Fatou's theorem for symmetric spaces: I*

By A. W. KNAPP

Introduction

Let $P(r, t)$ denote the Poisson kernel

$$\frac{1 - r^2}{1 - 2r \cos t + r^2} .$$

The Poisson integral of an integrable function f on the circle is the harmonic function h in the disc given by

$$h(re^{ix}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, x - t) f(t) dt ,$$

and Fatou's theorem is the assertion that, as r tends to 1, h tends to f almost everywhere.

If the disc and the circle are viewed as homogeneous spaces of the group $SL(2, R)$, the setting for this theorem may be generalized as follows. Let G be any connected non-compact semi-simple Lie group with finite center, and let K be a maximal compact subgroup. G/K is the symmetric space of G, and a complex-valued function on G/K is harmonic if it is annihilated by every G-invariant differential operator on G/K. In [3] Furstenberg exhibited a Poisson integral formula for the bounded harmonic functions on G/K, and he generalized his results to positive harmonic functions as part of [4]. Furstenberg knew that the boundary (analogous to the circle) was a homogeneous space of G (and actually of K), and he wrote the Poisson kernel as Radon-Nikodym derivatives of the action of G on the K-invariant measure on the boundary. Moore [9] identified the boundary explicitly, and a concrete formula for the kernel followed from calculations of Harish-Chandra in [5].

Now a symmetric space admits polar coordinates, in which the radial direction is indexed by a cone in a euclidean space and the other coordinate is indexed by the boundary. A theorem of Fatou type would say that, as the radial coordinate tends to ∞ in some fashion, the Poisson integral of an integrable function on the boundary tends to the function at almost every point of the boundary.

Theorems of this sort are known in several special cases. In addition to

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Fatou's result for $SL(2, R)$, there is a proof for $SL(2, C)$ implicit in Ahlfors [1], Helgason (unpublished) has proved the theorem for the Lorentz groups $SO_e(n, 1)$, Korányi [7] has treated the case of the groups $SU(n, 1)$, and Marcinkiewicz and Zygmund [8] proved the theorem for the direct sum of finitely many copies of $SL(2, R)$. Helgason's proof is interesting in that it does not use an explicit formula for the kernel, though the symmetric space for $SO_e(n, 1)$ is the n-ball, and the kernel is easily seen to be

$$P(r, \mathbf{x}) = \left(\frac{1 - r^2}{1 - 2rx_1 + r^2}\right)^{n-1}, \qquad |\mathbf{x}| = 1 .$$

With $SU(n,1)$ the symmetric space is the unit ball in complex n-space and the kernel is

$$P(r, \mathbf{z}) = \left(\frac{1 - r^2}{|1 - rz_1|^2}\right)^{n}, \qquad |\mathbf{z}| = 1 .$$

In this paper we prove a Fatou theorem for all symmetric spaces of rank one. Some results for the general case will appear later. The individual ideas in the argument given here extend to the general case, but the general proofs are more elaborate and the results taken together are insufficient for proving a Fatou theorem in the general case. The difficulty is that the Poisson kernel, as the radial coordinate tends to ∞, has a one-point singularity for rank-one spaces and a higher-dimensional singularity for spaces of rank greater than one. The complications introduced by this fact are suggested by a comparison of the simple proof for the disc (rank one) and the involved proof of Marcinkiewicz and Zygmund for the product of two discs (rank two).

The theorem we prove is stated precisely in § 1, and the contents of the paper are outlined at the end of that section.

I am indebted to S. Helgason for helpful conversations in connection with this paper. The line of proof in § 5 arose from joint work with Richard Williamson on the case of $SO_e(n, 1)$, before we knew of Helgason's proof for this case.

1. Statement of Fatou's theorem

The following notation will be in force throughout the paper. See [6] for details of the results quoted. Let G be a connected non-compact semi-simple Lie group with finite center and with identity e, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra of G, let θ be the corresponding Cartan involution (and the exponential of it), and let B_θ be the positive definite form $B_\theta(X, Y) = -B(X, \theta Y)$, where B is the Killing form. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subspace, and fix an open Weyl chamber $\mathfrak{a}^+$ as positive.

If $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and $\mathfrak{n}$ is the sum of the positive restricted root spaces $\mathfrak{g}_\lambda$, we have the decompositions

$$\mathfrak{g} = \theta\mathfrak{n} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \quad \text{and} \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} .$$

If A and N are the exponentials of $\mathfrak{a}$ and $\mathfrak{n}$, then $G = KAN$. The log of the A-component of an element g is denoted $H(g)$, and the K-component is denoted $k(g)$. G/K is the symmetric space of G, M is the centralizer of A in K, K/M is the boundary, and dk is the normalized K-invariant measure on K/M. G acts transitively on K/M by $g(kM) = k(g)M$, and this action is isomorphic in the obvious way to the action of G on G/MAN. Finally 2ρ is the sum of the positive restricted roots on $\mathfrak{a}$, counted according to their multiplicities, and A^+ is the exponential of $\mathfrak{a}^+$.

It is known that $(a, kM) \to kaK$ is a map of $A \times K/M$ onto G/K. Restricted to $A^+ \times K/M$, this map is one-one onto an open subset whose complement is of lower dimension. That is, all points of G/K are of the form kaK with k in K and a in the closure $\mathrm{cl}(A^+)$, and for most points k is unique up to M and a is unique and is in A^+. For such points, kM and a are the polar coordinates of kaK.

The Poisson kernel is the function on $G/K \times K/M$ given by

$$P(gK, kM) = e^{-2\rho H(g^{-1}k)} ,$$

and the Poisson integral of an integrable f on K/M is the function h on G/K defined by

$$h(gK) = \int_{K/M} P(gK, kM) f(kM) dk .$$

If gK is a point of G/K with polar coordinates k_0M and a, then use of the identity $P(k_0aK, kM) = P(aK, k_0^{-1}kM)$ and a change of variables give

$$h(k_0aK) = \int_{K/M} P(aK, kM) f(k_0kM) dk . \tag{1}$$

Formula (1) shows that, for our current purposes, the Poisson kernel can be viewed as a function on $A^+ \times K/M$, and it was this point of view that was adopted in the introduction.

The rank of G/K is the dimension of $\mathfrak{a}$. If G/K has rank one, there are either one or two positive roots. We denote them α and 2α (or just α if there is only one). We can now state the main theorem.

THEOREM 1.1. *Let G/K have rank one, and let H_0 be a member of $\mathfrak{a}^+$. If f is in $L^1(K/M)$, and if h is its Poisson integral, then*

$$\lim_{t \to +\infty} h(k_0 \exp(tH_0)K) = f(k_0M)$$

almost everywhere with respect to dk.

The proof will be modeled after one proof of Fatou's theorem for the disc. The proof we have in mind is in five steps. First, one observes that the Poisson kernel in the disc satisfies the inequalities

$$P(r, x) \leqq \frac{C_1}{1 - r} \quad \text{and} \quad P(r, x) \leqq \frac{C_2(1 - r)}{x^2} . \tag{2}$$

Next, the second of these inequalities implies that $P(r, x)$ is an approximate identity, and Fatou's theorem follows for continuous boundary values. Third, one proves a form of the Hardy-Littlewood *maximal theorem*: *If f is an integrable function on the circle, and if*

$$f^*(x) = \sup_{0<y\leqq\pi} \frac{1}{2y} \int_{x-y}^{x+y} |f(t)|\, dt ,$$

then the measure of the set where $f^*(x) > \xi$ is $\leqq C\xi^{-1} \| f \|_1$ with C independent of f and ξ. Fourth, if h is the Poisson integral of f, and if $f_*(x) = \sup_{0\leqq r<1} | h(re^{ix}) |$, then f_* is shown to be $\leqq$ some constant times $f^*(x)$, and f_* therefore satisfies the same kind of maximal inequality as f^*. Fatou's theorem follows easily from this maximal inequality and the theorem for continuous boundary values.

For the proof of Theorem 1.1, the inequalities for the kernel are derived in § 3 from some more general inequalities proved in § 2. The theorem for continuous boundary values appears also in § 3. The main step in the proof of the maximal theorem is a covering theorem which is the subject of § 4. The maximal inequalities themselves, together with the argument that completes the proof of Theorem 1.1, appear in § 5. In § 6 we extend the statement and proof of Theorem 1.1 to the case of signed measures, as opposed to integrable functions, as boundary values.

2. Inequalities for the Poisson kernel

We begin by obtaining inequalities which generalize inequalities (2) for the disc. The main result of this section is Theorem 2.2, which gives a system of inequalities valid for any symmetric space. Appropriate positive combinations of these inequalities will give in Theorem 3.1 the generalizations of (2) to spaces of rank one. Throughout this section G/K is allowed to be of arbitrary rank.

