THE ROLE OF BASIC CASES IN CLASSIFICATION:

Theorems about Unitary Representations Applicable to SU(N,2)

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We propose in this paper a nontrivial subdivision of the problem of classifying the irreducible unitary representations of semisimple Lie groups. It is known that the classification problem comes down to deciding which of certain standard representations induced from cuspidal parabolic subgroups and having a unique irreducible quotient admit a semidefinite inner product that makes the irreducible quotient unitary. The idea of the subdivision is to separate matters into a consideration of a small number of "basic cases" and a conjectural reduction step.

We confine ourselves to the situation that the underlying group G is linear and has rank equal to the rank of a maximal compact subgroup K. The standard representations that one has to consider are of the form

$$U(MAN,\sigma,\nu) = \operatorname{ind}_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes 1) , \qquad (0.1)$$

where MAN is a cuspidal parabolic subgroup, σ is a discrete series or nondegenerate limit of discrete series representation of M, and ν is a real-valued linear functional on the Lie algebra of A in the closed positive Weyl chamber. (When ν is on the boundary of the Weyl chamber, some additional conditions are imposed on ν so that (0.1) has a unique irreducible quotient. See [11].)

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For each such MAN we shall give an existence result for certain σ's that we call "basic cases." We construct the basic cases explicitly when MAN is minimal. To any other σ we shall associate a proper reductive subgroup L with rank L = rank(L ∩ K) and a basic case σ^L of L, and a conjectural reduction step will describe the unitarity of the σ series in terms of the unitarity of the σ^L series. Part of the conjecture is closely related to conjectures by D. A. Vogan ([18], p. 408); the conjecture says also that L is large enough for a comparison of unitarity at all A parameters.

The discussion of the basic cases and the reduction conjecture are in §§1 and 3-5. They form the core of the paper. In §§2 and 6-10 we give a number of theorems that can be regarded as evidence for the conjecture or as treatment of basic cases. What these theorems have in common is that they all give new nontrivial information about unitarity in SU(N,2). Some of these results have been announced by us earlier ([11], [8]).

Of particular interest are two general results in §§6-7. One way of viewing basic cases is as minimal elements under translation of the M parameter toward the walls of the Weyl chambers of G, in the sense of the appendix of [13]. It follows from Conjecture 5.1 that this operation must preserve unitarity, and we state such a result for MAN minimal as Theorem 6.1.

In §7 we address a consistency question for the conjecture in the situation that dim A = real-rank(L) and σ^L is trivial on the M of the derived group L' of L. In this situation the trivial representation of L' occurs for a certain parameter $\nu = \nu_0$, and parameters ν with $|\nu| > |\nu_0|$ cannot lead to unitary representations of L. According to Conjecture 5.1, parameters ν with $|\nu| > |\nu_0|$ should not give unitary representations of G either, and this we verify as Theorem 7.1.

Our notion of basic cases evolved from the theorems in §§6-7 mentioned above and from a detailed study of the groups SU(N,2) and Sp(n,1). We obtained the general definition and theorem for MAN minimal only afterward. Upon seeing our constructive proof when MAN is minimal, Vogan was able to give an existential proof that applies also when MAN is nonminimal. The conjectural reduction was adjusted to take into account some examples supplied by Vogan for MAN nonminimal. We are grateful to Vogan for his suggestions and for permission to include his existence proof. We are grateful also to Welleda Baldoni Silva for highlighting her results [1] about Sp(n,1) in various ways for us at our request so that we could guess what the basic cases are in Sp(n,1).

Contents

1. Basic cases, minimal MAN

To keep the ideas clear, we shall begin with the situation of a minimal parabolic subgroup MAN of G. We shall define "basic cases" as certain representations of the compact group M. To do so, we first introduce the notion of a format. Recall that we are assuming G is linear and $\operatorname{rank} G = \operatorname{rank} K$.

We shall use the notation of [9], which we summarize briefly and incompletely here. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, let $\mathfrak{b} \subseteq \mathfrak{k}$ be a compact Cartan subalgebra of $\mathfrak{g}$, and let Δ and Δ_K be the sets of roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$ and $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$, respectively.

Fix a sequence $\alpha_1, \ldots, \alpha_\ell$ of strongly orthogonal noncompact members of Δ. In order to arrive at a minimal parabolic subgroup of G, we assume that

$$\ell = \text{real-rank}(G)\,. \tag{1.1}$$

Let

$$\mathfrak{b}_r = \sum \mathbb{R} i H_{\alpha_j}$$
$$\mathfrak{b}_- = \text{orthocomplement in } \mathfrak{b} \text{ to } \mathfrak{b}_r\,.$$

Anticipating a Cayley transform, we say that a root in Δ is real if it is carried on $\mathfrak{b}_r$, imaginary if it is carried on $\mathfrak{b}_-$, and complex otherwise. Let Δ_r and Δ_- be the subsets of real and imaginary roots, respectively. We construct a split subgroup G_r of G from $\mathfrak{b}$ and the members of Δ_r, and we let $K_r = K \cap G_r$ be its maximal compact subgroup. Let E be the orthogonal projection of $(i\mathfrak{b})'$ on $(i\mathfrak{b}_r)'$.

We build a Cayley transform $\underset{\sim}{c}$ from the roots $\alpha_1, \ldots, \alpha_\ell$ and use it as in [9] to form MA. If $\mathfrak{m}$ denotes the Lie algebra of M, we can regard Δ_- as the system of roots of $(\mathfrak{m}^{\mathbb{C}}, \mathfrak{b}_-^{\mathbb{C}})$. In [9] we defined

$$M^{\#} = M_e M_r\,, \tag{1.2}$$

where M_e is the analytic subgroup corresponding to $\mathfrak{m}$ and M_r is a finite abelian group built from the real roots. Because of (1.1), we have

$$M = M^{\#} . \tag{1.3}$$

Fix a positive system $(\Delta_-)^+$ for Δ_-. An irreducible representation σ of M is then determined by the highest weight λ of $\sigma|_{M_e}$ and a compatible character χ of M_r. By means of the ordered basis (1.2) in [9], we can introduce a positive system Δ^+ in which $\lambda + \rho_-$ is Δ^+ dominant, $(i\mathfrak{b}_r)'$ is spanned by the real simple roots, and some other conditions are satisfied. Let μ (in $(i\mathfrak{b}_r)'$) be the highest weight of a fine K_r-type whose restriction to M_r contains the translate of χ given by (2.2) of [9]. We shall say that λ has <u>format</u> $(\{\alpha_j\},\Delta^+,\chi,\mu)$ if the linear form

$$\Lambda = \lambda - E(2\rho_K) + 2\rho_{K_r} + \mu \tag{1.4}$$

given in Theorem 1 of [9] is Δ_K^+ dominant; Theorem 2 of [9] provides checkable necessary and sufficient conditions on μ for deciding this dominance.

<u>Theorem 1.1</u>. Suppose the group G with rank G = rank K has $G^{\mathbb{C}}$ simply connected. Fix a format $(\{\alpha_j\},\Delta^+,\chi,\mu)$ corresponding to a minimal parabolic MAN. Among all highest weights λ with this format, there is a unique one λ_b such that any other λ with this format has $\lambda - \lambda_b$ dominant for Δ^+ and G-integral.

We call λ_b (or the associated representation of M) the <u>basic case</u> for the format $(\{\alpha_j\},\Delta^+,\chi,\mu)$. If $G^{\mathbb{C}}$ is not simply connected, the basic case can still be defined as a member of $(i\mathfrak{b}_-)'$ by taking it to be the basic case for that format in the covering group that has a simply connected complexification.

Our constructive proof of Theorem 1.1 is too long to give here. It consists in writing down a formula for λ_b and verifying all the properties with the aid of some of the lemmas used to prove the theorems of [9]. However, Vogan has given a short existence proof that does not attempt to derive the formula, and we can include that. We shall therefore give some examples, followed by the formula for λ_b, followed by Vogan's proof.

Examples.

1) G of real rank one. Denote the real positive root by α. The fine K_r-type μ is 0 or $+\frac{1}{2}\alpha$ or $-\frac{1}{2}\alpha$, and it determines χ. Often two different formats (one with $\mu = +\frac{1}{2}\alpha$ and one with $\mu = -\frac{1}{2}\alpha$) will lead to the same basic case.

a) $\widetilde{SO}(2n,1)$, $n \geq 2$. Here $M = \widetilde{SO}(2n-1)$. The basic cases (as representations of M) are the trivial representation and the spin representation.

b) $SU(n,1)$, $n \geq 2$. Here

$$M = \left\{ \begin{pmatrix} \omega & & \\ & e^{i\theta} & \\ & & e^{i\theta} \end{pmatrix},\ \omega \in U(n-1) \text{ and total determinant} = 1 \right\}.$$

The basic cases are $\sigma(\text{this}) = e^{ik\theta}$ with $|k| \leq n$.

c) $Sp(n,1)$, $n \geq 2$. Here $M = SU(2) \times Sp(n-1)$. The basic cases are $\sigma = (k \times \text{fundamental}) \otimes 1$ with $0 \leq k \leq 2n-1$.

d) Real form of F_4. Here $M = \widetilde{SO}(7)$. There are five basic cases——the trivial representation, the first three multiples of the spin representation, and the Cartan composition of the spin representation with the standard representation. In classical notation their highest weights are $(0,0,0)$, $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$, $(1,1,1)$, $(\frac{3}{2},\frac{3}{2},\frac{3}{2})$, and $(\frac{3}{2},\frac{1}{2},\frac{1}{2})$.

2) $SU(N,2)$, $N \geq 3$. Here

$$M = \left\{ \begin{pmatrix} \omega & & & & \\ & e^{i\theta} & & & \\ & & e^{i\varphi} & & \\ & & & e^{i\varphi} & \\ & & & & e^{i\theta} \end{pmatrix}, \ \omega \in U(N-2) \text{ and total determinant } = 1 \right\}.$$

The basic cases are $\sigma(\text{this}) = e^{i(m\theta + n\varphi)}$ with $|m| \leq N-1$ and $|n| \leq N-1$.

