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Classification of irreducible tempered representations of semisimple groups

By A. W. KNAPP* and GREGG J. ZUCKERMAN*

Part I

For each connected linear real semisimple Lie group G, we shall establish constructively a list without repetitions of all the irreducible tempered representations of G and of their characters. "Tempered" here means that the character, as a distribution on the smooth functions of compact support on G, extends continuously to Harish-Chandra's Schwartz space [4] on G. Equivalently ([29], p. 71), "tempered" means that the K-finite matrix coefficients, for K a maximal compact subgroup, are in $L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$. Our list of irreducible tempered representations and characters will be expressed ultimately in terms of the representations and characters of the discrete series of G and its subgroups.

Such a list was announced in [18] and [19], and the present paper gives the detailed proofs of the announced results.

The significance of our list results from a theorem of Langlands [21] that provides a classification of all irreducible admissible representations of G (up to infinitesimal equivalence) once one knows the irreducible tempered representations. In essence our list therefore finishes a classification of irreducible representations of G apart from questions of unitarity. The two classifications together do not go so far as to classify irreducible unitary representations; one simply has the inclusions

$$\left\{\begin{matrix}\text{irreducible}\\ \text{tempered}\\ \text{representations}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{irreducible}\\ \text{unitary}\\ \text{representations}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{irreducible}\\ \text{admissible}\\ \text{representations}\end{matrix}\right\}$$

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and a less-than-satisfactory criterion ([19], p. 147) for deciding whether an irreducible admissible representation can be made unitary. For an alternative approach to the classification theory of irreducible admissible representations and the nature of irreducible tempered representations, see Vogan [33].

Tempered representations arose historically first in the work of Harish-Chandra [4]. The irreducible tempered representations are the ones needed [5] to decompose $L^2(G)$, and Harish-Chandra worked with all of them except for a set of Plancherel measure zero. More detailed knowledge of irreducible tempered representations plays a role in the work of Arthur [1] on analysis of the Schwartz space of G and the space of tempered distributions on G, and also in the work of Herb and Sally [10] on inversion formulas for orbital integrals. Tempered representations play a special role in the Langlands correspondence [21] between the homomorphisms of the Weil group of $\mathbf{R}$ into the L group of G, on the one hand, and L-packets of irreducible admissible representations of G, on the other hand; the tempered representations are the ones for which the homomorphism into the L group has relatively compact image.

Before describing our classification, we introduce enough notation to be able to define representations of G "induced from discrete series." Fix a maximal compact subgroup K of G. A representation of G will be called *induced from discrete series* if it is a unitarily induced representation of the form

$$\operatorname{ind}_{MAN}^{G}(\xi \otimes e^{i\nu} \otimes 1), \tag{0.1}$$

where MAN is the Langlands decomposition of a cuspidal parabolic subgroup of G, ξ is a discrete series representation of M, and $e^{i\nu}$ is a unitary character of A.

A preliminary classification of irreducible tempered representations is obtained by giving a complete reduction into irreducible constituents of each representation induced from discrete series. In fact, one knows that

(i) Every irreducible tempered representation is (equivalent with) a constituent of some representation induced from discrete series. See [28] and [21]. (This result is also implicit in [5], though not explicitly stated there.)

(ii) Two representations induced from discrete series either are equivalent or else have no constituents in common. See Langlands [21].

(iii) Two representations induced from discrete series are equivalent if and only if their data (except for their respective N groups) are conjugate within G. See Harish-Chandra [5].

(iv) The irreducible constituents of any representation induced from discrete series all have multiplicity one. See [13].

Combining these facts, we see how to obtain a complete list of irreducible tempered representations, with no repetitions, from a complete reduction of all representations induced from discrete series. Much of this paper will be devoted to just this reducibility question, and the final result will be given as Theorem 8.7.

However, it is desirable to have a more intrinsic classification. For this purpose we have to deal with a wider class of induced representations (0.1) in which ξ is allowed to be either a discrete series representation of M or a so-called "limit of discrete series" representation. (See § 1 for the definition of "limit of discrete series.") Such induced representations of G will be called *basic*. In settling questions of irreducibility and equivalence of basic representations, we are led inevitably to the notion of a basic representation "given with nondegenerate data." (See § 12 for the definition of "nondegenerate data," and see § 10 for the definition of the R group that controls the reducibility.)

Our main theorem, given as Theorem 14.2, is stated in terms of this notion. Roughly the theorem says that the irreducible tempered representations, up to equivalence, are exactly the basic representations given with nondegenerate data and having trivial R group and that two such basic representations are equivalent if and only if their data (except for their respective N groups) are conjugate within G.

We list here the other main results of the paper (besides Theorem 14.2):

(i) The fundamental properties of limits of discrete series (Theorem 1.1), including criteria for nonvanishing, irreducibility, and equivalence.

(ii) The complete reduction of representations induced from discrete series, i.e., the preliminary classification (Theorem 8.7). A special case of Theorem 8.7 is worth noting: In the case that MAN is minimal parabolic, M is compact and the theory of discrete series is consequently not needed to describe the irreducible representations of M; the problem of determining the characters of the irreducible constituents in this case, which is solved by Theorem 8.7, is an old problem from the 1950's.

(iii) An extension of the theory of the R group and the standard intertwining operators of [16] to basic representations (Theorems 11.1 and 12.6). We develop the same level of understanding of the standard intertwining operators for basic representations as was previously known for representations induced from discrete series. In particular we extend the Harish-Chandra completeness theorem to basic representations and we obtain good algebraic invariants for working with the standard intertwining operators for these representations.

(iv) The complete reduction of basic representations into irreducible constituents (Corollary 12.2 and Theorem 13.3).

(v) Further development beyond [31] of the theory of tensor products of finite- and infinite-dimensional representations (Appendices).

Our work makes repeated and extensive use of three well-developed theories. When we bring them all to bear on our problem, we find it necessary to generalize each of them significantly. The three theories are:

(i) Character identities of Schmid [24] and Hecht-Schmid [8]. The Schmid identity exhibits the sum of two characters of limits of discrete series as equal to a certain induced character, and the Hecht-Schmid identity shows that certain limits of discrete series are zero. We introduce generalized Schmid identities in Theorem 4.3; these are induced versions of the Schmid identity that take into account the possible disconnectedness of M. In Theorem 6.1 we give conditions under which they can be inverted. We need also a converse to the Hecht-Schmid identity and prove the converse as part of Theorem 1.1.

(ii) The theory of the R group and the standard intertwining operators for representations induced from discrete series, as in [16] and [13]. This theory gives a basis of the commuting algebra for representations induced from discrete series in terms of a computable finite group R, and it shows that the commuting algebra is always commutative and that R is always a sum of two-element groups. Our generalization extends this theory to basic representations; the extended definitions are not the obvious ones, and the full results are valid only when the data are nondegenerate.

(iii) The theory of tensor products of finite- and infinite-dimensional representations, as in [31]. This theory provides a means for coherent translation of parameters appearing in suitable series of representations. The theory is enlarged in three ways in an appendix to the present paper. This appendix, written by the second author, forms the promised sequel to [31].

Our approach to classification begins with reducibility of representations induced from discrete series. Essentially there is a local theory in which one does each step of reducibility by means of a generalized Schmid identity, and there is a global theory in which one sorts out the results of a succession of generalized Schmid identities. The R group provides the global control in the process. In fact, the R group does even more, since the explicit form of its elements points directly to the final result.

As soon as one applies the first generalized Schmid identity, one is led outside the realm of representations induced from discrete series to the wider class of basic representations. The concept of a basic representation appears to be new, but its chief ingredient—the limit of discrete series representation—had been discovered and developed earlier in some special cases. In [14], limits of discrete series were introduced in instances in which G/K is Hermitian. For connected groups, Schmid [25] introduced general limits of discrete series involving singularities with respect to noncompact roots and later proved their irreducibility, and [31] simplified this theory and introduced limits of discrete series with arbitrary singularities and showed that their characters are multiples of irreducible characters. Still for connected groups, Enright [32] treated limits of discrete series with arbitrary singularities, showed which ones are nonzero, and showed that the nonzero ones are irreducible. Realizations of these representations under suitable hypotheses appear in the work of Wallach [30] and in [17]. To get complete results concerning general limits of discrete series, we shall develop limits of discrete series and generalized Schmid identities simultaneously. The main result about limits of discrete series is stated as Theorem 1.1, but the proof is not completed until the end of Section 5.

The general plan of the paper starting with Section 6 is as follows. In Sections 6–8 we exhibit the reduction into irreducible constituents of representations induced from discrete series. In Sections 9–11, the theory of the R group and of the standard intertwining operators is extended to basic representations, and results special to the case of nondegenerate data are collected in Sections 12–13. The main theorem is the subject of Section 14.

Contents

1. Limits of discrete series

For this section only, G will be assumed to be a linear real reductive group satisfying the axioms of Section 1 of [13]. The disconnectedness of G is limited considerably by these axioms, one of the requirements being that G is a matrix group contained in the product of a connected complexification $G^{\mathbf{C}}$ and the centralizer $Z(G)$ of G. Our intention is to define limits of discrete series representations for such a group G and to list their properties.

Let K be a maximal compact subgroup of G, let θ be the corresponding Cartan involution, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. Following the notation of [13], let G_0 be the identity component of G, let Z_G be the center of G, and define $G^{\#} = G_0 Z_G$.

In this section we shall assume that rank G = rank K. Let $\mathfrak{b}$ be a maximal abelian subspace of $\mathfrak{k}$, and let B be the corresponding (compact) analytic subgroup. Then $\mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{g}$, and we can speak of compact roots and noncompact roots of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ in the usual way. Note that B may not be the full centralizer of $\mathfrak{b}$ in G and hence may not be the full Cartan subgroup corresponding to $\mathfrak{b}$.

The "Weyl group" $W(B:G)$ is defined to be the quotient of the normalizer $N_G(B)$ by the centralizer $Z_G(B)$. (More generally $W(X:Y)$ is defined as $N_Y(X)/Z_Y(X)$.) It is clear that

$$W(B:G_0) \subseteq W(B:G) \subseteq W(B:G^{\mathbf{C}}Z(G)) = W(B:G^{\mathbf{C}}). \tag{1.1}$$

Proposition 4.5 of [13], together with the known identity $W(B:K) = W(B:G)$ in the connected case, implies here that

$$W(B:K) = W(B:G). \tag{1.2}$$

Let $\mathfrak{t}$ be any θ-stable Cartan subalgebra of $\mathfrak{g}$. We identify complex-valued real-linear functionals on $\mathfrak{t}$ with complex-linear functionals on $\mathfrak{t}^{\mathbf{C}}$, denoting these spaces $\mathfrak{t}'$ or $\mathfrak{t}^{\mathbf{C}\prime}$. A linear form in $\mathfrak{t}'$ is called *integral* (or $\mathfrak{g}$-*integral*) if it is the differential of a holomorphic character of $\exp \mathfrak{t}^{\mathbf{C}}$. If λ is in $\mathfrak{t}'$ and ρ denotes half the sum of the positive roots in some ordering, then the question of whether $\lambda - \rho$ is integral does not depend on the choice of the ordering that determines ρ.

In the case of $\mathfrak{t} = \mathfrak{b}$, an integral form is imaginary-valued on $\mathfrak{b}$. To each nonsingular λ in $\mathfrak{b}'$ such that $\lambda - \rho$ is integral, one can associate ([4], [24]) a discrete series (irreducible) representation $\pi^{G_0}(\lambda)$ of G_0 with Harish-Chandra parameter λ. All discrete series representations of G_0 are obtained in this way, and $\pi^{G_0}(\lambda)$ is equivalent with $\pi^{G_0}(\lambda')$ if and only if $\lambda' = w\lambda$ for some w in $W(B:G_0)$. On $B \cap Z_{G_0}$, $\pi^{G_0}(\lambda)$ agrees with $e^{\lambda-\rho}$ times the identity operator.

The character of $\pi^{G_0}(\lambda)$ will be denoted $\Theta^{G_0}(\lambda, C)$, where C is the (unique) Weyl chamber of $i\mathfrak{b}$ with respect to which λ is dominant. This character is given by a well-known formula [4] on B. The above equivalence criterion translates as: $\Theta^{G_0}(\lambda, C) = \Theta^{G_0}(\lambda', C')$ if and only if there is some w in $W(B:G_0)$ such that $\lambda' = w\lambda$ and $C' = wC$.

If $\tilde{w}$ represents an element w of $W(B : N_{G^{\mathbb{C}}}(G_0))$ and if $\tilde{w}\pi(g)$ means $\pi(\tilde{w}^{-1}g\tilde{w})$, then

$$\tilde{w}\pi^{G_0}(\lambda) \text{ has character } \Theta^{G_0}(w\lambda, wC). \tag{1.3}$$

In fact, $\tilde{w}\pi^{G_0}(\lambda)$ is a discrete series representation, and it is enough to see that its character matches $\Theta^{G_0}(w\lambda, wC)$ on B.

The discrete series of a general G is obtained as follows: We start with a discrete series representation $\pi^{G_0}(\lambda)$ of G_0 and a character χ of Z_G such that χ agrees with $e^{\lambda-\rho}$ on $B \cap Z_G$, we form $\pi^{G_0}(\lambda) \otimes \chi$ on $G^{\#} = G_0 Z_G$, and then we induce to G, obtaining

$$\operatorname{ind}_{G^{\#}}^{G}(\pi^{G_0}(\lambda) \otimes \chi). \tag{1.4}$$

(See [6] in general, [13] for features special to linear G.) The character of (1.4) is denoted

$$\Theta^G(\lambda, C, \chi) = \operatorname{ind}_{G^{\#}}^{G}(\Theta(\lambda, C) \otimes \chi), \tag{1.5}$$

where C is the (unique) Weyl chamber of $i\mathfrak{b}$ with respect to which λ is dominant.

In practice, we will have at our disposal a well understood finite abelian subgroup Z'_G of Z_G such that

$$G^{\#} = G_0 Z'_G. \tag{1.6}$$

In this case the characters χ in (1.5) stand in one-one correspondence via restriction with characters of Z'_G that equal $e^{\lambda-\rho}$ on $B \cap Z'_G$, since

$$BZ'_G = BZ_G.$$

Hence when convenient, we may take the domain of χ in (1.5) to be Z'_G.

In any event, the characters (1.5) are irreducible and exhaust the discrete series; moreover $\Theta^G(\lambda, C, \chi(= \Theta^G(\lambda', C', \chi')$ if and only if $\chi = \chi'$ and there is some w in $W(B : G)$ such that $\lambda' = w\lambda$ and $C' = wC$. The equivalence criterion and irreducibility of discrete series for disconnected G follow from the results in the connected case; the proof that we give in Theorem 1.1 of the corresponding implication for limits of discrete series works equally well for discrete series. In either case we obtain also the identity

$$\Theta^G(\lambda, C, \chi)\,|_{G_0} = \sum_{w \in W(B\,:\,G)/W(B\,:\,G_0)} \Theta^{G_0}(w\lambda, wC). \tag{1.7}$$

Let us recall the functor ψ introduced in [31] as an operation on compatible $(\mathcal{G}, K_0)$ Harish-Chandra modules for G_0.[1] The definition makes sense in the context of any θ-stable Cartan subalgebra $\mathfrak{t}$ of G_0. Let $T = \exp \mathfrak{t}$. Fix a Weyl

[1]In [31], it is assumed that the group is connected semisimple with finite center, but the theorems and proofs apply to the more general class of connected reductive groups with compact center.

chamber C, and suppose that λ in $\mathfrak{t}^{\mathbf{C}\prime}$ satisfies

$$\langle \operatorname{Re} \lambda, \alpha \rangle \geq 0 \quad \text{for all } C\text{-positive roots } \alpha.^2$$

Let μ be integral and C-dominant, let $F_{-\mu}$ be the finite-dimensional irreducible representation of $\mathfrak{g}^{\mathbf{C}}$ with lowest weight $-\mu$, and let p_λ denote projection according to the infinitesimal character (with parameter) λ. The functor $\psi_\lambda^{\lambda+\mu}$ is defined on compatible quasisimple Harish-Chandra modules with infinitesimal character $\lambda + \mu$ by

$$\psi_\lambda^{\lambda+\mu}(\cdot) = p_\lambda[(\cdot) \otimes F_{-\mu}]. \tag{1.8}$$

Let us observe that changing C leads to the same functor:

$$\psi_\lambda^{\lambda+\mu}(\cdot) = \psi_{w\lambda}^{w(\lambda+\mu)}(\cdot) \quad \text{for } w \text{ in } W(T : G^{\mathbf{C}}). \tag{1.9}$$

This equation follows from the facts that (i) two parameters conjugate under $W(T : G^{\mathbf{C}})$ yield the same infinitesimal character and (ii) the set of weights of a finite-dimensional representation is closed under the operation of $W(T : G^{\mathbf{C}})$. Notice that the same functor $\psi_\lambda^{\lambda+\mu}$ can arise also from a different Cartan subgroup, provided the parameters match appropriately.

Now we can define limits of discrete series for G_0. The parameters will be a Weyl chamber C in $i\mathfrak{b}$ and a member λ of $\mathfrak{b}'$ such that $\lambda - \rho$ is integral and λ is C-dominant. Let μ be a nonsingular integral form on $\mathfrak{b}$ that is C-dominant. Then the discrete series representation $\pi^{G_0}(\lambda + \mu)$ is defined. The Harish-Chandra module

$$\pi^{G_0}(\lambda, C) = \psi_\lambda^{\lambda+\mu}(\pi^{G_0}(\lambda + \mu)) \tag{1.10}$$

is independent of μ and equals $\pi^{G_0}(\lambda)$ if λ is nonsingular, by Theorem 1.2 and Corollary 5.5 of [31]. Then Theorem A.1 of the appendix shows it is independent of μ if λ is singular. In the singular case, at this stage there is no natural globalization of (1.10), but it will follow from Theorem 1.1 that the Harish-Chandra module (1.10) is irreducible and infinitesimally unitary, and we can then use the notation of (1.10) to denote also a unitary globalization. In any event we call (1.10) a *limit of discrete series* representation if λ is singular, and for any λ we denote its character by $\Theta^{G_0}(\lambda, C)$. In combination with (1.8), the known formula for $\pi^{G_0}(\lambda + \mu)$ on $B \cap Z_{G_0}$ shows that $\pi^{G_0}(\lambda, C)$ equals $e^{\lambda - \rho} I$ on $B \cap Z_{G_0}$.

Let λ and μ be C-dominant, and suppose $\lambda - \rho$ and μ are integral. Then

$$\pi^{G_0}(\lambda, C) = \psi_\lambda^{\lambda+\mu}(\pi^{G_0}(\lambda + \mu, C)) \tag{1.11}$$

[2] A condition is imposed in [31] on $\langle \operatorname{Im} \lambda, \alpha \rangle$ as well, but the condition plays no role in the proofs and it will be important for us to drop it.

even if $\lambda + \mu$ is singular. Equation (1.11) follows by choice of a nonsingular dominant integral μ' and application to $\pi^{G_0}(\lambda + \mu + \mu')$ of both sides of the identity

$$\psi_{\lambda}^{\lambda+\mu+\mu'} = \psi_{\lambda}^{\lambda+\mu}\psi_{\lambda+\mu}^{\lambda+\mu+\mu'}, \tag{1.12}$$

which is given as Theorem A.1 in the appendix.

Limits of discrete series for the general group G are parametrized by

(i) a Weyl chamber C in $i\mathfrak{b}$,

(ii) a member λ of $\mathfrak{b}'$ such that $\lambda - \rho$ is integral and λ is C-dominant,

(iii) a character χ of Z_G that agrees with $e^{\lambda-\rho}$ on $B \cap Z_G$.

The *limit of discrete series* representation with these parameters is

$$\operatorname{ind}_{G^{\#}}^{G}\left(\pi^{G_0}(\lambda, C) \otimes \chi\right),$$

and its character is denoted $\Theta^G(\lambda, C, \chi)$. As with discrete series, when (1.6) holds, it is enough to take the domain of χ to be Z'_G, and then the compatibility condition in (iii) is that χ agree with $e^{\lambda-\rho}$ on $B \cap Z'_G$.

THEOREM 1.1. *Limit of discrete series characters have the following properties*:

(a) $\Theta^G(\lambda, C, \chi)$ *is irreducible or zero.*

(b) (*partly due to Hecht-Schmid* [8]) $\Theta^G(\lambda, C, \chi)$ *is zero if and only if* $\langle\lambda, \alpha\rangle = 0$ *for some C-simple compact root* α.

(c) *When* $\Theta^G(\lambda, C, \chi)$ *is nonzero, an equality*

$$\Theta^G(\lambda, C, \chi) = \Theta^G(\lambda', C', \chi')$$

holds if and only if $\chi = \chi'$ *and there exists* w *in* $W(B : G)$ *with* $w\lambda = \lambda'$ *and* $wC = C'$.

(d) *When* $\Theta^G(\lambda, C, \chi)$ *is nonzero, it is the character of a tempered unitary representation.*

(e) $\Theta^G(\lambda, C, \chi)|_{G_0} = \Sigma_{w \in W(B\,:\,G)/W(B\,:\,G_0)}\Theta^{G_0}(w\lambda, wC)$.

Remarks in the connected case. Most of the proof in the connected case is deferred to Section 5. Much attention has focused on limits of discrete series when $\langle\lambda, \alpha\rangle = 0$ only for noncompact roots α. In this special case if G is connected, then the character does not vanish on B, a number of techniques are available, and most of the theorem was known previously. See Section 5 of [31] for a discussion of this case. In the present context, we place no such restriction on λ. For parts (a) and (b) in the connected case, see Enright [32], Theorem 4 on p. 5 and Theorem 16.2 on p. 79.

We can give part of the proof in the connected case now. In (a), the character is at worst a multiple of an irreducible character, by Theorem 1.3 of

[31]. We prove in Section 5 that the multiple is one or zero. The sufficiency in (b) is due to Hecht and Schmid [8]; the necessity will be proved in Section 5. The sufficiency in (c) follows immediately from (1.9) and the corresponding condition for discrete series; the necessity will be proved in Section 5. Result (d) for the connected case will be proved in Section 5, and (e) is trivial in the connected case.

Proof for general G, given theorem in connected case. Results (b) and (d) follow immediately from (b) and (d) for the connected case. Result (d) allows us to work with limits of discrete series as global unitary representations.

To prove (a) and (c), we shall apply Theorem 3′ of Mackey [22]. Let ξ and ξ' on $G^\#$ be defined by $\xi = \pi^{G_0}(\lambda, C) \otimes \chi$ and $\xi' = \pi^{G_0}(\lambda', C') \otimes \chi'$, and suppose ξ is not 0. By assumption ξ and ξ' are unitary, and $G^\#$ is an open normal subgroup of G of finite index. Mackey's Theorem implies that $\mathrm{ind}_{G^\#}^G \xi$ is irreducible if and only if (i) the representation ξ of $G^\#$ is irreducible and (ii) $g\xi$ is inequivalent with ξ for g not in $G^\#$. Condition (i) follows from (a) for the connected case. For (ii), it is shown in the course of the proof of Lemma 4.4 of [13] that the mapping $w \in N_G(B) \to wG^\#$ yields an isomorphism

$$G/G^\# \cong W(B:G)/W(B:G_0). \tag{1.13}$$

Consequently (ii) follows from result (c) in the connected case if it is shown for w in $W(B:G)$ that

$$w\Theta^{G_0}(\lambda, C) = \Theta^{G_0}(w\lambda, wC); \tag{1.14}$$

this equation will be proved below.

To prove (c), let w be in $W(B:G)$. Then

$$w\Theta^{G_0}(\lambda, C) \otimes \chi \quad \text{and} \quad \Theta^{G_0}(\lambda, C) \otimes \chi$$

lead to the same character of G upon induction from $G^\#$; thus (1.14) yields

$$\Theta^G(w\lambda, wC, \chi) = \Theta^G(\lambda, C, \chi).$$

Conversely suppose

$$\Theta^G(\lambda, C, \chi) = \Theta^G(\lambda', C', \chi'). \tag{1.15}$$

The scalar value of the representations in question on Z_G must match χ on the one hand and χ' on the other hand. So $\chi = \chi'$. Mackey's Theorem implies that

$$\Theta^{G_0}(\lambda', C') = g\Theta^{G_0}(\lambda, C)$$

for some g in G. By (1.13), we may assume g normalizes B. Then we can apply (1.14) and result (c) in the connected case to obtain (c) in the general case.

Now we derive (1.14) from (1.3). Let $\tilde{w}$ represent w in $W(B:G)$. Then

$$\begin{aligned}
\tilde{w}\pi^{G_0}(\lambda, C) &= \tilde{w}\psi_{\lambda}^{\lambda+\mu}\pi^{G_0}(\lambda+\mu) \\
&= \tilde{w}p_{\lambda}\left[\pi^{G_0}(\lambda+\mu)\otimes F_{-\mu}\right] \\
&= p_{w\lambda}\left[\tilde{w}\pi^{G_0}(\lambda+\mu)\otimes \tilde{w}F_{-\mu}\right] \\
&= p_{w\lambda}\left[\pi^{G_0}(w\lambda+w\mu)\otimes F_{-w\mu}\right] \quad \text{by (1.3)} \\
&= \psi_{w\lambda}^{w\lambda+w\mu}\pi^{G_0}(w\lambda+w\mu) \\
&= \pi^{G_0}(w\lambda, wC),
\end{aligned}$$

and (1.14) follows by passing to characters.

We are left with (e). Each w in $W(B:G)$ leaves stable the set of compact roots. Thus by (a) and (b) in the connected case, every term on the right side of (e) is zero or else every term on the right side of (e) is nonzero irreducible. We may assume the latter. Because of (d) in the connected case, the character on the left side of (e) is the character of a unitary representation of the form $U = \operatorname{ind}_{G^\#}^{G}\xi$. It is trivial that

$$U|_{G^\#} \cong \sum_{g_0 \in G/G^\#} g_0\xi.$$

Using (1.13) and (1.14), we therefore obtain

$$\Theta^G(\lambda, C, \chi)|_{G^\#} = \sum_{w \in W(B\,:\,G)/W(B\,:\,G_0)} \Theta^{G_0}(w\lambda, wC)\otimes\chi,$$

and (e) is obtained by restricting both sides to G_0.

2. Center of M

Henceforth we shall assume G is a connected reductive Lie group with compact center and with a faithful matrix representation. The notation of Section 1 still applies since G does satisfy the axioms of [13].

It is possible to construct an abelian group F that is useful in describing the disconnectedness of parabolic subgroups of G. Because G is connected, we can use a group F that is often smaller than the corresponding group in [13];[3] in particular, F will always be finite here.

[3] The group F defined in [13] becomes $G \cap (Z_{G^c}\exp i\mathfrak{a}_p)$ when G is connected. For SU(2, 1) this group contains the three-element center of G. However, the group defined by equation (2.1) in the present paper has only two elements when $G = \mathrm{SU}(2,1)$.

To define F, let $\mathfrak{a}_p$ be a maximal abelian subspace of $\mathfrak{p}$, and let $M_p = Z_K(\mathfrak{a}_p)$. The definition of F is

$$F = K \cap \exp i\mathfrak{a}_p, \tag{2.1}$$

and it is easy to see that F is a finite subgroup of the center of M_p. The proof of Lemma 9 of [23] shows that

$$M_p = (M_p)_0 F. \tag{2.2}$$

One can describe F more explicitly as follows. (Cf. [14], Lemma 3.7.) Form roots of $(\mathfrak{g}, \mathfrak{a}_p)$. For each such root α, let H_α be the corresponding member of $\mathfrak{a}_p$, and set

$$\gamma_\alpha = \exp 2\pi i \,|\, \alpha \,|^{-2} H_\alpha. \tag{2.3}$$

Computations in $SL(2, \mathbf{R})$ show that γ_α is in F and has order at most 2. The theorem is that the elements γ_α together generate F.

The Cartan involution leads to a Langlands decomposition for any parabolic subgroup of G. Let MAN be the Langlands decomposition of such a parabolic subgroup P, and let $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the corresponding decomposition of the Lie algebra. Although M need not be connected, it does satisfy the axioms of [13], by Lemma 1.3 of that paper. Let $\mathfrak{a}_M$ be a maximal abelian subspace of $\mathfrak{m} \cap \mathfrak{p}$, and form the maximal abelian subspace $\mathfrak{a}_p$ of $\mathfrak{p}$ given by

$$\mathfrak{a}_p = \mathfrak{a} \oplus \mathfrak{a}_M.$$

The above remarks then apply, and the proof of Lemma 1.3 of [13] then shows that (2.2) implies

$$M = M_0 F.$$

It follows from this equation that

$$MA \subseteq (MA)^{\mathbf{C}}, \tag{2.4}$$

where $(MA)^{\mathbf{C}}$ is the analytic subgroup of $G^{\mathbf{C}}$ with Lie algebra $(\mathfrak{m} \oplus \mathfrak{a})^{\mathbf{C}}$. Equation (2.4) will allow us later to use the result in Section B of the appendix. Although F is central in M_p, it is not necessarily central in M. However, if as in [13] we let

$$M^{\#} = M_0 Z_M,$$

then the proof of Lemma 4.2 of [13] shows that

$$M^{\#} = M_0(Z_M \cap F).$$

The group $K_M = K \cap M$ is a maximal compact subgroup of M. We shall assume hereafter that $\operatorname{rank}(\mathfrak{k} \cap \mathfrak{m}) = \operatorname{rank} \mathfrak{m}$, a condition summarized by saying that the parabolic MAN is *cuspidal*. Let $\mathfrak{b}$ be a maximal abelian subalgebra of $\mathfrak{k} \cap \mathfrak{m}$. Then $\mathfrak{b}$ is a compact Cartan subalgebra of $\mathfrak{m}$, and $\mathfrak{a} \oplus \mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{g}$. Let B be the analytic subgroup with Lie algebra $\mathfrak{b}$.

We can form roots relative to $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}})$, $(\mathfrak{g}, \mathfrak{a})$, and $(\mathfrak{g}, \mathfrak{a}_p)$. Some relationships among these systems are described in detail in [12] and in Section 2 of [13].

One such relationship is worth special note. A root of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}})$ is *real* if it is real-valued on $\mathfrak{a} \oplus \mathfrak{b}$, i.e., if it vanishes on $\mathfrak{b}$. By Lemma 2.1 of [13], the restriction of a real root α to $\mathfrak{a}$ is an $\mathfrak{a}$-root of odd multiplicity, and its extension to $\mathfrak{a}_p$ (by 0 on $\mathfrak{a}_M$) is an $\mathfrak{a}_p$-root of odd multiplicity. In whatever context we view α, its corresponding vector H_α is the same member of $\mathfrak{a}$. These identifications[4] allow us to use equation (2.3) to define γ_α for any real root α. Let

$$F(B) = \operatorname{span}\left\{\gamma_\alpha \mid \alpha \text{ is a real root of } \left(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}}\right)\right\}.$$

Then $F(B)$ is a subgroup of $Z_M \cap F$.

LEMMA 2.1. *If MAN is the Langlands decomposition of a cuspidal parabolic, then*

(a) $Z_G(\mathfrak{a} \oplus \mathfrak{b}) = ABF(B)$,
(b) $Z_M = (Z_M \cap B)F(B) = Z_{M_0}F(B)$,
(c) $M^\# = M_0F(B)$.

Proof. We first show that the centralizer $Z_G(\mathfrak{b})$ is connected.[5] In fact, let $k \exp Y$ be the Cartan decomposition of a member of $Z_G(\mathfrak{b})$ according to $G = K \exp \mathfrak{p}$. Then $\theta(k \exp Y) = k(\exp Y)^{-1}$ and $\theta(k \exp Y)^{-1} k \exp Y = \exp 2Y$ are in this centralizer since $\theta\mathfrak{b} = \mathfrak{b}$. Thus $\operatorname{Ad}(\exp 2Y)X = X$ for all X in $\mathfrak{b}$. Now $\operatorname{ad} 2Y$ is self-adjoint in the usual inner product on $\mathfrak{g}$ and has real eigenvalues. Decompose $X = \Sigma X_\lambda$ into the sum of eigenvectors under the action of $\operatorname{ad} 2Y$. Then we have

$$\sum X_\lambda = X = \operatorname{Ad}(\exp 2Y)X = \sum e^\lambda X_\lambda,$$

from which $e^\lambda = 1$ whenever $X_\lambda \neq 0$. Thus $\lambda = 0$ is the only possibility and $[Y, X] = 0$. That is, $\exp Y$ is in $Z_G(\mathfrak{b})$.

In other words k and $\exp Y$ are each separately in $Z_G(\mathfrak{b})$, and also we see from above that

$$(\exp \mathfrak{p}) \cap Z_G(\mathfrak{b}) = \exp\left(\mathfrak{p} \cap Z_{\mathfrak{g}}(\mathfrak{b})\right).$$

---

[4]Under these identifications the set of real roots is identified in equation (3.10) of [13] with a root system Δ_0 that will be important to us in Section 8.

[5]Our proofs of this statement and of (a) amount to a shorter version of the proofs on pp. 62–64 of [24].

Thus we have

$$Z_G(\mathfrak{b}) = Z_K(\mathfrak{b}) \cdot \exp(\mathfrak{p} \cap Z_{\mathfrak{g}}(\mathfrak{b})),$$

and each factor on the right is connected, from p. 247 of [9], since G connected implies K connected. Hence $Z_G(\mathfrak{b})$ is connected.

Consequently $Z_G(\mathfrak{b})$ is the analytic group corresponding to

$$\begin{aligned} Z_{\mathfrak{g}}(\mathfrak{b}) &= \mathfrak{a} + \mathfrak{b} + \sum_{\alpha \text{ real}} \mathfrak{g}_\alpha \\ &= (\mathfrak{a} + \mathfrak{b}) + \Big(\sum_{\alpha \text{ real}} \mathbf{R} H_\alpha + \sum_{\alpha \text{ real}} \mathfrak{g}_\alpha \Big). \end{aligned}$$

The second grouped term on the right is an $\mathbf{R}$-split semisimple Lie algebra. Hence its F group is generated by the elements γ_α, α real. Applying (2.2) to the minimal parabolic subgroup of this $\mathbf{R}$-split algebra, we conclude that the centralizer of the Cartan subalgebra $\mathfrak{a} \oplus \mathfrak{b}$ in $Z_G(\mathfrak{b})$ is

$$ABF(B).$$

This proves (a).

From (a) it follows that $Z_K(\mathfrak{a} + \mathfrak{b}) = BF(B)$. Intersecting both sides with Z_M and using the fact that $F(B) \subseteq Z_M$, we obtain the first equality in (b). The second equality follows since Z_{M_0} is contained in every maximal torus of $K \cap M_0$. Substituting (b) into the definition $M^\# = M_0 Z_M$ and regrouping, we obtain (c).

We shall relate extensively representations induced from different cuspidal parabolic subgroups. Following Koranyi-Wolf [20] and Schmid [24], we shall use "Cayley transforms" to pass back and forth between parabolics. We assemble the standard facts about such Cayley transforms here. First let $P = MAN$ be the Langlands decomposition of a cuspidal parabolic subgroup, let $\mathfrak{b} \subseteq \mathfrak{k} \cap \mathfrak{m}$ be a compact Cartan subalgebra, and let α be a real root of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus i\mathfrak{b})^{\mathbf{C}})$. Fix a root vector X_α in $\mathfrak{g}$ in the root space for α such that

$$[X_\alpha, \theta X_\alpha] = -2|\alpha|^{-2} H_\alpha.$$

Note that $\{H_\alpha, X_\alpha, \theta X_\alpha\}$ spans a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbf{R})$. Let $\mathbf{d}_\alpha$ be the unitary automorphism of $\mathfrak{g}^{\mathbf{C}}$ given by

$$\mathbf{d}_\alpha = \mathrm{Ad}\Big(\exp \frac{\pi i}{4}(X_\alpha - \theta X_\alpha)\Big). \tag{2.5}$$

The function $\mathbf{d}_\alpha$ carries $(\mathfrak{a} \oplus i\mathfrak{b})^{\mathbf{C}}$ to a new Cartan subalgebra $(\mathfrak{a}^* \oplus i\mathfrak{b}^*)^{\mathbf{C}}$ of $\mathfrak{g}^{\mathbf{C}}$, where

$$\begin{aligned} \mathfrak{a} &= \mathfrak{a}^* \oplus \mathbf{R} H_\alpha, \\ \mathfrak{b}^* &= \mathfrak{b} \oplus \mathbf{R}(X_\alpha + \theta X_\alpha) \qquad \text{(orthogonal sums)} \end{aligned}$$

are in $\mathfrak{g}$. The function satisfies

$$\mathbf{d}_\alpha(2|\alpha|^{-2}H_\alpha) = -i(X_\alpha + \theta X_\alpha), \tag{2.6a}$$

$$\mathbf{d}_\alpha(H) = H \quad \text{for } H \in \mathfrak{a}^* \oplus \mathfrak{b}, \tag{2.6b}$$

$$\mathbf{d}_\alpha(i(X_\alpha + \theta X_\alpha)) = 2|\alpha|^{-2}H_\alpha. \tag{2.6c}$$

The transpose mapping, also denoted $\mathbf{d}_\alpha$, carries roots of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus i\mathfrak{b})^{\mathbf{C}})$ to roots of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a}^* \oplus i\mathfrak{b}^*)^{\mathbf{C}})$. From $\mathfrak{a}^*$ we can construct a group M^* in a standard way [5], and the various choices of positive systems for the roots of $(\mathfrak{g}, \mathfrak{a}^*)$ lead to cuspidal parabolic subgroups $M^*A^*N_i^*$. Here $\mathfrak{b}^*$ is a compact Cartan subalgebra of $\mathfrak{m}^*$ (and is in $\mathfrak{k} \cap \mathfrak{m}^*$), $\mathfrak{m}^* \supseteq \mathfrak{m}$, and $\mathfrak{a}^* \subseteq \mathfrak{a}$. And $\mathbf{d}_\alpha(\alpha)$ is a noncompact root of $(\mathfrak{m}^{*\mathbf{C}}, \mathfrak{b}^{*\mathbf{C}})$.

The function $\mathbf{d}_\alpha^{-1}$ leads us from the data $(\mathfrak{m}^*, \mathfrak{a}^*, \mathfrak{b}^*)$ to the data $(\mathfrak{m}, \mathfrak{a}, \mathfrak{b})$ and can be constructed in its own right. Starting from $(\mathfrak{m}^{*\mathbf{C}}, \mathfrak{b}^{*\mathbf{C}})$, let β be a noncompact root and fix E_β and $E_{-\beta}$ in the root spaces for β and $-\beta$ such that

(i) $C(E_\beta, E_{-\beta}) = 2|\beta|^{-2}$, where C is the θ-invariant $\mathfrak{g}$-invariant symmetric bilinear form used as a substitute for the (possibly degenerate) Killing form.

(ii) $i(E_\beta - E_{-\beta})$ and $E_\beta + E_{-\beta}$ are in $\mathfrak{g}$.

(See Lemma 3.1 on p. 219 of [9] for a discussion of how to achieve this construction.) If $H'_\beta = 2|\beta|^{-2}H_\beta$, then we have the bracket relations

$$[H'_\beta, E_\beta] = 2E_\beta, \qquad [H'_\beta, E_{-\beta}] = -2E_{-\beta}, \qquad [E_\beta, E_{-\beta}] = H'_\beta.$$

Note that $\{H'_\beta, E_\beta, E_{-\beta}\}$ spans a Lie subalgebra of $\mathfrak{g}^{\mathbf{C}}$ over $\mathbf{C}$ isomorphic to $\mathfrak{sl}(2, \mathbf{C})$; the intersection of this subalgebra with $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2, \mathbf{R})$. Let $\mathbf{c}_\beta$ be the unitary automorphism of $\mathfrak{g}^{\mathbf{C}}$ given by

$$\mathbf{c}_\beta = \operatorname{Ad}\left(\exp\frac{\pi}{4}(E_\beta - E_{-\beta})\right). \tag{2.7}$$

The function $\mathbf{c}_\beta$ carries $(\mathfrak{a}^* \oplus i\mathfrak{b}^*)^{\mathbf{C}}$ to a new Cartan subalgebra $(\mathfrak{a} \oplus i\mathfrak{b})^{\mathbf{C}}$ of $\mathfrak{g}^{\mathbf{C}}$, where

$$\mathfrak{a} = \mathfrak{a}^* \oplus \mathbf{R}(E_\beta + E_{-\beta}),$$
$$\mathfrak{b}^* = \mathfrak{b} \oplus i\mathbf{R}H_\beta \qquad \text{(orthogonal sums)}$$

are in $\mathfrak{g}$. The function satisfies

$$\mathbf{c}_\beta(2|\beta|^{-2}H_\beta) = -(E_\beta + E_{-\beta}), \tag{2.8a}$$

$$\mathbf{c}_\beta(H) = H \qquad \text{for } H \in \mathfrak{a}^* \oplus \mathfrak{b}, \tag{2.8b}$$

$$\mathbf{c}_\beta(E_\beta + E_{-\beta}) = 2|\beta|^{-2}H_\beta. \tag{2.8c}$$

The transpose mapping, also denoted $\mathbf{c}_\beta$, carries roots of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a}^* \oplus i\mathfrak{b}^*)^{\mathbf{C}})$ to roots of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus i\mathfrak{b})^{\mathbf{C}})$. From $\mathfrak{a}$ we can construct a group M and cuspidal parabolic subgroups MAN_i. Here $\mathfrak{b}$ is a compact Cartan subalgebra of $\mathfrak{m}$ (and is in $\mathfrak{k} \cap \mathfrak{m}$), $\mathfrak{m}^* \supseteq \mathfrak{m}$, and $\mathfrak{a}^* \subseteq \mathfrak{a}$. Also $\mathbf{c}_\beta(\beta)$ is a real root of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus i\mathfrak{b})^{\mathbf{C}})$.

The Cayley transforms $\mathbf{d}_\alpha$ and $\mathbf{c}_\beta$ are inverses of one another if $\beta = \mathbf{d}_\alpha(\alpha)$, or equivalently $\alpha = \mathbf{c}_\beta(\beta)$, and if the X's and E's are chosen suitably.

LEMMA 2.2. *Let $M^*A^*N^*$ be a cuspidal parabolic subgroup, let $\mathfrak{b} \subseteq \mathfrak{k} \cap \mathfrak{m}^*$ be a compact Cartan subalgebra of $\mathfrak{m}^*$, and let $\tilde{\alpha}$ be a noncompact root of $(\mathfrak{m}^{*\mathbf{C}}, \mathfrak{b}^{*\mathbf{C}})$. Suppose that the Cayley transform $\mathbf{c}_{\tilde{\alpha}}$ leads from the data $(\mathfrak{m}^*, \mathfrak{a}^*, \mathfrak{b}^*)$ to data $(\mathfrak{m}, \mathfrak{a}, \mathfrak{b})$ and that $\alpha = \mathbf{c}_{\tilde{\alpha}}(\tilde{\alpha})$. Then the inclusion mapping of Z_M into $Z_M(M \cap M^{*\#})$ yields an isomorphism*

$$Z_M/\{1, \gamma_\alpha\}Z_{M_0}Z_{M^*} \cong Z_M(M \cap M^{*\#})/(M \cap M^{*\#}). \tag{2.9}$$

The groups on either side of (2.9) have order at most 2. They have order exactly 2 if and only if the reflection $p_{\tilde{\alpha}}$ is in $W(B^* : M^*)$.

Proof. Clearly Z_M maps onto the right side of (2.9). We have

$$\{1, \gamma_\alpha\} \subseteq M_p \cap M_0, \qquad Z_{M_0} \subseteq M_0, \qquad Z_{M^*} \subseteq M_p \cap M^{*\#},$$

and all three right sides are contained in $M \cap M^{*\#}$. Hence

$$\{1, \gamma_\alpha\}Z_{M_0}Z_{M^*} \subseteq M \cap M^{*\#},$$

and our map yields a quotient homomorphism of the left side of (2.9) onto the right side. Suppose z is in $Z_M \cap (M \cap M^{*\#})$. We have

(2.10)

$$Z_M \cap (M \cap M^{*\#}) = Z_M \cap M^{*\#} = Z_M \cap (M_0^* \cdot Z_{M^*}) = (Z_M \cap M_0^*)Z_{M^*}.$$

The group M_0^* is connected reductive with compact center, and we can apply Lemma 2.1b with G replaced by M_0^* and M replaced by $M \cap M_0^*$. There is only one real root in this situation, and the lemma gives

$$Z_M \cap M_0^* = Z_{M_0}\{1, \gamma_\alpha\}.$$

Substituting into (2.10), we see that z is in $\{1, \gamma_\alpha\}Z_{M_0}Z_{M^*}$. Hence our quotient homomorphism is one-one, and (2.9) follows.

Let z be in Z_M, and consider the $\mathfrak{sl}(2, \mathbf{R})$ algebra corresponding to the real root α. This has a basis $\{H_\alpha(\text{in } \mathfrak{a}), X_\alpha, X_{-\alpha}(= -\theta X_\alpha)\}$, and $\operatorname{ad}(\mathfrak{a} \oplus \mathfrak{b})$ acts by $\pm\alpha$ on $X_{\pm\alpha}$. Let H be in $\mathfrak{a} \oplus \mathfrak{b}$; since z is in Z_M,

$$[H, \operatorname{Ad}(z)X_\alpha] = \operatorname{Ad}(z)[\operatorname{Ad}(z)^{-1}H, X_\alpha] = \operatorname{Ad}(z)[H, X_\alpha] = \alpha(H)\operatorname{Ad}(z)X_\alpha.$$

Thus $\operatorname{Ad}(z)X_\alpha = \pm X_\alpha$, and similarly $\operatorname{Ad}(z)X_{-\alpha} = \pm X_{-\alpha}$. The two signs must be the same since $\operatorname{Ad}(z)$ acts on $[X_\alpha, X_{-\alpha}] = 2|\alpha|^{-2}H_\alpha$ by $+1$.

Suppose $\mathrm{Ad}(z)X_\alpha = X_\alpha$ and $\mathrm{Ad}(z)X_{-\alpha} = X_{-\alpha}$. Then $\mathrm{Ad}(z)$ fixes $X_\alpha + \theta X_\alpha$, which by (2.6a) is a multiple of $H_{\tilde{\alpha}}$. By Lemma 4.3 of [13], z is in $M^{*\#}$. Hence z maps to the identity in the quotient group (2.9).

On the other hand, any two elements z and z' of Z_M mapping $X_{\pm\alpha}$ to $-X_{\pm\alpha}$ would have quotient fixing $X_{\pm\alpha}$ and (by the previous paragraph) mapping to 1 in the group (2.9). Consequently the group (2.9) has order at most 2.

If the group (2.9) has order 2, then we have seen that there is an element z in Z_M with $\mathrm{Ad}(z)X_{\pm\alpha} = -X_{\pm\alpha}$. This element satisfies $\mathrm{Ad}(z)(X_\alpha + \theta X_\alpha) = -(X_\alpha + \theta X_\alpha)$ and $\mathrm{Ad}(z) = 1$ on $\mathfrak{b}$. Since $X_\alpha + \theta X_\alpha$ is a multiple of $H_{\tilde{\alpha}}$ by (2.6a), z exhibits the reflection $p_{\tilde{\alpha}}$ as existing in $W(B^* : M^*)$.

