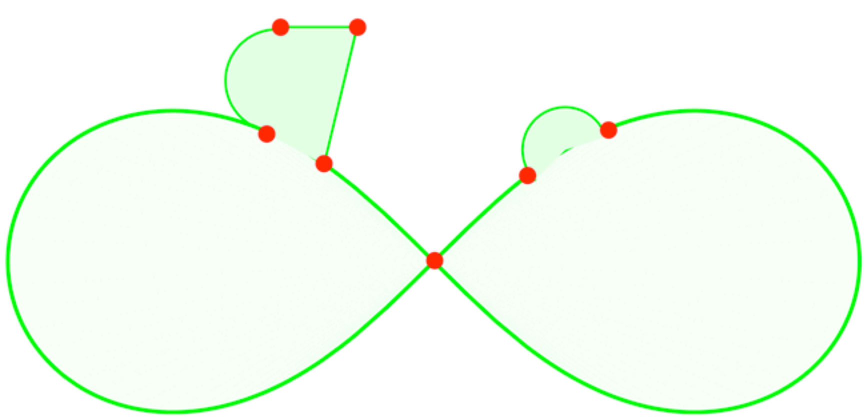
STOKES'S THEOREM and WHITNEY MANIFOLDS

Anthony W. Knapp



Anthony W. Knapp

Stokes's Theorem and Whitney Manifolds

A Sequel to *Basic Real Analysis*

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Anthony W. Knapp
81 Upper Sheep Pasture Road
East Setauket, N.Y. 11733–1729, U.S.A.
Email to: aknapp@math.stonybrook.edu
Homepage: www.math.stonybrook.edu/∼aknapp

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Cover: An example of a Whitney domain in two-dimensional space. The green portion is a manifold-with-boundary for which Stokes's Theorem applies routinely. The red dots indicate exceptional points of the boundary where a Whitney condition applies that says Stokes's Theorem extends to the whole domain.

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To Susan

and

To My Children, Sarah and William,

and

To My Grandchildren, Michelle and Scott

CONTENTS

CONTENTS OF *BASIC REAL ANALYSIS*

CONTENTS OF *BASIC ALGEBRA*

PREFACE

This book is a sequel to the author's *Basic Real Analysis*, which systematically developed concepts and tools in real analysis that are vital to every mathematician, whether pure or applied, aspiring or established. The intention was that it and its companion volume, *Advanced Real Analysis*, together would contain what the young mathematician needs to know about real analysis in order to communicate well with colleagues in all branches of mathematics.

The first editions of these books consciously omitted a few topics, the most notable of which were advanced topics in the calculus of several real variables, particularly the integration theorems that relate an integral over a region or surface to an integral over the boundary. These integration theorems go under the general name "Stokes's Theorem" because of the history that will be explained in the Introduction, and they too are tools in real analysis that are vital to every mathematician.

This book aims to treat that topic. Actually the digital second edition of *Basic Real Analysis* dealt with low dimensional aspects of the topic somewhat by addressing arc length, line integrals, and Green's Theorem in the plane in Chapter III. The spirit of the treatment of these matters was the same as the treatment in that book of Riemann integration in one and several variables, careful and thorough, the expectation being that the reader had earlier seen this material presented in a utilitarian fashion. When it comes to surface integrals, however, the method used for addressing arc length breaks down, as was shown toward the end of Section III.13 of *Basic Real Analysis*. Unlike the length of a curve, the area of a surface cannot be defined as the supremeum of some obvious inscribed approximations, and a different approach to the whole subject is needed.

The different approach that we follow is to use material that lies at the beginning of the study of both differentiable topology and algebraic topology. The material in question is the topic of differential forms, including integration of differential forms. The spirit of the treatment is quite different from that of *Basic Real Analysis*, and Chapter I of the present book takes some time to develop differential forms and tools for working with them.

By way of prerequisites, this book relies in part on some real analysis that is treated in Chapters III, V, VI, and X of *Basic Real Analysis*. In addition, it makes use of elementary linear algebra and a certain amount of multilinear algebra that can be found in the author's *Basic Algebra*, Chapter VI, Sections 1–7.

The key theorems that are needed from real analysis are the Inverse and Implicit Function Theorems and the change-of-variables formula for the Lebesgue integral in Euclidean space. The Riemann integral could be used in place of the Lebesgue integral in most circumstances, but at a cost of making certain statements more cumbersome. The key thing that is needed from algebra is some familiarity with the tensor algebra of a real finite dimensional vector space.

A philosophical problem arises in finding the right setting for the integration theorems that are collectively known as Stokes's Theorem and that relate an integral over a region or surface to a integral over the boundary. The integration theorems are most transparent when the sets of integration are rectangular, and we indicate the simple idea in the Introduction. On the other hand, the proofs are most natural when the sets and functions are smooth, as they are for a circle or a ball. Rectangular sets are not smooth. The setting in which the sets and functions are smooth is that of "manifolds-with-boundary," which are defined in Chapter II of the present book. To handle both settings at the same time—rectangular sets and smooth manifolds-with-boundary—the traditional approach is to break the sets of integration into pieces by some kind of triangulation or other cutting of regions into parts. Then one establishes Stokes's Theorem for each piece and adds the results. Pedagogically this approach is unsatisfactory.

A more modern approach is to use "manifolds-with-corners," which are defined and used in the first half of Chapter III. Manifolds-with-corners handle a great many cases without any cutting of regions into pieces, but they are still insufficient to handle all cases of practical interest without additional effort. The second half of Chapter III treats Stokes's Theorem in a still broader context due to Hassler Whitney. Whitney worked with what he called "standard manifolds" but which are more aptly called "Whitney manifolds." Whitney manifolds do indeed handle all cases of practical interest.

Some years ago, aware of the tension between the two standard approaches to Stokes's Theorem via rectangular sets and smooth manifolds, I asked my colleague Blaine Lawson whether one could now finally give a satisfactory exposition of the theorem. At that time he introduced me to manifolds-with-corners and explained to me how one could often use them to avoid the traditional cutting of manifolds into concrete pieces. The resulting situation, although better, was still not satisfactory in my view.

More recently, to help cope with restrictions because of the COVID-19 pandemic, I decided to look at the matter again. Libraries were closed. But during my online reading I encountered Whitney's book *Geometric Integration Theory*, which proves a version of Stokes's Theorem that seems to handle all examples of practical interest without any need at all to cut manifolds into concrete pieces. In response to emailed questions about some passages in Whitney's book, my colleague Chris Bishop introduced me to various notions of dimension and explained to me the relationships among them, pointing to his book *Fractals in Probability and Analysis* written with Yuval Perez for some of the details. I am grateful to both colleagues for sharing information with me.

The problems at the ends of chapters are an important part of the book. Some of them are really theorems, some are significant examples, and a few are just exercises. The reader gets no indication which problems are of which type, nor of which ones are relatively easy. Each problem except perhaps the last one can be solved with tools developed up to that point in the book, plus any additional prerequisites that are noted. Detailed hints appear at the end of the book.

The typesetting was by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX, and the figures were drawn with help from Mathematica.

I invite corrections and other comments from readers. I plan to maintain a list of known corrections on my own Web page.

A. W. KNAPP
November 2020

INTRODUCTION

Stokes's Theorem is a generalization of the Fundamental Theorem of Calculus from one dimension to higher dimensions. In an easy formulation the Fundamental Theorem of Calculus says that

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

on the closed interval $[a, b]$ if F is a real-valued function with a continuous derivative F'. In thinking how to generalize this theorem while keeping the ideas clear, we shall not be looking for the best possible hypotheses and will be content with assuming in the statement merely that F is smooth (i.e., infinitely differentiable). At any rate the formula relates the integral of the derivative of F over an interval to a linear combination of the values of F at the endpoints.

We encountered two qualitatively similar results in Chapter III of *Basic Real Analysis*, as follows:

(1) One such result was the formula in Proposition 3.47 for the line integral of the gradient of a smooth function over a smooth curve γ in $\mathbb{R}^n$ with domain $[a, b]$, namely

$$\int_\gamma \nabla f \cdot ds = f(\gamma(b)) - f(\gamma(a)).$$

Again the formula relates an integral of a derivative of f over a curve to a linear combination of the values of f at the endpoints of the curve.

(2) The other such result was the formula in Section III.13 concerning Green's Theorem in the plane, namely

$$\iint_U \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dx\,dy = \int_\gamma P\,dx + Q\,dy.$$

Here U is a region in $\mathbb{R}^2$, the curve γ traces out its boundary with the region on the left, and U and γ are assumed to be suitably nice. This formula involves a two-component real-valued function with entries P and Q, and it relates an integral over the region involving first derivatives of P and Q to an integral over the boundary of the values of P and Q.

The first of these results is simply a matter of applying the Fundamental Theorem of Calculus component by component, and it is not mysterious.

Let us consider Green's Theorem in more detail. The idea behind the theorem is clearest for the special case that U is a rectangle with sides parallel to the axes, a case that was considered in Example 1 of Section III.13 of *Basic Real Analysis*. In that case Theorem 3.49 is proved by considering P and Q separately. To handle P, one applies the Fundamental Theorem of Calculus in the x variable and integrates the result in the y variable; to handle Q, one applies the Fundamental Theorem of Calculus in the y variable and integrates the result in the x variable.

Unfortunately the style of proof that works well for a rectangle already runs into technical problems if one tries to prove the theorem for a closed disk in $\mathbb{R}^2$. Example 2 in Section III.13 of *Basic Real Analysis* gives the details. There are two technical problems—(a) the need to impose new parametrizations on a curve and see that its line integrals are unchanged and (b) the need to use Lebesgue integration or some other device to cope with unbounded integrands. Example 3 in Section III.13 shows that for a washer (or annulus) in $\mathbb{R}^2$, further difficulties arise, and the argument uses a decomposition of the region into a number of parts. For a more complicated region, the corresponding decomposition may be expected to be more difficult to describe, and it is not at all apparent how to make a general argument.

Classical treatments of calculus in three variables, or particularly of what is sometimes given the more advanced-sounding name vector analysis, discuss two further theorems of this kind, known respectively as the Divergence Theorem (or the Gauss–Ostrogradsky Theorem) and the Kelvin–Stokes Theorem (or simply Stokes's Theorem).

The Divergence Theorem in $\mathbb{R}^3$ concerns a solid bounded region U in $\mathbb{R}^3$ with a 2 dimensional boundary ∂U. In classical notation it says that

$$\iiint_U \Big(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\Big)\,dx\,dy\,dz = \iint_{\partial U} P\,dy\wedge dz+Q\,dz\wedge dx+R\,dx\wedge dy.$$

Evaluation of a term on the right side involves parametrizing the surface in (x, y, z) space by parameters s and t, and then $dy \wedge dz$, $dz \wedge dx$, and $dx \wedge dy$ are given formally by substituting the product of a two-by-two determinant times $ds\,dt$, specifically

$$dy\wedge dz = \frac{\partial(y,z)}{\partial(s,t)}\,ds\,dt,\qquad dz\wedge dx = \frac{\partial(z,x)}{\partial(s,t)}\,ds\,dt,\qquad dx\wedge dy = \frac{\partial(x,y)}{\partial(s,t)},$$

and carrying out the double integrations. Some important questions concerning orientations and signs need to be sorted out, but we skip over those for the time being.

In the case that U is a rectangular solid with faces parallel to the axes, the formula can be verified one term at a time by using the Fundamental Theorem of Calculus in the differentiated variable and then integrating in the other two variables, carefully managing the signs that appear from the integrated terms. This computation is the expected generalization of the computation in Example 1 of Section III.13 of *Basic Real Analysis* on Green's Theorem. For more general solids U, one can attempt a similar argument after breaking the original integral into a number of pieces. Once again, it is not at all apparent how to describe such a decomposition of a complicated region, and thus it is not at all apparent how to give a general proof of the Divergence Theorem in this style.

The Kelvin–Stokes Theorem,[1] sometimes known simply as Stokes's Theorem, concerns an oriented 2 dimensional surface S having a 1 dimensional boundary given by a curve γ, the whole manifold plus boundary embedded in $\mathbb{R}^3$. The formula is

$$\iint_S \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\Big)\, dy \wedge dz + \Big(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\Big)\, dz \wedge dx + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\, dx \wedge dy$$
$$= \int_\gamma P\, dx + Q\, dy + R\, dz.$$

When a sketch of proof is given in an elementary text for this theorem in special cases, it often goes by reducing the theorem to Green's Theorem in the plane. When necessary, the surface is cut into pieces and canceling pieces of boundary curve are adjoined.

From an expository point of view, the whole matter is rather unsatisfactory. In 1934 the young French mathematicians André Weil and Henri Cartan had the joint responsibility in Strasbourg for teaching a course on "differential and integral calculus," and they consulted each other frequently. In his autobiography Weil writes of this interaction, saying,[2]

> One point that concerned him [Cartan] was the degree to which we should generalize Stokes' formula in our teaching. This formula is written as follows:
> $$\int_{b(X)} \omega = \int_X d\omega,$$

[1]The theorem was discovered by William Thomson (Lord Kelvin) and communicated to Stokes by letter in 1850.

[2]André Weil, *The Apprenticeship of a Mathematician*, Birkhäuser, Basel, 1992, pp. 99-100.

> where ω is a differential form, $d\omega$ is its derivative, X its domain of integration, and $b(X)$ the boundary of X. There is nothing difficult about this if for example X is the infinitely differentiable image of an oriented sphere and if ω is a form with infinitely differentiable coefficients. Particular cases of this formula appear in classical treatises, but we were not content to make do with these.

Weil goes on to describe how this interaction led a group of young French mathematicians over a period of years to explain systematically much of elementary mathematics in a series of published books going under the title *Eléments de Mathématique* and written with the pseudonym Nicolas Bourbaki.[3]

Ironically although Bourbaki's books eventually developed a wide swath of mathematics rigorously, especially in the 1950s and 1960s, they had not yet treated Stokes's Theorem as of 2018. Possibly the reason was that a suitable framework, conveniently handling all shapes of interest at once, was not developed until well after World War II. Let us elaborate somewhat on the history.

Building on his own work from much earlier and on some work of H. Poincaré and E. Goursat, Elie Cartan[4] had brought a degree of unity to the subject by showing that Green's Theorem, the Divergence Theorem, and the Kelvin–Stokes Theorem were really special cases of the same general theorem. In a course in 1936–1937, whose notes were published as a book in 1945, he showed how to view all three classical theorems as instances of a result about "differential forms" and "exterior differentiation," the unifying formula being the one in the quotation above from Weil's book. Moreover, the theory, which dealt with smooth "manifolds-with-boundary," was not limited to cases in $\mathbb{R}^3$, and the final proof took little more than a couple of pages. The cost of having such a tidy final result for smooth manifolds-with-boundary was that the hard work was transferred into the definitions and verifications necessary to set up the theory. The 1965 book by M. Spivak, *Calculus on Manifolds*, proves Stokes's Theorem just for smooth manifolds-with-boundary,[5] it does so in exactly this way, and it makes the point that the difficulty occurs in setting up the theory. We shall see this cost first hand in the present book in that all of Chapter I and part of Chapter II are devoted to setting up the theory.

In practical applications unfortunately, physicists and engineers need a version of Stokes's Theorem that holds for rectangular sets and other polyhedral sets, as

[3] Although the original six members of the Bourbaki group were all French, mathematicians of other nationalities joined the group later. Members were expected to retire from the group about at age 50.

[4] Father of Henri.

[5] In Spivak's book the manifolds-with-boundary are always embedded in some Euclidean space for the sake of concreteness, but working in such a setting merely adds one unnecessary parameter to the mix and obscures the simplicity of the final formula.

well as for smooth manifolds-with-boundary. This is the matter that concerned H. Cartan in the quotation above. Even as late as the 1950s, rectangular solids and polyhedral sets were best treated directly, essentially by cutting the set into pieces and making an explicit calculation for each piece, while round shapes were best treated as manifolds-with-boundary to which E. Cartan's theory could be applied directly.

In 1961 J. Cerf and A. Douady introduced smooth "manifolds-with-corners," which included solid balls and also rectangular solids. In other words, smooth manifolds-with-corners offered a step toward further unifying the treatment of Stokes's Theorem. The present book will give a proof of Stokes's Theorem for smooth manifolds-with-corners in Sections 1–3 of Chapter III. The argument is really no harder than the argument for smooth manifolds-with-boundary, and one can perhaps regard the setting of manifolds-with-corners as giving a sufficient answer to H. Cartan's question about pedagogy.

It may be a sufficient answer, but it is not completely satisfactory. The corners in the theory of smooth manifolds-with-corners turn out to be of really limited scope. In $\mathbb{R}^3$, for example, when three planes come together at a point, the result is a corner in the sense of the theory, but when four planes come together at a point, the resulting intersection point no longer fits the theory. Thus, for example, the theory applies to a tetrahedron in $\mathbb{R}^3$ but not to a square pyramid.