<u>Formula</u> for λ_b. We define λ_b by giving its inner product with each Δ^+ simple root β. Namely

$$\frac{2\langle \lambda_b, \beta \rangle}{|\beta|^2} = \begin{cases} 0 & \text{if } \beta \text{ real or imaginary} \\ -\dfrac{2\langle \rho_-, \beta \rangle}{|\beta|^2} + \text{correction}(\beta) & \text{if } \beta \text{ complex.} \end{cases} \tag{1.5}$$

Here correction(β) is always 0, $\frac{1}{2}$, or 1, depending on the form of β. Let ε denote a member of $(i\mathfrak{b}_-)'$. Then

$$\text{correction}(\beta) = \begin{cases} \frac{1}{2} & \text{if } \beta = \varepsilon - \frac{1}{2}\alpha_j,\ |\beta| = |\alpha_j|, \text{ and } \mu \perp \alpha_j \\ \frac{1}{2}(1 - \operatorname{sgn}\langle \mu, \gamma \rangle) & \text{if } \beta = \varepsilon - \frac{1}{2}\alpha_j,\ |\beta| = |\alpha_j|, \ \mu \not\perp \alpha_j, \text{ and a sign } \pm \text{ is fixed so that } \gamma = \varepsilon \pm \frac{1}{2}\alpha_j \text{ is compact} \\ \frac{1}{2} + \dfrac{2\langle \mu, \beta \rangle}{|\beta|^2} & \text{if } \beta = \varepsilon - \alpha_j \text{ with } |\beta|^2 = 2|\alpha_j|^2 \\ \dfrac{2\langle \mu, \gamma \rangle}{|\gamma|^2} & \text{if } \beta = \varepsilon - \frac{1}{2}\alpha_i - \frac{1}{2}\alpha_j,\ |\beta| = |\alpha_i| = |\alpha_j|, \text{ and a sign } \pm \text{ is fixed so that } \gamma = \varepsilon - \frac{1}{2}\alpha_i \pm \frac{1}{2}\alpha_j \text{ is compact.} \end{cases}$$

Proof of Theorem 1.1. Uniqueness is trivial. For existence, fix some λ corresponding to the format $(\{\alpha_j\}, \Delta^+, \chi, \mu)$. By adding suitable fundamental weights for G to λ, we may assume $\lambda + \rho_-$ is nonsingular with respect to all nonreal roots. Let σ be the representation of M determined by λ and χ, and form the unitary principal series representation $U = U(MAN, \sigma, 0)$. The μ in the format picks out one minimal K-type Λ of U by (1.4), and Theorem 1.1 of [17] says that Λ lies in a unique irreducible constituent π of U. For each non-real simple root β for Δ^+, let Λ_β be the fundamental weight corresponding to β and define n_β to be the greatest integer

$$n_\beta = \left[\frac{2\langle \lambda + \rho_- , \beta \rangle}{|\beta|^2} \right] .$$

Then $\lambda + \rho_- - n_\beta \Lambda_\beta$ is Δ^+ dominant and we can apply the Zuckerman ψ functor [19] to π, obtaining

$$\psi^{\lambda+\rho_-}_{\lambda+\rho_- - n_\beta \Lambda_\beta}(\pi) . \tag{1.6}$$

Define n'_β by

$$n'_\beta = \begin{cases} n_\beta & \text{if (1.6) is not } 0 \\ n_\beta - 1 & \text{if (1.6) is } 0 . \end{cases}$$

Then

$$\psi^{\lambda+\rho_-}_{\lambda+\rho_- - n'_\beta \Lambda_\beta}(\pi)$$

is not 0. By Theorem 6.18 of Speh-Vogan [15],

$$\psi^{\lambda+\rho_-}_{\lambda+\rho_- - \Sigma n'_\beta \Lambda_\beta}(\pi) \tag{1.7}$$

is not 0. Put $\lambda_b = \lambda - \Sigma\, n'_\beta \Lambda_\beta$. Theorem B.1 of [13] shows that (1.7) is contained in an induced representation $U_b = U(MAN, \sigma_b, 0)$

and that the infinitesimal character of σ_b is obtained by moving the infinitesimal character $\lambda + \rho_-$ of σ by $-\Sigma n'_\beta \Lambda_\beta$. The parameter χ is not changed. The highest weight of σ_b therefore has to be λ_b, and so λ_b is $(\Delta_-)^+$ dominant. We have arranged that $\lambda_b + \rho_-$ is Δ^+ dominant and that

$$0 \leq \frac{2\langle \lambda_b + \rho_-, \beta\rangle}{|\beta|^2} \leq 1.$$

The minimal K-type of (1.7) is easily seen to be $\Lambda - \Sigma n'_\beta \Lambda_\beta$.

Now let λ' correspond to the format $(\{\alpha_j\}, \Delta^+, \chi, \mu)$. It is not hard to see that $\lambda' - \lambda_b$ is G-integral. The only way that $\lambda' - \lambda_b$ can fail to be Δ^+ dominant is if some simple β_0 has

$$\frac{2\langle \lambda_b + \rho_-, \beta_0\rangle}{|\beta_0|^2} = 1 \quad \text{and} \quad \frac{2\langle \lambda' + \rho_-, \beta_0\rangle}{|\beta_0|^2} = 0. \tag{1.8}$$

So assume (1.8). We do not affect the definition of λ_b if we add enough fundamental weights for G to our initial λ so that $\lambda - \lambda'$ is Δ^+ dominant. The first equality in (1.8) implies that $n'_{\beta_0} = n_{\beta_0} - 1$. Hence (1.6) is 0 for $\beta = \beta_0$. The second equality in (1.8) allows us to compose (1.6) with a further ψ functor to obtain

$$\psi_{\lambda'+\rho_-}^{\lambda+\rho_-}(\pi) = 0. \tag{1.9}$$

If the form Λ' obtained by using λ' in (1.4) were Δ_K^+ dominant, we could construct a nonzero element in the space for (1.9) from the Λ K-type of π. Hence Λ' is not Δ_K^+ dominant, and (1.8) has led us to a contradiction.

2. Unitarity for some basic cases

We define the Langlands quotient

$$J(MAN,\sigma,\nu) \tag{2.1}$$

to be the unique irreducible quotient of (0.1) under the conditions on the (real-valued) element ν in the introduction. For the basic cases σ in the examples of §1, we shall describe those ν for which $J(MAN,\sigma,\nu)$ is infinitesimally unitary. Our description will be complete except for certain undecided isolated points in the case of $SU(N,2)$.

For groups of real rank one, the classification of irreducible unitary representations is known, with the final work appearing in [1] and [2]. In the notation of the examples of §1, denote the parameter on the Lie algebra of A by $\nu = t\underset{\sim}{c}(\alpha)$, $t > 0$, with the understanding that $\underset{\sim}{c}(\alpha)$ is a positive restricted root, and let ρ_A be half the sum of the positive restricted roots with multiplicities counted.

a) $\widetilde{SO}(2n,1)$, $n \geq 2$. Here ρ_A corresponds to $t = n - \frac{1}{2}$. For σ trivial, the unitary points are $0 < t \leq n - \frac{1}{2}$. For σ equal to the spin representation, no $t > 0$ gives a unitary point.

b) $SU(n,1)$, $n \geq 2$. Here ρ_A corresponds to $t = \frac{1}{2}n$. For $\sigma \leftrightarrow e^{ik\theta}$ with $|k| \leq n$, the unitary points are $0 < t \leq \frac{1}{2}(n - |k|)$.

c) $Sp(n,1)$, $n \geq 2$. Here ρ_A corresponds to $t = n + \frac{1}{2}$. For σ trivial, the unitary points are $0 < t \leq n - \frac{1}{2}$ and $t = n + \frac{1}{2}$. For $\sigma = (k \times \text{fundamental}) \otimes 1$ with $1 \leq k \leq 2n - 1$, the unitary points are $0 < t \leq \frac{1}{2}(2n - 1 - k)$.

d) Real form of F_4. Here ρ_A corresponds to $t = \frac{11}{2}$. For σ trivial, the unitary points are $0 < t \leq \frac{5}{2}$ and $t = \frac{11}{2}$. For σ equal to k times the spin representation with $1 \leq k \leq 3$, the unitary points are $0 < t \leq \frac{1}{2}(3 - k)$. For σ equal to the Cartan composition of the spin representation with the standard representation, no $t > 0$ gives a unitary point.

We turn to $SU(N,2)$, $N \geq 3$. With the notation for M as in §1, we have

$$A = \exp \begin{pmatrix} 0 & & & \\ & & & s \\ & & t & \\ & t & & \\ & s & & \end{pmatrix},$$

and we let f_1 and f_2 of the Lie algebra matrix here be s and t, respectively. We write $\nu = af_1 + bf_2$ and choose the positive Weyl chamber to be $a \geq b \geq 0$. Then ρ_A has $a = N+1$ and $b = N-1$. The basic cases have $\sigma \leftrightarrow e^{i(m\theta+n\varphi)}$ with $|m| \leq N-1$ and $|n| \leq N-1$. Since complex conjugation is an outer automorphism of $SU(N,2)$ fixing A and sending (m,n) to $(-m,-n)$, it is enough to understand $m \geq n$.

Theorem 2.1. In $SU(N,2)$ the unitary points $\nu = (a,b)$ in the positive Weyl chamber for the basic cases $\sigma \leftrightarrow e^{i(m\theta+n\varphi)}$ with $m \geq n$ are as follows:

(a) If $|m| \leq N-2$ and $|n| \leq N-2$, the unitary points within the closed rectangle

$$0 \leq a \leq N-1-|m|, \quad 0 \leq b \leq N-1-|n| \qquad (2.2)$$

are exactly the points

(i) in the triangle $a+b \leq m-n+2$

(ii) in any of the triangles

$$a-b \geq m-n+2k, \quad a+b \leq m-n+2k+2$$

for an integer $k \geq 1$.

(iii) on any of the lines

$$a - b = m - n + 2k$$

for an integer $k \geq 1$.