Conversely suppose $p_{\tilde{\alpha}}$ exists in $W(B^* : M^*)$. Choose, by equation (1.2), a representative $u_{\tilde{\alpha}}$ in $K \cap M^*$. Let $E_{\tilde{\alpha}}$ and $E_{-\tilde{\alpha}}$ be the root vectors used in constructing the Cayley transform $\mathbf{c}_{\tilde{\alpha}}$. Then $\mathrm{Ad}(u_{\tilde{\alpha}})$ satisfies

$$\mathrm{Ad}(u_{\tilde{\alpha}})H_{\tilde{\alpha}} = -H_{\tilde{\alpha}}$$

and

$$\mathrm{Ad}(u_{\tilde{\alpha}})E_{\tilde{\alpha}} = e^{i\varphi}E_{-\tilde{\alpha}} \quad \text{and} \quad \mathrm{Ad}(u_{\tilde{\alpha}})E_{-\tilde{\alpha}} = e^{i\theta}E_{\tilde{\alpha}}.$$

Since $[E_{\tilde{\alpha}}, E_{-\tilde{\alpha}}]$ is a multiple of $H_{\tilde{\alpha}}$, we must have $\varphi = -\theta$. Thus

$$\mathrm{Ad}(u_{\tilde{\alpha}})(E_{\tilde{\alpha}} + E_{-\tilde{\alpha}}) = e^{i\theta}E_{\tilde{\alpha}} + e^{-i\theta}E_{-\tilde{\alpha}}.$$

Let $b_\theta = \exp(i\theta\,|\,\alpha\,|^{-2}H_{\tilde{\alpha}})$. Then b_θ is in $K \cap M^*$ and satisfies

$$\mathrm{Ad}(b_\theta)(E_{\tilde{\alpha}} + E_{-\tilde{\alpha}}) = e^{i\theta}E_{\tilde{\alpha}} + e^{-i\theta}E_{-\tilde{\alpha}}.$$

Hence $w_{\tilde{\alpha}} = b_\theta^{-1}u_{\tilde{\alpha}}$ is another representative of $p_{\tilde{\alpha}}$, and $\mathrm{Ad}(w_{\tilde{\alpha}})$ fixes $E_{\tilde{\alpha}} + E_{-\tilde{\alpha}}$, which is a multiple of H_α. Since $w_{\tilde{\alpha}}$ is in M^*, $\mathrm{Ad}(w_{\tilde{\alpha}})$ fixes $\mathfrak{a}^*$. The fact that $w_{\tilde{\alpha}}$ represents $p_{\tilde{\alpha}}$ means that $w_{\tilde{\alpha}}$ fixes $\mathfrak{b}$. Thus $w_{\tilde{\alpha}}$ centralizes $\mathfrak{a} \oplus \mathfrak{b}$ and is in K. By Lemma 2.1a, $w_{\tilde{\alpha}}$ is in

$$BF(B) \subseteq M_0 Z_M \subseteq (M \cap M^{*\#})Z_M.$$

On the other hand, $w_{\tilde{\alpha}}$ is not in $M \cap M^{*\#}$ because it is not in $M^{*\#}$; in fact,

$$W(B^* : M^{*\#}) = W(B^* : M_0^*),$$

and the right side contains no reflections in noncompact roots (like $p_{\tilde{\alpha}}$) since M_0^* is connected. Thus $Z_M(M \cap M^{*\#})/(M \cap M^{*\#})$ has a nontrivial element and must be of order 2.

3. Basic characters

As in Section 2, let G be a linear connected reductive group with compact center, let $P = MAN$ be a cuspidal parabolic subgroup, and let $\Theta^M(\lambda, C, \chi)$ be a discrete series or limit of discrete series character. If ν is a real-valued linear functional on $\mathfrak{a}$, we let

$$\Theta^{MA}(\lambda, C, \chi, \nu) = \Theta^{M}(\lambda, C, \chi) \otimes e^{i\nu},$$

which is a character of MA. If we tensor with the trivial character of N and induce, we obtain characters

$$\operatorname{ind}_P^G \Theta^{MA}(\lambda, c, \chi, \nu), \tag{3.1}$$

which we call *basic characters*. When we want to insist that Θ^M is a discrete series character, we shall call (3.1) a *basic character induced from discrete series*.

After Theorem 1.1 is proved, it will follow that (3.1) is a tempered unitary character. It need not be irreducible, however. We should point out that (3.1) does not depend on which group N is used to make MA into a parabolic subgroup; this fact follows from Proposition 8.5(v) of [16] once we have seen in Theorem 1.1 that limits of discrete series are irreducible unitary.

We shall want to be able to apply the functor ψ to basic characters. As preparation for such applications, we have the following lemma, which uses the functor ${}^M\psi$ defined in Section B of the appendix. The parabolic $P = MAN$ is assumed cuspidal, with notation as in Section 2.

LEMMA 3.1. *Let a positive system $\Delta_{\mathfrak{m}}^+$ of roots for $((\mathfrak{m} \oplus \mathfrak{a})^{\mathbf{C}}, (\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}})$ be specified, and let λ be a $\Delta_{\mathfrak{m}}^+$-dominant linear form on $(\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}}$ vanishing on $\mathfrak{a}^{\mathbf{C}}$. Then there exists a positive system $\Delta_{\mathfrak{g}}^+$ of roots for $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}})$ such that*

(i) $\Delta_{\mathfrak{m}}^+ \subseteq \Delta_{\mathfrak{g}}^+$,

(ii) *λ is $\Delta_{\mathfrak{g}}^+$-dominant,*

(iii) *the intersection of $i\mathfrak{b}$ with the closure of the Weyl chamber in $\mathfrak{a} \oplus i\mathfrak{b}$ determined by $\Delta_{\mathfrak{g}}^+$ has nonempty interior in $i\mathfrak{b}$.*

Under these conditions, suppose C is the Weyl chamber of $i\mathfrak{b}$ determined by $\Delta_{\mathfrak{m}}^+$, suppose $\Theta^M(\lambda, C, \chi)$ is a discrete series or limit of discrete series character, and suppose ν is any real-valued linear function on $\mathfrak{a}$. Then

$$\Theta^{MA}(\lambda, C, \chi, \nu) = {}^M\psi_{\lambda+i\nu}^{\lambda+i\nu+\mu} \Theta^{MA}(\lambda+\mu, C, \chi e^{\mu}, \nu)$$

and

$$\operatorname{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu) = \psi_{\lambda+i\nu}^{\lambda+i\nu+\mu} \operatorname{ind}_P^G \Theta^{MA}(\lambda+\mu, C, \chi e^{\mu}, \nu)$$

for every integral[6] $\Delta_{\mathfrak{g}}^+$-dominant linear form μ on $(\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}}$ such that μ vanishes on $\mathfrak{a}$. Moreover there exists such a μ satisfying

$$\langle \mu, \alpha \rangle > 0 \quad \text{for } \alpha \text{ in } \Delta_{\mathfrak{m}}^+ \tag{3.2}$$

and even

$$\langle \mu, \alpha \rangle = 0 \quad \text{for } \alpha \text{ in } \Delta_{\mathfrak{g}}^+ \text{ implies } \alpha|_{\mathfrak{b}} = 0. \tag{3.3}$$

[6] "Integral" is defined after Equation (1.2).

Proof. First we show that the $\mathfrak{g}$-integral points in $(i\mathfrak{b})'$ form a lattice in $(i\mathfrak{b})'$. Set $\mathfrak{b}_1 = \mathfrak{b} \cap [\mathfrak{g}, \mathfrak{g}]$ and form the orthogonal sums $\mathfrak{b} = \mathfrak{b}_1 \oplus Z_{\mathfrak{g}}$ and $(i\mathfrak{b})' = (i\mathfrak{b}_1)' \oplus (iZ_{\mathfrak{g}})'$. Every $\mathfrak{g}^{\mathbf{C}}$-root α in $(\mathfrak{a} \oplus i\mathfrak{b})'$ is integral, and conjugation leaves the set of roots stable. Thus $\alpha - \bar{\alpha}$ is integral, and it is in $(i\mathfrak{b})'$. Since the α's span $(\mathfrak{a} \oplus i\mathfrak{b}_1)'$ and since $\alpha \to \frac{1}{2}(\alpha - \bar{\alpha})$ is the orthogonal projection of $(\mathfrak{a} \oplus i\mathfrak{b})'$ on $(i\mathfrak{b})'$, the differences $\alpha - \bar{\alpha}$ span $(i\mathfrak{b}_1)'$.

We still need to see that the $\mathfrak{g}$-integral points in $(iZ_{\mathfrak{g}})'$ form a lattice. The points in $(iZ_{\mathfrak{g}})'$ corresponding to characters of the identity component $(Z_G)_0$ of the center of G form a lattice L in $(iZ_{\mathfrak{g}})'$. If G_s is the analytic subgroup of G with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$, then G_s is a semisimple subgroup of a matrix group and is closed in G, by [2]. Consequently $(Z_G)_0 \cap G_s$ is finite. Those members of L corresponding to characters that are trivial on $(Z_G)_0 \cap G_s$ form a sublattice and all correspond to characters that extend to holomorphic characters on $\exp(\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}}$ that are trivial on $\exp(\mathfrak{a} \oplus \mathfrak{b}_1)^{\mathbf{C}}$. Thus the $\mathfrak{g}$-integral points in $(iZ_{\mathfrak{g}})'$ form a lattice, and the same thing is true of $(i\mathfrak{b})'$.

Next we produce the required ordering. Let H_λ be the element of $i\mathfrak{b}$ dual to λ, and let H^+ be an element of $i\mathfrak{b}$ in the open positive Weyl chamber relative to $\mathfrak{m}$. We sort the closed Weyl chambers in $\mathfrak{a} \oplus i\mathfrak{b}$ relative to $\mathfrak{g}$ into three types. The first type consists of those containing H_λ and meeting $i\mathfrak{b}$ in a subset of $i\mathfrak{b}$ with nonempty interior. The second type consists of those containing H_λ and meeting $i\mathfrak{b}$ in a set with empty interior. The third type consists of those not containing H_λ. Observe that the first and third type together cover $i\mathfrak{b}$. Choose $\varepsilon > 0$ small enough so that the segment from H_λ to $H_\lambda + \varepsilon H^+$ does not meet any of the third type. Then $H_\lambda + \varepsilon H^+$ must lie in the union of the closed Weyl chambers of the first type; choose one containing $H_\lambda + \varepsilon H^+$, and let $C_{\mathfrak{g}}$ be its interior in $\mathfrak{a} \oplus i\mathfrak{b}$. We take $\Delta_{\mathfrak{g}}^+$ to be the positive system determined by $C_{\mathfrak{g}}$: A root is positive if it is positive on some element of the closure $\bar{C}_{\mathfrak{g}}$, and every positive root is ≥ 0 on every element of $\bar{C}_{\mathfrak{g}}$.

If α is in $\Delta_{\mathfrak{m}}^+$, then

$$\alpha(H_\lambda + \varepsilon H^+) = \alpha(H_\lambda) + \varepsilon\alpha(H^+) \geq \varepsilon\alpha(H^+) > 0.$$

Since $H_\lambda + \varepsilon H^+$ is in $\bar{C}_{\mathfrak{g}}$, α is in $\Delta_{\mathfrak{g}}^+$. Thus (i) holds. By construction $\bar{C}_{\mathfrak{g}}$ contains H_λ; hence (ii) holds. Condition (iii) holds by definition of $\Delta_{\mathfrak{g}}^+$.

Now suppose we are given $\Delta_{\mathfrak{g}}^+$ satisfying (i), (ii), and (iii). We shall prove the character identities. Let μ be given. For the first identity, we notice that the group A acts as a scalar in ${}^MF_{-\mu}$ since ${}^MF_{-\mu}$ is irreducible on MA, and then

$$ {}^MF_{-\mu}(a) = I \quad \text{for } a \text{ in } A \tag{3.4}$$

since μ vanishes on $\mathfrak{a}$. Let $\pi^M(\cdot)$ denote a Harish-Chandra module with character

$\Theta^M(\cdot)$. In the notation of Section B of the appendix, we have

$$
\begin{aligned}
&{}^M\psi_{\lambda+i\nu}^{\lambda+i\nu+\mu}\big(\pi^M(\lambda+\mu, C, \chi e^\mu)\otimes e^{i\nu}\big)\\
&\quad = p_{\lambda+i\nu}\Big[\big\{\mathrm{ind}_{M^\#}^M\big(\pi^{M_0}(\lambda+\mu, C)\otimes \chi e^\mu\big)\otimes e^{i\nu}\big\}\otimes {}^M F_{-\mu}\Big]\\
&\quad = p_{\lambda+i\nu}\Big[\mathrm{ind}_{M^\#}^M\Big(\big(\pi^{M_0}(\lambda+\mu, C)\otimes {}^M F_{-\mu}|_{M_0}\big)\otimes\chi\Big)\otimes e^{i\nu}\Big] \quad \text{by (3.4)}\\
&\quad = p_\lambda \mathrm{ind}_{M^\#}^M\Big[\big(\pi^{M_0}(\lambda+\mu, C)\otimes {}^M F_{-\mu}|_{M_0}\big)\otimes\chi\big)\Big]\otimes e^{i\nu}\\
&\quad = \mathrm{ind}_{M^\#}^M p_\lambda\Big[\pi^{M_0}(\lambda+\mu, C)\otimes {}^M F_{-\mu}|_{M_0}\otimes\chi\Big]\otimes e^{i\nu}
\end{aligned}
$$

since $\mathrm{Ad}(M)$ is the identity on the center of the universal enveloping algebra of $(\mathfrak{m}\otimes\mathfrak{a})^{\mathbb{C}}$, and this last expression is

$$= \mathrm{ind}_{M^\#}^M\big[\pi^{M_0}(\lambda, C)\otimes\chi\big]\otimes e^{i\nu}.$$

Taking the character of both sides, we obtain the first character identity of the lemma. The second character identity of the lemma follows from the first one and from Theorem B.1 of the appendix.

Finally we produce an integral $\Delta_{\mathfrak{g}}^+$-dominant linear μ on $(\mathfrak{a}\oplus\mathfrak{b})^{\mathbb{C}}$ such that μ vanishes on $\mathfrak{a}$ and satisfies (3.2) and (3.3). By (iii) find $\varepsilon > 0$ and H' in $i\mathfrak{b}$ such that the ε-ball in $i\mathfrak{b}$ about H' lies in the closure of the Weyl chamber in $\mathfrak{a}\oplus i\mathfrak{b}$ determined by $\Delta_{\mathfrak{g}}^+$. Let μ' in $(i\mathfrak{b})'$ be the element dual to H'. Then every member of the ε-ball in $(i\mathfrak{b})'$ about μ' satisfies all the properties required of μ except (3.3) and the integrality. Multiplying this ball by all positive scalars, we obtain an open cone in $(i\mathfrak{b})'$ of elements satisfying all the properties required of μ except the integrality. We have seen that the $\mathfrak{g}$-integral points in $(i\mathfrak{b})'$ form a lattice in $(i\mathfrak{b})'$, and an open cone contains lattice points. In this way we can obtain μ satisfying all the required properties except (3.3). Moreover, not every lattice point in the open cone can lie in one of the hyperplanes in $(i\mathfrak{b})'$ given by

$$\langle\mu, \alpha\rangle = 0 \quad \text{for } \alpha \text{ in } \Delta_{\mathfrak{g}}^+ \text{ with } \alpha|_{\mathfrak{b}} \neq 0.$$

Thus we can even find μ integral so that (3.3) is valid. This completes the proof of Lemma 3.1.

4. Generalized Schmid identities

Let G be a linear connected reductive group with compact center. The purpose of this section is to relate basic characters associated to two different cuspidal parabolic subgroups of G that are obtained from each other by the Cayley transforms (2.5) and (2.7).

Let MAN be a cuspidal parabolic subgroup. In the notation $\Theta^M(\lambda, C, \chi)$ of Section 1, the character χ is initially defined on Z_M. However, we mentioned in connection with equation (1.6) that it is sufficient to have χ defined on a subgroup Z'_M of Z_M as long as $M^\# = M_0 Z'_M$. We examine this remark in the light of Lemmas 2.1b and 2.2.

First suppose rank G = rank K and $\mathfrak{b}^* \subseteq \mathfrak{k}$ is a compact Cartan subalgebra of $\mathfrak{g}$. Let M be built from a Cayley transform $\mathbf{c}_{\tilde{\alpha}}$ with respect to a noncompact root $\tilde{\alpha}$ of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{b}^{*\mathbf{C}})$. Then Lemma 2.1b says that $Z_M = Z_{M_0} F(B)$, and we therefore can take $Z'_M = F(B) = \{1, \gamma_\alpha\}$. This choice of notation is made in the statement of Theorem 4.1 below.

If we have a cuspidal parabolic $M^*A^*N^*$ and build MA from M^*A^* by a Cayley transform $\mathbf{c}_{\tilde{\alpha}}$ with respect to a noncompact root $\tilde{\alpha}$ of $(\mathfrak{m}^{*\mathbf{C}}, (\mathfrak{a} \oplus i\mathfrak{b}^*)^{\mathbf{C}})$, then Lemma 2.2 says that

$$\{1, \gamma_\alpha\} Z_{M_0} Z_{M^*}$$

has index 1 or 2 in Z_M. In this case if the index is 1, we can take $Z'_M = \{1, \gamma_\alpha\} Z_{M^*}$. Otherwise $\{1, \gamma_\alpha\} Z_{M^*}$ determines a subgroup of index 2 in Z_M. These choices of notation are made in the statement of Theorem 4.3 below.

We can now state one of Schmid's character identities. (See Theorem 9.4 of [24] and page 135 of [8]. See also [27].)

THEOREM 4.1 (Schmid). *Let G be linear connected reductive with rank G = rank K, and let $\mathfrak{b}^*$ be a compact Cartan subalgebra of $\mathfrak{g}$ contained in K. Suppose that C is a Weyl chamber in $i\mathfrak{b}^*$ and $\tilde{\alpha}$ is a C-simple noncompact root of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{b}^{*\mathbf{C}})$. Suppose that the Cayley transform $c_{\tilde{\alpha}}$ leads from the data $(\mathfrak{g}, 0, \mathfrak{b}^*)$ to data $(\mathfrak{m}, \mathfrak{a}, \mathfrak{b})$ and that $\alpha = c_{\tilde{\alpha}}(\tilde{\alpha})$. If λ is a member of $\mathfrak{b}^{*\prime}$ such that*

(i) *$\lambda - \rho$ is integral, where ρ is half the sum of the positive roots,*

(ii) *λ is C-dominant,*

(iii) $\langle \lambda, \alpha \rangle = 0$,

(iv) *λ is nonsingular with respect to compact roots,*

then

$$\Theta^G(\lambda, C) + \Theta^G(\lambda, p_{\tilde{\alpha}} C) = \operatorname{ind}_P^G \Theta^{MA}(\lambda|_{\mathfrak{b}}, C^{p_{\tilde{\alpha}}}, \zeta, 0), \tag{4.1}$$

where

(4.2) $C^{p_{\tilde{\alpha}}}$ = *unique Weyl chamber*[7] *of $i\mathfrak{b}$ containing the orthogonal projection* $\operatorname{Proj}_{i\mathfrak{b}} C = \operatorname{Proj}_{i\mathfrak{b}} p_{\tilde{\alpha}} C$

ζ = *character of $\{1, \gamma_\alpha\}$ given by* $\zeta(\gamma_\alpha) = (-1)^{2\langle \rho_\alpha, \alpha\rangle/|\alpha|^2}$

ρ_α = *half sum of roots of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}})$ whose restriction to $\mathfrak{a}$ is $c\alpha$ with $c > 0$.*

[7] The set $\operatorname{Proj}_{i\mathfrak{b}} C$ is open and connected in $i\mathfrak{b}$, and no root of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ vanishes anywhere on it; thus it is contained in a single Weyl chamber of $i\mathfrak{b}$. However, it need not exhaust the Weyl chamber of $i\mathfrak{b}$, as is seen in SU(2, 1).

COROLLARY 4.2. *In Theorem* 4.1, *assumption* (iv) *may be dropped.*

Proof. We shall specify consistent systems of positive roots for $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}})$ and $((\mathfrak{m} \oplus \mathfrak{a})^{\mathbf{C}}, (\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}})$ and then apply Lemma 3.1. The chamber C determines a positive system for $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{b}^*)^{\mathbf{C}})$, and we carry that notion of positivity over to the roots of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}})$ by the Cayley transform $\mathbf{c}_{\tilde{\alpha}}$. Restricting the notion of positivity to roots of $((\mathfrak{m} \oplus \mathfrak{a})^{\mathbf{C}}, (\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}})$, we obtain (i) in Lemma 3.1. Then the corresponding positive Weyl chamber in $i\mathfrak{b}$ is $C^{p_{\tilde{\alpha}}}$. The form $\lambda|_{\mathfrak{b}}$, extended by 0 on $\mathfrak{a}$, is just $c_{\tilde{\alpha}}(\lambda)$, which is C-dominant. Consequently $\lambda|_{\mathfrak{b}}$ satisfies (ii) in Lemma 3.1. We verify (iii) by showing that $\mathrm{Proj}_{i\mathfrak{b}} C \subseteq \bar{C}$. Thus suppose H_i is in $\mathrm{Proj}_{i\mathfrak{b}} C$. Choose $H_{\mathfrak{a}}$ in $\mathfrak{a}$ so that $H_{i\mathfrak{b}} + H_{\mathfrak{a}}$ is in C. Let β be a C-positive root other than $\tilde{\alpha}$. Then $p_{\tilde{\alpha}}\beta$ is also C-positive since $\tilde{\alpha}$ is simple, and hence

$$2\beta(H_{i\mathfrak{b}}) = \beta(H_{i\mathfrak{b}} + H_{\mathfrak{a}}) + p_{\tilde{\alpha}}\beta(H_{i\mathfrak{b}} + H_{\mathfrak{a}})$$

is > 0. Also $\tilde{\alpha}(H_{i\mathfrak{b}}) = 0$. Hence $H_{i\mathfrak{b}}$ is in $\bar{C}$, and (iii) holds.

Apply Lemma 3.1 to find a $\mathfrak{g}^{\mathbf{C}}$-dominant integral form $c_{\tilde{\alpha}}(\mu)$ on $(\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}}$ that vanishes on $\mathfrak{a}$ and satisfies (3.3). The lemma says that the right side of (4.1) equals

$$\psi_{c_{\tilde{\alpha}}(\lambda)}^{c_{\tilde{\alpha}}(\lambda+\mu)} \mathrm{ind}_P^G \Theta^{MA}((\lambda + \mu)|_{\mathfrak{b}}, C^{p_{\tilde{\alpha}}}, \zeta, 0). \tag{4.3}$$

Equation (3.3) implies that $\langle \lambda + \mu, \beta \rangle = 0$ only for $\beta = \pm\alpha$, hence that (iv) holds in Theorem 4.1. The theorem then says that (4.3) equals

$$\psi_{\lambda}^{\lambda+\mu} \Theta^G(\lambda + \mu, C) + \psi_{\lambda}^{\lambda+\mu} \Theta^G(\lambda + \mu, p_{\tilde{\alpha}} C),$$

because ψ does not depend on the choice of Cartan or the ordering. (See (1.9).) The corollary then follows from (1.11).

We shall generalize the character identity (4.1) by replacing the connected group G by the possibly disconnected group M^*. The situation splits into two cases, depending on whether the reflection $p_{\tilde{\alpha}}$ is in the Weyl group $W(B^* : M^*)$. We call (4.4a) and (4.4b) below *generalized Schmid identities.*

THEOREM 4.3. *Within a linear connected reductive Lie group G, let $M^*A^*N^*$ be a cuspidal parabolic subgroup, and let $\mathfrak{b}^* \subseteq \mathfrak{k} \cap \mathfrak{m}^*$ be a compact Cartan subalgebra of $\mathfrak{m}^*$. Suppose that C is a Weyl chamber in $i\mathfrak{b}^*$ and $\tilde{\alpha}$ is a C-simple noncompact root of $(\mathfrak{m}^{*\mathbf{C}}, \mathfrak{b}^{*\mathbf{C}})$. Suppose that the Cayley transform $\mathbf{c}_{\tilde{\alpha}}$ leads from the data $(\mathfrak{m}^*, \mathfrak{a}^*, \mathfrak{b}^*)$ to data $(\mathfrak{m}, \mathfrak{a}, \mathfrak{b})$ and that $\alpha = \mathbf{c}_{\tilde{\alpha}}(\tilde{\alpha})$. Finally suppose that data λ, χ, and ν for $\Theta^{M^*A^*}(\lambda, C, \chi, \nu)$ are such that*

(i) *$\lambda - \rho_{M^*}$ is $\mathfrak{b}^*$-integral, and $e^{\lambda - \rho_{M^*}}$ agrees with χ on $Z_{M^*} \cap B^*$,*

(ii) *$\langle \lambda, \tilde{\alpha} \rangle = 0$,*

(iii) λ *is C-dominant (and hence also* $p_{\tilde{\alpha}}C$*-dominant).*
There are the following cases:

(a) *If* $p_{\tilde{\alpha}}$ *is not in* $W(B^* : M^*)$, *then* $Z_M = \{1, \gamma_\alpha\} Z_{M_0} Z_{M^*}$ *and*

$$(4.4a)\quad \Theta^{M^*A^*}(\lambda, C, \chi, \nu) + \Theta^{M^*A^*}(\lambda, p_{\tilde{\alpha}} C, \chi, \nu) = \mathrm{ind}_{P\cap M^*A^*}^{M^*A^*} \Theta^{MA}(\lambda|_{\mathfrak{b}}, C^{p_{\tilde{\alpha}}}, \zeta \otimes \chi, \nu \oplus 0),$$

where $C^{p_{\tilde{\alpha}}}$ *and* ζ *are defined as in* (4.2).

(b) *If* $p_{\tilde{\alpha}}$ *is in* $W(B^* : M^*)$, *then* $|Z_M/\{1, \gamma_\alpha\} Z_{M_0} Z_{M^*}| = 2$. *Let* $C^{p_{\tilde{\alpha}}}$ *and* ζ *be defined as in* (4.2), *and let* $(\zeta \oplus \chi)^+$ *and* $(\zeta \otimes \chi)^-$ *denote the two extensions of* $\zeta \otimes e^{(\lambda - \rho_M)|_{\mathfrak{b}}} \otimes \chi$ *to* Z_M. *Then*

$$(4.4b)\quad \Theta^{M^*A^*}(\lambda, C, \chi, \nu) = \mathrm{ind}_{P\cap M^*A^*}^{M^*A^*} \Theta^{MA}\big(\lambda|_{\mathfrak{b}}, C^{p_{\tilde{\alpha}}}, (\zeta \otimes \chi)^+, \nu \oplus 0\big)$$
$$= \mathrm{ind}_{P\cap M^*A^*}^{M^*A^*} \Theta^{MA}\big(\lambda|_{\mathfrak{b}}, C^{p_{\tilde{\alpha}}}, (\zeta \otimes \chi)^-, \nu \oplus 0\big).$$

Remarks. This theorem is intimately connected with Theorem 1.1. We shall use the "if" part of Theorem 1.1c, which was proved completely in Section 1, in the proof. In turn we shall use Theorem 4.3 and the Hecht-Schmid identity (stated as the "if" part of Theorem 1.1b) to prove the remaining parts of Theorem 1.1; this step will be carried out in Section 5.

Once Theorem 1.1 is proved, it follows that the two characters on the left of (4.4a) are distinct unless both are zero. We can observe now that the two characters on MA that are induced in (4.4b) are distinct; in fact, the corresponding representations of MA are scalar on Z_M and equal to the distinct characters $(\zeta \otimes \chi)^{\pm}$.

Proof. We shall suppress some of the variables to simplify the notation. In both cases (a) and (b) we begin with an application of Theorem 4.1 or Corollary 4.2 to M_0^*, obtaining

$$(4.5)\qquad \Theta^{M_0^*}(C) + \Theta^{M_0^*}(p_{\tilde{\alpha}} C) = \mathrm{ind}_{P\cap M_0^*}^{M_0^*} \big(\Theta^{M\cap M_0^*} \otimes 1^{A\cap M^*}\big).$$

We intend to use χ on Z_{M^*} to extend both sides of (4.5) to $M^{*\#}$. There are compatibility conditions to check: The representations in question are all scalar on $Z_{M_0^*}$, and we have to check that χ agrees with these scalars on $Z_{M_0^*}$. By assumption (i), χ agrees with the scalar on the left, and it is implicit in the statement of Theorem 4.1 or Corollary 4.2 that the scalar on the left agrees with the scalar on the right. Thus we have the required consistency, and we obtain

$$\Theta^{M^{*\#}}(C) + \Theta^{M^{*\#}}(p_{\tilde{\alpha}} C) = \mathrm{ind}_{P\cap M^{*\#}}^{M^{*\#}} \big(\Theta^{M\cap M^{*\#}} \otimes 1^{A\cap M^*}\big).$$

Induce to M^*, using the double induction formula on the right, and then rewrite the right side by further applications of double induction. Since

$M(A \cap M^*)$ is the reductive part of $P \cap M^*$, the result is that
(4.6)

$$\Theta^{M^*}(C) + \Theta^{M^*}(p_{\tilde{\alpha}} C) = \operatorname{ind}_{P\cap M^{*\#}}^{M^*}\left(\Theta^{M\cap M^{*\#}} \otimes 1^{A\cap M^*}\right)$$
$$= \operatorname{ind}_{P\cap M^*}^{M^*}\left[\left(\operatorname{ind}_{Z_M(M\cap M^{*\#})}^{M} \operatorname{ind}_{M\cap M^{*\#}}^{Z_M(M\cap M^{*\#})}\Theta^{M\cap M^{*\#}}\right) \otimes 1^{A\cap M^*}\right].$$

Here $Z_M(M \cap M^{*\#})$ is a group intermediate between $M^\#$ and M, and it satisfies the axioms of Section 1 of [13]. In view of our definitions,

$$\Theta^{Z_M(M\cap M^{*\#})} = \operatorname{ind}_{M^\#}^{Z_M(M\cap M^{*\#})}\Theta^{M^\#}. \tag{4.7}$$

We now divide matters into the two cases distinguished in the theorem.

(a) Suppose $p_{\tilde{\alpha}}$ is not in $W(B^* : M^*)$. Then $Z_M = \{1, \gamma_\alpha\} Z_{M_0} Z_{M^*}$ by Lemma 2.2. Also Lemma 2.2 says that $M \cap M^{*\#} = Z_M(M \cap M^{*\#})$, so that the inner induction in (4.6) is trivial and we may write $\Theta^{Z_M(M\cap M^{*\#})}$ in place of $\Theta^{M\cap M^{*\#}}$. Substituting from (4.7) into (4.6) and using the double induction formula, we find that

$$\Theta^{M^*}(C) + \Theta^{M^*}(p_{\tilde{\alpha}} C) = \operatorname{ind}_{P\cap M^*}^{M^*}\left[\left(\operatorname{ind}_{M^\#}^{M}\Theta^{M^\#}\right) \otimes 1^{A\cap M^*}\right]$$
$$= \operatorname{ind}_{P\cap M^*}^{M^*}\left[\Theta^{M} \otimes 1^{A\cap M^*}\right].$$

Tensoring both sides by $e^{i\nu}$ on A^*, we obtain (4.4a).

(b) Suppose $p_{\tilde{\alpha}}$ is in $W(B^* : M^*)$. Then $|Z_M/\{1, \gamma_\alpha\} Z_{M_0} Z_{M^*}| = 2$ by Lemma 2.2. Since $p_{\tilde{\alpha}}$ is in $W(B^* : M^*)$, $\Theta^{M^*}(C) = \Theta^{M^*}(p_{\tilde{\alpha}} C)$, by the (already proved) "if" part of Theorem 1.1c. Thus the left side of (4.6) equals $2\Theta(C)$.

Now $M \cap M^{*\#}$ has index 2 in $Z_M(M \cap M^{*\#})$, by Lemma 2.2, and therefore the inner induction in (4.6) may be written as

$$\operatorname{ind}_{M\cap M^{*\#}}^{Z_M(M\cap M^{*\#})}\Theta^{M\cap M^{*\#}}(\zeta \otimes \chi) \tag{4.8}$$
$$= \Theta^{Z_M(M\cap M^{*\#})}\left((\zeta \otimes \chi)^+\right) + \Theta^{Z_M(M\cap M^{*\#})}\left((\zeta \otimes \chi)^-\right).$$

Substituting from (4.7) into (4.8) and from (4.8) into (4.6), we obtain
(4.9)

$$2\Theta^{M^*}(C) = \operatorname{ind}_{P\cap M^*}^{M^*}\left\{\left[\operatorname{ind}_{Z_M(M\cap M^{*\#})}^{M}\operatorname{ind}_{M^\#}^{Z_M(M\cap M^{*\#})}\left(\Theta^{M^\#}\left((\zeta \otimes \chi)^+\right)\right.\right.\right.$$
$$\left.\left.\left.+\Theta^{M^\#}\left((\zeta \otimes \chi)^-\right)\right)\right] \otimes 1^{A\cap M^*}\right\}$$
$$= \operatorname{ind}_{P\cap M^*}^{M^*}\left\{\left[\operatorname{ind}_{M^\#}^{M}\left(\Theta^{M^\#}\left((\zeta \otimes \chi)^+\right) + \Theta^{M^\#}\left((\zeta \otimes \chi)^-\right)\right)\right] \otimes 1^{A\cap M^*}\right\}.$$

Let p_α be the reflection in α in $W((A \cap M^*) : M^*)$. We shall show that

$$p_\alpha\Theta^{M^\#}\left((\zeta \otimes \chi)^+\right) = \Theta^{M^\#}\left((\zeta \otimes \chi)^-\right). \tag{4.10}$$

Then it will follow that the two terms in (4.9) each contribute the same thing to the sum after the outside induction, and hence (4.4b) is obtained by tensoring with $e^{i\nu}$ on A^* and dividing by 2.

Thus we are to prove (4.10). On M_0 the two characters are equal since Lemma 3.2a of [13] shows p_α has a representative in $K \cap M_0^*$ that centralizes M_0. Thus we are done if we can prove that the characters $(\zeta \otimes \chi)^{\pm}$ of Z_M are related by

$$p_\alpha(\zeta \otimes \chi)^+ = (\zeta \otimes \chi)^- . \tag{4.11}$$

Choose the standard representative $w_\alpha = \exp\frac{\pi}{2}(X_\alpha + \theta X_\alpha)$ for p_α. The element w_α commutes with each element of the subgroup $\{1, \gamma_\alpha\}Z_{M_0}Z_{M^*}$ of Z_M (since w_α lies in the $\mathrm{SL}(2, \mathbf{R})$ corresponding to α and since w_α centralizes B and lies in M^*), and $(\zeta \otimes \chi)^+$ and $(\zeta \otimes \chi)^-$ agree on this subgroup. Let z be in Z_M but not $\{1, \gamma_\alpha\}Z_{M_0}Z_{M^*}$. Then

$$(\zeta \otimes \chi)^+(z) = -(\zeta \otimes \chi)^-(z). \tag{4.12}$$

We shall prove that

$$w_\alpha^{-1} z w_\alpha = z\gamma_\alpha \tag{4.13}$$

and

$$\zeta(\gamma_\alpha) = -1. \tag{4.14}$$

Then we will have

$$\begin{aligned} w_\alpha(\zeta \otimes \chi)^+(z) &= (\zeta \otimes \chi)^+\left(w_\alpha^{-1} z w_\alpha\right) \\ &= (\zeta \otimes \chi)^+(z)(\zeta \otimes \chi)^+(\gamma_\alpha) && \text{by (4.13)} \\ &= -(\zeta \otimes \chi)^-(z)\zeta(\gamma_\alpha) && \text{by (4.12)} \\ &= (\zeta \otimes \chi)^-(z) && \text{by (4.14)}, \end{aligned}$$

and (4.11) and (4.4b) follow.

Thus we are to prove (4.13) and (4.14). For (4.13) we know that z commutes with A (z being in M) and with B (z being in Z_M). Therefore z is in $Z_{MA}(AB)$, which is contained in $Z_{(MA)^{\mathbf{C}}}(AB)$ by (2.4). Since $(MA)^{\mathbf{C}}$ is connected, z lies in $\exp(\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}}$. Thus let $z = \exp H$ with H in $(\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}}$. Then

$$\left(w_\alpha^{-1} z w_\alpha\right)z^{-1} = \exp(p_\alpha H - H) = \exp\left(-2\,|\alpha|^{-2}\alpha(H)H_\alpha\right). \tag{4.15}$$

To identify $\alpha(H)$, we recall from the proof of Lemma 2.2 that $\mathrm{Ad}(z)X_{\pm\alpha} = -X_{\pm\alpha}$ since z is not in $\{1, \gamma_\alpha\}Z_{M_0}Z_{M^*}$. Hence

$$-X_\alpha = \mathrm{Ad}(z)X_\alpha = \exp(\mathrm{ad}\, H)X_\alpha = e^{\alpha(H)}X_\alpha,$$

and it follows that $\alpha(H)$ is an odd multiple of πi. Substituting this fact into (4.15), we obtain (4.13).

For (4.14) the fact that $p_{\tilde\alpha}$ exists in $W(B^* : M^*)$ means that $p_{\tilde\alpha}$ preserves type for roots of $(\mathfrak{m}^{*\mathbf{C}}, \mathfrak{b}^{*\mathbf{C}})$—compact vs. noncompact. Since $p_{\tilde\alpha}\tilde\beta$ must be of the same type as $\tilde\beta$ and since $\tilde\alpha$ is noncompact, it follows that $2\langle \tilde\alpha, \tilde\beta\rangle/|\tilde\alpha|^2$ must be even for every root $\tilde\beta$ of $(\mathfrak{m}^{*\mathbf{C}}, \mathfrak{b}^{*\mathbf{C}})$. Let $\rho_{\tilde\alpha}$ be half the sum of all the roots of

$(\mathfrak{m}^{*\mathbf{C}}, \mathfrak{b}^{*\mathbf{C}})$ having positive inner product with $\tilde{\alpha}$. If $\tilde{\beta} \neq \tilde{\alpha}$ is a root contributing to this sum, then so is

$$-p_{\tilde{\alpha}}\tilde{\beta} = \frac{2\langle \tilde{\alpha}, \tilde{\beta}\rangle}{|\tilde{\alpha}|^2}\tilde{\alpha} - \tilde{\beta},$$

and the sum of these two roots is $2|\tilde{\alpha}|^{-2}\langle \tilde{\alpha}, \tilde{\beta}\rangle \tilde{\alpha}$. Hence

$$2\rho_{\tilde{\alpha}} = \tilde{\alpha} + \sum_{\substack{\text{pairs} \\ \tilde{\beta}, -p_{\tilde{\alpha}}\tilde{\beta}}} \frac{2\langle \tilde{\alpha}, \tilde{\beta}\rangle}{|\tilde{\alpha}|^2}\tilde{\alpha}.$$

The sum over the pairs is an even multiple of $\tilde{\alpha}$, and thus $2\langle \rho_{\tilde{\alpha}}, \tilde{\alpha}\rangle / |\tilde{\alpha}|^2$ is an odd integer. Since $\rho_\alpha = \mathbf{c}_{\tilde{\alpha}}(\rho_{\tilde{\alpha}})$, it follows that

$$\zeta(\gamma_\alpha) \equiv (-1)^{2\langle \rho_\alpha, \alpha\rangle/|\alpha|^2} = -1,$$

and (4.14) is proved. This completes the proof of Theorem 4.3.

Corollary 4.4. *Every basic character is contained in a basic character induced from discrete series* (*in the sense that the basic character induced from discrete series is the sum of the given basic character and some other character*).

Proof. For basic characters of the form $\operatorname{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu)$, the proof proceeds by induction downward on the dimension of A, with G fixed. The first step of the induction is for a minimal parabolic. In this case, M is compact; thus λ nonsingular with respect to the roots of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ gives a finite-dimensional representation of M, and λ singular gives 0.

Now suppose $\Theta^{M^*A^*}(\lambda, C, \chi, \nu)$ is given and the result is known when $\dim A > \dim A^*$. If λ is nonsingular with respect to all C-simple roots of $(\mathfrak{m}^{*\mathbf{C}}, \mathfrak{b}^{*\mathbf{C}})$, then $\Theta^{M^*}(\lambda, C, \chi)$ is in the discrete series and there is nothing to prove. If λ is singular with respect to some C-simple compact root, then $\Theta^{M^*}(\lambda, C, \chi) = 0$ by the Hecht-Schmid identity (see Theorem 1.1b). Finally if λ is singular with respect to some C-simple noncompact root $\tilde{\alpha}$, then the conditions of Theorem 4.3 are satisfied and either $\Theta^{M^*A^*}(\lambda, C, \chi, \nu)$ or

$$\Theta^{M^*A^*}(\lambda, C, \chi, \nu) + \Theta^{M^*A^*}(\lambda, p_{\tilde{\alpha}}C, \chi, \nu)$$

is the left side of a generalized Schmid identity. Inducing both sides from P^* to G, we can apply our inductive hypothesis to the right side to complete the proof.

5. Limits of discrete series, continued

We can now complete the proof of Theorem 1.1. In view of the partial proof given in Section 1, we may assume that G is linear connected reductive with compact center. We still have to prove

(a′) $\Theta^G(\lambda, C)$ is irreducible or zero.

(b′) $\Theta^G(\lambda, C)$ is zero only if $\langle \lambda, \alpha \rangle = 0$ for some C-simple compact root α.

(c′) When $\Theta^G(\lambda, C)$ is nonzero, then $\Theta^G(\lambda, C) = \Theta^G(\lambda', C')$ implies there exists w in $W(B : G)$ with $w\lambda = \lambda'$ and $wC = C'$.

(d′) When $\Theta^G(\lambda, C)$ is nonzero, it is the character of a tempered unitary representation.

For (a′) and (d′), we shall use the following Multiplicity One Theorem.

THEOREM 5.1 ([13], Theorem 7.1). *The irreducible characters that occur in any basic character induced from discrete series all occur with multiplicity one.*

Proof of (a′) *and* (d′). We saw in Section 1 that $\Theta^G(\lambda, C)$ is a multiple of an irreducible character. Corollary 4.4 shows that $\Theta^G(\lambda, C)$ occurs in a basic character induced from discrete series. By Theorem 5.1 we conclude that $\Theta^G(\lambda, C)$ is multiplicity-free. This proves (a′). Hence $\Theta^G(\lambda, C)$ is the character of an irreducible constituent of a tempered unitary representation (one induced from discrete series), and (d′) follows.

Proof of (b′). The proof is by induction on the rank of G. Let $\mathfrak{b}^* \subseteq \mathfrak{k}$ be a compact Cartan subalgebra of $\mathfrak{g}$, and let $\Theta^G(\lambda, C)$ be given with λ C-dominant. Define

$$\Delta_{C,\lambda} = \{\alpha \text{ root of } (\mathfrak{g}^{\mathbf{C}}, \mathfrak{b}^{*\mathbf{C}}) \mid \langle \alpha, \lambda \rangle = 0\}.$$

If $\Delta_{C,\lambda}$ contains only noncompact roots, then $\Theta^G(\lambda, C)$ is given by the usual nonzero formula on B^* and so is nonzero. (See also Theorem 5.7 of [31].) If $\Delta_{C,\lambda}$ contains a compact root but no compact C-simple root, then there exist simple noncompact β and δ in $\Delta_{C,\lambda}$ that are nonorthogonal, since $\Delta_{C,\lambda}$ is a root system spanned by the simple roots that it contains. Say $|\beta| \geq |\delta|$. Then $p_\beta\delta = \delta + \beta$, which is compact. Also $p_\beta\delta$ is simple for $p_\beta C$. By the Hecht-Schmid identity (the "if" part of Theorem 1.1b), $\Theta(\lambda, p_\beta C) = 0$. By the Schmid identity (Theorem 4.1 or Corollary 4.2),

$$\Theta(\lambda, C) = \operatorname{ind}_P^G \Theta^M(\lambda|_{\mathfrak{b}}, C^{p_\beta}, \zeta) \otimes e^0;$$

here $(\mathfrak{m}, \mathfrak{a}, \mathfrak{b})$ is obtained from $(\mathfrak{g}, 0, \mathfrak{b}^*)$ by means of the Cayley transform $\mathbf{c}_\beta$. Our result will follow by induction if we show that $\lambda|_{\mathfrak{b}}$ is nonsingular with respect to the $\mathfrak{m}$-compact simple roots of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ in the ordering given by C^{p_β}. (The disconnectedness of M does not affect whether a limit of discrete series character is zero.)

Let us emphasize the distinction between $\mathfrak{g}$-compactness and $\mathfrak{m}$-compactness. To decide $\mathfrak{m}$-compactness of a root γ of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$, one is to regard γ as a root of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}})$ that vanishes on $\mathfrak{a}$; to decide $\mathfrak{g}$-compactness of γ, one uses

$\mathbf{c}_\beta^{-1}(\gamma)$. Although $\mathbf{c}_\beta^{-1}$ does not move the vector H_γ in $i\mathfrak{b}$, $\mathbf{c}_\beta^{-1}$ will move the root vector associated with γ if and only if γ and β are orthogonal but not strongly orthogonal. By Lemma 5.4 of [17],

(5.1a) γ and β strongly orthogonal

$$\Rightarrow (\gamma \text{ is } \mathfrak{m}\text{-compact} \Leftrightarrow \gamma \text{ is } \mathfrak{g}\text{-compact});$$

(5.1b) γ and β orthogonal but not strongly orthogonal

$$\Rightarrow (\gamma \text{ is } \mathfrak{m}\text{-compact} \Leftrightarrow \gamma \text{ is } \mathfrak{g}\text{-noncompact}).$$

Thus let

$$\Delta_\beta = \{\text{roots for } C^{p_\beta}\} = \left\{\varepsilon \text{ root of } \left(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}}\right) \mid \langle \varepsilon, \beta \rangle = 0\right\},$$

and let γ be a C^{p_β}-simple root in Δ_β such that $\langle \gamma, \lambda \rangle = 0$. Our proof by induction will go through if we prove γ is $\mathfrak{m}$-noncompact. We expand γ in terms of the C-simple roots of $\Delta_{C,\lambda}$:

$$\gamma = \sum m_i \alpha_i, \quad \alpha_i \text{ simple in } \Delta_{C,\lambda}, \quad m_i \geq 0.$$

By assumption the C-simple roots in $\Delta_{C,\lambda}$ are all noncompact, and thus the "height" Σm_i determines whether γ is $\mathfrak{g}$-compact or $\mathfrak{g}$-noncompact.

