $\mathfrak{g}_0$ relative to θ as $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. Our θ-stable parabolic subalgebra of $\mathfrak{g}$ is $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, with $\mathfrak{u}$ the unipotent radical. With bar denoting the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$, we have $\mathfrak{g} = \bar{\mathfrak{u}} \oplus \mathfrak{l} \oplus \mathfrak{u}$. The group L is connected.

Extend $\mathfrak{t}_0$ to a maximal abelian subspace $\mathfrak{b}_0$ of $\mathfrak{k}_0$, and let $B = \exp \mathfrak{b}_0$. The centralizer $\mathfrak{h}_0$ of $\mathfrak{b}_0$ in $\mathfrak{g}_0$ is of the form $\mathfrak{h}_0 = \mathfrak{b}_0 \oplus \mathfrak{a}_0'$ with $\mathfrak{a}_0' \subseteq \mathfrak{p}_0$ and is a maximally compact Cartan subalgebra of $\mathfrak{g}_0$. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The Cartan involution θ acts on roots by $+1$

on $\mathfrak{b}$ and -1 on $\mathfrak{a}'$. No root vanishes on $\mathfrak{b}$ [13, p. 420]. Within the subset Δ_B of roots vanishing on $\mathfrak{a}'$, we say that a root is compact or noncompact according as its root vector lies in $\mathfrak{k}$ or $\mathfrak{p}$, and we write $\Delta_{B,\mathrm{c}}$ and $\Delta_{B,\mathrm{n}}$ for the sets of compact and noncompact roots within Δ_B. Let r be the restriction map from the dual $\mathfrak{h}^*$ to the dual $\mathfrak{b}^*$. For members β of Δ_B, it will be convenient to use the notation β and $r(\beta)$ interchangeably.

Let $\Delta_K = \Delta(\mathfrak{k}, \mathfrak{b})$ be the roots of $\mathfrak{k}$ with respect to $\mathfrak{b}$, and let $\Delta_n = \Delta(\mathfrak{p}, \mathfrak{b})$ be the set of nonzero weights of $\mathfrak{p}$ with respect to $\mathfrak{b}$.

LEMMA 1.1. (a) *The restriction map r carries $\Delta - \Delta_{B,\mathrm{n}}$ onto Δ_K. The preimage of a member γ of Δ_K is either one member of $\Delta_{B,\mathrm{c}}$ or two members β and $\theta\beta$ of $\Delta - \Delta_B$.*

(b) *The restriction map r carries $\Delta - \Delta_{B,\mathrm{c}}$ onto Δ_{n}. All members of Δ_n have multiplicity one. The preimage of a member γ of Δ_{n} is either one member of $\Delta_{B,\mathrm{n}}$ or two members β and $\theta\beta$* of $\Delta - \Delta_B$.

Proof. Let $\mathfrak{g} = \mathfrak{h} + \sum_{\beta \in \Delta} \mathbb{C} E_\beta$ be the root space decomposition of $\mathfrak{g}$. Then we have

$$\mathfrak{g} = \Bigg(\mathfrak{b} + \sum_{\gamma \in \Delta_{B,\mathrm{c}}} \mathbb{C} E_\gamma + \sum_{\{\beta, \theta\beta\} \subseteq \Delta - \Delta_B} \mathbb{C}(E_\beta + \theta E_\beta)\Bigg)$$
$$\oplus \Bigg(\mathfrak{a}' + \sum_{\gamma \in \Delta_{B,\mathrm{n}}} \mathbb{C} E_\gamma + \sum_{\{\beta, \theta\beta\} \subseteq \Delta - \Delta_B} \mathbb{C}(E_\beta - \theta E_\beta)\Bigg).$$

The two terms on the right side are contained in $\mathfrak{k}$ and $\mathfrak{p}$, respectively, and hence must equal $\mathfrak{k}$ and $\mathfrak{p}$. Thus r is onto as required. The rest will follow as soon as it is shown that roots β and β' with $r(\beta) = r(\beta')$ have $\beta' = \beta$ or $\beta' = \theta\beta$. We have

$$\langle \beta + \theta\beta, \beta' \rangle = 2\, |r(\beta)|^2 > 0,$$

from which it follows that $\langle \beta, \beta' \rangle > 0$ or $\langle \theta\beta, \beta' \rangle > 0$. Therefore either $\beta' - \beta$ or $\beta' - \theta\beta$ is 0 (and we are done), or one of these is a root vanishing on $\mathfrak{b}$. But no root vanishes on $\mathfrak{b}$.

For the moment let Δ^+ be any positive system for Δ such that $\theta\Delta^+ = \Delta^+$. We impose further conditions on Δ^+ below. In view of Lemma 1.1, we get well defined positive sets $\Delta_K^+ \subseteq \Delta_K$ and $\Delta_{\mathrm{n}}^+ \subseteq \Delta_{\mathrm{n}}$ by saying that $r(\beta)$ is positive if and only if β is positive.

Since $\mathfrak{h}_0$ centralizes $\mathfrak{t}_0$, $\mathfrak{h}_0$ is contained in $\mathfrak{l}_0$. Thus we can speak of sets of roots $\Delta(\mathfrak{u}, \mathfrak{h})$, $\Delta(\mathfrak{u} \cap \mathfrak{k}, \mathfrak{b})$, and $\Delta(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{b})$, as well as similar sets for $\mathfrak{l}$ and $\bar{\mathfrak{u}}$. The roots of $\bar{\mathfrak{u}}$ are the negatives of the roots of $\mathfrak{u}$. In the choice of Δ^+ to be made presently, we insist that $\Delta(\mathfrak{u}, \mathfrak{h}) \subseteq \Delta^+$. The set Δ^+ then consists of $\Delta(\mathfrak{u}, \mathfrak{h})$ together with a choice of $\Delta^+(\mathfrak{l}, \mathfrak{h})$. For any of these sets of roots, we let $\delta(\cdot)$ be half the sum of the positive members. From Lemma 1.1 it follows that

$$\delta(\mathfrak{u} \cap \mathfrak{k}) + \delta(\mathfrak{u} \cap \mathfrak{p}) = \delta(\mathfrak{u}), \tag{1.1}$$

with $\delta(\mathfrak{u})$ equal to 0 on $\mathfrak{a}'$.

LEMMA 1.2. *Let Ω be the set of weights of a finite-dimensional representation, let m_ω be the multiplicity of ω in Ω, and let Γ be a subset of Ω. Suppose that α is a root such that $\gamma \in \Gamma$ and $\alpha + \gamma \in \Omega$ imply $\alpha + \gamma \in \Gamma$. Then $\langle \sum_{\gamma \in \Gamma} m_\gamma \gamma, \alpha \rangle \geqslant 0$. Strict inequality holds when Ω is the set of roots and α is in Γ and $-\alpha$ is not in Γ.*

Proof. With s_α denoting reflection in α, we have

$$\sum_{\gamma \in \Gamma} m_\gamma \gamma = \sum_{\substack{\gamma \in \Gamma \\ \langle \gamma, \alpha \rangle < 0}} m_\gamma (\gamma + s_\alpha \gamma) + \sum_{\substack{\gamma \in \Gamma \\ \langle \gamma, \alpha \rangle = 0}} m_\gamma \gamma + \sum_{\substack{\gamma \in \Gamma, s_\alpha \gamma \notin \Gamma \\ \langle \gamma, \alpha \rangle > 0}} m_\gamma \gamma.$$

The inner product of α with the first two sums on the right is 0, and the inner product of α with the third sum is term-by-term positive. When Ω is the set of roots, if $\alpha \in \Gamma$ and $-\alpha \notin \Gamma$, then α occurs in the third sum and gives a positive inner product.

COROLLARY 1.3. *If α is in Δ^+, then*

$$\langle \delta(\mathfrak{u}), \alpha \rangle \text{ is } \begin{cases} = 0 & \text{if } \alpha \in \Delta(\mathfrak{l}, \mathfrak{h}) \\ > 0 & \text{if } \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}). \end{cases}$$

If α is in Δ_K^+, then

$$\langle \delta(\mathfrak{u} \cap \mathfrak{k}), \alpha \rangle \text{ is } \begin{cases} = 0 & \text{if } \alpha \in \Delta(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{b}) \\ > 0 & \text{if } \alpha \in \Delta(\mathfrak{u} \cap \mathfrak{k}, \mathfrak{b}) \end{cases}$$

and

$$\langle \delta(\mathfrak{u} \cap \mathfrak{p}), \alpha \rangle \text{ is } \begin{cases} = 0 & \text{if } \alpha \in \Delta(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{b}) \\ \geqslant 0 & \text{if } \alpha \in \Delta(\mathfrak{u} \cap \mathfrak{k}, \mathfrak{b}). \end{cases}$$

Remarks. See [32, p. 124] for some of these.

Proof. This is immediate from Lemma 1.2. For the cases with equality the lemma is to be applied to both α and $-\alpha$.

For our given one-dimensional representation ξ of L, let $\lambda + \nu$ be the unique weight relative to $\mathfrak{h}$. Here λ is the part on $\mathfrak{b}$, and ν is the part on

$\mathfrak{a}'$. (Thus λ is a discrete parameter, and ν is a continuous parameter.) Since ξ is one-dimensional, $\lambda+\nu$ is orthogonal to the roots of $\mathfrak{l}$. The representation $\xi^{\#}=\xi\otimes\bigwedge^{\mathrm{top}}\mathfrak{u}$ of L has weight $(\lambda+2\delta(\mathfrak{u}))+\nu$; we write $\mathbb{C}^{\#}_{\lambda+\nu}$ for the space $\mathbb{C}$ in which it acts. The holomorphic line bundle of interest is

$$G\times_L \mathbb{C}^{\#}_{\lambda+\nu}\to G/L, \tag{1.2}$$

with G/L as a complex manifold having $\mathfrak{u}$ as antiholomorphic tangent space at the identity. The holomorphic bundle structure is exhibited in [30]. Meanwhile, $K/(L\cap K)$ is a compact complex manifold having $\mathfrak{u}\cap\mathfrak{k}$ as antiholomorphic tangent space at the identity, and the natural map $K/(L\cap K)\to G/L$ is one–one and holomorphic. The pullback of (1.2) is the holomorphic line bundle

$$K\times_{L\cap K}\mathbb{C}^{\#}_{\lambda+\nu}\to K/(L\cap K). \tag{1.3}$$

Let $s=\dim_{\mathbb{C}} K/(L\cap K)=\dim(\mathfrak{u}\cap\mathfrak{k})$.

The bundle-valued $(0, m)$ forms for (1.2) are the smooth sections of

$$G\times_L(\mathbb{C}^{\#}_{\lambda+\nu}\oplus(\textstyle\bigwedge^m\mathfrak{u})^*),$$

where $(\cdot)^*$ denotes dual. We write this space of sections variously as $C^{0,m}(G/L, \xi^{\#})$ or $C^{0,m}(G/L, \mathbb{C}^{\#}_{\lambda+\nu})$. It is the same space as

$$C^\infty(G/L, \mathbb{C}^{\#}_{\lambda+\nu}\otimes(\textstyle\bigwedge^m\mathfrak{u})^*)=\{f\colon G\to\mathbb{C}^{\#}_{\lambda+\nu}\otimes(\bigwedge^m\mathfrak{u})^*\text{ of class }C^\infty \\ \mid f(xl)=(\xi^{\#}(l)^{-1}\otimes\mathrm{Ad}^*(l)^{-1})f(x)\}. \tag{1.4}$$

An equivalent way of writing it is as the space of L-invariants

$$(C^\infty(G)\otimes\mathbb{C}^{\#}_{\lambda+\nu}\otimes(\textstyle\bigwedge^m\mathfrak{u})^*)^L,$$

with L acting on $C^\infty(G)$ in the last case by the right regular representation. For an invariant definition of the $\bar\partial$ operator (0.1), see Wells [35, Chapter I]; an explicit formula for it in terms of root vectors and their duals appears in Griffiths–Schmid [6]. We do not need the explicit formula. For handling $\mathscr{S}$ we need only the facts that $\bar\partial$ is a map between the spaces (0.1), commutes with the left action of G, is given by a local expression (involving various derivatives on the right of the G variable, as well as operations on alternating tensors), and satisfies $\bar\partial^2=0$. Let $H^{0,m}(G/L, \mathbb{C}^{\#}_{\lambda+\nu})$ denote the cohomology in degree m.

The dominance condition (0.2) on ξ translates into

$$\langle\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p}), \alpha\rangle\geqslant 0\qquad\text{for all}\quad\alpha\in\Delta_K^+, \tag{1.5}$$

as an easy consequence of the Borel–Weil–Bott Theorem. This condition is implied by the more usual dominance condition

$$\langle \lambda, \alpha \rangle \geqslant 0 \qquad \text{for all} \quad \alpha \in \Delta^+, \tag{1.6}$$

as a consequence of Corollary 1.3. (Alternatively see [32, p. 364].)

When rank $G = \operatorname{rank} K$, so that ν is not present, the representations $H^{0,s}(G/L, \mathbb{C}^{\#}_{\lambda+\nu})$ are supposed to be analogs of the derived functor modules $A_{\mathfrak{q}}(\lambda)$ defined in [14]. The condition (1.6) ensures that $A_{\mathfrak{q}}(\lambda)$ is irreducible. We do not need $A_{\mathfrak{q}}(\lambda)$ in this paper, and it will be sufficient to assume (1.5). *We assume* (1.5) *for the remainder of this paper.*

There is a formal adjoint $\bar{\partial}^*$ to

$$\bar{\partial}\colon C^{0,m}(G/L, \mathbb{C}^{\#}_{\lambda+\nu}) \to C^{0,m+1}(G/L, \mathbb{C}^{\#}_{\lambda+\nu})$$

given by the operator

$$\bar{\partial}^*\colon C^{0,m+1}(G/L, \mathbb{C}^{\#}_{\lambda-\bar{\nu}}) \to C^{0,m}(G/L, \mathbb{C}^{\#}_{\lambda-\bar{\nu}}). \tag{1.7}$$

In more detail, let C be the Killing form of $\mathfrak{g}$. The Hermitian form $\langle X, Y \rangle = C(X, \bar{Y})$ on $\mathfrak{g}$ is G-invariant and induces an L-invariant Hermitian form $\langle \cdot, \cdot \rangle_m$ on $(\bigwedge^m \mathfrak{u})^*$. In turn this induces an L-invariant sesquilinear pairing

$$\langle z \otimes X, w \otimes Y \rangle = z\bar{w} \langle X, Y \rangle_m$$

of $\mathbb{C}^{\#}_{\lambda+\nu} \otimes (\bigwedge^m \mathfrak{u})^*$ with $\mathbb{C}^{\#}_{\lambda-\bar{\nu}} \otimes (\bigwedge^m \mathfrak{u})^*$. Thus if f is in $C^{0,m}(G/L, \mathbb{C}^{\#}_{\lambda+\nu})$ and g is in $C^{0,m+1}(G/L, \mathbb{C}^{\#}_{\lambda-\bar{\nu}})$, $\langle \bar{\partial} f(x), g(x) \rangle$ is a well defined scalar-valued function on G/L. We define $\bar{\partial}^*$ by the expected formula

$$\int_{G/L} \langle f(x), \bar{\partial}^* g(x) \rangle \, d\dot{x} = \int_{G/L} \langle \bar{\partial} f(x), g(x) \rangle \, d\dot{x},$$

with f running through the forms of compact support modulo L.