It turns out that there is a more all-encompassing theory, and it was already known by 1960. Hassler Whitney developed the theory and published it in a book in 1957. The present book concludes Chapter III with the relevant parts of Whitney's theory. Qualitatively Whitney's theory looks at a manifold and boundary and divides the boundary into two sets. One set consists of nice points like those in the E. Cartan theory of smooth manifolds-with-boundary. The other set consists of exceptional points. Whitney's theorem is that if the set of exceptional points is small in a certain precise sense, then everything is fine and the Stokes formula is valid. The theorem handles all smooth manifolds-with-corners. In fact, the theorem appears to handle all situations that might be of interest to physicists and engineers, as well as all those that are of interest to most mathematicians. The proof still takes only a few pages, with its complications concealed in the definitions. One cannot ask for more.

LIST OF FIGURES

STANDARD NOTATION

Item	**Meaning**
$\varnothing$	empty set
$\{x \in E \mid P\}$	the set of x in E such that P holds
E^c	complement of the set E
$E \cup F,\ E \cap F,\ E - F$	union, intersection, difference of sets
$\bigcup_\alpha E_\alpha,\ \bigcap_\alpha E_\alpha$	union, intersection of the sets E_α
$E \subseteq F,\ E \supseteq F$	E is contained in F, E contains F
$E \times F$	product of sets
$(a_1, \ldots, a_n),\ \{a_1, \ldots, a_n\}$	ordered n-tuple, unordered n-tuple
$f : E \to F,\ x \mapsto f(x)$	function, effect of function
$f \circ g,\ f\big\|_E$	composition of f following g, restriction to E
$f(\,\cdot\,, y)$	the function $x \mapsto f(x, y)$
$f(E),\ f^{-1}(E)$	direct and inverse image of a set
δ_{ij}	Kronecker delta: 1 if $i = j$, 0 if $i \neq j$
c positive, c negative	$c > 0$, $c < 0$
$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	integers, rationals, reals, complex numbers
max (and similarly min)	maximum of finite subset of a totally ordered set
$\sum$	sum, possibly with a limit operation
countable	finite or in one-one correspondence with $\mathbb{Z}$
1 or I	identity matrix or function or operator
$\dim V$	dimension of vector space
$\mathbb{R}^n$, $\mathbb{C}^n$	spaces of column vectors
$\det A$	determinant of A
A^{tr}	transpose of A
$\mathrm{Hom}_{\mathbb{R}}(U, V)$	space of linear functions from U to V
E^o	interior of set E
E^{cl}	closure of set E

ACKNOWLEDGMENTS

The author acknowledges the sources below as the main ones he used in preparing the notes from which this book evolved. The descriptions below have been abbreviated. Full descriptions of the items may be found in the section "Selected References" at the end of the book.

This list is not to be confused with a list of recommended present-day reading for these topics; in some cases newer books deserve attention.

CHAPTER I. Section 1 of Chapter VIII of the author's *Advanced Real Analysis* for Section 1 of the present book. Sections 1–7 of Chapter VI of the author's *Basic Algebra* for Section 2. Spivak's *Calculus on Manifolds*, Warner's *Foundations of Differentiable Manifolds*, and Bott–Tu's *Differential Forms in Algebraic Topology* for Sections 3 and 4. Spivak's *Calculus on Manifolds*, Warner's *Foundations of Differentiable Manifolds* and De Rham's *Variétés différentiables* for Section 5. Ghomi's *Lecture Notes on Differential Geometry* and Lee's *Introduction to Smooth Manifolds* for orientation in Section 6. Chevalley's *Theory of Lie Groups* and Helgason's *Differential Geometry and Symmetric Spaces* for integration in Section 6.

CHAPTER II. Spivak's *Calculus on Manifolds*, Warner's *Foundations of Differentiable Manifolds*, and Bott–Tu's *Differential Forms in Algebraic Topology* for Sections 1 and 2. Lee's *Introduction to Smooth Manifolds* for Section 3. Bott–Tu's *Differential Forms in Algebraic Topology* for Section 4. Spivak's *Calculus on Manifolds* for Section 5.

CHAPTER III. Joyce's paper "On manifolds with corners," Conrad's undated course notes concerning manifolds with corners, and Chapter 1 of Melrose's unfinished book *Differential Analysis on Manifolds* for Sections 1–3. Whitney's *Geometric Integration Theory* and Bishop–Perez's *Fractals in Probability and Analysis* for Sections 4–6.

CHAPTER I

Smooth Manifolds

Abstract. This chapter introduces just enough differential topology to serve as a suitable framework for Stokes's Theorem. The subject matter is the elementary structure of smooth manifolds, which is a topic in real analysis that sits at the intersection of algebraic topology and differential geometry.

Section 1 presents the beginning definitions and results about smooth manifolds, tangent vectors and vector fields, cotangent vectors and differential 1 forms, derivatives of smooth mappings, and differentials.

Section 2 defines the exterior algebra of a finite dimensional real vector space. Tensor algebras, which are discussed in Chapter VI of the author's *Basic Algebra*, are taken as known.

Section 3 introduces differential forms and their pullbacks under smooth maps. It shows how to compute pullbacks, and it establishes some properties of them.

Section 4 introduces the exterior derivative, which is the differentiation operator to be used with differential forms, and shows that it satisfies a number of properties.

Section 5 contains the construction of a smooth partition of unity, which is a device making it unnecessary in many cases to cut manifolds into pieces when treating integration problems.

Section 6 introduces the notion of an oriented smooth manifold and integration of top-degree differential forms on it. The section shows also the relationship between integration and pullback.

1. Smooth Manifolds, Vector Fields, Derivatives, and Differentials

This section introduces smooth manifolds, and it briefly develops the notions of smooth function, tangent and cotangent space, vector field, derivative, differential 1 form, and differential. For a more thorough presentation of this material, the reader may wish to consult the author's *Advanced Real Analysis*, particularly Sections VIII.1–4.

"Manifolds" in our treatment are built from "charts," each manifold has a uniform dimension, and each manifold will be assumed to be separable in the sense of having a countable base for its topology. The term "smooth" is used interchangeably with the term C^∞. The prototype for a manifold is the surface of a sphere in three dimensions. Let us discuss this case informally first and then return to develop the formal mathematics.

In the real world one describes the surface of the earth by means of "charts," with each chart containing a likeness of part of the earth's surface and with all the charts together describing the whole surface. The collection of charts is an "atlas." The sense in which a chart contains a likeness of part of the surface is that there is an understood one-one function ("map") from the one onto the other. In mathematics this function goes from a part of the surface into a likeness; in the real world it tends to go in the opposite direction, namely from the likeness into the surface.

Let M be a separable[1] Hausdorff topological space, and fix an integer $m \geq 0$. A **chart** (M_α, α) on M of **dimension** m is a homeomorphism $\alpha : M_\alpha \to \alpha(M_\alpha)$ of a nonempty open subset M_α of M onto an open subset $\alpha(M_\alpha)$ of $\mathbb{R}^m$; the chart is said to be **about** a point p in M if p is in the domain M_α of α. We say that M is a **manifold** if there is an integer $m \geq 0$ such that each point of M has a chart of dimension m about it.

A **smooth structure** of dimension m on a manifold M is a family $\mathcal{F}$ of m dimensional charts with the following three properties:

(i) any two charts (M_α, α) and (M_β, β) in $\mathcal{F}$ are smoothly **compatible** in the sense that $\beta \circ \alpha^{-1}$, as a mapping of the open subset $\alpha(M_\alpha \cap M_\beta)$ of $\mathbb{R}^m$ to the open subset $\beta(M_\alpha \cap M_\beta)$ of $\mathbb{R}^m$, is smooth and has a smooth inverse,

(ii) the system of compatible charts (M_α, α) is an **atlas** in the sense that the domains M_α together cover M,

(iii) $\mathcal{F}$ is maximal among families of compatible charts on M.

A **smooth manifold** of **dimension** m is a manifold together with a smooth structure of dimension m. In the presence of an understood atlas, a chart will be said to be **compatible** if it is compatible with all the members of the atlas.

Once we have an atlas of compatible m dimensional charts for a manifold M, i.e., once (i) and (ii) are satisfied, then the family of all compatible charts satisfies (i) and (iii), as well as (ii), and therefore is a smooth structure. In other words, an atlas of compatible charts determines one and only one smooth structure. As a practical matter we can thus construct a smooth structure for a manifold by finding an atlas satisfying (i) and (ii), and the extension of the atlas for (iii) to hold is automatic. Particularly in discussing orientability in Section 6, it will be convenient to work with atlases that are not maximal.

EXAMPLE. The unit sphere $M = S^n$ in $\mathbb{R}^{n+1}$, the set of vectors of Euclidean norm 1, can be made into a smooth manifold of dimension n by using two charts defined as follows. One of these charts is (M_φ, φ) with

$$\varphi(x_1, \dots, x_{n+1}) = \Big(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}\Big)$$

[1] The treatment in Sections VIII.1–4 of *Advanced Real Analysis* does not insist on separability of manifolds.

and with domain $M_\varphi = S^n - \{(0, \dots, 0, 1)\}$, and the other is (M_ψ, ψ) with

$$\psi(x_1, \dots, x_{n+1}) = \Big(\frac{x_1}{1 + x_{n+1}}, \dots, \frac{x_n}{1 + x_{n+1}}\Big)$$

and with domain $M_\psi = S^n - \{(0, \dots, 0, -1)\}$. We need to check that the two charts are smoothly compatible. The set $M_\varphi \cap M_\psi$ is $S^n - \{(0, \dots, 0, \pm 1)\}$, and the image of this under φ and ψ is $\mathbb{R}^n - \{(0, \dots, 0)\}$. Put $y_j = x_j/(1 - x_{n+1})$, so that $\varphi^{-1}(y_1, \dots, y_n) = (x_1, \dots, x_{n+1})$. Then $\psi \circ \varphi^{-1}(y_1, \dots, y_n)$ is

$$\begin{aligned} &= \big(x_1/(1 + x_{n+1}), \dots, x_n/(1 + x_{n+1})\big) \\ &= \big(y_1(1 - x_{n+1})/(1 + x_{n+1}), \dots, y_n(1 - x_{n+1})/(1 + x_{n+1})\big). \end{aligned}$$

To compute $(1 - x_{n+1})/(1 + x_{n+1})$, we take $\sum\limits_{j=1}^{n+1} x_j^2 = 1$ into account and write $1 = \sum\limits_{j=1}^{n+1} x_j^2 = x_{n+1}^2 + \sum\limits_{j=1}^{n} y_j^2 (1 - x_{n+1})^2$. Then $\sum\limits_{j=1}^{n} y_j^2 = (1 - x_{n+1}^2)/(1 - x_{n+1})^2 = (1 + x_{n+1})/(1 - x_{n+1})$, and

$$\psi \circ \varphi^{-1}(y_1, \dots, y_n) = \Big(y_1 \Big/ \sum_{j=1}^{n} y_j^2, \dots, y_n \Big/ \sum_{j=1}^{n} y_j^2\Big).$$

The entries on the right are smooth functions of y since $y \neq 0$. Similarly if we put $z_j = x_j/(1 + x_{n+1})$, we calculate that

$$\varphi \circ \psi^{-1}(z_1, \dots, z_n) = \Big(z_1 \Big/ \sum_{j=1}^{n} z_j^2, \dots z_n \Big/ \sum_{j=1}^{n} z_j^2\Big).$$

Again the entries on the right are smooth functions of z since $z \neq 0$. Thus the two charts are smoothly compatible, and S^n is a smooth manifold.

Euclidean space $\mathbb{R}^m$ itself is of course a smooth manifold of dimension m, with an atlas consisting of the single chart $(\mathbb{R}^m, 1)$, where 1 is the identity function on $\mathbb{R}^m$. Real projective spaces, which are defined in Problem 3 at the end of the chapter, give further straightforward examples. A number of interesting manifolds arise as a part of the space of simultaneous solutions of some equations, often polynomial equations in several variables. The technical device that shows the solution space to be part of a smooth manifold is normally the Implicit Function Theorem (Theorem 3.16 of *Basic Real Analysis*), as is explained in Problem 30 at the end of the chapter.

Another simple example of a smooth manifold M of dimension m is any nonempty open subset U of M. The subset U becomes a smooth manifold of dimension m if we define an atlas for it to consist of all restrictions $(U \cap M_\alpha, \alpha\big|_{U\cap M_\alpha})$ of members of the atlas $\{(M_\alpha, \alpha)\}$ for M; then we must discard occurrences of the empty set. We shall often use this observation without special notice, in effect making definitions and deducing conclusions for nonempty open subsets of a manifold M from the corresponding definitions and conclusions about all manifolds.

Most manifolds, however, are constructed globally out of other manifolds or are pieced together from local data. The Hausdorff condition often has to be checked, is often subtle, and is always important. The first place that the Hausdorff condition plays a role is in Lemma 1.2 below.

Any manifold is a locally compact Hausdorff space. The separability implies that there exists an **exhausting sequence** in M, i.e., an increasing sequence of compact sets with union all of M and with each set contained in the interior of the next member of the sequence. This is Proposition 10.25 of *Basic Real Analysis*.

Let us mention that because of the separability and Theorem 10.45 of *Basic Real Analysis*, the topology of a manifold can always be realized by a metric; this fact turns out to be more comforting than useful.

Although manifolds have a global definition, it is often convenient to work with them by referring matters to local coordinates. If p is a point of the smooth manifold M of dimension m, then a compatible chart (M_α, α) about p can be viewed as giving a **local coordinate system** near p. Specifically if the Euclidean coordinates in $\alpha(M_\alpha)$ are $(u_1, \dots, u_m)$, then $q = \alpha^{-1}(u_1, \dots, u_m)$ is a general point of M_α, and we define m real-valued functions $q \mapsto x_j(q)$ on M_α by $x_j(q) = u_j$, $1 \le j \le m$. Then $\alpha = (x_1, \dots, x_m)$. To refer the functions x_j to Euclidean space $\mathbb{R}^m$, we use $x_j \circ \alpha^{-1}$, which carries $(u_1, \dots, u_m)$ to u_j.

The way that the functions x_j are referred to Euclidean space mirrors how a more general real-valued function on an open subset of M may be referred to Euclidean space, and then we can define a real-valued function on M to be smooth if it is smooth in the sense of Euclidean differential calculus when referred to Euclidean space.

Therefore a **smooth function** $f : M \to \mathbb{R}$ on the smooth manifold M is by definition a function such that for each $p \in M$ and each compatible chart (M_α, α) about p, the function $f \circ \alpha^{-1}$ is smooth as a function from the open subset $\alpha(M_\alpha)$ of $\mathbb{R}^m$ into $\mathbb{R}$. A smooth function is necessarily continuous.

In verifying that a real-valued function f on M is smooth, it is sufficient, for each point in M, to check smoothness within only one compatible chart about that point. The reason is the compatibility of the charts: if (M_α, α) and (M_β, β) are two compatible charts about p, then $f \circ \beta^{-1}$ is the composition of the smooth function $\alpha \circ \beta^{-1}$ followed by $f \circ \alpha^{-1}$.

The space of smooth real-valued functions on the nonempty open set U of M will be denoted by $C^\infty(U)$. The space $C^\infty(U)$ is an associative algebra over $\mathbb{R}$

under the pointwise operations, and it contains the constants. The **support** of a real-valued function is the *closure* of the set where the function is nonzero. We write $C^\infty_{\text{com}}(U)$ for the subset of $C^\infty(U)$ of functions whose support is a compact subset of U.

The space $C^\infty_{\text{com}}(U)$ is not 0. This fact is a consequence of the following result for Euclidean space that appeared as Proposition 8.12 in *Basic Real Analysis*.

Lemma 1.1. If K and U are subsets of $\mathbb{R}^m$ with K compact, U open, and $K \subseteq U$, then there exists $\varphi \in C^\infty_{\text{com}}(U)$ with values in $[0, 1]$ such that φ is identically 1 on K.

Lemma 1.2. If U is a nonempty open subset of a smooth manifold M and if f is in $C^\infty_{\text{com}}(U)$, then the function F defined on M so as to equal f on U and to equal 0 off U is in $C^\infty_{\text{com}}(M)$ and has support contained in U,

PROOF. The set $S = \text{support}(f)$ is a compact subset of U and is compact as a subset of M since the fact that U gets the relative topology means that the inclusion of U into M is continuous. *Since M is Hausdorff*, S is closed in M. The function F is smooth at all points of U and in particular at all points of S, and we need to prove that it is smooth at all points of the open complement V of S in M. If p is in V, we can find a compatible chart (M_α, α) about p with $M_\alpha \subseteq V$. The function F is 0 on $M_\alpha \cap U \subseteq V \cap U = S^c \cap U$ because it equals f on U and f is 0 on the complement of S in U. The function F is 0 on $M_\alpha \cap U^c$ since it is 0 everywhere on U^c. Therefore F is identically 0 on M_α and is exhibited as smooth in a neighborhood of p. Thus F is smooth. $\square$

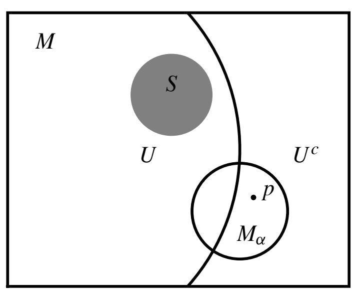

FIGURE 1.1. Diagram for Lemma 1.2 with p shown outside $M_\alpha \cap U$.