Let ρ be half the sum of all the C-positive roots. It is well known that $2\langle \rho, \alpha_i \rangle / |\alpha_i|^2 = 1$ for α_i simple (see [11], p. 248). We introduce the dual root system (of co-roots)

$$a^\vee = \frac{2}{|\alpha|^2}\alpha,$$

and this relation becomes

$$\langle \rho, \alpha_i^\vee \rangle = 1 \quad \text{for } \alpha_i \text{ simple}.$$

If we dualize this equation, we obtain

$$\langle \rho^\vee, \alpha_i \rangle = 1 \quad \text{for } \alpha_i \text{ simple}. \tag{5.2}$$

Consequently

$$\langle \rho^\vee, \gamma \rangle = \left\langle \rho^\vee, \sum m_i \alpha_i \right\rangle = \sum m_i.$$

Since γ is $\mathfrak{m}$-simple,

$$\langle \gamma, \rho^\vee - (\rho_M)^\vee \rangle = \sum m_i - 1. \tag{5.3}$$

We consider the co-roots $\alpha^\vee$ contributing to the left side of (5.3). Each such is C-positive and nonorthogonal to β and γ. For any such $\alpha^\vee$, choose a sign $\pm$ so that $\pm p_\gamma \alpha^\vee$ is positive. We cannot have $\pm p_\gamma \alpha^\vee = \beta^\vee$ since γ and β are orthogonal, and therefore the simplicity of β implies that $\pm p_\beta p_\gamma \alpha^\vee$ is positive. We shall consider the set of positive co-roots

$$\left\{\alpha^\vee, p_\beta\alpha^\vee, \pm p_\gamma\alpha^\vee, \pm p_\beta p_\gamma\alpha^\vee\right\}. \tag{5.4}$$

The four co-roots in (5.4) are distinct unless $-p_\gamma\alpha^\vee$ is positive and $-p_\gamma\alpha^\vee = p_\beta\alpha^\vee$, in which case $-p_\beta p_\gamma\alpha^\vee = \alpha^\vee$ also. The exceptional case occurs when

$$\frac{2\langle\alpha^\vee,\gamma\rangle}{|\gamma|^2}\gamma - \alpha^\vee = \alpha^\vee - \frac{2\langle\alpha^\vee,\beta\rangle}{|\beta|^2}\beta,$$

and this condition simplifies to

$$\alpha = \frac{\langle\alpha,\gamma\rangle}{|\gamma|^2}\gamma = \frac{\langle\alpha,\beta\rangle}{|\beta|^2}\beta. \tag{5.5}$$

If α does not satisfy (5.5), then the four co-roots in (5.4) are distinct, and their contribution to the left side of (5.3) is

$$\frac{2}{|\alpha|^2}\cdot\frac{1}{2}\left[(\langle\gamma,\alpha\rangle + \langle\gamma,p_\beta\alpha\rangle) \pm (\langle\gamma,p_\gamma\alpha\rangle + \langle\gamma,p_\beta p_\gamma\alpha\rangle)\right],$$

which simplifies (when we move the reflections to the other side of the inner products) to

$$|\alpha|^{-2}[2\langle\gamma,\alpha\rangle \mp 2\langle\gamma,\alpha\rangle],$$

which is an even integer. Thus the roots α not satisfying (5.5) do not affect the parity of the left side of (5.3).

Now consider roots α satisfying (5.5). Since β and γ are orthogonal, equation (5.5) corresponds to recognizable situations in rank-two root systems. One possibility is B_2. Then $|\beta| = |\gamma|$.

If β and γ are long roots in B_2, then α (in order to be nonorthogonal to β and γ) is $\pm\frac{1}{2}\gamma \pm \frac{1}{2}\beta$, and the positive choices for α are $\frac{1}{2}\gamma \pm \frac{1}{2}\beta$. The contribution to the left side of (5.3) is

$$\begin{aligned}&\tfrac{1}{2}\langle\gamma, (\tfrac{1}{2}(\gamma-\beta))^\vee + (\tfrac{1}{2}(\gamma+\beta))^\vee\rangle \\ &\qquad = \frac{1}{2}\frac{2}{|\frac{1}{2}(\gamma-\beta)|^2}\left\langle\gamma, \frac{1}{2}(\gamma-\beta) + \frac{1}{2}(\gamma+\beta)\right\rangle = \frac{|\gamma|^2}{|\frac{1}{2}(\gamma-\beta)|^2} = 2,\end{aligned}$$

and the parity of the left side of (5.3) is not affected.

If β and γ are short in B_2, then α is $\pm\gamma\pm\beta$, and the positive choices for α are $\gamma\pm\beta$. The contribution to the left side of (5.3) is

$$\begin{aligned}&\tfrac{1}{2}\langle\gamma, (\gamma-\beta)^\vee + (\gamma+\beta)^\vee\rangle \\ &\qquad = \frac{1}{2}\frac{2}{|\gamma-\beta|^2}\langle\gamma, (\gamma-\beta) + (\gamma+\beta)\rangle = \frac{2|\gamma|^2}{|\gamma-\beta|^2} = 1,\end{aligned}$$

and the parity of the left side of (5.3) is reversed.

The remaining possibility is G_2. Let β and ε be the simple roots. The two cases are $\gamma = \beta + 2\varepsilon$, with height 3, and $\gamma = 3\beta + 2\varepsilon$, with height 5. In both cases, the left side of (5.3) is even.

Putting our computations together, we conclude from (5.3) that γ has odd height if and only if the orthogonal roots β and γ are strongly orthogonal. Since the α_i's contributing to the height of γ are all $\mathfrak{g}$-noncompact, γ is $\mathfrak{g}$-noncompact if and only if the orthogonal roots β and γ are strongly orthogonal. Combining this statement with (5.1), we see that γ is $\mathfrak{m}$-noncompact in every case. This completes the proof of (b′).

Preparation for proof of (c′). We shall use the theory in Section C of the appendix. We are assuming G is connected. Let us observe that the identity component of the center of G contributes only an exponential to a character on G, so that we may assume G is semisimple. Then without loss of generality, we may pass to a covering group and assume that $G^{\mathbf{C}}$ is simply-connected.

We shall make critical use of Schmid's Coherent Translation Theorem [26], given below as Theorem 5.2. Schmid observed [24, 8] that in the formula for $\Theta^G(\lambda, C)$ on each Cartan subgroup, the formula continues to make sense when λ is no longer C-dominant but $\lambda - \rho$ is still integral. The result is an invariant eigendistribution on G for every value of λ, and it will still be denoted $\Theta(\lambda, C)$.

THEOREM 5.2 (Schmid [26]). *Let G be linear connected semisimple, and suppose $\mathfrak{b} \subseteq \mathfrak{k}$ is a compact Cartan subalgebra of $\mathfrak{g}$. Let C be a Weyl chamber in $i\mathfrak{b}$, and let λ_1 be a nonsingular C-dominant form such that $\lambda_1 - \rho$ is integral. If α is a noncompact C-simple root, then*

$$\Theta(p_\alpha\lambda_1, C) = \Theta(\lambda_1, C) + \Theta_0,$$

where Θ_0 is a true character.

Proof of (c′). As we saw above, we may assume that G is contained in a simply connected complexification. We shall use freely the notation of Section C of the appendix. Suppose

$$\Theta(\lambda, C) = \Theta(\lambda', C') \neq 0 \tag{5.6}$$

for a C-dominant λ and a C'-dominant λ'. Choose w_0 in $W(B : G^{\mathbf{C}})$ so that $C' = w_0 C$. Then it follows from consideration of infinitesimal characters that $\lambda' = w_0\lambda$. We are to prove that w_0 is in $W(B : G)$.

Choose a dominant integral nonsingular form μ and put

$$\lambda_1 = \lambda + \mu \quad \text{and} \quad \lambda_1' = \lambda' + w_0\mu. \tag{5.7}$$

By definition,

$$
\begin{aligned}
\Theta(\lambda, C) &= \psi_{\lambda}^{\lambda_1}\Theta(\lambda_1, C), \\
\Theta(\lambda', C') &= \psi_{\lambda'}^{\lambda'_1}\Theta(\lambda'_1, C').
\end{aligned} \tag{5.8}
$$

Let α be a noncompact C-simple root. Comparing Schmid's definition of $\Theta(p_\alpha\lambda_1, C)$ with the statement of Theorem C.1 of the appendix, we see that $\Theta(p_\alpha\lambda_1, C)$ equals $p_\alpha\Theta(\lambda_1, C)$, in the notation of the appendix. Thus Theorem 5.2 translates as

$$p_\alpha\Theta(\lambda_1, C) = \Theta(\lambda_1, C) + \Theta_0, \tag{5.9}$$

with Θ_0 a true character. Applying ψ_α to both sides of (5.9) and using the identity $\psi_\alpha p_\alpha = \psi_\alpha$ of (C.8), we obtain

$$\psi_\alpha\Theta_0 = 0.$$

Consequently Lemma C.4 shows that

$$\langle\lambda, \alpha\rangle = 0 \quad \text{implies} \quad \Theta_0 \text{ is in } J_{\lambda}^{\lambda_1}. \tag{5.10}$$

In view of (5.10), equation (5.9) implies

$$\langle\lambda, \alpha\rangle = 0 \quad \text{implies} \quad p_\alpha\Theta(\lambda_1, C) \equiv \Theta(\lambda_1, C) \bmod J_{\lambda}^{\lambda_1}. \tag{5.11}$$

Let $W(\lambda)$ be the subgroup of $W(B : G^{\mathbf{C}})$ that fixes λ. In the notation of Theorem C.2, we shall prove that

$$w \in W(\lambda) \quad \text{implies} \quad w\Theta(\lambda_1, C) \equiv \Theta(\lambda_1, C) \bmod J_{\lambda}^{\lambda_1}. \tag{5.12}$$

Since λ is dominant, $W(\lambda)$ is a Weyl group generated by the simple reflections that it contains. We shall prove (5.12) by induction on the length $l(w)$. The group $W(\lambda)$ contains no simple reflections in compact roots, by the Hecht-Schmid identity (the "if" part of Theorem 1.1b). Thus the case $l(w) = 1$ is exactly handled by (5.11). Proceeding inductively, suppose (5.12) is known for w and suppose p_α is a simple reflection in $W(\lambda)$ with $l(p_\alpha w) > l(w)$. Then α is noncompact and we have

$$
\begin{aligned}
p_\alpha w\Theta(\lambda_1, C) &= p_\alpha\Theta(\lambda_1, C) + p_\alpha j && \text{by induction} \\
&= \Theta(\lambda_1, C) + j' + p_\alpha j && \text{by (5.11)} \\
&= \Theta(\lambda_1, C) + j'' && \text{by Theorem C.5.}
\end{aligned}
$$

Thus (5.12) follows.

Then we compute

$$\begin{aligned}\varphi_{\lambda_1}^{\lambda}\Theta(\lambda, C) &= \varphi_{\lambda_1}^{\lambda}\psi_{\lambda}^{\lambda_1}\Theta(\lambda_1, C) && \text{by (5.8)}\\ &= \sum_{w\in W(\lambda)} w\Theta(\lambda_1, C) && \text{by Theorem C.3}\\ &= |W(\lambda)|\,\Theta(\lambda_1, C) + j''' && (5.13)\end{aligned}$$

by (5.12). In view of (5.7), $\varphi_{\lambda_1}^{\lambda}$ and $\varphi_{\lambda_1'}^{\lambda'}$ are the same functor. Thus there is an equation analogous to (5.13) for $\varphi_{\lambda_1'}^{\lambda'}$ and (5.6) gives

$$|W(\lambda)|\,\Theta(\lambda_1, C) = |W(\lambda)|\,\Theta(\lambda_1', C') + j''''.$$

Decompose $j'''' = p - n$ into the difference of two true characters. Then we have

$$p + |W(\lambda)|\,\Theta(\lambda_1', C') = n + |W(\lambda)|\,\Theta(\lambda_1, C).$$

Every irreducible constituent of p and of n is in $J(\psi_{\lambda}^{\lambda_1})$, and the discrete series characters $\Theta(\lambda_1', C')$ and $\Theta(\lambda_1, C)$ are not in $J(\psi_{\lambda}^{\lambda_1})$. Since distinct irreducible characters are linearly independent, we conclude that

$$\Theta(\lambda_1', C') = \Theta(\lambda_1, C). \tag{5.14}$$

The characters in (5.14) are discrete series characters, and thus the element w_0 in $W(B : G^{\mathbf{C}})$ with $C' = w_0 C$ has to be in $W(B : G)$. We have seen that $\lambda' = w_0\lambda$. Thus the proof of Theorem 1.1 is complete.

Proof. If $\theta_2(w/\sqrt{r})\mu((w - y)/\varepsilon) \neq 0$, $|w| < 2\sqrt{r}/3$ and $|w - y| < \varepsilon < \sqrt{r}/6$. This is case (i). Assume now that $\theta_1(w/\sqrt{r})\mu(w, (w - y)/\varepsilon) \neq 0$. Since $\mu(w, u)$ is invariant under the map $(w, u) \to (gw, gu)$ for any orthogonal g by (23), the condition (i) (22) implies that $\langle u, w\rangle \geq 2|u||w|/3$ provided $\mu(w, u) \neq 0$. Hence, by our assumption, $\langle w - y, w\rangle \geq 2|w - y||w|/3$. Then

generalized Schmid identities to exhibit reducibility. There is an obstruction to inverting the identities, having to do with integrality.

THEOREM 6.1. *Let $P = MAN$ be a cuspidal parabolic subgroup of a linear connected reductive Lie group G with compact center. Let $\mathfrak{b} \subseteq \mathfrak{k} \cap \mathfrak{m}$ be a compact Cartan subalgebra of $\mathfrak{m}$, and let α be a real root of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}})$. Suppose that the Cayley transform $\mathbf{d}_\alpha$ leads from the data $(\mathfrak{m}, \mathfrak{a}, \mathfrak{b})$ to data $(\mathfrak{m}^*, \mathfrak{a}^*, \mathfrak{b}^*)$ and that $\tilde{\alpha} = \mathbf{d}_\alpha(\alpha)$. Then the character*

$$\operatorname{ind}_{P\cap M^*A^*}^{M^*A^*}\Theta^{MA}(\lambda_M, C_M, \chi_M, \nu_M)$$

is the right side of a generalized Schmid identity (4.4a) or (4.4b) *obtained from $\tilde{\alpha}$ if and only if $\langle \nu_M, \alpha\rangle = 0$ and*

$$\chi_M(\gamma_\alpha) = (-1)^{2\langle \rho_\alpha, \alpha\rangle/|\alpha|^2}, \tag{6.1}$$

where ρ_α is given by (4.2). When these conditions are satisfied, the Weyl chamber C in $i\mathfrak{b}^*$ can be taken to be any chamber such that

(i) $\lambda_M \oplus 0$ *is C-dominant,*

(ii) $(\bar{C} \cap i\mathfrak{b}) \cap C_M$ *has nonempty interior, and*

(iii) $\tilde{\alpha}$ *is C-positive.*

Proof. The necessity of the conditions is obvious, in view of the statement of Theorem 4.3. For the sufficiency, we are to construct

$$\Theta^{M^*A^*}(\lambda, C, \chi, \nu)$$

with the properties in Theorem 4.3. The definitions of λ, χ, and ν have to be

$$\lambda = \begin{cases} \lambda_M & \text{on } \mathfrak{b} \\ 0 & \text{on } H_{\tilde{\alpha}}, \end{cases}$$

$$\chi = \chi_M|_{Z_{M^*}}, \quad \text{and} \quad \nu = \nu_M|_{\mathfrak{a}^*}.$$

The sufficiency will then be proved if we show that

(a) any chamber C satisfying (i), (ii), and (iii) has $\tilde{\alpha}$ simple and $C_M = C^{p_{\tilde{\alpha}}}$,

(b) there exists a chamber C satisfying (i), (ii), and (iii),

(c) $\lambda - \rho_{M^*}$ is $\mathfrak{b}^*$-integral, and $e^{\lambda-\rho_{M^*}}$ agrees with χ on $Z_{M^*} \cap B^*$.

Note that ζ and χ_M will have consistent values on γ_α by the assumption (6.1).

First we prove (a). For each root $\tilde{\beta}$ of $(\mathfrak{m}^{*\mathbf{C}}, \mathfrak{b}^{*\mathbf{C}})$ that is not identically zero on $\mathfrak{b}$ (i.e., $\tilde{\beta} \neq \pm\tilde{\alpha}$), $\ker\tilde{\beta} \cap i\mathfrak{b}$ has empty interior in $i\mathfrak{b}$. Hence there is an open dense set C_M' in C_M on which $\tilde{\beta}$ is nonvanishing for every root $\tilde{\beta}$ of $(\mathfrak{m}^{*\mathbf{C}}, \mathfrak{b}^{*\mathbf{C}})$ other than $\pm\tilde{\alpha}$. By (ii), $(\bar{C} \cap i\mathfrak{b}) \cap C_M'$ is nonempty; let $H_{i\mathfrak{b}}$ be a member of this set. Since $H_{i\mathfrak{b}}$ is in $\bar{C}$, $\tilde{\beta}(H_{i\mathfrak{b}})$ is ≥ 0 for all positive roots $\tilde{\beta} \neq \tilde{\alpha}$. Since $\tilde{\beta}(H_{i\mathfrak{b}}) \neq 0$ for all roots $\tilde{\beta} \neq \pm\tilde{\alpha}$, we actually have $\tilde{\beta}(H_{i\mathfrak{b}}) > 0$ for all positive roots $\tilde{\beta} \neq \tilde{\alpha}$.

Now let $\tilde{\beta} \neq \tilde{\alpha}$ be > 0 and form $p_{\tilde{\alpha}}\tilde{\beta}$. We have

$$p_{\tilde{\alpha}}\tilde{\beta}\,(H_{i\mathfrak{b}}) = \tilde{\beta}\,(H_{i\mathfrak{b}}) > 0,$$

and hence $p_{\tilde{\alpha}}\tilde{\beta} > 0$. Thus the only positive root sent into a negative root by $p_{\tilde{\alpha}}$ is $\tilde{\alpha}$, and $\tilde{\alpha}$ must be simple.

In addition, $H_{i\mathfrak{b}} + \varepsilon H_{\tilde{\alpha}}$ must be in C if $\varepsilon > 0$ is sufficiently small. Then $H_{i\mathfrak{b}}$ is in $\operatorname{Proj}_{i\mathfrak{b}} C$ (in the notation of (4.2)) and is in $C_M' \subseteq C_M$. Since $C_M \cap \operatorname{Proj}_{i\mathfrak{b}} C$ is not empty, it follows that $\operatorname{Proj}_{i\mathfrak{b}} C \subseteq C_M$ and that $C^{p_{\tilde{\alpha}}} = C_M$. This proves (a).

For (b), we choose C by means of Lemma 3.1. Then (ii) of the lemma yields (i) here, and (i) and (iii) of the lemma yield (ii) here. If (iii) here fails, we replace C by $p_{\tilde{\alpha}}\, C$, and then (iii) will be valid.

For (c), the ordering that defines ρ_{M^*} is at our disposal. Form roots of $(\mathfrak{m}^{*\mathbf{C}}, (\mathbf{R}H_\alpha + \mathfrak{b})^{\mathbf{C}})$, order $\mathbf{R}H_\alpha$ before $i\mathfrak{b}$, and transfer the resulting positive system to roots of $(\mathfrak{m}^{*\mathbf{C}}, \mathfrak{b}^{*\mathbf{C}})$ by the Cayley transform $\mathbf{d}_\alpha$. Then we obtain

$$\rho_{M^*} = \rho_{\tilde{\alpha}} + \rho_M,$$

where $\rho_{\tilde{\alpha}} = \mathbf{d}_\alpha(\rho_\alpha)$.

Let $B_{\tilde{\alpha}} = \exp i\mathbf{R}H_{\tilde{\alpha}}$; this is a circle group in $SL(2, \mathbf{R})$ built from the real root α. We then have $B^* = BB_{\tilde{\alpha}}$ on the group level and

$$\lambda - \rho_{M^*} = (\lambda_M - \rho_M) + (-\rho_{\tilde{\alpha}}) \tag{6.2}$$

on the Lie algebra. It is implicit in the existence of Θ^{MA} that $\lambda_M - \rho_M$ exponentiates to B. Thus $\lambda - \rho_{M^*}$ is $\mathfrak{b}^*$-integral if and only if

(c′) $-\rho_{\tilde{\alpha}}$ exponentiates to $B_{\tilde{\alpha}}$ and

(c″) $e^{\lambda_M - \rho_M}$ and $e^{-\rho_{\tilde{\alpha}}}$ agree on $B \cap B_{\tilde{\alpha}}$.

In (c), we are to prove also that $e^{\lambda - \rho_{M^*}}$ agrees with χ on $Z_{M^*} \cap B^*$. By Lemma 2.1b, we have

$$Z_{M^*} \cap B^* \subseteq Z_M \cap M_0^* = \{1, \gamma_\alpha\}(Z_M \cap B).$$

It is implicit in the existence of Θ^{MA} that $e^{\lambda_M - \rho_M}$ agrees with χ_M on $Z_M \cap B$. If (c′) and (c″) hold, then (6.2) shows that $e^{\lambda - \rho_{M^*}}$ agrees with χ_M on $Z_M \cap B$. Also

$$\begin{aligned} e^{\lambda - \rho_{M^*}}(\gamma_\alpha) &= \exp 2\pi i\,|\tilde{\alpha}|^{-2}\langle \lambda - \rho_{M^*}, \tilde{\alpha}\rangle \\ &= \exp 2\pi i\,|\tilde{\alpha}|^{-2}\langle -\rho_{\tilde{\alpha}}, \tilde{\alpha}\rangle && \text{by (6.2)} \\ &= \zeta(\gamma_\alpha) && \text{since } \mathbf{d}_\alpha \text{ is unitary} \\ &= \chi_M(\gamma_\alpha). && \text{by (6.1).} \end{aligned}$$

Hence $e^{\lambda - \rho_{M^*}}$ agrees with χ on $Z_{M^*} \cap B^*$. Thus (c) will follow if we prove (c′) and (c″).

In (c′), $\rho_{\tilde{\alpha}}$ is a half-integral multiple of $\tilde{\alpha}$, and $\tilde{\alpha}$ exponentiates to B^*, hence to $B_{\tilde{\alpha}}$, since $\tilde{\alpha}$ is a root. Thus the only way that $\rho_{\tilde{\alpha}}$ can fail to exponentiate to $B_{\tilde{\alpha}}$ is if both

$$\rho_{\tilde{\alpha}} = (n + \tfrac{1}{2})\tilde{\alpha} \quad \text{for some integer } n$$

and

$$\gamma_\alpha = 1,$$

and these two equations together are incompatible with (6.1). This proves (c′).

In (c″), the most general element of $B_{\tilde{\alpha}}$ is $\exp(i\theta\,|\tilde{\alpha}|^{-2}H_{\tilde{\alpha}})$, and we have

$$\mathrm{Ad}(\exp i\theta\,|\tilde{\alpha}|^{-2}H_{\tilde{\alpha}})E_{\tilde{\alpha}} = e^{i\theta}E_{\tilde{\alpha}}.$$

Since $\mathrm{Ad}(b)$ fixes $E_{\tilde{\alpha}}$ for b in B, the only elements of $B_{\tilde{\alpha}}$ that can be in B have θ a multiple of 2π. That is

$$B \cap B_{\tilde{\alpha}} \subseteq \{1, \gamma_\alpha\}.$$

If γ_α is in $B \cap B_{\tilde{\alpha}}$, then $e^{\lambda_M - \rho_M}$ and $e^{-\rho_{\tilde{\alpha}}}$ agree on γ_α by (6.1). This proves (c″). The proof of Theorem 6.1 is complete.

7. Plancherel factors

Let $P = MAN$ be the Langlands decomposition of a cuspidal parabolic subgroup in the linear group G. For any irreducible unitary representation ξ of M and character $e^{i\nu}$ of A, we have associated in Section 7 of [16] various "η-factors" to the induced representation

$$U_P(\xi, i\nu) = \mathrm{ind}_P^G(\xi \otimes e^{i\nu}).$$

These η-factors arise from intertwining operators in a way that will not concern us at present, and they are related to the reducibility of $U_P(\xi, i\nu)$. We shall study an η-factor $\eta_{\xi,\alpha}(i\nu)$ attached to each (positive) reduced $\mathfrak{a}$-root α.

The formula for $\eta_{\xi,\alpha}(i\nu)$ in [16] is fairly complicated,[8] and it is our intention to simplify it when ξ is a discrete series or limit of discrete series representation. The final result is Proposition 7.1. Let us recall the formula in [16]; it is not evident that it depends only on ξ, ν, and α. Let $\mathfrak{a}_M$ be a maximal abelian subspace of $\mathfrak{m} \cap \mathfrak{p}$, and let $\mathfrak{a}_p = \mathfrak{a} \oplus \mathfrak{a}_M$. Choose a positive system of $\mathfrak{a}_p$-roots so that $\mathfrak{a}$ comes before $\mathfrak{a}_M$, and form the corresponding minimal parabolic subgroups $M_pA_pN_p$ of G and $M_pA_MN_M$ of M, where $N_p = NN_M$. The representation ξ imbeds as a subrepresentation of a nonunitary principal series representation of M, say the one with parameters (σ, λ_M), where σ is an irreducible unitary representation of M_p and λ_M is a member of $\mathfrak{a}_M'$. Then the formula in [16] given by (7.7) and Theorem 7.6(v) is

$$\eta_{\xi,\alpha}(i\nu) = \prod_{\substack{\beta = \text{reduced } \mathfrak{a}_p\text{-root} \\ \beta|_{\mathfrak{a}} = c\alpha \quad \text{with } c>0}} \eta_{\sigma,\beta}(\lambda_M \oplus i\nu). \tag{7.1}$$

Here, apart from a multiplicative constant independent of ξ, $i\nu$, σ, and λ_M, we have

$$\eta_{\sigma,\beta}(\lambda_M \oplus i\nu)^{-1} = p_{\sigma,\beta}(\lambda_M \oplus i\nu), \tag{7.2}$$

where the product of the degree of σ and the function $p_{\sigma,\beta}$ gives a multiple of the

[8] In [16], the actual notation was $\eta^{(\alpha)}(\theta P^{(\alpha)} : P^{(\alpha)} : \xi : i\nu|_{\mathfrak{a}^{(\alpha)}})$.

Plancherel density for the σ-principal series in the Plancherel formula for a group $G^{(\beta)}$. See Propositions 7.3, 7.4, and 10.1 of [16].

When ξ is a discrete series representation, $\eta_{\xi,\alpha}$ itself is related to a Plancherel formula; namely, the product of the formal degree of ξ by the function $\eta_{\xi,\alpha}^{-1}$ gives a multiple of the Plancherel density for the ξ-induced series in the Plancherel formula for a group $G^{(\alpha)}$. This interpretation breaks down when ξ is a limit of discrete series representation because the formal degree of ξ is 0, but $\eta_{\xi,\alpha}$ is still defined. Accordingly we define[9] a *Plancherel factor* $\mu_{\xi,\alpha}(i\nu)$ by

$$\mu_{\xi,\alpha}(i\nu) = \eta_{\xi,\alpha}(i\nu)^{-1}. \tag{7.3}$$

Then (7.1) and (7.2) give

$$\mu_{\xi,\alpha}(i\nu) = \prod_{\substack{\beta \text{ reduced } \mathfrak{a}_p\text{-root} \\ \beta|_{\mathfrak{a}} = c\alpha \quad \text{with } c>0}} p_{\sigma,\beta}(\lambda_M \oplus i\nu). \tag{7.4}$$

Formulas for $p_{\sigma,\beta}(\omega)$ were derived by several authors, and the results and references are compiled in Section 12 of [15]. We shall need the formulas and hence reproduce them here. Let $\mathfrak{b}_p \subseteq \mathfrak{m}_p$ be a maximal abelian subspace. Then $\mathfrak{a}_p + \mathfrak{b}_p$ is a Cartan subalgebra of $\mathfrak{g}$. Form roots of $(\mathfrak{g}^{\mathbb{C}}, (\mathfrak{a}_p + \mathfrak{b}_p)^{\mathbb{C}})$, introduce a positive system so that $\mathfrak{a}_p$ comes before $\mathfrak{b}_p$, let ρ_{M_p} be half the sum of the positive roots that vanish on $\mathfrak{a}_p$, and let Λ^- be the highest weight of σ. There are two cases, each involving a product over roots ε of $(\mathfrak{g}^{\mathbb{C}}, (\mathfrak{a}_p + \mathfrak{b}_{\mathbb{C}})^{\mathbb{C}})$:

(a) If β is an even $\mathfrak{a}_p$-root, then

$$p_{\sigma,\beta}(\omega) = \prod_{\varepsilon|_{\mathfrak{a}_p} = \beta} \langle \omega \oplus (\Lambda^- + \rho_{M_p}), \varepsilon \rangle. \tag{7.5a}$$

(b) If β is an odd $\mathfrak{a}_p$-root, and $\beta' = 2\beta$ or β according to whether 2β is an $\mathfrak{a}_p$-root or not, then

$$p_{\sigma,\beta}(\omega) = \left(\prod_{\varepsilon|_{\mathfrak{a}_p} = \beta \text{ or } 2\beta} \langle \omega \oplus (\Lambda^- + \rho_{M_p}), \varepsilon \rangle \right) f_{\sigma,\beta'}(\omega), \tag{7.5b}$$

where

$$f_{\sigma,\beta'}(\omega) = \begin{cases} i\tan(\pi\langle \omega, \beta' \rangle / |\beta'|^2) & \text{if } \sigma(\gamma_{\beta'}) = -(-1)^{2\langle \rho_\beta, \beta' \rangle / |\beta'|^2} \\ -i\cot(\pi\langle \omega, \beta' \rangle / |\beta'|^2) & \text{if } \sigma(\gamma_{\beta'}) = +(-1)^{2\langle \rho_\beta, \beta' \rangle / |\beta'|^2}. \end{cases} \tag{7.5c}$$

When ξ is in the discrete series, (7.4) has to simpify to the formula in Proposition 7.1 below because of the connection of $\mu_{\xi,\alpha}$ to the Plancherel formula

[9] This definition is off from the one in [16] for discrete series by a nonzero multiplicative constant independent of ξ and ν; this constant need not be positive.

and because of the results of [7]. The content of the proposition is that this result persists when ξ is a limit of discrete series representation.

PROPOSITION 7.1. *Let the discrete series or limit of discrete series representation ξ have character $\Theta^M(\lambda, C, \chi)$, and let α be a reduced $\mathfrak{a}$-root. Then $\mu_{\xi,\alpha}(i\nu)$ is given by two cases, each involving a product over roots ε of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a}+\mathfrak{b})^{\mathbf{C}})$:*

(a) *If α is an even $\mathfrak{a}$-root, then*

$$\mu_{\xi,\alpha}(i\nu) = \prod_{\varepsilon|_{\mathfrak{a}}=\alpha} \langle \lambda + i\nu, \varepsilon\rangle. \tag{7.6a}$$

(b) *If α is an odd $\mathfrak{a}$-root and $\alpha' = 2\alpha$ or α according to whether 2α is an $\mathfrak{a}$-root or not, then*

$$\mu_{\xi,\alpha}(i\nu) = \left(\prod_{\varepsilon|_{\mathfrak{a}}=c\alpha,\, c>0} \langle \lambda + i\nu, \varepsilon\rangle\right) f_{\xi,\alpha'}(i\nu), \tag{7.6b}$$

where

$$f_{\xi,\alpha'}(i\nu) = \begin{cases} i\tan\left(\dfrac{\pi\langle i\nu, \alpha'\rangle}{|\alpha'|^2}\right) = -\tanh\left(\dfrac{\pi\langle \nu, \alpha'\rangle}{|\alpha'|^2}\right) \\ \qquad\qquad \textit{if } \chi(\gamma_{\alpha'}) = -(-1)^{2\langle\rho_\alpha, \alpha'\rangle/|\alpha'|^2} \\ -i\cot\left(\dfrac{\pi\langle i\nu, \alpha'\rangle}{|\alpha'|^2}\right) = -\coth\left(\dfrac{\pi\langle \nu, \alpha'\rangle}{|\alpha'|^2}\right) \\ \qquad\qquad \textit{if } \chi(\gamma_{\alpha'}) = +(-1)^{2\langle\rho_\alpha, \alpha'\rangle/|\alpha'|^2}. \end{cases} \tag{7.6c}$$

The proof requires some preparation. Let us observe that we have at our disposal how $\mathfrak{a}_M$ is chosen to extend $\mathfrak{a}$ to $\mathfrak{a}_p$ and how $\mathfrak{b}_p$ is chosen to extend $\mathfrak{a}_p$ to $\mathfrak{a}_p \oplus \mathfrak{b}_p$, together with the orderings on $\mathfrak{a}_M$ and $\mathfrak{a}_M \oplus \mathfrak{b}_p$. The positive system for $\mathfrak{b} \subseteq \mathfrak{k} \cap \mathfrak{m}$ is determined by C (given in the notation for the character of ξ). Then in addition we have the imbedding $\xi \hookrightarrow (\sigma, \lambda_M)$ at our disposal. The choices that we make will be those dictated by the construction in [17], and we now describe that construction.

We thus start with λ and the positive system of noncompact roots for $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$. Following [17], we choose any "fundamental sequence" of positive noncompact roots $\tilde{\delta}_1, \ldots, \tilde{\delta}_m$ of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$. This sequence is strongly orthogonal and leads to a definition of $\mathfrak{a}_M$. The Cayley transforms $\mathbf{c}_{\tilde{\delta}_1}, \ldots, \mathbf{c}_{\tilde{\delta}_m}$ commute, and the product $\mathbf{c} = \mathbf{c}_{\tilde{\delta}_1} \cdot \ldots \cdot \mathbf{c}_{\tilde{\delta}_m}$ enables us to define $\mathfrak{a}_M$ and $\mathfrak{b}_p$ by

$$\mathfrak{a}_M = \mathbf{c}\left(\sum_{j=1}^m \mathbf{R} H_{\tilde{\delta}_j}\right)$$

and

$$\mathfrak{b}_p = \text{orthocomplement in } \mathfrak{b} \text{ of } \sum_{j=1}^{m} i\mathbf{R}H_{\tilde{\delta}_j}.$$

For the positive system of roots of $(\mathfrak{m}^{\mathbf{C}}, (\mathfrak{a}_M + \mathfrak{b}_p)^{\mathbf{C}})$, we say that $\varepsilon > 0$ if either

$$\langle \varepsilon, \tilde{\delta}_1 \rangle = \cdots = \langle \varepsilon, \tilde{\delta}_{j-1} \rangle = 0 \quad \text{and} \quad \langle \varepsilon, \tilde{\delta}_j \rangle > 0 \quad \text{for some } j$$

or

$$\langle \varepsilon, \tilde{\delta}_1 \rangle = \cdots = \langle \varepsilon, \tilde{\delta}_m \rangle = 0 \quad \text{and} \quad \mathbf{c}^{-1}(\varepsilon) > 0 \quad \text{for } (\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}}).$$

Let ρ_M^+ be half the sum of the positive roots of $(\mathfrak{m}^{\mathbf{C}}, (\mathfrak{a}_M + \mathfrak{b}_p)^{\mathbf{C}})$. The images $\mathbf{c}(\tilde{\delta}_j)$ under the Cayley transform will be denoted δ_j.

Roots of $(\mathfrak{g}, \mathfrak{a})$ are restrictions to $\mathfrak{a}$ or $\mathfrak{a}^{\mathbf{C}}$ of roots of $(\mathfrak{g}, \mathfrak{a}_p)$, which in turn are restrictions of roots of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a}_p \oplus \mathfrak{b}_p)^{\mathbf{C}})$.

LEMMA 7.2. *The discrete series or limit of discrete series representation ξ of M has an imbedding (infinitesimally) as a subrepresentation of the nonunitary principal series of M with parameters σ on M_p and λ_M on $\mathfrak{a}_M$ such that each $\tilde{\delta} = \tilde{\delta}_j$ satisfies*

$$\frac{2\langle \lambda_M, \delta \rangle}{|\delta|^2} = \frac{2\langle \lambda, \tilde{\delta} \rangle}{|\tilde{\delta}|^2} \tag{7.7}$$

and

$$\sigma(\gamma_\delta) = (-1)^{2\langle \lambda_M - \rho_M^+, \delta \rangle / |\delta|^2}. \tag{7.8}$$

Proof. We apply the results of [17] to M_0. Theorem 10.8 of [17a] and the proof of Theorem A of [17b] apply as long as the Blattner parameter of ξ (there called λ, here to be called λ_B) is integral and $K \cap M_0$ dominant and the Harish-Chandra parameter of ξ (there called Λ, here called λ) is M_0 dominant. We need to check that $\lambda_B = \lambda - \rho_k + \rho_n$ is $K \cap M_0$ dominant if ρ_k and ρ_n are the half sums of the positive compact and noncompact roots of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$, respectively.

Write $\lambda_B = \lambda + \rho - 2\rho_k$, and let ε be a simple root of $((\mathfrak{k} \cap \mathfrak{m})^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$. If ε is simple for $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$, then Theorem 1.1b says $2\langle \lambda, \varepsilon \rangle / |\varepsilon|^2 \geq 1$. Hence

$$\frac{2\langle \lambda_B, \varepsilon \rangle}{|\varepsilon|^2} = \frac{2\langle \lambda, \varepsilon \rangle}{|\varepsilon|^2} + 1 - 2 \geq 0.$$

If ε is not simple for $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$, then $2\langle \rho, \varepsilon \rangle / |\varepsilon|^2 \geq 2$, and hence

$$\frac{2\langle \lambda_B, \varepsilon \rangle}{|\varepsilon|^2} \geq 0 + 2 - 2 \geq 0.$$

Thus λ_B is $K \cap M_0$ dominant.

The results of [17] therefore give us an imbedding of ξ as a quotient of a nonunitary principal series representation. The $\mathfrak{a}$ parameter of the imbedding is there called $\rho^+ - \nu$ with $\tilde{\delta} = \tilde{\delta}_j$ satisfying

$$\frac{2\langle \nu, \delta \rangle}{|\delta|^2} = \frac{2\langle \lambda_B, \tilde{\delta} \rangle}{|\tilde{\delta}|^2} + 2n_j \qquad \text{by (6.5a) of [17a],}$$

and

$$2n_j = \frac{2\langle \rho^+, \delta \rangle}{|\delta|^2} + \frac{2\langle \rho_k - \rho_n, \tilde{\delta} \rangle}{|\tilde{\delta}|^2}$$

by Lemma 8.5 of [17a].

Thus

$$\frac{2\langle \rho^+ - \nu, \delta \rangle}{|\delta|^2} = -\frac{2\langle \lambda_B + \rho_k - \rho_n, \tilde{\delta} \rangle}{|\tilde{\delta}|^2} = -\frac{2\langle \lambda, \tilde{\delta} \rangle}{|\tilde{\delta}|^2}$$

for the quotient. The $\mathfrak{a}$ parameter for a subrepresentation is the negative of this, and (7.7) follows for M_0.

According to Theorem C of [17b], the $(M_0)_p$ parameter σ_0 of the corresponding quotient imbedding or subrepresentation satisfies

$$\sigma_0(\gamma_\delta) = (-1)^{2\langle \lambda_B, \tilde{\delta} \rangle / |\tilde{\delta}|^2}. \tag{7.9}$$

Now $\lambda_B = \lambda - \rho_k + \rho_n$, and Lemma 8.5 of [17a] gives

$$\frac{2\langle \rho_M^+, \delta \rangle}{|\delta|^2} = 1 + m_j + n_j,$$

$$\frac{2\langle \rho_k - \rho_n, \tilde{\delta} \rangle}{|\tilde{\delta}|^2} = -1 - m_j + n_j$$

for certain integers m_j and n_j. Then

$$\frac{2\langle -\rho_k + \rho_n, \tilde{\delta} \rangle}{|\tilde{\delta}|^2} \equiv \frac{2\langle \rho_M^+, \delta \rangle}{|\delta|^2} \equiv -\frac{2\langle \rho_M^+, \delta \rangle}{|\delta|^2} \bmod 2,$$

and (7.8) follows for σ_0 and M_0, by applying (7.7).

Now we pass to M. According to the proof of Theorem 5.4 of [16], ξ imbeds infinitesimally as a subrepresentation in a nonunitary principal series representation of M with parameters (σ, λ_M), where σ is some irreducible constituent of $\mathrm{ind}_{M_p \cap M_0}^{M_p} \sigma_0$. Thus (7.7) is still valid, and (7.8) is valid for σ since it is valid for σ_0 and since γ_δ is central in M_p.

LEMMA 7.3. *In the notation of Lemma* 7.2, *the scalars* $\xi(z)$ *and* $\sigma(z)$ *match for* z *in* Z_M.

Proof. The result reflects a well known property of induced representations.

In proving Proposition 7.1, we are to combine (7.4) and (7.5) and simplify the result to obtain (7.6). When we combine (7.4) and (7.5), we obtain a polynomial part and a tangent/cotangent part. The polynomial part is

$$\prod_{\substack{\beta \text{ reduced} \\ \beta|_{\mathfrak{a}}=c\alpha,\, c>0}} \prod_{\substack{\varepsilon|_{\mathfrak{a}_p}=c'\beta \\ c'>0}} \langle \lambda_M + i\nu + \Lambda^- + \rho_{M_p}, \varepsilon\rangle \tag{7.10}$$

$$= \prod_{\varepsilon|_{\mathfrak{a}}=c\alpha,\, c>0} \langle \lambda_M + i\nu + \Lambda^- + \rho_{M_p}, \varepsilon\rangle.$$

Here $\lambda_M + \Lambda^- + \rho_{M_p}$ is a parameter for the infinitesimal character of

$$\operatorname{ind}_{M_pA_MN_M}^{M}(\sigma \otimes e^{\lambda_M} \otimes 1), \tag{7.11}$$

and λ is a parameter for the infinitesimal character of ξ, which is a subrepresentation of (7.11). Hence

$$\lambda_M + \Lambda^- + \rho_{M_p} = w\mathbf{c}(\lambda)$$

for some w in $W(A_MB_p : (MA)^{\mathbf{C}})$. Thus (7.10) is

$$= \prod_{\varepsilon|_{\mathfrak{a}}=c\alpha,\, c>0} \langle w\mathbf{c}(\lambda) + i\nu, \varepsilon\rangle = \prod_{\varepsilon'|_{\mathfrak{a}}=c\alpha,\, c>0} \langle \mathbf{c}(\lambda) + i\nu, \varepsilon'\rangle.$$

If we reinterpret the roots ε' on the right as roots of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a}+\mathfrak{b})^{\mathbf{C}})$, then we may replace $\mathbf{c}(\lambda)$ by λ, and the right side matches the polynomial part of (7.6). In other words, the problem in Proposition 7.1 is to show that

$$\prod_{\substack{\beta \text{ reduced, } \beta \text{ odd} \\ \beta|_{\mathfrak{a}}=c\alpha,\, c>0}} f_{\sigma,\beta'}(\lambda_M + i\nu) = \begin{cases} f_{\xi,\alpha'}(i\nu) & \text{if } \alpha \text{ is odd} \\ 1 & \text{if } \alpha \text{ is even.} \end{cases} \tag{7.12}$$

The proof of (7.12) is the heart of the proof of Proposition 7.1. Ultimately we shall divide matters into a number of cases depending on α and the β's that restrict to it. Until the end, we assume that α does not arise as the reduced $\mathfrak{a}$-root of a maximal parabolic of the exceptional group split G_2, or as a version of this situation imbedded in a non-simple reductive group.

Define

$$p_{+1}(z) = i\tan\tfrac{1}{2}\pi z \quad \text{and} \quad p_{-1}(z) = -i\cot\tfrac{1}{2}\pi z$$

for z in $\mathbf{C}$. These functions jointly satisfy the identities

$$p_{+1}(z)p_{-1}(z) = 1 \tag{7.13}$$

and

$$p_{\pm 1}(z) = p_{\pm(-1)^n}(n+z) \quad \text{for } n \text{ in } \mathbf{Z}. \tag{7.14}$$

LEMMA 7.4. *Let α be a reduced $\mathfrak{a}$-root other than for a maximal parabolic in split G_2. Suppose $\beta = \alpha + \alpha_I$ is an extension to an $\mathfrak{a}_p$-root, and define $\bar\beta = \alpha - \alpha_I$. If $\alpha_I \neq 0$, then there are only the following possibilities for α and β.*

(a) *α is even, and $2\langle \beta, \bar\beta\rangle/|\beta|^2 = +1$. Then $\beta = \alpha \pm \frac{1}{2}\delta$ for some $\delta = \delta_j$ with $|\delta| = |\beta|$ and $2\langle \rho_M^+, \delta\rangle/|\delta|^2$ odd.*

(b) *α is even, and $2\langle \beta, \bar\beta\rangle/|\beta|^2 = 0$. Then $\beta = \alpha \pm \frac{1}{2}\delta' \pm \frac{1}{2}\delta$ for some $\delta' = \delta_i$ and $\delta = \delta_j$ with $i < j$, $2\langle \rho_M^+, \delta\rangle/|\delta|^2$ odd, and $|\delta'| = |\delta| = |\beta|$.*

(c) *α is odd, α is a root, and $2\langle \beta, \bar\beta\rangle/|\beta|^2 = 0$. Then either*

(1) *$\beta = \alpha \pm \delta$ for some $\delta = \delta_j$ with $2\langle \rho_M^+, \delta\rangle/|\delta|^2$ odd and $2|\delta|^2 = |\beta|^2$, or*

(2) *$\beta = \alpha \pm \frac{1}{2}\delta' \pm \frac{1}{2}\delta$ for some $\delta' = \delta_i$ and $\delta = \delta_j$ with $i < j$, $2\langle \rho_M^+, \delta\rangle/|\delta|^2$ odd, and $|\delta'| = |\delta| = |\beta|$.*

(d) *α is odd, 2α is a root, and $2\langle \beta, \bar\beta\rangle/|\beta|^2 = 0$. Then either*

(1) *$\beta = \alpha \pm \frac{1}{2}\delta$ for some $\delta = \delta_i$ with $|\delta|^2 = |2\alpha|^2 = 2|\beta|^2$, or*

(2) *$\beta = \alpha \pm \frac{1}{2}\delta' \pm \frac{1}{2}\delta$ for some $\delta' = \delta_i$ and $\delta = \delta_j$ with $i < j$, $2\langle \rho_M^+, \delta\rangle/|\delta|^2$ odd, and $|\delta'| = |\delta| = |\beta|$.*

(e) *α is odd, 2α is a root, and $2\langle \beta, \bar\beta\rangle/|\beta|^2 = -1$. Then*

(1) *$\beta = \alpha \pm \frac{1}{2}\delta \pm \frac{1}{2}\delta'$ for some $\delta' = \delta_i$ and $\delta = \delta_j$ with $2\langle \rho_M^+, \delta\rangle/|\delta|^2$ odd and $|\delta'|^2 = 2|\delta|^2 = 2|\beta|^2$, or*

(2) *$\beta = \alpha \pm \frac{1}{2}\delta' \pm \delta$ for some $\delta' = \delta_i$ and $\delta = \delta_j$ with $2\langle \rho_M^+, \delta\rangle/|\delta|^2$ odd and $|\delta'|^2 = 2|\delta|^2 = |\beta|^2$, or*

(3) *$\beta = \alpha \pm \frac{1}{2}\delta'' \pm \frac{1}{2}\delta' \pm \frac{1}{2}\delta$ for some $\delta'' = \delta_l$, $\delta' = \delta_i$, and $\delta = \delta_j$ with $l < i < j$, $2\langle \rho_M^+, \delta\rangle/|\delta|^2$ odd, and $|\delta''| = |\delta'| = |\delta| = |\beta|$.*

Moreover, β has multiplicity one in cases (a), (b), and (c).