Next we introduce normalized root vectors for Δ. Following [9, p. 181], we choose root vectors E_α so that the Killing form satisfies

$$C(E_\alpha, E_{-\alpha}) = 1 \tag{1.8}$$

and so that $E_\alpha - E_{-\alpha}$ and $i(E_\alpha + E_{-\alpha})$ lie in the compact form $\mathfrak{k}_0 \oplus i\mathfrak{p}_0$. For roots in Δ_B,

$E_\alpha - E_{-\alpha}$	and	$i(E_\alpha + E_{-\alpha})$	are in $\mathfrak{k}_0$	if $\alpha \in \Delta_{B,\mathrm{c}}$
$E_\alpha + E_{-\alpha}$	and	$i(E_\alpha - E_{-\alpha})$	are in $\mathfrak{p}_0$	if $\alpha \in \Delta_{B,\mathrm{n}}$.

We define H_α to be the member of $\mathfrak{h}$ dual to α under C. From (1.8) it follows that $[E_\alpha, E_{-\alpha}] = H_\alpha$.

We recall that roots α and β in Δ are said to be *strongly orthogonal* if $\beta \neq \pm\alpha$ and if neither $\alpha+\beta$ nor $\alpha-\beta$ is a root. In this case we write $\alpha \perp\!\!\perp \beta$. Strongly orthogonal implies orthogonal.

Finally we give the remaining conditions to be satisfied by our positive system Δ^+. So far, we have insisted that $\theta\Delta^+ = \Delta^+$ and that $\Delta(\mathfrak{u}) \subseteq \Delta^+$. We need $\Delta^+(\mathfrak{l}, \mathfrak{h})$, and then $\Delta^+ = \Delta(\mathfrak{u}) \cup \Delta^+(\mathfrak{l}, \mathfrak{h})$. Let us say that a positive system for $\Delta(\mathfrak{l}, \mathfrak{h})$ and an ordered sequence $\alpha_1, ..., \alpha_l$ from $\Delta(\mathfrak{l}, \mathfrak{h}) \cap \Delta_{B,\mathrm{n}}$ are *compatible* if

(i) the α_i are strongly orthogonal,

(ii) $\mathfrak{a}_0'' = \sum_{j=1}^{l} \mathbb{R}(E_{\alpha_j} + E_{-\alpha_j})$ has $\mathfrak{a}_0 = \mathfrak{a}_0'' \oplus \mathfrak{a}_0'$ maximal abelian in $\mathfrak{l}_0 \cap \mathfrak{p}_0$ (and therefore maximal abelian in $\mathfrak{p}_0$, since real rank G and real rank L are assumed equal),

(iii) each α_j is (positive and) simple in the subsystem of roots of $\Delta(\mathfrak{l}, \mathfrak{h}) \cap \Delta_B$ strongly orthogonal to $\alpha_1, ..., \alpha_{j-1}$.

Since any choice of $\Delta^+(\mathfrak{l}, \mathfrak{h})$ is generated by the Δ^+ simple roots that it contains (as a consequence of $\Delta(\mathfrak{u}) \subseteq \Delta^+$), it follows that

(iii′) each α_j is (positive and) simple in the subsystem of roots of Δ_B strongly orthogonal to $\alpha_1, ..., \alpha_{j-1}$.

Our final condition on Δ^+ is that $\Delta^+(\mathfrak{l}, \mathfrak{h})$ be given along with a compatible sequence $\alpha_1, ..., \alpha_l$. It is not immediately evident that any such Δ^+ exists. However, when rank G = rank K, any choice of $\Delta^+(\mathfrak{l}, \mathfrak{b})$ has a compatible set $\alpha_1, ..., \alpha_l$, as a consequence of [15, §4]. In §11, we show the existence of compatible $\Delta^+(\mathfrak{l}, \mathfrak{h})$ and $\{\alpha_j\}$ in general. A feature of our construction in §11 is that a further desirable condition is satisfied (ρ_L dominant for G, in the notation of §2).

2. Cayley Transform

Our choice of Δ^+ carried with it a choice of a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{p}_0$ and a sequence of strongly orthogonal roots $\alpha_1, ..., \alpha_l$ in $\Delta(\mathfrak{l}, \mathfrak{h}) \cap \Delta_B$ used in defining $\mathfrak{a}_0$. In terms of these roots, we can define a Cayley transform.

For $\alpha \in \Delta_{B,\mathrm{n}}$, let $E'_\alpha = (\sqrt{2}/|\alpha|)\, E_\alpha$ and $E'_{-\alpha} = (\sqrt{2}/|\alpha|)\, E_{-\alpha}$. Then

$$u_\alpha = \exp \frac{\pi}{4} (E'_{-\alpha} - E'_\alpha)$$

is a member of $G^{\mathbb{C}}$ that normalizes the $\mathfrak{sl}(2, \mathbb{C})$ corresponding to α and interchanges the two standard Cartan subalgebras [13, pp. 417–419]. Specifically

$$\operatorname{Ad}(u_\alpha)\left(\frac{2H_\alpha}{|\alpha|^2}\right) = E'_\alpha + E'_{-\alpha}.$$

The different u_{α_j} commute, because of the strong orthogonality, and we put $u = u_{\alpha_1} \cdots u_{\alpha_l}$ and $\mathbf{c} = \operatorname{Ad}(u)$. The map $\mathbf{c}\colon \mathfrak{g} \to \mathfrak{g}$ is the Cayley transform. Let

$$\mathfrak{b}_0'' = \sum_{j=1}^{l} i\mathbb{R}H_{\alpha_j} \subseteq \mathfrak{b}_0$$

$$\mathfrak{b}_0^- = \text{orthocomplement of } \mathfrak{b}_0'' \text{ in } \mathfrak{b}_0.$$

On $\mathfrak{b}$ the formula for $\mathbf{c}$ is

$$\mathbf{c}\left(\frac{2H_{\alpha_j}}{|\alpha_j|^2}\right) = \frac{\sqrt{2}}{|\alpha_j|}(E_{\alpha_j} + E_{-\alpha_j})$$

$$\mathbf{c}(H) = H \qquad \text{for} \quad H \in \mathfrak{b}^- \oplus \mathfrak{a}'.$$

Thus $\mathbf{c}(i\mathfrak{b}_0'') = \mathfrak{a}_0''$. So $\mathbf{c}$ carries $\mathfrak{h} = \mathfrak{b}^- \oplus \mathfrak{b}'' \oplus \mathfrak{a}'$ to a new Cartan subalgebra $\mathfrak{h}' = \mathfrak{b}^- \oplus \mathfrak{a}'' \oplus \mathfrak{a}'$, acting as the identity on $\mathfrak{b}^- \oplus \mathfrak{a}'$ and carrying $\mathfrak{b}''$ to $\mathfrak{a}''$. Under the definition $(\mathbf{c}\beta)(H) = \beta(\mathbf{c}^{-1}H)$, $\mathbf{c}$ carries $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ to $\mathbf{c}\Delta = \Delta(\mathfrak{g}, \mathfrak{h}')$.

Since each α_j is in $\Delta(\mathfrak{l}, \mathfrak{h})$, $\mathbf{c}$ is in $\operatorname{Ad}(L^{\mathbb{C}})$. Therefore $\mathbf{c}$ normalizes $\bar{\mathfrak{u}}$, $\mathfrak{l}$, and $\mathfrak{u}$. Moreover, $\mathfrak{h}' = \mathbf{c}\mathfrak{h}$ is in $\mathfrak{l}$.

We shall define a positive system $(\mathbf{c}\Delta)^+$ for $\mathbf{c}\Delta$ different from the image of Δ^+ under $\mathbf{c}$. Namely we list $H_{\alpha_1}, \ldots, H_{\alpha_l}$ as an ordered orthogonal basis of $i\mathfrak{b}_0''$ and extend it to an orthogonal basis of $i\mathfrak{b}_0'' \oplus \mathfrak{a}_0'$ by adjoining elements at the end. We use this basis in lexicographic fashion to determine positivity for members of $\mathbf{c}\Delta$ that do not vanish identically on $\mathfrak{a}'' \oplus \mathfrak{a}'$. (Namely $\mathbf{c}\beta$ is in $(\mathbf{c}\Delta)^+$ if $\beta(H_{\alpha_1}) > 0$, or if $\beta(H_{\alpha_1}) = 0$ and $\beta(H_{\alpha_2}) > 0$, etc.) For roots $\mathbf{c}\beta$ supported on $\mathfrak{b}^-$, we say that $\mathbf{c}\beta$ is in $(\mathbf{c}\Delta)^+$ if β is in Δ^+. (It is not necessary for theoretical purposes to define $(\mathbf{c}\Delta)^+$ quite so rigidly, but the above definition is convenient for computing examples with the aid of the formulas in [15, §5].)

If $\beta \in \Delta_B$ is orthogonal to α_j, Lemma 5.4 of [15] gives

$$\operatorname{Ad}(u_{\alpha_j})E_\beta = \begin{cases} E_\beta & \text{if} \quad \beta \perp\perp \alpha_j \\ \frac{1}{2}([E'_{-\alpha_j}, E_\beta] - [E'_{\alpha_j}, E_\beta]) & \text{if} \quad \beta \perp/\perp \alpha_j, \end{cases} \tag{2.1}$$

where $E'_{\alpha_j} = (\sqrt{2}/|\alpha_j|)E_{\alpha_j}$.

LEMMA 2.1. *Suppose $\beta \in \Delta_B$ is orthogonal to $\alpha_1, \ldots, \alpha_l$. Then β fails to be strongly orthogonal to at most one α_i. If $\beta \perp\perp \alpha_i$ for all i, then β is in $\Delta_{B,\mathrm{c}}$* and $\mathbf{c}(E_\beta) = E_\beta$. *If $\beta \perp/\perp \alpha_j$, then β is in $\Delta_{B,\mathrm{n}}$ and*

$$\mathbf{c}(E_\beta) = \tfrac{1}{2}([E'_{-\alpha_j}, E_\beta] - [E'_{\alpha_j}, E_\beta]). \tag{2.2}$$

Proof. Suppose $\beta \perp/\perp \alpha_i$ and $\beta \perp/\perp \alpha_j$. Computing both sides of

$$\mathrm{Ad}(\mathrm{u}_{\alpha_i})\,\mathrm{Ad}(\mathrm{u}_{\alpha_j})E_\beta = \mathrm{Ad}(\mathrm{u}_{\alpha_j})\,\mathrm{Ad}(\mathrm{u}_{\alpha_i})E_\beta$$

by means of (2.1), we see that at least one of the four expressions $\beta \pm \alpha_i \pm \alpha_j$ is a root. This root, together with β and $\beta \pm \alpha_i$, exhibits roots of three different lengths, contradiction. Thus $\beta \perp/\perp \alpha_i$ for at most one index i.

If $\beta \perp\perp \alpha_i$ for all i, then $\mathbf{c}(E_\beta) = E_\beta$ by (2.1). If β were in $\Delta_{B,\mathrm{n}}$, then $E_\beta + E_{-\beta}$ would be a member of $\mathfrak{p}_0$ outside $\mathfrak{a}_0$ commuting with $\mathfrak{a}_0$, in contradiction to (ii) in §1. Thus β is in $\Delta_{B,\mathrm{c}}$.

Finally suppose $\beta \perp/\perp \alpha_j$. Then $\mathbf{c}(E_\beta)$ is given by (2.2), as a result of (2.1). If β is in $\Delta_{B,c}$, then $\mathbf{c}(E_\beta)$ is in $\mathfrak{p}$ but not $\mathfrak{a}$. Since E_β commutes with $\mathfrak{b}''$ and $\mathfrak{a}'$, $\mathbf{c}(E_\beta)$ commutes with $\mathfrak{a}$, in contradiction to (ii) in §1. Thus β is in $\Delta_{B,\mathrm{n}}$.

The Cayley transform allows us to define a minimal parabolic subgroup MAN. Let $A = \exp \mathfrak{a}_0$, and let M be the centralizer of A in K. Then $M = M_0 F$ with

$$F = M \cap \exp i\mathfrak{a}_0, \tag{2.3}$$

by Lemma 9 of [24] and Lemmas 1 and 3 of [22]. We have $A \subseteq L$, and it follows that

$$F \subseteq L, \tag{2.4}$$

since $F \subseteq G \cap \exp \mathfrak{a} \subseteq G \cap \exp \mathfrak{l} \subseteq G \cap L^{\mathbb{C}} = L$.

The subalgebra $\mathfrak{h}'_0 = \mathfrak{b}_0^- \oplus \mathfrak{a}_0$ is a maximally noncompact Cartan subalgebra of $\mathfrak{g}_0$, and we take the positive roots of $\mathbf{c}\Delta = \Delta(\mathfrak{g}, \mathfrak{h}')$ to be those of $(\mathbf{c}\Delta)^+$. This positive system has the property that a root that is not identically 0 on $\mathfrak{a}$ has its positivity decided by its restriction to $\mathfrak{a}$. Hence we can consistently define positive restricted roots by saying that $\mathbf{c}\beta|_{\mathfrak{a}} > 0$ if $\mathbf{c}\beta|_{\mathfrak{a}} \neq 0$ and $\beta \in (\mathbf{c}\Delta)^+$. Then we define $\mathfrak{n}_0$ to be the sum of the root spaces in $\mathfrak{g}_0$ for the positive restricted roots, $\bar{\mathfrak{n}}_0$ to be $\theta\mathfrak{n}_0$, and N and $\bar{N}$ to be the corresponding analytic subgroups. (The notation $\bar{\mathfrak{n}}_0$ and $\bar{N}$ is traditional and does not refer to conjugation.) Then MAN is a minimal parabolic subgroup of G. Note that $A \subseteq L$ implies $L \cap MAN = (L \cap M)\,A(L \cap N)$.

For $\beta \in \Delta$, define $X_\beta = \mathfrak{c}(E_\beta)$. Let $\Delta_- = \{\beta \in \Delta \mid \beta|_{\mathfrak{b}'' + \mathfrak{a}'} = 0\}$. The subalgebra $\mathfrak{b}_0^-$ is a Cartan subalgebra of $\mathfrak{m}_0$. The root system $\Delta(\mathfrak{m}, \mathfrak{b}^-)$ may be viewed as $\mathfrak{c}(\Delta_-)$, the root space decomposition of $\mathfrak{m}$ being

$$\mathfrak{m} = \mathfrak{b}^- \oplus \sum_{\beta \in \Delta_-} \mathbb{C} X_\beta. \tag{2.5}$$

Positivity for these roots is the same, whether obtained from Δ^+ or from $(\mathfrak{c}\Delta)^+$.