(b) If $|m| \leq N-2$ and $|n| \leq N-2$, the only possible unitary point outside the closed rectangle is

$$(a,b) = (N+1-|m|\ ,\ N-1-|n|)\ . \qquad (2.3)$$

This can be a unitary point only if $m = n$ or $0 > m > n$. If $m = n = 0$, this point corresponds to the trivial representation.

(c) If $|m| = N-1$ or $|n| = N-1$ and if $m > n$, there are no unitary points in the positive Weyl chamber.

(d) If $|m| = N-1$ and $m = n$, the unitary points in the positive Weyl chamber are exactly the points $a+b \leq 2$.

This theorem will be proved in §10. Pictures of the unitary points for the cases $(m,n) = (0,0)$, $(0,-1)$, and $(2,2)$ appear in [11] and [8]. In situation (d) in the theorem, as well as in situation (c) when $n = -(N-1)$, the axis $b = 0$ is disallowed since $U(MAN,\sigma,\nu)$ does not have a unique irreducible quotient.

Vogan has shown us some computations indicating that at the undecided isolated points in (b) the representation $J(MAN,\sigma,\nu)$ has a highest weight vector. The unitary representations with a highest weight vector are known ([4] and [6]), and N. Wallach has given us information that suggests that these isolated points actually do correspond to unitary representations.

3. Basic cases, general MAN

Only minor modifications are needed to define "basic cases" for a general cuspidal parabolic subgroup MAN. The notation of [9] needs little adjustment. The system of strongly orthogonal noncompact roots $\{\alpha_j\}$ no longer need satisfy (1.1). The group $M^\#$ in (1.2) no longer need be all of M, but every discrete series or limit of discrete series σ of M is induced from a representation $\sigma^\#$ of the same type for $M^\#$.

Such a representation σ of M is therefore determined by a triple (λ_0, C, χ), where (λ_0, C) is a Harish-Chandra parameter of $\sigma^\#$ and χ is the scalar value of σ on $M_r \subseteq Z_M$. The chamber C for Δ_- determines $(\Delta_-)^+$, and the Blattner parameter (minimal $(K \cap M)$-type) of $\sigma^\#|_{M_e}$ is given by

$$\lambda = \lambda_0 - \rho_{-,c} + \rho_{-,n}. \tag{3.1}$$

We can define Δ^+ and μ as in §1, and we say λ_0 has <u>format</u> $(\{\alpha_j\}, \Delta^+, \chi, \mu)$ if the linear form Λ in (1.4) is Δ_K^+ dominant. (Observe that we have switched from the highest weight to the infinitesimal character λ_0 as reference parameter.)

<u>Theorem 3.1</u>. Suppose the group G with rank G = rank K has $G^{\mathbb{C}}$ simply connected. Fix a format $(\{\alpha_j\}, \Delta^+, \chi, \mu)$ corresponding to a general cuspidal parabolic subgroup MAN. Among all infinitesimal characters λ_0 of discrete series or limits (degenerate or nondegenerate) with this format, there is a unique one $\lambda_{b,0}$ such that any other λ_0 with this format has $\lambda_0 - \lambda_{b,0}$ dominant for Δ^+ and G-integral.

We call $\lambda_{b,0}$ (or its associated $\sigma = \sigma_b$) the <u>basic case</u> for the format $(\{\alpha_j\}, \Delta^+, \chi, \mu)$. Again $\lambda_{b,0}$ still makes sense if $G^{\mathbb{C}}$ is not simply connected.

The proof of Theorem 3.1 is the same as the proof we gave of Vogan's for Theorem 1.1 except for minor modifications. The σ_b that results is often, but not always, a limit of discrete series representation if MAN is not minimal. In fact, σ_b may even be degenerate as a limit of discrete series, and the theorem will fail if we look for $\lambda_{b,0}$ only among nondegenerate cases.

4. Associated subgroup L

Fix an infinitesimal character λ_0 for M and a format $(\{\alpha_j\},\Delta^+,\chi,\mu)$ for it. Let $\lambda_{b,0}$ be the basic case for this format given by Theorem 3.1, and let $\mathfrak{q} = \mathfrak{l}^{\mathbb{C}} \oplus \mathfrak{u}$ be the θ-stable parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$ defined by the Δ^+ dominant form $\lambda_0 - \lambda_{b,0}$:

$$\begin{aligned} &\mathfrak{q} \quad \text{is built from } \mathfrak{b}^{\mathbb{C}} \text{ and all } \beta \in \Delta \text{ with } \langle \lambda_0 - \lambda_{b,0}, \beta \rangle \geq 0, \\ &\mathfrak{l}^{\mathbb{C}} \quad \text{is built from } \mathfrak{b}^{\mathbb{C}} \text{ and all } \beta \in \Delta \text{ with } \langle \lambda_0 - \lambda_{b,0}, \beta \rangle = 0, \\ &\mathfrak{u} \quad \text{is built from all } \beta \in \Delta \text{ with } \langle \lambda_0 - \lambda_{b,0}, \beta \rangle > 0. \end{aligned} \tag{4.1}$$

Here $\mathfrak{l}^{\mathbb{C}}$ is the complexification of $\mathfrak{l} = \mathfrak{l}^{\mathbb{C}} \cap \mathfrak{g}$. The analytic subgroup of G corresponding to $\mathfrak{l}$ will be denoted L; it is the centralizer in G of a suitable torus.

From the definition, $\mathfrak{b}^{\mathbb{C}}$ is contained in $\mathfrak{l}^{\mathbb{C}}$, and the root system of $(\mathfrak{l}^{\mathbb{C}},\mathfrak{b}^{\mathbb{C}})$ is

$$\Delta^L = \{\beta \in \Delta \mid \langle \lambda_0 - \lambda_{b,0}, \beta \rangle = 0\}.$$

Moreover, each α_j in our strongly orthogonal sequence of noncompact roots is in Δ^L. Hence $\mathfrak{g}_r \subseteq \mathfrak{l}$ and $A \subseteq L$. We shall associate to the Langlands quotient $J(MAN,\sigma,\nu)$ for G, given in (2.1), a Langlands quotient

$$J^L((M \cap L)A(N \cap L),\sigma^L,\nu) \tag{4.2}$$

for L. For brevity we shall write $J^G(\sigma,\nu)$ and $J^L(\sigma^L,\nu)$ for such a corresponding pair of representations of G and L.

To specify (4.2) we need to define σ^L, and we do so by giving its infinitesimal character λ_0^L and a compatible format

$$(\{\alpha_j\}, \Delta^+ \cap \Delta^L, \chi^L, \mu) . \tag{4.3}$$

With superscripts "L" referring to objects in L and to the positive system $\Delta^+ \cap \Delta^L$ for L, we define

$$\begin{aligned} \rho(\mathfrak{u}) &= \rho - \rho^L \\ \rho(\mathfrak{u} \cap \mathfrak{k}^{\mathbb{C}}) &= \rho_K - \rho_K^L \\ \rho(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}}) &= \rho(\mathfrak{u}) - \rho(\mathfrak{u} \cap \mathfrak{k}^{\mathbb{C}}) . \end{aligned}$$

The form $\rho(\mathfrak{u})$ is orthogonal to every root in Δ^L, and $\rho(\mathfrak{u} \cap \mathfrak{k}^{\mathbb{C}})$ and $\rho(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}})$ are orthogonal to every root in Δ_K^L.

Because of this orthogonality, $E(2\rho(\mathfrak{u} \cap \mathfrak{k}^{\mathbb{C}}))$ is the differential of a one-dimensional representation of K_r, and we can define

$$\chi^L = \chi \cdot [\exp E(2\rho(\mathfrak{u} \cap \mathfrak{k}^{\mathbb{C}}))]|_{M_r} . \tag{4.4}$$

Let

$$\lambda_0^L = \lambda_0 - \rho(\mathfrak{u}) . \tag{4.5}$$

These definitions are motivated by the theory of [15] and [18] in a way that we shall describe in §5.

Proposition 4.1. The definitions (4.4) and (4.5) of χ^L and λ_0^L consistently define σ^L, and (4.3) is a compatible format. The corresponding form (1.4) for σ^L is given by

$$\Lambda^L = \Lambda - 2\rho(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}}) . \tag{4.6}$$

Proof. It is clear that λ_0^L is dominant for $(\Delta_-^L)^+$. Writing the formula of Lemma 3 of [9] for G and for L and subtracting, we have

$$2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}}) - 2\rho_{-,c} + 2\rho^L_{-,c} = \rho(\mathfrak{u}) - \rho_- + \rho^L_- + E(2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}})) .$$

Therefore the Blattner parameter (3.1) for λ^L_0 can be transformed as

$$\begin{aligned}\lambda^L &= \lambda^L_0 - \rho^L_{-,c} + \rho^L_{-,n} \\ &= [\lambda_0 - \rho(\mathfrak{u})] + [2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}}) - E(2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}})) - \rho(\mathfrak{u}) + \rho_{-,n} - \rho_{-,c}] \\ &= \lambda - 2\rho(\mathfrak{u}) + [2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}}) - E(2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}}))] . \end{aligned} \tag{4.7}$$

Each of the terms on the right is analytically integral on $\exp\mathfrak{b}_-$, and hence λ^L is analytically integral on $\exp\mathfrak{b}_-$. We shall prove that

$$\langle\lambda^L, \beta\rangle = \langle\lambda, \beta\rangle \quad \text{for } \beta\in(\Delta^L_{-,c})^+ , \tag{4.8}$$

and then it follows that λ^L_0 is the infinitesimal character of a discrete series or limit of discrete series of $(M\cap L)_e$.