The proof of Theorem 2.1 below was simplified to its present form by Helgason and Kostant in a seminar.

THEOREM 2.1. *Let the rank of G/K be arbitrary. Then*

$$e^{2\rho H(g)} = \det [P_{\mathfrak{k}} \mathrm{Ad}(g^{-1}) \mid \mathfrak{k}] ,$$

where $|\mathfrak{k}$ *means restriction to* $\mathfrak{k}$, *and where* $P_{\mathfrak{k}}$ *is the projection of* $\mathfrak{g}$ *on* $\mathfrak{k}$ *along* $\mathfrak{a}+\mathfrak{n}$. *Consequently*

$$P(aK, kM)^{-1} = \det[P_{\mathfrak{k}}\mathrm{Ad}(k^{-1}a) \mid \mathfrak{k}]\ . \tag{3}$$

PROOF. Write $g = kan$. Since K is compact connected, and since $\mathrm{Ad}(k^{-1})$ maps $\mathfrak{k}$ into $\mathfrak{k}$, we have $\det \mathrm{Ad}(k^{-1})|\mathfrak{k} = 1$. Since determinant is multiplicative, we can assume $k = e$ and $g = an$. Now $\mathrm{Ad}(an)^{-1}$ preserves $\mathfrak{a}+\mathfrak{n}$, and $\det \mathrm{Ad}(an)^{-1} = 1$. Thus

$$\det[P_{\mathfrak{k}}\,\mathrm{Ad}(an)^{-1} \mid \mathfrak{k}] \det[\mathrm{Ad}(an)^{-1} \mid \mathfrak{a}+\mathfrak{n}] = \det \mathrm{Ad}(an)^{-1} = 1$$

or

$$\det[P_{\mathfrak{k}}\,\mathrm{Ad}(an)^{-1} \mid \mathfrak{k}] = \det[\mathrm{Ad}(an) \mid \mathfrak{a}+\mathfrak{n}]\ .$$

The right side of the last equality is $e^{2\rho \log a}$, and the proof is complete.

We wish to describe a special system of vectors in $\mathfrak{g}$. Orthogonality will be with respect to the inner product B_θ. Since the restricted root spaces are mutually orthogonal, we can let $\{X_i\}$ be an orthogonal basis of $\mathfrak{m}+\mathfrak{n}$ compatible with the decomposition

$$\mathfrak{m}+\mathfrak{n} = \mathfrak{m} + \textstyle\sum_{\lambda>0}\mathfrak{g}_\lambda\ .$$

For uniformity of notation, we shall let X_i be in $\mathfrak{g}_{\lambda_i}$. Here λ_i is $\geqq 0$, and the λ_i's may be repeated. With the X_i's constructed this way, the vectors $X_i + \theta X_i$ are an orthogonal basis of $\mathfrak{k}$. If X is any vector in $\mathfrak{g}$, the notation $\{X\}_{\mathfrak{k}}$ will mean the column vector which represents $P_{\mathfrak{k}}(X)$ in the basis consisting of the vectors $X_i + \theta X_i$.

Let H be in $\mathfrak{a}$, and let $a = \exp H$. From equation (3) we have

$$P(aK, kM)^{-1} = \det[\{e^{\lambda_i(H)}\mathrm{Ad}(k^{-1})X_i + e^{-\lambda_i(H)}\mathrm{Ad}(k^{-1})\theta X_i\}_{\mathfrak{k}}]\ . \tag{4}$$

This formula is motivation for the discussion which follows.

Let ν be the dimension of $\mathfrak{k}$, and consider the determinants $D_R(g)$ of the 2^ν possible matrices with i^{th} column either $\{\mathrm{Ad}(g^{-1})X_i\}_{\mathfrak{k}}$ or $\{\mathrm{Ad}(g^{-1})\theta X_i\}_{\mathfrak{k}}$. Here R is an index on the subsets of $1, \cdots, \nu$ telling for which i's to use $\mathrm{Ad}(g^{-1})X_i$ rather than $\mathrm{Ad}(g^{-1})\theta X_i$. If $\mathfrak{m} \neq 0$, different R's may give identical matrices, but this fact is irrelevant. The relevance of the determinants $D_R(g)$ is that the right side of (4) is a positive combination of the determinants $D_R(k)$, and the coefficients depend only on H, not on k.

THEOREM 2.2. $D_R(g) \geqq 0$ *for all subsets* R.

The proof of this theorem will be given after two preliminary lemmas.

LEMMA 2.3. *A real polynomial* $x^n + px^{n-2} + \cdots$ *with* $p > 0$ *and* $n > 1$ *cannot have all roots real.*

PROOF. The roots of the $n-2^{\text{nd}}$ derivative are non-zero and pure imaginary, and they lie in the convex hull of the roots of the given polynomial.

LEMMA 2.4. *If X and Y are in $\mathfrak{k}$ and a is in A with*

$$B_\theta(\mathrm{Ad}(a)X, \mathrm{Ad}(a)Y) = B_\theta(\mathrm{Ad}(a^{-1})X, \mathrm{Ad}(a^{-1})Y) = 0 ,$$

then $\mathrm{ad}(\mathrm{Ad}(a)X + \mathrm{Ad}(a^{-1})Y)$ *cannot have all eigenvalues real unless* $X = Y = 0$.

PROOF. Let the dimension of $\mathfrak{g}$ be n. Choose an orthonormal basis of $\mathfrak{g}$ compatible with $\mathfrak{g} = \theta\mathfrak{n}\oplus\mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n}$. In this basis the matrices ad $\mathfrak{k}$ are skew, ad $\mathfrak{p}$ are symmetric, and ad $\mathfrak{a}$ are diagonal. This fact means θ is negative transpose and $\mathrm{Ad}(a)$ is conjugation by a diagonal matrix. Thus if Z is in $\mathfrak{g}$, then

$$B_\theta(Z, \theta Z) = \textstyle\sum_{i,j} (\mathrm{ad}\, Z)_{ij}(\mathrm{ad}\, Z)_{ji} .$$

Also ad of

$$Z = \mathrm{Ad}(a)X + \mathrm{Ad}(a^{-1})Y \tag{5}$$

has 0's on the diagonal.

Now if T is any n-by-n matrix with 0's on the diagonal, then

$$\det(xI - T) = x^n + (\textstyle\sum_{i,j} T_{ij}T_{ji})x^{n-2} + \cdots .$$

With Z as in (5),

$$\det(xI - \mathrm{ad}\, Z) = x^n + B_\theta(Z, \theta Z)x^{n-2} + \cdots .$$

But the given orthogonality conditions, the invariance of the Killing form, and the fact that $\theta a = a^{-1}$ give

$$\begin{aligned} B_\theta(Z, \theta Z) &= B_\theta(\mathrm{Ad}(a)X, \mathrm{Ad}(a^{-1})X) + B_\theta(\mathrm{Ad}(a^{-1})Y, \mathrm{Ad}(a)Y) + 0 + 0 \\ &= B_\theta(X, X) + B_\theta(Y, Y) . \end{aligned}$$

By Lemma 2.3, either $X = Y = 0$ or $\det(xI - \mathrm{ad}\, Z)$ has a non-real root.

PROOF OF THEOREM 2.2. To see that $D_R(g) \geqq 0$, let H be any vector in $\mathrm{cl}(\mathfrak{a}^+)$ and change the i^{th} column of the matrix defining $D_R(g)$ to

$$\{e^{\lambda_i(H)}\mathrm{Ad}(g^{-1})X_i + e^{-\lambda_i(H)}\mathrm{Ad}(g^{-1})\theta X_i\}_{\mathfrak{k}}$$

if it was $\{\mathrm{Ad}(g^{-1})X_i\}_{\mathfrak{k}}$ or

$$\{e^{-\lambda_i(H)}\mathrm{Ad}(g^{-1})X_i + e^{\lambda_i(H)}\mathrm{Ad}(g^{-1})\theta X_i\}_{\mathfrak{k}}$$

if it was $\{\mathrm{Ad}(g^{-1})\theta X_i\}_{\mathfrak{k}}$. If the new determinant is called $D_R(g, H)$, and if $\mu = \dim \mathfrak{m}$, then

$$D_R(g) = 2^{-\mu} \lim_{\mathrm{All}\,\lambda_i(H)\to\infty} e^{-2\rho(H)} D_R(g, H) .$$

Since H runs through a connected set, and since Theorem 2.1 shows that $D_R(g, 0) = e^{2\rho H(g)}$, it suffices to prove that $D_R(g, H) \neq 0$.