Proof. For $n = +1$, 0, or -1, we have

$$n = \frac{2\langle \beta, \bar\beta\rangle}{|\beta|^2} = \frac{2(|\alpha|^2 - |\alpha_I|^2)}{|\beta|^2} = \frac{2|\beta|^2 - 4|\alpha_I|^2}{|\beta|^2}$$

and so

$$4|\alpha_I|^2 = (2-n)|\beta|^2. \tag{7.15}$$

Since

$$\alpha_I = \sum_{j=1}^{m} \frac{\langle \beta, \delta_j\rangle}{|\delta_j|^2}\delta_j,$$

we have

$$|\alpha_I|^2 = \sum_{j=1}^{m} \frac{\langle \beta, \delta_j \rangle^2}{|\delta_j|^2},$$

and thus (7.15) implies

$$\sum_{j=1}^{m} \frac{4\langle \beta, \delta_j \rangle^2}{|\delta_j|^2} = 2 - n. \tag{7.16}$$

The partitioning of the possibilities according to the first sentences of (a) through (e) is by [12] and Lemma 2.7 of [13], and then (7.16) shows the form of α_I in each case in terms of the δ_j's, including the lengths. We have to show that $2\langle \rho_M^+, \delta \rangle / |\delta|^2$ is odd in all cases except (d1) and that β has the stated multiplicities.

In (a), Proposition 10a of [12] shows that δ_j is a simple $\mathfrak{a}_p$-root. Lemma 2.3 of [13] shows that β and δ_j have multiplicity one as $\mathfrak{a}_p$-roots. Thus

$$2\langle \rho_M^+, \delta_j \rangle / |\delta_j|^2 = 1.$$

Let us consider (b) and (c2) at once. For the statement about multiplicities, suppose $\gamma \neq 0$ in $\mathfrak{b}_p'$ makes $\beta \pm \gamma$ a root. Then

$$\langle \beta + \gamma, \bar{\beta} - \gamma \rangle < \langle \beta, \bar{\beta} \rangle = 0,$$

and $(\beta + \gamma) + (\bar{\beta} - \gamma) = 2\alpha$ is a root, a contradiction. Hence β has multiplicity one. Next, we consider the parity of

$$2\langle \rho_M^+, \delta_j \rangle / |\delta_j|^2. \tag{7.17}$$

Let $\varepsilon > 0$ be an $\mathfrak{a}_p$-root contributing to ρ_M^+ such that $\langle \varepsilon, \delta_j \rangle = 0$. Any effect of ε is determined by $\langle \varepsilon, \delta_k \rangle$ for some $k \leq j$. If $k < j$, then ε and $p_{\delta_j}\varepsilon$ are both > 0, and $\langle \varepsilon + p_{\delta_j}\varepsilon, \delta_j \rangle = 0$. Thus (7.17) is affected only by δ_j itself (which contributes an odd integer) and by $\mathfrak{a}_p$-roots

$$\varepsilon = +\delta_j + \sum_{k>j} c_k \delta_k \quad \text{and} \quad \varepsilon = +\tfrac{1}{2}\delta_j + \sum_{k>j} c_k \delta_k. \tag{7.18}$$

The first kind sum in pairs ε and $-p_{\delta_j}\varepsilon$ and then do not change the parity of (7.17). With the second kind, let us say $\beta = \alpha + \frac{1}{2}\delta_i + \frac{1}{2}\delta_j$. Then $\beta - \varepsilon$ is an $\mathfrak{a}_p$-root and must be of the form (b) or (c2). We conclude there are $\mathfrak{a}_p$-roots

$$\alpha + \tfrac{1}{2}\delta_i + \tfrac{1}{2}\delta_j \quad \text{and} \quad \alpha + \tfrac{1}{2}\delta_i \pm \tfrac{1}{2}\delta_k, \qquad |\delta_i| = |\delta_j| = |\delta_k|.$$

From $\alpha + \frac{1}{2}\delta_i + \frac{1}{2}\delta_j$ and $\alpha - \frac{1}{2}\delta_i \pm \frac{1}{2}\delta_k$, we see that $\delta_i + \frac{1}{2}\delta_j \pm \frac{1}{2}\delta_k$ is an $\mathfrak{a}_p$-root. However, its length squared works out to $\frac{3}{2}|\delta_i|^2$, a contradiction. Consequently there are no $\mathfrak{a}_p$-roots of the second kind in (7.18) and (7.17) is odd.

Consider (c1). If $\alpha \pm \delta \pm \gamma$ is a root with $\gamma \neq 0$ in $\mathfrak{b}_p'$, then again $\langle \alpha + \delta + \gamma, \alpha - \delta - \gamma \rangle$ is < 0 and 2α has to be a root, a contradiction. So β has multiplicity one. In considering the parity of (7.17), we argue as in the previous case and have only to consider $\mathfrak{a}_p$-roots of the form

$$\varepsilon = \tfrac{1}{2}\delta_j + \sum_{k>j} c_k\delta_k. \tag{7.19}$$

But there are no such $\mathfrak{a}_p$-roots, since otherwise $\alpha + \delta_j - \varepsilon$ would have to be an $\mathfrak{a}_p$-root but would not be of the form (c). Thus (7.17) is odd.

Next consider (d2). Again the only $\mathfrak{a}_p$-roots that can affect the parity of (7.17) are those of the form (7.19). Forming the inner product of both sides of (7.19) with δ_j, we see that $\langle \varepsilon, \delta_j \rangle = \frac{1}{2} | \delta_j |^2$ and hence $| \varepsilon | \leq | \delta_j |$. On the other hand, $| \delta_j | < | 2\alpha |$. We claim that $| \varepsilon | = | \delta_j |$.

Thus suppose that $| \varepsilon | < | \delta_j |$. The $\mathfrak{a}_p$-roots in question then all lie in a BC-type root system, and 2ε is an $\mathfrak{a}_p$-root. Let us say that $\beta = \alpha + \frac{1}{2}\delta_i + \frac{1}{2}\delta_j$. Then $p_\beta(2\varepsilon) = 2\varepsilon - 2\beta$ is an $\mathfrak{a}_p$-root of the same length as 2α, not equal to $\pm 2\alpha$, and not orthogonal to 2α, a contradiction.

Hence $| \varepsilon | = | \delta_j |$. Then it follows that $| \Sigma c_k\delta_k |^2 = \frac{3}{4} | \varepsilon |^2$, and hence there are at least two k's for which $c_k \neq 0$. Taking into account the possible signs of the c_k's in (7.19), we see that the number of $\mathfrak{a}_p$-roots of the form (7.19) is a multiple of four. Thus the roots (7.19) do not affect the parity of (7.17), and (7.17) is odd.

We argue similarly with (e1). The $\mathfrak{a}_p$-roots $\alpha + \frac{1}{2}\delta_j + \frac{1}{2}\delta_i$ and $\alpha + \frac{1}{2}\delta_j - \frac{1}{2}\delta_i$ are orthogonal, and their difference is an $\mathfrak{a}_p$-root. Hence their sum $2\alpha + \delta_j$ is an $\mathfrak{a}_p$-root. As with (d2), we have $| \varepsilon | \leq | \delta_j |$. On the other hand, $| \delta_j | < | 2\alpha + \delta_j |$. Since ε is not orthogonal to $2\alpha + \delta_j$ and is not $\pm \frac{1}{2}(2\alpha + \delta_j)$, we must have $| \varepsilon | = | \delta_j |$. Then $| \Sigma c_k\delta_k |^2 = \frac{3}{4} | \varepsilon |^2$ in (7.19), and the number of $\mathfrak{a}_p$-roots of the form (7.19) is a multiple of four. Thus the roots (7.19) do not affect the parity of (7.17), and (7.17) is odd.

We use the same argument with (e2). Here we have $| \varepsilon | \leq | \delta_j | < | \beta |$. Since ε is not orthogonal to β and is not $\pm \frac{1}{2}\beta$, we must have $| \varepsilon | = | \delta_j |$. Then $| \Sigma c_k\delta_k |^2 = \frac{3}{4} | \varepsilon |^2$ in (7.19), and the number of roots of the form (7.19) is a multiple of four. Thus the roots (7.19) do not affect the parity of (7.17), and (7.17) is odd.

Finally we consider (e3). Once again the only $\mathfrak{a}_p$-roots that can affect the parity of (7.17) are those of the form (7.19). Let us say

$$\beta = \alpha + \tfrac{1}{2}\delta_l + \tfrac{1}{2}\delta_i + \tfrac{1}{2}\delta_j.$$

Then $\beta - \varepsilon$ is an $\mathfrak{a}_p$-root of the form

$$\beta - \varepsilon = \alpha + \tfrac{1}{2}\delta_l + \tfrac{1}{2}\delta_i - \sum_{k>j} c_k\delta_k,$$

with $| 2\alpha | = | \delta_l | = | \delta_i | = | \delta_j |$. Referring to our classification into cases (d) and (e), we see that $\beta - \varepsilon$ must be of type (e3); hence there is just one $k > j$ with

$c_k \neq 0$, c_k is $\pm \frac{1}{2}$, and $|\delta_k| = |\delta_j|$. Thus we may assume

$$\beta - \varepsilon = \alpha + \tfrac{1}{2}\delta_l + \tfrac{1}{2}\delta_i + \tfrac{1}{2}\delta_k.$$

Then

$$\frac{2\langle \beta, \beta - \varepsilon \rangle}{|\beta|^2} = \frac{2|\delta_j|^2(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + 0 + 0)}{|\delta_j|^2} = \tfrac{3}{2},$$

a contradiction. Thus there are no $\mathfrak{a}_p$-roots of the form (7.19), and (7.17) is odd.

LEMMA 7.5. *Let α be a reduced $\mathfrak{a}$-root other than for a maximal parabolic in split* G_2. *Suppose 2α is an $\mathfrak{a}$-root and $\beta = 2\alpha + \alpha_I$ is an extension to an $\mathfrak{a}_p$-root. If $\alpha_I \neq 0$, then $2\langle \beta, \bar{\beta} \rangle / |\beta|^2 = 0$ and there are only the following possibilities for β.*

(1) *$\beta = 2\alpha \pm \delta$ for some $\delta = \delta_j$ with $2\langle \rho_M^+, \delta \rangle / |\delta|^2$ odd and $2|\delta|^2 = |\beta|^2$.*

(2) *$\beta = 2\alpha \pm \frac{1}{2}\delta' \pm \frac{1}{2}\delta$ for some $\delta' = \delta_i$ and $\delta = \delta_j$ with $i < j$, $2\langle \rho_M^+, \delta \rangle / |\delta|^2$ odd, and $|\delta'| = |\delta| = |\beta|$.*

Proof. From Lemma 2.7 of [13], we know that α is odd, 2α is useful, and 4α is not an $\mathfrak{a}$-root. Hence $2\langle \beta, \bar{\beta} \rangle / |\beta|^2 = 0$. Then (7.16) follows with $n = 0$, and the only possibilities are those analogous to (c) in Lemma 7.4. Then we can argue as in the proof of Lemma 7.4c to see that $2\langle \rho_M^+, \delta \rangle / |\delta|^2$ is odd in the cases at hand.

LEMMA 7.6. *Let α be a reduced $\mathfrak{a}$-root such that 2α is an $\mathfrak{a}$-root. Suppose $\beta = \alpha + \alpha_I$ is an extension to an $\mathfrak{a}_p$-root, and suppose 2β is an $\mathfrak{a}_p$-root. Then $\alpha_I = 0$.*

Proof. We may disregard split G_2 since twice an $\mathfrak{a}_p$-root cannot be an $\mathfrak{a}_p$-root in that case. Suppose $\alpha_I \neq 0$. Applying Lemma 7.5 to 2β, we see that $2\langle \beta, \bar{\beta} \rangle / |\beta|^2 = 0$. Choose γ in $\mathfrak{b}_p'$ so that $\beta + \gamma$ is a root of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a}_p + \mathfrak{b}_p)^{\mathbf{C}})$. If $\gamma \neq 0$, then we have

$$-1 \leq \frac{2\langle \beta + \gamma, \bar{\beta} - \gamma \rangle}{|\beta + \gamma|^2} = \frac{2\langle \beta + \gamma, \beta - \gamma \rangle}{|\beta + \gamma|^2} - \frac{4|\alpha_I|^2}{|\beta + \gamma|^2} < \frac{2\langle \beta + \gamma, \beta - \gamma \rangle}{|\beta + \gamma|^2}.$$

Thus $\langle \beta + \gamma, \beta - \gamma \rangle \geq 0$. The sum $2\beta = (\beta + \gamma) + (\beta - \gamma)$ is a root, by Lemma 3 of [12], and the difference $2\gamma = (\beta + \gamma) - (\beta - \gamma)$ is not a root, by Lemma 1 of [12]. Thus we have a contradiction and must have $\gamma = 0$. But then β and 2β are both roots, in contradiction to the fact that twice a root is not a root in the system of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a}_p + \mathfrak{b}_p)^{\mathbf{C}})$.

Proof of Proposition 7.1. We continue to leave aside the cases in which α arises as the reduced root of a maximal parabolic of split G_2. We are to prove (7.12).

We first dispose of the $\mathfrak{a}_p$-roots β of the form $\alpha + \alpha_I$ or $2\alpha + \alpha_I$ with $\alpha_I \neq 0$ and with β not of type (d1) in Lemma 7.4. We may assume that β is odd. Neither $\frac{1}{2}\beta$ nor 2β is an $\mathfrak{a}_p$-root, by Lemma 7.6, and hence $\beta' = \beta$. Since β is odd, $2\rho_\beta$ is an odd multiple of β. Thus

$$f_{\sigma,\beta'}(\lambda_M + i\nu) = p_{\sigma(\gamma_\beta)}\left(\frac{2\langle \lambda_M + i\nu, \beta\rangle}{|\beta|^2}\right) \tag{7.20}$$

in the notation of (7.5c) and (7.13). Lemmas 7.4 and 7.5 associate to each β under consideration a certain root δ such that

$$\frac{2\langle \rho_M^+, \delta\rangle}{|\delta|^2} \text{ is odd.} \tag{7.21}$$

We shall use (7.13) to show that the product of the factors (7.20) for β and $p_\delta\beta$ is one. In every case $|\beta| \geq |\delta|$ and hence

$$\gamma_{p_\delta\beta} = \gamma_\beta\gamma_\delta^{2\langle\beta,\delta\rangle/|\beta|^2} = \gamma_\beta\gamma_\delta. \tag{7.22}$$

Let $p_\delta\beta = \beta - r\delta$ where $|r| = |\beta|^2/|\delta|^2$. Then

$$\begin{aligned}
p_{\sigma(\gamma_{p_\delta\beta})}\left(\frac{2\langle \lambda_M + i\nu, p_\delta\beta}{|p_\delta\beta|^2}\right) &= p_{\sigma(\gamma_\beta)\sigma(\gamma_\delta)}\left(\frac{2\langle \lambda_M + i\nu, \beta - r\delta\rangle}{|\beta|^2}\right) \qquad \text{by (7.22)}\\
&= p_{\sigma(\gamma_\beta)\sigma(\gamma_\delta)}\left(\frac{2\langle \lambda_M + i\nu, \beta\rangle}{|\beta|^2} - \frac{2\langle \lambda_M, \delta\rangle}{r^{-1}|\beta|^2}\right)\\
&= p_{\sigma(\gamma_\beta)\sigma(\gamma_\delta)}\left(\frac{2\langle \lambda_M + i\nu, \beta\rangle}{|\beta|^2} \pm \frac{2\langle \lambda_M, \delta\rangle}{|\delta|^2}\right)\\
&= p_{\sigma(\gamma_\beta)\sigma(\gamma_\delta)(-1)^{2\langle\lambda_M,\delta\rangle/|\delta|^2}}\left(\frac{2\langle \lambda_M + i\nu, \beta\rangle}{|\beta|^2}\right)\\
&\qquad\qquad \text{by (7.14)}\\
&= p_{\sigma(\gamma_\beta)(-1)^{2\langle\rho_M^+,\delta\rangle/|\delta|^2}}\left(\frac{2\langle \lambda_M + i\nu, \beta\rangle}{|\beta|^2}\right) \qquad \text{by (7.8)}\\
&= p_{-\sigma(\gamma_\beta)}\left(\frac{2\langle \lambda_M + i\nu, \beta\rangle}{|\beta|^2}\right) \qquad \text{by (7.21).}
\end{aligned}$$

Hence (7.13) shows the factors (7.20) for β and $p_\varepsilon\beta$ cancel.

In particular, this proves (7.12) if α is even. Suppose now that α is a root. Then $2\langle \rho_\alpha, \alpha' \rangle / |\alpha'|^2$ is just the multiplicity of α as an $\mathfrak{a}$-root, which is odd. But α also has odd multiplicity as an $\mathfrak{a}_p$-root. Hence

$$\begin{aligned} f_{\xi,\alpha'}(i\nu) &= p_{\xi(\gamma_\alpha)}\left(\frac{2\langle i\nu, \alpha\rangle}{|\alpha|^2}\right) \\ &= p_{\sigma(\gamma_\alpha)}\left(\frac{2\langle \lambda_M + i\nu, \alpha\rangle}{|\alpha|^2}\right) \qquad \text{by Lemma 7.3} \\ &= f_{\sigma,\alpha'}(\lambda_M + i\nu). \end{aligned}$$

In view of the cancellation in the previous paragraph, this completes the proof of (7.12) when α is a root.

Finally suppose 2α is a root. Then

$$\frac{2\langle \rho_\alpha, \alpha' \rangle}{|\alpha'|^2} = \frac{1}{2}\text{ multiplicity } (\alpha \text{ as } \mathfrak{a}\text{-root}) + \text{multiplicity } (2\alpha \text{ as } \mathfrak{a}\text{-root}). \tag{7.23}$$

The multiplicity of 2α is odd. The multiplicity of α is understood by looking at cases (d) and (e) in Lemma 7.4. All possible signs are allowed in the indicated cases, and hence the contribution to the multiplicity from (d2) and from (e) is divisible by 4 and does not affect the parity of (7.23). Even $\mathfrak{a}_p$-roots β of type (d1) similarly do not affect the parity of (7.23). Thus let

$$C = \#\left(\text{odd } \mathfrak{a}_p\text{-root pairs } \beta = \alpha \pm \tfrac{1}{2}\delta\right).$$

Then we conclude that

$$\frac{2\langle \rho_\alpha, \alpha' \rangle}{|\alpha'|^2} \equiv 1 + C + \frac{1}{2}\text{multiplicity } (\alpha \text{ as } \mathfrak{a}_p\text{-root}) \mod 2. \tag{7.24}$$

One reduced $\mathfrak{a}_p$-root β is a multiple of α, say $\beta = \beta_0$. Whether β_0 equals α or 2α, β_0' is 2α. Moreover,

$$\begin{aligned} \frac{2\langle \rho_{\beta_0}, 2\alpha \rangle}{|2\alpha|^2} &= \frac{1}{2}\text{multiplicity } (\alpha \text{ as } \mathfrak{a}_p\text{-root}) \\ &\quad + \text{multiplicity } (2\alpha \text{ as } \mathfrak{a}_p\text{-root}) \\ &\equiv \frac{1}{2}\text{multiplicity } (\alpha \text{ as } \mathfrak{a}_p\text{-root}) + 1 \mod 2. \end{aligned}$$

Hence in either case we have

$$\frac{2\langle \rho_\alpha, \alpha' \rangle}{|\alpha'|^2} \equiv C + \frac{2\langle \rho_{\beta_0}, \beta_0' \rangle}{|\beta_0'|^2} \mod 2. \tag{7.25}$$

Let us now consider the contribution to the left side of (7.12) of an odd $\mathfrak{a}_p$-root pair $\beta = \alpha + \frac{1}{2}\delta$ and $\bar\beta = \alpha - \frac{1}{2}\delta$. Here $\gamma_\beta = \gamma_{\bar\beta} = \gamma_{2\alpha}\gamma_\delta = \gamma_{\alpha'}\gamma_\delta$ and thus

$$\begin{aligned} f_{\sigma,\beta'}(\lambda_M + i\nu) &= p_{\sigma(\gamma_\beta)}\left(\frac{2\langle\lambda_M + i\nu, \beta\rangle}{|\beta|^2}\right) \\ &= p_{\sigma(\gamma_{\alpha'})\sigma(\gamma_\delta)}\left(\frac{2\langle\lambda_M + i\nu, \alpha + \frac{1}{2}\delta\rangle}{|\beta|^2}\right) \\ &= p_{\xi(\gamma_{\alpha'})\sigma(\gamma_\delta)}\left(\frac{2\langle i\nu, \alpha'\rangle}{|\alpha'|^2} + \frac{2\langle\lambda_M, \delta\rangle}{|\delta|^2}\right) \\ &\qquad\qquad\text{by Lemma 7.3} \\ &= p_{\xi(\gamma_{\alpha'})\sigma(\gamma_\delta)(-1)^{2\langle\lambda_M,\delta\rangle/|\delta|^2}}\left(\frac{2\langle i\nu, \alpha'\rangle}{|\alpha'|^2}\right) \qquad \text{by (7.14)} \\ &= p_{\xi(\gamma_{\alpha'})(-1)^{2\langle\rho_M^+,\delta\rangle/|\delta|^2}}\left(\frac{2\langle i\nu, \alpha'\rangle}{|\alpha'|^2}\right) \qquad \text{by (7.8)} \\ &= \left[p_{\xi(\gamma_{\alpha'})}\left(\frac{2\langle i\nu, \alpha'\rangle}{|\alpha'|^2}\right)\right]^{(-1)^{2\langle\rho_M,\delta\rangle/|\delta|^2}} \end{aligned}$$

by (7.13). We get the same result for $\bar\beta$.

The only other contribution to the left side of (7.12) is from β_0. Here $\beta_0' = \alpha'$, and

$$f_{\sigma,\beta_0'}(\lambda_M + i\nu) = p_{-\sigma(\gamma_{\alpha'})(-1)^{2\langle\rho_{\beta_0},\beta_0'\rangle/|\beta_0'|^2}}\left(\frac{2\langle\lambda_M + i\nu, \beta_0'\rangle}{|\beta_0'|^2}\right),$$

which is

$$= \left[p_{\xi(\gamma_{\alpha'})}\left(\frac{2\langle i\nu, \alpha'\rangle}{|\alpha'|^2}\right)\right]^{(-1)^{C+1+2\langle\rho_\alpha,\alpha'\rangle/|\alpha'|^2}}$$

by Lemma 7.3, (7.13) and (7.25). Similarly the right side of (7.12) is

$$\left[p_{\xi(\gamma_{\alpha'})}\left(\frac{2\langle i\nu, \alpha'\rangle}{|\alpha'|^2}\right)\right]^{(-1)^{1+2\langle\rho_\alpha,\alpha'\rangle/|\alpha'|^2}}.$$

Thus (7.12) boils down to a question of proving

$$(7.26)\quad (-1)^{C+1+2\langle\rho_\alpha,\alpha'\rangle/|\alpha'|^2} + 2\sum_{\substack{\delta \text{ in (d1)},\\ \beta \text{ odd}}} (-1)^{2\langle\rho_M^+,\delta\rangle/|\delta|^2} = (-1)^{1+2\langle\rho_\alpha,\alpha'\rangle/|\alpha'|^2}.$$

If $C = 0$, (7.26) is clearly valid. If $C \geq 1$, then there exists an odd $\beta = \alpha + \frac{1}{2}\delta$, and 2α is an $\mathfrak{a}_p$-root. By Proposition 7, parts (a) and (b), of [12], α cannot be an $\mathfrak{a}_p$-root. Thus β_0 is 2α, not α, and $2\langle \rho_{\beta_0}, \beta_0' \rangle / |\beta_0'|^2$ is just the $\mathfrak{a}_p$-multiplicity of 2α, which is odd. By (7.25)

$$\frac{2\langle \rho_\alpha, \alpha' \rangle}{|\alpha'|^2} \equiv C + 1 \mod 2.$$

Thus we can rewrite the statement (7.26) to be proved as

$$(7.27) \qquad 1 + 2 \sum_{\substack{\delta \text{ in (d1)}, \\ \beta \text{ odd}}} (-1)^{2\langle \rho_M^+, \delta \rangle / |\delta|^2} = (-1)^C,$$

for $C \geq 1$.

Let us determine the $\mathfrak{a}_p$-roots that make a contribution to the parity of (7.17) under the assumption that $\beta = \alpha + \frac{1}{2}\delta_j$ is an odd $\mathfrak{a}_p$-root of type (d1). As usual, we are to consider $\mathfrak{a}_p$-roots ε of the form (7.19). When two or more indices k are present, the number of such roots is a multiple of four, and the parity of (7.17) is unaffected. Thus the parity is influenced only by

$$(7.28a) \qquad \varepsilon = \tfrac{1}{2}\delta_j$$

and

$$(7.28b) \qquad \varepsilon = \tfrac{1}{2}\delta_j \pm c_k \delta_k .$$

If (7.28a) is an $\mathfrak{a}_p$-root, then so is α, and we saw above that this is not possible because β is an odd $\mathfrak{a}_p$-root. If (7.28b) is an $\mathfrak{a}_p$-root, then $\beta - \varepsilon$ is an $\mathfrak{a}_p$-root, and Lemma 7.4 shows that $c_k = \pm \frac{1}{2}$. Hence the parity is influenced only by $\mathfrak{a}_p$-roots

$$(7.28c) \qquad \varepsilon = \tfrac{1}{2}\delta_j \pm \tfrac{1}{2}\delta_k, \qquad k > j.$$

Such choices of k clearly stand in one-one correspondence with the choices of $k > j$ such that $\alpha + \frac{1}{2}\delta_k$ is an odd $\mathfrak{a}_p$-root of type (d1).

Since the other δ's do not matter, let us renumber the relevant δ's from 1 to C. Then modulo 2, we have

$$\frac{2\langle \rho_M^+, \delta_1 \rangle}{|\delta_1|^2}$$
$$\equiv \frac{\langle \delta_1 + \frac{1}{2}(\delta_1 + \delta_2) + \frac{1}{2}(\delta_1 - \delta_2) + \cdots + \frac{1}{2}(\delta_1 + \delta_C) + \frac{1}{2}(\delta_1 - \delta_C), \delta_1 \rangle}{|\delta_1|^2} = C,$$

and so on. In general, $2\langle \rho_M^+, \delta_i \rangle / |\delta_i|^2 \equiv C + 1 - i$. Thus the left side of (7.27) is

$$1 + 2(-1)^C + 2(-1)^{C-1} + 2(-1)^{C-2} + \cdots + 2(-1)^1,$$

which equals $(-1)^C$. This proves (7.27) and completes the proof of Proposition 7.1 except when α is the reduced root in a maximal parabolic in split G_2.

In split G_2 there are two really distinct choices for a maximal parabolic, and in both cases 2α is an $\mathfrak{a}_p$-root. First consider the case when 2α is a short $\mathfrak{a}_p$-root. Take δ to be the unique positive root of $(\mathfrak{m}, \mathfrak{a}_M)$. Then δ is simple for $(\mathfrak{m}, \mathfrak{a}_M)$, and (7.21) holds. The $\mathfrak{a}_p$-roots restricting to $c\alpha$ with $c > 0$, except for 2α, are paired by p_δ as

$$\alpha \pm \tfrac{1}{2}\delta \quad \text{and} \quad 3\alpha \pm \tfrac{1}{2}\delta.$$

For each pair, (7.22) is still valid. For the second pair, $|\beta| = |\delta|$ and the argument following (7.22) is valid. For the first pair, $p_\delta\beta$ is $\beta \pm \delta$ and r should be taken as 1 in the argument. However, $|\beta|^2 = \frac{1}{3}|\delta|^2$, and the effect is that the cube of $(-1)^{2\langle\lambda_M, \delta\rangle/|\delta|^2}$ occurs instead of the first power. Thus the argument following (7.22) is valid with minor changes, and the factors $f_{\sigma,\beta'}(\lambda_M + i\nu)$ all cancel except for $\beta = 2\alpha$.

For the factor $f_{\sigma,2\alpha}(\lambda_M + i\nu)$, we have $2\rho_\alpha = 10\alpha$ and $2\langle \rho_\alpha, \alpha'\rangle / |\alpha'|^2 = 5$. Thus

$$f_{\xi,\alpha'}(i\nu) = p_{\xi(\gamma_{2\alpha})}\left(\frac{2\langle i\nu, 2\alpha\rangle}{|2\alpha|^2}\right) = f_{\sigma,2\alpha}(\lambda_M + i\nu) \tag{7.29}$$

by Lemma 7.3, and (7.12) follows.

In the other case with split G_2, 2α is a long $\mathfrak{a}_p$-root. Then δ is short, but (7.21) still holds because δ is simple. The $\mathfrak{a}_p$-roots restricting to $c\alpha$ with $c > 0$, except for 2α, are paired by p_δ as

$$\alpha \pm \tfrac{1}{2}\delta \quad \text{and} \quad \alpha \pm \tfrac{3}{2}\delta.$$

For each pair, (7.22) is still valid and $|\beta| \geq |\delta|$. Then the argument following (7.22) is valid, and the factors $f_{\sigma,\beta'}(\lambda_M + i\nu)$ all cancel except for $\beta = 2\alpha$. In this case, $2\langle \rho_\alpha, \alpha'\rangle / |\alpha'|^2 = 3$ and (7.29) is again valid. Then (7.12) follows, and the proof of Proposition 7.1 is complete.

8. Complete reduction of characters induced from discrete series

We come to the first form of a classification theorem. The setting is a cuspidal parabolic subgroup MAN in a linear connected reductive group G with compact center. We are given a basic character induced from discrete series

$$\operatorname{ind}_P^G \Theta^{MA}(\lambda_M, C_M, \chi, \nu), \tag{8.1}$$

with $\Theta^M(\lambda_M, C_M, \chi)$ the character of the discrete series representation ξ of M. In Theorem 8.7 we shall decompose (8.1) into irreducible characters.

Let W be the subgroup of the Weyl group of A that leaves $\xi \otimes e^{i\nu}$ fixed:

$$W = W_{\xi,\nu} = \{w \in W(A:G) \mid w\xi \cong \xi \text{ and } w\nu = \nu\}$$
$$= \{w \in W(A:G) \mid w\Theta^{MA} = \Theta^{MA}\}.$$

The group W corresponds to standard self-intertwining operators of the induced representation [16]. We shall recall from [16] a semidirect product decomposition $W = W'R$, where W' is the Weyl group of a (nonreduced) root system Δ' and W' is normal in W.

Namely let Δ be the (nonreduced) root system of "useful" roots[10] of $(\mathfrak{g}, \mathfrak{a})$. The main theorem of [12] identifies $W(A:G)$ as exactly the Weyl group of Δ. Let

(8.2) $$\Delta' = \{\alpha \in \Delta \mid \mu_{\xi,\alpha}(i\nu) = 0\}.^{11}$$

Then W carries Δ' into itself, and it follows that Δ' is a root system. Let W' be its Weyl group. It is a nontrivial fact ([5], Lemma 19) that $W' \subseteq W$; we shall give a different proof of this fact in Corollary 10.6 that applies also when ξ is a limit of discrete series and (8.2) is generalized suitably. In the case at hand, the definition

(8.3) $$R = \{w \in W \mid w\alpha > 0 \text{ for all } \alpha > 0 \text{ in } \Delta'\}$$

leads to the semidirect product decomposition $W = W'R$.

This decomposition of W has the following properties:

(a) The dimension of the commuting algebra for the induced representation with character (8.1) is exactly the order $|R|$ of R, according to Theorem 13.4 of [16]. (In terms of the operators of [16], W' corresponds to trivial operators, and R corresponds to independent operators. The operators for W span the commuting algebra by Harish-Chandra's completeness theorem ([7], Theorem 38.1) and a supplementary argument; see Sections 9–13 of [16].)

(b) The $\mathfrak{a}$-roots of odd multiplicity[12] form a reduced root system $\Delta_0 \subseteq \Delta$ whose Weyl group $S \subseteq W(A:G)$ can be characterized (see § 3 of [13]): The group $W(A:G)$ has an action on $\mathfrak{b}'$ preserving C_M-dominance such that $w\xi|_{M_0} \cong \xi|_{M_0}$ if and only if $w\lambda_M = \lambda_M$, by Theorem 3.7 of [13], and S is the

[10] "Useful" is defined as follows, in terms of the notation of Section 7. Define a conjugation of $\mathfrak{a}_p$ to be 1 on $\mathfrak{a}$ and -1 on $\mathfrak{a}_M$. Except in a maximal parabolic of split G_2, call a root α of $(\mathfrak{g}, \mathfrak{a})$ *useful* if it is the restriction of some $\mathfrak{a}_p$ root β with $2\langle\beta, \bar{\beta}\rangle/|\beta|^2 \neq +1$. In a maximal parabolic of split G_2, with reduced root α, we take α and 2α to be useful, 3α to be not useful. See [12] and Section 2 of [13] for properties of useful roots.

[11] By convention, $\mu_{\xi,\alpha}$ means $\mu_{\xi,\beta}$ for the unique reduced $\mathfrak{a}$ root β that is a multiple of α.

[12] By Lemma 2.1 of [13], the $\mathfrak{a}$-roots of odd multiplicity are exactly the restrictions to $\mathfrak{a}$ of the real roots of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} + \mathfrak{b})^{\mathbf{C}})$.

group that operates trivially on $\mathfrak{b}'$ in this action. In terms of this action, define

$$W_{\lambda_M,\nu} = \left\{ w \in W(A:G) \,\middle|\, \begin{matrix} w\lambda_M = \lambda_M & \text{in the action on } \mathfrak{b}', \text{ and} \\ w\nu = \nu & \text{in the action on } \mathfrak{a}' \end{matrix} \right\}.$$

Then $W_{\lambda_M,\nu}$ is the Weyl group of a root system

$$\Delta_{\lambda_M,\nu} = \left\{ \alpha \text{ root of } (\mathfrak{g}, \mathfrak{a}) \,\middle|\, \begin{matrix} p_\alpha \text{ is in } W_{\lambda_M,\nu}, \text{ and } \alpha, \\ \text{if odd, has odd multiplicity} \end{matrix} \right\},$$

and we have the following inclusions:

$$W' \subseteq W \subseteq W_{\lambda_M,\nu}.$$

See Section 6 of [13] for details.[13]

(c) The group R is a subgroup of the group S defined in (b), by Lemma 7.3 of [13].[13]

(d) $R = \Sigma \mathbf{Z}_2$, by Theorem 6.1 of [13].

LEMMA 8.1. *Suppose α and β are nonproportional nonorthogonal $\mathfrak{a}$-roots in $\Delta_{\lambda_M,\nu}$ with $|\alpha| \geq |\beta|$. If α and β are not in Δ', then $p_\beta\alpha$ is in Δ'.*

Remarks. For ξ in the discrete series and $\nu = 0$, this is an immediate consequence of (6.2) and Lemma 6.5 of [13]. A different proof will be given in Lemma 10.17 that applies also to limits of discrete series and general ν.

LEMMA 8.2. *Let q be the linear transformation on $\mathfrak{a}'$ given by*

$$q = \frac{1}{|R|} \sum_{r \in R} r. \tag{8.4}$$

Then ν is orthogonal to $(1-q)\mathfrak{a}'$, and no $\mathfrak{a}$-root in $(1-q)\mathfrak{a}'$ lies in Δ'.

Proof. The operator q is the orthogonal projection on the simultaneous $+1$ eigenspace of the members of R. Thus the image of $1-q$ is the same as the kernel of q. If α is in Δ', then $r\alpha$ has the same sign as α, for each r in R, and hence $q\alpha$ cannot be 0. Thus α cannot be in the image of $1-q$. Moreover $q\nu = \nu$, and hence ν is orthogonal to $(1-q)\mathfrak{a}'$.

A set of $\mathfrak{a}$-roots $\{\alpha_1, \ldots, \alpha_n\}$ in $\Delta_{\lambda_M,\nu}$ will be said to be *superorthogonal* (relative to $\Delta_{\lambda_M,\nu}$) if the only way that $\Sigma c_j\alpha_j$ can be in $\Delta_{\lambda_M,\nu}$ is for all but one of the c_j's to be 0.

[13]Statements (b) and (c) need to be treated in [13] only for the case $\nu = 0$, but minor additional arguments handle general ν. See the end of the present Section 10, starting with Lemma 10.12, for an indication of details in the context that ξ is a discrete series or limit of discrete series representation of M.

LEMMA 8.3. *With q as in (8.4), the positive members of $\Delta_{\lambda_M,\nu}$ that lie in the space $(1-q)\mathfrak{a}'$ are superorthogonal.*

Proof. Otherwise we can find nonproportional nonorthogonal $\mathfrak{a}$-roots α and β in $(1-q)\mathfrak{a}' \cap \Delta_{\lambda_M,\nu}$ with $|\alpha| \geq |\beta|$. Lemma 8.2 shows that α and β cannot be in Δ'. By Lemma 8.1, $p_\beta \alpha$ is in Δ'. But $p_\beta \alpha$ is also in $(1-q)\mathfrak{a}'$, and we have a contradiction to Lemma 8.2.

PROPOSITION 8.4. *With q as in (8.4), the set $\mathcal{H} = \{\alpha_1, \ldots, \alpha_q\}$ of positive $\mathfrak{a}$-roots of odd multiplicity in $(1-q)\mathfrak{a}'$ is superorthogonal and spans $(1-q)\mathfrak{a}'$. Moreover,*

(a) *each r in R is of the form $p_{\alpha_{j_1}} \cdot \ldots \cdot p_{\alpha_{j_n}}$ with $\{\alpha_{j_1}, \ldots, \alpha_{j_n}\} \subseteq \mathcal{H}$ and*
(b) *each α_j in $\mathcal{H}$ satisfies*

$$\langle \nu, \alpha_j \rangle = 0, \quad \textit{and} \tag{8.5}$$

(c) *each α_j occurs in the decomposition of some r in R.*

Remark. Since $\mathcal{H} \subseteq \Delta_{\lambda_M}$, this use of "superorthogonality" lies within the scope of our definition.

Proof. The set $\mathcal{H}$ is superorthogonal by Lemma 8.3. If r is in R, then r fixes $q\mathfrak{a}'$. According to property (c) of the group R, R is contained in the Weyl group S of Δ_0, and hence Chevalley's Lemma says that r is the product of reflections in roots of Δ_0 each fixing $q\mathfrak{a}'$. Such roots of Δ_0 must lie in $(1-q)\mathfrak{a}'$, hence are members of $\mathcal{H}$ up to sign. Thus $\mathcal{H}$ satisfies (a). Conclusion (b) follows from Lemma 8.2.

To prove conclusion (c), we show that the roots needed for (a), as r varies, span $(1-q)\mathfrak{a}'$. Assume the contrary, and let v be a member of $(1-q)\mathfrak{a}'$ orthogonal to all such roots. On the one hand, v must be fixed by every element of R, hence by q, so that v is in the image of q. On the other hand, v is in the image of $(1-q)$ and hence the kernel of q. Since the image and kernel of q are disjoint, $v = 0$. This completes the proof.

Each of the $\mathfrak{a}$-roots α_i of Proposition 8.4 can be regarded as a real root of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a}+\mathfrak{b})^{\mathbf{C}})$. (See footnote 12.) Let us form the Cayley transforms $\mathbf{d}_{\alpha_1}, \ldots, \mathbf{d}_{\alpha_q}$. The product $\mathbf{d}_{\mathcal{H}} = \prod_{j=1}^{q} \mathbf{d}_{\alpha_j}$ does not depend on what order the roots α_j are written in, since the α_j are strongly orthogonal, and $\mathbf{d}_{\mathcal{H}}$ leads us to data $(\mathfrak{m}^*, \mathfrak{a}^*, \mathfrak{b}^*)$ with

$$\mathfrak{b} \oplus \sum i\mathbf{R} H_{\tilde{\alpha}_j} = \mathfrak{b}^*, \quad \text{where } \tilde{\alpha}_j = \mathbf{d}_{\mathcal{H}}(\alpha_j),$$

$$\mathfrak{a}^* \oplus \sum \mathbf{R}\left(E_{\tilde{\alpha}_j} + E_{-\tilde{\alpha}_j}\right) = \mathfrak{a}.$$

Define

$$\lambda = \begin{Bmatrix} \lambda_M & \text{on } \mathfrak{b} \\ 0 & \text{on } \sum i\mathbf{R}H_{\tilde{\alpha}_j} \end{Bmatrix} \quad \text{on } \mathfrak{b}^*,$$

$$\chi^* = \chi|_{Z_{M^*}} \quad \text{on } Z_{M^*},$$

$$\nu^* = \nu|_{\mathfrak{a}^*} \quad \text{on } \mathfrak{a}^*.$$

Let $W_{\mathcal{K}}$ be the abelian subgroup of order 2^q in $W(B^* : (M^*)^{\mathbf{C}})$ generated by the reflections $p_{\tilde{\alpha}_j}$, and let $E_{\mathcal{K}}$ be the subgroup of members of $W_{\mathcal{K}}$ realizable in M^*:

$$E_{\mathcal{K}} = W_{\mathcal{K}} \cap W(B^* : M^*).$$

LEMMA 8.5. *The parameter λ satisfies $\langle \lambda, \tilde{\beta} \rangle \neq 0$ for all roots $\tilde{\beta}$ of $(\mathfrak{m}^{*\mathbf{C}}, \mathfrak{b}^{*\mathbf{C}})$ other than $\pm\tilde{\alpha}_j$, $1 \leq j \leq q$. Consequently there exist exactly 2^q Weyl chambers C in $i\mathfrak{b}^*$ such that $\bar{C}$ contains an open neighborhood of λ_M in $i\mathfrak{b}$. All such chambers are obtained from one of them, C_0, as $\{wC_0 \mid w \in W_{\mathcal{K}}\}$. For any such chamber the positive roots from among the $\pm\tilde{\alpha}_j$, $1 \leq j \leq q$, are all simple.*

Remark. A reference chamber C_0 can be constructed explicitly as the chamber containing a specific small perturbation of λ. See the proof below, especially (8.7).

Proof. First we show $\langle \lambda, \tilde{\beta} \rangle \neq 0$ under the circumstances stated. Regard $\tilde{\beta}$ as a root of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a}^* + \mathfrak{b}^*)^{\mathbf{C}})$ that vanishes on $\mathfrak{a}^*$ and form $\mathbf{d}_{\mathcal{K}}^{-1}(\tilde{\beta})$. We may assume this has nonzero $\mathfrak{a}$ component since λ_M is $\mathfrak{m}$-nonsingular. Let β_R be the $\mathfrak{a}$-root $\mathbf{d}_{\mathcal{K}}^{-1}(\tilde{\beta})|_{\mathfrak{a}}$. Then β_R is a linear combination of the $\mathfrak{a}$-roots α_j and hence is in $(1-q)\mathfrak{a}'$. By Lemma 8.2, β_R is not in Δ'. However, according to Proposition 7.1, we have

$$\mu_{\xi,\beta_R}(i\nu) = \left(\prod_{\varepsilon|_{\mathfrak{a}} = c\beta_R,\, c>0} \langle \lambda_M + i\nu, \varepsilon \rangle \right) f_{\xi,\beta_R'}(i\nu), \tag{8.6}$$

where $f_{\xi,\beta_R'}$ is 1 or is given by (7.6c). One of the factors in the polynomial part of this expression comes from

$$\varepsilon = \mathbf{d}_{\mathcal{K}}^{-1}(\tilde{\beta}) = \beta_R + \beta_I,$$

where β_I is the restriction to $i\mathfrak{b}$. If $\beta_I \neq 0$, another factor comes from

$$\bar{\varepsilon} = \beta_R - \beta_I.$$

The factors in question area

$$\langle \lambda_M + i\nu, \beta_R \pm \beta_I \rangle = \pm\langle \lambda_M, \beta_I \rangle + i\langle \nu, \beta_R \rangle = \pm\langle \lambda, \tilde{\beta} \rangle + i\langle \nu, \beta_R \rangle.$$

By equation (8.5), $\langle \nu, \beta_R \rangle = 0$. Then $\langle \lambda, \tilde{\beta} \rangle = 0$ and $\beta_I \neq 0$ together mean that

(8.6) vanishes at ν, since $f_{\xi, \beta_R'}$ contributes at most a simple pole. Since β_R is not in Δ', we conclude either β_R is not useful or else $\beta_I = 0$.

We can rule out β_R as not useful since Lemma 7.4a shows in this case that $2\beta_I$ is a root of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$. Thus

$$\langle \lambda_M, \beta_I \rangle = \langle \lambda, \tilde{\beta} \rangle = 0$$

would contradict the $\mathfrak{m}$-nonsingularity of λ_M.

Thus $\beta_I = 0$ and β_R is in $\Delta_0 \subseteq \Delta_{\lambda_M}$. By Lemma 8.3, $\beta_R = \pm\alpha_j$ for some j, and hence $\tilde{\beta} = \pm\tilde{\alpha}_j$. This completes the argument about $\langle \lambda, \tilde{\beta} \rangle$.

Now we prove the statements about Weyl chambers. We have just shown that $\langle \lambda, \tilde{\beta} \rangle \neq 0$ for $\tilde{\beta} \neq \pm\tilde{\alpha}_j$, and we know $\langle \lambda, \tilde{\alpha}_j \rangle = 0$. It therefore follows that any choice of q signs $s_j = \pm 1$ makes

$$\lambda + \sum_{j=1}^{q} \varepsilon_j s_j \tilde{\alpha}_j \tag{8.7}$$

nonsingular as long as the ε_j are sufficiently small and positive. Each choice of signs $\{s_j\}$ therefore leads to a chamber as asserted, and there are no other chambers whose closure meets λ_M. We obtain 2^q chambers, and the members of $W_{\mathcal{K}}$ clearly permute them simply transitively.

If C is such a chamber, the positive roots are those $\tilde{\beta}$ such that $\langle v, \tilde{\beta} \rangle > 0$ for some v in C. Suppose $\tilde{\beta}$ is not one of the $\pm\tilde{\alpha}_j$ and $\tilde{\beta}$ is positive. Take v as in (8.7). Then

$$\begin{aligned} \langle v, p_{\tilde{\alpha}_k} \tilde{\beta} \rangle &= \langle v, \tilde{\beta} \rangle - \frac{2\langle \tilde{\alpha}_k, \tilde{\beta} \rangle}{|\tilde{\alpha}_k|^2} \langle v, \tilde{\alpha}_k \rangle \\ &= \langle v, \tilde{\beta} \rangle - 2\langle \tilde{\alpha}_k, \tilde{\beta} \rangle \varepsilon_k s_k, \end{aligned}$$

which is positive for ε_k sufficiently small. Hence $p_{\tilde{\alpha}_k}\tilde{\beta}$ is positive, and it follows that one of $\tilde{\alpha}_k$ and $-\tilde{\alpha}_k$ is simple. The proof of the lemma is complete.

Let C be one of the chambers described in Lemma 8.5. The case $i = q$ in Lemma 8.9 below will show that λ satisfies the appropriate integrality condition so that we can speak of

$$\Theta^{M^*A^*}(\lambda, C, \chi^*, \nu^*).$$

In that context, the statement of the following lemma makes sense.