Thus let β be in $\Delta^L_{-,c}$. Then $\langle 2\rho(\mathfrak{u}), \beta\rangle = 0$ since β is in Δ^L, and $\langle E(2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}})), \beta\rangle = 0$ since β is in Δ_-. Thus we are to show that

$$\langle 2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}}), \beta\rangle = 0 . \tag{4.9}$$

If β is compact for $\mathfrak{g}^{\mathbb{C}}$, (4.9) is clear. If β is noncompact for $\mathfrak{g}^{\mathbb{C}}$, then β is orthogonal but not strongly orthogonal to some member α of the sequence $\{\alpha_j\}$. Then $\beta+\alpha$ and $\beta-\alpha$ are in Δ^L_K, and the product $s_{\beta+\alpha}s_{\beta-\alpha}$ of two reflections fixes $2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}})$. But then

$$\begin{aligned}2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}}) &= s_{\beta+\alpha}s_{\beta-\alpha}(2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}})) \\ &= s_\alpha s_\beta(2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}})) \\ &= 2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}}) - \frac{2\langle 2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}}),\alpha\rangle}{|\alpha|^2}\alpha - \frac{2\langle 2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}}),\beta\rangle}{|\beta|^2}\beta ,\end{aligned}$$

and (4.9) follows. This proves (4.8).

Next we show that $\exp\lambda^L$ and χ^L agree on $(\exp\mathfrak{b}_-)\cap(\exp\mathfrak{b}_r)$, so that we obtain a well defined representation of $(M\cap L)^{\#}$, then of

$M \cap L$. In view of (4.7) and (4.4), we are to show that the character

$$\xi_{-2\rho(\mathfrak{u})}\xi_{[2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}})-E(2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}}))]} \tag{4.10}$$

of $\exp \mathfrak{b}_-$ and the character

$$\xi_{E(2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}}))} \tag{4.11}$$

of $\exp \mathfrak{b}_r$ agree on $(\exp \mathfrak{b}_-) \cap (\exp \mathfrak{b}_r)$. The first factor of (4.10) is well defined on all of $\exp \mathfrak{b}$ and is trivial on $\exp \mathfrak{b}_r$. The second factor of (4.10) and the character (4.11) are the respective restrictions of the character $\xi_{2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}})}$ of $\exp \mathfrak{b}$. The required consistency is therefore proved.

To see that (4.3) is a compatible format, we check that the representation τ_μ of K_r with highest weight μ contains

$$\chi^L \cdot \exp(E(2\rho_K^L) - 2\rho_{K_r})|_{M_r} . \tag{4.12}$$

Using (4.4), we see that (4.12) equals

$$\chi \cdot \exp(E(2\rho_K) - 2\rho_{K_r})|_{M_r} ,$$

and τ_μ contains this by assumption.

Finally we combine (1.4) and Lemma 3 of [9] to write

$$\Lambda = \lambda_0 + \rho - \rho_r - 2\rho_K + 2\rho_{K_r} + \mu . \tag{4.13}$$

Writing the corresponding expression for Λ^L and subtracting, we have

$$\begin{aligned} \Lambda^L - \Lambda &= (\lambda_0^L - \lambda_0) - (\rho - \rho^L) + (2\rho_K - 2\rho_K^L) \\ &= -\rho(\mathfrak{u}) - \rho(\mathfrak{u}) + 2\rho(\mathfrak{u}\cap\mathfrak{k}^{\mathbb{C}}) \\ &= -2\rho(\mathfrak{u}\cap\mathfrak{p}^{\mathbb{C}}) . \end{aligned}$$

This proves (4.6) and completes the proof of the proposition.

The group L is reductive, not necessarily semisimple, and we have to adjust the definitions of §§1-3 to speak of "basic cases" for L. Let us agree that a <u>basic case</u> for L is one whose restriction to the semisimple part of L is basic. In terms of a comparison of infinitesimal characters with a given format, one therefore fixes the restriction to the central torus of L.

<u>Proposition 4.2</u>. The infinitesimal character λ_0^L given in (4.5) is a basic case for the format (4.3) for L.

Proof. We may assume that $G^{\mathbb{C}}$ is simply connected. Proposition 4.1 shows that λ_0^L does correspond to a nonzero representation with (4.3) as format. Suppose λ_0^L is not a basic case, i.e., that there is some Δ^+ dominant integral ξ not orthogonal to Δ^L such that $\lambda_0^L - \xi$ corresponds to a nonzero representation with (4.3) as format. For each Δ^+ simple β in Δ outside Δ^L, let Λ_β be the fundamental weight, and let η be the sum of such Λ_β. Then we claim that

$$\lambda_0' = \lambda_0 + n\eta - \xi$$

corresponds to a nonzero representation with $(\{\alpha_j\}, \Delta^+, \chi, \mu)$ as format, provided n is sufficiently large.

In fact, the integrality condition is no problem. The other conditions are that certain inner products of λ_0' or its translates with certain members of Δ^+ are to be ≥ 0. When these members are in Δ^L, we have the same inner product as for $\lambda_0^L - \xi$. When they are outside Δ^L, the n dominates (if n is sufficiently large) and makes the inner product ≥ 0.

Now choose β simple for Δ^L so that $\langle \xi, \beta \rangle > 0$. Then

$$\begin{aligned} \langle \lambda_0' - \lambda_{b,0}, \beta \rangle &= \langle \lambda_0 - \lambda_{b,0}, \beta \rangle - \langle \xi, \beta \rangle \\ &= -\langle \xi, \beta \rangle < 0, \end{aligned}$$

and we have a contradiction to the fact that $\lambda_{b,0}$ is basic for G. The proposition follows.

5. Conjecture about reduction

Conjecture 5.1. Let σ be a discrete series or nondegenerate limit of discrete series representation of M, given by the infinitesimal character λ_0 and a compatible format $(\{\alpha_j\},\Delta^+,\chi,\mu)$. Let L be defined from (4.1), and let σ^L be defined as in (4.4) and (4.5). If ν is real-valued, then the Langlands quotient $J^G(\sigma,\nu)$ for G is infinitesimally unitary if and only if the Langlands quotient $J^L(\sigma^L,\nu)$ for L is infinitesimally unitary.

If true, the conjecture reduces the classification question for irreducible unitary representations to a consideration of $J(\sigma,\nu)$ with σ basic and ν real-valued, under the assumption that G is linear and rank G = rank K. (This follows from Proposition 4.2.)

The conjecture is true if G has real rank one. For $G = SU(N,2)$, there are two proper cuspidal parabolic subgroups MAN to consider: When dim A = 1, the conjecture is true at least when σ is a discrete series representation of M and L has real rank one; this follows essentially from Proposition 9.1 below. When dim A = 2, the conjecture is true inside a certain rectangle of ν's (by Propositions 8.1 and 8.2 below), and it is often true also outside a slightly larger circle of ν's (cf. Theorem 7.1), but in general the conjecture is not settled.

Actually the conjecture should be regarded as suggesting more than just a correspondence of unitary parameters. It should suggest that certain functors going back and forth between representations of L and representations of G preserve unitarity. These functors in many situations coincide with the ones of Vogan ([17],[18]) and

Speh and Vogan [15], and the choice of parameters in (4.4), (4.5), and (4.6) is in fact motivated by the Vogan and Speh-Vogan functors.

The correspondence between our work and that in [15], [17], and [18] is not immediately evident, since the orderings are different. The ordering in [17] is obtained by associating to the minimal K-type Λ a form $\tilde{\lambda}$ via Proposition 4.1 of [17]. Then one builds a smallest permissible θ-stable parabolic subalgebra for the theory from $\{\beta \in \Delta \mid \langle \tilde{\lambda}, \beta \rangle \geq 0\}$. The point is that $\tilde{\lambda}$ is just the infinitesimal character λ_0, so that our $\mathfrak{q}$ is indeed a permissible θ-stable parabolic subalgebra.

It is implicit in §7 of [17] that $\tilde{\lambda} = \lambda_0$, and we shall here write out a direct proof. From (4.13) we have

$$\Lambda + 2\rho_K = \lambda_0 + (\rho - \rho_r) + (\mu + 2\rho_{K_r}) . \tag{5.1}$$

We introduce a new positive system $(\Delta^+)'$ by changing the notion of positivity on Δ_r (and only there) so that $\mu + 2\rho_{K_r}$ is $(\Delta_r^+)'$ dominant. With "primes" referring to objects in the new ordering, we have

$$\Delta_{K,r}^+ = (\Delta_{K,r}^+)' \quad \text{and} \quad \Delta_K^+ = (\Delta_K^+)'$$

$$\rho_{K_r} = \rho'_{K_r} \quad \text{and} \quad \rho_K = \rho'_K$$

$$\rho - \rho_r = \rho' - \rho'_r . \tag{5.2}$$

By Proposition 4.1 of [17], we can write

$$\mu + 2\rho_{K_r} - \rho'_r = \tilde{\lambda}_\mu - \tfrac{1}{2}\Sigma \beta_i , \tag{5.3}$$

where the β_i are in $(\Delta_r^+)'$ and have certain properties listed in the proposition. Since μ is fine and "fine" is equivalent with "small" ([18], p. 294), $\tilde{\lambda}_\mu = 0$. Combining (5.1), (5.2), and (5.3), we obtain

$$\Lambda + 2\rho_K - \rho' = \lambda_0 - \tfrac{1}{2}\Sigma\,\beta_i\,. \qquad (5.4)$$

The claim is that $\Lambda + 2\rho_K$ is dominant for $(\Delta^+)'$ and hence that $\tilde{\lambda} = \lambda_0$ by the uniqueness in Proposition 4.1 of [17].

In fact, let β be simple for $(\Delta^+)'$. If β is not in $(\Delta_r^+)'$, then $\langle\beta,\rho'\rangle \geq 0$ and $\langle\beta,\beta_i\rangle \leq 0$ for all i. Also β is in Δ^+, and hence $\langle\lambda_0,\beta\rangle \geq 0$. Thus $\langle\Lambda + 2\rho_K, \beta\rangle \geq 0$ by (5.4). On the other hand, if β is in $(\Delta_r^+)'$, then β is orthogonal to the first two terms on the right of (5.1) and has inner product ≥ 0 with the third term by construction. Thus $\langle\Lambda + 2\rho_K, \beta\rangle \geq 0$.