Let Σ_L and Σ_G be the sets of restricted roots for L and G, and let Σ_L^+ and Σ_G^+ be the subsets of positive elements. Define ρ_L and ρ_G to be the half sums of the members of Σ_L^+ and Σ_G^+, with multiplicities counted. We can regard ρ_L and ρ_G as members of $\mathfrak{h}'^*$ when necessary, by extending them so as to be 0 on $\mathfrak{b}^{-*}$. In any event, ρ_L and $\rho_G - \rho_L$ are Σ_L^+ dominant (the latter by Lemma 1.2, for example), and ρ_G is Σ_G^+ dominant.

Our hypotheses do not force ρ_L to be Σ_G^+ dominant. This matter is of some significance when reinterpreting our results in terms of Langlands parameters, as in the introduction. In §11 when we show the existence of compatible $\Delta^+(\mathfrak{l}, \mathfrak{h})$ and $\alpha_1, ..., \alpha_l$, our construction leads to a situation in which ρ_L is Σ_G^+ dominant.

3. Principal Series Parameters

We use the $\mathfrak{a}^*$ element $\rho_L + \nu$ and an irreducible representation σ of M defined below to form the nonunitary principal series representation $\operatorname{ind}_{MAN}^G(\sigma \otimes e^{\rho_L + \nu} \otimes 1)$ of G. This is a representation in which G acts by the left regular representation in the space

$$\{f: G \to V^\sigma \mid f(xman) = a^{-(\rho_L + \rho_G + \nu)} \sigma(m)^{-1} f(x)\}, \tag{3.1}$$

where V^σ is the space in which σ acts. Letting

$$\tilde{\sigma}(man) = a^{\rho_L + \rho_G + \nu} \sigma(m),$$

we shall find it convenient to refer to the space of smooth functions in (3.1) by $C^\infty(G/MAN, \tilde{\sigma})$.

Let μ be an irreducible representation of K with highest weight $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$, acting in a space V^μ and having ϕ for a nonzero unit highest weight vector.

Proposition 3.1. *The cyclic span of ϕ in V^μ under M is irreducible under M. Namely*

(a) *ϕ is a highest weight vector under* $\mathfrak{m}$, *with highest weight*

$$[\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})]|_{\mathfrak{b}^-}$$

(b) *$\mathbb{C}\phi$ is a one-dimensional subspace stable under the group $L \cap K$, which contains F.*

Warning. Contrast this result with the need for the addendum to [15] because of the disconnectedness of M.

Proof. Since M_0 and F commute, (a) and (b) will prove the proposition. Conclusion (b) follows from the fact that $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is orthogonal to $\Delta(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{b})$, as a consequence of Corollary 1.3.

To prove (a), again define $X_\beta = \mathbf{c}(E_\beta)$ for $\beta \in \Delta$. We need to prove, for $\beta \in \Delta^+ \cap \Delta_-$, that $\mu(X_\beta)(\phi) = 0$. Since $\Delta_- \subseteq \Delta_B$, we can use Lemma 2.1. If $\beta \perp\perp \alpha_i$ for all i, then $X_\beta = E_\beta$, and $\mu(E_\beta)(\phi) = 0$ since ϕ is highest for K. If $\beta \perp/\perp \alpha_j$, then X_β is a linear combination of $E_{\beta - \alpha_j}$ and $E_{\beta + \alpha_j}$, and $\beta \pm \alpha_j$ are in $\Delta_{B,\mathrm{c}}$. Since members of $\Delta_{B,\mathrm{c}}$ yield members of Δ_K (Lemma 1.1a), it is enough to see that $\beta + \alpha_j$ and $\beta - \alpha_j$ are positive. For $\beta + \alpha_j$, the positivity is automatic. For $\beta - \alpha_j$, it follows from (iii') at the end of §1.

We let σ be the irreducible representation of M acting in the M-cyclic span V^σ of ϕ, and we let τ be the one-dimensional representation of $L \cap K$ acting in $\mathbb{C}\phi$.

4. Subrepresentations of $\bigwedge^s \mathfrak{u}$

In this section we define a subset S of $\Delta(\mathfrak{u}, \mathfrak{h})$ by means of the positive system $(\mathbf{c}\Delta)^+$. The set S has s members and hence defines a member E_S of $\bigwedge^s \mathfrak{u}$. We shall see that E_S generates an irreducible representation π_1 of L, and we shall identify the parameters that characterize π_1.

We let

$$S = \{\gamma \in \Delta(\mathfrak{u}, \mathfrak{h}) \mid \mathbf{c}\gamma|_{\mathfrak{a}} \leqslant 0\}.$$

In more detail we include in S all roots γ of $\mathfrak{u}$ for which $\mathbf{c}\gamma|_{\mathfrak{a}}$ is a negative restricted root, as well as all roots γ of $\mathfrak{u}$ for which $\mathbf{c}\gamma$ vanishes on $\mathfrak{a}$. Let $\gamma_1, \gamma_2, \ldots$ be the members of S, and let $E_S = E_{\gamma_1} \wedge E_{\gamma_2} \wedge \cdots$ as a member of the exterior algebra of $\mathfrak{u}$.

Proposition 4.1. $|S| = s$, *so that E_S is in $\bigwedge^s \mathfrak{u}$.*

Proof. Again we let $X_\beta = \mathbf{c}(E_\beta)$ for $\beta \in \Delta$, so that X_β is a root vector for $\mathbf{c}\beta \in \mathbf{c}\Delta$. Then θX_β is a root vector for $\theta(\mathbf{c}\beta)$, which has the same sign on

$\mathfrak{b}^-$ and the opposite sign on $\mathfrak{a}$. In all sums below, β is to run through all members of $\Delta(\mathfrak{u}, \mathfrak{h})$ that satisfy the indicated conditions. Since $\mathbf{c}$ is in $\mathrm{Ad}(L^{\mathbb{C}})$ and $\mathfrak{u} = \sum \mathbb{C} E_\beta$, we have

$$\mathfrak{u} = \sum \mathbb{C} X_\beta = \Big(\sum_{\mathbf{c}\beta|_\mathfrak{a} = 0} \mathbb{C} X_\beta \oplus \sum_{\mathbf{c}\beta|_\mathfrak{a} < 0} \mathbb{C}(X_\beta + \theta X_\beta) \Big) \oplus \Big(\sum_{\mathbf{c}\beta|_\mathfrak{a} < 0} \mathbb{C}(X_\beta - \theta X_\beta) \Big).$$

In the first sum on the right, X_β is in $\mathfrak{m}$ and hence is in $\mathfrak{k}$. Therefore the first two sums contribute to $\mathfrak{u} \cap \mathfrak{k}$, and the third sum contributes to $\mathfrak{u} \cap \mathfrak{p}$. But

$$\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{k}) \oplus (\mathfrak{u} \cap \mathfrak{p}),$$

and therefore

$$\mathfrak{u} \cap \mathfrak{k} = \sum_{\mathbf{c}\beta|_\mathfrak{a} = 0} \mathbb{C} X_\beta \oplus \sum_{\mathbf{c}\beta|_\mathfrak{a} < 0} \mathbb{C}(X_\beta + \theta X_\beta) \tag{4.1}$$

and

$$\mathfrak{u} \cap \mathfrak{p} = \sum_{\mathbf{c}\beta|_\mathfrak{a} < 0} \mathbb{C}(X_\beta - \theta X_\beta). \tag{4.2}$$

In (4.1), the dimension of the left side is s, and the dimension of the right side is $|S|$. The result follows.

Thus $E_S = E_{\gamma_1} \wedge \cdots \wedge E_{\gamma_s}$. Motivated by the proof of Proposition 4.1, we put

$$X_S = \mathbf{c} E_S = X_{\gamma_1} \wedge \cdots \wedge X_{\gamma_s}.$$

PROPOSITION 4.2. *The* $\mathrm{Ad}(L)$ *cyclic span of* E_S *in* $\bigwedge^s \mathfrak{u}$ *is an irreducible representation* π_1 *of* L *with* E_S *as an extreme weight vector relative to the Cartan subalgebra* $\mathfrak{h}$. *With respect to the system* Σ_G^+ *of positive restricted roots,* X_S *is a lowest restricted weight vector of* π_1, *and its restricted weight is* $\rho_L - \rho_G$.

Proof. We show that E_S is lowest relative to the positive system $\mathbf{c}^{-1}(\mathbf{c}\Delta)^+ \cap \Delta(\mathfrak{l}, \mathfrak{h})$ for $\mathfrak{l}$ with respect to $\mathfrak{h}$. Thus suppose $\beta \in \Delta(\mathfrak{l}, \mathfrak{h})$ has $\mathbf{c}\beta$ in $-(\mathbf{c}\Delta)^+$. If γ is in S, then $\mathbf{c}\gamma|_\mathfrak{a} \leqslant 0$. If $\beta + \gamma$ is a root, then $\mathbf{c}(\beta + \gamma)|_\mathfrak{a} \leqslant 0$ and $\beta + \gamma$ is in S. Hence $(\bigwedge_{\gamma_j \neq \gamma} E_{\gamma_j}) \wedge \mathrm{ad}(E_\beta) E_\gamma = 0$. Then it follows that $\mathrm{ad}(E_\beta)(E_S) = 0$, and E_S is lowest for the indicated system.

Applying $\mathbf{c}$, we see that X_S is lowest relative to the positive system $(\mathbf{c}\Delta)^+ \cap \Delta(\mathfrak{l}, \mathfrak{h}')$ for $\mathfrak{l}$ with respect to $\mathfrak{h}'$. In particular, it is a lowest restricted weight vector. Its restricted weight is

$$\begin{aligned}\sum_{j=1}^{s} \mathbf{c}\gamma_j|_{\mathfrak{a}} &= \sum_{\substack{\beta \in \Delta(\mathfrak{u}, \mathfrak{h}') \\ \beta|\mathfrak{a} \leqslant 0}} \beta|_{\mathfrak{a}} = \sum_{\substack{\beta \in \Delta(\mathfrak{u}, \mathfrak{h}') \\ \beta|\mathfrak{a} < 0}} \beta|_{\mathfrak{a}} \\ &= \frac{1}{2} \sum_{\substack{\beta \in \Delta(\mathfrak{u}+\bar{\mathfrak{u}}, \mathfrak{h}') \\ \beta|\mathfrak{a} < 0}} \beta|_{\mathfrak{a}} \\ &= \frac{1}{2} \sum_{\substack{\beta \in \Delta(\mathfrak{g}, \mathfrak{h}') \\ \beta|\mathfrak{a} < 0}} \beta|_{\mathfrak{a}} - \frac{1}{2} \sum_{\substack{\beta \in \Delta(\mathfrak{l}, \mathfrak{h}') \\ \beta|\mathfrak{a} < 0}} \beta|_{\mathfrak{a}} \\ &= -\rho_G + \rho_L,\end{aligned}$$

as asserted.

PROPOSITION 4.3. *Let X_C be a nonzero member of the one-dimensional subspace $\bigwedge^s(\mathfrak{u} \cap \mathfrak{k})$ of $\bigwedge^s \mathfrak{u}$. Relative to the natural Hermitian inner product on $\bigwedge^s \mathfrak{u}$, $\langle X_S, X_C \rangle \neq 0$.*

Proof. Let $\gamma_1, ..., \gamma_t$ have $\mathbf{c}\gamma_j|_{\mathfrak{a}} = 0$ while $\gamma_{t+1}, ..., \gamma_s$ have $\mathbf{c}\gamma_j|_{\mathfrak{a}} < 0$. Then (4.1) shows that we may take X_C to be

$$X_{\gamma_1} \wedge \cdots \wedge X_{\gamma_t} \wedge (X_{\gamma_{t+1}} + \theta X_{\gamma_{t+1}}) \wedge \cdots \wedge (X_{\gamma_s} + \theta X_{\gamma_s}).$$

This has nonzero inner product with

$$X_S = X_{\gamma_1} \wedge \cdots \wedge X_{\gamma_s},$$

and the result follows.

Let τ_1 be an abstract copy of the representation of $L \cap K$ with highest weight $2\delta(\mathfrak{u} \cap \mathfrak{k})$ relative to $\mathfrak{b}$. Then τ_1 is one-dimensional, by Corollary 1.3, and $\bigwedge^s(\mathfrak{u} \cap \mathfrak{k})$ is a concrete realization of it.

COROLLARY 4.4. *The one-dimensional representation τ_1 of $L \cap K$ occurs in $\pi_1|_{L \cap K}$, necessarily with multiplicity one.*

Proof. Let χ be the character of τ_1. Proposition 4.3 produces an X in the space of π_1 ($X = X_S$ actually) with $\langle X, X_C \rangle \neq 0$. Then we have

$$
\begin{aligned}
0 \neq \langle X, X_C\rangle &= \langle X, \overline{\chi(k)}\,\mathrm{Ad}(k)X_C\rangle \qquad \text{for all} \quad k \in L\cap K \\
&= \int_{L\cap K} \langle X, \overline{\chi(k)}\,\mathrm{Ad}(k)X_C\rangle\, dk \\
&= \int_{L\cap K} \langle \chi(k)\,\mathrm{Ad}(k)^{-1} X, X_C\rangle\, dk \\
&= \int_{L\cap K} \langle \overline{\chi(k)}\,\mathrm{Ad}(k)X, X_C\rangle\, dk \qquad \text{under} \quad k \to k^{-1} \\
&= \left\langle \int_{L\cap K} \overline{\chi(k)}\,\mathrm{Ad}(k)X\, dk, X_C \right\rangle.
\end{aligned}
$$

Therefore $\int_{L\cap K} \overline{\chi(k)}\,\mathrm{Ad}(k)X\,dk$ is a nonzero vector within the space of π_1, and it is certainly of $L\cap K$ type τ_1. It has multiplicity one by [13, p. 206, item (2)].

Lemma 4.5. *In a finite-dimensional representation R of a compact group $\mathscr{K}$, suppose v_0 is cyclic under $\mathscr{K}$. If P_ω denotes the projection operator to the isotypic component of $\mathscr{K}$ type ω, then $P_\omega v_0$ is cyclic for that isotypic component.*

Proof. The most general v, by cyclicity, is $v = \sum c_j R(k_j) v_0$. Then $P_\omega v = \sum c_j P_\omega R(k_j) v_0 = \sum c_j R(k_j)\, P_\omega v_0$. Taking v in image P_ω, we see that $P_\omega v_0$ is cyclic within image P_ω.