Thus suppose $D_R(g, H) = 0$. Then for suitable constants c_i and d_j not all 0, the vector

$$\mathrm{Ad}(g^{-1})\left[\sum_{i\in R} c_i(e^{\lambda_i(H)}X_i + e^{-\lambda_i(H)}\theta X_i) + \sum_{j\notin R} d_j(e^{-\lambda_j(H)}X_j + e^{\lambda_j(H)}\theta X_j)\right]$$

is in $\mathfrak{a} + \mathfrak{n}$. If $a = \exp H$, this fact means

$$\mathrm{Ad}(g^{-1})[\mathrm{Ad}(a)X + \mathrm{Ad}(a^{-1})Y]$$

is in $\mathfrak{a} + \mathfrak{n}$, where

$$(3)\qquad \begin{aligned} X &= \sum_{i\in R} c_i(X_i + \theta X_i) \in \sum_{i\in R} \mathrm{span}\,(X_i, \theta X_i) \\ Y &= \sum_{j\notin R} d_j(X_j + \theta X_j) \in \sum_{j\notin R} \mathrm{span}\,(X_j, \theta X_j)\ , \end{aligned}$$

and where the subspaces on the right sides of (6) are orthogonal and $\mathrm{Ad}(A)$-invariant. That is, X and Y satisfy the orthogonality conditions of Lemma 2.4. Thus either $X = Y = 0$ (and all c_i's and d_j's are 0, contradiction) or

$$\mathrm{ad}\big(\mathrm{Ad}(a)X + \mathrm{Ad}(a^{-1})Y\big)$$

has a non-real eigenvalue. In the latter case the same is true of the conjugate matrix

$$\mathrm{ad}\big(\mathrm{Ad}(g^{-1})\big(\mathrm{Ad}(a)X + \mathrm{Ad}(a^{-1})Y\big)\big)\ ,$$

in contradiction to the fact that the members of $\mathrm{ad}(\mathfrak{a} + \mathfrak{n})$ have all eigenvalues real.

3. Special case with continuous boundary values

Return to the case that G/K has rank one. Theorem 3.1 will finish the construction of inequalities generalizing (2). Actually for $SL(2, R)$ and the disc, inequalities (7) and (8) below are

$$P(r, x) \leqq \frac{1+r}{1-r} \quad \text{and} \quad P(r, x) \leqq \frac{1-r}{1+r}\frac{2}{1-\cos x}\ ,$$

which are slightly better estimates than (2).

THEOREM 3.1. *Let G/K have rank one. Then*

$$(7)\qquad P(aK, kM) \leqq e^{2\rho\log a} \qquad \textit{for } a \textit{ in } \mathrm{cl}(A^+)\ ,$$

and

$$(8)\qquad P(aK, kM) \leqq e^{-2\rho\log a}F(kM)^{-1} \qquad \textit{for } a \textit{ in } \mathrm{cl}(A^+)\ ,$$

where $F(kM)$ is a continuous function vanishing only at eM.

PROOF. Part of the content of Theorem 2.2 is that the determinant of any matrix whose i^{th} column, for each i, is a non-negative combination (depending on i) of $\{\mathrm{Ad}(k^{-1})X_i\}_{\mathfrak{k}}$ and $\{\mathrm{Ad}(k^{-1})\theta X_i\}_{\mathfrak{k}}$ is decreased if the coefficients in the combination are decreased. This is so because any such determinant is a non-negative combination of the 2^ν determinants $D_R(k)$, and the coefficients are products of the coefficients of the $\{\mathrm{Ad}(k^{-1})X_i\}_{\mathfrak{k}}$ and $\{\mathrm{Ad}(k^{-1})\theta X_i\}_{\mathfrak{k}}$.

We apply this observation to equation (4). Inequality (7) results if we replace the coefficients $e^{\lambda_i(H)}$ by $e^{-\lambda_i(H)}$, and leave the coefficients $e^{-\lambda_i(H)}$ alone. The coefficients do not increase since $\lambda_i(H) \geqq 0$ for each i. A factor of $e^{-\lambda_i(H)}$ comes out of the i^{th} column, and the total factor is $e^{-2\rho \log a}$. The remaining determinant is $\det[\mathrm{Ad}(k^{-1}) \mid \mathfrak{k}]$, which is 1.

To obtain inequality (8), we replace, in (4), $e^{-\lambda_i(H)}$ by 0 if $\lambda_i > 0$, and we leave it alone if $\lambda_i = 0$. Also we leave $e^{\lambda_i(H)}$ alone in either case. Define $E(k)$ to be the determinant of a matrix whose i^{th} column is $\{\mathrm{Ad}(k^{-1})X_i\}_{\mathfrak{k}}$ if $\lambda_i > 0$, and is $\{\mathrm{Ad}(k^{-1})(X_i + \theta X_i)\}_{\mathfrak{k}}$ if $\lambda_i = 0$. Then

$$P(aK, kM) \leqq e^{-2\rho \log a} E(k)^{-1} ,$$

and we are left with proving that E is right-invariant under M and that, if $F(kM) = E(k)$, then F vanishes only at eM. The right invariance of E under M is so because $\mathrm{Ad}(m^{-1})$, for m in M, commutes with $P_{\mathfrak{k}}$ and $\mathrm{Ad}(m^{-1}) \mid \mathfrak{k}$ has determinant one.

Let us treat the second problem more generally, defining $E(g)$ to be the function obtained from replacing k by g in the formula for $E(k)$. $E(g)$ is $\geqq 0$ by Theorem 2.2, and the claim is that sign $E(g)$ is right and left invariant under MAN. In fact, the vectors X_i with $\lambda_i > 0$, and the vectors $X_i + \theta X_i$ with $\lambda_i = 0$, together form a basis of $\mathfrak{n} + \mathfrak{m}$, and thus sign $E(g) = 0$ if and only if there is a non-zero vector Z in $\mathfrak{n} + \mathfrak{m}$ such that $\mathrm{Ad}(g^{-1})Z$ is in $\mathfrak{a} + \mathfrak{n}$. The right and left invariance under MAN then follow from the fact that both $\mathfrak{n} + \mathfrak{m}$ and $\mathfrak{a} + \mathfrak{n}$ are invariant under $\mathrm{Ad}(MAN)$.

Choose m' as a member of the normalizer of A in K so that $\mathrm{Ad}(m')$ maps $\mathfrak{a}^+$ into $-\mathfrak{a}^+$. $\mathrm{Ad}(m')$ maps $\mathfrak{n}$ into $\theta\mathfrak{n}$ and maps $\mathfrak{m}$ into itself. Thus the criterion in the preceding paragraph shows that $E(m') \neq 0$. Now since G/K is assumed to be of rank one, the Bruhat *decomposition theorem* gives

$$G = MAN \cup MANm'MAN .$$

Since sign $E(g)$ is right and left invariant under MAN, $E(g)$ can vanish only if g is in MAN. Specializing to g a member of K, we see that $E(k)$ vanishes only for k in M. The proof is complete.

Inequality (7) was obtained by Harish-Chandra in [5] in a slightly different form. We shall remark on this further in the proof of Lemma 5.2.

COROLLARY 1. *Let G/K have rank one. Then*
(a) $P(aK, kM) \geqq 0$,

(b) $\int_{K/M} P(aK, kM)dk = 1$,

(c) *for any neighborhood U of eM in K/M*

$$\lim_{\rho(\log a)\to+\infty} \sup_{kM\notin U} P(aK, kM) = 0 .$$

Result (a) is trivial, result (b) follows from the representation of the harmonic function 1, and result (c) is immediate from inequality (8). Together these three statements show that $P(aK, kM)$ is an approximate identity. The style of the proofs in [12, Ch. 3, 4] yield the following corollary.

COROLLARY 2. *Let G/K have rank one, let f be an integrable fuuction on K/M, and let $h(gK)$ be its Poisson integral. With the indicated limits being as $\rho(\log a)\to+\infty$, the following things happen.*

(a) *If f is continuous,* $\lim h(kaK) = f(kM)$ *uniformly.*

(b) *If f is in L^p and $1 \leqq p < \infty$,* $\lim h(kaK) = f(kM)$ *in the norm topology of L^p.*

(c) *If f is in L^∞,* $\lim h(kaK) = f(kM)$ *weak-$*$ against L^1.*

4. A covering theorem

The heart of the proof of Theorem 1.1 is a differentiation theorem, which we prove in the form of a maximal inequality as Theorem 5.1. The maximal inequality is an easy consequence of a covering theorem, something like Vitali's, and this we give as Theorem 4.1.

We require some information about the relationship between θN and K/M. (Typically a member of θN will be denoted $\bar{n}$.) The map γ which sends $\bar{n}$ into $k(\bar{n})M$ is known to be a real analytic diffeomorphism of θN onto an open subset of K/M whose complement is of lower dimension. If K/M and G/MAN are identified, the image of $\bar{n}$ is $\bar{n}MAN$. If kM is in the image of γ, we write $\bar{n}(k)$ for $\gamma^{-1}(kM)$. In this case, $\bar{n}(k) = kman$, where man is some member of MAN.