Hence $\tilde{\lambda} = \lambda_0$, and our $\mathfrak{q}$ is permissible in the theory of [15], [17], and [18]. Conjecture 5.1 is thus closely related to the two conjectures on page 408 of [18]. One additional thing that Conjecture 5.1 says is that our L is large enough to capture all the unitary points in G.

6. Preservation of unitarity under tensoring

In [19] and the appendix of [13], G. Zuckerman began a systematic investigation of one technique for moving a parameter by a discrete step through a series of representations of a connected semisimple Lie group. The technique consists of tensoring with a suitable finite-dimensional representation and projecting according to a particular value of the infinitesimal character. This is done in two distinct ways——by a ψ functor that makes the parameter smaller and by a φ functor that makes the parameter larger.

Since finite-dimensional representations are generally not unitary, this technique need not carry unitary representations to unitary representations in general. However, Conjecture 5.1 predicts that unitarity will be preserved for Langlands quotients when ψ or

φ moves only the M parameter. (For example, $\lambda_0 - \lambda_{b,0}$ is the highest weight of a finite-dimensional representation that moves the parameter this way.) The point of this section will be to verify this prediction under the additional assumption that MAN is minimal.

Let G be a linear connected semisimple group with maximal compact subgroup K. For this section only, we do not assume rank G = rank K. Let MAN be a minimal parabolic subgroup of G, and let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{a}$, and $\mathfrak{n}$ be the various Lie algebras corresponding to our Lie groups. Let $\mathfrak{b}_- \subseteq \mathfrak{m}$ be a maximal abelian subspace (so that $\mathfrak{a} \oplus \mathfrak{b}_-$ is a Cartan subalgebra of $\mathfrak{g}$), let $B_- = \exp \mathfrak{b}_-$, let Δ be the roots of $(\mathfrak{g}^{\mathbb{C}}, (\mathfrak{a} \oplus \mathfrak{b}_-)^{\mathbb{C}})$, and let $\Delta_- \subseteq \Delta$ be the roots of $(\mathfrak{m}^{\mathbb{C}}, \mathfrak{b}_-^{\mathbb{C}})$. A positive system Δ^+ for Δ will be said to be compatible with a positive system $(\Delta_-)^+$ for Δ_- if $(\Delta_-)^+ \subseteq \Delta^+$. (No compatibility of Δ^+ with $\mathfrak{n}$ is assumed.)

If $(\Delta_-)^+$ is specified, let ρ_- denote half the sum of the members of $(\Delta_-)^+$. An irreducible unitary (finite-dimensional) representation σ of M is determined by a pair (λ, χ), where

λ is a dominant analytically integral member of $(i\mathfrak{b}_-)'$

χ is a character of $M \cap \exp i\mathfrak{a}$ that agrees with e^{λ} on $B_- \cap \exp i\mathfrak{a}$,

the correspondence being that λ is the highest weight of $\sigma|_{M_e}$ and χ is the scalar value of σ on $M \cap \exp i\mathfrak{a} \subseteq Z_M$. We call χ the central character of σ.

Suppose ν in $(\mathfrak{a}')^{\mathbb{C}}$ has Re ν in the closed positive Weyl chamber of $\mathfrak{a}'$ determined by $\mathfrak{n}$. (We do not need to assume ν is real-valued.) Theorem 1.1 of [11] recalls a necessary and sufficient condition for the induced representation $U(MAN, \sigma, \nu)$ to have a unique irreducible quotient, which we define to be $J(MAN, \sigma, \nu)$. For an application in Proposition 8.2, we note now that $J(MAN,\sigma,\nu)$ is always defined if $\mathrm{Re}\,\nu$ is in the interior of the positive Weyl chamber.

The two theorems to follow concern the effect on unitarity of moving the σ parameter of $J(MAN,\sigma,\nu)$. In the notation of [19], Theorem 6.1 deals with the ψ functor and Theorem 6.2 deals with the φ functor.

Theorem 6.1. Let MAN be a minimal parabolic subgroup of G, fix a positive system $(\Delta_-)^+$ for M, and let σ and σ' be irreducible unitary representations of M with respective highest weights λ and λ' and with a common central character χ. Let ν be in $(\mathfrak{a}')^{\mathbb{C}}$ with $\mathrm{Re}\,\nu$ in the closed positive Weyl chamber, and let Δ^+ be a positive system of roots of $(\mathfrak{g}^{\mathbb{C}}, (\mathfrak{a}\oplus\mathfrak{b}_-)^{\mathbb{C}})$ compatible with $(\Delta_-)^+$. Suppose that

(i) $J(MAN,\sigma',\nu)$ is defined

(ii) $\lambda' + \rho_- + \mathrm{Re}\,\nu$ and $\lambda + \rho_- + \mathrm{Re}\,\nu$ are Δ^+ dominant

(iii) $\lambda' - \lambda$ is Δ^+ dominant and G-integral.

Then

(iv) $J(MAN,\sigma,\nu)$ is defined

(v) $J(MAN,\sigma',\nu)$ infinitesimally unitary implies $J(MAN,\sigma,\nu)$ infinitesimally unitary.

Theorem 6.2. Let MAN be a minimal parabolic subgroup of G, fix a positive system $(\Delta_-)^+$ for M, and let σ and σ' be irreducible unitary representations of M with respective highest weights λ and λ' and with a common central character χ. Let ν be in $(\mathfrak{a}')^{\mathbb{C}}$ with $\mathrm{Re}\,\nu$ in the closed positive Weyl chamber, and let Δ^+ be a positive system of roots of $(\mathfrak{g}^{\mathbb{C}}, (\mathfrak{a}\oplus\mathfrak{b}_-)^{\mathbb{C}})$ compatible with $(\Delta_-)^+$. Suppose that

(i) $J(MAN,\sigma',\nu)$ is defined

(ii) $\lambda' + \rho_- + \text{Re}\,\nu$ and $\lambda + \rho_- + \text{Re}\,\nu$ are Δ^+ dominant and are equisingular (i.e., singular with respect to the same roots of $(\mathfrak{g}^{\mathbb{C}}, (\mathfrak{a} \oplus \mathfrak{b}_-)^{\mathbb{C}})$)

(iii) $\lambda - \lambda'$ is Δ^+ dominant and G-integral.

Then

(iv) $J(MAN,\sigma,\nu)$ is defined

(v) $J(MAN,\sigma',\nu)$ infinitesimally unitary implies $J(MAN,\sigma,\nu)$ infinitesimally unitary.

These theorems will be proved on another occasion. Each proof consists in tracking down what happens to the relevant intertwining operator and seeing that positivity is preserved. From Theorem 6.1 and the results for the basic cases listed in Theorem 2.1, we can exclude many representations in $SU(N,2)$ from being unitary; we state a precise result in this direction as Proposition 8.2.

7. Zuckerman triples

In this section we give a general theorem applicable when G is linear and $\text{rank } G = \text{rank } K$ that says that $J(MAN,\sigma,\nu)$ cannot be infinitesimally unitary for real ν outside a certain radius, for a wide class of σ. Motivation for the theorem in terms of a construction of Zuckerman appears in [8] and will not be repeated here.

Thus let $\mathfrak{b} \subseteq \mathfrak{k}$ be a compact Cartan subalgebra of $\mathfrak{g}$, and let Δ be the set of roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$. We say that (Δ^+, Σ, χ) is a <u>Zuckerman triple</u> if

Δ^+ = a positive root system for Δ

Σ = a root system in Δ generated by Δ^+ simple roots

χ = an analytically integral form on $\mathfrak{b}^{\mathbb{C}}$ orthogonal to Σ with $\chi - \rho_\Delta + 2\rho_\Sigma$ dominant for Δ^+.

Let Δ_K and Σ_K be the subsystems of compact roots in Δ and Σ. We let $\rho_{\Delta,K}$, $\rho_{\Delta,n}$, $\rho_{\Sigma,K}$, and $\rho_{\Sigma,n}$ denote the half sums of the indicated positive compact or noncompact roots, and we let w_Σ and $w_{\Sigma,K}$ denote the long elements of the Weyl groups of Σ and Σ_K, respectively. We say that (Δ^+,Σ,χ) is <u>nondegenerate</u> if $\chi - \rho_\Delta + 2\rho_\Sigma$ is nonorthogonal to every root $w_\Sigma\beta$ with β in Δ_K.

<u>Theorem 7.1</u>. With rank G = rank K, suppose that a Langlands quotient $J(P,\sigma,\nu_0)$ is such that there is a Zuckerman triple (Δ^+,Σ,χ) for which $J(P,\sigma,\nu_0)$ has the real infinitesimal character $\chi - \rho_\Delta + 2\rho_\Sigma$ and a (Δ_K^+ dominant) minimal K-type $\chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K}$. Suppose further that (Δ^+,Σ,χ) is nondegenerate; this condition is satisfied in particular if $\chi - \rho_\Delta + 2\rho_\Sigma$ is nonsingular. Then $J(P,\sigma,\nu)$ is not infinitesimally unitary for any real ν with $|\nu_0| < |\nu|$.

Remarks. This theorem was our first clue about basic cases. Its relevance is as follows: Under the assumptions in the theorem if also $\chi - \rho_\Delta + 2\rho_\Sigma$ is nonsingular, then the group L attached to σ by §4 often has Δ^L essentially equal to Σ.

The proof will use the Dirac inequality in the following form. See §4 of Baldoni Silva [1] for a proof of this inequality.

<u>Lemma 7.2</u>. If Δ^+ is any positive system for Δ and if π is an irreducible unitary representation of G with real infinitesimal character $\chi(\pi)$ and a minimal K-type Λ, then

$$|\chi(\pi)| \leq |w(\Lambda - \rho_{\Delta,n}) + \rho_{\Delta,K}| , \tag{7.1}$$

where w is chosen in the Weyl group of Δ_K to make $w(\Lambda - \rho_{\Delta,n})$ be Δ_K^+ dominant.

Proof of Theorem 7.1. It is enough to prove that equality holds in (7.1) for $\pi = J(MAN,\sigma,\nu_0)$. We proceed in several steps.

(1) $w_{\Sigma,K}$ fixes $\chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K}$.