Proposition 4.6. *The lowest restricted weight space of π_1 is one-dimensional, and $L\cap M$ acts in it by $\tau_1|_{L\cap M}$.*

Proof. Let χ be the character of τ_1, and let P_{τ_1} be the projection of the space of π_1 to the $L\cap K$ isotypic component of type τ_1. Since $L\cap M \subseteq L\cap K$, we have, for $m \in L\cap M$,

$$P_{\tau_1}\pi_1(m)X_S = \tau_1(m)\, P_{\tau_1}X_S = \chi(m)\, P_{\tau_1}X_S$$

and

$$P_{\tau_1}(\pi_1(m)X_S - \chi(m)X_S) = 0. \tag{4.3}$$

By Proposition 4.2, $v_0 = \pi_1(m)X_S - \chi(m)X_S$ is a lowest restricted weight vector for π_1. Suppose $v_0 \neq 0$. Since

$$L = (L\cap K)\, A(L\cap \bar{N})$$

and since π_1 is irreducible under L, v_0 is cyclic for $\pi_1|_{L\cap K}$. By Lemma 4.5, $P_{\tau_1}v_0$ is cyclic for the τ_1 subspace, which is nonzero by Corollary 4.4. But this cyclicity contradicts (4.3), and we conclude $v_0 = 0$. Therefore $\mathbb{C}X_S$ is an invariant subspace under $L \cap M$, necessarily of type τ_1. By [13, Lemma 9.12], $\mathbb{C}X_S$ is the entire lowest restricted weight space.

5. Finite-Dimensional Representation π of L

In §6, we introduce the operator $\mathscr{S}$ from a nonunitary principal series representation of G into $C^{0,s}(G/L, \mathbb{C}^{\#}_{\lambda+\nu})$. The functions on G in the image of $\mathscr{S}$ have their values not in the whole space $\mathbb{C}^{\#}_{\lambda+\nu} \otimes (\bigwedge^s \mathfrak{u})^*$, but in a subspace that is irreducible under L. In the present section we shall construct this irreducible representation of L, calling it π.

Within the dual $\mathfrak{u}^*$, let $\{\omega_\beta\}$ be the basis dual to $\{X_\beta,\ \beta \in \Delta(\mathfrak{u}, \mathfrak{h})\}$. Under the isomorphism $(\bigwedge^s \mathfrak{u})^* \cong \bigwedge^s(\mathfrak{u}^*)$, we may regard elements $\omega_{\beta_1} \wedge \cdots \wedge \omega_{\beta_s}$ as belonging to $(\bigwedge^s \mathfrak{u})^*$. With $S = \{\gamma_1, ..., \gamma_s\}$, define

$$\omega_S = \omega_{\gamma_1} \wedge \cdots \wedge \omega_{\gamma_s}. \tag{5.1}$$

Recall from §3 that τ is a one-dimensional representation of $L \cap K$ of weight $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$.

Proposition 5.1. *The L-cyclic span of $1 \otimes \omega_S$ in $\mathbb{C}^{\#}_{\lambda+\nu} \otimes (\bigwedge^s \mathfrak{u})^*$ is an irreducible representation π of L with $1 \otimes \omega_S$ as a highest weight vector relative to $(\mathfrak{c}\Delta)^+$. The highest restricted weight of π relative to Σ_G^+ is $\rho_G - \rho_L + \nu$, the highest restricted weight space is the one-dimensional space $\mathbb{C}(1 \otimes \omega_S)$, and the representation of $L \cap M$ in the highest restricted weight space is $\tau|_{L \cap M}$.*

Proof. In passing from X_S in $\bigwedge^s \mathfrak{u}$ to $1 \otimes \omega_S$ in $\mathbb{C}^{\#}_{\lambda+\nu} \otimes (\bigwedge^s \mathfrak{u})^*$, we have taken the contragradient and then tensored with $\mathbb{C}^{\#}_{\lambda+\nu} = \mathbb{C}_{\lambda+2\delta(\mathfrak{u})+\nu}$. So most of this proposition follows directly from Propositions 4.2 and 4.6. The highest restricted weight here is obtained by taking the negative of the lowest $(\rho_L - \rho_G)$ in Proposition 4.2 and adding the contribution from $\mathbb{C}_{\lambda+2\delta(\mathfrak{u})+\nu}$, which comes from the part on $\mathfrak{a}'$, namely ν. The same argument as for Proposition 4.6 shows that the $L \cap M$ behavior on the highest restricted weight space is forced by the presence in $\pi|_{L \cap K}$ of the one-dimensional representation $\mathbb{C}_{\lambda+2\delta(\mathfrak{u})} \otimes \bar{\tau}_1$, with τ_1 as in Corollary 4.4. The weight of this representation is

$$(\lambda + 2\delta(\mathfrak{u})) - 2\delta(\mathfrak{u} \cap \mathfrak{k}),$$

which equals $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ by (1.1), and hence this representation is of type τ.

PROPOSITION 5.2. *The representation* $\operatorname{ind}_{L\cap MA\bar{N}}^{G}(\tau|_{L\cap M}\otimes e^{\rho_G+\nu}\otimes 1)$ *of the nonunitary principal series of L has a unique irreducible subrepresentation* π', *and this subrepresentation is finite-dimensional of type* π.

Proof. The uniqueness of the irreducible subrepresentation is part of the Langlands classification [19]; it is dual to [13, Theorem 7.24]. The relevant observations are that ρ_G is strictly dominant for Σ_L^+ and that the exponential of H_ν (the element of $\mathfrak{a}'$ dual to ν) is in the noncompact part of the center of L. (This property of ν is seen as follows. For every $\beta\in\Delta(\mathfrak{l},\mathfrak{h})$, we have $\langle\lambda+\nu,\beta\rangle=0$ and $\langle\lambda+\nu,\theta\beta\rangle=0$. Half the difference of these equations gives $\langle\nu,\beta\rangle=0$, from which the property follows.)

Thus it is enough to see that a representation of type π occurs as a subrepresentation of the induced representation. This trick is a central idea of Lepowsky–Wallach [20]. We define an L-equivariant map φ of the space V^π of π into a space of functions on L by

$$\varphi(v)(l)=p(\pi(l)^{-1}v) \qquad \text{for} \quad v\in V^\pi, \tag{5.2}$$

p being the projection to the highest restricted weight space. Using Proposition 5.1, we readily check that $\varphi(v)(\cdot)$ satisfies the correct transformation laws under $L\cap MA\bar{N}$ on the right so as to be in the space of the induced representation. This completes the proof.

6. OPERATOR $\mathscr{S}$

The finite-dimensional representation π of L and the highest weight vector $1\otimes\omega_S$ allow us to define the operator $\mathscr{S}$ as in the following theorem. Recall from §3 that $V^\sigma\subseteq V^\mu$ with ϕ as a common highest weight vector.

THEOREM 6.1. *The operator* $\mathscr{S}$ *given by*

$$(\mathscr{S}f)(x)=\int_{L\cap K}\langle f(xk),\phi\rangle\,\pi(k)(1\otimes\omega_S)\,dk \tag{6.1}$$

carries $\operatorname{ind}_{MAN}^{G}(\sigma\otimes e^{\rho_L+\nu}\otimes 1)$ *continuously in G-equivariant fashion into*

$$\operatorname{ind}_L^G(\pi)\subseteq\operatorname{ind}_L^G(\mathbb{C}^{\#}_{\lambda+\nu}\otimes(\textstyle\bigwedge^s\mathfrak{u})^*).$$

Proof. It is clear that the integral is convergent, $\mathscr{S}$ is continuous and G-equivariant, and $(\mathscr{S}f)(x)$ is in the space of π. We are to show that

$$(\mathscr{S}f)(xl)=\pi(l)^{-1}((\mathscr{S}f)(x)) \qquad \text{for} \quad l\in L. \tag{6.2}$$

To do so, let us write $g=\kappa(g)e^{H(g)}n$ for the decomposition of $g\in G$ according to $G=KAN$. We shall apply the change of variables $k\to\kappa(l^{-1}k)$ to (6.1); then dk is replaced by $e^{-2\rho_L H(l^{-1}k)}dk$, according to [13, (7.4)]. Since

$$f(xl\kappa(l^{-1}k))=e^{(\rho_L+\nu+\rho_G)H(l^{-1}k)}f(xk) \tag{6.3}$$

by definition of the induced representation, and since

$$\pi(\kappa(l^{-1}k))(1\otimes\omega_S)=e^{-(\rho_G-\rho_L+\nu)H(l^{-1}k)}\pi(l^{-1}k)(1\otimes\omega_S) \tag{6.4}$$

by Proposition 5.1, we obtain

$$\begin{aligned}(\mathscr{S}f)(xl)&=\int_{L\cap K}\langle f(xlk),\phi\rangle\,\pi(k)(1\otimes\omega_S)\,dk\\&=\int_{L\cap K}\langle f(xl\kappa(l^{-1}k),\phi\rangle\,\pi(\kappa(l^{-1}k))(1\otimes\omega_S)\,e^{-2\rho_L H(l^{-1}k)}dk\\&=\int_{L\cap K}\langle f(xk),\phi\rangle\,\pi(l^{-1}k)(1\otimes\omega_S)\,dk\qquad\text{by (6.3) and (6.4)}\\&=\pi(l)^{-1}\,(\mathscr{S}f)(x)\end{aligned}$$

as required.

Although (6.1) is the only formula we need for $\mathscr{S}$, it is illuminating to see $\mathscr{S}$ from a more general point of view. According to Proposition 5.2, the Langlands quotient mapping J^L for L, defined in [13, Theorem 7.24], is an operator

$$J^L\colon \operatorname{ind}_{L\cap MAN}^{L}(\tau|_{L\cap M}\otimes e^{\rho_G+\nu}\otimes 1)\to\pi'\cong\pi.$$

Therefore the induced mapping $\operatorname{ind}_L^G J^L$ is formally an operator

$$\operatorname{ind}_L^G J^L\colon \operatorname{ind}_{L\cap MAN}^{G}(\tau|_{L\cap M}\otimes e^{\rho_G+\nu}\otimes 1)\to\operatorname{ind}_L^G\pi'\cong\operatorname{ind}_L^G\pi. \tag{6.5}$$

The domain of $\operatorname{ind}_L^G(J^L)$ can be seen formally to contain as a subrepresentation

$$\operatorname{ind}_{MAN}^{G}(\operatorname{ind}_{L\cap M}^{M}(\tau|_{L\cap M})\otimes e^{\rho_L+\nu}\otimes 1) \tag{6.6}$$

and as a further subrepresentation

$$\operatorname{ind}_{MAN}^{G}(\sigma\otimes e^{\rho_L+\nu}\otimes 1). \tag{6.7}$$

Apart from isomorphisms, $\mathscr{S}$ is just the restriction of $\operatorname{ind}_L^G J^L$ to (6.7), as the next proposition shows. It was the recognition of (6.6) as a subrepresentation of the domain of $\operatorname{ind}_L^G J^L$ that led to the discovery of the explicit formula for $\mathscr{S}$.

PROPOSITION 6.2. *Let $\varphi: \pi \to \pi'$ be the L-equivalence of (5.2), let $\iota: \mathbb{C}(1 \otimes \omega_S) \to \mathbb{C}\phi$ be the $(L \cap M)$-isomorphism given by $\iota(1 \otimes \omega_S) = \phi$, and let P_τ be the orthogonal projection of V^σ on $\mathbb{C}\phi$. If f is in the space for $\operatorname{ind}_{MAN}^G(\sigma \otimes e^{\rho_L + \nu} \otimes 1)$, then*

$$\iota(\varphi(\mathscr{S}f(x))(l)) = \int_{L \cap \bar{N}} P_\tau(f(xl\bar{n}))\, d\bar{n}, \tag{6.8}$$

provided $d\bar{n}$ is normalized so that $\int_{L \cap \bar{N}} e^{-2\rho H(\bar{n})} d\bar{n} = 1$.

Proof. The formula for φ is

$$\varphi(v)(l) = |1 \otimes \omega_S|^{-2} \langle \pi(l)^{-1} v, 1 \otimes \omega_S \rangle (1 \otimes \omega_S),$$

and the formula for P_τ is

$$P_\tau(u) = \langle u, \phi \rangle \phi.$$

The known invariance properties of $\mathscr{S}$ and φ imply that

$$\begin{aligned} \iota(\varphi(\mathscr{S}f(x))(l)) &= \iota(\varphi(\mathscr{S}f(xl))(1)) \\ &= \iota(|1 \otimes \omega_S|^{-2} \langle \mathscr{S}f(xl), 1 \otimes \omega_S \rangle (1 \otimes \omega_S)) \\ &= |1 \otimes \omega_S|^{-2} \langle \mathscr{S}f(xl), 1 \otimes \omega_S \rangle \phi. \end{aligned}$$

Substituting from (6.1) and making the change of variables that passes from $L \cap K$ to $(L \cap \bar{N}) \times (L \cap M)$, given in [13, (5.25)], we see that the above expression is

$$\begin{aligned} &= |1 \otimes \omega_S|^{-2} \int_{L \cap K} \langle \pi(k)(1 \otimes \omega_S), 1 \otimes \omega_S \rangle\, P_\tau(f(xlk))\, dk \\ &= |1 \otimes \omega_S|^{-2} \int_{L \cap \bar{N}} \int_{L \cap M} \langle \pi(\kappa(\bar{n})m)(1 \otimes \omega_S), 1 \otimes \omega_S \rangle \\ &\qquad \times P_\tau(f(xl\kappa(\bar{n})m))\, e^{-2\rho_L H(\bar{n})}\, dm\, d\bar{n}. \end{aligned}$$

The $L \cap M$ integration goes away, and this expression collapses to (6.8) because $1 \otimes \omega_S$ transforms under $L \cap M$ according to τ.

7. INVARIANT DISTRIBUTIONS

The proof that $\bar{\partial}$ and $\bar{\partial}^*$ give 0 on the image of $\mathscr{S}$ uses just invariance properties of these operators, not explicit formulas. Ultimately the effect of the operators will be captured in terms of invariant distributions on spaces of smooth functions. We work abstractly with such distributions in this section and give a characterization of them that is related to results of Bruhat [4]. Lemma 7.1 is elementary, and its proof is omitted.

LEMMA 7.1. *There exists a closed submanifold $\bar{N}'$ of $\bar{N}$ that contains* 1 *and is diffeomorphic to Euclidean space such that $\bar{N} = (L \cap \bar{N})\bar{N}'$ in the sense that multiplication $(L \cap \bar{N}) \times \bar{N}' \to \bar{N}$ is a diffeomorphism onto.*

If V is a finite-dimensional complex vector space, a *V-distribution* on $\bar{N}$ is a continuous linear functional on the space $C^{\infty}_{\mathrm{com}}(\bar{N}, V)$ of compactly supported smooth functions on $\bar{N}$ with values in V. Let $U(\bar{\mathfrak{n}})$ be the universal enveloping algebra of $\bar{\mathfrak{n}}^{\mathbb{C}}$.