Both M and A act on θN on the left by inner automorphism, and M and A also act on G/MAN on the left in the natural way. Helgason pointed out (oral communication) that these actions correspond. That is, with obvious notation

$$m\gamma(\bar{n}) = \gamma(m\bar{n}m^{-1}) \quad \text{and} \quad a\gamma(\bar{n}) = \gamma(a\bar{n}a^{-1}) .$$

Again let us identify G/MAN and K/M, and let us denote the action of G on K/M by a dot. That is, $g\cdot kM = k(gk)M$. Then we have

$$(9) \qquad m\cdot k(\bar{n})M = k(m\bar{n}m^{-1})M \quad \text{and} \quad a\cdot k(\bar{n})\, M = k(a\bar{n}a^{-1})M .$$

Recall that in the rank-one case $\theta\mathfrak{n} = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$, where α is the smaller positive root. We now introduce some coordinates on $\theta\mathfrak{n}$ and transfer them to K/M, where they will be used to give an appropriate system of neighborhoods for the statement of the differentiation theorem. For the Lorentz groups, $\mathfrak{g}_{-2\alpha} = 0$ and K/M is a sphere; the set $\bar{B}_t$ defined below is the set of points whose distance (the usual distance on the sphere) from eM is less than a certain constant. For groups in which $\mathfrak{g}_{-2\alpha}$ is not 0, $\bar{B}_t$ is harder to visualize.

Choose an $\mathrm{Ad}(M)$-invariant inner product on $\theta\mathfrak{n} = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$, let $\|\cdot\|$ be the corresponding norm, and let $(\,,)$ denote the decomposition of $\theta\mathfrak{n}$ into $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-2\alpha}$. Fix the vector H_0 in $\mathfrak{a}^+$, which appears in the statement of Theorem 1.1, and let

$$B_t = \{(X, Y) \mid \max\{\|X\|^2, \|Y\|\} < e^{-2\alpha(H_0)t}\} .$$

Since α is positive, these sets shrink to 0 as $t \to +\infty$, and they expand to $\theta\mathfrak{n}$ as $t \to -\infty$. B_t is defined in such a way that $B_t = \mathrm{Ad}(\exp tH_0)B_0$. Let S_t be the boundary of B_t, and let $\bar{B}_t$ and $\bar{S}_t$ be the images under γ of $\exp B_t$ and $\exp S_t$, respectively.

If kM is in K/M, let $\bar{B}_{t,k} = k\bar{B}_t$ and $\bar{S}_{t,k} = k\bar{S}_t$. Notice that these sets depend only on the coset of k modulo M. That is, if m is in M, then

$$\bar{B}_{t,k} = \bar{B}_{t,km} \quad \text{and} \quad \bar{S}_{t,k} = \bar{S}_{t,km} . \tag{10}$$

It suffices to show the first equality. We have

$$\bar{B}_{t,k} = k\bar{B}_t = km(m^{-1}\bar{B}_t) = km\bar{B}_t = \bar{B}_{t,km} ,$$

the next-to-last equality following from (9) and the $\mathrm{Ad}(M)$-invariance of B_t.

In writing measures of sets, we shall write $m_{K/M}(E)$ for the dk-measure on the set E in K/M, and we let $m_{\theta N}$ be a Haar measure on θN; $m_{\theta N}$ can be taken as the image of Lebesgue measure under the exponential map.

THEOREM 4.1. *There is a positive real number C_1 with this property. If E is any Borel set in K/M, and if to each point kM in E, there is associated a set $\bar{B}_{t,k}$ (with t perhaps depending on k), then there is a finite or infinite disjoint sequence of these associated $\bar{B}$'s, say $\bar{B}_{t_1,k_1}, \bar{B}_{t_2,k_2}, \cdots$, such that*

$$\sum_{j=1}^{\infty} m_{K/M}(\bar{B}_{t_j,k_j}) \geqq C_1^{-1} m_{K/M}(E) .$$

Theorem 4.1 has a euclidean analog. K/M is replaced by a euclidean space, E is assumed to have finite measure, and $\bar{B}_{t,k}$ is replaced by a cube centered at the point in question, and having edges parallel to the axes. A similar conclusion is valid. Both Theorem 4.1 and the euclidean analog depend on a specific geometric fact about the group in question. For the euclidean plane, this fact is that no matter how small a square we start with, the union of all squares which are translates of the given square, and which intersect the given square, has area ≤ 9 times the area of the given square.

The analogous geometric fact for K/M is given as Theorem 4.2. Once the geometric fact is known, Theorem 4.1 follows from a standard kind of proof, which is given at the end of this section. See Marcinkiewicz and Zygmund [8], Zygmund [11], and Edwards and Hewitt [2] for closely related proofs.

By the *K-hull* of $\bar{B}_t$ is meant the union of all K-translates of $\bar{B}_t$ which have non-empty intersection with $\bar{B}_t$. The *θN-hull* of $\exp B_t$ is the union of all θN-translates of $\exp B_t$ which have non-empty intersection with $\exp B_t$. Both kinds of hulls are open sets since B_t is open.

THEOREM 4.2. $$\sup_{-\infty<t<\infty} \frac{m_{K/M}(K\text{-}hull\ of\ \bar{B}_t)}{m_{K/M}(\bar{B}_t)} < \infty .$$

The proof of Theorem 4.2 will occupy most of the rest of this section. Our procedure is first to prove the analog of Theorem 4.2 for the θN-hull of $\exp B_t$, and then to show that the map of θN into K/M comes close enough to preserving the group actions that the result for K-hulls follows.

A word is in order about where the difficulty lies in the proof. In the case of $SU(2,1)$, the set B_t looks like a wafer in three dimensions. Its radius is of the order of e^{-t} and its height is of order e^{-2t}. Hence its measure is of order e^{-4t}. The θN-translates of $\exp B_t$ tilt somewhat, and it is conceivable that the θN-hull of $\exp B_t$ had all sides of the order of e^{-t}, hence measure of order e^{-3t}. In this case the ratio of the measures in question would be e^t and the supremum would be infinite. Thus the problem is to show that the tilt is not so serious. But what about the passage to K/M? The K-translates of $\bar{B}_t$, when pulled back to θN under γ^{-1}, look like the θN-translates of $\exp B_t$, except that the former bend slightly. Again the problem is to show the contributions to the bending are mostly in the directions in which the wafer is big (the $\mathfrak{g}_{-\alpha}$ directions) and not in the direction of the small dimension of the wafer (the $\mathfrak{g}_{-2\alpha}$ direction).

For the remainder of this section, we shall identify $\theta\mathfrak{n}$ and θN *via* the exponential map, and we omit writing exp. Then $\theta\mathfrak{n}$ has two multiplications, its bracket operation and the multiplication $\cdot$ inherited from θN. We can view the map $\gamma\colon \theta N \to K/M$ as defined on $\theta\mathfrak{n}$. Under the identification of $\theta\mathfrak{n}$ and θN, the multiplication rule in $\theta\mathfrak{n}$ is

$$(A_\alpha, A_{2\alpha})\cdot(B_\alpha, B_{2\alpha}) = \left(A_\alpha + B_\alpha, A_{2\alpha} + B_{2\alpha} + \frac{1}{2}[A_\alpha, B_\alpha]\right),$$

where A_α and B_α are in $\mathfrak{g}_{-\alpha}$, and $A_{2\alpha}$ and $B_{2\alpha}$ are in $\mathfrak{g}_{-2\alpha}$.