In fact, we have

$$\begin{aligned} w_{\Sigma,K}\rho_{\Delta,K} &= \rho_{\Delta,K} - 2\rho_{\Sigma,K} \\ w_{\Sigma,K}\rho_{\Sigma,K} &= -\rho_{\Sigma,K} \\ w_{\Sigma,K}\chi &= \chi , \end{aligned} \tag{7.2}$$

and (1) follows.

(2) $w_{\Sigma,K}w_\Sigma(\chi - \rho_\Delta + 2\rho_\Sigma) = (\chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K}) - w_{\Sigma,K}\rho_{\Delta,n} + \rho_{\Delta,K}$.

In fact, the left side is

$$\begin{aligned} &= \chi - w_{\Sigma,K}w_\Sigma(\rho_\Delta - \rho_\Sigma) + w_{\Sigma,K}w_\Sigma\rho_\Sigma \\ &= \chi - w_{\Sigma,K}(\rho_\Delta - \rho_\Sigma) - w_{\Sigma,K}\rho_\Sigma \\ &= \chi - w_{\Sigma,K}\rho_\Delta \\ &= \chi - w_{\Sigma,K}\rho_{\Delta,K} - w_{\Sigma,K}\rho_{\Delta,n} \\ &= \chi - \rho_{\Delta,K} + 2\rho_{\Sigma,K} - w_{\Sigma,K}\rho_{\Delta,n} \quad \text{by (7.2)} \\ &= (\chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K}) - w_{\Sigma,K}\rho_{\Delta,n} + \rho_{\Delta,K}. \end{aligned}$$

(3) $|\chi - \rho_\Delta + 2\rho_\Sigma| = |w_{\Sigma,K}((\chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K}) - \rho_{\Delta,n}) + \rho_{\Delta,K}|$.

In fact, we can take the magnitude of both sides of (2) and apply (1).

(4) $\langle w_{\Sigma,K}((\chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K}) - \rho_{\Delta,n}), \beta\rangle \geq 0$ for $\beta \in \Sigma_K^+$.

In fact, the left side by (1) is

$$= \langle \chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K}, \beta\rangle + \langle \rho_{\Delta,n}, -w_{\Sigma,K}\beta\rangle$$

$\geq \langle \rho_{\Delta,n}, -w_{\Sigma,K}\beta\rangle$ by the assumed Δ_K^+ dominance

≥ 0

since $-w_{\Sigma,K}\beta$ is in Σ_K^+ and $\rho_{\Delta,n}$ is Δ_K^+ dominant.

(5) $\langle w_{\Sigma,K}((\chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K}) - \rho_{\Delta,n}), \beta\rangle \geq 0$ for $\beta \in \Delta_K^+$.

In fact, we may assume that β is Δ_K^+ simple, and by (4) we may assume β is not in Σ_K^+. By (1) and (2), we have

$$\langle w_{\Sigma,K}((\chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K}) - \rho_{\Delta,n}) + \rho_{\Delta,K}, \beta\rangle$$
$$= \langle \chi - \rho_\Delta + 2\rho_\Sigma, w_\Sigma w_{\Sigma,K}\beta\rangle . \qquad (7.3)$$

Since β is in Δ_K^+ but not Σ, $w_{\Sigma,K}\beta$ is in Δ_K^+ but not Σ. Then $w_\Sigma(w_{\Sigma,K}\beta)$ is in Δ^+, and the dominance of $\chi - \rho_\Delta + 2\rho_\Sigma$ implies that (7.3) is ≥ 0. Since $w_{\Sigma,K}\beta$ is in Δ_K^+, the assumed nondegeneracy implies (7.3) is $\neq 0$. Therefore

$$\frac{2\langle w_{\Sigma,K}((\chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K}) - \rho_{\Delta,n}) + \rho_{\Delta,K}, \beta\rangle}{|\beta|^2} \geq 1 .$$

Since β is simple for Δ_K^+, $2\langle \rho_{\Delta,K}, \beta\rangle/|\beta|^2 = 1$. Then (5) follows.

(6) Comparing (3) and (5) with the statement of Lemma 7.2, we see that equality holds in (7.1) for $J(MAN,\sigma,\nu_0)$, and the theorem is proved.

Remark. For any nondegenerate Zuckerman triple, it is automatic that $\chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K}$ is Δ_K^+ dominant. In fact, if β is in Σ_K^+, then $\langle \chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K}, \beta\rangle = 0$. If β is in Δ_K^+ but not Σ_K, we have

$$\begin{aligned}\langle \chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K}, \beta\rangle &= \langle \chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K} - w_{\Sigma,K}\rho_{\Delta,n}, \beta\rangle + \langle w_{\Sigma,K}\rho_{\Delta,n}, \beta\rangle \\ &= \langle \chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K} - w_{\Sigma,K}\rho_{\Delta,n}, \beta\rangle + \langle \rho_{\Delta,n}, \beta'\rangle \\ &\geq \langle \chi - 2\rho_{\Delta,K} + 2\rho_{\Sigma,K} - w_{\Sigma,K}\rho_{\Delta,n}, \beta\rangle ,\end{aligned}$$

and this is ≥ 0 by the same calculation as in (5).

8. Unitary degenerate series

We turn now to results that we shall formulate specifically only for $SU(N,2)$, $N \geq 3$. In this section we shall identify some unitary representations attached to the minimal parabolic subgroup. The new ones will be degenerate series, induced from a finite-dimensional representation of a noncuspidal maximal parabolic subgroup, and they have the striking feature that the finite-dimensional representation of the M of the maximal parabolic is usually nonunitary.

For $SU(N,2)$ with $N \geq 3$, we have already fixed a choice of the M and the A of a minimal parabolic subgroup in §§1-2, and we defined linear functionals f_1 and f_2 on the Lie algebra of A. We continue to write $\nu = af_1 + bf_2$, and we return to the assumption that ν is real-valued. The positive Weyl chamber is given by $a \geq b \geq 0$. An irreducible representation σ of M can be written (nonuniquely) as

$$\sigma\begin{pmatrix} \omega & & & & \\ & e^{i\theta} & & & \\ & & e^{i\varphi} & & \\ & & & e^{i\varphi} & \\ & & & & e^{i\theta} \end{pmatrix} = e^{i(m\theta + n\varphi)}\sigma_0(\omega) ,$$

where σ_0 is an irreducible representation of $U(N-2)$. If σ_0 has highest weight $\sum_{j=1}^{N-2} c_j e_j$, then the infinitesimal character of $U(MAN,\sigma,\nu)$ is

$$\sum_{j=1}^{N-2}(c_j+\tfrac{1}{2}(N-2j-1))e_j+\tfrac{m}{2}(e_{N-1}+e_{N+2})+\tfrac{n}{2}(e_N+e_{N+1})$$
$$+\tfrac{a}{2}(e_{N-1}-e_{N+2})+\tfrac{b}{2}(e_N-e_{N+1}). \quad (8.1)$$

We shall define a "fundamental rectangle" in the ν space. If we restrict σ to the subgroup of M where $\varphi = 0$, we obtain a representation σ_1 of the M for a subgroup $SU(N-1,1)$ of $SU(N,2)$. The corresponding A for this $SU(N-1,1)$ has $b = 0$. Let a_0 be the first point ≥ 0 such that the infinitesimal character of the representation of $SU(N-1,1)$ induced from σ_1 and af_1 is integral and fails to be singular with respect to two linearly independent roots. Operationally a_0 is the first value ≥ 0 of a in $n-1-m+2\mathbb{Z}$ such that $\frac{1}{2}(m+a)$ and $\frac{1}{2}(m-a)$ do not both appear among the numbers $c_j+\frac{1}{2}(N-2j-1)$ for $1 \leq j \leq N-2$. Similarly the condition $\theta = 0$ leads us to a different subgroup $SU(N-1,1)$ and to a representation σ_2 of its M, and we define b_0 relative to σ_2 and bf_2. The <u>fundamental rectangle</u> is then given by

$$0 \leq a \leq a_0 \quad \text{and} \quad 0 \leq b \leq b_0.$$

In this section we shall identify some points in the fundamental rectangle that correspond to unitary representations. In Proposition 8.2 we shall see that the remaining points in the fundamental rectangle do not correspond to unitary representations. Because of Theorem 2.1, Conjecture 5.1 would imply that there is at most one ν (for fixed σ) outside the fundamental rectangle that corresponds to a unitary representation unless $a_0 = b_0 = 0$ and $m = n$.

If $a_0 = 0$ or $b_0 = 0$, then there are no unitary points at all unless $m = n$ (cf. [1], Theorem 6.1), and in this case the points with $b = 0$ do not have well defined Langlands quotients. Thus we shall assume $a_0 > 0$ and $b_0 > 0$ in our analysis.

We can determine which $U(MAN,\sigma,\nu)$ are reducible as in [10] by decomposing the standard intertwining operator for the large element of the 8-element Weyl group. The result is that the only reducibility within the fundamental rectangle occurs on the lines

$$a + b = |m - n| + 2\ell\ , \quad \ell \text{ an integer} \geq 1$$

$$a - b = |m - n| + 2k\ , \quad k \text{ an integer} \geq 1\,.$$

In view of Proposition 3.1 of [11], the representation $U(MAN,\sigma,\nu)$ at $b = 0$ with $0 \leq a \leq a_0$ is unitarily induced from a complementary series of $SU(N-1,1)$ and hence is unitary. Since the standard intertwining operator is the identity at $\nu = 0$, it follows from a familiar continuity argument that the following ν's in the positive Weyl chamber within the fundamental rectangle correspond to unitary representations:

(i) the triangle $\quad a + b \leq |m - n| + 2 \qquad (8.2)$

(ii) the triangles

$$a - b \geq |m - n| + 2k\ , \quad a + b \leq |m - n| + 2k + 2 \qquad (8.3)$$

for each integer $k \geq 1$.

These unitary points had been recognized earlier. (Cf. Knapp-Stein [12] for (i) and Guillemonat [5] for (ii).) Further unitary points are given in the following proposition, which was announced in [11]. Some of these points were recognized independently by Schlichtkrull [14].