PROPOSITION 7.2. *Let V be a finite-dimensional vector space, and let D be a V-distribution on $\bar{N}$ that is left invariant under $L \cap \bar{N}$ and is supported on $L \cap \bar{N}$. If $\{v_j\}_{j=1}^n$ is a basis of V with dual basis $\{v_j^*\}_{j=1}^n$ for V^*, then D is of the form*

$$D(F) = \sum_{j=1}^{n} \int_{L \cap \bar{N}} \langle (u_j F)(x), v_j^* \rangle \, dx$$

for suitable left invariant differential operators $u_j \in U(\bar{\mathfrak{n}})$.

Proof. Write $\bar{N} = (L \cap \bar{N})\bar{N}'$ as in Lemma 7.1. For $f \in C^{\infty}_{\mathrm{com}}(L \cap \bar{N}, V)$ and $g \in C^{\infty}_{\mathrm{com}}(\bar{N}', \mathbb{C})$, define $f \otimes g$ in $C^{\infty}_{\mathrm{com}}(\bar{N}, V)$ by $(f \otimes g)(x, y) = f(x)\, g(y)$. For fixed g, define

$$D_g(f) = D(f \otimes g).$$

Letting $\mathscr{L}$ be the left regular representation by $L \cap \bar{N}$, we have

$$\begin{aligned} D_g(\mathscr{L}(x_0) f) &= D(\mathscr{L}(x_0) f \otimes g) = D(\mathscr{L}(x_0)(f \otimes g)) \\ &= D(f \otimes g) = D_g(f) \end{aligned}$$

for $x_0 \in L \cap \bar{N}$, by the hypothesis of invariance. By [4, p. 123], D_g is of the form

$$D_g(f) = \int_{L \cap \bar{N}} \langle f(x), v_g^* \rangle \, dx$$

for some v_g^* in V^*.

Fix f_0 in $C^{\infty}_{\mathrm{com}}(L \cap \bar{N}, \mathbb{C})$ with $\int_{L \cap \bar{N}} f_0(x)\, dx \neq 0$, and form the function

$f_0(\cdot)v_j$. The expression $g \to D(f_0(\cdot)v_j \otimes g)$ is a $\mathbb{C}$-distribution on $\bar{N}'$ and is given by

$$D(f_0(\cdot)v_j \otimes g) = D_g(f_0(\cdot)v_j) = \left(\int_{L \cap \bar{N}} f_0(x)\, dx\right) \langle v_j, v_g^* \rangle.$$

Consequently $g \to \langle v_j, v_g^* \rangle$ is a distribution on $\bar{N}'$. Since D is supported on $L \cap \bar{N}$, $g \to \langle v_j, v_g^* \rangle$ is supported at 1 in $\bar{N}$. By [10, Theorem 2.3.4], $\langle v_j, v_g^* \rangle = D_j g(1)$, where D_j is some linear combination of partial derivatives of various orders. Then we have

$$\begin{aligned} D(f \otimes g) &= \int_{L \cap \bar{N}} \langle f(x), v_g^* \rangle\, dx \\ &= \sum_{j=1}^{n} \left(\int_{L \cap \bar{N}} \langle f(x), v_j^* \rangle\, dx\right) \langle v_j, v_g^* \rangle \\ &= \sum_{j=1}^{n} \left(\int_{L \cap \bar{N}} \langle f(x), v_j^* \rangle\, dx\right) D_j g(1) \\ &= \sum_{j=1}^{n} \int_{L \cap \bar{N}} \langle (f \otimes D_j g)(x, 1), v_j^* \rangle\, dx, \end{aligned}$$

which we can rewrite as

$$= \sum_{j=1}^{n} \int_{L \cap \bar{N}} \langle u_j(f \otimes g)(x, 1), v_j^* \rangle\, dx.$$

By [10, Theorem 5.1.1], linear combinations of the functions $f \otimes g$ are dense in $C^\infty_{\mathrm{com}}(\bar{N}, V)$. Thus the result follows.

8. Effect of $\bar{\partial}$

Our objective in this section is to prove that $\bar{\partial} \circ \mathscr{S} = 0$. Let $(\tilde{\sigma}, V^\sigma)$ be the representation of MAN in (3.2). For this section we define $E = \mathbb{C}^{\#}_{\lambda+\nu} \otimes (\bigwedge^{s+1}\mathfrak{u})^*$, and we let Φ: $C^\infty(G/MAN, \tilde{\sigma}) \to E$ be the composition $\Phi = e \circ \bar{\partial} \circ \mathscr{S}$, where e is evaluation at 1. Here $\bar{\partial} \circ \mathscr{S}$ is G-equivariant, and e is L-equivariant; thus Φ is L-equivariant. Also Φ is continuous.

The space $C^\infty_{\mathrm{com}}(\bar{N}, V^\sigma)$ is a representation space for $\bar{N}$ under the left regular representation $\mathscr{L}$. In addition, MA acts on $C^\infty_{\mathrm{com}}(\bar{N}, V^\sigma)$ by

$$(maf)(\bar{n}) = \tilde{\sigma}(ma) f(a^{-1}m^{-1}\bar{n}ma),$$

and the combined action of $MA\bar{N}$ is a representation of $MA\bar{N}$. Let i be the natural inclusion

$$i: C^{\infty}_{\mathrm{com}}(\bar{N}, V^{\sigma}) \to C^{\infty}(G/MAN, \tilde{\sigma}).$$

Then i is $\bar{N}$-equivariant, and one checks readily that it is MA-equivariant.

We work with the transpose maps between continuous duals

$$\Phi^{\mathrm{tr}}: E^* \to C^{\infty}(G/MAN, \tilde{\sigma})^*$$

and

$$i^{\mathrm{tr}}: C^{\infty}(G/MAN, \tilde{\sigma})^* \to C^{\infty}_{\mathrm{com}}(\bar{N}, V^{\sigma})^*.$$

Here Φ^{tr} is L-equivariant, and i^{tr} is $MA\bar{N}$-equivariant. The action of L on E is fully reducible, and thus the same thing is true of E^*. Hence E^* is spanned by weight vectors under the action of $\exp \mathfrak{h}'_0 = AB^-$, B^- being the Cartan subgroup $\exp \mathfrak{b}'_0$ of $L \cap M_0$. For any such vector e^*, $(\Phi \circ i)^{\mathrm{tr}}(e^*)$ is an AB^- weight vector since $(\Phi \circ i)^{\mathrm{tr}}$ is $(L \cap MA)$-equivariant. If e^* happens to be a lowest restricted weight vector for L (under the action of A), then e^* is fixed by $L \cap \bar{N}$. In this case, $(\Phi \circ i)^{\mathrm{tr}}(e^*)$ is fixed by $L \cap \bar{N}$, too, since $(\Phi \circ i)^{\mathrm{tr}}$ is $(L \cap \bar{N})$-equivariant.

PROPOSITION 8.1. *Suppose $e^* \in E^*$ is an AB^- weight vector such that $(\Phi \circ i)^{\mathrm{tr}}(e^*) \neq 0$. If $(\Phi \circ i)^{\mathrm{tr}}(e^*)$ is fixed by $L \cap \bar{N}$, then it is acted upon by A with a restricted weight of the form*

$$-(\rho_G - \rho_L + \nu) - \sum_{\alpha \in \Sigma_G^+} n_\alpha \alpha, \qquad n_\alpha \geqslant 0. \tag{8.1}$$

Proof. The V^{σ}-distribution $(\Phi \circ i)^{\mathrm{tr}}(e^*)$ on $\bar{N}$ is acted upon trivially by $L \cap \bar{N}$ by hypothesis, and it is acted upon by AB^- according to some weight, as a consequence of equivariance. For f in $C^{\infty}_{\mathrm{com}}(\bar{N}, V^{\sigma})$, we find that

$$(\Phi \circ i)^{\mathrm{tr}}(e^*)(f) = \left\langle \bar{\partial} \left(\int_{L \cap K} \langle i(f(xk)), \phi \rangle\, \pi(k)(1 \otimes \omega_S)\, dk \right)_{x=1}, e^* \right\rangle.$$

Here $i(f(xk))$ is well defined since x is close to the identity. Now $\bar{\partial}$ involves differentiations on the right of x and some manipulations with alternating tensors. Then x is put equal to 1. Hence the V^{σ}-distribution in question is supported on $L \cap \bar{N}$. By Proposition 7.2, there exist members v_j^* of $(V^{\sigma})^*$ and left-invariant differential operators u_j in $U(\bar{\mathfrak{n}})$ such that

$$(\Phi \circ i)^{\mathrm{tr}}(e^*)(f) = \sum_j \int_{L \cap \bar{N}} \langle (u_j f)(x), v_j^* \rangle\, dx.$$

Let us see the effect of $a \in A$ on our V^σ-distribution. For use in §9, we carry along the effect also of $m \in B^-$. The effect of $am \in AB^-$ is

$$am((\Phi \circ i)^{\mathrm{tr}}(e^*))(f) = (\Phi \circ i)^{\mathrm{tr}}(e^*)(m^{-1}a^{-1}f)$$

$$= \sum_j \int_{L \cap \bar{N}} \langle (u_j(m^{-1}a^{-1}f))(x), v_j^* \rangle \, dx. \tag{8.2}$$

The integrand of (8.2) is, in obvious notation,

$$\begin{aligned}
&= \langle m^{-1}a^{-1}f(xu_j), v_j^* \rangle \\
&= \langle \tilde{\sigma}(m^{-1}a^{-1})\, f(amxu_jm^{-1}a^{-1}), v_j^* \rangle \\
&= a^{-(\rho_L + \nu + \rho_G)} \langle \sigma(m)^{-1} f((amxm^{-1}a^{-1})\, amu_jm^{-1}a^{-1}), v_j^* \rangle \\
&= a^{-(\rho_L + \nu + \rho_G)} \langle \sigma(m)^{-1}\, (\mathrm{Ad}(am)u_j)\, f(amxm^{-1}a^{-1}), v_j^* \rangle.
\end{aligned}$$

Thus (8.2) is

$$= \sum_j \int_{L \cap \bar{N}} a^{-(\rho_L + \nu + \rho_G)} \langle \sigma(m)^{-1}\, (\mathrm{Ad}(am)u_j)\, f(amxm^{-1}a^{-1}), v_j^* \rangle \, dx$$

$$= a^{-(\rho_G - \rho_L + \nu)} \sum_j \int_{L \cap \bar{N}} \langle \sigma(m)^{-1}\, (\mathrm{Ad}(am)u_j)\, f(x), v_j^* \rangle \, dx, \tag{8.3}$$

the latter equality following from the change of variables $amxm^{-1}a^{-1} \to x$. In turn, since $(\Phi \circ i)^{\mathrm{tr}}(e^*)$ is a weight vector, (8.3) must be an exponentiated weight (as a function of am) times

$$\sum_j \int_{L \cap \bar{N}} \langle u_j f(x), v_j^* \rangle \, dx. \tag{8.4}$$

Now let us put $m = 1$. Since $\mathrm{Ad}(a)$ acts on $U(\bar{\mathfrak{n}})$ by sums of negative restricted roots, comparison of (8.3) and (8.4) shows that the restricted weight is of the form in (8.1).

PROPOSITION 8.2. *The restricted weight of any member of E^* is*

$$\geqslant -(\rho_G - \rho_L + \nu) + \alpha_0,$$

where α_0 denotes the smallest member of Σ_G^+ in the lexicographic ordering.

Proof. The restricted weights of E^* are the negatives of those of E, which in turn are the sum of ν with the restricted weights of $(\bigwedge^{s+1}\mathfrak{u})^*$, since $\mathfrak{a}''$ acts trivially on $\mathbb{C}^\#_{\lambda+\nu}$ and $\mathfrak{a}'$ acts by ν. Hence the restricted weights of E^* are the sum of $-\nu$ and the restricted weights of $\bigwedge^{s+1}\mathfrak{u}$. We are thus to prove that the restricted weights of $\bigwedge^{s+1}\mathfrak{u}$ are $\geqslant -(\rho_G-\rho_L)+\alpha_0$.

A basis of restricted weight vectors is given by all monomials $X_{\beta_1}\wedge\cdots\wedge X_{\beta_{s+1}}$, where $\{\beta_1, ..., \beta_{s+1}\}$ is a set of $s+1$ members of $\Delta(\mathfrak{u}, \mathfrak{h})$. One such monomial is $X_S\wedge X_\alpha$, where α is any member of $\Delta(\mathfrak{u}, \mathfrak{h})$ with $\mathbf{c}\alpha|_\mathfrak{a}=\alpha_0$; Proposition 4.2 shows that this has restricted weight $-(\rho_G-\rho_L)+\alpha_0$. To get a different weight vector, one must drop some members of S, replacing them by roots $\beta\in\Delta(\mathfrak{u}, \mathfrak{h})$ with $\mathbf{c}\beta|_\mathfrak{a}>0$, or one must replace α by some other root β with $\mathbf{c}\beta|_\mathfrak{a}>0$. The results of each of these operations do not decrease the restricted weight. Therefore $-(\rho_G-\rho_L)+\alpha_0$ is the lowest restricted weight.

LEMMA 8.3. $(\Phi\circ i)^{\mathrm{tr}}=0$.

Proof. Assuming the contrary, fix an L-invariant irreducible direct summand of E^* on which $(\Phi\circ i)^{\mathrm{tr}}$ is not 0, and let e^* be a weight vector under AB^- with lowest possible restricted weight such that $(\Phi\circ i)^{\mathrm{tr}}(e^*)\neq 0$. Then $(\Phi\circ i)^{\mathrm{tr}}(Xe^*)=0$ for all $X\in\mathfrak{l}\cap\bar{\mathfrak{n}}$, since Xe^* has lower restricted weight. Since $(\Phi\circ i)^{\mathrm{tr}}$ is $(L\cap\bar{N})$-equivariant, $X((\Phi\circ i)^{\mathrm{tr}}(e^*))=0$. Thus $L\cap\bar{N}$ fixes $(\Phi\circ i)^{\mathrm{tr}}(e^*)$. By Proposition 8.1, the restricted weight of $(\Phi\circ i)^{\mathrm{tr}}(e^*)$ is of the form

$$-(\rho_G-\rho_L+\nu)-\sum_{\alpha\in\Sigma_G^+} n_\alpha\alpha$$

with all $n_\alpha\geqslant 0$. On the other hand, the A-equivariance of $(\Phi\circ i)^{\mathrm{tr}}$ implies that the restricted weight of e^* is of this form, too. But this conclusion contradicts Proposition 8.2, and the lemma follows.

THEOREM 8.4. $\bar{\partial}\circ\mathscr{S}=0$.