Lemma 1.3. Suppose that p is a point in a smooth manifold M, that (M_α, α) is a compatible chart about p, and that K is a compact subset of M_α containing p. Then there is a smooth function $f : M \to \mathbb{R}$ with compact support contained in M_α such that f has values in $[0, 1]$ and f is identically 1 on K.

PROOF. The set $\alpha(K)$ is a compact subset of the open subset $\alpha(M_\alpha)$ of Euclidean space, and Lemma 1.1 produces a smooth function g in $C^\infty_{\text{com}}(\alpha(M_\alpha))$ with values in $[0, 1]$ that is identically 1 on $\alpha(K)$. If f is defined to be $g \circ \alpha$ on M_α, then f is in $C^\infty_{\text{com}}(M_\alpha)$. Extending f to be 0 on the complement of M_α in M and applying Lemma 1.2, we see that the extended f satisfies the required conditions. $\square$

Proposition 1.4. Let p be a point of a smooth manifold M, let U be an open neighborhood of p, and let f be in $C^\infty(U)$. Then there is a function g in $C^\infty(M)$ such that $g = f$ in a neighborhood of p.

PROOF. Possibly by shrinking U, we may assume that U is the domain of some compatible chart (M_α, α) about p. Let K be a compact neighborhood of p contained in U, and use Lemma 1.3 to find h in $C^\infty(M)$ with compact support in U such that h is identically 1 on K. Define g to be the pointwise product hf on U and to be 0 off U. Then g equals f on the neighborhood K of p, and Lemma 1.2 shows that g is everywhere smooth. $\square$

In the same way that we defined smoothness of real-valued functions on smooth manifolds by means of local coordinates, we define smoothness for a *continuous* function from an m dimensional manifold M into an n dimensional manifold N. Namely let p be in M, so that $F(p)$ is in N. Assuming that F is continuous at p, let a local coordinate system be given at $F(p)$ by means of a chart (N_β, β), and choose a local coordinate system at p given by a chart (M_α, α) such that $F(M_\alpha) \subseteq N_\beta$. The local version of F is the function $\beta \circ F \circ \alpha^{-1}$, which carries $\alpha(M_\alpha)$ into $\beta(N_\beta)$. If we write $\alpha = (x_1, \ldots, x_m)$ and $\beta = (y_1, \ldots, y_n)$, then we obtain an expression of the form

$$(y_1, \ldots, y_n) = \beta \circ F \circ \alpha^{-1}(x_1, \ldots, x_m),$$

and we see that $\beta \circ F \circ \alpha^{-1}$ is the local function in the Euclidean setting that corresponds to F in the manifold setting. The function $F : M \to N$ is said to be **smooth** if it is continuous and all the functions $\beta \circ F \circ \alpha^{-1}$ are smooth, more precisely if it is continuous and for each p in E and each compatible chart β about $F(p)$, there is some compatible chart α about p such that $\beta \circ F \circ \alpha^{-1}$ is defined and smooth. In this case we often call F a **smooth map**. A smooth function between smooth manifolds with a smooth inverse is called a **diffeomorphism**.

In this way all questions about smoothness of functions in the manifold setting can be translated into questions about smoothness of functions in the Euclidean setting. One consequence, by means of the Inverse Function Theorem,[2] is that the dimension of a smooth manifold is well defined. More specifically the same underlying topological space cannot have two compatible atlases of distinct dimensions.

[2]Theorem 3.17 of *Basic Real Analysis*.

We turn to a discussion of tangent spaces and vector fields. Let M be a smooth manifold of dimension m. The idea is that the tangent space to M at p is the space of all first-order derivatives at p. To make this notion precise, one introduces the space of **germs** $\mathcal{C}_p(M)$ of smooth functions at p. These are equivalence classes formed from pairs (f, U), each pair consisting of an open set U containing p and a smooth real-valued function f defined on that open set, two such being equivalent if their restrictions are equal on some subneighborhood of p. The set $\mathcal{C}_p(M)$ of equivalence classes inherits arithmetic operations that make it into an associative algebra over $\mathbb{R}$. Evaluation at p is a well defined linear functional e on $\mathcal{C}_p(M)$. A **derivation** of $\mathcal{C}_p(M)$ is a linear function $L : \mathcal{C}_p(M) \to \mathbb{R}$ such that $L(fg) = L(f)e(g) + e(f)L(g)$. Each such L annihilates constant functions because

$$L(1) = L(1 \cdot 1) = L(1)e(1) + e(1)L(1) = 2L(1)$$

forces $L(1) = 0$. The set of derivations of $\mathcal{C}_p(M)$ forms a real vector space that is denoted by $T_p(M)$ and is called the **tangent space** of M at p. If a local coordinate system at p is given by means of a chart (M_α, α) with $\alpha = (x_1, \dots, x_m)$, then m examples of members of $T_p(M)$ are given by the derivations $\left[\frac{\partial}{\partial x_j}\right]_p$ defined by

$$\left[\frac{\partial f}{\partial x_j}\right]_p = \frac{\partial(f \circ \alpha^{-1})}{\partial u_j}\bigg|_{(u_1,\dots,u_m)=(x_1(p),\dots,x_m(p))} \qquad \text{for} \quad j = 1, \dots, m.$$

These derivations satisfy

$$\left[\frac{\partial x_i}{\partial x_j}\right]_p = \frac{\partial u_i}{\partial u_j}\bigg|_{(u_1,\dots,u_m)=(x_1(p),\dots,x_m(p))} = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. It follows that the m derivations $\left[\frac{\partial}{\partial x_j}\right]_p$ are linearly independent. Actually these m derivations form a vector-space basis of $T_p(M)$, as is shown in the following proposition. Spanning follows from an expansion formula established by the proposition for all members of $T_p(M)$.

Proposition 1.5. If M is a smooth manifold and if a compatible chart (M_α, α) about a point p in M is given by $\alpha = (x_1, \dots, x_m)$, then each member L of $T_p(M)$ is given on $\mathcal{C}_p(M)$ by

$$L = \sum_{j=1}^{m} L(x_j)\left[\frac{\partial}{\partial x_j}\right]_p.$$

Consequently the m derivations $\left[\frac{\partial}{\partial x_j}\right]_p$ form a vector-space basis of $T_p(M)$.

PROOF. Let L be a derivation of $\mathcal{C}_p(M)$, and let (f, U) represent a member of $\mathcal{C}_p(M)$, U being an open neighborhood of p in M. Without loss of generality, we may assume that $U \subseteq M_\alpha$ and that $\alpha(U)$ is an open ball in $\mathbb{R}^m$. Put $u_0 = (u_{0,1}, \ldots, u_{0,m}) = \alpha(p)$, let q be a variable point in U, and define $u = (u_1, \ldots, u_n) = \alpha(q)$. Taylor's Theorem[3] applied to $f \circ \alpha^{-1}$ on $\alpha(U)$ gives

$$f \circ \alpha^{-1}(u) = f \circ \alpha^{-1}(u_0) + \sum_{j=1}^{m} (u_j - u_{0,j}) \frac{\partial (f \circ \alpha^{-1})}{\partial u_j}(u_0) + \sum_{i,j} (u_i - u_{0,i})(u_j - u_{0,j}) R_{ij}(u)$$

with each R_{ij} in $C^\infty(\alpha(U))$. Referring this formula to M, we obtain

$$f(q) = f(p) + \sum_{j=1}^{m} (x_j(q) - x_j(p)) \Big[\frac{\partial f}{\partial x_j}\Big]_p + \sum_{i,j} (x_i(q) - x_i(p))(x_j(q) - x_j(p)) r_{ij}(q)$$

on U, where $r_{ij} = R_{ij} \circ \alpha$ on U. Because L annihilates constant functions and has the derivation property and satisfies $e(x_j) = x_j(p)$ for $1 \leq j \leq m$, application of L yields

$$\begin{aligned} L(f) &= \sum_{j=1}^{m} L(x_j) \Big[\frac{\partial f}{\partial x_j}\Big]_p + \sum_{i,j} \big(L(x_i)(e(x_j) - x_j(p)) e(r_{ij}) \\ &\qquad + (e(x_i) - x_i(p)) L(x_j) e(r_{ij}) + (e(x_i) - x_i(p))(e(x_j) - x_j(p)) L(r_{ij})\big) \\ &= \sum_{j=1}^{m} L(x_j) \Big[\frac{\partial f}{\partial x_j}\Big]_p, \end{aligned}$$

as asserted. $\square$

Still with M as a smooth manifold, form the set $T(M)$ of all pairs (p, L) such that p is in M and L is in $T_p(M)$. The set $T(M)$ can be topologized and given a smooth manifold structure in a natural way, and then the pair consisting of $T(M)$ together with the projection-to-the-first-component function is called the **tangent bundle** of M. For current purposes we do not need to know what the topology and manifold structure on $T(M)$ are, and we shall ignore them.[4] A **vector field** X on M is a function from M into $T(M)$ that selects a member of $T_p(M)$ for each p in M; in other words, a vector field is any right inverse to the projection-to-the-first-component function under composition.[5] An immediate consequence of Proposition 1.5 is the following expansion of any vector field.

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[3] In the form of Theorem 3.11 of *Basic Real Analysis*.

[4] The construction is a chore to carry out. Not needing it, we skip the details. The reader who would like to see a careful construction of the tangent bundle may wish to look at Proposition 8.14 and the remarks after it in Section VIII.4 of *Advanced Real Analysis*.

[5] In the terminology of tangent bundles, a vector field is any section of the tangent bundle.

Corollary 1.6. Let M be a smooth manifold of dimension m. If (M_α, α) is any compatible chart for M, say with $\alpha = (x_1, \dots, x_m)$, and if X is a vector field on M_α, then

$$Xf(p) = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(p)\,(Xx_i)(p)$$

for all p in M_α and f in $C^\infty(M_\alpha)$.

For vector fields we satisfy ourselves with the following definition of smoothness: the vector field X on M is defined to be **smooth** on M if Xx_i is smooth for each coordinate function x_i of each compatible chart[6] on M. From Corollary 1.6 it is apparent that the set of smooth vector fields on M is closed under addition and scalar multiplication and is also closed under multiplication by members of $C^\infty(M)$. It is therefore a $C^\infty(M)$ module.

Next we discuss derivatives. Let $F : M \to N$ be a smooth function from a smooth manifold M of dimension m into a smooth manifold N of dimension n. For any p in M, the function F carries any tangent vector L in $T_p(M)$ into a tangent vector $(DF)_p(L)$ in $T_{F(p)}(N)$ by the formula $(DF)_p(L)(g) = L(g \circ F)$ for g in the space $\mathcal{C}_{F(p)}(N)$ on which a tangent vector in $T_{F(p)}(N)$ operates. The result is a linear function $(DF)_p : T_p(M) \to T_{F(p)}(N)$ called the **derivative** of F at p. The name "derivative" and the notation $(DF)_p$ are a change from *Advanced Real Analysis*.[7]

Proposition 1.7. Let M and N be smooth manifolds of respective dimensions m and n, and let $F : M \to N$ be a smooth function. Fix p in M, let $\alpha = (x_1, \dots, x_m)$ be a compatible chart in M about p, and let $\beta = (y_1, \dots, y_n)$ be a compatible chart in N about $F(p)$. Define $F_i = y_i \circ F$ for $1 \le i \le n$. Relative to the bases $\left[\dfrac{\partial}{\partial x_j}\right]_p$ of $T_p(M)$ and $\left[\dfrac{\partial}{\partial y_i}\right]_{F(p)}$ of $T_{F(p)}(N)$, the matrix of the linear function $(DF)_p : T_p(M) \to T_{F(p)}(N)$ has size n by m, and its $(i, j)^{\text{th}}$ entry is $\left[\dfrac{\partial F_i}{\partial u_j}\bigg|_{(u_1,\dots,u_m)=(x_1(p),\dots,x_m(p))}\right]$.

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[6]Section VIII.4 of *Advanced Real Analysis* shows that the smooth vector fields are exactly the sections of the tangent bundle that are smooth. We never need to use this fact.

[7]In *Advanced Real Analysis* the name was "differential," and the notation was $(dF)_p$. The need for a change may be seen in the case that F is a real-valued function, i.e., $N = \mathbb{R}$. In this case, $(DF)_p$ is a member of $\mathrm{Hom}_{\mathbb{R}}(T_p(M), T_{F(p)}(\mathbb{R}))$, and it is being called the "derivative." The word "differential" will acquire a different standard meaning later in this section in such a way that $(dF)_p$ is a member of $\mathrm{Hom}_{\mathbb{R}}(T_p(M), \mathbb{R})$. The two range spaces, $T_{F(p)}(\mathbb{R})$ and $\mathbb{R}$, are isomorphic, but confusion easily arises when the isomorphism is not made explicit. Some authors use the term "push-forward" in referring to what is being called the the derivative here.

REMARKS. In other words the matrix in question is the usual derivative matrix or Jacobian matrix of the set of coordinate functions of the function obtained by referring F to Euclidean space. Hence the derivative at a point is the object for smooth manifolds that generalizes the multivariable derivative at a point for Euclidean space. Accordingly, let us make the definition

$$\Big[\frac{\partial F_i}{\partial x_j}\Big]_p = \Big[\frac{\partial F_i}{\partial u_j}\Big|_{(u_1,\dots,u_n)=(x_1(p),\dots,x_n(p))}\Big].$$

PROOF. Application of the definitions gives

$$\begin{aligned}(DF)_p\Big(\Big[\frac{\partial}{\partial x_j}\Big]_p\Big)(y_i) &= \Big[\frac{\partial}{\partial x_j}\Big]_p(y_i\circ F)\\ &= \frac{\partial(y_i\circ F\circ\alpha^{-1})}{\partial u_j}\,(x_1(p),\dots,x_n(p))\\ &= \frac{\partial F_i}{\partial u_j}\Big|_{(u_1,\dots,u_m)=(x_1(p),\dots,x_m(p))}.\end{aligned}$$

The formula in Proposition 1.5 allows us to express any member of $T_{F(p)}(N)$ in terms of its values on the local coordinate functions y_i, and therefore

$$(DF)_p\Big(\Big[\frac{\partial}{\partial x_j}\Big]_p\Big) = \sum_{i=1}^n \frac{\partial F_i}{\partial u_j}\Big|_{(u_1,\dots,u_m)=(x_1(p),\dots,x_m(p))}\Big[\frac{\partial}{\partial y_i}\Big]_p \quad \text{for } 1\le j\le m.$$

Thus the matrix is as asserted. $\square$

Proposition 1.8 (chain rule). Let M, N, and R be smooth manifolds, and let $F : M \to N$ and $G : N \to R$ be smooth functions. If p is in M, then

$$(D(G\circ F))_p = (DG)_{F(p)}\circ(DF)_p.$$

PROOF. If L is in $T_p(M)$ and h is in $\mathcal{C}_{G(F(p))}(R)$, then the definitions give

$$\begin{aligned}(D(G\circ F))_p(L)(h) &= L(h\circ G\circ F)\\ &= (DF)_p(L)(h\circ G) = (DG)_{F(p)}(DF)_p(L)(h),\end{aligned}$$

as asserted. $\square$

Finally we discuss differential 1 forms and differentials. Still with M as smooth manifold, for each $p\in M$, let $T_p^*(M)$ be the dual vector space of $T_p(M)$, i.e., the real vector space of all linear functionals on $T_p(M)$. Members of $T_p^*(M)$ are called **cotangent vectors** at p. Consider the set $T^*(M)$ of all pairs (p,ℓ) such that

p is in M and ℓ is in $T_p^*(M)$. As with $T(M)$, the set $T^*(M)$ can be topologized and given a smooth manifold structure in a natural way,[8] and then the pair consisting of $T^*(M)$ together with the projection-to-the-first-component function is called the **cotangent bundle** of M. Once again we do not need to know what the topology and manifold structure are, and we shall ignore them. A **differential 1 form** on M is a function from M into $T^*(M)$ that selects a member of $T_p^*(M)$ for each p in M; in other words, a differential 1 form is any right inverse to the projection-to-the-first-component function under composition.[9]

To get some first examples of differential 1 forms, fix $p \in M$ and let f be any member of $C^\infty(M)$. Then f carries any germ L in $\mathcal{C}_p(M)$ into the germ $(Df)_p(L)$ in $\mathcal{C}_{f(p)}(\mathbb{R})$ by the formula

$$(Df)_p(L)(g) = L(g \circ f) \qquad \text{for all } g \in \mathcal{C}_{f(p)}(\mathbb{R}).$$

Let us take g to be the identity function $g_0(t) = t$ on $\mathbb{R}$, no matter what f is. For this choice of g, the formula reduces to $(Df)_p(L)(g_0) = Lf$ for all L in $T_p(M)$. If we suppress g_0 in this formula and write $(df)_p(L)$ for the left side, the formula becomes

$$(df)_p(L) = Lf \qquad \text{for all } L \in T_p(M),$$

and we obtain a linear functional on $T_p(M)$. This linear functional $(df)_p$ is called the **differential** of f at p. As p varies, the result is a differential 1 form df on M.