Proposition 6.1. Within the fundamental rectangle when $a_0 > 0$ and $b_0 > 0$, the points ν on the lines

$$a - b = |m-n| + 2k, \quad k \text{ an integer} \geq 1, \tag{8.4}$$

correspond to unitary representations.

Proof. On any line (8.4), the argument with the intertwining operator that detected reducibility of $U(MAN,\sigma,\nu)$ shows also, just as in [10], that the Langlands quotient is an irreducible degenerate series representation as long as ν is in the interior of the fundamental rectangle and ν is not at a point where the line (8.4) crosses a line

$$a + b = |m-n| + 2\ell, \quad \ell \text{ an integer} \geq 1. \tag{8.5}$$

The idea is to show that the irreducibility of the degenerate series persists at the crossing points. Then it follows by a continuity argument that the unitarity established by (8.3) at one end of the line (8.4) extends along the line to the other end at the edge of the fundamental rectangle.

Fix k and ℓ, and let us reparametrize the lines (8.4) and (8.5) about the crossing point by

$$(a,b) = (|m-n| + k + \ell + s,\ \ell - k + s) \quad \text{in the case of (8.4)}$$

$$(a,b) = (|m-n| + k + \ell + t,\ \ell - k - t) \quad \text{in the case of (8.5)}.$$

We denote the respective full induced representations along these lines by $U_1^+(s)$ and $U_1^-(t)$, the sign referring to the slope of the line. For any admissible representation π, let $\Theta(\pi)$ denote the global character.

We treat only $m \geq n$. Use of the intertwining operators (including knowledge of decompositions in $SL(2,\mathbb{C})$ and the fact that the image of the Langlands intertwining operator is irreducible) implies in deleted neighborhoods of $s = 0$ and $t = 0$ that we have

$$\Theta(U_1^+(s)) = \Theta(D_1^+(s)) + \Theta(U_2(s)) \tag{8.6a}$$

$$\Theta(U_1^-(t)) = \Theta(D_1^-(t)) + \Theta(U_3(t)), \tag{8.6b}$$

with the characters on the right irreducible. Here $D_1^+(s)$ and $D_1^-(t)$ are degenerate series induced from finite-dimensional representations of a noncuspidal maximal parabolic subgroup, and $U_2(s)$ and $U_3(t)$ are induced from the minimal parabolic subgroup with data as follows:

$U_2(s)$: same σ_0 but $[m,n,a,b]$ replaced by $[m+k\,,\, n-k\,,\, m-n+\ell+s\,,\, \ell+s]$

$U_3(t)$: same σ_0 but $[m,n,a,b]$ replaced by $[m+\ell\,,\, n-\ell\,,\, m-n+k+t\,,\, k+t]$.

The decompositions (8.6) persist for $s = 0$ and $t = 0$, but the characters on the right may become reducible.

Similar analysis of the intertwining operators for $U_2(s)$ and $U_3(t)$ in neighborhoods of $s = 0$ and $t = 0$ (including 0 this time) shows that

$\Theta(U_2(s))$ is irreducible for $s \neq 0$

$\Theta(U_3(t))$ is irreducible for all t

$$\Theta(U_2(0)) = \Theta(D_2(0)) + \Theta(U_3(0)) \quad \text{irreducibly}. \tag{8.7}$$

Let $J(0)$ be the Langlands quotient of $U_1^+(0) = U_1^-(0)$.

We shall show shortly that the only irreducible composition factors that can occur in $U_1^+(0)$ are $J(0)$, $D_2(0)$, and $U_3(0)$, and we know $J(0)$ occurs with multiplicity one. Consideration of Gelfand-Kirillov dimension (see Lemma 2.3 of [16]) then shows that

$$\Theta(D_1^+(0)) = \Theta(J(0)) + u\Theta(D_2(0)) \tag{8.8}$$

for an integer $u \geq 0$. We are to show that $u = 0$.

Let ψ denote the effect on characters of tensoring with the finite-dimensional representation of $SU(N,2)$ with extreme weight ke_{N+2} and then projecting according to the infinitesimal character given by the sum of (8.1) and ke_{N+2}. Direct computation with the aid of Corollary 5.10 of Speh-Vogan [15] shows that

$$\psi\Theta(U_1^+(0)) = \psi\Theta(U_2(0)) = \Theta(U_2'(0)) \tag{8.9}$$
$$\psi\Theta(U_3(0)) = \Theta(U_3'(0)),$$

where $U_2'(0)$ and $U_3'(0)$ are induced from the minimal parabolic subgroup with data as follows:

$U_2'(0)$: same σ_0 but $[m,n,a,b]$ replaced by $[m+k\,,\,n\,,\,m-n+\ell\,,\,\ell-k]$

$U_3'(0)$: same σ_0 but $[m,n,a,b]$ replaced by $[m+\ell\,,\,n-\ell+k\,,\,m-n+k\,,\,0]$.

Just as in (8.7), we have

$$\Theta(U_2'(0)) = \Theta(D_2'(0)) + \Theta(U_3'(0)) \quad \text{irreducibly} \tag{8.10}$$

with $D_2'(0) \neq 0$. Applying ψ to (8.6a) at $s = 0$ and using (8.9), we see that $\psi\Theta(D_1^+(0)) = 0$. But then ψ applied to (8.8) shows that $u\Theta(D_2'(0)) = 0$ (and also $\psi\Theta(J(0)) = 0$). Since $D_2'(0) \neq 0$, we conclude $u = 0$.

We are left with showing that the only irreducible composition factors that can occur in $U_1^+(0)$ are $J(0)$, $D_2(0)$, and $U_3(0)$. If the infinitesimal character is integral at our crossing point, then the fact that the crossing point is inside the fundamental rectangle implies the infinitesimal character is orthogonal to four mutually orthogonal roots. Four is too many singularities for any representation attached to G or to the cuspidal maximal parabolic subgroup, and four implies that e_{N-1}, e_N, e_{N+1}, and e_{N+2} participate in the singularities in the case of the minimal parabolic subgroup. Then it follows that $J(0)$, $D_2(0)$, and $U_3(0)$ are the only irreducible representations with the same infinitesimal character as $U_1^+(0)$. When the infinitesimal character is not integral at the crossing point, then it is so far from being integral that it is not the infinitesimal character of any representation attached to G or the cuspidal maximal parabolic subgroup. Moreover, the only way it can be the infinitesimal character of a representation attached to the minimal parabolic is if the coefficients of e_{N-1}, e_N, e_{N+1}, and e_{N+2} are merely permuted among themselves (in which case we are led to $J(0)$, $D_2(0)$, and $U_3(0)$) or if $N \leq 6$. For $N \leq 4$ there are no crossing points under study, for $N = 5$ only σ trivial is of concern and it is handled by inspection of the integrality, and for $N = 6$, interchange of the first 4 entries of the infinitesimal character with the last 4 entries leads to a nontrivial change of the representation on the 8-element center. Proposition 8.1 follows.

Applying Theorems 6.1 and 2.1, we immediately obtain the following complementary result.

Proposition 8.2. For $SU(N,2)$ with the minimal parabolic, with σ such that the fundamental rectangle has $a_0 > 0$ and $b_0 > 0$, and with ν real, no points ν within the closed fundamental rectangle and the closed positive Weyl chamber correspond to unitary representations except those listed in (8.2), (8.3), and Proposition 8.1.

9. Series associated with cuspidal maximal parabolic

Let MAN be the cuspidal maximal parabolic subgroup of $SU(N,2)$, $N \geq 3$. Let $\mathfrak{b} \subseteq \mathfrak{k}$ be a compact Cartan subalgebra, let Δ be the roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$, and suppose that A is constructed by Cayley transform $\underset{\sim}{c}$ from a noncompact root α.

Proposition 9.1. Let σ be a discrete series representation of the M of the cuspidal maximal parabolic subgroup of $SU(N,2)$, $N \geq 3$, and let λ_0 be its infinitesimal character. Let $t_0 \geq 0$ be the least number such that $\lambda_0 + t_0\alpha$ is integral and fails to be orthogonal to at least one compact root and one noncompact root. For $t > 0$, $J(MAN, \sigma, t\underset{\sim}{c}(\alpha))$ is infinitesimally unitary for $0 < t \leq t_0$ and not otherwise.

Sketch of proof. Either $\lambda_0 + \frac{1}{2}\alpha$ is integral (the "tangent case") or λ_0 is integral (the "cotangent case"). We sketch the proof only in the tangent case.

Fix $(\Delta_-)^+$ to make λ_0 dominant for it. Replacing α by $-\alpha$ if necessary, we can arrange that $\lambda_0 + t_0\alpha$ is nonsingular with respect to the noncompact roots in Δ. Then we can introduce $\Delta^+ = (\Delta^+)_1$ as in §3 so that λ_0 is Δ^+ dominant and α is simple. Let $r = t_0 + \frac{1}{2}$. For $1 \leq j \leq r$, we shall introduce recursively positive systems $(\Delta^+)_j$, subsystems Σ_j generated by simple roots, and forms χ_j so that $((\Delta^+)_j, \Sigma_j, \chi_j)$ is a Zuckerman triple (see §7) with parameters corresponding to $J(MAN, \sigma, (j-\frac{1}{2})\underset{\sim}{c}(\alpha))$ and so that $((\Delta^+)_r, \Sigma_r, \chi_r)$ is nondegenerate. Theorem 7.1 then says that unitarity does not extend beyond $t_0\underset{\sim}{c}(\alpha)$.