Proof. Assuming the contrary, choose F in $C^\infty(G/MAN, \tilde{\sigma})$ with $\bar{\partial}\mathscr{S}(F)\neq 0$. Say $\bar{\partial}\mathscr{S}(F)(x)\neq 0$ for some $x\in G$. Then

$$\begin{aligned}\Phi(\mathscr{L}(x)^{-1}F)&=e\circ\bar{\partial}\circ\mathscr{S}(\mathscr{L}(x)^{-1}F)=\bar{\partial}\circ\mathscr{S}(\mathscr{L}(x)^{-1}F)(1)\\&=\mathscr{L}(x)^{-1}(\bar{\partial}\mathscr{S}F)(1)=\bar{\partial}\mathscr{S}F(x)\neq 0\end{aligned}$$

shows $\Phi\neq 0$. Changing notation, let us suppose that F is a member of $C^\infty(G/MAN, \tilde{\sigma})$ with $\Phi(F)\neq 0$.

Let U be the open image of $\bar{N}$ in G/MAN. As l varies through $L\cap K$, the open sets lU cover the compact set of cosets in G/MAN corresponding to $L\cap K$. Let $l_1U, ..., l_rU$ be a finite subcover of this compact set, and let U' be the open complement of this compact set. Then $U', l_1U, ..., l_rU$ is an open cover of G/MAN. Let φ', $\varphi_1, \ldots, \varphi_r$ be a C^∞ partition of unity subordinate to this open cover, so that

$$F = \varphi' F + \sum_{j=1}^{r} \varphi_j F,$$

with each summand in $C^\infty(G/MAN, \tilde{\sigma})$. Since $\varphi' F$ vanishes in a neighborhood of the image of $L \cap K$ in G/MAN, $\Phi(\varphi' F) = 0$. Thus

$$\sum_{j=1}^{r} \Phi(\varphi_j F) \neq 0,$$

and we may assume by renumbering that $\Phi(\varphi_1 F) \neq 0$. Consequently

$$0 \neq \mathrm{Ad}^*(l_1)^{-1} (\Phi(\varphi_1 F)) = \Phi(\mathscr{L}(l_1)^{-1} (\varphi_1 F)).$$

Here $\mathscr{L}(l_1)^{-1} (\varphi_1 F)$ is compactly supported in U, and thus $\mathscr{L}(l_1)^{-1} (\varphi_1 F) = i(f)$ for some $f \in C^\infty_{\mathrm{com}}(\bar{N}, V^\sigma)$. Thus $\Phi \circ i(f) \neq 0$, and $\Phi \circ i \neq 0$. Since our continuous duals separate points, it follows that $(\Phi \circ i)^{\mathrm{tr}} \neq 0$, in contradiction to Lemma 8.3.

9. Effect of $\bar{\partial}^*$

The operator $\bar{\partial}^*$ was defined in (1.7) and just afterward, and we take it now to be an operator

$$\bar{\partial}^*\colon C^{0,s}(G/L, \mathbb{C}^{\#}_{\lambda+\nu}) \to C^{0,s-1}(G/L, \mathbb{C}^{\#}_{\lambda+\nu}).$$

In this section we prove that $\bar{\partial}^* \circ \mathscr{S} = 0$. The proof has much in common with §8. But there is one additional twist: the analogs of Propositions 8.1 and 8.2, as well as the way they are used, are more complicated.

We define $E = \mathbb{C}^{\#}_{\lambda+\nu} \otimes (\bigwedge^{s-1} \mathfrak{u})^*$, and we let $\Phi\colon C^\infty(G/MAN, \tilde{\sigma}) \to E$ be the composition $\Phi = e \circ \bar{\partial}^* \circ \mathscr{S}$, where e is evaluation at 1. Here $\bar{\partial}^* \circ \mathscr{S}$ is G-equivariant, and e is L-equivariant; thus Φ is L-equivariant. Again Φ is continuous. The inclusion mapping

$$i\colon C^\infty_{\mathrm{com}}(\bar{N}, V^\sigma) \to C^\infty(G/MAN, \tilde{\sigma})$$

is unchanged from §8 and is $MA\bar{N}$-equivariant.

Proposition 9.1. *Suppose $e^* \in E^*$ is an AB^- weight vector such that $(\Phi \circ i)^{\mathrm{tr}} (e^*) \neq 0$. If $(\Phi \circ i)^{\mathrm{tr}} (e^*)$ is fixed by $L \cap \bar{N}$, then it is acted upon by A with a restricted weight of the form*

$$-(\rho_G - \rho_L + \nu) - \sum_{\alpha \in \Sigma_G^+} n_\alpha \alpha, \qquad n_\alpha \geqslant 0. \tag{9.1}$$

Moreover, if $\sum n_\alpha \alpha = 0$, *then it is acted upon by* B^- *with a weight of the form*

$$\Big[-\lambda - 2\delta(\mathfrak{u} \cap \mathfrak{p}) + \sum_{\gamma \in \Delta^+(\mathfrak{m}, \mathfrak{b}^-)} m_\gamma \gamma\Big]\Big|_{\mathfrak{b}^-}. \tag{9.2}$$

Proof. We argue as in Proposition 8.1, replacing $\bar\partial$ by $\bar\partial^*$, and (9.1) follows. Now suppose $\sum n_\alpha \alpha = 0$. From (8.4) we see that all the u_j that make a contribution must be scalars c_j. In (8.3) let us write v^* for $\sum c_j v_j^*$, and let us put $a = 1$. The result is

$$m((\Phi \circ i)^{\mathrm{tr}}(e^*))(f) = \int_{L \cap \bar N} \langle \sigma(m)^{-1}(f(x)), v^* \rangle \, dx.$$

It follows that the B^- weight in question is a weight of the conjugate $\bar\sigma$. Now σ has highest weight $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$, according to Proposition 3.1, and thus $\bar\sigma$ has lowest weight $-\lambda - 2\delta(\mathfrak{u} \cap \mathfrak{p})$. Thus the B^- weight in question is of the form (9.2).

PROPOSITION 9.2. *The restricted weight of any member of* E^* *is*

$$\geqslant -(\rho_G - \rho_L + \nu). \tag{9.3}$$

Any member of E^* *of restricted weight* $-(\rho_G - \rho_L + \nu)$ *that is a weight vector under* B^- *has weight*

$$[-\lambda - 2\delta(\mathfrak{u} \cap \mathfrak{p}) - \beta_0]|_{\mathfrak{b}^-} \tag{9.4}$$

for some $\beta_0 \in \Delta^+(\mathfrak{m}, \mathfrak{b}^-)$.

Proof. The proof of (9.3) is an easy adaptation of the proof of Proposition 8.2. Equality in (9.3) for a weight vector under B^- corresponds to having a monomial $X_{\beta_1} \wedge \cdots \wedge X_{\beta_{s-1}}$ involving all members β_j of S with $\mathbf{c}\beta_j|_{\mathfrak{a}} < 0$ and all but one member β_j of S with $\mathbf{c}\beta_j|_{\mathfrak{a}} = 0$. Let the missing root be β_0; β_0 is positive since it contributes to $\mathfrak{u}$.

Since $E^* \cong (\mathbb{C}^{\#}_{\lambda+\nu})^* \otimes \bigwedge^{s-1}\mathfrak{u}$, the corresponding weight for this monomial is the sum of the weight of 1 in $(\mathbb{C}^{\#}_{\lambda+\nu})^*$ and weight$(X_S) - \beta_0$. By Proposition 4.6, the restriction of this weight to $\mathfrak{b}^-$ is

$$[-(\lambda + 2\delta(\mathfrak{u})) + 2\delta(\mathfrak{u} \cap \mathfrak{k}) - \beta_0]|_{\mathfrak{b}^-},$$

and this is just (9.4), by (1.1).

LEMMA 9.3. $(\Phi \circ i)^{\mathrm{tr}} = 0$.

Proof. We begin with e^* as in Lemma 8.3 and again see that $L \cap \bar{N}$ fixes $(\Phi \circ i)^{\mathrm{tr}}(e^*)$. By Proposition 9.1, the restricted weight of $(\Phi \circ i)^{\mathrm{tr}}(e^*)$ is of the form

$$-(\rho_G - \rho_L + \nu) - \sum_{\alpha \in \Sigma_G^+} n_\alpha \alpha$$

with all $n_\alpha \geqslant 0$, and by Proposition 9.2 the restricted weight is $\geqslant -(\rho_G - \rho_L + \nu)$. Therefore the restricted weight is equal to $-(\rho_G - \rho_L + \nu)$. Proposition 9.1 says that its weight under B^- is of the form

$$\left[-\lambda - 2\delta(\mathfrak{u} \cap \mathfrak{p}) + \sum_{\gamma \in \Delta^+(\mathfrak{m}, \mathfrak{b}^-)} m_\gamma \gamma\right]\Bigg|_{\mathfrak{b}^-}, \tag{9.5}$$

while Proposition 9.2 says that its weight under B^- is of the form

$$[-\lambda - 2\delta(\mathfrak{u} \cap \mathfrak{p}) - \beta_0]|_{\mathfrak{b}^-}. \tag{9.6}$$

Since (9.5) and (9.6) are incompatible, the lemma follows.

THEOREM 9.4. $\bar{\partial}^* \circ \mathscr{S} = 0.$

Proof. This is derived from Lemma 9.3 in the same way that Theorem 8.4 is derived from Lemma 8.3.

10. OPERATOR $\mathscr{P}$

Our goal in this section is to prove that when $\mathscr{S}$ is followed by the quotient map of cocycles into cohomology, the image is not zero. The tool for the proof is an operator $\mathscr{P}$ that carries $C^{0,s}(G/L, \mathbb{C}^\#_{\lambda+\nu})$ into $C^\infty(G/K, V^\mu)$ and annihilates coboundaries; thus $\mathscr{P}$ is well defined on $H^{0,s}(G/L, \mathbb{C}^\#_{\lambda+\nu})$. This operator was introduced by Schmid [25] in the case that rank $G =$ rank K and L is a maximal torus, and later it was developed further by Wells and Wolf [35]. When G/K is Hermitian and suitable compatibility conditions are satisfied by complex structures, it is an instance of the Penrose transform described in [2].

The operator $\mathscr{P}$ is defined in terms of an operator P that can be seen to implement the Bott–Borel–Weil isomorphism in the direction

$$H^{0,s}(K/(L \cap K), \mathbb{C}^\#_\lambda) \cong V^\mu,$$

where $\mathbb{C}^\#_\lambda$ is the space of the one-dimensional representation of $L \cap K$ with weight $\lambda + 2\delta(\mathfrak{u})$ and where μ is as in §3. Namely let $\{\phi_i\}$ be an orthonormal basis of V^μ, and let $\bar{\omega}_C$ be the complex conjugate of ω_C; $\bar{\omega}_C$ is thus a nonzero element of $\bigwedge^s(\bar{\mathfrak{u}} \cap \mathfrak{k})^*$. To ϕ_i we make correspond an $(s, 0)$ form for the dual line bundle over $K/(L \cap K)$, namely a function

$$\varphi_i \colon K \to (\mathbb{C}_\lambda^\#)^* \otimes \textstyle\bigwedge^s(\bar{\mathfrak{u}} \cap \mathfrak{k})^*$$

transforming under $L \cap K$ on the right according to $(\mathbb{C}_\lambda^\#)^* \otimes \bigwedge^s(\bar{\mathfrak{u}} \cap \mathfrak{k})^*$, by means of the definition

$$\varphi_i(k) = \langle \mu(k)\phi, \phi_i \rangle \bar{\omega}_C. \tag{10.1}$$

If F is in $C^{0,s}(K/(L \cap K), \mathbb{C}_\lambda^\#)$, so that F is a function

$$F \colon K \to \mathbb{C}_\lambda^\# \otimes \textstyle\bigwedge^s(\mathfrak{u} \cap \mathfrak{k})^*$$

transforming under $L \cap K$ on the right according to $\mathbb{C}_\lambda^\# \otimes \bigwedge^s(\mathfrak{u} \cap \mathfrak{k})^*$, then $F \otimes \varphi_i$ is a function

$$F \otimes \varphi_i \colon K \to \textstyle\bigwedge^{2s}(\mathfrak{k}/(\mathfrak{l} \cap \mathfrak{k}))^*$$

right-invariant under $L \cap K$. Thus $F \otimes \varphi_i$ is a volume form on $K/(L \cap K)$. We define

$$P \colon C^{0,s}(K/(L \cap K), \mathbb{C}_\lambda^\#) \to V^\mu \tag{10.2a}$$

by

$$P(F) = \sum_i \left(\int_{K/(L \cap K)} F \otimes \varphi_i \right) \phi_i. \tag{10.2b}$$

PROPOSITION 10.1. *The operator P in (10.2) is independent of the orthonormal basis ϕ_i, is K-equivariant, and annihilates the image of*

$$\bar{\partial}_K \colon C^{0,s-1}(K/(L \cap K), \mathbb{C}_\lambda^\#) \to C^{0,s}(K/(L \cap K), \mathbb{C}_\lambda^\#).$$

Proof. It is straightforward to verify that P is independent of $\{\phi_i\}$ and is K-equivariant. Suppose $F = \bar{\partial}_K f$ for some

$$f \in C^{0,s-1}(K/(L \cap K), \mathbb{C}_\lambda^\#).$$

We shall use Stokes' Theorem to prove that $\int_{K/(L \cap K)} (F \otimes \varphi_i) = 0$ for all i. Thus introduce the deRham d_K and the operator ∂_K with $d_K = \partial_K + \bar{\partial}_K$. (See [34, Chapter I].) Now $f \otimes \varphi_i$ is an $(s, s-1)$ form on $K/(L \cap K)$, and so ∂_K of it is 0. By Stokes' Theorem

$$0 = \int_{K/(L \cap K)} d_K(f \otimes \varphi_i) = \int_{K/(L \cap K)} \bar{\partial}_K(f \otimes \varphi_i).$$

In terms of operators $\bar{\partial}_K$ on bundle-valued forms, the product rule for $\bar{\partial}_K$ gives

$$0 = \int_{K/(L\cap K)} (F \otimes \varphi_i) + (-1)^{s-1} \int_{K/(L\cap K)} (f \otimes \bar{\partial}_K \varphi_i).$$

Referring to (10.1), we see that the coefficient of $\bar{\omega}_C$ in φ_i is holomorphic (since $E_\alpha \langle \mu(k)\phi, \phi_i \rangle = 0$ for $\alpha \in \Delta_K^+$), and thus $\bar{\partial}_K \varphi_i = 0$. The proposition follows.