LEMMA 4.3. *If C_2 is the real number* max $\|[R, S]\|$ *with the maximum taken over all vectors R and S in $\mathfrak{g}_{-\alpha}$ of norm* $\leqq 1$, *then*

$$\theta N\text{-}hull(B_t) \subseteqq \left(3 + \frac{3}{2}C_2\right)B_t .$$

PROOF. Let $(X, Y)\cdot B_t \cap B_t \neq \emptyset$, and let $(A_\alpha, A_{2\alpha}) \in (X, Y)\cdot B_t$. Let $(B_\alpha, B_{2\alpha}) = (X, Y)\cdot(C_\alpha, C_{2\alpha})$ be in the intersection of B_t and $(X, Y)\cdot B_t$. Here $(B_\alpha, B_{2\alpha})$ and $(C_\alpha, C_{2\alpha})$ are to be in B_t. Then

$$(B_\alpha, B_{2\alpha}) = \left(X + C_\alpha,\ Y + C_{2\alpha} + \frac{1}{2}[X, C_\alpha]\right)$$

or

$$X = B_\alpha - C_\alpha \quad \text{and} \quad Y = B_{2\alpha} - C_{2\alpha} - \frac{1}{2}[B_\alpha, C_\alpha] .$$

From the definitions of B_t and C_2, and from the triangle inequality,

$$\|X\| \leqq 2e^{-\alpha(H_0)t} \quad \text{and} \quad \|Y\| \leqq \left(2 + \frac{1}{2}C_2\right)e^{-2\alpha(H_0)t} .$$

Now $(A_\alpha, A_{2\alpha})$ is of the form $(X, Y)\cdot(D_\alpha, D_{2\alpha})$ for some $(D_\alpha, D_{2\alpha})$ in B_t. That is,

$$(A_\alpha, A_{2\alpha}) = \left(X + D_\alpha,\ Y + D_{2\alpha} + \frac{1}{2}[X, D_\alpha]\right) ,$$

Hence

$$\|A_\alpha\| \leqq \|X\| + \|D_\alpha\| \leqq 3e^{-\alpha(H_0)t} ,$$

$$\begin{aligned} \|A_{2\alpha}\| &\leqq \|Y\| + \|D_{2\alpha}\| + \frac{1}{2}\|X, D_\alpha\| \\ &\leqq e^{-2\alpha(H_0)t}\left(2 + \frac{1}{2}C_2 + 1 + \frac{1}{2}C_2 \cdot 2\right) \\ &= e^{-2\alpha(H_0)t}\left(3 + \frac{3}{2}C_2\right) . \end{aligned}$$

Thus $(A_\alpha, A_{2\alpha}) \in \left(3 + \frac{3}{2}C_2\right)B_t$, and the proof is complete.

Now we pass to K/M. By joint continuity of the operation of K on K/M, choose open sets $\bar{U} \subseteqq K$ and $\bar{V} \subseteqq K/M$ with $e \in \bar{U}$, $eM \in \bar{V}$, and $\bar{U}\bar{V} \subseteqq \gamma(\theta N) \subseteqq K/M$. Let $V = \gamma^{-1}(\bar{V}) \subseteqq \theta N$; the function γ^{-1} is defined on all of $\bar{V}$ since $e\bar{V} \subseteqq \gamma(\theta N)$. For g in G and $\bar{n}$ in θN, we put

$$g\cdot\bar{n} = \gamma^{-1}(g\cdot\gamma(\bar{n}))$$

whenever the right side is defined. (Recall G acts on K/M when we identify K/M and G/MAN; the dot on the right refers to this action.) If k is in the subset $\bar{U}$ of K, and if $\bar{n}$ is in the subset V of θN, then $k\cdot\bar{n}$ is defined.

We shall study the real analytic map $\bar{U} \times V \to \theta N$ defined by

$$k \times \bar{n} \longrightarrow \bar{n}(k)^{-1}(k\cdot\bar{n}) . \tag{11}$$

This map is translation by K (but viewed in θN rather than K/M) followed by the inverse of the corresponding translation by θN, and hence the extent to which it does not send $\bar{U} \times V$ into $\{e\}$ measures the extent to which θN-hulls and K-hulls are different.

LEMMA 4.4. *There exist a neighborhood W_1 of e in θN, a compact neighborhood W_2 of e in K, and a positive real number C_3 such that, if $B_t \subseteqq W_1$ and $k \in W_2$, then $\bar{n}(k)^{-1}(k\cdot B_t) \subseteqq C_3 B_t$.*

PROOF. We require an expression for the form of the map (11) in local coordinates. The dimension of K is ν; let the dimension of θN be d. Choose real analytic coordinates on θN by picking an orthonormal basis of $\theta\mathfrak{n}$ consistently with the decomposition $\theta\mathfrak{n} = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$, and exponentiating to θN the coordinates $x_1, \cdots, x_d$ obtained from these vectors. Restrict the coordinates to the open set $V \subseteqq \theta N$. Also choose a real analytic chart on an open set $U_1 \subseteqq \bar{U}$ about e in k, and let $k_1, \cdots, k_\nu$ be local coordinates. The two systems of coordinates taken together give coordinates for the domain of the map (11), and the first system gives coordinates for the range.

Each coordinate of $\bar{n}(k)^{-1}(k\cdot\bar{n})$ has an expansion in a convergent power series in the x_i's and k_j's. Define W_1 and W_2 by the conditions that $W_1 \subseteqq V \cap \exp B_0$, $W_2 \subseteqq U_1$, W_1 has compact closure, and all d of these power series converge in an open neighborhood of the closure of $W_1 \times W_2$. On such a neighborhood we can rearrange the terms of these power series freely to write the l^{th} coordinate of $\bar{n}(k)^{-1}(k\cdot\bar{n})$ as

$$a_l(k) + \sum_{i=1}^{d} a_{li}(k)x_i + \sum_{i,j} a_{lij}(\bar{n}, k)x_i x_j , \qquad l = 1, \cdots, d ,$$

where $a_l(k)$, $a_{li}(k)$, and $a_{lij}(\bar{n}, k)$ are real analytic functions whose power series expansions converge when $\bar{n}$ is in W_1 and k is in W_2.

The term $a_l(k)$ is the l^{th} coordinate of $\bar{n}(k)^{-1}(k\cdot\bar{n})$ evaluated when $\bar{n} = e$; that is, $a_l(k) = 0$. We skip temporarily the first-order terms and consider the error terms. Let $\bar{n} = \exp X$ and let the functions $a_{lij}(\bar{n}, k)$ have a common bound C_4 on the compact closure of $W_1 \times W_2$. Then

$$\left| \sum_{i,j} a_{lij}(\bar{n}, k)x_i x_j \right| \leqq d^2 C_4 \| X \|^2$$

on the closure of $W_1 \times W_2$. That is, on this set $\bar{n}(k)^{-1}(k\cdot\bar{n})$ is the exponential of the sum of a vector with components $\sum a_{li}(k)x_i$ and a vector of norm $\leqq d^3 C_4 \| X \|^2$.

Now we examine the first-order terms. For fixed k in W_2, the matrix

$\{a_{li}(k)\}$ is clearly the jacobian matrix of the transformation

$$\bar{n} \longrightarrow \bar{n}(k)^{-1}(k\cdot\bar{n}) \ . \tag{12}$$

Since $k \in \bar{U}$ and $(U)(eM) \subseteqq \gamma(\theta N)$, the remarks at the beginning of this section show that we can write $\bar{n}(k) = kman$ with man in MAN. We shall show that the map (12) is the same as the map

$$\bar{n} \longrightarrow (man)^{-1}\cdot\bar{n} \ , \tag{13}$$

and we shall then compute its differential. To begin with, γ^{-1} is defined on $k\bar{n}MAN$, by construction of $\bar{U}$ and V, and the definition of γ^{-1} gives

$$k\bar{n}MAN = \gamma^{-1}(k\bar{n}MAN)MAN$$

or

$$\begin{aligned}(man)^{-1}\bar{n}MAN &= (man)^{-1}k^{-1}\gamma^{-1}(k\bar{n}MAN)MAN \\ &= \bar{n}(k)^{-1}\gamma^{-1}(k\bar{n}MAN)MAN \\ &= \gamma\big(\bar{n}(k)^{-1}\gamma^{-1}(k\bar{n}MAN)\big) \\ &= \gamma\big(\bar{n}(k)^{-1}\gamma^{-1}(k\cdot\gamma(\bar{n}))\big) \\ &= \gamma\big(\bar{n}(k)^{-1}(k\cdot\bar{n})\big) \ .\end{aligned}$$

Since γ^{-1} is defined on the right side, it is defined on the left side, $(man)^{-1}\cdot\bar{n}$ is defined, and

$$(man)^{-1}\cdot\bar{n} = \bar{n}(k)^{-1}(k\cdot\bar{n}) \ .$$

Next, we show that the differential of the map (13) is

$$X \longrightarrow P_{\theta\mathfrak{n}}\operatorname{Ad}(man)^{-1}X \ , \qquad X \in \theta\mathfrak{n} \ ,$$

where $P_{\theta\mathfrak{n}}$ is the projection of $\mathfrak{g}$ on $\theta\mathfrak{n}$ along $\mathfrak{m} + \mathfrak{a} + \mathfrak{n}$. The map (13) is the composition of conjugation by $(man)^{-1}$, the quotient map $G \longrightarrow G/MAN$, and the map γ^{-1}. The differential of conjugation by $(man)^{-1}$ is $\operatorname{Ad}(man)^{-1}$, and the differential of the composition of the other two maps is $P_{\theta\mathfrak{n}}$ because the composition is the identity on θN and sends MAN into e.