Lemma 8.6. *With C as in Lemma 8.5, let w be in $W_{\mathcal{K}}$ and e be in $E_{\mathcal{K}}$. Then*

$$\Theta^{M^*A^*}(\lambda, weC, \chi^*, \nu^*) = \Theta^{M^*A^*}(\lambda, wC, \chi^*, \nu^*).$$

Proof. We have $we = ew$, and λ is dominant with respect to wC. Also $e\lambda = \lambda$. By Theorem 1.1c,

$$\begin{aligned}\Theta^{M^*A^*}(\lambda, wC, \chi^*, \nu^*) &= \Theta^{M^*A^*}(e\lambda, ewC, \chi^*, \nu^*)\\ &= \Theta^{M^*A^*}(\lambda, weC, \chi^*, \nu^*).\end{aligned}$$

THEOREM 8.7. *Let* $\operatorname{ind}_P^G \Theta^{MA}(\lambda_M, C_M, \chi, \nu)$ *be induced from discrete series. In the context of the corresponding induced representation,* $E_{\mathcal{K}}$ *is given by*

$$E_{\mathcal{K}} = \left\{ w \in W_{\mathcal{K}} \,\middle|\, \begin{array}{l}\text{for each } r \text{ in } R,\ w \text{ and } r \text{ have an}\\ \text{even number of factors } p_{\alpha_i} \text{ in common}^{14}\end{array} \right\}, \tag{8.8}$$

and formula (8.8) *sets up a canonical isomorphism of* $W_{\mathcal{K}}/E_{\mathcal{K}}$ *onto the dual group* $\hat{R}$. *Moreover,*

$$\operatorname{ind}_P^G \Theta^{MA}(\lambda_M, C_M, \chi, \nu) = \sum_{w \in W_{\mathcal{K}}/E_{\mathcal{K}} \cong \hat{R}} \operatorname{ind}_{P^*}^G \Theta^{M^*A^*}(\lambda, wC, \chi^*, \nu^*) \tag{8.9}$$

for any choice of the chamber C as in Lemma 8.5. *The characters on the right side of* (8.9) *are all nonzero and irreducible.*

Remarks. It is shown in the introduction how Theorem 8.7 can be interpreted as a classification of irreducible tempered representations. Part of that argument leads to the following corollary.

COROLLARY 8.8. *Every irreducible tempered character is basic.*

Proof. It is known that every irreducible tempered character is a constituent of a basic character induced from discrete series. (See Langlands [21] and Trombi [28]; the result is also implicit in the work of Harish-Chandra [5], though not explicitly stated there.) Theorem 8.7 shows that these irreducible constituents are all basic.

The proof of Theorem 8.7 will be accomplished by the sequence of lemmas in the remainder of this section. We regard the roots $\alpha_1, \ldots, \alpha_q$ in $\mathcal{K}$ as listed in some particular enumeration, and we build intermediate data $(\mathfrak{m}_i, \mathfrak{a}_i, \mathfrak{b}_i)$ and subgroups

$$M = M_{(0)} \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_q = M^*$$

by means of the successive Cayley transforms $\mathbf{d}_{\alpha_i}$. For each i, let $C_j^{(i)}$ denote the various chambers in $i\mathfrak{b}_i$ given by $(wC)^{p_{\tilde{\alpha}_q} \cdots p_{\tilde{\alpha}_{i+1}}}$. (See (4.2) and footnote 7.)

[14] ... after identification of p_{α_i} with $p_{\tilde{\alpha}_i}$ by the Cayley transform.

Define

$$\lambda_i = \text{extension of } \lambda_M \text{ by } 0,$$
$$\chi_i = \text{restriction of } \chi \text{ to } Z_{M_i},$$
$$\nu_i = \text{restriction of } \nu \text{ to } \mathfrak{a}_i.$$

LEMMA 8.9. *For each i with $0 \le i \le q$,*

$$\operatorname{ind}_{P\cap M_iA_i}^{M_iA_i} \Theta^{MA}(\lambda_M, C_M, \chi, \nu) = \sum_{\text{some } j\text{'s}} \Theta^{M_iA_i}\big(\lambda_i, C_j^{(i)}, \chi_i, \nu_i\big), \tag{8.10}$$

with λ_i satisfying the appropriate conditions so that the right side makes sense.

Remark. As we observed earlier, the case $i = q$ implies that the characters in the statements of Lemma 8.6 and Theorem 8.7 are meaningful.

Proof. The proof is by induction on i by means of generalized Schmid identities and double induction. To use generalized Schmid identities, we apply Theorem 6.1. In view of the statement of that theorem, we have to check successive ρ_α type conditions like (6.1), and we have to see that the chambers are the ones asserted. (The orthogonality condition on ν_i follows from the fact that $\langle \nu, \alpha_j \rangle = 0$ for all j, which was noted in (8.5).)

Let us first check the ρ_α condition. Let $\rho_{\alpha_{i+1}}^{(i)}$ be half the sum of the roots of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a}_i + \mathfrak{b}_i)^{\mathbf{C}})$ whose restriction to $\mathfrak{a}_i$ is $c\alpha_{i+1}$ with $c > 0$. We are to check that

$$\chi(\gamma_{\alpha_{i+1}}) = (-1)^{2\langle \rho_{\alpha_{i+1}}^{(i)}, \alpha_{i+1} \rangle / |\alpha_{i+1}|^2}. \tag{8.11}$$

We know from Lemma 8.2 that α_{i+1} is not in Δ'. Since $\langle \lambda + i\nu, \alpha_{i+1} \rangle = 0$, we see from (7.6c) that

$$\chi(\gamma_{\alpha_{i+1}}) = (-1)^{2\langle \rho_{\alpha_{i+1}}^{(0)}, \alpha_{i+1} \rangle / |\alpha_{i+1}|^2}. \tag{8.12}$$

Hence (8.11) will follow if we show that

$$\frac{2\langle \rho_{\alpha_{i+1}}^{(i)} - \rho_{\alpha_{i+1}}^{(0)}, \alpha_{i+1} \rangle}{|\alpha_{i+1}|^2} \quad \text{is in } 2\mathbf{Z}. \tag{8.13}$$

That is, we are to show that the sum (with multiplicities) of all coefficients $c_{i+1} > 0$ of all roots β of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} + \mathfrak{b})^{\mathbf{C}})$ of the form

$$\beta \equiv c_{i+1}\alpha_{i+1} + c_i\alpha_i + \cdots + c_1\alpha_1 \mod\big((\mathfrak{a}^* + \mathfrak{b})^{\mathbf{C}}\big)'$$

with $c_i, \ldots, c_1$ not all zero is an even integer.

There are two situations. If β has a nonzero component β_I in $((\mathfrak{a}^* + \mathfrak{b})^{\mathbf{C}})'$, then we note that c_{i+1} is an integer or half-integer. If j is an index $\le i$ with $c_j \ne 0$, we can produce new roots by changing the sign of c_j and by changing the sign of β_I. Then β occurs in a set of $4n$ mates for some n, and $4nc_{i+1}$ is an even integer.

If β has zero component in $((\mathfrak{a}^* + \mathfrak{b})^{\mathbf{C}})'$, then Proposition 8.4 shows that $c_i = \cdots = c_1 = 0$, and there is no contribution to the expression in (8.13). Thus (8.13) holds, and (8.11) follows.

We are left with showing that the chambers in the generalized Schmid identities can be selected in such a way that (8.10) results. Fix a reference chamber C in $i\mathfrak{b}^*$ as in Lemma 8.5. To each w in $W_{\mathfrak{X}}$ corresponds a set of signs s_j, $1 \leq j \leq q$, such that

$$\lambda + \sum_{j=1}^{q} \varepsilon_j s_j \tilde{\alpha}_j \tag{8.14}$$

is in wC as long as the ε_j are sufficiently small and positive. Then $(wC)^{p_{\tilde{\alpha}_q} \cdots p_{\tilde{\alpha}_{i+1}}}$ is the unique Weyl chamber of $i\mathfrak{b}_i$ containing the orthogonal projection of (8.14) to $i\mathfrak{b}_i$, namely

$$\lambda + \sum_{j=1}^{i} \varepsilon_j s_j \tilde{\alpha}_j. \tag{8.15}$$

For $i = 0$, the chamber in question is C_M, the one in $i\mathfrak{b}$ containing λ. Suppose $C_j^{(i-1)} = (w_0C)^{p_{\tilde{\alpha}_q} \cdots p_{\tilde{\alpha}_i}}$ occurs at stage $i - 1$ on the right side of the lemma. We apply Theorem 6.1 to

$$\operatorname{ind}_{P_{i-1} \cap M_i A_i}^{M_i A_i} \Theta^{M_{i-1} A_{i-1}}\left(\lambda_{i-1}, C_j^{(i-1)}, \chi_{i-1}, \nu_{i-1}\right).$$

According to the theorem, this character is the right side of a generalized Schmid identity, and the chamber(s) on the left side can be taken to be C' and/or $p_{\tilde{\alpha}_i}C'$ in $i\mathfrak{b}_i$ such that

(a) λ_i is C'-dominant,

(b) $(\overline{C'} \cap i\mathfrak{b}_{i-1}) \cap C_j^{(i-1)}$ has nonempty interior.

Choose

$$C' = C_{j,1}^{(i)} = (w_0 C)^{p_{\tilde{\alpha}_q} \cdots p_{\tilde{\alpha}_{i+1}}},$$

$$p_{\tilde{\alpha}_i} C' = C_{j,2}^{(i)} = p_{\tilde{\alpha}_i} C_{j,1}^{(i)} = \left(p_{\tilde{\alpha}_i} w_0 C\right)^{p_{\tilde{\alpha}_q} \cdots p_{\tilde{\alpha}_{i+1}}}.$$

The C'-positive roots have positive inner product with (8.15), hence nonnegative inner product with λ; thus (a) holds. The set in (b) contains the sum of a small ball in $i\mathfrak{b}$ about λ plus $\sum_{j=1}^{i-1} \varepsilon_j s_j \tilde{\alpha}_j$ for all sufficiently small positive ε_j, and thus it has nonempty interior. A similar argument shows $p_{\tilde{\alpha}_i}C'$ satisfies (a) and (b). Then we can select $C_{j,1}^{(i)}$ and/or $C_{j,2}^{(i)}$ as the chambers that occur at stage i on the right side of the lemma, and the induction is complete.

Let $n(i)$, $0 \leq i \leq q$, denote the number of terms on the right side of the character identity (8.10) at stage i. It will turn out that none of the characters on the right side of (8.10) is zero. But until we know this fact, we count both zero and nonzero characters in computing $n(i)$. In passing from stage $i-1$ to stage i, we use a generalized Schmid identity of one of the two types (4.4a) or (4.4b), and the condition that decides which type to use does not depend on the chambers in question. Therefore

$$\begin{aligned} n(0) &= 1, \\ n(i) &= 2n(i-1) && \text{if identity (4.4a) is used in passing} \\ &&& \quad \text{from stage } i-1 \text{ to stage } i, \\ n(i) &= n(i-1) && \text{if identity (4.4b) is used in passing} \\ &&& \quad \text{from stage } i-1 \text{ to stage } i. \end{aligned}$$

The decisive property of the R group is that it controls completely when $n(i)$ jumps in value.

LEMMA 8.10. *The function $n(i)$ satisfies $n(i) = 2n(i-1)$ if there exists an element r of R of the form*

$$r = \Big(\prod_{\text{some } j<i} p_{\alpha_j} \Big) p_{\alpha_i}. \tag{8.16}$$

Proof. Assume the contrary, so that identity (4.4b) is used in passing from stage $i-1$ to stage i. From the statement of Theorem 4.3, and especially equations (4.11) and (4.12) in the proof, it follows that there exists z in $Z_{M_{i-1}}$ such that

$$p_{\alpha_i}\chi_{i-1}(z) \neq \chi_{i-1}(z).$$

The other factor $\prod p_{\alpha_j}$ of r in (8.16) has a representative in M_{i-1}, and hence it fixes χ_{i-1}. Thus we obtain $r\chi_{i-1} \neq \chi_{i-1}$, and we cannot have $r\chi = \chi$, a contradiction.

LEMMA 8.11. *The function $n(i)$ satisfies $n(q) = |R|$. Consequently*

(a) *the characters* $\operatorname{ind}_P^G \Theta^{MA}(\lambda, C_j^{(q)}, \chi^*, \nu^*)$, *induced from those on the right side of* (8.10) *when $i = q$, are nonzero and irreducible, and*

(b) $n(i) = n(i-1)$ *if there exists no element r of R of the form* (8.16).

Proof. For each i, Lemma 8.5 shows that λ_i is nonsingular with respect to all roots of $(\mathfrak{m}_i^{\mathbf{C}}, \mathfrak{b}_i^{\mathbf{C}})$ except $\pm\tilde{\alpha}_1, \ldots, \pm\tilde{\alpha}_i$, and these roots are all noncompact. By Theorem 1.1b the characters on the right side of (8.10) are all nonzero. Hence the induced characters are nonzero. If there are $\geq |R|$ terms in the sum, then there are exactly $|R|$ and the induced characters are irreducible, by Theorem 13.4 of [16].

We shall show that $n(i)$ jumps at least $\log_2|R|$ times as a result of Lemma 8.10. Then $n(q) \geq |R|$, and it follows that $n(q) = |R|$ and (a) holds. Moreover, $n(q) = |R|$ means that all jumps of $n(i)$ have been accounted for by Lemma 8.10, and hence (b) holds.

Write out each element of R as the product of its reflections in order, and let $i_1 < i_2 < \cdots < i_k$ be the final indices that appear. Lemma 8.10 says that $n(i)$ jumps between $i_j - 1$ and i_j, and hence it is enough to prove that $k = \log_2|R|$. Let R_j be the subgroup of R of elements with final index $\leq i_j$, and let $R_0 = \{1\}$. Then $|R_j/R_{j-1}| \geq 2$ since i_j is assumed to be a possible last index, and $|R_j/R_{j-1}| \leq 2$ since the product of two elements of R with last index i_j has last index $\leq i_{j-1}$. Thus $|R_j/R_{j-1}| = 2$ for $1 \leq j \leq k$, and $R_k = R$. Consequently $k = \log_2|R|$. This proves the lemma.

Lemma 8.11a shows that the character identity (8.10) for $i = q$ will be close to the desired identity (8.9) after we induce to G. We still have to identify the chambers $C_j^{(q)}$ that occur. For this purpose, we turn matters around, start with λ on $\mathfrak{b}^*$, and use generalized Schmid identities to work toward $\mathfrak{b}$. Let C be any chamber in $i\mathfrak{b}^*$ as in Lemma 8.5, and define

$$W_i = \text{subgroup of } W_{\mathfrak{K}} \text{ generated by } p_{\tilde{\alpha}_{i+1}}, \ldots, p_{\tilde{\alpha}_q},$$
$$E_i = E_{\mathfrak{K}} \cap W_i.$$

LEMMA 8.12. *For* $0 \leq i \leq q$,

$$\sum_{w \in W_i} \Theta^{M^*A^*}(\lambda, wC, \chi^*, \nu^*) = |E_i| \operatorname{ind}_{P_i \cap M^*A^*}^{M^*A^*} \Theta^{M_iA_i}\left(\lambda_i, C^{p_{\tilde{\alpha}_q} \cdots p_{\tilde{\alpha}_{i+1}}}, \chi_i, \nu_i\right).$$

Proof. Lemmas 8.9 and 9.5 show that the left side is defined for each i. We shall prove the identity by induction on i downward, the case $i = q$ being obvious. Suppose inductively that the identity is known for i for any reference chamber C; we prove it for $i - 1$.

Let us suppress some of the variables in the notation. We interpret Theorem 4.3 as saying

$$\Theta^{M_iA_i}\left(C^{p_{\tilde{\alpha}_q} \cdots p_{\tilde{\alpha}_{i+1}}}, \chi_i\right) + \Theta^{M_iA_i}\left(p_{\tilde{\alpha}_i} C^{p_{\tilde{\alpha}_q} \cdots p_{\tilde{\alpha}_{i+1}}}, \chi_i\right)$$
$$= c_i \operatorname{ind}_{P_{i-1} \cap M_iA_i}^{M_iA_i} \Theta^{M_{i-1}A_{i-1}}\left(C^{p_{\tilde{\alpha}_q} \cdots p_{\tilde{\alpha}_{i+1}} p_{\tilde{\alpha}_i}}, \chi_{i-1}\right), \tag{8.17}$$

where

$$c_i = \begin{cases} 2 & \text{if } n(i) = n(i-1) \\ 1 & \text{if } n(i) = 2n(i-1). \end{cases}$$

In fact, there are two cases. If the relevant generalized Schmid identity is (4.4a), then (8.17) holds once we observe from (8.12) that $\chi_{i-1}(\gamma_{\alpha_i})$ has the value required by (4.2). If the relevant generalized Schmid identity is (4.4b), then (8.17) is one of the versions of (4.4b) multiplied through by 2; since $\chi_{i-1}(\gamma_{\alpha_i})$ has the value required by (4.2), either $(\zeta \otimes \chi_i)^+$ or $(\zeta \otimes \chi_i)^-$ matches χ_{i-1}.

Now we apply our inductive hypothesis to write

$$\sum_{w \in W_i} \Theta^{M^*A^*}(wC) = |E_i| \operatorname{ind}_{P_i \cap M^*A^*}^{M^*A^*} \Theta^{M_iA_i}\left(C^{p_{\tilde{\alpha}_q} \cdots p_{\tilde{\alpha}_{i+1}}}, \chi_i\right)$$

and

$$\begin{aligned}\sum_{w \in W_i} \Theta^{M^*A^*}(wp_{\tilde{\alpha}_i}C) &= |E_i| \operatorname{ind}_{P_i \cap M^*A^*}^{M^*A^*} \Theta^{M_iA_i}\left((p_{\tilde{\alpha}_i}C)^{p_{\tilde{\alpha}_q} \cdots p_{\tilde{\alpha}_{i+1}}}, \chi_i\right) \\ &= |E_i| \operatorname{ind}_{P_i \cap M^*A^*}^{M^*A^*} \Theta^{M_iA_i}\left(p_{\tilde{\alpha}_i}C^{p_{\tilde{\alpha}_q} \cdots p_{\tilde{\alpha}_{i+1}}}, \chi_i\right).\end{aligned}$$

We add, substitute from (8.17), and use the double induction formula on the right side. The result completes our inductive proof of the lemma, provided we prove the formula in Lemma 8.13 below.

LEMMA 8.13. *For* $1 \le i \le q$,

$$|E_{i-1}| = \begin{cases} |E_i| & \text{if } n(i) = 2n(i-1) \\ 2|E_i| & \text{if } n(i) = n(i-1). \end{cases}$$

Remark: The proof of this lemma will be given after Lemma 8.15.

LEMMA 8.14. *Suppose* $w = (\prod_{\text{some } j > i} p_{\tilde{\alpha}_j}) p_{\tilde{\alpha}_i}$ *is in* $W(B^* : M^*)$ *and* x *is a representative in* $K \cap M^*$. *Then* $p_{\tilde{\alpha}_i}$ *is in* $W(B_i : M_i)$, *and there exists* b^* *in* B^* *such that* b^*x *is a representative.*

Proof. We shall use root vectors $E_{\tilde{\alpha}_j}$ relative to $(\mathfrak{m}^{*\mathbf{C}}, \mathfrak{b}^{*\mathbf{C}})$, with the normalization as in the definition of $\mathbf{c}_{\tilde{\alpha}_j}$ in (2.7). For any b^* in B^*, b^*x is in M^* and so $\operatorname{Ad}(b^*x)$ is 1 on $\mathfrak{a}^*$. We want to choose b^* so that $\operatorname{Ad}(b^*x)$ is 1 also on the remaining generators of $\mathfrak{a}_i$, which are $E_{\tilde{\alpha}_j} + E_{-\tilde{\alpha}_j}$ for $i+1 \le j \le q$. Then b^*x will be in M_i, $\operatorname{Ad}(b^*x)$ will act as $p_{\tilde{\alpha}_i}$ on $\mathfrak{b}_i$, and the proof will be complete.

First suppose j is not one of the indices with $p_{\tilde{\alpha}_j}$ appearing in w. Then

$$\begin{aligned}\operatorname{Ad}(x)E_{\tilde{\alpha}_j} &= cE_{\tilde{\alpha}_j} \quad \text{with } |c| = 1, \\ \operatorname{Ad}(x)E_{-\tilde{\alpha}_j} &= dE_{-\tilde{\alpha}_j} \quad \text{with } |d| = 1, \\ \operatorname{Ad}(x)H'_{\tilde{\alpha}_j} &= H'_{\tilde{\alpha}_j}.\end{aligned}$$

Since $[E_{\tilde{\alpha}_j}, E_{-\tilde{\alpha}_j}] = H'_{\tilde{\alpha}_j}$, it follows that $cd = 1$. Writing $c = e^{i\theta_j}$, we have

$$\operatorname{Ad}(x)E_{\tilde{\alpha}_j} = e^{i\theta_j}E_{\tilde{\alpha}_j} \quad \text{and} \quad \operatorname{Ad}(x)E_{-\tilde{\alpha}_j} = e^{-i\theta_j}E_{-\tilde{\alpha}_j}.$$

If we set

$$b_j^* = \exp \tfrac{1}{2} i\theta_j H'_{\tilde{\alpha}_j}, \tag{8.18}$$

then we have

$$\mathrm{Ad}(b_j^*)E_{\tilde{\alpha}_j} = e^{i\theta_j}E_{\tilde{\alpha}_j} \quad \text{and} \quad \mathrm{Ad}(b_j^*)E_{-\tilde{\alpha}_j} = e^{-i\theta_j}E_{-\tilde{\alpha}_j},$$

and

$$\mathrm{Ad}\big(b_j^{*-1}x\big)\big(E_{\tilde{\alpha}_j} + E_{-\tilde{\alpha}_j}\big) = E_{\tilde{\alpha}_j} + E_{-\tilde{\alpha}_j}. \tag{8.19}$$

Next suppose that $j \geq i+1$ is one of the indices with $p_{\tilde{\alpha}_j}$ appearing in w. Then

$$\mathrm{Ad}(x)E_{\tilde{\alpha}_j} = cE_{-\tilde{\alpha}_j} \quad \text{with } |c| = 1$$
$$\mathrm{Ad}(x)E_{-\tilde{\alpha}_j} = dE_{\tilde{\alpha}_j} \quad \text{with } |d| = 1$$
$$\mathrm{Ad}(x)H'_{\tilde{\alpha}_j} = -H'_{\tilde{\alpha}_j},$$

and again we conclude $cd = 1$. Writing $c = e^{-i\theta_j}$ and defining b_j^* by (8.18), we again obtain (8.19). Then

$$\mathrm{Ad}\big(b_{i+1}^{*-1} \cdots b_q^{*-1}x\big)\big(E_{\tilde{\alpha}_j} + E_{-\tilde{\alpha}_j}\big) = E_{\tilde{\alpha}_j} + E_{-\tilde{\alpha}_j}$$

for $j \geq i+1$, and $b^* = b_{i+1}^{*-1} \cdots b_q^{*-1}$ is the required element.

LEMMA 8.15. *Suppose $p_{\tilde{\alpha}_i}$ is in $W(B_i : M_i)$. Then some product*

$$w = \Big(\prod_{\text{some } j>i} p_{\tilde{\alpha}_j}\Big)p_{\tilde{\alpha}_i} \tag{8.20}$$

is in $W(B^* : M^*)$.

Proof. Let x be a member of $K \cap M_i$ representing $p_{\tilde{\alpha}_i}$. For $j > i$ we use root vectors $X_{\tilde{\alpha}_j}$ relative to $(\mathfrak{m}^{*\mathbf{C}}, (\mathfrak{a}_i + \mathfrak{b}_i)^*)$, with the normalization as in the definition of $\mathbf{d}_{\alpha_j}$ in (2.5). Since x is in M_i, $\mathrm{Ad}(x) = 1$ on $\mathfrak{a}_i$. Thus the vector H_{α_j} in $\mathfrak{a}_i$ satisfies

$$\mathrm{Ad}(x)H_{\alpha_j} = H_{\alpha_j} \quad \text{for } j > i. \tag{8.21}$$

Then we have

$$\mathrm{Ad}(x)X_{\alpha_j} = c_jX_{\alpha_j} \quad \text{with } |c_j| = 1.$$

Since x is in G and X_{α_j} is in $\mathfrak{g}$, c_j is real. Thus

$$\mathrm{Ad}(x)X_{\alpha_j} = s_jX_{\alpha_j}$$

with $s_j = \pm 1$. By (8.21) we have also

$$\mathrm{Ad}(x)\theta X_{\alpha_j} = s_j\theta X_{\alpha_j},$$

so that

$$\mathrm{Ad}(x)(X_{\alpha_j} + \theta X_{\alpha_j}) = s_j(X_{\alpha_j} + \theta X_{\alpha_j}), j > i.$$

The elements here are generators of the complement of $\mathfrak{b}_i$ in $\mathfrak{b}^*$. Thus if we define

$$w = \left(\prod_{\substack{j>i \text{ with} \\ s_j = -1}} p_{\tilde{\alpha}_j} \right) p_{\tilde{\alpha}_i},$$

then w is in $W(B^* : M^*)$ with x as a representative.

Proof of Lemma 8.13. First we observe that $E_i \subseteq E_{i-1}$ with index at most 2 because the product of two elements in E_{i-1} but not E_i does not involve $p_{\tilde{\alpha}_i}$ and therefore lies in E_i.

If $|E_{i-1}| = 2|E_i|$, then choose an element

$$w = \left(\prod_{\text{some } j>i} p_{\tilde{\alpha}_j} \right) p_{\tilde{\alpha}_i}$$

in E_{i-1} but not E_i. Lemma 8.14 shows that $p_{\tilde{\alpha}_i}$ is in $W(B_i : M_i)$. According to Theorem 4.3, the relevant generalized Schmid identity in passing between M_{i-1} and M_i is (4.4b), and hence $n(i) = n(i-1)$.

Conversely if $n(i) = n(i-1)$, then the generalized Schmid identity is (4.4b), so that $p_{\tilde{\alpha}_i}$ is in $W(B_i : M_i)$. By Lemma 8.15 some element of the form (8.20) is in $W(B^* : M^*)$. This element is in E_{i-1} but not E_i. Hence $|E_{i-1}| = 2|E_i|$.

LEMMA 8.16. *$W_{\mathcal{K}}/E_{\mathcal{K}}$ and R have the same number of elements, and*

$$(8.22)\quad \mathrm{ind}_P^G \Theta^{MA}(\lambda_M, C_M, \chi, \nu) = \sum_{w \in W_{\mathcal{K}}/E_{\mathcal{K}}} \mathrm{ind}_{P^*}^G \Theta^{M^*A^*}(\lambda, wC, \chi^*, \nu^*),$$

with the characters on the right nonzero and irreducible.

Remark: With this lemma, the proof of Theorem 8.7 is complete except for the formula (8.8) and the isomorphism $W_{\mathcal{K}}/E_{\mathcal{K}} \cong \hat{R}$.

Proof. We start with the identity of Lemma 8.12 for $i = 0$ and induce from P^* to G. The result is

$$|E_{\mathcal{K}}| \, \mathrm{ind}_P^G \Theta^{MA}(\lambda_M, C_M, \chi, \nu) = \sum_{w \in W_{\mathcal{K}}} \mathrm{ind}_{P^*}^G \Theta^{M^*A^*}(\lambda, wC, \chi^*, \nu^*).$$

Lemma 8.6 allows us to rewrite this identity in the form (8.22). Now we compare (8.22) with the identity referred to in Lemma 8.11a. The latter identity has all its terms nonzero irreducible, and each of its terms is one of the ones on the right of

(8.22), distinct terms going into distinct terms by the Multiplicity One Theorem (Theorem 5.1). Hence the right side of (8.22) contains exactly the same terms as in the identity referred to in Lemma 8.11a, with no terms zero by Theorem 1.1b and Lemma 8.5. The lemma follows.

It is important to notice that our lemmas so far work for all enumerations of $\alpha_1,\ldots,\alpha_q$. In the final lemmas we shall several times select convenient enumerations.

We now identify each element r in R with the subset of elements of $\{\alpha_1,\ldots,\alpha_q\}$ mapped by r to their negatives. An element of r will be called *minimal* if the corresponding subset is minimal under inclusion, by comparison with the other elements of R.

LEMMA 8.17. *Suppose r in R is minimal and r is a consecutive product $r = p_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_i}$. Then $|W_{\mathcal{K}}/(W_iE_{\mathcal{K}})| = 2$.*

Remark. Recall that W_i is the group generated by $p_{\tilde{\alpha}_{i+1}},\ldots,p_{\tilde{\alpha}_q}$.

Proof. Lemma 8.16 gives

$$|W_0/E_0| = |W_{\mathcal{K}}/E_{\mathcal{K}}| = |R|\ ,$$

and Lemma 8.13 and the minimality of r imply that

$$|W_i| = 2^{-i}|W_0| \quad \text{and} \quad |E_i| = |E_{i-1}| = 2^{-(i-1)}|E_0|\ .$$

Thus

$$|W_i/E_i| = \tfrac{1}{2}|W_0/E_0| = \tfrac{1}{2}|R|\ .$$

Taking the quotient, we obtain

$$|W_{\mathcal{K}}/(W_iE_{\mathcal{K}})| = |(W_0/E_0)/(W_i/E_i)| = 2.$$

LEMMA 8.18. *Suppose r in R is minimal and e is any member of $E_{\mathcal{K}}$. After identification of each α_j and $\tilde{\alpha}_j$ by the Cayley transform, r and e have an even number of root reflections in common.*

Proof. Without loss of generality, we may take r to be a consecutive product $r = p_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_i}$. We show first that $p_{\tilde{\alpha}_j}$ is not in $W_iE_{\mathcal{K}}$ for $j \le i$. Since W_i does not depend on the enumeration of $\alpha_1,\ldots,\alpha_i$, we may assume $j = i$ without loss of generality. So suppose on the contrary that $p_{\tilde{\alpha}_i}$ is in $W_iE_{\mathcal{K}}$, say with

$$p_{\tilde{\alpha}_i} = w_ie \quad \text{and} \quad w_i = \prod_{\text{some } j>i} p_{\tilde{\alpha}_j}.$$

Then we can write $w_ip_{\tilde{\alpha}_i} = e \in W(B^* : M^*)$ and apply Lemma 8.14 to see that

$p_{\tilde{\alpha}_i}$ is in $W(B_i : M_i)$. Then the generalized Schmid identity from step $i-1$ to step i is of type (4.4b), and we have a contradiction to Lemma 8.10.

Next, we note that if j_1 and j_2 are $\leq i$, then $p_{\tilde{\alpha}_{j_1}} p_{\tilde{\alpha}_{j_2}}$ is in $W_i E_{\mathcal{K}}$. In fact, $W_{\mathcal{K}}$ has two cosets modulo $W_i E_{\mathcal{K}}$, by Lemma 8.17. Each of $p_{\tilde{\alpha}_{j_1}}$ and $p_{\tilde{\alpha}_{j_2}}$ is in the nontrivial coset, by the result of the previous paragraph, and their product must then be in the identity coset $W_i E_{\mathcal{K}}$.

Consequently every product of an even number of root reflections of r is in $W_i E_{\mathcal{K}}$, and it then follows from the result of the first paragraph that every product of an odd number of root reflections of r is not in $W_i E_{\mathcal{K}}$.

Now let e be given in $E_{\mathcal{K}}$, and write $e = e_r e_{\text{other}}$, where e_r is the product of the root reflections common to e and r. Then $e_r = e_{\text{other}} e$ exhibits e_r as in $W_i E_{\mathcal{K}}$, and it follows from the result of the previous paragraph that e_r is the product of an even number of root reflections.

LEMMA 8.19. *Suppose r is in R and e is any member of $E_{\mathcal{K}}$. After identification of each α_j and $\tilde{\alpha}_j$ by the Cayley transform, r and e have an even number of root reflections in common.*

Proof. First we show that the minimal elements of R generate R. More generally, let S be any subgroup of the group $W(p_1, \ldots, p_n)$ generated by n commuting reflections. We prove (inductively on n) that S has a $\mathbf{Z}_2$-basis of minimal elements. This is trivial for $n = 1$; assume it for $n - 1$.

Let φ be the homomorphism of $W(p_1, \ldots, p_n)$ onto $W(p_1, \ldots, p_{n-1})$ given by mapping p_n to 1, and let $\bar{\varphi}$ be the restriction of φ to S. Suppose $\bar{\varphi}$ is one–one. By inductive hypothesis, let $\{v_j, 1 \leq j \leq k\}$ be a basis of minimal elements of $\bar{\varphi}(S)$. Define $u_j = \bar{\varphi}^{-1}(v_j)$, $1 \leq j \leq k$. Then $\{u_j\}$ is a basis of S, and each u_j is minimal since φ preserves "inclusions." Thus the induction goes through if $\bar{\varphi}$ is one - one.

Let the above φ be called φ_n, and treat similarly φ_i, $1 \leq i < n$, where φ_i maps p_i to 1. The induction goes through unless $\bar{\varphi}_i$ fails to be one - one for all i. If $\bar{\varphi}_i$ is not one - one, then p_i is in S; if this happens for every i, then $\{p_1, p_2, \ldots, p_n\}$ is the required basis of minimal elements. This completes the argument that S has a $\mathbf{Z}_2$-basis of minimal elements.

Thus the minimal elements of R generate R. To prove the lemma, it is enough, in view of Lemma 8.18, to prove that the product of any two elements r_1 and r_2 of R having an even number of root reflections in common with e also has an even number of root reflections in common with e. In terms of the sets of reflections involved, group multiplication corresponds to symmetric difference, and passage to the common root reflections corresponds to intersection. Using set-theoretic notation, we are to show $e \cap (r_1 \triangle r_2)$ has an even number of elements when $e \cap r_1$ and $e \cap r_2$ do. But

$$\begin{aligned}|e \cap (r_1 \triangle r_2)| &= |(e \cap r_1) \triangle (e \cap r_2)| \\ &= |e \cap r_1| + |e \cap r_2| - 2|e \cap r_1 \cap r_2| \\ &\equiv |e \cap r_1| + |e \cap r_2| \mod 2,\end{aligned}$$

and the result follows.

Let us identify R canonically with a subgroup of $W_{\mathcal{K}}$ by means of the Cayley transform $\alpha_j \to \tilde{\alpha}_j$. Each w in $W_{\mathcal{K}}$ gives a character χ_w of $W_{\mathcal{K}}$ whose value on w' is $+1$ or -1 according as the number of root reflections that w and w' have in common is even or odd. The mapping $w \to \chi_w$ carries $W_{\mathcal{K}}$ onto the dual group $\hat{W}_{\mathcal{K}}$. Restriction of these characters χ_w to R (viewed as $\subseteq W_{\mathcal{K}}$) gives us a homomorphism $w \to \chi_w$ of $W_{\mathcal{K}}$ onto $\hat{R}$.

LEMMA 8.20. *The homomorphism $w \to \chi_w$ of $W_{\mathcal{K}}$ onto $\hat{R}$ sets up an isomorphism of $W_{\mathcal{K}}/E_{\mathcal{K}}$ onto $\hat{R}$, and $E_{\mathcal{K}}$ is given by* (8.8).

Remark. This lemma and Lemma 8.16 together prove Theorem 8.7.

Proof. Lemma 8.19 shows that $E_{\mathcal{K}}$ maps to 1 under the map $w \to \chi_w$ of $W_{\mathcal{K}}$ onto $\hat{R}$. Hence $W_{\mathcal{K}}/E_{\mathcal{K}}$ maps onto $\hat{R}$. By Lemma 8.16, $W_{\mathcal{K}}/E_{\mathcal{K}}$ and R have the same number of elements, and the same thing is then true of $W_{\mathcal{K}}/E_{\mathcal{K}}$ and $\hat{R}$. Consequently the map of $W_{\mathcal{K}}/E_{\mathcal{K}}$ onto $\hat{R}$ is an isomorphism.

In (8.8), $E_{\mathcal{K}}$ is contained in the right side of that formula, according to Lemma 8.19. The fact that $W_{\mathcal{K}}/E_{\mathcal{K}} \to \hat{R}$ is one-one then says that equality holds in (8.8). This completes the proof.

9. Final basic characters

A basic character is *final* if it is not the right side of any generalized Schmid identity (4.4a) or (4.4b). According to Theorem 6.1, $\mathrm{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu)$ is final if there is no real root α of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} + \mathfrak{b})^{\mathbf{C}})$ such that $\langle \nu, \alpha \rangle = 0$ and $\chi(\gamma_\alpha) = (-1)^{2\langle \rho_\alpha, \alpha \rangle / |\alpha|^2}$.

The following theorem uses some of the initial ideas of Section 8 but not Theorem 8.7 itself. It provides an alternate proof of Corollary 8.8.

THEOREM 9.1. *Every final basic character is irreducible* (*or zero*).

Proof. Let $\mathrm{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu)$ be final. Choose a positive system of roots for $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} + \mathfrak{b})^{\mathbf{C}})$ meeting the conditions of Lemma 3.1, and choose an $\mathfrak{m}$-nonsingular linear form μ as in that lemma such that (3.2) holds. The lemma says that

$$\operatorname{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu) = \psi_{\lambda+i\nu}^{\lambda+i\nu+\mu} \operatorname{ind}_P^G \Theta^{MA}(\lambda+\mu, C, \chi e^{\mu}, \nu). \tag{9.1}$$

Suppose that the induced representation on the right side of (9.1) is reducible. Then its R group is not $\{1\}$, by Theorem 13.4 of [16], and the superorthogonal set $\mathcal{H}$ for it is not empty. If α is in the set $\mathcal{H}$, then $\langle \nu, \alpha \rangle = 0$ by (8.5) and

$$\chi e^{\mu}(\gamma_\alpha) = (-1)^{2\langle \rho_\alpha, \alpha\rangle/|\alpha|^2}$$

by (8.12). Since e^{μ} is a character of $(\mathfrak{a}+\mathfrak{b})^{\mathbf{C}}$ and $\langle \mu, \alpha \rangle = 0$, $e^{\mu}(\gamma_\alpha) = 1$. Thus

$$\chi(\gamma_\alpha) = (-1)^{2\langle \rho_\alpha, \alpha\rangle/|\alpha|^2},$$

and the left side of (9.1) is not final. Consequently we conclude that the induced representation on the right side of (9.1) is irreducible.

By Theorem 1.3 of [31], the character on the left side of (9.1) is a multiple of an irreducible character. However, the character on the left side of (9.1) is contained in a basic character induced from discrete series, by Corollary 4.4, which is multiplicity-free by Theorem 5.1. Thus the character on the left side of (9.1) is irreducible or zero.

COROLLARY 9.2. *A nonzero basic character* $\operatorname{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu)$ *is irreducible if* $\langle \nu, \alpha \rangle \neq 0$ *for every* $\mathfrak{a}$*-root* α.

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Part II of this paper will appear in the next issue.

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Classification of irreducible tempered representations of semisimple groups

By A. W. Knapp and Gregg J. Zuckerman

Part II

10. Modified Plancherel factors

In Sections 10–11, we shall extend the theory of the R group to basic characters. The definition of Δ' in (8.2) for discrete series on M requires a non-obvious generalization for limits of discrete series on M. In order to give this generalization, we introduce *modified Plancherel factors* μ' as below. Modifying the original Plancherel factors μ of Section 7 will allow us to obtain the key property of μ' given in Corollary 10.6.

Let the basic character $\operatorname{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu)$ be given, with $\Theta^M(\lambda, C, \chi)$ the character of a discrete series or limit of discrete series representation ξ of M. Let α be an $\mathfrak{a}$-root; ε will denote a root of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} + \mathfrak{b})^{\mathbf{C}})$. If α is even, we define

$$\mu'_{\xi,\alpha}(i\nu) = \begin{cases} 1 & \text{if } \alpha \text{ not useful} \\ \prod_{\varepsilon|_{\mathfrak{a}}=\alpha} \langle \lambda + i\nu, \varepsilon\rangle & \text{if } \alpha \text{ useful.} \end{cases} \tag{10.1}$$

If α is odd and not an $\mathfrak{a}$-root for a maximal parabolic subgroup of split G_2 we define

$$\mu'_{\xi,\alpha}(i\nu) = \left(\prod_{\substack{\varepsilon|_{\mathfrak{a}}=c\alpha,\, c>0 \\ |\varepsilon| \le |\alpha'|}} \langle \lambda + i\nu, \varepsilon\rangle \right) f_{\xi,\alpha'}(i\nu), \tag{10.2}$$

where $\alpha' = 2\alpha$ or α according to whether 2α is an $\mathfrak{a}$-root or not and where $f_{\xi,\alpha'}$ is given by (7.6c).

Finally for a maximal parabolic subgroup of split G_2, there is exactly one pair of roots ε and $\bar{\varepsilon}$ such that $2\langle \varepsilon, \bar{\varepsilon}\rangle / |\varepsilon|^2 = +1$, and we take $\mu'_{\xi,\alpha}$ to be the same as $\mu_{\xi,\alpha}$ except that the factors contributed by this pair ε and $\bar{\varepsilon}$ are to be dropped.

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LEMMA 10.1. *The notation $\mu'_{\xi,\alpha}$ is unambiguous in the sense that the formula is unchanged if λ is replaced by $w\lambda$ with w in $W(B: M)$.*

Proof. We have
$$\langle w\lambda + i\nu, \varepsilon\rangle = \langle \lambda + i\nu, w^{-1}\varepsilon\rangle,$$
and $w^{-1}\varepsilon$ ranges over the same collection of roots in the definition as ε does.

By way of preparation for the next two propositions, let us recall that once C is fixed, $W(A: G)$ has an action on $\mathfrak{b}'$ preserving C-dominance. This action is defined in Theorem 3.7 of [13], and some of its properties are listed here under property (b) of the decomposition $W = W'R$ near the start of Section 8. A number of results connecting this action with discrete series extend to limits of discrete series.

LEMMA 10.2. *Let ξ be a discrete series or limit of discrete series representation of M with character $\Theta^M(\lambda, C, \chi)$, and define the action of $W(A: G)$ on $\mathfrak{b}'$ by Theorem 3.7 of [13] so as to preserve C-dominance. If p is in $W(A: G)$ and w is a representative of p in $N_K(\mathfrak{a})$, then $w\xi$ is a discrete series or limit of discrete series representation, respectively, and its character is $\Theta^M(p\lambda, C, p\chi)$.*

Proof. Without loss of generality we may assume that w is the special representative of p given in Theorem 3.7a of [13] with the property that $\mathrm{Ad}(w)$ agrees on $\mathfrak{b}'$ with the action of p. Taking φ to be the automorphism of M_0 given by $\varphi(m) = w^{-1}mw$ and applying Proposition 4.1 of [13], we see that

$$w\Theta^{M_0}(\lambda + \mu, C) = \Theta^{M_0}(p(\lambda + \mu), C) \tag{10.3}$$

whenever μ is C-dominant, M_0-integral, and M_0-nonsingular. (Cf. the proof of Proposition 4.7 of [13].) Fix such a μ and apply the functor $\psi_{p\lambda}^{p\lambda+p\mu}$ to both sides of (10.3). We shall show that

$$\psi_{p\lambda}^{p\lambda+p\mu} w\Theta = w\psi_{\lambda}^{\lambda+\mu}\Theta, \tag{10.4}$$

and then the result will be

$$\begin{aligned}\Theta^{M_0}(p\lambda, C) &= \psi_{p\lambda}^{p\lambda+p\mu} w\Theta^{M_0}(\lambda + \mu, C) \quad \text{by (10.3)}\\ &= w\psi_{\lambda}^{\lambda+\mu}\Theta^{M_0}(\lambda + \mu, C) \quad \text{by (10.4)}\\ &= w\Theta^{M_0}(\lambda, C).\end{aligned}$$

Extending both sides to $M^{\#}$ by $p\chi$ and inducing to M, we obtain the conclusion of the lemma.

Thus we are to prove (10.4). Let π be a representation with character Θ and with infinitesimal character $\chi_{\lambda+\mu}$ with parameter $\lambda + \mu$. We claim that

$$w\pi \text{ has infinitesimal character } p(\lambda + \mu) \tag{10.5}$$

and

$$wF_{-\mu} = F_{-p\mu}. \tag{10.6}$$

If so, then

$$\begin{aligned} w\psi_\lambda^{\lambda+\mu}\pi &= w(\pi \otimes F_{-\mu})_\lambda && \text{by definition} \\ &= \big(w(\pi \otimes F_{-\mu})\big)_{p\lambda} && \text{by (10.5) with } \lambda \text{ in place of } \lambda+\mu \\ &= (w\pi \otimes wF_{-\mu})_{p\lambda} \\ &= (w\pi \otimes F_{-p\mu})_{p\lambda} && \text{by (10.6)} \\ &= \psi_{p\lambda}^{p\lambda+p\mu} w\pi && \text{by (10.5) and definition.} \end{aligned}$$

Thus (10.4) follows if we prove (10.5) and (10.6). Equation (10.6) follows since $\mathrm{Ad}(w)$ acts on $\mathfrak{b}'$ the same way p does, so that $-p\mu$ is the lowest weight of $wF_{-\mu}$. For (10.5) we need to observe that, for z in the center of the universal enveloping algebra of $\mathfrak{m}^{\mathbf{C}}$, we have

$$\chi_{\lambda+\mu}\big(\mathrm{Ad}(w)^{-1}z\big) = \chi_{p(\lambda+\mu)}(z). \tag{10.7}$$

Then

$$w\pi(z) = \pi\big(\mathrm{Ad}(w)^{-1}z\big) = \chi_{\lambda+\mu}\big(\mathrm{Ad}(w)^{-1}z\big)I = \chi_{p(\lambda+\mu)}(z)I$$

and (10.5) follows. Equation (10.7) can be phrased more generally as a statement about automorphisms of $\mathfrak{m}^{\mathbf{C}}$ that map $\mathfrak{b}^{\mathbf{C}}$ into itself and preserve the positive system of roots and can then be proved readily by using Verma modules. This completes the proof of the lemma.

LEMMA 10.3. *Let ξ be a discrete series or limit of discrete series representation of M with character $\Theta^M(\lambda, C, \chi)$, and define the action of $W(A\colon G)$ on $\mathfrak{b}'$ by Theorem 3.7 of [13] so as to preserve C-dominance. If p is in $W(A\colon G)$ and w is a representative of p in $N_K(\mathfrak{a})$, then*

(a) *$w\xi$ is equivalent with ξ if and only if $p\lambda = \lambda$ and $p\chi = \chi$, and*

(b) *$w\xi|_{M_0}$ is equivalent with $\xi|_{M_0}$ if and only if $p\lambda = \lambda$.*

Proof. Result (a) is immediate from Theorem 1.1c and Lemma 10.2. For (b), we use Lemma 10.2 and Theorem 1.1e to see that

$$\xi|_{M_0} \text{ has character } \sum_{s \in W(B\colon M)/W(B\colon M_0)} \Theta^{M_0}(s\lambda, sC) \tag{10.8a}$$

and

$$w\xi|_{M_0} \text{ has character } \sum_{s \in W(B\colon M)/W(B\colon M_0)} \Theta^{M_0}(sp\lambda, sC). \tag{10.8b}$$

Each of $w\xi|_{M_0}$ and $\xi|_{M_0}$ is fully reducible, and they are equivalent if and only if their characters are equal. Thus we can read off conclusion (b) from (10.8) and Theorem 1.1c.

PROPOSITION 10.4. *Let ξ be a discrete series or limit of discrete series representation of M with character $\Theta^M(\lambda, C, \chi)$, and define the action of $W(A: G)$ on $\mathfrak{b}'$ by Theorem 3.7 of [13] so as to preserve C-dominance. If α is an even $\mathfrak{a}$-root and ν is real-valued in $\mathfrak{a}'$, then the following are equivalent:*

a) $\mu'_{\xi,\alpha}(i\nu) = 0$,

b) $p_\alpha\lambda = \lambda$ *in the action of* $W(A: G)$ *on* $\mathfrak{b}'$, *and* $\langle \nu, \alpha\rangle = 0$,

c) $p_\alpha\xi \cong \xi$ *and* $p_\alpha\nu = \nu$.