Let R: $K/(L\cap K) \to G/L$ be the holomorphic map induced by inclusion, and let

$$R^*\colon C^{0,m}(G/L, \mathbb{C}^\#_{\lambda+\nu}) \to C^{0,m}(K/(L\cap K), \mathbb{C}^\#_\lambda)$$

be the pullback. This amounts to restricting functions in

$$C^\infty(G/L, \mathbb{C}^\#_{\lambda+\nu} \otimes (\textstyle\bigwedge^m \mathfrak{u})^*)$$

from G to K and projecting their values from $(\bigwedge^m \mathfrak{u})^*$ to $(\bigwedge^m(\mathfrak{u}\cap\mathfrak{k}))^*$. Since R is holomorphic,

$$\bar{\partial}_K \circ R^* = R^* \circ \bar{\partial}. \tag{10.3}$$

For $F \in C^{0,s}(G/L, \mathbb{C}^\#_{\lambda+\nu})$, we define the G-equivariant operator $\mathscr{P}$ with

$$\mathscr{P}\colon C^{0,s}(G/L, \mathbb{C}^\#_{\lambda+\nu}) \to C^\infty(G/K, V^\mu) \tag{10.4a}$$

by

$$\mathscr{P}F(x) = P(R^*(\mathscr{L}(x)^{-1} F)), \tag{10.4b}$$

where $\mathscr{L}$ is the left regular representation. Recall from §3 that $V^\sigma \subseteq V^\mu$ and that the unit vector ϕ is a highest weight vector for each.

Proposition 10.2. *$\mathscr{P}$ annihilates the image of $\bar{\partial}$.*

Proof. We have

$$\begin{aligned}\mathscr{P}(\bar{\partial}f)(x) &= P(R^*(\mathscr{L}(x)^{-1}\bar{\partial}f)) = P(R^*(\bar{\partial}\mathscr{L}(x)^{-1}f)) \\ &= P(\bar{\partial}_K(R^*\mathscr{L}(x)^{-1}f)) \qquad \text{by (10.3)} \\ &= 0 \qquad \text{by Proposition 10.1.}\end{aligned}$$

Theorem 10.3. *There exist nonzero constants c_1 and c_2 such that*

$$(\mathscr{P} \circ \mathscr{S}) f(x) = c_1 \int_K \langle f(xk), \phi \rangle \, \mu(k)\phi \, dk = c_2 \int_K \mu(k)(f(xk))\, dk$$

for all $f \in C^\infty(G/MAN, \bar{\sigma})$ and all $x \in G$.

Remark. The expression $\int_K \mu(k)(f(xk))\,dk$ is the Szegö integral of f, as used in [5], [21], and [15]. Thus the theorem gives a factorization of the Szegö operator into $\mathscr{P} \circ \mathscr{S}$ under our standing equal real rank hypothesis.

Proof. By G-equivariance we may take $x = 1$. Let c denote a nonzero constant whose value may change at each appearance. Then

$$\begin{aligned}
\mathscr{P}\mathscr{S}f(1) &= P(R^*(\mathscr{S}f)) \\
&= cP(\langle \mathscr{S}f(\cdot), 1 \otimes \omega_C\rangle(1 \otimes \omega_C)) \qquad \text{with} \quad (\cdot) \text{ in } K \\
&= c\sum_i \left(\int_{K/L \cap K} \langle \mathscr{S}f(k), 1 \otimes \omega_C\rangle \langle \mu(k)\phi, \phi_i\rangle (1 \otimes \omega_C \otimes \bar{\omega}_C) \right) \phi_i \\
&= c\sum_i \left(\int_{K/L \cap K} \langle \mathscr{S}f(k), 1 \otimes \omega_C\rangle \langle \mu(k)\phi, \phi_i\rangle \, dk \right) \phi_i \\
&= c\int_{K/L \cap K} \mu(k) \langle \mathscr{S}f(k), 1 \otimes \omega_C\rangle \phi \, dk.
\end{aligned}$$

Now

$$\begin{aligned}
\langle \mathscr{S}f(k), 1 \otimes \omega_C\rangle \phi &= \int_{L \cap K} \langle f(kl), \phi\rangle \langle \pi(l)(1 \otimes \omega_S), 1 \otimes \omega_C\rangle \phi \, dl \\
&= \int_{L \cap K} \langle f(kl), \phi\rangle \langle 1 \otimes \omega_S, \pi(l)^{-1} (1 \otimes \omega_C)\rangle \phi \, dl \\
&= \int_{L \cap K} \langle f(kl), \phi\rangle \langle 1 \otimes \omega_S, 1 \otimes \omega_C\rangle \, \tau(l)\phi \, dl \\
&= c\int_{L \cap K} \langle f(kl), \phi\rangle \, \tau(l)\phi \, dl,
\end{aligned}$$

the last equality holding by Proposition 4.3. Substituting, we obtain

$$\begin{aligned}
\mathscr{P}\mathscr{S}f(1) &= \int_{K/L \cap K} \int_{L \cap K} \langle f(kl), \phi\rangle \, \mu(k)\, \mu(l) \, dl \, dk \\
&= c\int_K \langle f(k), \phi\rangle \, \mu(k)\phi \, dk,
\end{aligned}$$

in agreement with the first assertion of the theorem.

For the second assertion, let us introduce, for each $v \in V^\mu$, the member f_v of $C^\infty(G/MAN, \tilde{\sigma})$ given by

$$f_v(k) = P_\sigma(\mu(k)^{-1} v), \tag{10.5}$$

where P_σ is the orthogonal projection of V^μ on V^σ. Let $\{\psi_i\}$ be an orthonormal basis of the subspace V^σ. For any $v \in V^\mu$, we have

$$\begin{aligned}
\left\langle \int_K \langle f(k), \phi\rangle\, \mu(k)\phi\, dk, v \right\rangle &= \int_K \langle f(k), \phi\rangle \langle \mu(k)\phi, v\rangle\, dk \\
&= \int_K \langle f(k), \phi\rangle \langle \phi, \mu(k)^{-1} v\rangle\, dk \\
&= \int_K \langle f(k), \phi\rangle \langle \phi, P_\sigma \mu(k)^{-1} v\rangle\, dk \\
&= \int_K \langle f(k), \phi\rangle \langle \phi, f_v(k)\rangle\, dk \\
&= \int_K \langle f(k), \phi\rangle \overline{\langle f_v(k), \phi\rangle}\, dk.
\end{aligned}$$

For each $m \in M$, this expression is

$$\begin{aligned}
&= \int_K \langle f(km), \phi\rangle \overline{\langle f_v(km), \phi\rangle}\, dk \\
&= \int_K \langle f(k), \sigma(m)\phi\rangle \overline{\langle f_v(k), \sigma(m)\phi\rangle}\, dk,
\end{aligned}$$

and therefore it is equal to the average

$$\begin{aligned}
&= \int_K \int_M \langle f(k), \sigma(m)\phi\rangle \overline{\langle f_v(k), \sigma(m)\phi\rangle}\, dm\, dk \\
&= \frac{1}{\deg \sigma} \int_K \langle f(k), f_v(k)\rangle\, dk \qquad \text{by Schur orthogonality} \\
&= \frac{1}{\deg \sigma} \int_K \langle f(k), P_\sigma \mu(k)^{-1} v\rangle\, dk \\
&= \frac{1}{\deg \sigma} \int_K \langle \mu(k) f(k), v\rangle\, dk.
\end{aligned}$$

Since $v \in V^\mu$ is arbitrary,

$$\int_K \langle f(k), \phi\rangle\, \mu(k)\phi\, dk = \frac{1}{\deg \sigma} \int_K \mu(k) f(k)\, dk,$$

and the theorem follows.

COROLLARY 10.4. *The image of $\mathscr{S}$ is nonzero when regarded as in $H^{0,s}(G/L, \mathbb{C}^{\#}_{\lambda+\nu})$.*

Proof. Proposition 10.2 says that $\mathscr{P}$ descends to an operator on $H^{0,s}(G/L, \mathbb{C}^{\#}_{\lambda+\nu})$. It is therefore enough to prove that $\mathscr{P} \circ \mathscr{S} \neq 0$. In fact, we show that $\mathscr{P}\mathscr{S}f_\phi(1) \neq 0$, where f_ϕ is the function in (10.5) with $v = \phi$. According to Theorem 10.3,

$$\mathscr{P}\mathscr{S}f_\phi(1) = c_1 \int_K \langle f_\phi(k), \phi \rangle\, \mu(k)\phi\, dk.$$

Thus

$$\begin{aligned}\langle \mathscr{P}\mathscr{S}f_\phi(1), \phi \rangle &= c_1 \int_K \langle P_\sigma \mu(k)^{-1}\phi, \phi \rangle \langle \mu(k)\phi, \phi \rangle\, dk \\ &= c_1 \int_K |\langle \mu(k)\phi, \phi \rangle|^2\, dk \\ &= \frac{c_1}{\deg \mu} \neq 0.\end{aligned}$$

11. EXISTENCE OF SPECIAL ORDERINGS

In the course of defining Δ^+ in §1, we assumed that we were given a positive system for $\Delta(\mathfrak{l}, \mathfrak{h})$ and an ordered sequence $\alpha_1, \ldots, \alpha_l$ from $\Delta(\mathfrak{l}, \mathfrak{h}) \cap \Delta_{B,\mathrm{n}}$ that were compatible in a sense defined at the end of §1. We address existence of compatible $\Delta^+(\mathfrak{l}, \mathfrak{h})$ and $\alpha_1, \ldots, \alpha_l$ in this section.

Actually the existence by itself is easy to prove. First let us prove that there exists a strongly orthogonal sequence $\alpha_1, \ldots, \alpha_l$ in $\Delta(\mathfrak{l}, \mathfrak{h}) \cap \Delta_{B,\mathrm{n}}$ with $\sum_{j=1}^{l} \mathbb{R}(E_{\alpha_j} + E_{-\alpha_j}) \oplus \mathfrak{a}_0'$ maximal abelian in $\mathfrak{l}_0 \cap \mathfrak{p}_0$. If G and K have equal rank, such a sequence $\alpha_1, \ldots, \alpha_l$ exists as a consequence of results of [15, §4] applied to L in place of G. If G and K have unequal rank, let $\mathfrak{l}_0'$ be the centralizer in $\mathfrak{l}_0$ of $\mathfrak{a}_0'$, and let $\mathfrak{l}_0''$ be the orthogonal complement of $\mathfrak{a}_0'$ in $\mathfrak{l}_0'$ relative to the Killing form C of $\mathfrak{g}_0$. The corresponding analytic subgroup L'' is reductive and has B as a compact Cartan subgroup. Moreover, the root system $\Delta(\mathfrak{l}'', \mathrm{b})$ is just $\Delta(\mathfrak{l}, \mathfrak{h}) \cap \Delta_B$. This construction reduces matters to the equal rank case settled above.

To complete the proof that compatibility can be achieved, we work with the sequence $\alpha_1, \ldots, \alpha_l$ just constructed (without renumbering it), and we construct a compatible $\Delta^+(\mathfrak{l}, \mathfrak{h})$. To do so, let $H_{\beta_1}, \ldots, H_{\beta_r}$ be an orthogonal basis of $i\mathfrak{b}_0^-$, and let $H_{\gamma_1}, \ldots, H_{\gamma_t}$ be an orthogonal basis of $\mathfrak{a}_0'$. We define $\Delta^+(\mathfrak{l}, \mathfrak{h})$ via the lexicographic ordering relative to the basis

$$H_{\beta_1}, \ldots, H_{\beta_r}, H_{\alpha_1}, \ldots, H_{\alpha_l}, H_{\gamma_1}, \ldots, H_{\gamma_t} \tag{11.1}$$

of $i\mathfrak{b}_0 \oplus \mathfrak{a}_0'$. Then the α_j are positive, and it is easy to see that (iii) holds in §1. Moreover, θ preserves positivity, since $\mathfrak{a}_0'$ comes last. Hence $\Delta^+(\mathfrak{l}, \mathfrak{h})$ and $\alpha_1, ..., \alpha_l$ together have all the desired properties.

This completes the proof that compatibility can be achieved. However, it is reasonable to demand more. The philosophy suggested in the introduction was that $\mathscr{S}$, when regarded as a map into cohomology, should induce an isomorphism of the derived functor module $A_{\mathfrak{q}}(\lambda)$ with the cohomology. For this purpose, our parameters should be set up so that $A_{\mathfrak{q}}(\lambda)$ is obviously a quotient of the domain of $\mathscr{S}$. Thus it is natural to demand that our nonunitary principal series representation have the real part of its A parameter $\rho_L + \nu$ dominant for Σ_G^+, so that the Langlands theory [19] is applicable. Actually ν is orthogonal to Σ_G^+, as we saw in the proof of Proposition 5.2, so that it is the Σ_G^+ dominance of ρ_L that is at issue. The proposition below says that this dominance can be achieved along with everything else.

PROPOSITION 11.1. *Let $\alpha_1, ..., \alpha_l$ be a strongly orthogonal sequence in $\Delta(\mathfrak{l}, \mathfrak{h}) \cap \Delta_{B,\mathfrak{n}}$ with $\sum_{j=1}^l \mathbb{R}(E_{\alpha_j} + E_{-\alpha_j}) \oplus \mathfrak{a}_0'$ maximal abelian in $\mathfrak{l}_0 \cap \mathfrak{p}_0$. Then $\alpha_1, ..., \alpha_l$ can be renumbered in such a way that the new $\Delta^+(\mathfrak{l}, \mathfrak{h})$ and the renumbered α's are still compatible and the renumbered α's, together with a suitable basis of $(\mathfrak{a}_0')^*$, make ρ_L dominant relative to Σ_G^+.*

Of course, any renumbering of $\alpha_1, ..., \alpha_l$ still results in a compatible $\Delta^+(\mathfrak{l}, \mathfrak{h})$ and sequence. Before coming to the proof that ρ_L can be made Σ_G^+-dominant, we introduce the renumbering and make a definition to simplify the notation. Let our given collection of l roots be $\{\alpha\}$. Enumerate these roots inductively as follows: If $\alpha_1, ..., \alpha_{j-1}$ have been defined, let α_j be one of the remaining α's for which

$$\sum_{\substack{\langle \beta, \alpha_1 \rangle = \cdots = \langle \beta, \alpha_{j-1} \rangle = 0 \\ \langle \beta, \alpha \rangle > 0,\ \beta \in \Delta(\mathfrak{l}, \mathfrak{h})}} \langle \beta, \alpha \rangle \tag{11.2}$$

is a maximum. For $\beta \in \Delta(\mathfrak{l}, \mathfrak{h})$, let $\beta_j = \langle \beta, \alpha_j \rangle$. All sums that appear below are sums over all $\beta \in \Delta(\mathfrak{l}, \mathfrak{h})$ with the indicated properties.