Let us look more closely at this construction for a moment. For f in $C^\infty(M)$, we passed from $(Df)_p$, which is a member of $\mathrm{Hom}_{\mathbb{R}}(T_p(M), T_{f(p)}(\mathbb{R}))$, to $(df)_p$, which is a member of $\mathrm{Hom}_{\mathbb{R}}(T_p(M), \mathbb{R})$. We did so, in effect, by following the member of $\mathrm{Hom}_{\mathbb{R}}(T_p(M), T_{f(p)}(\mathbb{R}))$ by a particular isomorphism of $T_{f(p)}(\mathbb{R})$ with $\mathbb{R}$.

We just saw that the differentials at p of members of $C^\infty(M)$ are examples of members of $T_p^*(M)$. The proposition below identifies all members of $T_p^*(M)$.

Proposition 1.9. Let M be a smooth manifold of dimension m, fix p in M, and let (M_α, α) be a compatible chart about p with $\alpha = (x_1, \dots, x_m)$. Then the differentials $(dx_1)_p, \dots, (dx_m)_p$ form the dual basis in $T_p^*(M)$ to the basis $\big[\frac{\partial}{\partial x_1}\big]_p, \dots, \big[\frac{\partial}{\partial x_m}\big]_p$ of $T_p(M)$. Also if $f : M \to \mathbb{R}$ is any smooth function on M, then

$$(df)_p = \sum_{i=1}^m \Big(\frac{\partial f}{\partial x_i}\Big)_p (dx_i)_p \qquad \text{for all } p \in M_\alpha.$$

[8] The details appear in Section VIII.4 of *Advanced Real Analysis*.

[9] In the terminology of cotangent bundles, a differential 1 form is any section of the cotangent bundle.

PROOF. Taking $f = x_i$ and $L = \left[\frac{\partial}{\partial x_j}\right]_p$ in the formula $(df)_p(L) = Lf$, we obtain $(dx_i)_p\big(\left[\frac{\partial}{\partial x_j}\right]_p\big) = \left(\frac{\partial x_i}{\partial x_j}\right)_p = \delta_{ij}$. Hence $(dx_1)_p, \dots (dx_m)_p$ indeed forms the dual basis to the basis $\left[\frac{\partial}{\partial x_1}\right]_p, \dots, \left[\frac{\partial}{\partial x_m}\right]_p$ of $T_p(M)$.

To prove the displayed equality in the proposition, it is enough to prove that equality is maintained when both sides are applied to each basis vector $\left[\frac{\partial}{\partial x_j}\right]_p$ of $T_p(M)$. On the left side we have

$$(df)_p\left[\frac{\partial}{\partial x_j}\right]_p = \left(\frac{\partial f}{\partial x_j}\right)_p,$$

and on the right side we have

$$\sum_{i=1}^m \left(\frac{\partial f}{\partial x_i}\right)_p (dx_i)_p\Big(\Big(\frac{\partial f}{\partial x_j}\Big)_p\Big) = \sum_{i=1}^m \left(\frac{\partial f}{\partial x_i}\right)_p \delta_{ij} = \left(\frac{\partial f}{\partial x_j}\right)_p.$$

These are equal, and the proof is complete. □

According to Proposition 1.9, any differential 1 form $\omega(p)$ on the smooth manifold M expands as

$$\omega(p) = \sum_{i=1}^m a_i(p)(dx_i)_p$$

in each compatible chart (M_α, α) with $\alpha = (x_1, \dots, x_m)$. We say that the differential 1 form ω is **smooth** if all coefficient functions a_i for all compatible charts are smooth functions.[10] Part of the content of Proposition 1.9 is that every differential 1 form df with $f \in C^\infty(M)$ is smooth.[11]

2. Properties of Exterior Algebras

If one looks carefully at the classical integration theorems stated in the Introduction, one sees that minus signs play an important role in the theory. Why is it that the right side of the Fundamental Theorem of Calculus reads $F(b) - F(a)$ and not $F(a) - F(b)$? Why is it in Green's Theorem that the region is to lie on the left of the boundary curve as the curve is traced out? And what are these "important questions of orientations" that need to be sorted out in the Divergence Theorem?

[10] Section VIII.4 of *Advanced Real Analysis* shows that the smooth sections of the cotangent bundle are exactly the differential 1 forms that are smooth.

[11] It turns out that not every smooth differential form on a smooth manifold M need be given as df for some smooth f. See Problem 8 at the end of the chapter for an example.

It turns out that all such questions can be resolved by augmenting the heuristic interpretation of dx that one is often taught. Instead of its being an element of numerical length, it is to be a one dimensional vector element of length, with both a magnitude (the usual notion of length) and direction (its sign). In two dimensions similarly, $dx\,dy$ is to be thought of as incorporating information about the angle between the vector dx and the vector dy, thus akin to the area of the parallelogram spanned by the two vectors in $\mathbb{R}^2$, namely the product of their magnitudes by the sine of the angle between them. As soon as one makes this adjustment, one is led to think of dx and dy not as commuting objects but as anticommuting objects.[12] This section takes up the algebraic preliminaries for dealing with a multiplication that is anticommutative but is still associative.

Chapter VI of *Basic Algebra* defines the **tensor algebra** $T(V)$ of a vector space V over $\mathbb{R}$ to be the direct sum over $n \geq 0$ of the n-fold tensor product $T^n(V)$ of V with itself, the 0-fold tensor product being understood to consists just of the scalars $\mathbb{R}$. The operation of multiplication is written as $\otimes$. The space $T^n(V)$ is a vector space with a universal mapping property relative to n-linear functions on V. The full tensor algebra $T(V)$ is an associative algebra with a universal mapping property relative to any linear mapping of V into an associative algebra A with identity: the linear map extends uniquely to an algebra homomorphism of $T(V)$ into A carrying 1 into 1. We take all this as known.

Chapter VI of *Basic Algebra* speaks also of multilinear forms that are **alternating** in the sense that their value is 0 whenever two of the arguments are equal. Alternating forms are skew symmetric in the sense that if two of the arguments are interchanged, then the value of the form is multiplied by -1. Alternating forms will play an important role in what follows.

We shall introduce "exterior algebras" over the field $\mathbb{R}$. If E is a vector space over $\mathbb{R}$, the exterior algebra $\bigwedge(E)$ is to be an associative algebra, and the elements of $\bigwedge(E)$ are to include the members of $\mathbb{R}$ and all the members of E itself. The algebra $\bigwedge(E)$ will be defined as a quotient of the tensor algebra $T(E)$, with all those members of $T(E)$ mapped to 0 that are to represent 0 in the quotient. Its product operation is written as $\wedge$. To force skew symmetry (i.e., $y \wedge x = -x \wedge y$) for multiplication in the quotient of the embedded members of E, we require that $v \otimes v$ maps to 0 in $\bigwedge(E)$ whenever v is in $T^1(E)$. To arrange that the quotient algebra is as large as possible, we factor out nothing more than is necessary from $T(E)$. Thus we define the **exterior algebra**[13] of E by the formula

$$\textstyle\bigwedge(E) = T(E)/I',$$

where $$I' = \begin{pmatrix} \text{two-sided ideal in } T(E) \text{ generated} \\ \text{by all } v \otimes v \text{ with } v \text{ in } T^1(E) \end{pmatrix}.$$

[12] With this thought in mind, we shall be writing $dx \wedge dy$ instead of $dx\,dy$. The notation with the symbol $\wedge$ was already used in the Introduction.

[13] Sometimes known as **Grassmann algebras** for historical reasons.

Then $\bigwedge(E)$ is an associative algebra with identity.

It is clear that I' is homogeneous in the sense that $I' = \bigoplus_{n=0}^{\infty} (I' \cap T^n(E))$. Consequently we can write

$$\bigwedge(E) = \bigoplus_{n=0}^{\infty} T^n(E)/(I' \cap T^n(E)).$$

We write $\bigwedge^n(E)$ for the n^{th} summand on the right side, so that

$$\bigwedge(E) = \bigoplus_{n=0}^{\infty} \bigwedge^n(E).$$

Since $I' \cap T^0(E) = 0$, $\bigwedge^0(E)$ consists of just the scalar multiples of the identity. Since $I' \cap T^1(E) = 0$, the map of E into first-order elements $\bigwedge^1(E)$ is one-one onto and is just a copy of E. The product operation in $\bigwedge(E)$ is called the **exterior product** or **wedge product** and is denoted by $\wedge$ rather than $\otimes$. Thus the image in $\bigwedge^n(E)$ of the element $v_1 \otimes \cdots \otimes v_n$ of $T^n(E)$ can be written as $v_1 \wedge \cdots \wedge v_n$. If a is in $\bigwedge^m(E)$ and b is in $\bigwedge^n(E)$, then $a \wedge b$ is in $\bigwedge^{m+n}(E)$. Moreover, $\bigwedge^n(E)$ is generated by elements $v_1 \wedge \cdots \wedge v_n$ with all v_j in $\bigwedge^1(E) \cong E$, since $T^n(E)$ is generated by corresponding elements $v_1 \otimes \cdots \otimes v_n$. The defining relations for $\bigwedge(E)$ force the condition of **skew symmetry**, $v_i \wedge v_j = -v_j \wedge v_i$ for v_i and v_j in $\bigwedge^1(E)$. Writing members of $\bigwedge(E)$ as linear combinations of monomials and making repeated use of the skew symmetry of multiplication for members of $\bigwedge^1(E)$, we obtain the following result.

Proposition 1.10. If E is a vector space over $\mathbb{R}$, then

$$a \wedge b = (-1)^{mn} b \wedge a \qquad \text{for } a \in \textstyle\bigwedge^m(E) \text{ and } b \in \textstyle\bigwedge^n(E).$$

PROOF. By linearity in each variable in wedge product, it is enough the prove the conclusion when a and b are monomials, say $a = a_1 \wedge \cdots \wedge a_m$ and $b = b_1 \wedge \cdots \wedge b_n$. We induct on m, the base case for the induction being $m = 1$. With $a \in \bigwedge^1(E)$, the skew symmetry allows us to start from $a \wedge b$ and commute a to the right one step at a time, until a is on the right side of b. Then we are introducing n sign changes, and the base case is established. In the general case we write $a = a' \wedge a_m$ with $a' \in \bigwedge^{m-1}(E)$ and $a_m \in \bigwedge^1(E)$. Applying the base case and then the induction hypothesis, we obtain

$$a \wedge b = a' \wedge a_m \wedge b = (-1)^n a' \wedge b \wedge a_m = (-1)^n (-1)^{(m-1)n} b \wedge a' \wedge a_m,$$

and $a \wedge b = (-1)^{mn} b \wedge a$ as required. $\square$

Proposition 1.11. Let E be a real vector space.

(a) Let ι be the n-multilinear function $\iota(v_1, \ldots, v_n) = v_1 \wedge \cdots \wedge v_n$ of $E \times \cdots \times E$ into $\bigwedge^n(E)$. Then $(\bigwedge^n(E), \iota)$ has the following **universal mapping property**: whenever l is any alternating n-multilinear map of $E \times \cdots \times E$ into a vector space U, then there exists a unique linear map $L : \bigwedge^n(E) \to U$ such that the diagram

$$\begin{tikzcd} E \times \cdots \times E \arrow[r, "l"] \arrow[d, "\iota"'] & U \\ \bigwedge^n(E) \arrow[ur, dashed, "L"'] & \end{tikzcd}$$

commutes.

(b) Let ι be the function that embeds E as $\bigwedge^1(E) \subseteq \bigwedge(E)$. Then $(\bigwedge(E), \iota)$ has the following **universal mapping property**: whenever l is any linear map of E into an associative algebra A with identity such that $l(v)^2 = 0$ for all $v \in E$, then there exists a unique algebra homomorphism $L : \bigwedge(E) \to A$ with $L(1) = 1$ such that the diagram

$$\begin{tikzcd} E \arrow[r, "l"] \arrow[d, "\iota"'] & A \\ \bigwedge(E) \arrow[ur, dashed, "L"'] & \end{tikzcd}$$

commutes.

PROOF. In both cases uniqueness is trivial. For existence we use the universal mapping properties of $T^n(E)$ and $T(E)$ to produce $\widetilde{L}$ on $T^n(E)$ or $T(E)$. If we can show that $\widetilde{L}$ annihilates the appropriate subspace so as to descend to $\bigwedge^n(E)$ or $\bigwedge(E)$, then the resulting map can be taken as L, and we are done. For (a), we have $\widetilde{L} : T^n(E) \to U$, and we are to show that $\widetilde{L}(T^n(E) \cap I') = 0$, where I' is generated by all $v \otimes v$ with v in $T^1(E)$. A member of $T^n(E) \cap I'$ is thus of the form $\sum a_i \otimes (v_i \otimes v_i) \otimes b_i$ with each term in $T^n(E)$. Each term here is a sum of pure tensors

$$x_1 \otimes \cdots \otimes x_r \otimes v_i \otimes v_i \otimes y_1 \otimes \cdots \otimes y_s \tag{$*$}$$

with $r + 2 + s = n$. Since l by assumption takes the value 0 on

$$x_1 \times \cdots \times x_r \times v_i \times v_i \times y_1 \times \cdots \times y_s,$$

$\widetilde{L}$ vanishes on $(*)$, and it follows that $\widetilde{L}(T^n(E) \cap I') = 0$.

For (b) we are to show that $\widetilde{L} : T(E) \to A$ vanishes on I'. Since $\ker \widetilde{L}$ is an ideal, it is enough to check that $\widetilde{L}$ vanishes on the generators of I'. But $\widetilde{L}(v \otimes v) = l(v)l(v)$, and the right side is 0 by hypothesis. Thus $L(I') = 0$. $\square$

Corollary 1.12. If E and F are real vector spaces, then the vector space $\mathrm{Hom}_{\mathbb{R}}(\bigwedge^n(E), F)$ of linear mappings from $\bigwedge^n(E)$ into F is canonically isomorphic (via restriction to pure tensors) to the vector space of all F-valued alternating n-multilinear functions on $E \times \cdots \times E$.

PROOF. Restriction is linear and one-one. It is onto by Proposition 1.10a. □

Corollary 1.13. If E is a real vector space, then the vector space dual $(\bigwedge^n(E))'$ of $\bigwedge^n(E)$ is canonically isomorphic (via restriction to pure tensors) to the real vector space of alternating n-multilinear forms on $E \times \cdots \times E$.

PROOF. This is the special case $F = \mathbb{R}$ of Corollary 1.12. □

Up until now, it has been immaterial whether E is finite dimensional or infinite dimensional. That circumstance now changes.

Proposition 1.14. Let E be a real vector space of finite dimension N, and let n be an integer ≥ 0. Then

(a) $\dim \bigwedge^n(E) = \binom{N}{n}$ for $0 \leq n \leq N$ and $= 0$ for $n > N$,
(b) for each integer n with $1 \leq n \leq N$, there is a canonical linear mapping $L : \bigwedge^n(E') \to \bigwedge^n(E)'$ such that $(f_1 \wedge \cdots \wedge f_n)(w_1 \wedge \ldots w_n) = \det\{f_i(w_j)\}_{i,j=1}^n$ for all $f_i \in E'$ and $w_j \in E$,
(c) whenever $u_1, \ldots, u_N$ is a basis of E, then the monomials $u_{i_1} \wedge \cdots \wedge u_{i_n}$ with $1 \leq i_1 < \cdots < i_n \leq N$ form a basis of $\bigwedge^n(E)$,
(d) the linear mapping $L : \bigwedge^n(E') \to \bigwedge^n(E)'$ of (c) is an isomorphism onto,
(e) if $u_1, \ldots, u_N$ is a basis of E and $u'_1, \ldots, u'_N$ is the dual basis of E', then the dual basis for $\bigwedge^n(E')$ to the basis of monomials $u_{i_1} \wedge \cdots \wedge u_{i_n}$ with $1 \leq i_1 < \cdots < i_n \leq N$ as in (b) is the basis of monomials $u'_{i_1} \wedge \cdots \wedge u'_{i_n}$ with $1 \leq i_1 < \cdots < i_n \leq N$.

REMARK. A version of some parts of this proposition remains valid even if E is infinite dimensional, but we shall not pursue the details.