Remark. It is implicit in (b) and (c) that α is useful, since p_α exists.

Proof. Suppose (a) holds. By (10.1), α is useful. Since (a) holds, some root $\varepsilon' = \alpha + \alpha'$ has

$$0 = \langle \lambda + i\nu, \varepsilon'\rangle = \langle \lambda, \alpha'\rangle + i\langle \nu, \alpha\rangle.$$

Thus $\langle \lambda, \alpha'\rangle = \langle \nu, \alpha\rangle = 0$. Choose α'' to be a minimal positive element in $(i\mathfrak{b})'$ such that $\varepsilon'' = \alpha + \alpha''$ is a root. Then $\alpha' \pm \alpha''$ are roots of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ of the same sign, say positive. We have

$$\begin{aligned} 0 = \langle \lambda, \alpha'\rangle &= \langle \lambda, \alpha' \pm \alpha''\rangle \mp \langle \lambda, \alpha''\rangle \\ &\geq \mp\langle \lambda, \alpha''\rangle, \end{aligned}$$

since λ is C-dominant. Thus $\langle \lambda, \alpha''\rangle = 0$. According to Lemma 3.8 of [13], p_α acts on λ by $p_{\alpha''}$, and hence $p_\alpha\lambda = \lambda$. Thus (b) holds.

Suppose (b) holds. Since $\langle \nu, \alpha\rangle = 0$, we have $p_\alpha\nu = \nu$. Since $p_\alpha\lambda = \lambda$, Lemma 10.3 shows that (b) will follow if $p_\alpha\chi = \chi$. The equality $p_\alpha\chi = \chi$ is proved in this case exactly as in the discrete series case (Proposition 5.1a of [13]). Thus (c) holds.

Finally suppose (c) holds. Then $\langle \nu, \alpha\rangle = 0$ and it is enough to find a root $\varepsilon'' = \alpha + \alpha''$ such that $\langle \lambda, \alpha''\rangle = 0$. Lemma 10.3 above and Lemma 3.8 of [13] show that the minimal positive α'' in $(i\mathfrak{b})'$ such that $\alpha + \alpha''$ is a root will have $p_{\alpha''}\lambda = \lambda$ and hence $\langle \lambda, \alpha''\rangle = 0$. Thus (a) holds.

PROPOSITION 10.5. *Suppose α is an $\mathfrak{a}$-root of odd multiplicity and ξ is a discrete series or limit of discrete series representation of M. If ν is real and $\mu'_{\xi,\alpha}(i\nu) = 0$, then $\langle \nu, \alpha\rangle = 0$ and*

(a) $\xi(\gamma_\alpha) = +I$ *or*

(b) $2\langle \alpha, \beta\rangle/|\beta|^2$ *is even for every $\mathfrak{a}$-root β of odd multiplicity.*

Proof. Suppose $\mu'_{\xi,\alpha}(i\nu) = 0$. Then certainly $\langle \nu, \alpha \rangle = 0$. We are to prove (a) holds or (b) holds.[15] Suppose (a) fails. In this case we shall produce below a complex root γ with $\gamma|_{\mathfrak{a}} = \frac{1}{2}\alpha$. We can use such a root γ to prove (b). If β is as in (b), then

$$\frac{\langle \alpha, \beta \rangle}{|\beta|^2} = \frac{2\langle \frac{1}{2}\alpha, \beta \rangle}{|\beta|^2} = \frac{2\langle \gamma, \beta \rangle}{|B|^2} \text{ is in } \mathbf{Z}.$$

Hence $2\langle \alpha, \beta \rangle / |\beta|^2$ is even, and (b) holds.

Assuming (a) fails, we produce γ. Since $\mu'_{\xi,\alpha}(i\nu) = 0$, either $f_{\xi,\alpha}$ is given by a tangent in (7.6c) or there exists a complex root γ of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} + \mathfrak{b})^{\mathbf{C}})$ whose factor $\langle \lambda + i\nu, \gamma \rangle$ is present in $\mu'_{\xi,\alpha}$ and is 0. In the latter case, $\gamma|_{\mathfrak{a}}$ is a positive multiple of α and $|\gamma| \leq |\alpha|$; hence $\gamma|_{\mathfrak{a}} = \frac{1}{2}\alpha$. Thus without loss of generality we may assume that $f_{\xi,\alpha}$ is given by a tangent in (7.6c) and that (a) fails, i.e., that $\xi(\gamma_\alpha) = -I$. In view of (7.6c), we see that $2\langle \rho_\alpha, \alpha \rangle / |\alpha|^2$ is even. Now

$$\frac{2\langle \rho_\alpha, \alpha \rangle}{|\alpha|^2} = \frac{1}{2} \text{ multiplicity} \left(\frac{1}{2}\alpha \right) + \text{multiplicity}(\alpha).$$

The multiplicity of α is odd, and we conclude the multiplicity of $\frac{1}{2}\alpha$ is nonzero. Thus γ exists, and the result follows.

COROLLARY 10.6. *If α is an $\mathfrak{a}$-root, ν is a real-valued member of $\mathfrak{a}'$, and ξ is a discrete series or limit of discrete series representation such that $\mu'_{\xi,\alpha}(i\nu) = 0$, then $p_\alpha \xi \cong \xi$ and $p_\alpha \nu = \nu$.*

Proof. For α even, the result is contained in Proposition 10.4. Thus we may assume that α is odd and (after multiplying α by a constant) that α has odd multiplicity. Clearly $\langle \nu, \alpha \rangle = 0$ and hence $p_\alpha \nu = \nu$. Lemma 3.8 of [13] shows that p_α operates trivially on $\mathfrak{b}'$. Thus by Lemma 10.3, it is enough to prove that $p_\alpha \xi(z) = \xi(z)$ for z in Z_M. Applying Lemma 2.1b and recalling the discussion in connection with equation (1.6), we see that it is enough to prove that $p_\alpha \xi(z) = \xi(z)$ for z in $F(B)$, hence for $z = \gamma_\beta$ such that β is an $\mathfrak{a}$-root of odd multiplicity. Now

$$p_\alpha \xi(\gamma_\beta) = \xi(\gamma_{p_\alpha \beta}) = \xi(\gamma_\beta) \xi(\gamma_\alpha)^{2\langle \alpha, \beta \rangle / |\beta|^2},$$

and the right side is $\xi(\gamma_\beta)$ by Proposition 10.5. This completes the proof.

PROPOSITION 10.7. *If α is an $\mathfrak{a}$-root, ν is real-valued in $\mathfrak{a}'$, and ξ is a discrete series representation of M, then $\mu'_{\xi,\alpha}$ vanishes at $i\nu$ if and only if $\mu_{\xi,\alpha}$ vanishes at $i\nu$.*

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[15] In a maximal parabolic in split G_2, (b) holds. Thus we may and shall disregard such cases.

Remark. Even when ξ is a limit of discrete series representation, $\mu_{\xi,\alpha}$ is obtained from $\mu'_{\xi,\alpha}$ by multiplying by a polynomial factor. Hence the vanishing of $\mu'_{\xi,\alpha}$ implies the vanishing of $\mu_{\xi,\alpha}$.

Proof. Suppose α is even. If α is useful, then $\mu'_{\xi,\alpha} = \mu_{\xi,\alpha}$. If α is not useful, then $\mu'_{\xi,\alpha}$ is 1 and

$$\begin{aligned}\mu_{\xi,\alpha}(i\nu) &= \prod \langle \lambda + i\nu, \alpha + \tfrac{1}{2}\delta\rangle\langle \lambda + i\nu, \alpha - \tfrac{1}{2}\delta\rangle \\ &= \prod\left[-\langle \nu, \alpha\rangle^2 - \tfrac{1}{4}\langle \lambda, \delta\rangle^2\right],\end{aligned}$$

with the products taken over certain roots δ of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$, by Lemma 7.4a; $\mu_{\xi,\alpha}$ cannot vanish anywhere since λ is M-nonsingular. Thus $\mu'_{\xi,\alpha}$ and $\mu_{\xi,\alpha}$ are both nonvanishing.

Suppose α is odd. Scale α so that it is of odd multiplicity. Apart from the G_2 cases, which can be handled by inspection, the factors that appear in $\mu_{\xi,\alpha}$ and not in $\mu'_{\xi,\alpha}$ are $\langle \lambda + i\nu, \gamma\rangle$ when $|\gamma| > |\alpha|$ and $\gamma|_{\mathfrak{a}} = c\alpha$ with $c > 0$. Then $|\gamma|^2 = 2|\alpha|^2$ and it follows that $\gamma = \alpha + \delta$, where δ is a root of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$. For such a root γ,

$$\langle \lambda + i\nu, \gamma\rangle = i\langle \nu, \alpha\rangle + \langle \lambda, \delta\rangle,$$

and this cannot vanish since λ is M-nonsingular. The proposition follows.

Now we are in a position to imitate a certain amount of the theory of the R group. For ξ a discrete series or limit of discrete series representation of M and for ν real-valued in $\mathfrak{a}'$, we define

$$\begin{aligned}W = W_{\xi,\nu} &= \{w \in W(A: G) \mid w\xi \cong \xi \text{ and } w\nu = \nu\} \\ &= \{w \in W(A: B) \mid w\Theta^{MA} = \Theta^{MA}\},\end{aligned}$$

where $\Theta^{MA} = \Theta^{MA}(\lambda, C, \chi, \nu)$. Let

$$\Delta' = \{\alpha \text{ root of } (\mathfrak{g}, \mathfrak{a}) \mid \mu'_{\xi,\alpha}(i\nu) = 0 \text{ and } \tfrac{1}{3}\alpha \text{ not an } \mathfrak{a}\text{-root}\}.$$ [16]

Proposition 10.7 shows that the definitions of W and Δ' are consistent with the earlier definitions in the case that ξ is in the discrete series. Note that every member of Δ' is useful, i.e., Δ' is contained in Δ.

Lemma 10.8. *If w is in $W(A: G)$, then $\mu'_{\xi,w\alpha}(i\nu) = \mu'_{w^{-1}\xi,\alpha}(iw^{-1}\nu)$.*

Proof. Suppose $\varepsilon|_{\mathfrak{a}} = cw\alpha$ with $c > 0$. We let w^{-1} act on $\mathfrak{b}'$ by means of Theorem 3.7 of [13], and we identify the effect on ξ by Lemma 10.2. Then

$$\langle \lambda + i\nu, \varepsilon\rangle = \langle w^{-1}(\lambda + i\nu), w^{-1}\varepsilon\rangle \text{ and } w^{-1}\varepsilon|_{\mathfrak{a}} = c\alpha,$$

[16] The condition on $\frac{1}{3}\alpha$ eliminates one positive $\mathfrak{a}$-root in one of the maximal parabolics of split G_2.

and $w^{-1}\lambda$ is the parameter for $w^{-1}\xi$. Thus the polynomial parts of $\mu'_{\xi, w\alpha}(iw\nu)$ and $\mu'_{w^{-1}\xi, \alpha}(i\nu)$ match.

Now suppose α has odd multiplicity. We have to see that $f_{\xi, w\alpha}(i\nu) = f_{w^{-1}\xi, \alpha}(iw^{-1}\nu)$. The arguments of the two functions are

$$\frac{\pi\langle i\nu, w\alpha\rangle}{|w\alpha|^2} \text{ and } \frac{\pi\langle iw^{-1}\nu, \alpha\rangle}{|\alpha|^2}$$

and so are equal. The two functions are the same, tangent or cotangent, because

$$\xi(\gamma_{w\alpha}) = w^{-1}\xi(\gamma_\alpha)$$

and

$$(-1)^{2\langle \rho_{w\alpha}, w\alpha\rangle/|w\alpha|^2} = (-1)^{2\langle \rho_\alpha, \alpha\rangle/|\alpha|^2}.$$

Proposition 10.9. *If w is in W and α is in Δ', then $w\alpha$ is in Δ'.*

Proof. By Lemma 10.8 and by the unambiguity proved in Lemma 10.1, we have

$$\mu'_{\xi, w\alpha}(i\nu) = \mu'_{w^{-1}\xi, \alpha}(iw^{-1}\nu) = \mu'_{\xi, \alpha}(i\nu),$$

and the result follows.

Corollary 10.10. *Δ' is a root system (possibly nonreduced).*

Proof. We have to show that Δ', as a subset of Δ, is closed under its own reflections. Let α and β be in Δ'. Then p_β is in W by Corollary 10.6, and $p_\beta\alpha$ is in Δ' by Proposition 10.9.

Define

$$W' = \text{Weyl group of } \Delta',$$
$$R = \{w \in W \mid w\alpha > 0 \text{ for all } \alpha > 0 \text{ in } \Delta'\}.$$

Corollary 10.11. *W is a semidirect product $W = W'R$ with W' normal.*

Proof. If α is in Δ', then p_α is in W by Corollary 10.6, and it follows that $W' \subseteq W$. If w is in W, then $wp_\alpha w^{-1} = p_{w\alpha}$, so that Proposition 10.9 shows that $wp_\alpha w^{-1}$ is in W'. Hence $wW'w^{-1}$ is in W', and W' is normal. The semidirect product decomposition is then standard.

The basic algebraic properties of the R group generalize in the expected way. We shall give brief proofs, referring often to [13] and to Section 8 of the present paper; however, we allow ν to be nonzero.

LEMMA 10.12. *Let*

$W_\lambda = \{w \in W(A: G) \mid w\lambda = \lambda$ *in action of* $W(A: G)$ *on* $\mathfrak{b}'\}$,

$W_e =$ *subgroup of* $W(A: G)$ *generated by reflections in even simple roots in* Δ,

$W_{e,\lambda} = W_e \cap W_\lambda$.

Then $W_{e,\lambda}$ *is generated by the reflections that it contains; hence*

$$\Delta_{e,\lambda} = \{\beta \in \Delta \mid p_\beta \in W_{e,\lambda}\}$$

is a root system and $W_{e,\lambda}$ *is its Weyl group.*

Proof. W_e is a Weyl group, and Theorem 3.7 says that its action on $\mathfrak{b}'$ is isometric with its action on $\mathfrak{a}'$. Thus Chevalley's Lemma applies, and the subgroup of W_e that fixes λ is a Weyl group, hence is generated by its own reflections. The rest is clear.

Recall that the $\mathfrak{a}$-roots of odd multiplicity form a reduced root system Δ_0 in Δ, and the Weyl group of Δ_0 is denoted S.

LEMMA 10.13. $W_\lambda = W_{e,\lambda}S$. *Moreover,*

$$W_{e,\lambda} \subseteq W' \subseteq W \subseteq W_\lambda.$$

Proof. The equality $W_\lambda = W_{e,\lambda}S$ follows as in Proposition 6.2 of [13], since S acts trivially on $\mathfrak{b}'$, by Lemma 3.8 of [13].

LEMMA 10.14. *Let*

$$\Delta_\lambda = \Delta_0 \cup S\Delta_{e,\lambda}.$$

Then Δ_λ *is a reduced root system on (a subspace of)* $\mathfrak{a}$, *and its Weyl group is* W_λ.

Proof. This is proved in the same way as Lemma 6.4 of [13].

LEMMA 10.15. *Let*

$$W_{\lambda,\nu} = \{w \in W_\lambda \mid w\nu = \nu\}$$

and

$$\Delta_{\lambda,\nu} = \{\alpha \in \Delta_\lambda \mid \langle \nu, \alpha \rangle = 0\}.$$

Then $\Delta_{\lambda,\nu}$ *is a reduced root system on (a subspace of)* $\mathfrak{a}$ *and its Weyl group is* $W_{\lambda,\nu}$. *Moreover,* $W' \subseteq W \subseteq W_{\lambda,\nu}$.

Proof. The first statement follows immediately from Lemma 10.14 and Chevalley's Lemma. The inclusion $W' \subseteq W$ is part of Corollary 10.11, and the inclusion $W \subseteq W_{\lambda,\nu}$ is implied by Lemma 10.3.

LEMMA 10.16. *R is contained in* $S \cap W_{\lambda,\nu}$.

Proof. We know that $R \subseteq W \subseteq W_{\lambda,\nu}$ by Lemma 10.15, and we show that $R \subseteq S$. Here $W_{\lambda,\nu}$ is the Weyl group of

$$\Delta_{\lambda,\nu} = \nu^{\perp} \cap (\Delta_0 \cup S\Delta_{e,\lambda}) = (\nu^{\perp} \cap \Delta_0) \cup (\nu^{\perp} \cap S\Delta_{e,\lambda}).$$

We first prove that $\nu^{\perp} \cap S\Delta_{e,\lambda} \subseteq \Delta'$. If α is in $\nu^{\perp} \cap S\Delta_{e,\lambda}$, then α is an even $\mathfrak{a}$-root and $\langle \nu, \alpha \rangle = 0$. Also $\alpha = s\beta$ with s in S and β in $\Delta_{e,\lambda}$, so that Lemma 3.8 of [13] gives

$$p_{\alpha}\lambda = p_{s\beta}\lambda = sp_{\beta}s^{-1}\lambda = sp_{\beta}\lambda = s\lambda = \lambda.$$

By Proposition 10.4, $\mu'_{\xi,\alpha}(i\nu) = 0$. Thus α is in Δ'.

Let us notice also that the roots in $\nu^{\perp} \cap \Delta_0$ are odd and those in $\nu^{\perp} \cap S\Delta_{e,\lambda}$ are even. Thus the even roots in $\Delta_{\lambda,\nu}$ are in Δ'.

Now let r be in R, and write $p_{\alpha_1} \cdots p_{\alpha_n}$ as a minimal product of reflections that are simple for $\Delta_{\lambda,\nu}$. Suppose α_j is even. The minimality of the decomposition implies that $\gamma = p_{\alpha_n} \cdots p_{\alpha_{j+1}}(\alpha_j)$ is a positive root in $\Delta_{\lambda,\nu}$ such that $r\gamma < 0$. Since α_j is even, γ is even. Thus γ is in Δ', and we obtain the conclusion that r maps the positive member γ of Δ' into a negative root, a contradiction. We conclude that $\alpha_1, \ldots, \alpha_n$ are all odd, and the lemma follows.

LEMMA 10.17. *Suppose α and β are nonproportional nonorthogonal $\mathfrak{a}$-roots in $\Delta_{\lambda,\nu}$ with $|\alpha| \geq |\beta|$. If α and β are not in Δ', then $p_{\beta}\alpha$ is in Δ'.*

Proof. Proposition 10.4 shows that the even $\mathfrak{a}$-roots in $\Delta_{\lambda,\nu}$ are in Δ'. Since α and β are not in Δ', they must both be odd. The odd $\mathfrak{a}$-roots in $\Delta_{\lambda,\nu}$ are scaled so as to be of odd multiplicity. If we combine (7.6c) and the orthogonality of α and β to ν with the fact that $\mu'_{\xi,\alpha}(i\nu)$ and $\mu'_{\xi,\beta}(i\nu)$ are not 0, we conclude that α and β correspond to cotangent cases; i.e.,

$$\chi(\gamma_{\alpha}) = (-1)^{2\langle \rho_{\alpha}, \alpha \rangle / |\alpha|^2} \qquad \text{and} \qquad \chi(\gamma_{\beta}) = (-1)^{2\langle \rho_{\beta}, \beta \rangle / |\beta|^2}.$$

Now

$$\chi(\gamma_{p_{\beta}\alpha}) = \chi(\gamma_{\alpha})\chi(\gamma_{\beta})^{2\langle \alpha, \beta \rangle / |\alpha|^2} = \chi(\gamma_{\alpha})\chi(\gamma_{\beta}).$$

We shall show that $\mu'_{\xi, p_{\beta}\alpha}(i\nu) = 0$ by showing that $p_{\beta}\alpha$ corresponds to a tangent case. (Note that $\langle \nu, p_{\beta}\alpha \rangle = 0$.) In view of our computations and of (7.6c), we are to show that

$$\text{(10.9)} \qquad \frac{2\langle \rho_{p_{\beta}\alpha}, p_{\beta}\alpha \rangle}{|p_{\beta}\alpha|^2} \not\equiv \frac{2\langle \rho_{\alpha}, \alpha \rangle}{|\alpha|^2} + \frac{2\langle \rho_{\beta}, \beta \rangle}{|\beta|^2} \mod 2.$$

The term on the left of (10.9) equals the first term on the right. Thus we are to show that

$$\frac{2\langle \rho_\beta, \beta\rangle}{|\beta|^2} \equiv 1 \bmod 2. \tag{10.10}$$

Since β has odd multiplicity, (10.10) will follow if we show that $\frac{1}{2}\beta$ is not an $\mathfrak{a}$-root. If $\frac{1}{2}\beta$ is an $\mathfrak{a}$-root, then the simple component of Δ to which α and β belong is of type BC, and β is already a long root. Since $|\alpha| \geq |\beta|$, we must have $|\alpha| = |\beta|$. In a BC system of roots, any two long roots are proportional or orthogonal. We conclude that $\frac{1}{2}\beta$ cannot be an $\mathfrak{a}$-root and that (10.10) holds. This proves the lemma.

LEMMA 10.18. *Let q be the linear transformation on $\mathfrak{a}'$ given by* (8.4). *Then ν is orthogonal to* $(1-q)\mathfrak{a}'$, *and no $\mathfrak{a}$-root in* $(1-q)\mathfrak{a}'$ *lies in* Δ'.

Proof. This is the same as for Lemma 8.2.

LEMMA 10.19. *With q as in* (8.4), *the positive members of $\Delta_{\lambda,\nu}$ in the space* $(1-q)\mathfrak{a}'$ *are superorthogonal.*

Proof. This is the same as for Lemma 8.3.

PROPOSITION 10.20. $R = \Sigma \mathbf{Z}_2$. *In fact, with q as in* (8.4), *the set* $\mathcal{H} = \{\alpha_1, \ldots, \alpha_q\}$ *of positive $\mathfrak{a}$-roots of odd multiplicity in* $(1-q)\mathfrak{a}'$ *is superorthogonal and spans* $(1-q)\mathfrak{a}'$; *therefore,*

(a) *each r in R is of the form $p_{\alpha_{i_1}} \cdots p_{\alpha_{i_n}}$ with $\{\alpha_{i_1}, \ldots, \alpha_{i_n}\} \subseteq \mathcal{H}$, and*
(b) *each α_i in $\mathcal{H}$ satisfies $\langle \nu, \alpha_i\rangle = 0$, and*
(c) *each α_i occurs in the decomposition of some r in R.*

Proof. The second sentence is established as in Proposition 8.4. Then $R = \Sigma \mathbf{Z}_2$ follows from (a) and the superorthogonality of $\mathcal{H}$.

PROPOSITION 10.21. *Two basic characters* $\operatorname{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu)$ *and* $\operatorname{ind}_P^G \Theta^{MA}(\lambda, C', \chi, \nu)$, *with the same data except for chamber, have the same R group.*

Proof. The revised Plancherel factors μ' for each do not depend on the chamber and are thus equal. Hence Δ' is the same for each. For each one, the R group is contained in S, the Weyl group of the odd roots of $(\mathfrak{g}, \mathfrak{a})$, by Lemma 10.16. The action of S is trivial on the parameter λ, by Lemma 3.8b of [13], and the two basic characters have the same central character χ. By Lemma 10.3 the same members of S fix the two representations of M that are induced. Hence the R group is the same for both basic characters.

11. Intertwining operators

We shall now connect the extended theory of the R group with the theory of intertwining operators. With ξ as at the start of Section 10 and with ν real, let

$$U_P(\xi, i\nu) = \operatorname{ind}_P^G(\xi \otimes e^{i\nu} \otimes 1).$$

In [16], a construction is given of *normalized standard intertwining operators* $\mathcal{Q}_P(w, \xi, i\nu)$ for w in $N_K(\mathfrak{a})$ with the following properties:

(i) $\mathcal{Q}_P(w, \xi, i\nu)$ is holomorphic in ν (on K-finite vectors) for ν real and also is unitary for each real ν.

(ii) $U_P(w\xi, iw\nu)\mathcal{Q}_P(w, \xi, i\nu) = \mathcal{Q}_P(w, \xi, i\nu)U_P(\xi, i\nu)$.

(iii) $E\mathcal{Q}_P(w, \xi, i\nu)E^{-1} = \mathcal{Q}_P(w, E\xi E^{-1}, i\nu)$ if E is a unitary operator on the space on which ξ operates.

(iv) $\mathcal{Q}_P(w_1w_2, \xi, i\nu) = \mathcal{Q}_P(w_1, w_2\xi, iw_2\nu)\mathcal{Q}_P(w_2, \xi, i\nu)$ (11.1).

See Sections 6–8 of [16]. Under the assumption that $w\xi$ is equivalent with ξ, Lemma 7.9 of [16] shows that it is possible to define $\xi(w)$ and extend ξ, without enlarging its representation space, to be defined on the subgroup of G generated by M and w; the definition of $\xi(w)$ is unique up to a scalar factor equal to a root of unity. The composition $\xi(w)\mathcal{Q}_P(w, \xi, i\nu)$ then depends only on the class of w in $W(\mathfrak{a})$. If w represents a member of the stability group $W = W_{\xi,\nu}$, then $\xi(w)\mathcal{Q}_P(w, \xi, i\nu)$ commutes with the induced representation $U_P(\xi, i\nu)$ and will be called a *normalized standard self-intertwining operator* for $U_P(\xi, i\nu)$.

THEOREM 11.1. *Suppose ξ is a nonzero discrete series or limit of discrete series representation of M and ν is a real-valued member of $\mathfrak{a}'$. Then*

(a) *The normalized standard self-intertwining operators for $U_P(\xi, i\nu)$ corresponding to members of W' are all scalar.*

(b) *The normalized standard self-intertwining operators for $U_P(\xi, i\nu)$ corresponding to all members of W span the commuting algebra of $U_P(\xi, i\nu)$.*

(c) *The dimension of the commuting algebra of $U_P(\xi,i\nu)$ is $\leq |R|$.*

Remark. If ξ is in the discrete series, this result is contained in Theorem 13.4 of [16].

Proof of (c) *when* (a) *and* (b) *are known.* By Corollary 10.11, $W = W'R$. The operators for W span the commuting algebra, by (b), and the operators multiply according to the group law of W except for scalar factors, by Lemma 13.1b of [16]. The operators for W' are scalar, by (a), and hence the operators for R span the commuting algebra. Then (c) follows.

Proof of (a). The group W' is generated by the reflections p_α in the (necessarily useful) $\mathfrak{a}$-roots α in Δ'. Thus we are to show that if $\mu'_{\xi,\alpha}(i\nu) = 0$, then

$\xi(p_\alpha)\mathcal{A}_P(p_\alpha, \xi, i\nu)$ is scalar. In the orthogonal complement of α in $\mathfrak{a}'$, there is a dense set of members $\nu^\perp$ such that $\langle \nu + \nu^\perp, \beta \rangle \neq 0$ for all $\mathfrak{a}$-roots β that are not multiples of α. Fix such an element $\nu^\perp$. Then $p_\alpha(\nu + \nu^\perp) = \nu + \nu^\perp$, and we have

$$\mu'_{\xi,\alpha}(i(\nu + \nu^\perp)) = \mu'_{\xi,\alpha}(i\nu) = 0 \tag{11.2}$$

since $\mu'_{\xi,\alpha}$ factors through the projection on $\mathbf{R}\alpha$. We shall prove that $U_P(\xi, i(\nu + \nu^\perp))$ is irreducible. Then the self-intertwining operator $\xi(p_\alpha)\mathcal{A}_P(p_\alpha, \xi, i(\nu + \nu^\perp))$ must be scalar, and hence the limiting operator $\xi(p_\alpha)\mathcal{A}_P(p_\alpha, \xi, i\nu)$, obtained by letting $\nu^\perp$ tend to 0, must be scalar.

Let ξ have character $\Theta^M(\lambda, C, \chi)$. By Theorem 9.1, $U_P(\xi, i(\nu + \nu^\perp))$ is irreducible unless α is odd and

$$\chi(\gamma_{\alpha'}) = (-1)^{2\langle \rho_{\alpha'}, \alpha' \rangle / |\alpha'|^2}, \tag{11.3}$$

since the only $\mathfrak{a}$-roots orthogonal to $\nu + \nu^\perp$ are the multiples of α. Here α' is the unique multiple of α that is a positive real root of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} + \mathfrak{b})^{\mathbf{C}})$. Thus suppose α is odd and (11.3) holds. Let us normalize matters so that $\alpha' = \alpha$. Equations (11.2) and (11.3) together imply there exists some complex root β such that

$$\beta|_{\mathfrak{a}} = c\alpha \quad \text{with } c \neq 0, \tag{11.4a}$$

$$\langle \lambda, \beta \rangle = 0, \tag{11.4b}$$

$$|\beta| \leq |\alpha|, \tag{11.4c}$$

the last condition coming from the use of μ' in place of μ.

We use the real root α to pass by a Cayley transform $\mathbf{d}_\alpha$ from the data $(\mathfrak{m}, \mathfrak{a}, \mathfrak{b})$ to data $(\mathfrak{m}^*, \mathfrak{a}^*, \mathfrak{b}^*)$. By Theorem 6.1, the character

$$\operatorname{ind}_{P \cap M^*A^*}^{M^*A^*} \Theta^{MA}(\lambda, C, \chi, \nu + \nu^\perp) \tag{11.5}$$

is the right side of a generalized Schmid identity (4.4a) or (4.4b). If the identity is (4.4b) with only one term on the left, then we have

$$\operatorname{ind}_{P^*}^G \Theta^{M^*A^*}(\lambda^*, C^*, \chi^*, (\nu + \nu^\perp)|_{\mathfrak{a}^*}) = \operatorname{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu + \nu^\perp) \tag{11.6}$$

for suitable λ^*, C^*, and χ^*. On the left side of (11.6), $(\nu + \nu^\perp)|_{\mathfrak{a}^*}$ is $\mathfrak{a}^*$-regular, by construction, and the character on the left side of (11.6) is irreducible by Theorem 9.1. Thus the character on the right side of (11.6) is irreducible, and hence $U_P(\xi, i(\nu + \nu^\perp))$ is irreducible.

Thus we may suppose that (11.5) is the right side of a generalized Schmid identity (4.4a) with two terms on the left. Inducing to G, we obtain

$$\begin{aligned} &\operatorname{ind}_{P^*}^G \Theta^{M^*A^*}(\lambda^*, C^*, \chi^*, (\nu + \nu^\perp)|_{\mathfrak{a}^*}) \\ &\qquad + \operatorname{ind}_{P^*}^G \Theta^{M^*A^*}(\lambda^*, p_{\tilde{\alpha}} C^*, \chi^*, (\nu + \nu^\perp)|_{\mathfrak{a}^*}) \\ &\qquad = \operatorname{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu + \nu^\perp). \end{aligned} \tag{11.7}$$

We shall use the Hecht-Schmid identity (Theorem 1.1b) to show that one of the terms on the left side of (11.7) is zero. Then the irreducibility of $U_P(\xi, i(\nu + \nu^\perp))$ follows as in the previous paragraph by means of Theorem 9.1.

We may suppose that α is not a real root of a maximal parabolic of split G_2 since this case leads to a generalized Schmid identity (4.4b). Possibly by replacing β by $-\beta$ we may assume that the root β satisfying (11.4) is such that $\mathbf{d}_\alpha(\beta) = \tilde{\beta}$ is positive for C^*. We may assume also that $\tilde{\beta}$ is as small as possible, relative to C^*, so that (11.4) holds. We prove that $\tilde{\beta}$ is C^*-simple.

Thus let $\tilde{\beta} = \tilde{\gamma} + \tilde{\varepsilon}$ with $\tilde{\gamma} > 0$ and $\tilde{\varepsilon} > 0$. We know that λ^* is just λ extended by 0. The equality

$$\langle \lambda^*, \tilde{\beta} \rangle = \langle \lambda, \beta \rangle = 0$$

of (11.4b) implies $\langle \lambda^*, \tilde{\gamma} \rangle = \langle \lambda^*, \tilde{\varepsilon} \rangle = 0$ since λ^* is C^*-dominant. Set $\gamma = \mathbf{d}_\alpha^{-1}(\tilde{\gamma})$ and $\varepsilon = \mathbf{d}_\alpha^{-1}(\tilde{\varepsilon})$. By (11.4a) and (11.4c), $\beta|_\mathfrak{a} = \pm\frac{1}{2}\alpha$. Since we have eliminated the exceptional cases in G_2, it follows that

$$\gamma|_\mathfrak{a} = c'\alpha \quad \text{and} \quad \varepsilon|_\mathfrak{a} = c''\alpha \quad \text{with} \quad c', c'' = 0, \pm\tfrac{1}{2}, \text{or } \pm 1.$$

Thus one of c' or c'' is $\pm\frac{1}{2}$. Say $\gamma|_\mathfrak{a} = \pm\frac{1}{2}\alpha$. Then γ satisfies (11.4a) and (11.4b). Since $p_\alpha\gamma|_\mathfrak{a} = \mp\frac{1}{2}\alpha$, we have $p_\alpha\gamma = \gamma \pm \alpha$. Thus $2\langle\gamma, \alpha\rangle/|\alpha|^2$ equals ∓ 1, and so $|\gamma| \leq |\alpha|$. Hence γ satisfies (11.4c). Then $0 < \tilde{\gamma} < \tilde{\beta}$ and γ satisfies (11.4), contradicting the minimality of $\tilde{\beta}$. We conclude that $\tilde{\beta}$ is C^*-simple.

It follows that $p_{\tilde{\alpha}}\tilde{\beta}$ is $p_{\tilde{\alpha}}\, C^*$-simple. Since λ^* is orthogonal to $\tilde{\beta}$ and $p_{\tilde{\alpha}}\tilde{\beta}$, one of the two terms on the left side of (11.7) will be 0, by the Hecht-Schmid identity, if either $\tilde{\beta}$ or $p_{\tilde{\alpha}}\tilde{\beta}$ is compact. However, (11.4c) implies that

$$\tilde{\beta} - p_{\tilde{\alpha}}\tilde{\beta} = \pm\tilde{\alpha},$$

and $\tilde{\alpha}$ is noncompact. Hence one of $\tilde{\beta}$ and $p_{\tilde{\alpha}}\tilde{\beta}$ is compact, one of the terms on the left side of (11.7) vanishes, and the proof of irreducibility goes through.

Proof of (b). Let ξ have character $\Theta^M(\lambda, C, \chi)$. By Proposition 3.1, we can write

$$\text{(11.8)} \quad \operatorname{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu) = \psi_{\lambda+i\nu}^{\lambda+i\nu+\mu} \operatorname{ind}_P^G \Theta^{MA}(\lambda + \mu, C, \chi e^\mu, \nu)$$

and thereby exhibit the character of $U_P(\xi, i\nu)$ as ψ of a basic character induced from discrete series. By the Multiplicity One Theorem (quoted as Theorem 5.1), we can write

$$\text{(11.9)} \qquad \operatorname{ind}_P^G \Theta^{MA}(\lambda + \mu, C, \chi e^\mu, \nu) = \Theta_1 + \cdots + \Theta_n,$$

where Θ_j are distinct irreducible characters and

$$\psi\Theta_j = \psi_{\lambda+i\nu}^{\lambda+i\nu+\mu}\Theta_j \quad \text{is} \quad \begin{cases} \text{nonzero} & \text{for } 1 \leq j \leq m \\ \text{zero} & \text{for } m + 1 \leq j \leq n. \end{cases}$$

By Theorem 1.3 of [31], $\psi\Theta_j$ is a multiple of an irreducible character. On the other hand, it is a contributor to the left side of (11.8), which in turn is contained in a basic character induced from discrete series, by Corollary 4.4. By the Multiplicity One Theorem, $\psi\Theta_j$ is therefore multiplicity free. Consequently $\psi\Theta_j$ is irreducible for $1 \leq j \leq m$. Moreover, the $\psi\Theta_j$ for $1 \leq j \leq m$ are distinct characters by another application of Corollary 4.4 and the Multiplicity One Theorem. All the representations in question are unitary. Hence we can describe the commuting algebras of the representations with characters (11.9) and (11.8) as follows. The one for (11.9) is n-dimensional, generated by the projections corresponding to each Θ_j, $1 \leq j \leq n$. The one for (11.8) is m-dimensional, generated by the projections corresponding to each $\psi\Theta_j$, $1 \leq j \leq m$. If $1 \leq j \leq m$, then ψ carries the projection for Θ_j to the projection for $\psi\Theta_j$ since ψ carries the identity morphism to the identity morphism. We conclude that ψ carries the commuting algebra corresponding to (11.9) *onto* the commuting algebra corresponding to (11.8).

By Theorem 13.4 of [16], the commuting algebra corresponding to (11.9) is spanned by standard normalized self-intertwining operators. Thus conclusion (b) of Theorem 11.1 will follow if we prove

PROPOSITION 11.2. *With notation as in* (11.8), *let* ξ' *be a discrete series representation with character* $\Theta^M(\lambda + \mu, C, \chi e^\mu)$. *If* w *is in* $N_K(\mathfrak{a})$ *and* $w\xi'$ *is equivalent with* ξ', *then* $w\xi$ *is equivalent with* ξ *and*

$$\psi_{\lambda+i\nu}^{\lambda+i\nu+\mu}(\xi'(w)\mathcal{a}_P(w, \xi', i\nu)) = c\xi(w)\mathcal{a}_P(w, \xi, i\nu)$$

for a nonzero constant $c = c(\nu)$.

Remarks. Equality of the operators in the proposition is to be understood in the following sense: ψ carries the Harish-Chandra module $U_P(\xi', i\nu)$ to a module equivalent with $U_P(\xi, i\nu)$, and under this identification the two operators correspond.

LEMMA 11.3. *With notation as in* (11.8), *let* ξ' *be a discrete series representation with character* $\Theta^M(\lambda + \mu, C, \chi e^\mu)$, *and let* ν' *be real-valued on* $\mathfrak{a}'$. *If* w *is in* $N_K(\mathfrak{a})$, *then*

$$\psi_{\lambda+i\nu'}^{\lambda+i\nu'+\mu}(\mathcal{a}_P(w, \xi', i\nu')) = c(\nu')\mathcal{a}_P(w, \xi, i\nu') \tag{11.10}$$

for a nonzero constant $c(\nu')$.

Proof. First suppose ν' is $\mathfrak{a}$-regular. In this case Corollary 9.2 shows that

$$U_P(\xi', i\nu') \quad \text{and} \quad U_P(w\xi', iw\nu') \tag{11.11a}$$

and

(11.11b) $\qquad U_P(\xi, i\nu') \quad$ and $\quad U_P(\xi, iw\nu')$

are all irreducible. The operator $\mathcal{Q}_P(w, \xi', i\nu')$ intertwines the two representations in (11.11a), and the irreducibility implies the operator is characterized by this intertwining property, up to a constant. Similar remarks apply to $\mathcal{Q}_P(w, \xi, i\nu')$ and (11.11b). Since ψ carries isomorphisms to isomorphisms, (11.10) follows for ν' regular.

Now let ν' be general. To complete the proof, it is enough to show that the operator on the left side of (11.10), regarded as an operator in the compact picture of $U_P(\xi, i\nu')$, varies continuously with ν'.

We shall use notation consistent with Section B of the appendix. Let V be the space of K_M-finite vectors for ξ', and let $V^{(\nu')}$ and $V_w^{(w\nu')}$ denote the compatible $(\mathfrak{M}\mathcal{Q}, K_M)$ modules corresponding to the action of MA by $\xi' \otimes e^{i\nu'}$ and $w\xi' \otimes e^{iw\nu'}$, respectively. Since

$$\psi_{\lambda+i\nu'}^{\lambda+i\nu'+\mu} = p_{\lambda+i\nu'} \circ \left[(-) \otimes F_{-\mu}\right],$$

we have

$$\psi_{\lambda+i\nu'}^{\lambda+i\nu'+\mu}(\mathcal{Q}_P(w, \xi', i\nu')) = p_{\lambda+i\nu'}\left[\mathcal{Q}_P(w, \xi', i\nu') \otimes I\right]. \tag{11.12}$$

On the right side of (11.12), we can regard $p_{\lambda+i\nu'}$ as an operator, rather than a functor, and the right side thus denotes a composition. The operator $\mathcal{Q}_P(w, \xi', i\nu') \otimes I$ has domain $\mathrm{ind}(V^{(\nu')} \otimes F_{-\mu})$ in the compact picture, the dependence is continuous in ν', and the image is contained in (the compact picture of) $\mathrm{ind}(V_w^{(w\nu')} \otimes F_{-\mu})$. We apply $p_{\lambda+i\nu'}$ (which is the same as $p_{w\lambda+iw\nu'}$ by Theorem 3.7 of [13]) to this space. We shall show that the operator $p_{\lambda+i\nu'}$ on $\mathrm{ind}(V_w^{(w\nu')} \otimes F_{-\mu})$ is independent of ν' when we view matters in the compact picture. This independence of ν' will prove the continuity.

Since $p_{\lambda+i\nu'}$ is a projection, it is enough to show that its image and kernel are independent of ν'. We form a composition series of MAN modules

$$F_{-\mu} = F^{(1)} \supseteq F^{(2)} \supseteq \cdots \supseteq F^{(n)} \supseteq F^{(n+1)} \supseteq 0$$

with irreducible quotients F_i and ${}^MF_{-\mu}$ in some order, as in the proof of Theorem B.1. As a sequence of MA modules, this sequence splits, and we can write

$$F_{-\mu} = F_1 \oplus F_2 \oplus \cdots \oplus F_n \oplus {}^MF_{-\mu}.$$

The proof of Theorem B.1 shows that

$$p_{\lambda+i\nu'}\left[\mathrm{ind}_{MAN}^G\left(V_w^{(w\nu')} \otimes F_i\right)\right] = 0 \quad \text{for } 1 \le i \le n$$

and then that

$$(11.13)\quad p_{\lambda+i\nu'}\Big[\operatorname{ind}_{MAN}^{G}\big(V_w^{(w\nu')}\otimes F_{-\mu}\big)\Big] = p_{\lambda+i\nu'}\Big[\operatorname{ind}_{MAN}^{G}\big(V_w^{(w\nu')}\otimes^M F_{-\mu}\big)\Big]$$
$$= \operatorname{ind}_{MAN}^{G}\Big[p_{\lambda+i\nu'}\big(V_w^{(w\nu')}\otimes^M F_{-\mu}\big)\Big]$$
$$= \operatorname{ind}_{MAN}^{G}\Big[p_{\lambda}\big(V_w^{(0)}\otimes^M F_{-\mu}\big)\otimes e^{iw\nu'}\Big].$$

Thus the image is independent of ν'. There are two contributions to the kernel, one from

$$\sum_{i=1}^{n}\oplus\operatorname{ind}_{MAN}^{G}\big(V_w^{(w\nu')}\otimes F_i\big)$$

and one from the part of

$$\operatorname{ind}_{MAN}^{G}\Big[\big(V_w^{(0)}\otimes^M F_{-\mu}\big)\otimes e^{iw\nu'}\Big]$$

complementary to the right side of (11.13). Both are independent of ν', and the proof of the lemma is complete.

LEMMA 11.4. *With notation as in* (11.8), *let ξ' be a discrete series representation with character $\Theta^M(\lambda+\mu, C, \chi e^{\mu})$, and define the action of* $W(A\colon G)$ *on* $\mathfrak{b}'$ *by Theorem* 3.7 *of* [13] *so as to preserve C-dominance. Let p be in* $W(A\colon G)$, *and let w be a representative of p in $N_K(\mathfrak{a})$. If $w\xi'$ is equivalent with ξ', then $w\xi$ is equivalent with ξ and $p\mu=\mu$.*

Proof. Without loss of generality, we may assume that w is the special representative of p given in Theorem 3.7a of [13] with the property that $\operatorname{Ad}(w)$ agrees on $\mathfrak{b}'$ with the action of p. Then $\operatorname{Ad}(w)$ normalizes $\mathfrak{a}+i\mathfrak{b}$ and defines a member $\bar{w}$ of the complex Weyl group $W(\mathfrak{a}\oplus i\mathfrak{b}\colon G^{\mathbf{C}})$. By Lemma 10.3a, we have $\bar{w}(\lambda+\mu)=\lambda+\mu$ and $p(\chi e^{\mu})=\chi e^{\mu}$. Since $\lambda+\mu$, λ, and μ are all $\mathfrak{g}$-dominant, $\bar{w}$ fixes λ and μ separately. Thus $p\lambda=\lambda$ and $p\mu=\mu$. Then it follows that $p\chi=\chi$. Since $p\lambda=\lambda$ and $p\chi=\chi$, Lemma 10.3a shows that $w\xi$ is equivalent with ξ.

Proof of Proposition 11.2. Lemma 11.4 shows that $w\xi$ is equivalent with ξ and therefore that $\xi'(w)\mathcal{C}_P(w,\xi',i\nu)$ and $\xi(w)\mathcal{C}_P(w,\xi,i\nu)$ are defined. Lemma 11.3 shows that ${}^G\psi=\psi_{\lambda+i\nu}^{\lambda+i\nu+\mu}$ carries $\mathcal{C}_P(w,\xi',i\nu)$ to a nonzero multiple of $\mathcal{C}_P(w,\xi,i\nu)$. Hence it is enough to show that ${}^G\psi(\xi'(w))$ is a nonzero multiple of $\xi(w)$.

Here $\xi'(w)$ and $\xi(w)$ are operators on the level of representations of G; they act on the values of the functions in the induced representation spaces. These operators are the images under the induction functor of operators $\xi'(w)$ and $\xi(w)$ on the level of representations of M, and we show first that ${}^M\psi(\xi'(w))$ is a multiple of $\xi(w)$.

In fact, ξ' and $w\xi'$ are equivalent, and $\xi'(w)$ implements the equivalence. Since ${}^M\psi$ respects equivalence of Harish-Chandra modules, we can regard ${}^M\psi$ as carrying ξ' to ξ and $w\xi'$ to $w\xi$, and then ${}^M\psi$ carries $\xi'(w)$ to an operator implementing the equivalence of ξ and $w\xi$. This operator must be $\xi(w)$, up to a scalar factor, since ξ is irreducible.

Hence, on the M level, ${}^M\psi(\xi'(w))$ is a multiple of $\xi(w)$. Then it follows from Corollary B.2 that, on the G level, ${}^G\psi(\xi'(w))$ is a multiple of $\xi(w)$. Thus Proposition 11.2 is completely proved.

12. Basic characters with nondegenerate data

We say that a basic character $\mathrm{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu)$ is given by *nondegenerate data* if, for each root $\tilde{\alpha}$ of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ with $\langle\lambda, \tilde{\alpha}\rangle = 0$, the reflection $p_{\tilde{\alpha}}$ is not in $W(B\colon M)$. Let us notice that this definition depends on how a character is written, not just on the character itself. If a basic character is exhibited as induced from discrete series, then it is given by nondegenerate data because $\langle\lambda, \tilde{\alpha}\rangle$ is nonzero for each root $\tilde{\alpha}$ of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$. Much of the theory of basic characters induced from discrete series extends to the case of nondegenerate data.

PROPOSITION 12.1. *A nonzero basic character* $\mathrm{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu)$ *is given by nondegenerate data if and only if* $\Theta^{MA}(\lambda, C, \chi, \nu)$ *is not equal to the left side of some generalized Schmid identity* (4.4a) *or* (4.4b).

Remark. In the case of (4.4a) with two terms on the left side of the identity, this condition means that Θ^{MA} cannot be taken as one of the terms on the left in such a way that the other term equals 0.