Let $(X_\alpha, X_{2\alpha})$ be in B_t with $B_t \subseteqq W_1$, and let $k \in W_2$. Write $\bar{n}(k) = kman$. If $\operatorname{Ad}(m)^{-1}X_\alpha = Y_\alpha$ and $\operatorname{Ad}(m)^{-1}X_{2\alpha} = Y_{2\alpha}$, then $\|X_\alpha\| = \|Y_\alpha\|$ and $\|X_{2\alpha}\| = \|Y_{2\alpha}\|$. Let $\operatorname{Ad}(a)^{-1}Y_\alpha = c_1 Y_\alpha$ and $\operatorname{Ad}(a)^{-1}Y_{2\alpha} = c_2 Y_{2\alpha}$. Then

$$P_{\theta\mathfrak{n}}\operatorname{Ad}(man)^{-1}(X_\alpha, X_{2\alpha}) = \big(c_1 Y_\alpha - c_2\operatorname{ad}(\log n)Y_{2\alpha},\, c_2 Y_{2\alpha}\big) = (Z_\alpha, Z_{2\alpha}) \ .$$

If $\|\operatorname{ad}(\log n)\| = c_3$, then the inclusion $W_1 \subseteqq \exp B_0$ gives

$$\begin{aligned}\|Z_\alpha\| &\leqq c_1\|X_\alpha\| + c_2c_3\|X_{2\alpha}\| \leqq (c_1 + c_2c_3)e^{-\alpha(H_0)t} \ , \\ \|Z_{2\alpha}\| &\leqq c_2\|X_{2\alpha}\| \leqq c_2 e^{-2\alpha(H_0)t} \ .\end{aligned}$$

The error term in the expression whose exponential is $\bar{n}(k)^{-1}(k\cdot\bar{n})$ satisfies

$$\|\text{Error}\| \leqq d^3C_4\|X\|^2 \leqq d^3C_4(\|X_\alpha\| + \|X_{2\alpha}\|)^2 \leqq 4d^3C_4e^{-2\alpha(H_0)t} \ .$$

Consequently the projections of the error term on $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-2\alpha}$ satisfy inequalities with the same right side. Thus the $\mathfrak{g}_{-\alpha}$ component of $\log(\bar{n}(k)^{-1}(k\cdot\bar{n}))$ has norm

$$\leqq (c_1 + c_2c_3 + 4d^3C_4)e^{-\alpha(H_0)t} \tag{14a}$$

and the $\mathfrak{g}_{-2\alpha}$ component has norm

$$\leqq (c_2 + 4d^3C_4)e^{-2\alpha(H_0)t} . \tag{14b}$$

As k varies through the compact set W_2, the numbers c_1, c_2, and c_3 remain bounded. If we take C_3 to be the larger of the bounds for the coefficients of the exponentials in (14a) and (14b), then the lemma follows.

PROOF OF THEOREM 4.2. First we prove there is a C_5 such that, for sufficiently small B_t (large t),

$$K\text{-hull}(\bar{B}_t) \subseteqq \gamma(C_5B_t) \tag{15}$$

Sufficiently small here means that $B_t \subseteqq W_1$ in the notation of Lemma 4.4. Thus suppose $B_t \subseteqq W_1$, and suppose for the moment that k is a member of W_2 with $k\cdot B_t \cap B_t \neq \varnothing$. We can rewrite the conclusion of Lemma 4.4 as

$$\bar{n}(k)^{-1}(k\cdot B_t) \subseteqq B_{t-c}$$

for a positive number c independent of t. We then have

$$\bar{n}(k)B_{t-c} \cap B_{t-c} \supseteqq \bar{n}(k)[\bar{n}(k)^{-1}(k\cdot B_t)] \cap B_t = k\cdot B_t \cap B_t \neq \varnothing .$$

Therefore

$$k\cdot B_t = \bar{n}(k)[\bar{n}(k)^{-1}(k\cdot B_t)] \subseteqq \bar{n}(k)B_{t-c} \subseteqq \left(3 + \frac{3}{2}C_2\right)B_{t-c} ,$$

the last inclusion holding by Lemma 4.3. Write $B_{t-c} \subseteqq C_6B_t$ with C_6 independent of t, and put $C_5 = (3 + 3C_2/2)C_6$. Then $k\cdot B_t \subseteqq C_5B_t$ or

$$k\bar{B}_t \subseteqq \gamma(C_5B_t) . \tag{16}$$

This is the asserted result except that we have restricted k to lie in W_2.

Let σ be the quotient mapping of K onto K/M. Since $k\bar{B}_t = \bar{B}_{t,k}$, equation (10) shows that (16) holds if $k \in \sigma^{-1}(\sigma W_2)$, as long as $B_t \subseteqq W_1$. Now suppose there are sequences $t_n \to \infty$ and $k_n \in K$ such that (16) is false for k_n and B_{t_n}, though $B_{t_n} \subseteqq W_1$ and $\bar{B}_{t_n,k_n} \cap \bar{B}_{t_n} \neq \varnothing$. We have shown $\sigma(k_n)$ is not in the neighborhood $\sigma(W_2)$ of eM. Without loss of generality we may suppose k_n converges, say to k_0, since K is compact. We know $\sigma(k_0) \neq eM$. Let $p_n \in \bar{B}_{t_n,k_n} \cap \bar{B}_{t_n}$. Since $p_n \in \bar{B}_{t_n}$ and $t_n \to \infty$, $p_n \to eM$. But $p_n = k_nq_n$ with q_n in $\bar{B}_{t_n}$. Since $t_n \to \infty$, $q_n \to eM$. Passing to the limit, we obtain $eM = k_0eM$ or $\sigma(k_0) = eM$, a contradiction.

Hence (16) holds when $B_t \subseteqq W_1$, say when $t \geqq t_0$. For such t, (16) gives

$$\frac{m_{\theta N}(\gamma^{-1}(K\text{-hull } \bar{B}_t))}{m_{\theta N}(B_t)} \leqq \frac{m_{\theta N}(C_5 B_t)}{m_{\theta N}(B_t)} = (C_5)^{\dim \theta N}$$

with the right side independent of t as long as $t \geqq t_0$. On the bounded subset $C_5 B_{t_0}$ of θN, the measures $m_{\theta N}$ and $\gamma^{-1}(m_{K/M})$ are bounded by multiples of one another since γ is a diffeomorphism onto its image. Hence there is a constant C_7 such that $t \geqq t_0$ implies

$$\frac{m_{K/M}(K\text{-hull } \bar{B}_t)}{m_{K/M}(\bar{B}_t)} \leqq C_7 .$$

Since K/M is compact, $m_{K/M}(K\text{-hull } \bar{B}_t)/m_{K/M}(\bar{B}_t)$ is trivially bounded if $t \leqq t_0$. The proof of Theorem 4.2 is therefore complete.

PROOF OF THEOREM 4.1. Theorem 4.2 and the argument in the last paragraph of its proof imply the formally stronger result

$$\sup_{-\infty < t < \infty} \frac{m_{K/M}(K\text{-hull of } \bar{B}_t)}{m_{K/M}(\bar{B}_{t+1})} < \infty . \tag{17}$$

Let C_1 be the left side of (17). Then $1 < C_1 < \infty$.

Let T_1 be the infimum of all t's such that $\bar{B}_{t,k}$ is one of the associated sets for some k. If $T_1 = -\infty$, then we can find a $\bar{B}_{t,k}$ with measure as close to 1 as we like, and the conclusion of the theorem follows. We assume from now on that $T_1 > -\infty$. Let R_1 be the class of all the sets $\bar{B}_{t,k}$ in question. Pick one, say $\bar{B}_{t_1,k_1}$, with $t_1 \leqq T_1 + 1$, and let R_2 be the set of members of R_1 which are disjoint from $\bar{B}_{t_1,k_1}$. If R_2 is empty, we let all further T_j's be $+\infty$ and all further $\bar{B}_{t_j,k_j}$'s be empty. If not, let T_2 be the infimum of the t's for which some $\bar{B}_{t,k}$ is in R_2, and choose a set $\bar{B}_{t_2,k_2}$ in R_2 with $t_2 \leqq T_2 + 1$. Let R_3 be the set of members of R_2 disjoint from $\bar{B}_{t_2,k_2}$ and proceed inductively to construct R_n, T_n, and $\bar{B}_{t_n,k_n}$.

The numbers T_j tend to ∞ since otherwise K/M would have an infinite disjoint sequence of sets with measures bounded below. If V_n is the (open) union of the members of $R_n - R_{n+1}$ and V_0 is the (open) union of the members of R_1, then

$$V_0 = \bigcup_{n=1}^{\infty} V_n . \tag{18}$$

In fact, a $\bar{B}_{t,k}$ which occurs in R_1 cannot occur in every R_j since the numbers T_j tend to ∞, and hence $\bar{B}_{t,k}$ must be in some R_n and not in R_{n+1}. Equation (18) follows.