PROOF. Let $u_1, \ldots, u_N$ be a basis of E. For $n = 0$, $\bigwedge^0(E)$ consists of the scalar multiples of the identity, and $\dim \bigwedge^0(E) = 1$. We may assume therefore that $n > 0$. The monomials of degree n in the u_j's span $T^n(E)$, and the same thing is therefore true of the quotient $\bigwedge^n(E)$. Any such monomial in $\bigwedge^n(E)$ with two equal factors is 0 by the alternating condition and can be disregarded. For the remaining monomials we can permute the factors, using the identity $b \wedge a = -a \wedge b$ valid for members of $\bigwedge^1(E)$, to arrange that the indices on the factors of the monomial are in increasing order. As a result we see the monomials of degree n in $u_1, \ldots, u_N$ whose indices are in strictly increasing order span $\bigwedge^n(E)$. If $n > N$, there are no such monomials, and $\bigwedge^n(E) = 0$. If $0 < n \leq N$, the number of such monomials in $\binom{N}{n}$. Thus $\dim \bigwedge^n(E) \le \binom{N}{n}$. This gives part of (a) and allows us to assume that $1 \le n \le N$ from now on. Also it proves that the monomials in (c) form a spanning set for $\bigwedge^n(E)$.

For (b), fix $f_1, \ldots, f_n$ in E', let $w_1, \ldots, w_n$ be in E, and define

$$l_{f_1,\ldots,f_n}(w_1, \ldots, w_n) = \det\{f_i(w_j)\}_{i,j=1}^n.$$

Then $l_{f_1,\ldots,f_n}$ is an alternating n-multilinear form on $E \times \cdots \times E$ and extends by Proposition 1.10a to a linear functional $L_{f_1,\ldots,f_n} : \bigwedge^n(E) \to \mathbb{R}$. Next we let $f_1, \ldots, f_n$ vary, and the result is that $l(f_1, \ldots, f_n) = L_{f_1,\ldots,f_n}$ defines an alternating n-multilinear map of $E' \times \cdots \times E'$ into $\bigwedge^n(E)'$. Its linear extension L given by Proposition 1.11a maps $\bigwedge^n(E')$ into $\bigwedge^n(E)'$. This proves (b).

Before proceeding with the remaining parts, let us prove the displayed formula $(*)$ below. Let $\{u_1, \ldots, u_N\}$ be a basis of E, and let $\{u'_1, \ldots, u'_N\}$ be the dual basis of E'. Suppose that two strictly increasing sets of n-element indices $I = (i_s)_{s=1}^n$ and $J = (j_t)_{t=1}^n$ between 1 and N are given. The claim is that

$$\det\{u'_{i_s}(u_{j_t})\}_{s,t=1}^n = \begin{cases} 1 & \text{if } i_k = j_k \text{ for } 1 \le k \le n, \\ 0 & \text{otherwise.} \end{cases} \qquad (*)$$

To see this, assume that $i_k \ne j_k$ for some k, and let l be the least such k. If $i_l < j_l$, then $i_l \ne j_t$ for all t and it follows that $u'_{i_l}(u_{j_t}) = 0$ for $1 \le t \le n$. The matrix $\{u'_{i_s}(u_{j_t})\}_{s,t=1}^n$ has a row of zeros, and its determinant is 0. On the other hand, if $i_l > j_l$, then the matrix $\{u'_{i_s}(u_{j_t})\}_{s,t=1}^n$ has a column of zeros, and its determinant is 0. The only other possibility is that $i_k = j_k$ for $1 \le k \le n$. Then the matrix $\{u'_{i_s}(u_{j_t})\}_{s,t=1}^n$ is the identity, and its determinant is 1. This proves $(*)$.

With the sets of indices $I = (i_s)_{s=1}^n$ and $J = (j_t)_{t=1}^n$ as above, define

$$\begin{aligned} u'_I &= u'_{i_1} \wedge \cdots \wedge u'_{i_s} \wedge \cdots \wedge u'_{i_n} && \text{as a member of } \textstyle\bigwedge^n(E'), \\ u_J &= u_{j_1} \wedge \cdots \wedge u_{j_s} \wedge \cdots \wedge u_{j_n} && \text{as a member of } \textstyle\bigwedge^n(E). \end{aligned}$$

What $(*)$ says, in terms of the mapping L of conclusion (a), is that $L(u'_I)(u_J) = \delta_{IJ}$. It follows from $(*)$ that the set of all u'_I as I varies through n-element sets of indices is linearly independent in $\bigwedge^n(E')$ and that the set of all u_J as J varies through n-element set of indices is linearly independent in $\bigwedge^n(E)$. This conclusion for $\bigwedge^n(E)$ completes the proof of (a) and (c), spanning having been proved earlier.

In view of (c), the linear mapping in (d) carries a basis to a basis and is therefore an isomorphism. This proves (d). Conclusion (e) is then immediate from $(*)$. $\square$

In our applications of this algebraic theory to manifolds, we shall be interested in the case that E is a tangent space $T_p(M)$ and its dual is the cotangent space

$T_p^*(M)$. Let ξ and η be typical vector fields, so that ξ_p and η_p are members of the tangent space $T_p(M)$, and let ω and σ be typical differential 1 forms, so that ω_p and σ_p are members of the cotangent space $T_p^*(M)$. Then expressions like $\xi_p \mapsto \omega_p(\xi_p)$ and $\eta_p \mapsto \sigma_p(\eta_p)$ are meaningful, and we can multiply them, obtaining a bilinear form $(\xi_p, \eta_p) \mapsto \omega_p(\xi_p)\sigma_p(\eta_p)$. How is the bilinear form $(\xi_p, \eta_p) \mapsto \omega_p(\xi_p)\sigma_p(\eta_p)$ related to the bilinear form $(\xi_p, \eta_p) \mapsto (\omega_p \wedge \sigma_p)(\xi_p, \eta_p)$? The answer is given by the corollary of the following proposition, which strips away the unnecessary information about manifolds. The corollary will be proved by applying Proposition 1.15 below with V equal to the dual E' of E.

Let V be a finite dimensional real vector space. On $V \times \cdots \times V$, let us define an n-multilinear function with values in $T^n(V)$ by

$$(v_1, \dots, v_n) \mapsto \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} (\operatorname{sgn} \tau) v_{\tau(1)} \otimes \cdots \otimes v_{\tau(n)},$$

where $\mathfrak{S}_n$ is the symmetric group on n letters, and let $\mathcal{A} : T^n(E) \to T^n(E)$ be its linear extension. We shall call $\mathcal{A}$ the **antisymmetrizer** operator. The image of $\mathcal{A}$ in $T^n(V)$ will be denoted by $\widetilde{\bigwedge}^n(V)$, and the members of this subspace will be called **antisymmetrized** tensors.

Proposition 1.15. If V is a finite dimensional real vector space, then the antisymmetrizer operator $\mathcal{A}$ satisfies $\mathcal{A}^2 = \mathcal{A}$. The kernel of $\mathcal{A}$ on $T^n(E)$ is exactly $T^n(E) \cap I'$, where I' is the two-sided ideal of $T(V)$ generated by all elements $v \otimes v$ with $v \in T^1(V)$. Therefore $T^n(V)$ is the vector-space direct sum

$$T^n(V) = \widetilde{\bigwedge}^n(V) \oplus (T^n(V) \cap I').$$

REMARK. In view of this proposition, the quotient map $T^n(V) \to \bigwedge^n(V)$ carries $\widetilde{\bigwedge}^n(V)$ one-one onto $\bigwedge^n(V)$. Thus $\widetilde{\bigwedge}^n(V)$ can be viewed as a copy of $\bigwedge^n(V)$ embedded as a direct summand of $T^n(V)$.

PROOF. We have

$$\begin{aligned}
\mathcal{A}^2(v_1 \otimes \cdots \otimes v_n) &= \frac{1}{(n!)^2} \sum_{\sigma, \tau \in \mathfrak{S}_n} (\operatorname{sgn} \sigma\tau) v_{\sigma\tau(1)} \otimes \cdots \otimes v_{\sigma\tau(n)} \\
&= \frac{1}{(n!)^2} \sum_{\sigma \in \mathfrak{S}_n} \sum_{\substack{\rho \in \mathfrak{S}_n, \\ (\rho = \sigma\tau)}} (\operatorname{sgn} \rho) v_{\rho(1)} \otimes \cdots \otimes v_{\rho(n)} \\
&= \frac{1}{n!} \sum_{\rho \in \mathfrak{S}_n} \mathcal{A}(v_1 \otimes \cdots \otimes v_n) \\
&= \mathcal{A}(v_1 \otimes \cdots \otimes v_n).
\end{aligned}$$

Hence $\mathcal{A}^2 = \mathcal{A}$. Consequently $T^n(E)$ is the direct sum of image $\mathcal{A}$ and $\ker \mathcal{A}$, and we are left with identifying $\ker \mathcal{A}$ as $T^n(V) \cap I'$.

The subspace $T^n(V) \cap I'$ is spanned by elements

$$x_1 \otimes \cdots \otimes x_r \otimes v \otimes v \otimes y_1 \otimes \cdots \otimes y_s$$

with $r + 2 + s = n$, and the antisymmetrizer $\mathcal{A}$ certainly vanishes on such elements. Hence $T^n(V) \cap I' \subseteq \ker \mathcal{A}$. Arguing by contradiction, suppose that the inclusion is strict, say with t in $\ker \mathcal{A}$ but t not in $T^n(V) \cap I'$. Let q be the quotient map $T^n(V) \to \bigwedge^n(V)$. The kernel of q is $T^n(V) \cap I'$, and thus $q(t) \neq 0$. From Proposition 1.14c the monomials $T(V)$ in members of a basis of V that have strictly increasing indices map onto a basis of $\bigwedge(V)$. The antisymmetrized version of each of these monomials has to map to a multiple of the initial monomial, and that multiple has to be nonzero because Proposition 1.14d says that the basis maps to a basis. Consequently q carries $\widetilde{\bigwedge}^n(V) =$ image $\mathcal{A}$ onto $\bigwedge^n(V)$. Thus we can choose $t' \in \widetilde{\bigwedge}^n(V)$ with $q(t') = q(t)$. Then $t' - t$ is in $\ker q = T^n(V) \cap I' \subseteq \ker \mathcal{A}$. Since $\mathcal{A}(t) = 0$, we see that $\mathcal{A}(t') = 0$. Consequently t' is in $\ker \mathcal{A} \cap \text{image}\, \mathcal{A} = 0$, and we obtain $t' = 0$ and $q(t) = q(t') = 0$, contradiction. □

Corollary 1.16. Let E be a finite dimensional real vector space, and let E' be its dual. If $\omega_1, \ldots, \omega_n$ are members of E' and $v_1, \ldots, v_n$ are members of E, then

$$(\omega_1 \wedge \cdots \wedge \omega_n)(v_1, \ldots, v_n) = \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} (\operatorname{sgn} \tau) \omega_1(v_{\tau(1)}) \cdots \omega_n(v_{\tau(n)}).$$

PROOF. Let $\sigma : T(E') \to \bigwedge(E')$ be the quotient mapping, let I' be the kernel, and let $\mathcal{A}$ be the antisymmetrizer mapping of $T(E')$ into itself. If $\omega_1, \ldots, \omega_n$ are in E', then Proposition 1.15 shows that $\omega_1 \otimes \cdots \otimes \omega_n - \mathcal{A}(\omega_1 \otimes \cdots \otimes \omega_n)$ lies in I'. Since $\sigma(\omega_1 \otimes \cdots \otimes \omega_n) = \omega_1 \wedge \cdots \wedge \omega_n$ and since a similar equality holds for each of the terms in $\mathcal{A}(\omega_1 \otimes \cdots \otimes \omega_n)$, we obtain

$$\omega_1 \wedge \cdots \wedge \omega_n = \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} (\operatorname{sgn} \tau) \omega_{\tau(1)} \wedge \cdots \wedge \omega_{\tau(n)}.$$

Restricting to pure tensors, using the isomorphism of Corollary 1.13 with $E = V'$, and making a change a variables in the sum, we can write this conclusion as

$$\begin{aligned}(\omega_1 \wedge \cdots \wedge \omega_n)(v_1, \ldots, v_n) &= \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} (\operatorname{sgn} \tau) \omega_{\tau(1)}(v_1) \cdots \omega_{\tau(n)}(v_n) \\ &= \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} (\operatorname{sgn} \tau) \omega_1(v_{\tau(1)}) \cdots \omega_n(v_{\tau(n)}),\end{aligned}$$

as required. □

EXAMPLES.

(1) Just before Proposition 1.15, this question was raised: If ξ and η are vector fields and ω and σ are differential 1 forms, how is the bilinear form $(\xi_p, \eta_p) \mapsto \omega_p(\xi_p)\sigma_p(\eta_p)$ related to the bilinear form $(\xi_p, \eta_p) \mapsto (\omega_p \wedge \sigma_p)(\xi_p, \eta_p)$? Corollary 1.16 tells us that the formula is supposed to turn out to be

$$(\omega \wedge \sigma)(\xi, \eta) = \tfrac{1}{2}\big(\omega(\xi)\sigma(\eta)\big) - \tfrac{1}{2}\big(\omega(\eta)\sigma(\xi)\big). \qquad (*)$$

On the level of full tensors before passage to the quotient, the formula with ξ and η suppressed is

$$\begin{aligned} \omega \otimes \sigma &= \mathcal{A}(\omega \otimes \sigma) + \big(\text{error}\big) \\ &= \tfrac{1}{2}(\omega \otimes \sigma - \sigma \otimes \omega) + \big(\tfrac{1}{2}(\omega \otimes \sigma + \sigma \otimes \omega)\big) \\ &= \tfrac{1}{2}(\omega \otimes \sigma - \sigma \otimes \omega) + \tfrac{1}{2}\big((\omega + \sigma) \otimes (\omega + \sigma) - (\omega \otimes \omega) - (\sigma \otimes \sigma)\big), \end{aligned}$$

and it is plain that the term called "error" above is in the ideal I' and hence maps to 0 under passage to the quotient. Thus passage to the quotient indeed yields (*).

(2) This example elaborates on the heuristic interpretation near the beginning of this section concerning expressions like dx. With $M = \mathbb{R}^2$ and p equal to $(0, 0)$, let us use Corollary 1.16 to evaluate

$$\big((dx)_{(0,0)} \wedge dy_{(0,0)}\big)\big(a\big[\tfrac{\partial}{\partial x}\big]_{(0,0)} + b\big[\tfrac{\partial}{\partial y}\big]_{(0,0)},\ c\big[\tfrac{\partial}{\partial x}\big]_{(0,0)} + d\big[\tfrac{\partial}{\partial y}\big]_{(0,0)}\big).$$

The corollary says that this expression is $\frac{1}{2}$ times the sum of two terms, separated by a minus sign, namely that it is

$$\begin{aligned} = \tfrac{1}{2}\Big(&\Big((dx)_{(0,0)}\big(a\big[\tfrac{\partial}{\partial x}\big]_{(0,0)} + b\big[\tfrac{\partial}{\partial y}\big]_{(0,0)}\big) \times (dy_{(0,0)})\big(c\big[\tfrac{\partial}{\partial x}\big]_{(0,0)} + d\big[\tfrac{\partial}{\partial y}\big]_{(0,0)}\big)\Big) \\ &- \Big((dx)_{(0,0)}\big(c\big[\tfrac{\partial}{\partial x}\big]_{(0,0)} + d\big[\tfrac{\partial}{\partial y}\big]_{(0,0)}\big) \times (dy_{(0,0)})\big(a\big[\tfrac{\partial}{\partial x}\big]_{(0,0)} + b\big[\tfrac{\partial}{\partial y}\big]_{(0,0)}\big)\Big)\Big). \end{aligned}$$

Since $(dx)_{(0,0)}\big(\big[\frac{\partial}{\partial x}\big]_{(0,0)}\big) = (dy)_{(0,0)}\big(\big[\frac{\partial}{\partial y}\big]_{(0,0)}\big) = 1$ and $(dx)_{(0,0)}\big(\big[\frac{\partial}{\partial y}\big]_{(0,0)}\big) = (dy)_{(0,0)}\big(\big[\frac{\partial}{\partial x}\big]_{(0,0)}\big) = 0$, the expression reduces to

$$= \tfrac{1}{2}(ad - bc).$$

Except for a sign and the factor $\frac{1}{2}$, this is just the area of the rectangle in $\mathbb{R}^2$ spanned by the vectors $\binom{a}{c}$ and $\binom{b}{d}$. The factor $\frac{1}{2}$ means that the area is in fact the area of the triangle spanned by the two vectors rather than the rectangle. Thus the expression evaluates as the signed area of the spanned simplex.