Proof. Suppose that $\Theta^{MA}(\lambda, C, \chi, \nu)$ equals the left side of a generalized Schmid identity (4.4a) or (4.4b). If the identity is (4.4b), then the C-simple noncompact root $\tilde{\alpha}$ with respect to which (4.4b) is constructed is such that $\langle\lambda, \tilde{\alpha}\rangle = 0$ and $p_{\tilde{\alpha}}$ is in $W(B\colon M)$. If the identity is (4.4a), let the left side of the identity be

$$\Theta^{MA}(\lambda, C, \chi, \nu) + \Theta^{MA}(\lambda, p_{\tilde{\beta}} C, \chi, \nu), \tag{12.1}$$

where $\tilde{\beta}$ is a C-simple noncompact root $\tilde{\beta}$ with $\langle\lambda, \tilde{\beta}\rangle = 0$. Our assumption, as amplified in the remark, is that the second term of (12.1) is 0. By Theorem 1.1b this means that there is a compact root $\tilde{\alpha}$ that is simple for $p_{\tilde{\beta}} C$ and satisfies $\langle\lambda, \tilde{\alpha}\rangle = 0$. Since $\tilde{\alpha}$ is compact, $p_{\tilde{\alpha}}$ is in $W(B\colon M_0) \subseteq W(B; M)$. Hence $\tilde{\alpha}$ has the required properties.

Conversely suppose that the data are degenerate. Let $\tilde{\alpha}$ be a root of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ such that $\langle\lambda, \tilde{\alpha}\rangle = 0$ and $p_{\tilde{\alpha}}$ is in $W(B\colon M)$. Without loss of generality we may assume that $\tilde{\alpha}$ is the smallest C-positive root with these properties.

First suppose $\tilde{\alpha}$ is noncompact. If $\tilde{\alpha}$ were not C-simple, we could write $\tilde{\alpha} = \tilde{\beta} + \tilde{\gamma}$ with $\tilde{\beta}$ and $\tilde{\gamma}$ both C-positive and one of $\tilde{\beta}$ and $\tilde{\gamma}$, say $\tilde{\gamma}$, compact. The C-dominance of λ would imply $\langle \lambda, \tilde{\gamma} \rangle = 0$, and we would have $p_{\tilde{\gamma}}$ in $W(B\colon M)$ since $\tilde{\gamma}$ is compact. Thus $\tilde{\gamma}$ would contradict the minimality of $\tilde{\alpha}$.

Thus if $\tilde{\alpha}$ is noncompact, $\tilde{\alpha}$ is C-simple. Then we can form a generalized Schmid identity (4.4b) from $\tilde{\alpha}$, and the left side will be $\Theta^{MA}(\lambda, C, \chi, \nu)$, as required.

Now suppose $\tilde{\alpha}$ is compact. Since our given character is non-zero, the Hecht-Schmid identity (Theorem 1.1b) implies that $\tilde{\alpha}$ is not C-simple. Hence we can write $\tilde{\alpha} = \tilde{\beta} + \tilde{\gamma}$ with $\tilde{\beta}$ and $\tilde{\gamma}$ both C-positive. The argument just given when $\tilde{\alpha}$ is noncompact shows that both $\tilde{\beta}$ and $\tilde{\gamma}$ are noncompact and C-simple. The C-dominance of λ implies $\langle \lambda, \tilde{\beta} \rangle = \langle \lambda, \tilde{\gamma} \rangle = 0$. Say $|\tilde{\beta}| \leq |\tilde{\gamma}|$. Form a generalized Schmid identity from $\tilde{\gamma}$. If the identity is of type (4.4b), we are done. Thus we may assume it is of type (4.4a). The left side is then of the form

$$\Theta^{MA}(\lambda, C, \chi, \nu) + \Theta^{MA}(\lambda, p_{\tilde{\gamma}}C, \chi, \nu). \tag{12.2}$$

The root $p_{\tilde{\gamma}}\tilde{\beta} = \tilde{\alpha}$ is simple for $p_{\tilde{\gamma}}C$, and it is compact and orthogonal to λ. Therefore the second term of (12.2) is 0 by the Hecht-Schmid identity (Theorem 1.1b), and the generalized Schmid identity is of the required form.

COROLLARY 12.2. *Every nonzero basic character can be given with nondegenerate data.*

Proof. If the data are degenerate, Proposition 12.1 allows us to rewrite the character with data corresponding to a more noncompact Cartan subgroup. Iterating this construction, we arrive ultimately at nondegenerate data, since a nonzero basic character given in terms of data from a minimal parabolic subgroup is necessarily given with nondegenerate data.

PROPOSITION 12.3. *Let*

$$\operatorname{ind}_P^G \Theta^{MA}(\lambda_M, C_M, \chi, \nu) = \sum_{w \in W_{\mathcal{K}}/E_{\mathcal{K}} \cong \hat{R}} \operatorname{ind}_{P^*}^G \Theta^{M^*A^*}(\lambda, wC, \chi^*, \nu^*)$$

be the canonical decomposition given in Theorem 8.7 of a basic character induced from discrete series into irreducible basic characters. In this identity, the irreducible characters on the right side are given by nondegenerate data.

Proof. Let $\mathcal{K} = \{\alpha_1, \ldots, \alpha_q\}$. If $\tilde{\alpha}$ is a root of $(\mathfrak{m}^{*\mathbf{C}}, \mathfrak{b}^{*\mathbf{C}})$ such that $\langle \lambda, \tilde{\alpha} \rangle = 0$, then Lemma 8.5 shows that $\tilde{\alpha} = \pm \tilde{\alpha}_j$ for some j. The reflection $p_{\tilde{\alpha}_j}$ is not in $W(B^*\colon M^*)$ by Lemma 8.10, and hence the data are nondegenerate.

PROPOSITION 12.4. *A basic character* $\operatorname{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu)$ *is given by nondegenerate data if and only if all four of these conditions are satisfied*:

(a) λ *is nonsingular with respect to the compact roots of* $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$.

(b) *For each* $\mathfrak{a}$*-root* α *that is not useful,* $\langle \lambda, \alpha + \frac{1}{2}\tilde{\delta} \rangle \neq 0$ *for every root* $\tilde{\delta}$ *of* $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ *such that* $\alpha + \frac{1}{2}\tilde{\delta}$ *is a root of* $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} + \mathfrak{b})^{\mathbf{C}})$.

(c) *For each real root* α *of* $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} + \mathfrak{b})^{\mathbf{C}})$, $\langle \lambda, \alpha + \tilde{\beta} \rangle \neq 0$ *for every root* $\tilde{\beta}$ *of* $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ *such that* $\alpha + \tilde{\beta}$ *is a root of* $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} + \mathfrak{b})^{\mathbf{C}})$.

(d) *In any simple component of* $\mathfrak{g}$ *that is isomorphic to split* G_2 *and whose intersection with* $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ *is a maximal parabolic, the component of* λ *is nonzero.*

Proof. Suppose (a) fails. Then there is a compact root $\tilde{\alpha}$ of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ with $\langle \lambda, \tilde{\alpha} \rangle = 0$. Since $\tilde{\alpha}$ is compact, $p_{\tilde{\alpha}}$ is in $W(B: M)$. Hence the data are degenerate.

Suppose (b) fails. Then $\tilde{\delta}$ is a noncompact root of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ for which $\langle \lambda, \tilde{\delta} \rangle = 0$, and Proposition 10a of [12] shows it can be taken to be C-simple. Construct a generalized Schmid identity by means of $\tilde{\delta}$, passing from the data $(\mathfrak{m}, \mathfrak{a}, \mathfrak{b})$ to data $(\mathfrak{m}_*, \mathfrak{a}_*, \mathfrak{b}_*)$ and letting $\delta = \mathbf{c}_{\tilde{\delta}}(\tilde{\delta})$. Then $\alpha + \frac{1}{2}\delta$ is a real root of $(\mathfrak{m}_*^{\mathbf{C}}, \mathfrak{b}_*^{\mathbf{C}})$. Suppose the generalized Schmid identity is of type (4.4a) with two terms on the left. Then Theorem 4.3 shows that the M_* central element $\gamma_{\alpha+\frac{1}{2}\delta}$ is in $\{1, \gamma_\delta\} Z_{(M_*)_0} Z_M$ and hence centralizes $H_{\tilde{\delta}}$. But $\mathrm{Ad}(\gamma_{\alpha+\frac{1}{2}\delta}) H_{\tilde{\delta}} = -H_{\tilde{\delta}}$ by a computation in $\mathrm{SL}(2, \mathbf{C})$. We conclude the generalized Schmid identity from $\tilde{\delta}$ is of type (4.4b). Hence the data are degenerate by Proposition 12.1.

Suppose (c) fails in the case of $\alpha + \tilde{\beta}$. Then $\tilde{\beta}$ is a root of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ and $\langle \lambda, \tilde{\beta} \rangle = 0$, and we may assume $\tilde{\beta}$ is C-positive. We have seen that if λ is orthogonal to a compact root of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$, then the data are degenerate. Thus we may assume without loss of generality that λ is nonsingular with respect to all compact roots of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$. Then it follows that $\tilde{\beta}$ is noncompact and C-simple. Construct a generalized Schmid identity by means of $\tilde{\beta}$, passing from the data $(\mathfrak{m}, \mathfrak{a}, \mathfrak{b})$ to data $(\mathfrak{m}_*, \mathfrak{a}_*, \mathfrak{b}_*)$ and letting $\beta = \mathbf{c}_{\tilde{\beta}}(\tilde{\beta})$. Then $\alpha + \beta$ is a real root of $(\mathfrak{m}_*^{\mathbf{C}}, \mathfrak{b}_*^{\mathbf{C}})$. Suppose the generalized Schmid identity is of type (4.4a). Then Theorem 4.3 shows that the M_* central element $\gamma_{\alpha+\beta}$ is in $\{1, \gamma_\beta\} Z_{(M_*)_0} Z_M$ and hence centralizes $H_{\tilde{\beta}}$. But $\mathrm{Ad}(\gamma_{\alpha+\beta}) H_{\tilde{\beta}} = -H_{\tilde{\beta}}$ by a computation in $\mathrm{SL}(2, \mathbf{C})$. We conclude that the generalized Schmid identity is of type (4.4b). Hence the data are degenerate by Proposition 12.1.

Suppose (d) fails. Then there is a root $\alpha + \frac{1}{2}\tilde{\delta}$ of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} + \mathfrak{b})^{\mathbf{C}})$ with $\alpha|_{\mathfrak{b}} = 0$ and $\tilde{\delta}|_{\mathfrak{a}} = 0$ such that $\pm\tilde{\delta}$ is a noncompact C-simple root of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ with $\langle \lambda, \tilde{\delta} \rangle = 0$. Then we can argue as when (b) fails to see that the data are degenerate.

Conversely suppose that (a), (b), (c), and (d) hold but that the data are degenerate. By (a) and Theorem 1.1b, $\Theta^{MA}(\lambda, C, \chi, \nu)$ is nonzero. By Proposition 12.1, $\Theta^{MA}(\lambda, C, \chi, \nu)$ is the left side of some generalized Schmid identity.

This identity cannot be of type (4.4a) since, by (a), λ is nonsingular with respect to all compact roots of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$. Thus the identity is of type (4.4b). Suppose that the identity is built from a noncompact C-simple root $\tilde{\beta}$ and that the Cayley transform $\mathbf{c}_{\tilde{\beta}}$ leads from the data $(\mathfrak{m}, \mathfrak{a}, \mathfrak{b})$ to data $(\mathfrak{m}_*, \mathfrak{a}_*, \mathfrak{b}_*)$. Let $\beta = \mathbf{c}_{\tilde{\beta}}(\tilde{\beta})$. Since the identity is of type (4.4b), Theorem 4.3 says that

$$|Z_{M_*}/\{1, \gamma_\beta\}Z_{(M_*)_0}Z_M| = 2. \tag{12.3}$$

By Lemma 2.1b, $Z_{M_*} = Z_{(M_*)_0}F(B_*)$. Consequently (12.3) implies that

$$F(B_*) \not\subseteq \{1, \gamma_\beta\}Z_{(M_*)_0}Z_M.$$

Thus we can choose a real root δ of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a}_* \oplus \mathfrak{b}_*)^{\mathbf{C}})$ such that

$$\gamma_\delta \text{ is not in } \{1, \gamma_\beta\}Z_{(M_*)_0}Z_M. \tag{12.4}$$

It is clear from (12.4) that $\delta \neq \pm\beta$. Also $\mathbf{c}_{\tilde{\beta}}^{-1}(\delta)$ cannot be a real root of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}})$ since otherwise γ_δ would be in Z_M, in contradiction to (12.4). Thus δ is of the form

$$\delta = c\beta + \alpha \quad \text{with} \quad c \neq 0, \alpha \neq 0, \langle \beta, \alpha \rangle = 0. \tag{12.5}$$

If $c = \pm 1$ in (12.5), then α is a real root of both $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a}_* \oplus \mathfrak{b}_*)^{\mathbf{C}})$ and $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}})$, and $\alpha + \tilde{\beta}$ is a complex root of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}})$ with $\langle \lambda, \alpha + \tilde{\beta} \rangle = 0$. Thus (c) fails, and we have a contradiction.

If $c = \pm\frac{1}{2}$ in (12.5), then we may assume $\delta = \alpha + \frac{1}{2}\beta$. So $\alpha + \frac{1}{2}\tilde{\beta}$ is a complex root. Thus

$$\frac{2\langle \alpha + \frac{1}{2}\tilde{\beta}, \alpha - \frac{1}{2}\tilde{\beta} \rangle}{|\alpha + \frac{1}{2}\tilde{\beta}|^2} = 2 - \frac{2\langle \alpha + \frac{1}{2}\tilde{\beta}, \tilde{\beta} \rangle}{|\alpha + \frac{1}{2}\tilde{\beta}|^2} = 1 \text{ or } 0 \text{ or } -1. \tag{12.6}$$

If (12.6) is -1, then $|\tilde{\beta}|^2 = 3|\alpha + \frac{1}{2}\tilde{\beta}|^2$ and (d) fails, a contradiction. If (12.6) is 0, then $|\tilde{\beta}|^2 = 2|\alpha + \frac{1}{2}\tilde{\beta}|^2$, 2α is a root, and

$$\gamma_{2\alpha}\gamma_\beta = \gamma_{\alpha + \frac{1}{2}\beta} = \gamma_\delta,$$

so that γ_δ is exhibited as in $Z_M\{1, \gamma_\beta\}$, in contradiction with (12.4). Thus (12.6) is $+1$, and we obtain a contradiction to either (b) or (d).

All possibilities for the root δ thus lead to contradictions, and we conclude that no generalized Schmid identity was possible. This completes the proof of the proposition.

COROLLARY 12.5. *Suppose the basic character* $\operatorname{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu)$ *is given by nondegenerate data. Then each* $\mathfrak{a}$*-root* α *has the property that* $\mu_{\xi,\alpha}(i\nu)$ *and* $\mu'_{\xi,\alpha}(i\nu)$ *are both zero or both nonzero.*

Proof. Suppose the conclusion fails for α. In view of Proposition 7.1 and the definition of μ', we must have

$$\mu_{\xi,\alpha}(i\nu) = 0 \qquad \text{and} \qquad \mu'_{\xi,\alpha}(i\nu) \neq 0,$$

and one of three things must happen:

(i) α is not useful and $\langle \lambda + i\nu, \alpha + \frac{1}{2}\tilde{\delta} \rangle = 0$ for some extension $\alpha + \frac{1}{2}\tilde{\delta}$ of α to a root of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}})$.

(ii) After multiplication by a scalar, α extends by 0 to a real root and there exists a root $\tilde{\beta}$ of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ such that $\alpha + \tilde{\beta}$ is a root of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a} \oplus \mathfrak{b})^{\mathbf{C}})$ and $\langle \lambda + i\nu, \alpha + \tilde{\beta} \rangle = 0$, or

(iii) α is an $\mathfrak{a}$-root of a maximal parabolic in a split G_2 that occurs as a simple component in $\mathfrak{g}$, and the component of λ in that split G_2 is 0.

Then (i), (ii), or (iii) shows respectively that (b), (c), or (d) fails in Proposition 12.4. Hence the proposition shows that the data are degenerate. The corollary follows.

The Plancherel factor $\mu_{\xi,\alpha}(i\nu)$ is used to define normalizing factors for the standard intertwining operators in [16]. Corollary 12.5 therefore allows us to define the R group of Section 10 in the nondegenerate case directly in terms of normalizing factors. Going over the proofs of Lemma 11.6 and Theorem 12.1 of [16] and taking the present Theorem 11.1 into account, we obtain the following conclusion.

THEOREM 12.6. *Suppose ξ is a nonzero discrete series or limit of discrete series representation of M and ν is a real-valued member of $\mathfrak{a}'$, and suppose that the character $\operatorname{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu)$ of the induced representation $U_P(\xi, i\nu)$ is given by nondegenerate data. Then*

(a) *The normalized standard self-intertwining operators for $U_P(\xi, i\nu)$ corresponding to members of R are linearly independent.*

(b) *The dimension of the commuting algebra of $U_P(\xi, i\nu)$ is exactly $|R|$.*

Remark. Theorem 13.3 will give an independent proof of this theorem, but without the conclusion that the unnormalized standard self-intertwining operators corresponding to R are regular at ν.

13. Complete reduction of basic characters with nondegenerate data

For basic characters given by nondegenerate data, we can imitate the proof of the decomposition of Theorem 8.7 for basic characters induced from discrete series.

Let

$$\operatorname{ind}_P^G \Theta^{MA}(\lambda_M, C_M, \chi, \nu) \tag{13.1}$$

be a basic character. As in Section 10, we form the stability group W in $W(A: G)$, and we let $W = W'R$ be the semidirect product decomposition of Corollary 10.11. The fundamental properties of the R group have been assembled at the end of Section 10 and in Theorem 11.1. *Throughout this section, we shall assume that* (13.1) *is given by nondegenerate data.*

Let $\mathcal{H} = \{\alpha_1, \ldots, \alpha_q\}$ be the superorthogonal system of positive $\mathfrak{a}$-roots of odd multiplicity given in Proposition 10.20. Then we can make the same definitions as after Proposition 8.4 (which dealt with the special case of basic characters induced from discrete series). Namely $\mathbf{d}_{\mathcal{H}}$ is the Cayley transform built from $\mathcal{H}$, and we are led from the data $(\mathfrak{m}, \mathfrak{a}, \mathfrak{b})$ to data $(\mathfrak{m}^*, \mathfrak{a}^*, \mathfrak{b}^*)$ with $\mathfrak{m}^* \supseteq \mathfrak{m}$, $\mathfrak{a}^* \subseteq \mathfrak{a}$, and $\mathfrak{b}^* \supseteq \mathfrak{b}$. Let

$$\mathfrak{b}^{\perp} = \sum_{j=1}^{q} i\mathbf{R}H_{\tilde{\alpha}_j}.$$

Then $\mathfrak{b}^{\perp}$ is the orthogonal complement of $\mathfrak{b}$ in $\mathfrak{b}^*$. We still define

$$\lambda = \begin{cases} \lambda_M & \text{on } \mathfrak{b} \\ 0 & \text{on } \mathfrak{b}^{\perp}, \end{cases}$$

$$\chi^* = \chi|_{Z_{M^*}} \quad \text{on } Z_{M^*},$$

$$\nu^* = \nu|_{\mathfrak{a}^*} \quad \text{on } \mathfrak{a}^*.$$

Let ρ_M be half the sum of the C_M-positive roots of $(\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$, and extend ρ_M to $(\mathfrak{b}^*)^{\mathbf{C}}$ by taking it to be 0 on $\mathfrak{b}^{\perp}$. Define $W_{\mathcal{H}}$ and $E_{\mathcal{H}}$ as in Section 8.

The assumption of nondegenerate data comes in partly through the following lemma, which generalizes Lemma 8.5. Nondegeneracy will play a critical role when we prove that the component characters of Theorem 13.3 are nonzero and irreducible (see Lemma 13.6).

LEMMA 13.1. *Under the assumption that* (13.1) *is given by nondegenerate data, the parameter λ satisfies $\langle \lambda, \tilde{\beta} \rangle \neq 0$ for all roots $\tilde{\beta}$ of $(\mathfrak{m}^{*\mathbf{C}}, \mathfrak{b}^{*\mathbf{C}})$ other than* (1) $\pm\tilde{\alpha}_j$, $1 \leq j \leq q$, *and* (2) *certain roots that are orthogonal to $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_q$. Consequently any sufficiently small rectangular neighborhood $U_{\mathfrak{b}} \times U_{\mathfrak{b}^{\perp}}$ of λ in $(i\mathfrak{b})' \oplus (i\mathfrak{b}^{\perp})' = (i\mathfrak{b}^*)'$ has the property that there exist exactly 2^q Weyl chambers C in $i\mathfrak{b}^*$ such that $\bar{C} \cap (U_{\mathfrak{b}} \cap C_M) \neq \varnothing$, and this intersection has in each case nonempty interior. All such chambers are obtained from one of them C_0 as $\{wC_0 \mid w \in W_{\mathcal{H}}\}$. For any such chamber the positive roots from among the $\pm\tilde{\alpha}_j$, $1 \leq j \leq q$, are all simple.*

Proof. First we show $\langle \lambda, \tilde{\beta} \rangle \neq 0$ under the circumstances stated. Thus suppose $\langle \lambda, \tilde{\beta} \rangle = 0$ and $\tilde{\beta}$ is not orthogonal to all of $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_q$. We show

$\tilde{\beta} = \pm\tilde{\alpha}_j$ for some j. Regard $\tilde{\beta}$ as a root of $(\mathfrak{g}^{\mathbf{C}}, (\mathfrak{a}^* + \mathfrak{b}^*)^{\mathbf{C}})$ that vanishes on $\mathfrak{a}^*$ and form $\mathbf{d}_{\mathcal{K}}^{-1}(\tilde{\beta})$. Since $\tilde{\beta}$ is assumed nonorthogonal to at least one of $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_q$, $\mathbf{d}_{\mathcal{K}}^{-1}(\tilde{\beta})$ has nonzero $\mathfrak{a}$ component, say β_R. Then β_R is a linear combination of the $\mathfrak{a}$-roots α_j and hence is in $(1 - q)\mathfrak{a}'$. By Lemma 10.18, β_R is not in Δ'. Thus $\mu'_{\xi, \beta_R}(i\nu) \neq 0$. By nondegeneracy and by Corollary 12.5,

$$\mu_{\xi, \beta_R}(i\nu) \neq 0. \tag{13.2}$$

Now Proposition 7.1 gives

$$\mu_{\xi, \beta_R}(i\nu) = \prod_{\varepsilon|_{\mathfrak{a}} = c\beta_R,\ c>0} \langle \lambda_M + i\nu, \varepsilon \rangle f_{\xi, \beta'_R}(i\nu), \tag{13.3}$$

where f_{ξ, β'_R} is 1 or is given by (7.6c). One of the factors in the polynomial part of (13.3) comes from

$$\varepsilon = \mathbf{d}_{\mathcal{K}}^{-1}(\tilde{\beta}) = \beta_R + \beta_I,$$

where β_I is the restriction to $i\mathfrak{b}$. If $\beta_I \neq 0$, another factor comes from

$$\bar{\varepsilon} = \beta_R - \beta_I.$$

The factors in question are

$$\langle \lambda_M + i\nu, \beta_R \pm \beta_I \rangle = \pm\langle \lambda_M, \beta_I \rangle + i\langle \nu, \beta_R \rangle = \pm\langle \nu, \tilde{\beta} \rangle + i\langle \nu, \beta_R \rangle.$$

By Proposition 10.20b, $\langle \nu, \beta_R \rangle = 0$. Then $\langle \lambda, \tilde{\beta} \rangle = 0$ and $\beta_I \neq 0$ together mean that (13.3) vanishes at ν, since f_{ξ, β'_R} contributes at most a simple pole. Since the vanishing of (13.3) would contradict (13.2), we conclude that $\beta_I = 0$. Therefore β_R has odd multiplicity and is in $\Sigma\mathbf{R}\alpha_j$. Proposition 10.20 shows that $\beta_R = \pm\alpha_j$ for some j, and hence $\tilde{\beta} = \pm\tilde{\alpha}_j$.

This proves the part of the conclusion concerning $\langle \lambda, \tilde{\beta} \rangle$. Since there are only finitely many $\tilde{\beta}$'s, we can choose a rectangular neighborhood $V_{\mathfrak{b}} \times V_{\mathfrak{b}^\perp}$ around λ in $(i\mathfrak{b})' \oplus (i\mathfrak{b}^\perp)'$ on which $\langle \cdot\,, \tilde{\beta} \rangle$ continues to be nonvanishing for the same $\tilde{\beta}$'s. It follows that any choice of q signs $s_j = \pm 1$ makes

$$\lambda' + \sum_{j=1}^{q} \varepsilon_j s_j \tilde{\alpha}_j \tag{13.4}$$

nonsingular as long as λ' is in $V_{\mathfrak{b}} \cap C_M$ and the ε_j are sufficiently small and positive. Each choice of signs $\{s_j\}$ therefore leads to a chamber as asserted, and there are no other chambers whose closure meets $V_{\mathfrak{b}} \cap C_M$. We obtain 2^q chambers, and the members of $W_{\mathcal{K}}$ clearly permute them simply transitively. Then we can argue with (13.4) as in Lemma 8.5 to see, for any such chamber, that the positive roots from among the $\pm\tilde{\alpha}_j$, $1 \leq j \leq q$, are all simple.

Let C be one of the chambers described in Lemma 13.1. The case $i = q$ in Lemma 13.4 below will show that λ satisfies the appropriate integrality condition so that we can speak of

$$\Theta^{M^*A^*}(\lambda, C, \chi^*, \nu^*).$$

In that context, the statement of the following lemma makes sense.

LEMMA 13.2. *With C as in Lemma 13.1, let w be in $W_{\mathcal{H}}$ and let e be in $E_{\mathcal{H}}$. Then*

$$\Theta^{M^*A^*}(\lambda, weC, \chi^*, \nu^*) = \Theta^{M^*A^*}(\lambda, wC, \chi^*, \nu^*).$$

Proof. Same as for Lemma 8.6.

THEOREM 13.3. *Let $\mathrm{ind}_P^G \Theta^{MA}(\lambda_M, C_M, \chi, \nu)$ be a basic character given by nondegenerate data. In the context of the induced representation with these data, $E_{\mathcal{H}}$ is given by*

$$(13.5) \qquad E_{\mathcal{H}} = \left\{ w \in W_{\mathcal{H}} \;\middle|\; \begin{array}{l} \text{for each } r \text{ in } R,\ w \text{ and } r \text{ have an} \\ \text{even number of factors } p_{\alpha_j} \text{ in common}^{17} \end{array} \right\},$$

and formula (13.5) *sets up a canonical isomorphism of $W_{\mathcal{H}}/E_{\mathcal{H}}$ onto the dual group $\hat{R}$. Moreover,*

$$(13.6) \qquad \mathrm{ind}_P^G \Theta^{MA}(\lambda_M, C_M, \chi, \nu) = \sum_{w \in W_{\mathcal{H}}/E_{\mathcal{H}} \cong \hat{R}} \mathrm{ind}_{P^*}^G \Theta^{M^*A^*}(\lambda, wC, \chi^*, \nu^*)$$

for any choice of the chamber C as in Lemma 13.1. *The characters on the right side of* (13.6) *are all nonzero and irreducible, and they are given by nondegenerate data.*

The proof of Theorem 13.3 will be accomplished by the sequence of lemmas in the remainder of this section. We shall cite Section 8 for proofs whenever possible. Our emphasis here will be on how nondegeneracy is used.

As in Section 8, we enumerate the members of $\mathcal{H}$ as $\alpha_1, \ldots, \alpha_q$ and build intermediate data $\mathfrak{m}_i$, $\mathfrak{a}_i$, $\mathfrak{b}_i$, $C_j^{(i)}$, λ_i, χ_i, ν_i.

LEMMA 13.4. *For each i with $0 \le i \le q$,*

$$(13.7) \qquad \mathrm{ind}_{P \cap M_i A_i}^{M_i A_i} \Theta^{MA}(\lambda_M, C_M, \chi, \nu) = \sum_{\text{some } j\text{'s}} \Theta^{M_i A_i}\big(\lambda_i, C_j^{(i)}, \chi_i, \nu_i\big),$$

with λ_i satisfying the appropriate conditions so that the right side makes sense.

Proof. Most of the argument is the same as for Lemma 8.9. When we come to the last paragraph of the argument, in which we are to show that the chambers can be selected in such a way that (8.10) or (13.7) results, we replace λ

[17]. . . after identification of p_{α_j} with $p_{\tilde{\alpha}_j}$ by the Cayley transform.

by $\lambda + \varepsilon\rho_M$ for any sufficiently small $\varepsilon > 0$. In particular, (8.14) is replaced by

$$\lambda + \varepsilon\rho_M + \sum_{j=1}^{q} \varepsilon_j s_j \tilde{\alpha}_j,$$

and then the argument goes through.

As in Section 8 let $n(i)$, $0 \leq i \leq q$, denote the number of terms on the right side of the character identify (13.7) at stage i. It will turn out that none of the characters on the right of (13.7) is zero. But until we know this fact, we count both zero and nonzero characters in computing $n(i)$.

LEMMA 13.5. *The function $n(i)$ satisfies $n(i) = 2n(i-1)$ if there exists an element r of R of the form*

$$r = \Big(\prod_{\text{some } j<i} p_{\alpha_j} \Big) p_{\alpha_i}. \tag{13.8}$$

Proof. Same as for Lemma 8.10.

LEMMA 13.6. *Under the assumption that* (13.1) *is given by nondegenerate data*:

(a) *For each C as in Lemma* 13.1 *the character* $\mathrm{ind}_{P^*}^{G}\Theta(\lambda, C, \chi^*, \nu^*)$ *is well defined and is given by nondegenerate data, hence is nonzero.*

(b) *The characters* $\mathrm{ind}_{P^*}^{G}\Theta(\lambda, C_j^{(q)}, \chi^*, \nu^*)$, *induced from those on the right side of* (13.7) *when $i = q$, are nonzero and irreducible and are given by nondegenerate data.*

(c) *The function $n(i)$ satisfies $n(q) = |R|$.*

(d) *$n(i) = n(i-1)$ if there exists no element r of R of the form* (13.8).

Proof. We begin with (a). The case $i = q$ in Lemma 13.4 shows there is some chamber for which the character in question is well defined. Since the integrality and compatibility conditions needed for existence of the character do not depend on the chamber, the character is well defined for every C.

We shall prove the nondegeneracy from the definition, showing that $\langle \lambda, \tilde{\beta} \rangle = 0$ for a root $\tilde{\beta}$ of $(\mathfrak{m}^{*\mathbb{C}}, \mathfrak{b}^{*\mathbb{C}})$ implies that $p_{\tilde{\beta}}$ is not in $W(B^*: M^*)$. Thus suppose $\langle \lambda, \tilde{\beta} \rangle = 0$. Lemma 13.1 shows that either $\tilde{\beta}$ is orthogonal to $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_q$ or $\tilde{\beta} = \pm\tilde{\alpha}_i$ for some i.

Let $\tilde{\beta}$ be orthogonal to $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_q$, and suppose $p_{\tilde{\beta}}$ is in $W(B^*: M^*)$ with x as a representative in $K \cap M^*$. Put $\beta = \mathbf{d}_{\mathfrak{X}}^{-1}(\tilde{\beta})$. Then β is a root of $(\mathfrak{m}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$ with $\langle \lambda_M, \beta \rangle = 0$. We can produce b^* in B^* so that b^*x exhibits p_β as in $W(B: M)$, and we then have a contradiction to the nondegeneracy of the data in (13.1). To do so, we argue as in the first part of the proof of Lemma 8.14 where

$\mathrm{Ad}(x)H'_{\tilde\alpha_j} = H'_{\tilde\alpha_j}$. For $1 \le j \le q$ we define b_j^* by (8.18) and then (8.19) holds. Consequently if we take $b^* = b_1^{*-1} \cdots b_q^{*-1}$, then $\mathrm{Ad}(b^*x)$ fixes $\mathfrak{a}^*$ and each $E_{\tilde\alpha_j} + E_{-\tilde\alpha_j}$. Hence b^*x is in M. The element b^*x acts on $\mathfrak{b} \subseteq \mathfrak{b}^*$ as the restriction of $p_{\tilde\beta_2}$ hence as p_β. We obtain the contradiction asserted, and we conclude that $\langle\lambda, \tilde\beta\rangle = 0$ and $\tilde\beta$ orthogonal to $\tilde\alpha_1, \ldots, \tilde\alpha_q$ together imply $p_{\tilde\beta}$ is not in $W(B^*: M^*)$.

The other case in which $\langle\lambda, \tilde\beta\rangle = 0$ is possible is $\tilde\beta = \pm\tilde\alpha_j$. Renumbering $\tilde\alpha_1, \ldots, \tilde\alpha_q$, we may assume the root in question is $\tilde\alpha_q$. By Proposition 10.20c we can choose an element r in R that involves p_{α_q}. Then Lemma 13.5 shows that $p_{\tilde\alpha_q}$ is not in $W(B^*: M^*)$. (See the proof of Lemma 8.10.)

Now we can prove (b), (c), and (d). The nondegeneracy in (b) is a special case of (a). The rest of the argument proceeds as in the proof of Lemma 8.11. This time we refer to the present Theorem 11.1 in place of Theorem 13.4 of [16], and we refer to Lemma 13.5 in place of Lemma 8.10.

LEMMA 13.7. *If C is any chamber in $i\mathfrak{b}^*$ as in Lemma 13.1, then*

$$\sum_{w \in W_{\mathcal{K}}} \Theta^{M^*A^*}(\lambda, wC, \chi^*, \nu^*) = |E_{\mathcal{K}}| \, \mathrm{ind}_{P\cap M^*A^*}^{M^*A^*} \Theta^{MA}(\lambda_M, C_M, \chi, \nu).$$

Proof. The proof follows the same inductive pattern as in Lemmas 8.12 to 8.15, passing from $(\mathfrak{m}^*, \mathfrak{a}^*, \mathfrak{b}^*)$ to $(\mathfrak{m}, \mathfrak{a}, \mathfrak{b})$. No new steps are involved.

LEMMA 13.8. *$W_{\mathcal{K}}/E_{\mathcal{K}}$ and R have the same number of elements, and*

$$(13.9) \quad \mathrm{ind}_P^G \Theta^{MA}(\lambda_M, C_M, \chi, \nu) = \sum_{w \in W_{\mathcal{K}}/E_{\mathcal{K}}} \mathrm{ind}_{P^*}^G \Theta^{M^*A^*}(\lambda, wC, \chi^*, \nu^*)$$

with the characters on the right nonzero and irreducible, given by nondegenerate data.

Proof. We start with the identity of Lemma 13.7 and induce from P^* to G; Lemma 13.2 allows us to write the result in the form (13.9). Every term on the right side of (13.9) is nonzero and is given by nondegenerate data, by Lemma 13.6a. We shall show the terms are irreducible. Let us take (13.7) with $i = q$, induce from P^* to G, and compare the result with (13.9). The induced version of (13.7) has all its terms nonzero irreducible by Lemma 13.6b, and each of its terms is one of the terms on the right of (13.9). Distinct terms go into distinct terms since (13.1) decomposes with multiplicity one, by Corollary 4.4 and Theorem 5.1. Hence the right side of (13.9) contains exactly the same terms as in the identity induced from (13.7) with $i = q$. The lemma follows.

LEMMA 13.9. *Suppose r is in R and e is any member of $E_{\mathcal{K}}$. After identification of each α_j and $\tilde\alpha_j$ by the Cayley transform, r and e have an even number of root reflections in common.*

Proof. Same as for Lemmas 8.17 to 8.19.

LEMMA 13.10. *The homomorphism* $w \to \chi_w$ *of* $W_{\mathcal{K}}$ *onto* $\hat{R}$ *sets up an isomorphism of* $W_{\mathcal{K}}/E_{\mathcal{K}}$ *onto* $\hat{R}$, *and* $E_{\mathcal{K}}$ *is given by* (13.5).

Proof. Same as for Lemma 8.20.

Lemmas 13.8 and 13.10 together prove Theorem 13.3. With the aid of Theorem 13.3, we can decompose *any* basic character into its irreducible constituents. First we use the degeneracies and Proposition 12.1 to rewrite it, one step at a time by generalized Schmid identities, in terms of data for a smaller group MA. This step is noncanonical. When the data are finally nondegenerate, we decompose the character by means of Theorem 13.3. This step is canonical.

14. Classification of irreducible tempered representations

A first classification of irreducible tempered representations is implicit in Theorem 8.7. The results on nondegeneracy allow us to give in Theorem 14.2 a more intrinsic classification. In establishing the equivalence criterion in Theorem 14.2, we shall use the following Disjointness Theorem of Langlands. (See [21], pp. 65 and 76–78, and also Harish-Chandra [5].)

THEOREM 14.1 (Langlands). *If*

$$\operatorname{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu) \qquad \textit{and} \qquad \operatorname{ind}_{P'}^G \Theta^{M'A'}(\lambda', C', \chi', \nu')$$

are basic characters induced from discrete series that have an irreducible constituent in common, then the two induced characters are equal and there is an element w *in* K *such that* $M' = wMw^{-1}$, $A' = wAw^{-1}$, $B' = wBw^{-1}$, $\lambda' = w\lambda$, $C' = wC$, $\chi' = w\chi$, *and* $\nu' = w\nu$.

THEOREM 14.2. *Every irreducible tempered character is basic and can be written with nondegenerate data; when the character is written this way, its* R *group is trivial. Conversely every basic character with nondegenerate data and trivial* R *group is an irreducible tempered character. For two irreducible basic characters with nondegenerate data, an equality*

$$\operatorname{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu) = \operatorname{ind}_{P'}^G \Theta^{M'A'}(\lambda', C', \chi', \nu') \tag{14.1}$$

holds if and only if there is an element w *in* K *with*

$$M' = wMw^{-1}, \qquad A' = wAw^{-1}, \qquad B' = wBw^{-1}, \qquad \lambda' = w\lambda,$$
$$C' = wC, \qquad \chi' = w\chi, \qquad \textit{and} \qquad \nu' = w\nu.$$

Proof. Every irreducible tempered character is basic by Corollary 8.8 and can be written with nondegenerate data by Corollary 12.2. When a basic character is written with nondegenerate data, the dimension of the commuting algebra equals the order of the R group by Theorem 12.6, and hence irreducibility corresponds to trivial R group.

Conversely a basic character with nondegenerate data and trivial R group is irreducible by Theorem 12.6. According to Corollary 4.4, such a character imbeds in a basic character induced from discrete series and hence is tempered.

If two irreducible basic characters with nondegenerate data have data conjugate by an element w in K, then (14.1) holds. To complete the proof, we are to show conversely that (14.1) implies the data are conjugate.

The idea will be to work with one of the two sides of (14.1), obtaining a canonical imbedding of the character in a basic character induced from discrete series. We then show how to recover the original data, except for the chamber, from the data for the basic character induced from discrete series. Comparing the imbeddings for the two sides of (14.1) and applying Theorem 14.1, we obtain the conjugacy of the two sets of given data, except for the chambers. A supplementary argument will deal with the chambers.

Thus let

$$\operatorname{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu) \tag{14.2}$$

be a basic character given by nondegenerate data. Define

$$\mathcal{K} = \{\tilde{\alpha} \text{ root of } (\mathfrak{m}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}}) \mid \langle \lambda, \tilde{\alpha}\rangle = 0 \text{ and } \tilde{\alpha} \text{ is } C\text{-positive}\}.$$

By nondegeneracy, $\mathcal{K}$ contains only noncompact roots. The C-dominance of λ therefore implies that the members of $\mathcal{K}$ are C-simple. Since they are also all noncompact, they are orthogonal, hence superorthogonal. We form the Cayley transform

$$\mathbf{c}_{\mathcal{K}} = \prod_{\tilde{\alpha} \in \mathcal{K}} \mathbf{c}_{\tilde{\alpha}},$$

with $\mathbf{c}_{\tilde{\alpha}}$ as in (2.7), and we are led from the data $(\mathfrak{m}, \mathfrak{a}, \mathfrak{b})$ to data $(\mathfrak{m}_*, \mathfrak{a}_*, \mathfrak{b}_*)$. Let $W_{\mathcal{K}} \subseteq W(B: M^{\mathbf{C}})$ be the subgroup generated by the reflections $p_{\tilde{\alpha}}$ for $\tilde{\alpha}$ in $\mathcal{K}$. Number the members of $\mathcal{K}$ as $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_q$, let $\alpha_j = \mathbf{c}_{\mathcal{K}}(\tilde{\alpha}_j)$, and form successive generalized Schmid identities (4.4a) or (4.4b) by means of Theorem 4.3. Repeating the argument in Lemmas 8.12 to 8.15, we are led to the identity

$$\sum_{w \in W_{\mathcal{K}}} \Theta^M(\lambda, wC, \chi) = |E_{\mathcal{K}}| \operatorname{ind}_{P_* \cap M}^M \Theta^{M_* A_* \cap M}(\lambda|_{\mathfrak{b}^*}, C_*, \chi_*, 0), \tag{14.3}$$

where $C_* = C^{p_{\tilde{\alpha}_1} \cdots p_{\tilde{\alpha}_q}}$ and $E_{\mathcal{K}} = W_{\mathcal{K}} \cap W(B: M)$. The central character χ_* is an extension of χ (with apparent nonuniqueness occurring each time the generalized Schmid identity is of type (4.4b) with just one term on the left). We can rewrite (14.3) as

$$\sum_{w \in W_{\mathcal{K}}/E_{\mathcal{K}}} \Theta^M(\lambda, wC, \chi) = \operatorname{ind}_{P_* \cap M}^M \Theta^{M_* A_* \cap M}(\lambda|_{\mathfrak{b}^*}, C_*, \chi_*, 0). \tag{14.4}$$

Let us write $\mathfrak{a}_* = \mathfrak{a} \oplus \mathfrak{a}_{\mathcal{K}}$, orthogonal decomposition, with $A_* = AA_{\mathcal{K}}$ the corresponding decomposition of groups. Let $W_{\mathcal{K}_*}$ be the subgroup of $W(A_{\mathcal{K}}: M)$ generated by $p_{\alpha_1}, \ldots, p_{\alpha_q}$. The superorthogonality of $\mathcal{K}$ implies that $W_{\mathcal{K}_*}$ is the Weyl group of the odd roots of $(\mathfrak{m}, \mathfrak{a}_{\mathcal{K}})$.

The character on the right of (14.4) is a basic character of M induced from a discrete series ξ_* of M_* (and 0 on $\mathfrak{a}_{\mathcal{K}}$), and the theory of the R group as developed in [16] and [13] applies to it, since M satisfies the axioms of Section 1 of [13]. Let the usual Δ' and R for this situation be denoted Δ'_* and R_*. Lemma 7.3 of [13] shows that R_* is a subgroup of the Weyl group of the odd roots of $(\mathfrak{m}, \mathfrak{a}_{\mathcal{K}})$. Thus

$$(14.5) \qquad R_* = \left\{ w \in W_{\mathcal{K}_*} \mid w\xi_* \cong \xi_* \text{ and } w\Delta_*'^{+} \subseteq \Delta_*'^{+} \right\}.$$

By nondegeneracy, parts (a) and (b) of Theorem 1.1 show that the characters on the left of (14.4) are nonzero and irreducible. Thus Theorem 13.4 of [16] implies

$$(14.6) \qquad | W_{\mathcal{K}}/E_{\mathcal{K}} | = | R_* | .$$

Now we adjoin $\mathfrak{a}$ and order $\mathfrak{a}_*$ with $\mathfrak{a}$ before $\mathfrak{a}_{\mathcal{K}}$. We extend both sides of (14.4) by $i\nu$ on $\mathfrak{a}$ and induce to G. Then we obtain

$$(14.7) \qquad \sum_{w \in W_{\mathcal{K}}/E_{\mathcal{K}}} \operatorname{ind}_P^G \Theta^{MA}(\lambda, wC, \chi, \nu) = \operatorname{ind}_{P_*}^G \Theta^{M_*A_*}(\lambda\,|_{\mathfrak{b}^*}, C_*, \chi_*, \nu \oplus 0).$$

The right side of (14.7) is a basic character of G induced from the same discrete series ξ_* of M_* (and $i\nu$ on $\mathfrak{a}_*$). Let Δ' and R have the usual interpretation for it. (See § 8.)

Let us regard $W(A_{\mathcal{K}}: M)$ as a subgroup of $W(A_*: G)$. Under this identification, we shall show that

$$(14.8) \qquad R_* \subseteq R.$$

To do so, let w be in R_*. Then w fixes ξ_* by (14.5) and w fixes ν since w is 1 on $\mathfrak{a}$. Let $\beta > 0$ be in Δ', and form $w\beta$. Suppose by way of contradiction that $w\beta < 0$. Then the facts that $\mathfrak{a}$ is ordered before $\mathfrak{a}_{\mathcal{K}}$ and that w is 1 on $\mathfrak{a}$ imply that β vanishes on $\mathfrak{a}$, i.e., that β is a positive root of $(\mathfrak{m}, \mathfrak{a}_{\mathcal{K}})$. But then β is in Δ'_* since $\mu_{\xi_*, \beta}$ is the same whether computed in G or in M. Since w is assumed to be in R_*, $w\beta < 0$ is then impossible, and we have a contradiction. We conclude that $w\beta > 0$. This proves (14.8).

We return to (14.7). The induced terms on the left remain nonzero during induction. One of them, corresponding to $w = 1$, is by assumption irreducible. By Proposition 10.21, each term on the left has the same R group. By Theorem 12.6, irreducibility is controlled by the R group for each. Consequently every term on the left side of (14.7) is an irreducible character.

Thus the R group of the right side of (14.7) satisfies

$$|W_{\mathcal{H}}/E_{\mathcal{H}}| = |R|. \tag{14.9}$$

Putting together (14.6), (14.8), and (14.9), we conclude that

$$R_* = R. \tag{14.10}$$

Now we apply Theorem 8.7 to the right side of (14.7). The set $\mathcal{H}$ of that theorem, by (14.10), is a subset (apart from signs) of $\{\alpha_1, \ldots, \alpha_q\}$, and we claim it is exactly the full set. In fact, otherwise we can first form the decomposition of Theorem 8.7, passing from data $(\mathfrak{m}_*, \mathfrak{a}_*, \mathfrak{b}_*)$ to data $(\mathfrak{m}', \mathfrak{a}', \mathfrak{b}')$ and then we can do additional Cayley transforms $\mathbf{d}_{\alpha_i}$ and pass to the original data $(\mathfrak{m}, \mathfrak{a}, \mathfrak{b})$. The latter step gives us no new terms, since Theorem 8.7 has already provided a decomposition into irreducible characters, and thus we can reverse the latter step, using the given character $\Theta^{MA}(\lambda, C, \chi, \nu)$ as the left side of some generalized Schmid identity. In view of Proposition 12.1, this contradicts nondegeneracy. We conclude that $\mathcal{H} = \{\alpha_1, \ldots, \alpha_q\}$, apart from signs.