Since $E \subseteqq V_0$, equation (18) gives

$$m_{K/M}(E) \leqq \sum_{j=1}^{\infty} m_{K/M}(V_j) .$$

The proof will be complete if we show that $m_{K/M}(V_n) \leqq C_1 m_{K/M}(\bar{B}_{t_n,k_n})$. Thus let $\bar{B}_{t,k} \in R_n - R_{n+1}$. Then $t \geqq T_n$ and $\bar{B}_{t,k} \cap \bar{B}_{t_n,k_n} \neq \varnothing$. Consequently $\bar{B}_{T_n,k} \cap \bar{B}_{T_n,k_n} \neq \varnothing$, or $k_n^{-1}k\bar{B}_{T_n} \cap \bar{B}_{T_n} \neq \varnothing$, or $k_n^{-1}k\bar{B}_{T_n} \subseteqq K\text{-hull } \bar{B}_{T_n}$, or

$$\bar{B}_{t,k} \subseteqq k\bar{B}_{T_n} \subseteqq k_n(K\text{-hull } \bar{B}_{T_n}) .$$

That is, $V_n \subseteqq k_n(K\text{-hull } \bar{B}_{T_n})$. By the definition of C_1 and by the inequality $t_n \leqq T_n + 1$,

$$\begin{aligned} m_{K/M}(V_n) &\leqq m_{K/M}(k_n(K\text{-hull } \bar{B}_{T_n})) = m_{K/M}(K\text{-hull } \bar{B}_{T_n}) \\ &\leqq C_1 m_{K/M}(\bar{B}_{T_n+1}) \leqq C_1 m_{K/M}(\bar{B}_{t_n}) = C_1 m_{K/M}(\bar{B}_{t_n,k_n}) . \end{aligned}$$

5. Maximal theorems

With the notation of § 4 define, for each integrable f on K/M,

$$f^*(k_0M) = \sup_{-\infty<t<\infty} \frac{1}{m_{K/M}(\bar{B}_t)} \int_{\bar{B}_{t,k_0}} |f(kM)|\, dk .$$

The function f^* is measurable because the supremum over rational t gives the same answer.

THEOREM 5.1. *For any integrable f on K/M, and for any $\xi > 0$,*

$$m_{K/M}\{kM \mid f^*(kM) > \xi\} \leqq \frac{C_1}{\xi} \int_{K/M} |f(kM)|\, dk .$$

Remark. C_1 here is the same constant as in Theorem 4.1.

PROOF. Let $E = \{kM \mid f^*(kM) > \xi\}$. By definition of f^*, there is, associated to each k_0M in E, a set $\bar{B}_{t,k_0}$ such that

$$\int_{\bar{B}_{t,k_0}} |f(kM)|\, dk \geqq \xi m_{K/M}(\bar{B}_t) = \xi m_{K/M}(\bar{B}_{t,k_0}) .$$

Apply Theorem 4.1, and let the disjoint sets obtained from the theorem be $\bar{B}_{t_j,k_j}$. Then

$$\int_{K/M} |f(kM)|\, dk \geqq \sum_j \int_{\bar{B}_{t_j,k_j}} |f(kM)|\, dk \geqq \xi \sum_j m_{K/M}(\bar{B}_{t_j,k_j}) \geqq \xi C_1^{-1} m_{K/M}(E) ,$$

and the theorem is proved.

Let H_0 be the vector in $\mathfrak{a}^+$ appearing in the statement of Theorem 1.1 and in the definition of B_t, and put $a_t = \exp tH_0$. Let h be the Poisson integral of an integrable f on K/M and define

$$f_*(kM) = \sup_{0\leqq t<\infty} |h(ka_tK)| .$$

Recall the function $F(kM)$ of Theorem 3.1.

LEMMA 5.2. *If $0 \leqq s \leqq t$,*

$$P(a_tK, a_s\cdot kM) \leqq e^{-2\rho(tH_0)} e^{4\rho(sH_0)} F(kM)^{-1} .$$

PROOF. If $a \in \mathrm{cl}(A^+)$, then

$$e^{2\rho H(ak)} \leqq e^{2\rho(\log a)} . \tag{19}$$

This inequality appears in Harish-Chandra [5, p. 281]. It can also be obtained from inequality (7) and the fact that $H(ak) = -H(a^{-1}k_1)$ for some k_1 in K, or it can be derived directly by the method of Theorem 3.1.

To prove the lemma, we write

$$\begin{aligned} P(a_tK, a_s \cdot k_0M) &= \exp\{-2\rho H(a_t^{-1}k(a_sk_0))\} \\ &= \exp\{-2\rho H(a_t^{-1}a_sk_0a(a_sk_0)^{-1}n)\} , \end{aligned}$$

where $a(\cdot)$ means A-component and n is in N. The right side

$$\begin{aligned} &= \exp\{-2\rho H(a_t^{-1}a_sk_0)\}e^{2\rho H(a_sk_0)} \\ &\leqq e^{-2\rho(t-s)H_0}e^{2\rho H(a_sk_0)}F(k_0M)^{-1} && \text{by (8) of Theorem 3.1} \\ &\leqq e^{-2\rho(t-s)H_0}e^{2\rho(sH_0)}F(k_0M)^{-1} && \text{by (19)} . \end{aligned}$$

THEOREM 5.3. *There is a constant C_8 such that $f_*(kM) \leqq C_8f^*(kM)$ for all integrable f on K/M and for all kM in K/M. Consequently*

$$m_{K/M}\{kM \mid f_*(kM) > \xi\} \leqq \frac{C_1C_8}{\xi}\int_{K/M} |f(kM)|\, dk .$$

PROOF. Let $t \geqq 0$ and write

$$\begin{aligned} |h(k_0a_tM)| &\leqq \int_{K/M} P(a_tK, kM)\, |f(k_0kM)|\, dk \\ &= \int_{\bar B_t} + \int_{\bar B_0 - \bar B_t} + \int_{K/M - \bar B_0} = \mathrm{I} + \mathrm{II} + \mathrm{III} . \end{aligned}$$

We have $||f|| \leqq m_{K/M}(\bar B_{-\infty})f^*(k_0M) = f^*(k_0M)$ and hence

$$\begin{aligned} \mathrm{III} &\leqq e^{-2\rho(tH_0)} \sup_{kM \in K/M - \bar B_0} F(kM)^{-1}\, ||f||_1 && \text{by (8)} \\ &\leqq [\sup_{kM \in K/M - \bar B_0} F(kM)^{-1}]f^*(k_0M) . && \text{(recall } t \geqq 0) \end{aligned} \tag{20}$$

To estimate I and II, we need a lower bound for $f^*(k_0M)$. On the bounded set B_0, the pull-back of $m_{K/M}$ under γ^{-1} is dominated by a constant C_9 times $m_{\theta\mathfrak{n}}$. Thus for $t \geqq 0$,

$$m_{K/M}(\bar B_t) \leqq C_9 m_{\theta\mathfrak{n}}(B_t) = C_{10}e^{-2\rho(tH_0)} ,$$

where $C_{10}C_9^{-1} = m_{\theta\mathfrak{n}}(B_0)$. After a change of variable,

$$\begin{aligned} f^*(k_0M) &= \sup_{-\infty<t<\infty} \frac{1}{m_{K/M}(\bar B_t)} \int_{\bar B_t} |f(k_0kM)|\, dk \geqq \sup_{0\leqq t<\infty}(-) \\ &\geqq \sup_{0\leqq t<\infty} C_{10}^{-1}e^{2\rho(tH_0)} \int_{\bar B_t} |f(k_0kM)|\, dk \end{aligned}$$

or

$$e^{2\rho(tH_0)}\int_{\bar{B}_t}|f(k_0kM)|\,dk \leqq C_{10}f^*(k_0M) \qquad \text{for all } t \geqq 0\,. \tag{21}$$

Term I is estimated with (7) and (21). We have

$$\mathrm{I} \leqq e^{2\rho(tH_0)}\int_{B_t}|f(k_0kM)|\,dk \leqq C_{10}f^*(k_0M)\,. \tag{22}$$

For term II, we coordinatize $\theta N - \{e\}$ by the reals and $\exp S_0$ under the identification

$$(\omega, s) \in \exp S_0 \times \text{Reals} \longrightarrow a_s\omega a_s^{-1}\,.$$

Let $\Phi(t) = \int_{\bar{B}_t}|f(k_0kM)|\,dk$. Then

$$\begin{aligned}\Phi(t) &= \int_{\exp B_t}|f(k_0k(\bar{n})M)|\,\psi(\bar{n})d\bar{n}\,, && \psi \text{ a certain jacobian}\\ &= \int_{\exp B_t}|f(k_0k(a_s\omega a_s^{-1})M)|\,g(\omega, s)d\omega ds\,, && g \text{ a certain jacobian}\\ &= \int_t^\infty\left[\int_{\exp S_0}|f(k_0k(a_s\omega a_s^{-1})M)|\,g(\omega, s)d\omega\right]ds\,.\end{aligned}$$