LEMMA 11.2. *For $1 \leqslant j \leqslant l-1$,*

$$\sum_{\substack{\beta_j > 0 \\ \beta_1 = \cdots = \beta_{j-1} = 0}} \beta_j \geqslant \sum_{\substack{\beta_{j+1} > 0 \\ \beta_1 = \cdots = \beta_j = 0}} \beta_{j+1}.$$

Proof. All sums in this proof are over all $\beta \in \Delta(\mathfrak{l}, \mathfrak{h})$ with $\beta_1 = \cdots = \beta_{j-1} = 0$ and with the indicated properties. Let s_{α_j} denote the reflection in α_j. We have

$$\begin{aligned}\sum_{\beta_j>0} \beta_j &\geq \sum_{\beta_{j+1}>0} \beta_{j+1} \qquad \text{by maximality of (11.2)}\\
&= \sum_{\substack{\beta_{j+1}>0\\ \beta_j>0}} (\langle \beta, \alpha_{j+1}\rangle + \langle s_{\alpha_j}\beta, \alpha_{j+1}\rangle) + \sum_{\substack{\beta_{j+1}>0\\ \beta_j=0}} \beta_{j+1}\\
&= 2 \sum_{\substack{\beta_{j+1}>0\\ \beta_j>0}} \beta_{j+1} + \sum_{\substack{\beta_{j+1}>0\\ \beta_j=0}} \beta_{j+1}\\
&\geq \sum_{\substack{\beta_{j+1}>0\\ \beta_j=0}} \beta_{j+1}.\end{aligned}$$

LEMMA 11.3. *For* $1 \leq j \leq l$,

$$2\langle \mathbf{c}^{-1}\rho_L, \alpha_j\rangle = \sum_{\substack{\beta_j>0\\ \beta_1=\cdots=\beta_{j-1}=0}} \beta_j, \tag{11.3}$$

so that $\langle \mathbf{c}^{-1}\rho_L, \alpha_j\rangle \geq \langle \mathbf{c}^{-1}\rho_L, \alpha_{j+1}\rangle$ *for* $1 \leq j \leq l-1$.

Proof. The second conclusion follows from (11.3) and Lemma 11.2. Thus we are to prove (11.3). We have

$$2\mathbf{c}^{-1}\rho_L = \sum_{i=1}^{l} \sum_{\substack{\beta_i>0\\ \beta_1=\cdots=\beta_{i-1}=0}} \beta \mod \mathfrak{a}'^*$$

and therefore

$$\begin{aligned}2\langle \mathbf{c}^{-1}\rho_L, \alpha_j\rangle &= \sum_{i=1}^{j} \sum_{\substack{\beta_i>0\\ \beta_1=\cdots=\beta_{i-1}=0}} \langle \beta, \alpha_j\rangle\\
&= \sum_{i=1}^{j-1} \sum_{\substack{\beta_i>0\\ \beta_1=\cdots=\beta_{i-1}=0}} \tfrac{1}{2}\langle \beta + s_{\alpha_j}\beta, \alpha_j\rangle + \sum_{\substack{\beta_j>0\\ \beta_1=\cdots=\beta_{j-1}=0}} \langle \beta, \alpha_j\rangle.\end{aligned}$$

The first sum on the right is 0 term by term, and the lemma follows.

Proof of Proposition 11.1. Once we have fully defined $(\mathbf{c}\Delta)^+$, we are to prove that $\langle \mathbf{c}^{-1}\rho_L, \beta\rangle \geq 0$ for all β in $\mathbf{c}^{-1}(\mathbf{c}\Delta)^+$. This is automatic for $\beta \in \Delta(\mathfrak{l}, \mathfrak{h})$ since ρ_L is known to be Σ_L^+-dominant. Thus we may assume β is in $\Delta(\mathfrak{g}, \mathfrak{h})$ and is not in $\Delta(\mathfrak{l}, \mathfrak{h})$. Also β is in $\mathbf{c}^{-1}(\mathbf{c}\Delta)^+$.

Let us write $\beta = \beta_{\mathfrak{b}^-} + \beta_{\mathfrak{b}''} + \beta_{\mathfrak{a}'}$ in obvious notation. First we show $\beta_{\mathfrak{b}^-} \neq 0$. In fact, suppose β is in $\Delta(\mathfrak{u}, \mathfrak{h})$ and $\beta_{\mathfrak{b}^-} = 0$. The Weyl group element $s_{\alpha_1}, ..., s_{\alpha_l}$ leaves $\mathfrak{u}$ stable, and so does θ. But the composition maps

β to $-\beta$, contradiction. So $\beta \in \Delta(\mathfrak{u}, \mathfrak{h})$ and $\beta_{\mathfrak{b}^-} = 0$ is impossible. Similarly $\beta \in \Delta(\bar{\mathfrak{u}}, \mathfrak{h})$ and $\beta_{\mathfrak{b}^-} = 0$ is impossible. Hence $\beta_{\mathfrak{b}^-} \neq 0$.

If $\beta_{\mathfrak{b}''} + \beta_{\mathfrak{a}'} = 0$, then $\langle \mathbf{c}^{-1}\rho_L, \beta \rangle = 0$. So we may assume $\beta_{\mathfrak{b}''} + \beta_{\mathfrak{a}'} \neq 0$. We show that $\beta_{\mathfrak{b}''} = 0$ or $\beta_{\mathfrak{a}'} = 0$. The vector $[X_\beta, \theta X_\beta]$ is in $\mathfrak{p}$ and, if not 0, is a root vector for $\mathbf{c}(2\beta_{\mathfrak{b}^-})$, which corresponds to $\mathfrak{m}$. Since $\mathfrak{m} \cap \mathfrak{p} = 0$, we conclude that $2\beta_{\mathfrak{b}^-}$ is not a root. Thus

$$\langle \beta_{\mathfrak{b}^-} + \beta_{\mathfrak{b}''} + \beta_{\mathfrak{a}'}, \beta_{\mathfrak{b}^-} - \beta_{\mathfrak{b}''} - \beta_{\mathfrak{a}'} \rangle \tag{11.4}$$

is $\geqslant 0$. Suppose $\beta_{\mathfrak{b}''} \neq 0$. Then

$$\langle \beta_{\mathfrak{b}^-} + \beta_{\mathfrak{b}''} + \beta_{\mathfrak{a}'}, \beta_{\mathfrak{b}^-} + \beta_{\mathfrak{b}''} - \beta_{\mathfrak{a}'} \rangle$$

must be strictly larger than (11.4) and hence must be > 0. It follows that the difference of the two members of the latter inner product is a root or is 0. But the difference cannot be a root, since no root vanishes on $\mathfrak{b}$. Thus it is 0, and $\beta_{\mathfrak{a}'} = 0$. We conclude that $\beta_{\mathfrak{b}''} = 0$ or $\beta_{\mathfrak{a}'} = 0$.

Suppose $\beta_{\mathfrak{a}'} = 0$, so that $\beta = \beta_{\mathfrak{b}^-} + \beta_{\mathfrak{b}''}$. If (11.4) is > 0, then the difference of the two members of the inner product, namely $2\beta_{\mathfrak{b}''}$, is a root. We saw above that a root vanishing on $\mathfrak{b}^-$ has to be in $\Delta(\mathfrak{l}, \mathfrak{h})$. Now $\beta_{\mathfrak{b}''} \pm \beta_{\mathfrak{b}^-}$ are both positive for $\mathbf{c}^{-1}(\mathbf{c}\Delta)^+$, and hence so is $2\beta_{\mathfrak{b}''}$. Since $2\beta_{\mathfrak{b}''}$ is in $\Delta(\mathfrak{l}, \mathfrak{h})$, $\langle \mathbf{c}^{-1}\rho_L, 2\beta_{\mathfrak{b}''} \rangle \geqslant 0$. Thus

$$\langle \mathbf{c}^{-1}\rho_L, \beta \rangle = \tfrac{1}{2} \langle \mathbf{c}^{-1}\rho_L, 2\beta_{\mathfrak{b}''} \rangle \geqslant 0.$$

Therefore we may assume that (11.4) is 0. In this case, it follows that

$$|\beta_{\mathfrak{b}''}|^2 = |\beta_{\mathfrak{b}^-}|^2 = \tfrac{1}{2} |\beta|^2. \tag{11.5}$$

For such a β, we can write

$$\beta = c_1\alpha_1 + \cdots + c_l\alpha_l + \beta_{\mathfrak{b}^-}$$

with $c_i = \langle \beta, \alpha_i \rangle / |\alpha_i|^2$. From Parseval's equality and (11.5) we obtain

$$2 = \sum \frac{4\langle \beta, \alpha_i \rangle^2}{|\beta|^2 \, |\alpha_i|^2}. \tag{11.6}$$

It follows that at most two c_i are nonzero. If just one c_i is $\neq 0$, then $\beta_{\mathfrak{b}''} = c_i\alpha_i$ forces $c_i > 0$, and we have

$$\langle \mathbf{c}^{-1}\rho_L, \beta \rangle = c_i \langle \mathbf{c}^{-1}\rho_L, \alpha_i \rangle > 0.$$

If two terms are nonzero, say c_i and c_j with $i < j$, then (11.6) forces

$$\beta_{\mathfrak{b}''} = \tfrac{1}{2}\alpha_i \pm \tfrac{1}{2}\alpha_j.$$

Then

$$\langle \mathbf{c}^{-1}\rho_L, \beta\rangle = \tfrac{1}{2}\langle \mathbf{c}^{-1}\rho_L, \alpha_i\rangle \pm \tfrac{1}{2}\langle \mathbf{c}^{-1}\rho_L, \alpha_j\rangle,$$

and this is $\geqslant 0$ by Lemma 11.3.

We are left with the roots such that $\beta_{\mathfrak{b}''} = 0$. To handle them, we need to specify the promised adjustment to the part of the basis (11.1) that lies in $\mathfrak{a}'_0$. Our renumbered $H_{\alpha_1}, ..., H_{\alpha_l}$, followed by the elements $H_{\gamma_1}, ..., H_{\gamma_t}$ of (11.1), define one system of positive restricted roots for G, though not our final one. Let ρ_0 be the $(\mathfrak{a}')^*$ component of $\mathbf{c}^{-1}\rho_L$ for this system. If $\rho_0 \neq 0$, let $\rho_1 = \rho_0$ and extend H_{ρ_1} to an orthogonal basis $H_{\rho_1}, ..., H_{\rho_t}$ of $\mathfrak{a}'_0$; if $\rho_0 = 0$, let $H_{\rho_1}, ..., H_{\rho_t}$ be any orthogonal basis of $\mathfrak{a}'_0$. Then

$$H_{\alpha_1}, ..., H_{\alpha_l}, H_{\rho_1}, ..., H_{\rho_t}, H_{\beta_1}, ..., H_{\beta_r}$$

defines our ultimate positive system $\mathbf{c}^{-1}(\mathbf{c}\Delta)^+$ and, by restriction, also Σ_G^+.

The claim is that Σ_L^+ is unchanged in passing to this new system. This is clear for roots nonvanishing on $\mathfrak{b}''$. For a root γ that vanishes on $\mathfrak{b}''$ but not $\mathfrak{a}'$, $\langle \gamma, \rho_0\rangle$ equals the inner product of γ with $\mathbf{c}^{-1}$ of the old ρ_L, and this is not 0. Thus the old sign of γ forces the sign of $\langle \gamma, \rho_0\rangle$ to be the same, and this forces the new sign of γ to be the same. Hence Σ_L^+ is unchanged.

Finally we can return to our roots β with $\beta_{\mathfrak{b}''} = 0$. It follows from the previous paragraph that such roots have

$$\langle \mathbf{c}^{-1}\rho_L, \beta\rangle = \langle \beta, \rho_0\rangle \geqslant 0,$$

and Proposition 11.1 follows.

12. Nonlinear Groups

The main results of this paper remain valid, with only notational changes, for nonlinear groups. Thus let $\tilde{G}$ be connected semisimple with finite center, and let $\eta: \tilde{G} \to G$ be a covering homomorphism to a linear group. Form $\tilde{K}$ maximal compact in $\tilde{G}$, $\tilde{T}$ a torus in $\tilde{K}$, and $\tilde{L}$ the centralizer in $\tilde{G}$ of $\tilde{T}$. If $K = \eta(\tilde{K})$ and $T = \eta(\tilde{T})$, then $L = \eta(\tilde{L})$. Since $\tilde{L}$ is a centralizer, it contains the center of $\tilde{G}$, and it follows that $\tilde{G}/\tilde{L} \cong G/L$. Thus $\tilde{G}/\tilde{L}$ becomes a complex manifold as a consequence of the linear case. From a one-dimensional representation ξ of $\tilde{L}$, we can build our holomorphic line bundle and spaces of differential forms by means of the results of Tirao and Wolf [30].

The hypothesis of linearity entered in only two places after the initial construction—in the use of the Cayley transform and in the control of the disconnectedness of M. Inspection of the proofs shows that the Cayley transform could always be regarded as an operation on the Lie algebra, so that linearity was not needed for the applicability of the Cayley transform.

Controlling $\tilde{M}$ is trickier. Since $\tilde{A}$ and A are connected, we find that $\tilde{M}=\eta^{-1}(M)$. With F as in (2.3), define $\tilde{F}=\eta^{-1}(F)$. Then $\tilde{F}\subseteq\tilde{M}$ since $\mathrm{Ad}(\tilde{F})$ fixes the Lie algebra of $\tilde{A}$. The critical facts are that

$$\tilde{M}=\tilde{M}_0\tilde{F} \tag{12.1}$$

and that

$$\tilde{F}\subseteq\tilde{L}. \tag{12.2}$$

The group $\tilde{F}$ need no longer be abelian.

To prove (12.1), let $\tilde{m}\in\tilde{M}$ be given, and write $\eta(\tilde{m})=m_0 f$ with $m_0\in M_0$ and $f\in F$. Since $\eta(\tilde{M}_0)=M_0$, choose $\tilde{m}_0\in\tilde{M}_0$ with $\eta(\tilde{m}_0)=m_0$. Then $\eta(\tilde{m}_0^{-1}\tilde{m})=f$, $\tilde{m}_0^{-1}\tilde{m}=\tilde{f}\in\tilde{F}$, and $\tilde{m}=\tilde{m}_0\tilde{f}$ as required.

To prove (12.2), let $\tilde{f}$ be in $\tilde{F}$. Then $\eta(\tilde{f})$ is in L by (2.4). Since $\eta(\tilde{L})=L$, choose $\tilde{l}\in\tilde{L}$ with $\eta(\tilde{l})=\eta(\tilde{f})$. Then $\eta(\tilde{l}^{-1}\tilde{f})=1$, $\tilde{l}^{-1}\tilde{f}=\tilde{z}$ is in the center of $\tilde{G}$, and $\tilde{f}=\tilde{l}\tilde{z}$ exhibits $\tilde{f}$ as a member of $\tilde{L}$.

Thus (12.1) and (12.2) are valid for the nonlinear case, and the results of the paper extend without difficulty.

Note added in proof. After the submission of this paper, we learned of the 1992 Harvard Ph. D. dissertation of H.-W. Wong, which proves the Vogan–Zuckerman conjecture of the introduction when the inducing representation is finite-dimensional. This hypothesis is satisfied in the case under discussion in our paper. Wong's proof is an extension of the methods of [27] and [8] and does not give explicit formulas. Since our work appears to be helpful in such problems as understanding the unitarity of these representations from an analytic standpoint, his work and ours may be regarded as complementing each other.

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