In addition, each Σ_j will correspond to a subgroup of G of real rank one, Σ_{j+1} will be generated by Σ_j and the roots orthogonal to $\lambda_0 + (j - \frac{1}{2})\alpha$, and the members of $(\Delta^+)_j$ not in Σ_{j+1}

will remain in $(\Delta^+)_{j+1}$. Since $\lambda_0 + t\alpha$ can be orthogonal to roots only for t in $\mathbb{Z} + \frac{1}{2}$ and cannot be orthogonal to more than two distinct roots (up to sign) if $t > 0$, it follows that $\langle \lambda_0 + t\alpha, \beta \rangle$ is nonzero for $0 \leq t < t_0$ and β in Δ_r^+ but not Σ_r. Since $\langle \lambda_0, \beta \rangle > 0$, we see that

$$\langle \lambda_0 + t\alpha, \beta \rangle \geq 0 \quad \text{for } 0 \leq t \leq t_0 \text{ and } \beta \text{ in } \Delta_r^+ \text{ but not } \Sigma_r . \qquad (9.1)$$

Let L be the subgroup of G corresponding to Σ_r. Since L has real rank one, any nonunitary principal series representation of L that is orthogonal to two linearly independent roots is irreducible. Combining this fact, the inequality (9.1), and the theory of [15], we see that $U(MAN, \sigma, t\underset{\sim}{c}(\alpha))$ is irreducible for $0 \leq t < t_0$. Then a standard continuity argument shows that $J(MAN, \sigma, t\underset{\sim}{c}(\alpha))$ is unitary for $0 \leq t \leq t_0$.

Thus the whole issue is to construct $((\Delta^+)_j, \Sigma_j, \chi_j)$ and verify its properties. We define $\Sigma_1 = \{\pm\alpha\}$ and choose $(\Delta^+)_j$ to make $\lambda_0 + (j - \frac{1}{2} - \epsilon)\alpha$ dominant (for $\epsilon > 0$ small). The $(\Delta^+)_j$ positive roots that are orthogonal to $\lambda_0 + (j - \frac{1}{2})\alpha$ are $(\Delta^+)_j$ simple, and Σ_{j+1} is taken as the root system generated by Σ_j and these roots if $j < r$. We define χ_j by

$$\chi_j - \rho_{(\Delta^+)_j} + 2\rho_{\Sigma_j} = \lambda_0 + (j - \tfrac{1}{2})\alpha .$$

Then we can verify all the asserted properties.

The tricky step (and here we use real-rank$(G) \leq 2$), is to check recursively that the minimal K-type is given by

$$\Lambda_j = \chi_j - 2\rho_{(\Delta_K^+)_j} + 2\rho_{(\Sigma_K^+)_j} .$$

In passing from Σ_j to Σ_{j+1}, we adjoined two simple roots, one compact (say $\beta_{j,c}$) and one noncompact (say $\beta_{j,n}$). The fact that the real rank is ≤ 2 enters the proof of the identity

$$2\rho_{\Sigma_{j+1},n} = 2\rho_{\Sigma_j,n} - \beta_{j,n} + \beta_{j,c} + \alpha \quad \text{for } 1 \leq j < r,$$

which is proved at the same time as the fact that Σ_{j+1} corresponds to a group of real rank one. Then it follows easily that $\Lambda_j = \Lambda_{j+1}$ for $1 \leq j < r$, and one shows that Λ_1 is the minimal K-type by using Theorem 1 and Lemma 3 of [9]. As we remarked at the end of §7, Λ_r is necessarily $(\Delta_K^+)_r$ dominant.

10. Duflo's method and the basic cases for SU(N,2)

Duflo [3] succeeded in proving that certain nonunitary principal series representations in some complex groups are not unitary by computing some determinants associated to an intertwining operator explicitly and finding two K-types on whose sum the operator is indefinite. In [11] we indicated how Duflo's method can be adapted to real groups. In this section we shall apply the method to the basic cases in SU(N,2) in order to prove Theorem 2.1. We are indebted to P. Delorme for a useful suggestion that helped us in this analysis.

We have already established in §8, especially in Proposition 8.1, that the representations asserted to be unitary in Theorem 2.1a are indeed unitary. For the representations in part (d) of the theorem, the unitarity follows from the standard continuity argument for the intertwining operator, which is scalar at $\nu = 0$.

Also in §8 we noted that (c) of the theorem follows from results about minimal K-types. The claimed nonunitarity in (d) follows from the Dirac inequality (Lemma 7.2); Δ^+ is taken either from the standard ordering given by indices $(1,2,\ldots,N+2)$ or from the ordering $(N+1,N+1,1,2,\ldots,N)$. The remainder of Theorem 2.1 consists of assertions of nonunitarity that we shall prove with Duflo's method and supplementary applications of Lemma 7.2.

Sample detailed calculations appear in the case of $SU(2,2)$ in [10]. In dealing with $SU(N,2)$, we must replace $SL(2,\mathbb{R})$ in that kind of calculation by $SU(N-1,1)$. Thus we need to know the scalar value of an $SU(N-1,1)$ intertwining operator on a K-type in a nonunitary principal series representation. We shall give that information, then say what intertwining determinants arise from certain K-types of $SU(N,2)$, and finally tell what nonunitarity is established by each of the K-types. The actual calculations of the intertwining determinants, which are carried out in the style of [10], will be omitted.

In $G_1 = SU(N-1,1)$, write

$$M_1 = \begin{pmatrix} \omega & & \\ & e^{i\theta} & \\ & & e^{i\theta} \end{pmatrix},$$

and let $\sigma_1 \leftrightarrow e^{ip\theta}$ with $|p| \leq N-2$. The minimal K_1-type is one-dimensional, and we normalize the intertwining operator to act as 1 on it. The K_1-types in the induced representation are the ones of the form

$$\begin{matrix} \scriptstyle N-1 & \scriptstyle 1 \end{matrix}$$
$$\begin{pmatrix} u & \\ & e^{i\theta} \end{pmatrix} \longrightarrow e^{iq\theta}\, \tau_{(k,0,\ldots,0,\ell)}(u), \tag{10.1}$$

where $k \geq 0 \geq \ell$ and $q = p - k - \ell$. Let the A_1 parameter be af_1, where $2f_1$ is the real root. Reformulating some identities of Klimyk and Gavrilik [7] suitably, we find that the intertwining operator is given on the K_1-type (10.1) by the scalar

$$\Big[\prod_{j=0}^{k-1}\Big(\frac{2j-(a+p-N+1)}{2j+(a-p+N-1)}\Big)\Big]\Big[\prod_{j=\ell+1}^{0}\Big(\frac{2j+(a-p-N+1)}{2j-(a+p+N-1)}\Big)\Big]. \tag{10.2}$$

The denominators are nonvanishing for $a \geq 0$ and for our purposes can be discarded.

Returning to $SU(N,2)$, consider the basic case $\sigma \leftrightarrow e^{i(m\theta+n\varphi)}$ with $m \geq n$, and form the representations of K given by

$$\tau\begin{pmatrix}\overset{N}{\alpha} & \overset{2}{0}\\ 0 & \beta\end{pmatrix}(P\otimes Q)\begin{pmatrix}w_1\\ \\ w_N\end{pmatrix}\otimes\begin{pmatrix}z_1\\ z_2\end{pmatrix} = (\det\beta)^r P\Big(\alpha^{-1}\begin{pmatrix}w_1\\ \\ w_N\end{pmatrix}\Big)Q\Big(\beta^{-1}\begin{pmatrix}z_1\\ z_2\end{pmatrix}\Big).$$

We now give the functions of $\nu = af_1 + bf_2$ that arise as intertwining determinants from certain K-types. We neglect global constants and also irrelevant denominators like the ones in (10.2).

1) Fix an integer $\ell \geq 0$, let $r = m+\ell$, let $\{P\} = \mathbb{C}$, and let $\{Q\}$ be the holomorphic polynomials of degree $m-n+2\ell$. Then the intertwining determinant works out to be

$$\prod_{j=1}^{\ell}(a+b+n-m-2j)(a-b+n-m-2j).$$

2) Assume $m-n \geq 1$. Let $r = m$, let $\{P\}$ be the holomorphic polynomials of degree 1, and let $\{Q\}$ be the holomorphic polynomials of degree $m-n-1$. The intertwining determinant is

$$b-n-(N-1).$$

3) Assume $m-n \geq 1$. Let $r = m-1$, let $\{P\}$ be the antiholomorphic polynomials of degree 1, and let $\{Q\}$ be the holomorphic polynomials of degree $m-n-1$. The intertwining determinant is

$$a+m-(N-1).$$

4) Let $r = m - 1$, let $\{P\}$ be the antiholomorphic alternating tensors of rank 2, and let $\{Q\}$ be the holomorphic polynomials of degree $m - n$. The intertwining determinant is

$$[a + m - (N - 1)][b + n - (N - 1)] .$$

5) Let $r = m + 1$, let $\{P\}$ be the holomorphic alternating tensors of rank 2, and let $\{Q\}$ be the holomorphic polynomials of degree $m - n$. The intertwining determinant is

$$[a - m - (N - 1)][b - n - (N - 1)] .$$

We can use these determinants to exclude many representations from being unitary. If such a determinant has one sign in a region where unitary points occur, then no points are unitary in the region where the determinant takes on the opposite sign. From the determinants (1) it follows that no points with $a - b \neq m - n + 2k$ for an integer $k \geq 1$ are unitary except those in the triangles listed in Theorem 2.1a. If $m \geq 0$ and $m - n \geq 1$, then (2) and (3) exclude all points outside the fundamental rectangle if $n < -m$, and (3) suffices by itself if $n \geq -m$.

If $m = n$, then either (4) or (5) excludes points in the interior of the region to the right or above the fundamental rectangle (but not both), and one can exclude all the remaining points outside the fundamental rectangle except (2.3) by using a suitable Dirac inequality (Lemma 7.2). For the Dirac inequality one forms Δ^+ from the standard ordering $(1,2,\dots,N+2)$ or from the ordering $(N+1,N+2,1,2,\dots,N)$.

Finally if $0 > m > n$, then (2) excludes points strictly above the fundamental rectangle, and (5) excludes any other points to the right of the fundamental rectangle except those on the same horizontal as the top edge. One can then exclude all the remaining points outside the fundamental rectangle except (2.3) by using a suitable Dirac inequality (Lemma 7.2). For the Dirac inequality one forms Δ^+ from the standard ordering $(1,2,\ldots,N+2)$. This completes the proof of Theorem 2.1.

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