If we call the expression in brackets $\Psi(s)$, then

$$\begin{aligned}\mathrm{II} &= \int_{\bar{B}_0-\bar{B}_t}P(a_tK, kM)\,|f(k_0kM)|\,dk\\ &= \int_0^t\int_{\exp S_0}P(a_tK, k(a_s\omega a_s^{-1})M)\,|f(k_0k(a_s\omega a_s^{-1})M)|\,g(\omega, s)d\omega ds\\ &\qquad\qquad \text{with } g \text{ as above}\\ &\leqq \int_0^t\int_{\exp S_0}e^{-2\rho(tH_0)}e^{4\rho(sH_0)}[\sup_{kM\in\bar{S}_0}F(kM)^{-1}]\,|f(-)|\,g(\omega, s)d\omega ds\end{aligned}$$

by (9) and Lemma 5.2. If we call the expression in brackets C_{11}, then

$$\begin{aligned}\mathrm{II} &\leqq C_{11}e^{-2\rho(tH_0)}\int_0^t e^{4\rho(sH_0)}\Psi(s)ds\\ &= C_{11}e^{-2\rho(tH_0)}\left\{[-\Phi(s)e^{4\rho(sH_0)}]_0^t + 4\rho(H_0)\int_0^t e^{4\rho(sH_0)}\Phi(s)ds\right\}\\ &\leqq C_{11}e^{-2\rho(tH_0)}\left\{\Phi(0) + 4\rho(H_0)\int_0^t e^{2\rho(sH_0)}C_{10}f^*(k_0M)ds\right\} && \text{by (21)}\\ &\leqq C_{11}\Phi(0) + 2C_{11}C_{10}f^*(k_0M)\\ &\leqq 3C_{11}C_{10}f^*(k_0M) && \text{by (21)}\,. \qquad (23)\end{aligned}$$

Putting together (20), (22), and (23) and taking the supremum over $t \geqq 0$, we obtain the conclusion of the theorem.

Theorem 1.1 is an easy consequence of Theorem 5.3 and Corollary 2a of Theorem 3.1. Analogous arguments appear in abbreviated form in several places in [13], but we repeat the argument here for completeness.

PROOF OF THEOREM 1.1. Let $\varepsilon > 0$ be given and write $f = f_1 + f_2$, where f_1 is continuous and $\|f_2\|_1 < \varepsilon^2$. Let h_1 and h_2 be the Poisson integrals of f_1 and f_2. Choose, by Corollary 2a to Theorem 3.1, $T \geqq 0$ large enough so that $t \geqq T$ implies

$$|h_1(ka_tK) - f_1(kM)| < \varepsilon \qquad \text{for all } kM .$$

If E_1 is the set $\{|f - f_1| \geqq \varepsilon\}$, then $\varepsilon m_{K/M}(E_1) \leqq \|f_2\|_1 < \varepsilon^2$ or $m_{K/M}(E_1) < \varepsilon$. So, except in the set E_1 of measure $<\varepsilon$,

$$|h_1(ka_tK) - f(kM)| < 2\varepsilon \qquad \text{whenever } t \geqq T .$$

On the other hand, Theorem 3.1 shows that $|h_2(ka_tK)| \leqq \varepsilon$ for all $t \geqq 0$ except in a set $E_2 \subseteqq K/M$ of measure $\leqq C_1C_8\varepsilon^{-1}\varepsilon^2 = C_1C_8\varepsilon$. Hence, except in the set $E_1 \cup E_2$ of measure $\leqq (C_1C_8 + 1)\varepsilon$,

$$|h(ka_tK) - f(kM)| < 3\varepsilon \qquad \text{whenever } t \geqq T .$$

Replacing ε by $2^{-n}\varepsilon$ for each ε and taking the union of the $E_1^{(n)} \cup E_2^{(n)}$'s, we obtain a set E of measure $\leqq 2(C_1C_8 + 1)\varepsilon$, outside of which $\lim h(ka_tK) = f(kM)$. Since ε is arbitrary, the set where the convergence does not take place has measure 0.

6. Measures as boundary values

Fatou's theorem in the disc can be formulated more generally than for L^1 functions as boundary values. Namely there is a theorem with signed measures as boundary values. The obvious extension of the latter theorem to all symmetric spaces of rank one is true and generalizes Theorem 1.1. The statement is given below, and the proof is standard. (See [13, p. 313].)

THEOREM 6.1. *Let G/K have rank one, and let H_0 be a member of $\mathfrak{a}^+$. If μ is a signed measure on K/M, if h is its Poisson integral*

$$h(gK) = \int_{K/M} P(gK, kM)d\mu(kM) ,$$

and if $\mu = fdk + \mu_s$ is the Lebesgue decomposition of μ, then

$$\lim_{t\to+\infty} h\bigl(k_0 \exp(tH_0)K\bigr) = f(k_0M)$$

almost everywhere with respect to dk.

PROOF. Introducing maximal *functions*

$$\mu^*(k_0M) = \sup_{-\infty<t<\infty} \frac{1}{m_{K/M}(\bar{B}_t)} \int_{\bar{B}_{t,k_0}} d\,|\mu|\,(kM)$$

and

$$\mu_*(k_0M) = \sup_{0\leqq t<\infty} |h(k_0a_tK)| , \qquad a_t = \exp tH_0 ,$$

and checking that the arguments in Theorems 5.1 and 5.3 remain valid with f replaced by μ, we obtain

$$m_{K/M}\{kM \mid \mu_*(kM) > \xi\} \leqq \frac{C_1C_8}{\xi} \,||\, \mu \,|| \tag{24}$$

with C_1 and C_8 as in § 5. For proving Theorem 6.1, we may assume, by Theorem 1.1, that μ is singular with respect to dk, and clearly we may suppose $\mu \geqq 0$. We want $\lim h(k_0a_tK) = 0$ almost everywhere.

Let us notice also that, if $\mu(U) = 0$ for an open set U, and if $L \subseteqq U$ is compact, then $h(k_0a_tK) \to 0$ uniformly for $k_0M \in L$. In fact

$$h(k_0a_tK) = \int_{K/M} P(a_tK, k_0^{-1}kM)d\mu(kM) = \int_{kM \notin U} P(a_tK, k_0^{-1}kM)d\mu(kM)\,.$$

The k_0's such that $k_0M \in L$ form a compact set, and $K/M - U$ is compact. Hence the set of values of $k_0^{-1}kM$ is compact when $k_0M \in L$ and $kM \notin U$. Since $k_0^{-1}kM$ cannot equal eM, the expression $P(a_tK, k_0^{-1}kM)$, for these values of k_0 and k, tends to 0 uniformly as $t \to \infty$, by Corollary 1(c) of Theorem 3.1. The asserted uniform convergence follows.

Now let $\varepsilon > 0$ be given. Find, since μ is singular, an open $U \subseteqq K/M$ with $m_{K/M}(K/M - U) < \varepsilon^2$ and $\mu(U) < \varepsilon^2$. Put $\mu = \mu_1 + \mu_2$, where $\mu_1(E) = \mu(E - U)$ and $\mu_2(E) = \mu(E \cap U)$. Let h_1 and h_2 be the Poisson integrals of μ_1 and μ_2. Then μ_1 vanishes on U. By the preceding, we can find $T \geqq 0$ so that $t \geqq T$ implies $h_1(ka_tK) < \varepsilon$ for all kM in U except for those in a set E_1 of measure $<\varepsilon$, hence for all kM in K/M except for those in the set $E_1 \cup (K/M - U)$ of measure $< \varepsilon + \varepsilon^2 < 2\varepsilon$. But (24) shows that $h_2(ka_tK) < \varepsilon$ for all $t \geqq 0$ except in a set in K/M of measure $\leqq C_1C_8\varepsilon^{-1} \,||\, \mu_2 \,|| \leqq C_1C_8\varepsilon$. Thus $h(ka_tK) < 3\varepsilon$ for all $t \geqq T$ except for kM's in a set of measure $< (C_1C_8 + 2)\varepsilon$. This conclusion implies that $\lim h(ka_tK) = 0$ almost everywhere.

Remark added in proof. Recently S. Helgason and A. Korányi [Bull. Amer. Math. Soc. 74 (1968), 258–263] proved that any bounded harmonic function on a symmetric space of arbitrary rank has limits on almost every geodesic from a point. The common ground of their theorem and Theorem 1.1 is Theorem 1.1 for L^∞ boundary values.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY AND CORNELL UNIVERSITY

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