(3) This example notes the corresponding calculation for Example 2 when done in $\mathbb{R}^n$ for the point p equal to the origin 0. Here we are to evalaute

$$\big((dx_1)_0 \wedge \cdots \wedge (dx_n)_0\big)\Big(\sum_{j=1}^n a_{1j}\big[\tfrac{\partial}{\partial x_j}\big]_0, \dots, \sum_{j=1}^n a_{nj}\big[\tfrac{\partial}{\partial x_j}\big]_0\Big),$$

and a similar computation shows that the corollary gives $\frac{1}{n!}\det\{a_{ij}\}_{i,j=1}^n$, as will be shown in the proof of Proposition 1.17 below. Again the geometric significance of the coefficient $1/n!$ is that $n!$ is the ratio of the volume of the fundamental parallelepiped to the volume of the fundamental simplex.

3. Differential Forms and Pullbacks

We introduce differential k forms by analogy with how we introduced differential 1 forms. Still with M as smooth manifold, for each $p \in M$, let $\bigwedge^k(T_p^*(M))$ be the k^{th} exterior power of the cotangent space $T_p^*(M)$ at p on M. In view of Proposition 1.14d, we can regard this space as the vector space of all alternating k-linear forms on the product of k copies of $T_p(M)$ with itself. Let $\bigwedge^k T^*(M)$ be the set of all pairs (p, η) such that p is in M and η is in $\bigwedge^k(T_p^*(M))$. As with $T(M)$ and $T^*(M)$, the set $\bigwedge^k T^*(M)$ can be topologized and given a smooth manifold structure in a natural way, and then the pair consisting of $\bigwedge^k T^*(M)$ together with the projection-to-the-first-component function is called the **exterior** k **bundle** of M. Once again we do not need to know what this manifold structure is, and we shall ignore it. For $k > 0$, a **differential** k **form** on M is a function from M into $\bigwedge^k T^*(M)$ that selects, for each p in M, a member of $\bigwedge^k T^*(M)$ with first component p; in other words, a differential k form is any right inverse to the projection-to-the-first-component function under composition.[14] The integer k is called the **degree** of the differential form. The wedge product of any k differential 1 forms is an example of a differential k form. In any compatible chart (M_α, α) with $\alpha = (x_1, \dots, x_m)$, it follows from Propositions 1.9 and 1.14 that any differential k form ω has a unique local expansion

$$\omega(p) = \sum_{1 \le i_1 < \cdots < i_k \le m} a_{i_1,\dots,i_k}(p)(dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p.$$

The form[15] is said to be **smooth** on M if all the coefficient functions $p \mapsto a_{i_1,\dots,i_k}(p)$ in all such coordinate systems are smooth. As usual it is enough to

[14]In the terminology of vector bundles, a differential k form is any section of the exterior k bundle.

[15]The word **form** as a general matter refers to a scalar-valued function of several variables, always multilinear in this book but sometimes quadratic or homogenous of some other kind elsewhere in mathematics. In this book we shall follow the practice of freely using the word "form" as shorthand for "differential form" when there is no chance of ambiguity.

have smoothness relative to a family of compatible charts that covers M. We write $\Omega^k(M)$ for the real vector space of all smooth differential k forms on M. The space $\Omega^k(M)$ is a $C^\infty(M)$ module.

We extend the definition to the case $k = 0$ by saying that a **differential 0 form** on M is simply a real-valued function on M. The differential 0 form is smooth if is smooth as a real-valued function. We write $\Omega^0(M)$ for the space $C^\infty(M)$ of all smooth differential 0 forms on M.

Referring to the unique local expansion that differential forms have, we see that the wedge product of a member of $\Omega^k(M)$ and a member of $\Omega^l(M)$ is a member of $\Omega^{k+l}(M)$; in particular, the wedge product of two smooth differential forms is smooth. Sometimes we shall consider differential forms of all degrees at once, taking $\Omega(M) = \bigoplus_{k=0}^m \Omega^k(M)$. The space $\Omega(M)$ is a $C^\infty(M)$ module and an associative algebra. As a consequence of Proposition 1.10, wedge product in $\Omega(M)$ has the property that

$$\omega \wedge \sigma = (-1)^{kl} \sigma \wedge \omega$$

whenever ω is in $\Omega^k(M)$ and σ is in $\Omega^l(M)$.

The theory of differential forms makes crucial use of "pullbacks" of differential forms. The formulas for these are akin to, but more general than, certain change-of-variables formulas in advanced calculus. If $\Phi : M \to N$ is a smooth function between manifolds, we describe how Φ associates to each k form ω on N a certain k form $\Phi^*\omega$ on M that is known as the pullback of ω. In the case $k = 0$, a 0 form on N is nothing more than a real-valued function ω on N, and the **pullback** of the function ω is just the composition $\Phi^*\omega = \omega \circ \Phi$, which is a real-valued function on M.

EXAMPLE 1. Let M be a smooth manifold of dimension m, and let (M_α, α) be a compatible chart. If $(u_1, \ldots, u_m)$ are standard coordinates on $\mathbb{R}^m$, then the coordinates $(x_1, \ldots, x_m)$ on M_α given by $x_j = u_j \circ \alpha$ have the property that x_j is the pullback of u_j. In symbols, $x_j = \alpha^*(u_j)$. Similarly just before Proposition 1.5 we defined derivations $\big[\frac{\partial}{\partial x_j}\big]_p$ of $T_p(M)$ by

$$\Big[\frac{\partial f}{\partial x_j}\Big]_p = \frac{\partial (f \circ \alpha^{-1})}{\partial u_j}\Big|_{(u_1,\ldots,u_m)=(x_1(p),\ldots,x_m(p))} \qquad \text{for} \quad j = 1, \ldots, m.$$

In the present terminology, $\frac{\partial f}{\partial x_j}$ is therefore defined as the partial derivative with respect to the j^{th} variable of the pullback function $f \circ \alpha^{-1}$.

Pullback on 0 forms is $\mathbb{R}$ linear and carries smooth 0 forms to smooth 0 forms. If ω is a smooth 0 form and f is in $C^\infty(M)$, then

$$\Phi^*(f\omega) = (f\omega) \circ \Phi = f(\omega \circ \Phi) = f(\Phi^*\omega).$$

Hence pullback on 0 forms is $C^\infty(M)$ linear.

For $k \geq 1$, the notion of pullback involves the derivative of Φ. We start with the case $k = 1$. Let ω be a 1 form on N. The derivative $(D\Phi)_p$ of Φ at a point p of M is a linear function carrying the tangent space $T_p(M)$ at p into the tangent space $T_{\Phi(p)}(N)$ at the point $\Phi(p)$ in N. Thus $(D\Phi)_p(X_p)$ is in $T_{\Phi(p)}(N)$ whenever X_p is in $T_p(M)$. If we apply to this the value $\omega_{\Phi(p)}$ of the given 1 form at $\Phi(p)$, the result is a linear function from $T_p(M)$ into $\mathbb{R}$, hence a member of $T_p^*(M)$. Letting p move, we thus obtain a 1 form $\Phi^*\omega$ on M from the definition

$$(\Phi^*\omega)_p(X_p) = (\omega_{\Phi(p)})\big((D\Phi)_p(X_p)\big).$$

We take $\Phi^*\omega$ as the **pullback** of the 1 form ω from N to M.

Let us observe that the definition depends only on germs at p, specifically on $\Phi(p)$ and $(D\Phi)_p$, not otherwise on the behavior of Φ in a neighborhood of p. To underscore this point, we can introduce a more primitive notion of **pullback** as the linear function $\Phi_p^\# : T^*_{\Phi(p)}(N) \to T_p^*(M)$ defined by

$$(\Phi_p^\#(\omega_{\Phi(p)}))(X_p) = (\omega_{\Phi(p)})\big((D\Phi)_p(X_p)\big).$$

Then the pullback $\Phi^*\omega$ of a differential form ω on N is the differential form on M given by

$$(\Phi^*\omega)_p = \Phi_p^\#(\omega_{\Phi(p)}).$$

The definition of Φ^* via $\Phi^\#$ will play a role in Proposition 1.18 when we assemble a list of properties of pullback.

EXAMPLE 2. Let Φ be a smooth map from an open subset U of $\mathbb{R}^m$ into an open subset V of $\mathbb{R}^n$. Let us use the standard Euclidean coordinates $(x_1, \ldots, x_m)$ in $\mathbb{R}^m$ and $(y_1, \ldots, y_n)$ in $\mathbb{R}^n$, and let us write the entries of Φ as $(\Phi_1, \ldots, \Phi_n)$. This situation is an instance of the theory where $M = U$, $N = V$, and each of M and N is covered by a single chart. We shall compute the pullback $\Phi^*(dy_i)$ for $1 \leq i \leq n$, obtaining the result that $\Phi^*(dy_i) = d\Phi_i$. Since the set $\{(dy_1)_q, \ldots, (dy_n)_q\}$ is a basis of $T_q^*(V)$ for each q in V, we will in essence have computed the pullback of every differential 1 form on V.

By definition, $\Phi^*(dy_i)$ is the 1 form given by

$$(\Phi^*(dy_i))_p(X_p) = (dy_i)_{\Phi(p)}((D\Phi)_p(X_p)) \qquad \text{for every vector field } X \text{ on } U.$$

The right side is

$$\begin{aligned}
&= (dy_i)_{\Phi(p)}((D\Phi)_p\big(\sum_{j=1}^m (Xx_j)_p\big[\tfrac{\partial}{\partial x_j}\big]_p\big) && \text{by Proposition 1.5}\\
&= (dy_i)_{\Phi(p)}\big(\sum_{j=1}^m (Xx_j)_p(D\Phi)_p\big(\big[\tfrac{\partial}{\partial x_j}\big]_p\big)\big)
\end{aligned}$$

$$\begin{aligned}
&= (dy_i)_{\Phi(p)}\Big(\sum_{j=1}^{m}\sum_{k=1}^{n}(Xx_j)_p\big(\tfrac{\partial\Phi_k}{\partial x_j}\big)_p\big[\tfrac{\partial}{\partial y_k}\big]_{\Phi(p)}\Big) && \text{by Proposition 1.7}\\
&= \sum_{j=1}^{m}(Xx_j)_p\big(\tfrac{\partial\Phi_i}{\partial x_j}\big)_p && \text{since } (dy_i)_{\Phi(p)}\big(\big[\tfrac{\partial}{\partial y_k}\big]_{\Phi(p)}\big)=\delta_{ik}\\
&= X_p\Phi_i && \text{by Proposition 1.6}\\
&= (d\Phi_i)_p(X_p) && \text{by definition of } (d\Phi_i)_p.
\end{aligned}$$

Therefore $(\Phi^*(dy_i))_p = (d\Phi_i)_p$. In fact, the computation actually showed that $\Phi_p^{\#}(dy_i)_{\Phi(p)} = (d\Phi_i)_p$. Anyway, the final result is that $\Phi^*(dy_i) = d\Phi_i$.

EXAMPLE 3. Let $\Phi : M \to N$ be a smooth map from a smooth manifold M of dimension m to a smooth manifold N of dimension n. Let p be in M, and introduce local coordinates $(y_1, \ldots, y_n)$ about $\Phi(p)$ and $(x_1, \ldots, x_m)$ about p. The understanding is that (N_β, β) is a compatible chart about $\Phi(p)$ with $\beta = (y_1, \ldots, y_n)$ and that (M_α, α) is a compatible chart about p with M_α chosen small enough so that $\Phi(M_\alpha) \subseteq N_\beta$. We compute the pullback $\Phi^*(dy_i)$ to M_α of the 1 form dy_i on N_β for $1 \le i \le n$.

In fact, once we define $\Phi_i = y_i \circ \Phi$, both the result $\Phi^*(dy_i) = d\Phi_i$ and the computation, step by step, are the same as in Example 2. We have only to take into account the definitions of partial derivatives $\big[\frac{\partial}{\partial x_j}\big]_p$ and $\big[\frac{\partial}{\partial y_i}\big]_{\Phi(p)}$ that were given in Proposition 1.6 and its remark. Observe that as in Example 2, the computation is actually valid on the more primitive level of $\Phi^{\#}$; we shall use this observation later in this section in connection with Proposition 1.17.

Let us extend the definition of Φ^* from 1 forms to k forms for all positive integers k. We still assume that $\Phi : M \to N$ is a smooth map from a smooth manifold M of dimension m to a smooth manifold N of dimension n. For fixed p, the map

$$\omega \mapsto \Phi_p^{\#}(\omega_{\Phi(p)}) = (\Phi^*\omega)_p$$

is linear from $T^*_{\Phi(p)}(N)$ into $T^*_p(M)$, and we can regard it as a linear function ℓ from $T^*_{\Phi(p)}(N)$ into the associative algebra $\bigwedge(T^*_p(M))$ with the property that $\ell(v)^2 = 0$ for all v in $T^*_{\Phi(p)}(N)$. By Proposition 1.11b, ℓ extends uniquely to an algebra homomorphism $L : \bigwedge(T^*_{\Phi(p)}(N)) \to \bigwedge(T^*_p(M))$ sending 1 into 1 such that the diagram in Proposition 1.11b commutes. The resulting algebra homomorphism is the pullback $\Phi_p^{\#}$ on the full exterior algebra:

$$\Phi_p^{\#} : \bigwedge(T^*_{\Phi(p)}(N)) \to \bigwedge(T^*_p(M)).$$

By the nature of the construction, $\Phi_p^{\#}$ carries $\bigwedge^k(T^*_{\Phi(p)}(N))$ into $\bigwedge^k(T^*_p(M))$ for each integer $k \ge 0$. Letting p vary, we define $(\Phi^*\omega)_p = \Phi_p^{\#}(\omega_{\Phi(p)})$ for

$\omega \in \bigwedge^k(T^*_{\Phi(p)}(N))$, and we see that Φ^* carries the space $\Omega^k(N)$ of differential k forms on N into the space $\Omega^k(M)$ of differential k forms on M. For any differential k form ω on N, we call $\Phi^*\omega$ the **pullback** of ω to a differential form on M.

EXAMPLE 4. Let notation for a smooth map $\Phi : M \to N$ be as in Example 3. As a consequence of Propositions 1.9 and 1.14, any differential k form ω on N has a unique local expansion

$$\omega(\Phi(p)) = \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1,\dots,i_k}(\Phi(p))\,(dy_{i_1})_{\Phi(p)} \wedge \cdots \wedge (dy_{i_k})_{\Phi(p)}.$$

The pullback operation Φ^* is an algebra homomorphism of exterior algebras, it depends only on germs at p, it sends the function $\omega \circ \Phi$ into ω, and Example 3 shows that its value on $(dy_i)_p$ is $(d\Phi_i)_p$. Therefore

$$(\Phi^*\omega)(p) = \sum_{1 \le i_1 < \cdots < i_k \le n} (a_{i_1,\dots,i_k} \circ \Phi)(p)\,(d\Phi_{i_1})_p \wedge \cdots \wedge (d\Phi_{i_k})_p.$$

This is a perfectly fine way to write the answer for many purposes. On the other hand, if we want to involve the differentials $(dx_1)_p, \dots, (dx_m)_p$ on the right side, then we can substitute for each $(d\Phi_{i_r})_p$ and use the formula $(d\Phi_{i_r})_p = \sum_{j=1}^m \big[\frac{\partial \Phi_{i_r}}{\partial x_j}\big]_p (dx_j)_p$ to expand out the result in terms of expressions $(dx_{j_1})_p \wedge \cdots \wedge (dx_{j_k})$. Finally we can simplify. As a general rule, this computation is fairly messy. The following proposition isolates one important case in which the result is tidy.

Proposition 1.17. If Φ is a smooth map from M into N with $\dim M = \dim N = n$, if p is in M, and if $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are local coordinates about p and $\Phi(p)$, then

$$\Phi^*(dy_1 \wedge \cdots \wedge dy_n)_{\Phi(p)} = \det \left\{ \frac{\partial \Phi_i}{\partial x_j}\bigg|_p \right\}_{i,j=1,\dots,n} (dx_1)_p \wedge \cdots \wedge (dx_n)_p.$$

PROOF. Example 4 above shows that

$$\Phi^*(dy_1 \wedge \cdots \wedge dy_n)_{\Phi(p)} = (d\Phi_1)_p \wedge \cdots \wedge (d\Phi_n)_p, \tag{$*$}$$

and Proposition 1.9 shows that

$$(d\Phi_i)_p = \sum_{j=1}^m \left(\frac{\partial \Phi_i}{\partial x_j}\right)_p (dx_j)_p \qquad \text{for all } p \in M_\alpha. \tag{$**$}$$

Successively we use Corollary 1.16 and (∗∗) to calculate that

$$
\begin{aligned}
(d\Phi_1)_p \wedge \cdots \wedge (d\Phi_n)_p\Big(\Big[\frac{\partial}{\partial x_1}\Big]_p, \dots, \Big[\frac{\partial}{\partial x_n}\Big]_p\Big)& \\
&= \frac{1}{n!}\sum_{\tau\in\mathfrak{S}_n} (\operatorname{sgn}\tau)(d\Phi_1)_p\Big(\Big[\frac{\partial}{\partial x_{\tau(1)}}\Big]_p\Big)\cdots(d\Phi_n)_p\Big(\Big[\frac{\partial}{\partial x_{\tau(n)}}\Big]_p\Big) \\
&= \frac{1}{n!}\sum_{\tau\in\mathfrak{S}_n} (\operatorname{sgn}\tau)\Big(\frac{\partial\Phi_1}{\partial x_{\tau(1)}}\Big)_p\cdots\Big(\frac{\partial\Phi_n}{\partial x_{\tau(n)}}\Big)_p \\
&= \det\Big\{\frac{\partial\Phi_i}{\partial x_j}\Big|_p\Big\}_{i,j=1,\dots,n}. \qquad (\dagger)
\end{aligned}
$$

Since

$$(dx_1)_p \wedge \cdots \wedge (dx_n)_p\Big(\Big[\frac{\partial}{\partial x_1}\Big]_p, \dots, \Big[\frac{\partial}{\partial x_n}\Big]_p\Big) = 1$$

by Proposition 1.14e and since the space of alternating n-linear forms on $T_p(M)$ is 1 dimensional by Proposition 1.14a, we see from (†) that

$$(d\Phi_1)_p \wedge \cdots \wedge (d\Phi_n)_p = \det\Big\{\frac{\partial\Phi_i}{\partial x_j}\Big|_p\Big\}_{i,j=1,\dots,n}(dx_1)_p \wedge \cdots \wedge (dx_n)_p.$$

The proposition then follows from (∗). □

We conclude this section by giving another application of Example 4.