Let us summarize. Starting from the left side of (14.1), we used the given data to imbed the character in a basic character induced from the discrete series of some M_*. From a suitable ordering on $\mathfrak{a}_*$, depending on our given data, we were able to recover our original MA from the set $\mathcal{H}$ of superorthogonal roots obtained from the R group. If we start instead from the right side of (14.1), we must be led to the same basic character induced from discrete series, by the Langlands Disjointness Theorem (Theorem 14.1). Moreover, the data for the new version of the basic character induced from discrete series are G-conjugate to the data for the old version and hence may be assumed identical. Any two orderings for $\mathfrak{a}_*$ lead to W'-conjugate R groups and hence to superorthogonal systems $\mathcal{H}$ that are conjugate, apart from signs.

Thus without loss of generality we may assume $M'A' = MA$ in (14.1) and rewrite (14.1) as

$$\operatorname{ind}_P^G \Theta^{MA}(\lambda, C, \chi, \nu) = \operatorname{ind}_P^G \Theta^{MA}(\lambda', C', \chi', \nu). \tag{14.11}$$

We shall not attempt to track down the remaining G-conjugacy we have proved but shall instead relate (λ', χ', ν') and (λ, χ, ν) directly. The argument above imbeds the two characters of (14.11) into equal basic characters induced from discrete series:

$$\operatorname{ind}_{P_1}^G \Theta^{M_1A_1}\big(\lambda|_{\mathfrak{b}_1}, C_*, \chi_*, \nu \oplus 0\big) \tag{14.12a}$$

and

$$\operatorname{ind}_{P_2}^G \Theta^{M_2A_2}\big(\lambda'|_{\mathfrak{b}_2}, C'_*, \chi'_*, \nu' \oplus 0\big); \tag{14.12b}$$

also the data in (14.12b) are the transforms by some w in K of the data in (14.12a):

$$(14.13)\quad wM_1w^{-1} = M_2, \quad \mathrm{Ad}(w)\mathfrak{a}_1 = \mathfrak{a}_2, \quad \mathrm{Ad}(w)\mathfrak{b}_1 = \mathfrak{b}_2,$$
$$\mathrm{Ad}(w)(\lambda|_{\mathfrak{b}_1}) = \lambda'|_{\mathfrak{b}_2}, \quad w\chi_* = \chi'_*, \quad \mathrm{Ad}(w)(\nu \oplus 0) = \nu' \oplus 0.$$

Write $\mathfrak{a}_1 = \mathfrak{a} \oplus \mathfrak{a}_{\mathcal{K}_1}$ and $\mathfrak{a}_2 = \mathfrak{a} \oplus \mathfrak{a}_{\mathcal{K}_2}$. Order Δ'_1 so that $\mathfrak{a}$ comes before $\mathfrak{a}_{\mathcal{K}_1}$, and order Δ'_2 so that $\mathfrak{a}$ comes before $\mathfrak{a}_{\mathcal{K}_2}$. We have seen that R_1 determines $\mathfrak{a}_{\mathcal{K}_1}$ and R_2 determines $\mathfrak{a}_{\mathcal{K}_2}$. We know that $\mathrm{Ad}(w)\Delta'_1 = \Delta'_2$. Choose p in K representing a member of the Weyl group of Δ'_2 so that $\mathrm{Ad}(pw)\Delta'^+_1 = \Delta'^+_2$. Then $pwR_1w^{-1}p^{-1} = R_2$ and hence

$$(14.14)\qquad \mathrm{Ad}(pw)(\pm\mathcal{K}_1) = \pm\mathcal{K}_2$$

and $\mathrm{Ad}(pw)\mathfrak{a}_{\mathcal{K}_1} = \mathfrak{a}_{\mathcal{K}_2}$. Also $\mathrm{Ad}(pw)\mathfrak{a}_1 = \mathfrak{a}_2$, and thus $\mathrm{Ad}(pw)\mathfrak{a} = \mathfrak{a}$. Adjusting p by an element of $(M_2)_0$, we may assume that $\mathrm{Ad}(p)\mathfrak{b}_2 = \mathfrak{b}_2$.

Since p represents a member of the Weyl group of Δ'_2, p fixes the discrete series character $\Theta^{M_2}(\lambda'|_{\mathfrak{b}_2}, C'_*, \chi'_*)$ and p fixes $\nu' \oplus 0$. Hence p fixes χ'_*. Adjusting p by an element in the normalizer of $\mathfrak{b}_2$ in M_2, we may thus assume $p(\lambda'|_{\mathfrak{b}_2}) = \lambda'|_{\mathfrak{b}_2}$.

Combining this information about p with (14.13), we see that

$$(14.15)\quad pwM_1w^{-1}p^{-1} = M_2, \quad \mathrm{Ad}(pw)\mathfrak{a} = \mathfrak{a}, \quad \mathrm{Ad}(pw)\mathfrak{a}_{\mathcal{K}_1} = \mathfrak{a}_{\mathcal{K}_2},$$
$$\mathrm{Ad}(pw)\mathfrak{b}_1 = \mathfrak{b}_2, \quad \mathrm{Ad}(pw)(\lambda|_{\mathfrak{b}_1}) = \lambda'|_{\mathfrak{b}_2},$$
$$pw\chi_* = \chi'_*, \quad \mathrm{Ad}(pw)(\nu \oplus 0) = \nu' \oplus 0.$$

These equations imply that pw normalizes A, normalizes M, and satisfies $\mathrm{Ad}(pw)\nu = \nu'$ and $pw\chi = \chi'$.

We have

$$\mathfrak{b} = \mathfrak{b}_1 \oplus ic_{\mathcal{K}_1}^{-1}(\mathfrak{a}_{\mathcal{K}_1}).$$

Applying $\mathrm{Ad}(pw)$ and using (14.14) and (14.15), we have

$$(14.16)\qquad \mathrm{Ad}(pw)\mathfrak{b} = \mathfrak{b}_2 \oplus ic_{\mathcal{K}_2}^{-1}(\mathfrak{a}_{\mathcal{K}_2}).$$

But the right side of (14.16) is $\mathfrak{b}$, and hence $\mathrm{Ad}(pw)\mathfrak{b} = \mathfrak{b}$. Then it follows from (14.15) that $\mathrm{Ad}(pw)\lambda = \lambda'$. In short pw conjugates (λ, χ, ν) to (λ', χ', ν').

Thus without loss of generality we may rewrite (14.1) and (14.11) as

$$(14.17)\qquad \mathrm{ind}_P^G\Theta^{MA}(\lambda, C, \chi, \nu) = \mathrm{ind}_P^G\Theta^{MA}(\lambda, C', \chi, \nu).$$

The proof comes down to showing that an equality (14.17) implies $C' = wC$ with w in

$$W_{\mathcal{K}} \cap W(B\colon M) = E_{\mathcal{K}}.$$

Since λ has to be dominant, we certainly have $C' = wC$ with w in $W_{\mathcal{K}}$. Assume (14.17) and run through the argument leading to (14.7). If this w with $C' = wC$ is not in $E_{\mathcal{K}}$, then both sides of (14.17) contribute to the left side of (14.7). Hence the irreducible constituents of the right side of (14.7) do not occur with multiplicity one, in contradiction to Theorem 5.1. This contradiction completes the proof of Theorem 14.2.

15. Examples and counterexamples

1. *Two types of generalized Schmid identities.* Prototypes for the two generalized Schmid identities (4.4a) and (4.4b) occur with M^* as $SL(2, \mathbf{R})$ and $SL^{\pm}(2, \mathbf{R})$, respectively.

In $M^* = SL(2, \mathbf{R})$ there are two limits of discrete series; they correspond to $\lambda = 0$ and differ as to chamber. They occur as irreducible constituents of the unique reducible unitary principal series representation, which is induced from the extension to MAN of the signum character of the diagonal subgroup MA. The Cartan subalgebra $\mathfrak{b}^*$ is all multiples of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and the reflection $p_{\tilde{\alpha}}$ has no representative in the group M^*. The identity (4.4a) expresses the reducibility of this principal series representation.

In $M^* = SL^{\pm}(2, \mathbf{R})$ there is only one limit of discrete series (up to equivalence). Although we still have $\lambda = 0$ and there are still two chambers, $p_{\tilde{\alpha}}$ has $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as a representative in M^*, and Theorem 1.1c says the two chambers lead to the same character. The group M now has four elements $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$. The two characters of M that are -1 on $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ are what are denoted $(\zeta \otimes \chi)^+$ and $(\zeta \otimes \chi)^-$ in (4.4b). These two characters, extended by the trivial character on A, lead to equal principal series characters, which in turn are equal to the limit of discrete series character by (4.4b).

2. *Fundamental examples of reducibility.* Suppose G is split over $\mathbf{R}$ and $P = MAN$ is a minimal parabolic. When it turns out that the superorthogonal set $\mathcal{K}$ of Theorem 8.7 consists of simple $\mathfrak{a}$-roots, then the reducibility can be understood directly from the theory of intertwining operators for $SL(2, \mathbf{R})$ and related groups. In fact, the intertwining operator for a simple reflection p_α can be viewed (cf. [16], p. 23) as an intertwining operator for the group built from M and α, and the identity component of this group will be locally isomorphic to $SL(2, \mathbf{R})$. The operator for an element r in R of the form Πp_{α_i} with each α_i simple will be an operator for an analogous group involving several copies of $SL(2, \mathbf{R})$. The reducibility coming from r already comes from this group, and a

corresponding decomposition of the unitary principal series representation of G under study comes by inducing this decomposition to G.

We shall give the three examples of this phenomenon that led us to the general pattern of Theorem 8.7. These occur in the groups $\mathrm{Sp}(2, \mathbf{R})$, $\mathrm{SL}(2n, \mathbf{R})$, and $\widetilde{\mathrm{SO}}(4,4)$.

First we make some comments about reducibility in groups related to the sum of n copies of $\mathrm{SL}(2, \mathbf{R})$. When G is the direct sum of n copies of $\mathrm{SL}(2, \mathbf{R})$, the M for the minimal parabolic is itself a sum and has order 2^n. If we use the character of M that is signum in each of the n coordinates, then the corresponding principal series representation decomposes into 2^n pieces in an obvious fashion. If G is enlarged to a disconnected group by adjoining some n-tuples $(g_1, \ldots, g_n)$ with $\det g_i = \pm 1$ for each i, then the amount of reducibility for the corresponding principal series goes down. In terms of generalized Schmid identities, the additional elements force the use of identities (4.4b) in place of (4.4a) in the reduction, and fewer terms appear in the end.

a) $\mathrm{Sp}(2, \mathbf{R})$. In standard notation this group is of type C_2, and the simple roots of $(\mathfrak{g}, \mathfrak{a})$ are $e_1 - e_2$ and $2e_2$. Let

$$\xi(\gamma_{e_1 - e_2}) = 1 \qquad \text{and} \qquad \xi(\gamma_{2e_2}) = -1.$$

Then

$$\Delta'^{+} = \{e_1 + e_2, e_1 - e_2\}$$

and

$$R = \{1, p_{2e_2}\}.$$

Here $\mathcal{H}$ consists of a single root $2e_2$, and it is simple. The general theory says that reducibility already occurs within the group $M^* \cong \mathrm{SL}(2, \mathbf{R}) \oplus \mathbf{Z}_2$ built from M and the root $2e_2$. This fact can be seen directly on the level of intertwining operators, as noted above. The intertwining operator corresponding to the simple reflection p_{2e_2} can be viewed as an $\mathrm{SL}(2, \mathbf{R})$-type nontrivial self-intertwining operator (Hilbert transform) occurring in M^*, and thus the reducibility already occurs in M^*.

b) $\mathrm{SL}(n, \mathbf{R})$. In standard notation this group is of type A_n, and the simple roots of $(\mathfrak{g}, \mathfrak{a})$ are $e_1 - e_2, e_2 - e_3, \ldots, e_{2n-1} - e_{2n}$. Let

$$\xi(\gamma_{e_{2k-1} - e_{2k}}) = -1 \quad \text{for } 1 \le k \le n$$

and

$$\xi(\gamma_{e_{2k} - e_{2k+1}}) = +1 \quad \text{for } 1 \le k \le n-1.$$

Then Δ'^{+} is generated by all $e_{2k-1} - e_{2k+2}$ and $e_{2k} - e_{2k+1}$ for $1 \le k \le n-1$,

and

$$R = \left\{1, p_{e_1-e_2}p_{e_3-e_4} \cdot \cdots \cdot p_{e_{2n-1}-e_{2n}}\right\}.$$

Thus

$$\mathcal{H} = \{e_1 - e_2, e_3 - e_4, \ldots, e_{2n-1} - e_{2n}\},$$

and $\mathcal{H}$ consists entirely of simple roots. Again the reducibility that is explained by Theorem 8.7 can be predicted directly from intertwining operators. The group M^* built from M and the members of $\mathcal{H}$ is isomorphic to the sum of n copies of $SL(2, \mathbf{R})$, with all n-tuples of matrices $(g_1, \ldots, g_n)$ adjoined such that $\deg g_i = \pm 1$ for all i and $\Pi \det g_i = +1$. Intertwining operators show that the two-fold reducibility within this M^* accounts for the two-fold reducibility within G.

c) $\widetilde{SO}(4,4)$. In standard notation this group is of type D_4, and the simple roots of $(\mathfrak{g}, \mathfrak{a})$ are

$$e_1 - e_2, e_2 - e_3, e_3 - e_4, e_3 + e_4.$$

Let

$$\xi(\gamma_{e_1-e_2}) = \xi(\gamma_{e_2-e_3}) = \xi(\gamma_{e_3-e_4}) = \xi(\gamma_{e_3+e_4}) = -1.$$

Then

$$\Delta'^+ = \{e_1 \pm e_3, e_2 \pm e_4\}$$

and

$$R = \left\{1, p_{e_1-e_2}p_{e_3-e_4}, p_{e_1-e_2}p_{e_3+e_4}, p_{e_3-e_4}, p_{e_3+e_4}\right\}.$$

Thus

$$\mathcal{H} = \{e_1 - e_2, e_3 - e_4, e_3 + e_4\},$$

and $\mathcal{H}$ consists entirely of simple roots. Once again the reducibility that is explained by Theorem 8.7 can be seen directly. The group M^* built from M and the members of $\mathcal{H}$ is isomorphic to $SL(2,\mathbf{R}) \oplus SL(2,\mathbf{R}) \oplus SL(2,\mathbf{R})$ with $\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$ adjoined. The four-fold reducibility within M^* induces to the four-fold reducibility within G.

3. *Parametrization by* $\hat{R}$. It is natural to expect that the different chambers that appear in Theorem 8.7 are related by $C_j^* = r_j C_0^*$, with r_j in (the Cayley transform of) R, but this is not so. In the case of $SL(4,\mathbf{R})$ in Example 2b above, the chambers can be regarded as the quadrants in $\mathbf{R}^2$ with basis $e_1 - e_2$ and $e_3 - e_4$, and the nontrivial element of R is $p_{e_1-e_2}p_{e_3-e_4}$. This element carries a

chamber to its negative, which gives an equivalent representation. To get two inequivalent representations, we want to use two adjacent chambers. Adjacent chambers are the result of parametrizing by $\hat{R}$ instead of R.

4. *Degenerate intertwining operator.* Let $P = MAN$ be a cuspidal maximal parabolic subgroup in G, let α be a root of $(\mathfrak{g}, \mathfrak{a})$, and suppose that p_α exists in $W(A\colon G)$. If ξ is a discrete series representation of M and if $\nu = 0$ is the parameter on $\mathfrak{a}$, then there are only two possibilities for the behavior of the standard unnormalized self-intertwining operator for $U_P(\xi, 0)$ corresponding to p_α: (1) the operator has no pole at $\nu = 0$ and is obviously not scalar, in which case $U_P(\xi, 0)$ is reducible, or (2) the operator does have a pole at $\nu = 0$ and when normalized becomes scalar, in which case $U_P(\xi, 0)$ is irreducible. (See [16], § 11.)

If ξ is a limit of discrete series, this dichotomy breaks down. The unnormalized operator can have a pole at $\nu = 0$ and remain nonscalar after normalization. In such a case we necessarily have $\mu_{\xi,\alpha}(0) = 0$ (to get a pole) and $\mu'_{\xi,\alpha}(0) \neq 0$ (to have a chance of obtaining a nonscalar operator). For an example, take $G = \mathrm{Sp}(2, \mathbf{R})$ with notation as in Example 2a. Let $M_{e_1 - e_2}$ be built from the minimal M and the root $e_1 - e_2$. Then $M_{e_1 - e_2} \cong \mathrm{SL}^{\pm}(2, \mathbf{R})$. If ξ is the limit of discrete series representation of $M_{e_1 - e_2}$, then the standard unnormalized self-intertwining operator for the induced representation with $\nu = 0$ has the stated properties. We check readily that $\mu_{\xi,\alpha}(0) = 0$, from which it follows that the unnormalized operator has a pole, by (7.3) above and also Section 8 of [16].

To prove the normalized operator is not scalar, we apply the completeness result in Theorem 11.1b and find it is enough to show the induced representation is reducible. The data are degenerate, and we can rewrite the induced representation as coming from the minimal parabolic. One of the two ways of rewriting the representation leads exactly to the representation in Example 2a, and that is reducible. Hence the normalized operator under study is not scalar.

5. *Nonexistence of a unique "top" for characters.* There is a natural ordering on θ-stable Cartan subgroups in which a first one is smaller than a second one if the identity component of the compact part of the first one is conjugate to a subgroup of the compact part of the second one. It has been conjectured that to every irreducible character of a connected G is associated a unique "top", a unique largest Cartan subgroup (in this order) on which the character is nonvanishing. This conjecture is false, even for basic characters, as is shown by any of the irreducible constituents contained in the D_4 example treated as Example 2c.

At first glance these characters do appear to have a "top", namely the Cartan subgroup obtained by starting with the most noncompact Cartan and using

$$\mathbf{d}_{\mathcal{H}} = \mathbf{d}_{e_1-e_2}\mathbf{d}_{e_3-e_4}\mathbf{d}_{e_3+e_4}$$

to obtain a new one. But it is easily checked that these characters vanish on this Cartan subgroup and that the next lower Cartan subgroups in the ordering give three "tops". These are the Cartan subgroups obtained by instead using

$$\mathbf{d}_{e_1-e_2}\mathbf{d}_{e_3-e_4},\ \mathbf{d}_{e_1-e_2}\mathbf{d}_{e_3+e_4},\ \text{and } \mathbf{d}_{e_3-e_4}\mathbf{d}_{e_3+e_4}.$$

6. *Strict inequality may hold in Theorem* 11.1c. That is, for a degenerate basic character the dimension of the commuting algebra may be strictly less than the order of the R group. An example occurs in $G = \mathrm{Sp}(3, \mathbf{R})$. Let MAN be a minimal parabolic subgroup, and denote the simple roots by $e_1 - e_2$, $e_2 - e_3$, $2e_3$. Form $M_{e_1-e_2}$ from M and the root $e_1 - e_2$. Then $M_{e_1-e_2}$ is isomorphic with $\mathrm{SL}^{\pm}(2, \mathbf{R}) \oplus \mathbf{Z}_2$. We take $\lambda = 0$ and $\nu = 0$, and we define χ consistently by

$$\chi(\gamma_{e_1+e_2}) = -1 \qquad \text{and} \qquad \chi(\gamma_{2e_3}) = -1.$$

The basic character in question is the one induced from these data.

The positive roots for this parabolic are

$$e_1 + e_2,\quad 2e_3,\quad \tfrac{1}{2}(e_1 + e_2) \pm e_3,$$

and $\frac{1}{2}(e_1 + e_2) \pm e_3$ are not useful. We find that Δ' is empty and hence

$$R = \{1, p_{e_1+e_2}, p_{2e_3}, p_{e_1+e_2}p_{2e_3}\},$$

of order four.

However, the basic character splits into only two pieces. This fact can be seen in two ways. One is to rewrite the character in nondegenerate form, induced from the minimal parabolic, and to calculate the new R group. Another way is to use character theory directly. The usual superorthogonal set is $\mathcal{H} = \{e_1 + e_2, 2e_3\}$, and we can use generalized Schmid identities (first from $e_1 + e_2$ and then from $2e_3$) to decompose the character. The maximum number of constituents arises if all identities are of type (4.4a). In any event, we drop all indices for the characters except for the M of the parabolics and the simple roots of the chambers, and we obtain

$$\begin{aligned}
&\mathrm{ind}\Theta^{M_{e_1-e_2}}(\{e_1 - e_2\})\\
&\quad = \mathrm{ind}\Theta^{M_{e_1-e_2,2e_2}}(\{e_1 + e_2, -2e_2\}) + \mathrm{ind}\Theta^{M_{e_1-e_2,2e_2}}(\{-e_1 - e_2, 2e_1\})\\
&\quad = \Theta^G(\{e_1 + e_2, -e_2 - e_3, 2e_3\}) + \Theta^G(\{e_1 + e_2, -e_2 + e_3, -2e_3\})\\
&\qquad + \Theta^G(\{-e_1 - e_2, e_1 - e_3, 2e_3\}) + \Theta^G(\{-e_1 - e_2, e_1 + e_3, 2e_3\}).
\end{aligned}$$

The second and third terms on the right are zero by the Hecht-Schmid identity (Theorem 1.1b), and the reducibility is into only two pieces.

Appendix on tensor products

By GREGG J. ZUCKERMAN

The material in this appendix is a supplement to the paper [31], and we follow the notation of that paper. However, we shall restrict attention to connected semisimple Lie groups G having a faithful matrix representation.

The Lie algebra of G is denoted $\mathfrak{g}_0$, and the complexification is $\mathfrak{g}$. The group $G^{\mathbf{C}}$ is the analytic group with Lie algebra $\mathfrak{g}$.

A. Iteration of tensoring functors

Let H be a Cartan subgroup of G, $\mathfrak{h}_0$ its Lie algebra, $\mathfrak{h}$ the complexification of $\mathfrak{h}_0$, $H^{\mathbf{C}}$ the complexification of H in $G^{\mathbf{C}}$, and H' the set of regular elements in H. Suppose a system of positive roots has been selected. Let λ and μ be in $\mathfrak{h}^*$, suppose Re λ and μ are dominant, and suppose μ is integral (in the sense of being the differential of a holomorphic character ξ_μ of $H^{\mathbf{C}}$ into $\mathbf{C}^\times$). Then Definition 1.1 of [31] gives the ψ functor as

$$\psi_\lambda^{\lambda+\mu} = p_\lambda \circ \left[(-) \otimes F_{-\mu}\right] \circ p_{\lambda+\mu}$$

in the category of Harish-Chandra $(\mathcal{G}, K)$ modules.

THEOREM A.1. *If* Re λ *is dominant and* μ_1 *and* μ_2 *are dominant integral, there is a natural isomorphism*

$$\psi_\lambda^{\lambda+\mu_1+\mu_2} \cong \psi_\lambda^{\lambda+\mu_1} \circ \psi_{\lambda+\mu_1}^{\lambda+\mu_1+\mu_2}. \tag{A.1}$$

We shall observe presently that (A.1) is evident on the level of characters, but the formula says more—first that the two sides operate on any Harish-Chandra $(\mathcal{G}, K)$ module and lead to isomorphic modules, and second that the resulting system of isomorphisms is natural.

We begin with the appropriate observations from global character theory. Characters are locally integrable class functions on G analytic on the regular set of G, by [3]; they are thus determined by their values on the regular sets within each member of a system of nonconjugate Cartan subgroups of G. If such a Cartan subgroup is given and a positive system of roots is specified, then we may transform H and the parameters on ψ canonically, rewriting them in terms of this specified Cartan subgroup.[18] Changing notation, we may specify a character by giving its expression on H'.

The Weyl denominator ∇ is the function on $H^{\mathbf{C}}$ given by

$$\nabla(h) = \prod_{\beta>0} \left(\xi_{\beta/2}(h) - \xi_{-\beta/2}(h)\right). \tag{A.2}$$

If Θ is a character with infinitesimal character λ_1, then we have locally

$$\nabla\Theta(h_0 \exp X) = \sum_{s\in W} c(\Theta; s)(h_0)\exp\{s\lambda_1(X)\} \quad \text{in } H'. \tag{A.3}$$

Here the expressions $c(\Theta; s)(h_0)$ are locally constant functions on H'. (See equation (3.5) of [31] and the last two paragraphs of Section 3 of [31].)

LEMMA A.2. *Formula* (A.1) *is true at the level of characters.*

Proof. If Θ is a character with infinitesimal character $\lambda + \mu_1 + \mu_2$, then (A.3) gives

$$\nabla\Theta(h_0 \exp X) = \sum_{s\in W} c(\Theta; s)(h_0)\exp\{s(\lambda + \mu_1 + \mu_2)(X)\}. \tag{A.4}$$

By Formula (3.8) of [31], we have

$$\nabla\psi_{\lambda+\mu_1}^{\lambda+\mu_1+\mu_2}\Theta(h_0 \exp X) = \sum_{s\in W} c(\Theta; s)(h_0)\exp\{s(\lambda + \mu_1)(X)\}$$

and then

$$\nabla\psi_{\lambda}^{\lambda+\mu_1}\psi_{\lambda+\mu_1}^{\lambda+\mu_1+\mu_2}\Theta(h_0 \exp X) = \sum_{s\in W} c(\Theta; s)(h_0)\exp\{s\lambda(X)\}.$$

The right side of the latter expression equals $\nabla\psi_{\lambda}^{\lambda+\mu_1+\mu_2}\Theta(h_0 \exp X)$, again by Formula (3.8) of [31].

LEMMA A.3. *Suppose η is a dominant integral form with $\eta < \mu_1 + \mu_2$. Then the functor $p_\lambda \circ [(-) \otimes F_{-\eta}] \circ p_{\lambda+\mu_1+\mu_2}$ is naturally isomorphic to the zero functor.*

Proof. It is enough to prove that every character maps to 0. We argue as in the proof of Lemma 3.3 of [31]. Let

$$\theta(F_{-\eta})(h_0 \exp X) = \Sigma_{\nu\in P(F_{-\eta})}\xi_\nu(h_0)\exp\{\nu(X)\}.$$

[18]Any two Cartan subalgebras are conjugate by an inner automorphism of $G^{\mathbf{C}}$. As a map between Cartan subalgebras, this map is completely determined by specifying which positive systems of roots are to be matched.

If Θ denotes $p_{\lambda+\mu_1+\mu_2}$ of some character, then $\nabla\Theta$ is of the form (A.4) and

$$\nabla\Theta(F_{-\eta})\Theta(h_0 \exp X) = \sum_{\nu\in P(F_{-\eta})}\sum_{s\in W} \xi_\nu(h_0)c(\Theta; s)(h_0)\exp\{[s(\lambda+\mu_1+\mu_2)+\nu](X)\}.$$

To compute the image under p_λ, we must determine for which ν in $P(F_{-\eta})$ there exists t in W such that $t[s(\lambda+\mu_1+\mu_2)+\nu]=\lambda$. If t exists, then

$$\operatorname{Re}\lambda+\mu_1+\mu_2 = s^{-1}t^{-1}\operatorname{Re}\lambda - s^{-1}\nu.$$

However, $\operatorname{Re}\lambda \geq s^{-1}t^{-1}\operatorname{Re}\lambda$ since $\operatorname{Re}\lambda$ is dominant, and $\mu_1+\mu_2 > -s^{-1}\nu$ since

$$s^{-1}\nu \geq -\eta > -(\mu_1+\mu_2).$$

Hence there is no such t, and the image under p_λ is 0.

Proof of Theorem A.1. We introduce a new functor

$$\omega = p_\lambda \circ \left[(-)\otimes F_{-\mu_2}\otimes F_{-\mu_1}\right]\circ p_{\lambda+\mu_1+\mu_2}.$$

If we fix a decomposition

$$F_{-\mu_2}\otimes F_{-\mu_1} \cong F_{-\mu_1-\mu_2}\oplus \sum_{\eta<\mu_1+\mu_2} F_{-\eta}$$

into irreducibles, then ω is naturally isomorphic with

$$p_\lambda\circ\left[(-)\otimes\left(F_{-\mu_1-\mu_2}\oplus\sum_{\eta<\mu_1+\mu_2}F_{-\eta}\right)\right]\circ p_{\lambda+\mu_1+\mu_2},$$

which in turn is naturally isomorphic with $\psi_\lambda^{\lambda+\mu_1+\mu_2}$ by Lemma A.3. Write

$$\psi_\lambda^{\lambda+\mu_1}\circ\psi_{\lambda+\mu_1}^{\lambda+\mu_1+\mu_2} = p_\lambda\circ\left[(-)\otimes F_{-\mu_1}\right]\circ p_{\lambda+\mu_1}\circ\left[(-)\otimes F_{-\mu_2}\right]\circ p_{\lambda+\mu_1+\mu_2}.$$

There is a natural injection of $p_{\lambda+\mu_1}$ into the identity functor, and it follows from the exactness of the functors in question that there is a natural injection of $\psi_\lambda^{\lambda+\mu_1}\circ\psi_{\lambda+\mu_1}^{\lambda+\mu_1+\mu_2}$ into ω, hence into $\psi_\lambda^{\lambda+\mu_1+\mu_2}$.

Let us abbreviate $F=\psi_\lambda^{\lambda+\mu_1}\circ\psi_{\lambda+\mu_1}^{\lambda+\mu_1+\mu_2}$ and $G=\psi_\lambda^{\lambda+\mu_1+\mu_2}$. We know there is a natural injection of F into G. If A is any Harish-Chandra $(\mathcal{G}, K)$-module, then we have an exact sequence

$$0\to F(A)\to G(A)\to G(A)/F(A)\to 0.$$

Taking characters and using the fact given in Lemma A.2 that $\theta(F(A)) = \theta(G(A))$, we conclude that $\theta(G(A)/F(A))=0$. Hence $G(A)/F(A)=0$ and $F(A)\cong G(A)$. It follows that the natural injection of F into G is a natural isomorphism.

B. Parabolic induction

In this section we shall give conditions under which the ψ functor commutes with induction from a parabolic subgroup. The choice of a maximal compact subgroup K of G determines a Cartan involution of $\mathfrak{g}_0$, which in turn leads to a Langlands decomposition for any parabolic subgroup of G. Let MAN be the Langlands decomposition of such a parabolic subgroup, and let $\mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ be the corresponding decomposition of the Lie algebra. We write $\mathfrak{M}\mathfrak{A}$ for the universal enveloping algebra of $\mathfrak{m} \oplus \mathfrak{a}$, and we let $K_M = K \cap M$. We can define compatible $(\mathfrak{M}\mathfrak{A}, K_M)$-modules in analogy with compatible $(\mathcal{G}, K)$-modules. (See § 2 of [31].)

Fix a Cartan subalgebra $\mathfrak{b}_0$ of $\mathfrak{m}_0$. Then $\mathfrak{b}_0 \oplus \mathfrak{a}_0$ is a Cartan subalgebra (with complexification $\mathfrak{b} \oplus \mathfrak{a}$) of both $\mathfrak{m}_0 \oplus \mathfrak{a}_0$ and $\mathfrak{g}_0$. If $\Delta_{\mathfrak{m}}$ and $\Delta_{\mathfrak{g}}$ denote the sets of roots for $(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{b} \oplus \mathfrak{a})$ and $(\mathfrak{g}, \mathfrak{b} \oplus \mathfrak{a})$, then $\Delta_{\mathfrak{m}} \subseteq \Delta_{\mathfrak{g}}$. We suppose we are given positive systems, not necessarily related to $\mathfrak{n}_0$, such that

$$\Delta_{\mathfrak{m}}^+ \subseteq \Delta_{\mathfrak{g}}^+ . \tag{B.1}$$

If μ is an integral form that is $\mathfrak{g}$-dominant, then μ is also $\mathfrak{m}$-dominant, and we can form an irreducible finite-dimensional representation of $\mathfrak{m} \oplus \mathfrak{a}$ with lowest weight $-\mu$. Since $M \subseteq (MA)^{\mathbf{C}}$, the globalization to $(MA)^{\mathbf{C}}$ of this representation restricts to a well-defined representation ${}^M F_{-\mu}$ of MA. We regard ${}^M F_{-\mu}$ also as a representation of MAN with trivial N action. We shall use ${}^M F_{-\mu}$ to define a functor ψ for M.

Thus suppose λ is in the closed tube $\tilde{\mathfrak{T}}$ defined above Definition 2.3 of [31]. If V is a compatible $(\mathfrak{M}\mathfrak{A}, K_M)$-module whose infinitesimal character (relative to $\mathfrak{M}\mathfrak{A}$) has parameter $\lambda + \mu$, we define

$${}^M\psi_\lambda^{\lambda+\mu}(V) = p_\lambda\big[V \otimes {}^M F_{-\mu}\big],$$

where p_λ is projection according to the infinitesimal character λ relative to $\mathfrak{M}\mathfrak{A}$.

For emphasis we write ${}^G\psi_\lambda^{\lambda+\mu}$ for the usual ψ functor associated with G. Freedom for the imaginary component of λ will be important in applications, and we point out that the results of [31] apply for λ in the whole closed tube $\tilde{\mathfrak{T}}$, not just in the dense fundamental domain $\mathfrak{T}$. For a compatible $(\mathfrak{M}\mathfrak{A}, K_M)$-module V, let $\operatorname{ind}_{MAN}^G V$ denote the compatible $(\mathcal{G}, K)$-module obtained by induction, with parameters arranged so that unitary representations induce to unitary representations.

THEOREM B.1. *For any compatible $(\mathfrak{M}\mathfrak{A}, K_M)$-module V, the functors ${}^M\psi_\lambda^{\lambda+\mu}$ and ${}^G\psi_\lambda^{\lambda+\mu}$ are related by*

$${}^G\psi_\lambda^{\lambda+\mu}\big(\operatorname{ind}_{MAN}^G V\big) = \operatorname{ind}_{MAN}^G\big({}^M\psi_\lambda^{\lambda+\mu} V\big).$$

Proof. We know that

$$F_{-\mu} \otimes \operatorname{ind}_{MAN}^{G} V \cong \operatorname{ind}_{MAN}^{G}\left(F_{-\mu}|_{MAN} \otimes V\right). \tag{B.2}$$

An irreducible MAN-subquotient F_i of $F_{-\mu}|_{MAN}$ contributes the subquotient

$$p_\lambda\left[\operatorname{ind}_{MAN}^{G}(F_i \otimes V)\right] \tag{B.3}$$

to p_λ of the right side of (B.2). Thus the discussion on pages 300–301 of [31] shows that we can get a nonzero subquotient of p_λ of the left side of (B.2) only by having a weight $-\nu$ of F_i, a member t of the Weyl group W_M of $(MA)^{\mathbf{C}}$, and a member s of the Weyl group W_G of $G^{\mathbf{C}}$ such that

$$-\nu + t(\lambda + \mu) = s\lambda.$$

Here F_i has to be irreducible under MA, and N must operate trivially. Thus $-\nu' = -t^{-1}\nu$ is a weight of F_i, and the member $s' = t^{-1}s$ of W_G satisfies

$$\lambda + \mu = s'\lambda + \nu'. \tag{B.4}$$

Since $\operatorname{Re}\lambda$ is dominant and μ is the $\mathfrak{g}$-highest weight, we have $s'\operatorname{Re}\lambda \leq \operatorname{Re}\lambda$ and $\nu' \leq \mu$. Then (B.4) forces

$$s'\lambda = \lambda \text{ and } \nu' = \mu. \tag{B.5}$$

Thus the only contribution to p_λ of the left side of (B.2) can come from an irreducible subquotient of $F_{-\mu}|_{MAN}$ containing the weight $-\mu$. Since $F_{-\mu}$ is irreducible under $\mathfrak{g}$, there is at most one such subquotient.

On the other hand, $-\mu$ is a lowest weight of $F_{-\mu}$ relative to $\Delta_{\mathfrak{m}}^{+}$ since $\Delta_{\mathfrak{m}}^{+} \subseteq \Delta_{\mathfrak{g}}^{+}$, and hence ${}^{M}F_{-\mu}$ occurs, with trivial N action, as an irreducible MAN-subquotient of $F_{-\mu}|_{MAN}$. By (B.3),

$${}^{G}\psi_{\lambda}^{\lambda+\mu}\operatorname{ind}_{MAN}^{G} V = p_\lambda\left[\operatorname{ind}_{MAN}^{G}{}^{M}F_{-\mu} \otimes V\right],$$

and the right side here is

$$= \operatorname{ind}_{MAN}^{G}\left[p_\lambda\left({}^{M}F_{-\mu} \otimes V\right)\right] = \operatorname{ind}_{MAN}^{G}\left({}^{M}\psi_{\lambda}^{\lambda+\mu} V\right)$$

since $s'\lambda = \lambda$ in (B.5).

Corollary B.2. *If V_1 and V_2 are two compatible $(\mathcal{M}\mathcal{Q}, K_M)$-modules and T is a morphism from V_1 to V_2, then*

$${}^{G}\psi_{\lambda}^{\lambda+\mu}\left(\operatorname{ind}_{MAN}^{G} T\right) = \operatorname{ind}_{MAN}^{G}\left({}^{M}\psi_{\lambda}^{\lambda+\mu} T\right).$$

Proof. We trace through the proof of Theorem B.1, replacing spaces by morphisms, and we obtain a proof of the corollary.

C. Crossing walls

Now suppose that G is a connected semisimple Lie group with a faithful matrix representation and with G contained in a simply-connected complexification $G^{\mathbf{C}}$.

By a virtual character on G, we will mean a finite integral linear combination of characters of irreducible Harish-Chandra modules. The exact functors φ and ψ in [31] give rise to homomorphic maps, also denoted φ and ψ, from the additive group of virtual characters to itself. We shall study certain combinations of φ and ψ that leave stable the subgroup of virtual characters with a specified infinitesimal character. For simplicity we work only with virtual characters that have integral[19] infinitesimal character. With care, all our results can be generalized.

Fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and a positive system of roots for $(\mathfrak{g}, \mathfrak{h})$. Let α be a simple root, and let λ_1 be a nonsingular dominant integral form on $\mathfrak{h}$. Let Λ_α be the fundamental weight dual to α, and define

$$c = \frac{2\langle \lambda_1, \alpha \rangle}{|\alpha|^2}. \tag{C.1}$$

Then $\lambda_1 - c\Lambda_\alpha$ is dominant integral, and it is nonsingular with respect to all simple roots except α. Define

$$\psi_\alpha = \psi^{\lambda_1}_{\lambda_1 - c\Lambda_\alpha} \text{ and } \varphi_\alpha = \varphi^{\lambda_1 - c\Lambda_\alpha}_{\lambda_1}.$$

If Θ is a virtual character with infinitesimal character λ_1, let

$$p_\alpha\Theta = \varphi_\alpha\psi_\alpha\Theta - \Theta. \tag{C.2}$$

Then $p_\alpha\Theta$ is again a virtual character with infinitesimal character λ_1.

We shall study the homomorphism p_α of the additive group of virtual characters with given infinitesimal character λ_1 into itself. The mapping p_α has an interpretation in terms of reflection across the wall $\ker \alpha$, if we use global character theory. The precise statement is given as Theorem C.1.

The same remarks and notation concerning global character theory apply here as in Section A of this appendix before (A.2) and (A.3). In view of the identifications noted there, we can specify a virtual character by giving its expression on the set H' of regular elements of an arbitrary Cartan subgroup H of G. For any integral form μ on $\mathfrak{h}$, let ξ_μ be the holomorphic character of the connected complexification $H^{\mathbf{C}}$ into $\mathbf{C}^\times$. The Weyl denominator ∇ is the function on $H^{\mathbf{C}}$ given by (A.2). If Θ is a virtual character with infinitesimal

[19] Since $G^{\mathbf{C}}$ is simply-connected, the analytic definition of "integral" that we have been using coincides with the algebraic one: λ is *integral* if $2\langle \lambda, \beta \rangle / |\beta|^2$ is an integer for every root β.

character λ_1 and if h is in H', then our special assumptions on G and λ_1 allow us to simplify the expression for $\nabla\Theta$ in (A.3) to

$$\nabla(h)\Theta(h) = \sum_{s\in W} c(\Theta; s)(h)\xi_{s\lambda_1}(h), \tag{C.3}$$

where the expressions $c(\Theta; s)(h)$ are locally constant functions on H'.

THEOREM C.1. *Let a virtual character Θ with nonsingular dominant integral infinitesimal character λ_1 be given by equation* (C.3). *If α is a simple root, then $p_\alpha\Theta$ is given by*

$$\nabla(h)p_\alpha\Theta(h) = \sum_{s\in W} c(\Theta; s)(h)\xi_{sp_\alpha\lambda_1}(h). \tag{C.4}$$

Proof. Suppose that λ_1, λ_2, and μ are all dominant integral forms on $\mathfrak{h}$ and that $\lambda_1 = \lambda_2 + \mu$. Let Θ_1 and Θ_2 be virtual characters with infinitesimal characters λ_1 and λ_2, respectively. Formula (3.7) of [31] simplifies under our assumptions on G and λ_1 to

$$\nabla(h)\varphi_{\lambda_1}^{\lambda_2}\Theta_2(h) = \sum_{t\in\frac{W(\lambda_2)}{W(\lambda_1)}} \sum_{s\in W} c(\Theta_2; s)(h)\xi_{st^{-1}\lambda_1}(h). \tag{C.5}$$

Formula (3.8) of [31] simplifies to

$$\nabla(h)\psi_{\lambda_2}^{\lambda_1}\Theta_1(h) = \sum_{s\in W} c(\Theta_1; s)(h)\xi_{s\lambda_2}(h). \tag{C.6}$$

In our present situation, we combine (C.5) and (C.6) to compute $\varphi_{\lambda_1}^{\lambda_2}\psi_{\lambda_2}^{\lambda_1}$ in the case that $\lambda_1 = \lambda_1$ and $\lambda_2 = \lambda_1 - c\Lambda_\alpha$, with c as in (C.1). Since λ_1 is assumed nonsingular, we have $W(\lambda_1) = \{1\}$ and $W(\lambda_2) = \{1, p_\alpha\}$. Theorem C.1 then follows.

In [31], equations (C.5) and (C.6) were used to prove Lemma 3.3, a special case of which is the identity (on the level of characters)

$$\psi_\alpha\varphi_\alpha = 2. \tag{C.7}$$

Applying ψ_α to both sides of (C.2) and using (C.7), we obtain

$$\psi_\alpha p_\alpha\Theta = \psi_\alpha\Theta. \tag{C.8}$$

THEOREM C.2. *The mapping $\Theta \to p_\alpha\Theta$ extends* (*uniquely*) *to an action $\Theta \to w\Theta$ of W as automorphisms of the additive group of virtual characters having nonsingular dominant integral infinitesimal character λ_1. If Θ is given by equation* (C.3) *and if w is in W, then $w\Theta$ is given by*

$$\nabla(h)w\Theta(h) = \sum_{s\in W} c(\Theta; s)(h)\xi_{sw^{-1}\lambda_1}(h). \tag{C.9}$$

Proof. Write w in W as a product of simple reflections: $w = p_{\alpha_1}\cdots p_{\alpha_r}$. By induction on r, we see from Theorem C.1 that

$$\text{(C.10)} \qquad \begin{aligned} \nabla(h)p_{\alpha_1}\cdots p_{\alpha_r}\Theta(h) &= \sum_{s\in W} c(\Theta; s)(h)\xi_{sp_{\alpha_r}\cdots p_{\alpha_1}\lambda_1}(h) \\ &= \sum_{s\in W} c(\Theta; s)(h)\xi_{sw^{-1}\lambda_1}(h). \end{aligned}$$

The left side of (C.10) is $\nabla(h)$ times a value of a virtual character, hence of a locally integrable function on G. The right side of (C.10) depends only on w and not on the specific factorization into the product of simple reflections. The element h can be any regular element of G, and such elements exhaust G except for a set of measure zero. The result follows.

Lemma 3.3 of [31] gives a formula on the level of characters for $\psi_{\lambda_2}^{\lambda_1}\varphi_{\lambda_1}^{\lambda_2}$, and Theorem C.1 gives a formula for $\varphi_{\lambda_1}^{\lambda_2}\psi_{\lambda_2}^{\lambda_1}$ in a special case. We now consider $\varphi_{\lambda_1}^{\lambda_2}\psi_{\lambda_2}^{\lambda_1}$ more generally.

THEOREM C.3. *Let λ_1, λ_2, and μ be dominant integral forms on $\mathfrak{h}$ such that $\lambda_1 = \lambda_2 + \mu$ and λ_1 is nonsingular. If Θ is a virtual character with infinitesimal character λ_1, then*

$$\varphi_{\lambda_1}^{\lambda_2}\psi_{\lambda_2}^{\lambda_1}\Theta = \sum_{w\in W(\lambda_2)} w\Theta,$$

with $w\Theta$ defined as in Theorem C.2.

Proof. We combine (C.5) and (C.6) to obtain a pointwise equality on the regular set. The theorem then follows since characters on G are locally integrable functions.

We conclude this appendix by discussing the additive group $J_{\lambda_2}^{\lambda_1}$ of virtual characters generated by irreducible characters Θ such that Θ has infinitesimal character λ_1 and $\psi_{\lambda_2}^{\lambda_1}\Theta = 0$. We assume that λ_1, λ_2, and μ are dominant integral, that $\lambda_1 = \lambda_2 + \mu$, and that λ_1 is nonsingular. The group $J_{\lambda_2}^{\lambda_1}$ need not be zero.

LEMMA C.4. $\psi_{\lambda_2}^{\lambda_1} = \psi_{\lambda_2}^{\lambda_1 - c\Lambda_\alpha}\psi_{\lambda_1 - c\Lambda_\alpha}^{\lambda_1}$ *if α is a simple root such that $\langle \lambda_2, \alpha\rangle = 0$ and if c is given by* (C.1).

Proof. This result is a special case of Theorem A.1.

THEOREM C.5. *Suppose that λ_1, λ_2, and μ are dominant integral, that $\lambda_1 = \lambda_2 + \mu$, that λ_1 is nonsingular, and that $\langle \lambda_2, \alpha\rangle = 0$. Then*

$$p_\alpha J_{\lambda_2}^{\lambda_1} \subseteq J_{\lambda_2}^{\lambda_1}.$$

Proof. Let Θ be an irreducible character with infinitesimal character λ_1 and with $\psi_{\lambda_2}^{\lambda_1}\Theta = 0$. We have $p_\alpha\Theta = \varphi_\alpha\psi_\alpha\Theta - \Theta$, and $\varphi_\alpha\psi_\alpha\Theta$ and Θ are true characters. Since $\psi_{\lambda_2}^{\lambda_1}$ takes true characters to true characters, it is enough to prove that $\psi_{\lambda_2}^{\lambda_1}\varphi_\alpha\psi_\alpha\Theta = 0$ and $\psi_{\lambda_2}^{\lambda_1}\Theta = 0$. The latter formula is our assumption. The other formula holds because

$$\begin{aligned}\psi_{\lambda_2}^{\lambda_1}(\varphi_\alpha\psi_\alpha\Theta) &= \psi_{\lambda_2}^{\lambda_1 - c\Lambda_\alpha}\psi_\alpha(\varphi_\alpha\psi_\alpha\Theta) && \text{by Lemma C.4,}\\ &= 2\psi_{\lambda_2}^{\lambda_1 - c\Lambda_\alpha}\psi_\alpha\Theta && \text{by (C.7),}\\ &= 2\psi_{\lambda_2}^{\lambda_1}\Theta && \text{by Lemma C.4,}\\ &= 0 && \text{by assumption.}\end{aligned}$$

CORNELL UNIVERSITY, ITHACA, NEW YORK
YALE UNIVERSITY, NEW HAVEN, CONNECTICUT

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