Proposition 1.18. If $\Phi : M \to N$ is a smooth map between smooth manifolds, then pullbacks to M of differential forms on N have the following properties:

(a) for $k \geq 0$, $\Phi^*(\omega_1 + \omega_2) = \Phi^*\omega_1 + \Phi^*\omega_2$ whenever ω_1 and ω_2 are differential k forms on N,
(b) for $k \geq 0$, $\Phi^*(c\omega) = c\Phi^*(\omega)$ whenever c is in $\mathbb{R}$ and ω is a differential k form on N,
(c) for $k \geq 0$, $\Phi^*\omega$ is a smooth differential k form on M whenever ω is a smooth differential k form on N,
(d) for $k \geq 0$, $\Phi^*(f\omega) = f\Phi^*\omega$ whenever ω is a differential k form on N and $f : M \to \mathbb{R}$ is a real-valued function, and $f\omega$ is smooth if f and ω are both smooth,
(e) for $k \geq 0$ and $l \geq 0$, $\Phi^*(\omega_1 \wedge \omega_2) = \Phi^*\omega_1 \wedge \Phi^*\omega_2$ whenever ω_1 is a differential k form on N and ω_2 is a differential l form on N,
(f) $(\Psi \circ \Phi)^*\omega = \Phi^*(\Psi^*\omega)$ whenever $\Psi : N \to R$ is another smooth map between smooth manifolds and ω is a differential form on R.

PROOF. Conclusions (a), (b), and (e) are immediate consequences of the fact that Φ^* can be defined in terms of $\Phi^\#$, which is an algebra homomorphism. Conclusion (c) follows from the formula for pullback given in Example 4. In (d), the equality $\Phi^*(f\omega) = f\Phi^*\omega$ reflects the linearity over $\mathbb{R}$ of $\Phi^\#$ at each point. The conclusion about smoothness follows from the formula in Example 4. Conclusion (f) follows immediately by tracking down the definitions. $\square$

4. Exterior Derivative

The exterior derivative is an extension of the operator d, which so far carries smooth functions (i.e., 0 forms) into smooth 1 forms, to an operator sending smooth forms of any degree into smooth forms of the next higher degree. The original motivation for the definition of d on differential forms of degree ≥ 1 was from its appearance in Stokes's Theorem.

Even though we do not yet have Stokes's Theorem at hand, let us elaborate a bit. Recall from elementary calculus that the Fundamental Theorem of Calculus, saying that

$$\int_0^1 f'(x)\,dx = f(1) - f(0),$$

can be motivated heuristically by the approximations

$$\int_0^1 f(x)\,dx \approx \sum_{k=1}^{n} \tfrac{1}{n}\, f'(\tfrac{k}{n}) \approx \sum_{k=1}^{n} [f(\tfrac{k}{n}) - f(\tfrac{k-1}{n})] = f(1) - f(0).$$

Here the first $\approx$ refers to the approximation of the Riemann integral by a Riemann sum, the second $\approx$ uses the Mean Value Theorem and the continuity of f', and the equality on the right takes into account the telescoping nature of the sum. The equality of the Fundamental Theorem says that the aggregate of the infinitesimal change of f over the interval equals the difference between the values of f at the endpoints.

Nineteenth century mathematicians and physicists used this kind of reasoning in three dimensions to compute the total "flux" of a fluid or radiant energy across a given curve or surface, using an integral to express the aggregate of the infinitesimal flux and an integral in one less dimension to express the total. The infinitesimal changes were written in terms of the differential operators grad, curl, and div. Later it was seen that all three operators were instances of one operator that could be generalized to more dimensions. The relevant versions of Stokes's Theorem appear in the Introduction. The operator in question was the exterior derivative d, and we shall see its relation to grad, curl, and div momentarily.

Because of this convoluted history it would be somewhat artificial to begin with simple geometrically motivated axioms for the general operator d, derive what d must be, and then prove existence and uniqueness. Instead we shall start with an explicit formula for d in the context of $\mathbb{R}^m$ and its open subsets, derive certain properties of d, and then show how d can be defined on smooth manifolds.

Thus for now we work with the smooth manifold $\mathbb{R}^m$, which has an atlas consisting of one chart $(\mathbb{R}^m, 1)$, the mapping 1 being the identity mapping on $\mathbb{R}^m$. We can safely ignore 1 for the time being. The coordinates are $(x_1, \ldots, x_m)$. We saw in Section 3 for $k \geq 0$ that $\Omega^k(\mathbb{R}^m)$ consists exactly of all differential forms

$$\omega(p) = \sum_{1 \leq i_1 < \cdots < i_k \leq m} a_{i_1,\ldots,i_k}(p)(dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p$$

with all coefficients $a_{i_1,\ldots,i_k}(p)$ in $C^\infty(\mathbb{R}^m)$ and that the expansion of Ω in this way is unique.[16] Let us abbreviate this expansion[17] in obvious fashion as

$$\omega = \sum_I a_I dx_I,$$

the sum running over all strictly increasing sequences I of k integers between 1 and m.

We define an $\mathbb{R}$ linear operator $d : \Omega^k(\mathbb{R}^m) \to \Omega^{k+1}(\mathbb{R}^m)$ by

$$d\Big(\sum_I a_I dx_I\Big) = \sum_I (da_I) \wedge dx_I.$$

This operator is called **exterior differentiation**. For the special case $k = 0$, the operator d reduces to the passage from a smooth function f to its differential df as defined in Section 1.

The sum $\Omega(\mathbb{R}^m) = \bigoplus_{k=0}^m \Omega^k(\mathbb{R}^m)$ is the space of all smooth differential forms on $\mathbb{R}^m$. It is an associative algebra over $\mathbb{R}$ and is also a $C^\infty(\mathbb{R}^m)$ module. When it is convenient to do so, we can regard d as an $\mathbb{R}$ linear function from $\Omega(\mathbb{R}^m)$ into itself.

EXAMPLE 1. In $\mathbb{R}^2$, let us write (x, y) for the coordinates. The $C^\infty(\mathbb{R}^2)$ module $\Omega^k(\mathbb{R}^2)$ is nonzero for $k = 0, 1, 2$, and a free basis in the three cases consists of $\{1\}$, $\{dx, dy\}$, and $\{dx \wedge dy\}$. On 0 forms, d acts by $df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy$, on 1 forms, d acts by

$$d(p\, dx + q\, dy) = \big(\tfrac{\partial q}{\partial x} - \tfrac{\partial p}{\partial y}\big)\,(dx \wedge dy),$$

and on 2 forms, d acts as 0.

---

[16]For $k = 0$, the only such increasing sequence $(i_1, \ldots, i_m)$ with $1 \leq i_1 < \cdots < i_k \leq m$ is the empty sequence, and in this case the wedge product $(dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p$ is understood to be the identity element of $\Omega(\mathbb{R}^m)$.

[17]The existence and uniqueness of this expansion means in the terminology of Section VIII.1 of *Basic Algebra*, that $\Omega^k(\mathbb{R}^m)$ is a free $C^\infty(\mathbb{R}^m)$ module with free basis the various dx_I.

EXAMPLE 2. In $\mathbb{R}^3$, let us write (x, y, z) for the coordinates. The $C^\infty(\mathbb{R}^3)$ module $\Omega^k(\mathbb{R}^3)$ is nonzero for $k = 0, 1, 2, 3$, and a free basis in the four cases consists of $\{1\}$, $\{dx, dy, dz\}$, $\{dy \wedge dz, dz \wedge dx, dx \wedge dy\}$, and $\{dx \wedge dy \wedge dz\}$. On 0 forms, d acts by $df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$, and we can identify this with the vector-valued function

$$\operatorname{grad} f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}.$$

On 1 forms, d acts by

$$\begin{aligned} &d(p\,dx + q\,dy + r\,dz) \\ &\quad = \left(\tfrac{\partial r}{\partial y} - \tfrac{\partial q}{\partial z}\right)(dy \wedge dz) + \left(\tfrac{\partial p}{\partial z} - \tfrac{\partial r}{\partial x}\right)(dz \wedge dx) + \left(\tfrac{\partial q}{\partial x} - \tfrac{\partial p}{\partial y}\right)(dx \wedge dy), \end{aligned}$$

and we can identify this with the vector-valued function

$$\operatorname{curl}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial y} - \frac{\partial q}{\partial z} \\ \frac{\partial p}{\partial z} - \frac{\partial r}{\partial x} \\ \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \end{pmatrix}.$$

On 2 forms, d acts by

$$d\big(a\,dy \wedge dz + b\,dz \wedge dx + c\,dx \wedge dy\big) = \left(\tfrac{\partial a}{\partial x} + \tfrac{\partial b}{\partial y} + \tfrac{\partial c}{\partial z}\right)(dx \wedge dy \wedge dz),$$

and we can identify this with the real-valued function

$$\operatorname{div}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}.$$

Lemma 1.19. If I is a strictly increasing tuple of k integers from 1 to m and J is a strictly increasing tuple of l integers from 1 to m, then

$$dx_I \wedge dx_J = \begin{cases} 0 & \text{if } I \text{ and } J \text{ have an integer in common} \\ \varepsilon\,dx_K & \text{if } I \text{ and } J \text{ have no integer in common,} \end{cases}$$

where $\varepsilon = \pm 1$ and K is the union of I and J with the terms rearranged to be strictly increasing.

PROOF. If any factor of dx_I matches a factor of dx_J, then $dx_I \wedge dx_J = 0$ by the alternating property. Otherwise we can interchange individual terms of $dx_I \wedge dx_J$ repeatedly until the indices are in increasing order. Each interchange introduces a minus sign. $\square$

Proposition 1.20. The operator d on $\Omega(\mathbb{R}^m)$ is an antiderivation in the sense that if ω is in $\Omega^k(\mathbb{R}^m)$ and σ is in $\Omega^l(\mathbb{R}^m)$, then

$$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^k \omega \wedge d\sigma.$$

PROOF. Since d is $\mathbb{R}$ linear and wedge product is $\mathbb{R}$ linear in each variable, we may assume that $\omega = f_I\, dx_I$ and $\sigma = g_J\, dx_J$, where I is a strictly increasing tuple of k integers from 1 to m and J is a a strictly increasing tuple of l integers from 1 to m. By Lemma 1.19, $dx_I \wedge dx_J = \varepsilon dx_K$ for some strictly increasing $(k+l)$-tuple of integers, where ε is 0 or ± 1. Then we have

$$\begin{aligned}
&d(f_I\, dx_I \wedge g_J\, dx_J) \\
&\quad = d(f_I g_J\, dx_I \wedge dx_J) \\
&\quad = \varepsilon d(f_I g_J\, dx_K) && \text{by Lemma 1.19} \\
&\quad = \varepsilon d(f_I g_J) \wedge dx_K && \text{by definition of } d \\
&\quad = \varepsilon g_J\, df_I \wedge dx_K + \varepsilon f_I\, dg_J \wedge dx_K && \text{by the product rule for derivatives} \\
&\quad = g_J\, df_I \wedge dx_I \wedge dx_J + f_I\, dg_J \wedge dx_I \wedge dx_J \\
&\quad = (df_I \wedge dx_I) \wedge g_J\, dx_J + (-1)^k f_I\, dx_I \wedge dg_J \wedge dx_J && \text{by Proposition 1.16} \\
&\quad = d(f_I\, dx_I) \wedge (g_J\, dx_J) + (-1)^k f_I\, dx_I \wedge d(g_J \wedge dx_J)).
\end{aligned}$$
□

Lemma 1.21. For $k \geq 1$, whenever $u_1, \dots, u_k$ are members of $C^\infty(\mathbb{R}^m)$, then $d(du_1 \wedge \cdots \wedge du_k) = 0$.

PROOF. We induct on k. For $k = 1$, the fact that $u = \sum_j \frac{\partial u}{\partial x_j}\, dx_j$ means that we have

$$d(du) = \sum_j d\big(\tfrac{\partial u}{\partial x_j}\, dx_j\big) = \sum_{j,i} \tfrac{\partial}{\partial x_i} \tfrac{\partial u}{\partial x_j}\, dx_i \wedge dx_j.$$

On the right side the terms with $i = j$ are 0 since $dx_i \wedge dx_i = 0$, and a term with $i < j$ cancels a term with $i > j$ since $\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i}$ and since $dx_i \wedge dx_j = -dx_j \wedge dx_i$. This proves the lemma for $k = 1$. Inductively assuming the result for $k = r - 1$, we use Proposition 1.20 to write

$$d(du_1 \wedge \cdots \wedge du_r) = d(du_1 \wedge \cdots du_{r-1}) \wedge du_r - (du_1 \wedge \cdots \wedge du_{r-1}) \wedge d(du_r).$$

The first term on the right side is 0 by the case $k = r - 1$ of the lemma, and the second term on the right side is 0 by the case $k = 1$ of the lemma. This completes the induction and the proof. □

Theorem 1.22. The operator d on $\Omega(\mathbb{R}^m)$ is independent of coordinates in the following sense: Let $(u_1, \ldots, u_m)$ be any other system of coordinates on $\mathbb{R}^m$ related to $(x_1, \ldots, x_m)$ by a diffeomorphism of $\mathbb{R}^m$. For each strictly increasing sequence $I = \{i_1, \ldots, i_k\}$ of k integers between 1 and m, let $du_I = du_{i_1} \wedge \cdots \wedge du_{i_k}$. If $\omega = \sum_I a_I du_I$ is the expansion of a member ω of $\Omega^k(\mathbb{R}^m)$ for $k \geq 0$ in terms of the forms du_I, then $d\omega$ is given by $d\omega = \sum_I da_I \wedge du_I$.

PROOF. We have

$$\begin{aligned} d\omega &= \sum_I d(a_I du_I) \\ &= \sum_I da_I \wedge du_I + d(du_I) \quad \text{by Proposition 1.20} \\ &= \sum_I da_I \wedge du_I \quad \text{by Lemma 1.21.} \end{aligned}$$ □

The results we have just established for $\Omega(\mathbb{R}^m)$ in Lemma 1.19 through Theorem 1.22 remain valid for any nonempty subset U of $\mathbb{R}^m$ in place of $\mathbb{R}^m$ itself, and the proofs need no changes.

Of particular interest is what Theorem 1.22 is saying for the diffeomorphism that arises between two open subsets of $\mathbb{R}^m$ when two compatible charts of an m dimensional smooth manifold M overlap. Thus let (M_α, α) and (M_β, β) be compatible charts of M with $M_\alpha \cap M_\beta \neq \varnothing$. The compatibility condition is that $\beta \circ \alpha^{-1} : \alpha(M_\alpha \cap M_\beta) \to \beta(M_\alpha \cap M_\beta)$ is smooth and so is its inverse $\alpha \circ \beta^{-1} : \beta(M_\alpha \cap M_\beta) \to \alpha(M_\alpha \cap M_\beta)$. Theorem 1.22 says that d takes the same form in the coordinate systems of these two open sets. In other words, d can be consistently defined on M_α and M_β by the usual formula $d\big(\sum_I a_I dx_I\big) = \sum_I (da_I \wedge dx_I)$, and d becomes globally defined on M. In short, d extends to an operator on the smooth manifold M, carrying $\Omega^k(M)$ to $\Omega^{k+1}(M)$ for all $k \geq 0$. Let us summarize and collect the properties of d that follow at once.

Proposition 1.23. If M is a smooth manifold, then the exterior derivative operator d is well defined on M and carries $\Omega^k(M)$ into $\Omega^{k+1}(M)$ for all integers $k \geq 0$. It has the properties that

(a) $d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^k \omega \wedge d\sigma$ whenever ω is in $\Omega^k(M)$ and σ is in $\Omega^l(N)$,
(b) $d(d\omega) = 0$ whenever ω is in $\Omega^k(M)$.

PROOF. Conclusion (a) is an instance of Proposition 1.20. For conclusion (b), it is enough to consider d^2 of a form $f\, dx_I$. Conclusion (a) gives

$$d^2(f\, dx_I) = d(df \wedge dx_I) = d^2 f \wedge dx_I - f \wedge d(dx_I).$$

The term $d^2 f$ was shown to be 0 in the proof of Lemma 1.21, and the term $d(dx_I)$ equals 0 by the conclusion of Lemma 1.21. □

We shall need one further property of the exterior derivative.

Proposition 1.24. Exterior derivative commutes with pullback in the following sense: if $\Phi : M \to N$ is a smooth map between smooth manifolds and if ω is in $\Omega^k(N)$ with $k \geq 0$, then

$$d(\Phi^*\omega) = \Phi^*(d\omega).$$

PROOF. Let $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ be local coordinates about p in M and $\Phi(p)$ in N. We begin with the case $k = 0$, for which ω reduces to a member f of $C^\infty(N)$. Then

$$\begin{aligned}
(d(\Phi^* f))_p &= d((f \circ \Phi))_p && \text{by definition of } \Phi^* \text{ on functions} \\
&= \sum_j \tfrac{\partial(f\circ\Phi)(p)}{\partial x_j}\,(dx_j)_p && \text{by Proposition 1.9} \\
&= \sum_{i,j} \tfrac{\partial f}{\partial y_i}(\Phi(p))\tfrac{\partial \Phi_i}{\partial x_j}(p)\,(dx_j)_p && \text{by the chain rule} \\
&= \sum_i \tfrac{\partial f}{\partial y_i}(\Phi(p))\,(d\Phi_i)_p && \text{by Proposition 1.9} \\
&= \sum_i \tfrac{\partial f}{\partial y_i}(\Phi(p))\Phi_p^\#((dy_i)_{\Phi(p)}) && \text{by Example 2 in Section 3} \\
&= \Phi_p^\#\big(\sum_i \tfrac{\partial f}{\partial y_i}(\Phi(p))(dy_i)_{\Phi(p)}\big) && \text{by linearity of } \Phi_p^\# \\
&= \Phi_p^\#((df)_{\Phi(p)}) && \text{by Proposition 1.9} \\
&= \Phi^*(df)_p && \text{by definition of } \Phi^* \text{ on 1 forms} \\
& && \text{in terms of } \Phi^\#,
\end{aligned}$$

and the case $k = 0$ is proved. For general $k \geq 1$, let a member

$$\omega = \sum_{1\leq i_1<\cdots<i_k\leq n} a_{i_1,\ldots,i_k}(q)\,(dy_{i_1})_q \wedge \cdots \wedge (dy_{i_k})_q$$

be given in $\Omega^k(N)$, and abbreviate it as $\omega = \sum_I a_I(q)\,(dy_I)_q$. Then

$$\begin{aligned}
d(\Phi^*\omega)_p &= d\big(\sum_I a_I(\Phi(p))\,(d\Phi_I)_p\big) && \text{by Example 4 in Section 3} \\
&= \sum_I (da_I)_{\Phi(p)} \wedge (d\Phi_I)_p && \text{by Proposition 1.23} \\
&= \Phi^*\big(\sum_I (da_I)_p \wedge (dy_I)_p\big) && \text{by definition of } \Phi^* \\
&= \Phi^*(d\omega)_p. &&
\end{aligned}$$

□

5. Smooth Partitions of Unity

A **partition of unity** on a smooth manifold is a system of nonnegative real-valued smooth functions with sum one such that each point has a neighborhood on which only finitely many of the functions are not identically zero. The existence of this neighborhood for each point is a condition that removes all questions about limits from the construction.

Historically partitions of unity arose in an effort to make more flexible the requirement that a topological space be decomposed into disjoint subsets for some purpose. Triangulations of manifolds in the subject of topology were notable examples. A different example from *Basic Real Analysis* is the rendering in Section III.13 of an annulus as the union of four quarters of an annulus in order to be able to apply Green's Theorem. In any event a decomposition into disjoint subsets is in effect a system of indicator functions[18] with sum identically one. By allowing the use of other functions with values between 0 and 1, we get less precision in distinguishing the disjoint sets, but in compensation we are allowed to insist that the functions be smooth and hence enjoy nicer analytic properties.

Theorem 1.25. Let M be a smooth manifold, let K be a nonempty compact subset, and let $\{U_i \mid 1 \leq i \leq r\}$ be a finite open cover of K. Then there exist functions f_i in $C^\infty(M)$ for $1 \leq i \leq r$, taking values between 0 and 1 such that each f_i is identically 0 off a compact subset of U_i and $\sum_{i=1}^{r} f_i$ is identically 1 on K.

REMARK. The language that is used as shorthand for the conclusion of this theorem is that the set $\{f_i\}$ of functions is a **smooth partition of unity** of M **subordinate** to the finite open cover $\{U_i\}$ of K.

We shall use the following lemma, which was proved for $\mathbb{R}^n$ as Lemma 3.15 of *Basic Real Analysis* but is valid in any locally compact separable metric space with no essential change in proof.[19]

Lemma 1.26. In a smooth manifold M,

(a) if L is a compact set and U is an open set with $L \subseteq U$, then there exists an open set V with V^{cl} compact and $L \subseteq V \subseteq V^{\text{cl}} \subseteq U$,

(b) if K is a compact set and $\{U_1, \dots, U_r\}$ is a finite open cover of K, then there exists an open cover $\{V_1, \dots, V_r\}$ of K such that V_i^{cl} is a compact subset of U_i for each i.

[18]Indicator functions are real-valued functions taking only the values 0 and 1.

[19]A sufficiently large closed ball in the proof of Lemma 3.15 is to be replaced by a member of the exhausting sequence that is sufficiently far along in the sequence.

Lemma 1.27. Let M be a smooth manifold, K be a nonempty compact subset of M, and let U be an open subset of M containing K and having compact closure in M. Then there exists a function f in $C^\infty(M)$ such that f is everywhere positive on K and f vanishes off a compact subset of U.

PROOF. For each point p of K, choose a compatible chart $(M_{\alpha,p}, \alpha_p)$ about p. Without loss of generality, we may assume that $M_{\alpha,p} \subseteq U$ for all p. Then choose an open neighborhhood $M'_{\alpha,p}$ of p whose compact closure lies in $M_{\alpha,p}$.

As p varies in K, the sets $M'_{\alpha,p}$ form an open cover of K. By compactness of K, let $\{M'_{\alpha,p_1}, \dots, M'_{\alpha,p_l}\}$ be a finite subcover. Applying Lemma 1.3 to each chart M_{α,p_j}, choose a member f_j of $C^\infty(M)$ that has values in [0, 1], that vanishes off a compact subset of M_{α,p_j}, and that is identically 1 on the compact subset $(M'_{\alpha,p_j})^{\text{cl}}$. Then the sum $f = f_1 + \cdots + f_l$ is everywhere positive on the union of the sets $(M'_{\alpha,p_j})^{\text{cl}}$, hence is everywhere positive on K. Each f_j is 0 off a compact subset of M_{α,p_j}, hence is 0 off a compact subset of U. Therefore f is 0 off a compact subset of U. □

Lemma 1.28. Let M be a smooth manifold, and let K, V, L, and U be distinct nonempty subsets with K and L compact, V and U open, and $K \subseteq V \subseteq L \subseteq U$. Then there exists a function g in $C^\infty(M)$ such that g is identically 0 on K, is everywhere positive on $L - V$, and is compactly supported in U.

PROOF. For each point p of $L-V$, choose a compatible chart $(M_{\alpha,p}, \alpha_p)$ about p. Since p is not in K, we may without loss of generality assume that $M_{\alpha,p}$ does not meet K but is contained in U. Then choose an open neighborhhood $M'_{\alpha,p}$ of p whose closure is compact and lies in $M_{\alpha,p}$. As p varies in $L-V$, the sets $M'_{\alpha,p}$ form an open cover of $L-V$. By compactness of $L-V$, let $\{M'_{\alpha,p_1}, \dots, M'_{\alpha,p_l}\}$ be a finite subcover.

Applying Lemma 1.3 to each of the charts $(M_{\alpha,p_j}, \alpha_p)$, choose a member g_j of $C^\infty(M)$ that has values in [0, 1], that vanishes off a compact subset of M_{α,p_j}, and that is identically 1 on the compact set $(M'_{\alpha,p_j})^{\text{cl}}$. Then the sum $g = g_1 + \cdots + g_l$ is everywhere positive on the union of the sets $(M'_{\alpha,p_j})^{\text{cl}}$, hence is everywhere positive on $L-V$. Each g_j is 0 off a compact subset of M_{α,p_j}, and thus g is identically 0 on K. Each g_j is compactly supported in M_{α,p_j} and therefore in U. Thus g is compactly supported in U. □

PROOF OF THEOREM 1.25. Apply Lemma 1.26b to produce an open cover $\{W_1, \dots, W_r\}$ of K such that W_i^{cl} is compact and $W_i^{\text{cl}} \subseteq U_i$ for each i with $1 \le i \le r$. Then apply it a second time to produce an open cover $\{V_1, \dots, V_r\}$ of K such that V_i^{cl} is compact and $V_i^{\text{cl}} \subseteq W_i$ for each i. Put $V = V_1 \cup \cdots \cup V_r$ and $W = W_1 \cup \cdots \cup W_r$. Lemma 1.27 produces a function $h_i \ge 0$ in $C^\infty(M)$ that is everywhere positive on V_i^{cl} and is supported in a compact subset S_i of W_i. Then $h = h_1 + \cdots + h_r$ is smooth on M, is everywhere positive on V and hence on K, is ≥ 0 everywhere, and is identically 0 off the compact subset $S = S_1 \cup \cdots \cup S_r$ of W.

Put $L = W^{\mathrm{cl}}$. Using an exhausting sequence for M, choose an open set U containing L and having compact closure in M. Application of Lemma 1.28 produces a function g in $C^\infty(M)$ that is identically 0 on K, is everywhere positive on $L - V$, and is compactly supported in U. We wish to define

$$f_i = \begin{cases} h_i/(h+g) & \text{on } W \\ 0 & \text{on } S^c. \end{cases} \tag{$*$}$$

The denominator $h+g$ is nowhere 0 on W since h is everywhere positive on V and g is everywhere positive on the superset $L - V$ of $W - V$. The two expressions for f_i in $(*)$ are both smooth on their respective open domains W and S^c, and they agree on the overlap $W \cap S^c$ because h_i is identically 0 off S. Finally f_i is defined on all of M by $(*)$ because $S \subseteq W$. Therefore $(*)$ makes f_i into a well defined member of $C^\infty(M)$.

Plainly each f_i is ≥ 0 everywhere and is identically 0 off the compact subset S_i of $W_i \subseteq U_i$. The sum $\sum_{i=1}^{r} f_i$ equals $h/(h+g)$ on W. Since $W \supseteq K$ and since g vanishes on K, $\sum_{i=1}^{r} f_i$ is identically 1 on K. Thus the functions f_i have the required properties. $\square$

Two more general results are possible, but they will not really be needed for our purposes and we shall omit their proofs. They both construct smooth partitions of unity relative to an open cover $\{U_\alpha\}$ of a smooth manifold M with an index set I whose typical member is written as α. The partitions of unity are to be "locally finite" in the sense that each point p of M has an open neighborhood on which only finitely many of the functions are not identically 0. The following two situations are of interest:

(1) The functions in the partition of unity are indexed by the same set I, and the function f_α with index α has (closed) support contained in U_α.
(2) Each function in the partition of unity has compact support in some U_α, but the index set for the functions is allowed to be larger than the set I.

The example of $M = \mathbb{R}$ with cover $\{\mathbb{R}\}$ shows that we cannot insist on maintaining the same index set I for the members of the smooth of unity if we insist also on compact support for the functions. But we can insist on either condition (1) or condition (2). That is the combined conclusion of the two more general results.

6. Orientation and Integration on Smooth Manifolds

Let M be a smooth manifold of dimension m; we emphasize that M need not be connected. Our primary interest in this section will be in integrating smooth differential forms of the top degree m on M, since the content of Stokes's Theorem in that two specific integrals of such differential forms are equal. For this purpose we require an "orientation" on M. The orientation that is chosen can affect the value of the integral. If M has an orientation, we say that M is **orientable**.

Orientation refers eventually to a left vs. right kind of decision, or to a number of such decisions. For a smooth manifold M of dimension $m \geq 0$, the notion of orientation can be defined in a number of equivalent ways,[20] and we use a definition that leads to integration as quickly as possible.

Before getting started, let us observe that any manifold is locally connected because each point has arbitraily small neighborhoods that are homeomorphic with open Euclidean balls, hence connected. Consequently the connected components of a manifold are necessarily open. Charts about a point p are allowed to meet more than one component, but it will often be helpful to think of each chart as small enough so as to be connected and therefore to lie in a single connected component of M.

Let us set aside the special case $m = 0$ for now, returning to it after some examples, since some special remarks are appropriate for it. For M of dimension $m \geq 1$, we say that M is **oriented** if an atlas $\{(M_\alpha, \alpha)\}$ of compatible charts is given with the property that the m-by-m derivative matrix of each coordinate change

$$\beta \circ \alpha^{-1} : \alpha(M_\alpha \cap M_\beta) \to \beta(M_\alpha \cap M_\beta)$$

has everywhere positive determinant. Proposition 1.30 below will show that M can be oriented if and only if M admits a nowhere vanishing differential m form. Once that proposition is in hand, an "orientation" will be defined to be an equivalence class of such forms, two such being equivalent if the one is an everywhere positive function times the other. But we do not need Proposition 1.30 and the definition of orientation yet.

A smooth manifold that is oriented by some atlas is said to be **orientable**, otherwise **not orientable**. It is often easy to show that a certain manifold is orientable. Showing that a manifold is not orientable tends to be harder. Below we shall see examples of both situations.

[20]Our definition will be given after four examples below. A frequently used definition elsewhere involves singling out an equivalence class of ordered bases of the tangent space $T_p(M)$ at each p, two such bases being equivalent if the one is carried to the other by a linear function with positive determinant. Orientability means that this process can be carried out is a way that depends continuously on p in M, and an orientation is any such choice of continuously varying equivalence classes for all points of M.

When an atlas $\{(M_\alpha, \alpha)\}$ exhibits M as oriented, a compatible chart (U, φ) is said to be **positive** relative to $\{(M_\alpha, \alpha)\}$ if the derivative matrix of $\varphi \circ \alpha^{-1}$ has everywhere positive determinant for all α. We always have the option of adjoining to the given atlas of charts for an oriented M any or all other compatible charts (U, φ) that are positive relative to all (M_α, α), and M will still be oriented.

EXAMPLE 1. M equal to $\mathbb{R}^m$. The **standard atlas** for $\mathbb{R}^m$ has just one chart in it, consisting of the open set $\mathbb{R}^m$ and the identity mapping. The standard atlas makes $\mathbb{R}^m$ oriented, and the orientation is called the **standard orientation**. A compatible chart (U, φ) consists of a nonempty open set U of $\mathbb{R}^m$ and a diffeomorphism φ of U onto an open subset of $\mathbb{R}^m$. The chart is positive in the sense of the above definition if the Jacobian matrix $\left\{\frac{\partial \varphi_i}{\partial x_j}\right\}$ has everywhere positive determinant.

EXAMPLE 2. M equal to the circle $S_1 = \{(\cos\theta, \sin\theta) \in \mathbb{R}^2 \mid \theta \in \mathbb{R}\}$. The two charts (M_1, φ_1) and (M_2, φ_2) form an atlas under the definitions

$$M_1 = \{(\cos\theta, \sin\theta) \in \mathbb{R}^2 \mid -\pi < \theta < \pi\}, \quad \varphi_1(x, y) = \theta, \quad \varphi_1(M_1) = (-\pi, \pi)$$

$$M_2 = \{(\cos\theta, \sin\theta) \in \mathbb{R}^2 \mid 0 < \theta < 2\pi\}, \quad \varphi_2(x, y) = \theta, \quad \varphi_2(M_2) = (0, 2\pi).$$

With these definitions,

$$M_1 \cap M_2 = \{(\cos\theta, \sin\theta) \in \mathbb{R}^2 \mid -\pi < \theta < 0 \quad \text{or} \quad 0 < \theta < \pi\},$$

$$(\varphi_2 \circ \varphi_1^{-1})(\theta) = \begin{cases} \theta + 2\pi & \text{for } -\pi < \theta < 0 \\ \theta & \text{for } \quad 0 < \theta < 2\pi. \end{cases}$$

The derivative matrix is everywhere the 1-by-1 matrix (1). Thus this atlas of charts exhibits M as oriented.

EXAMPLE 3. M equal to a **Möbius band** or Möbius strip. This is a noncompact 2 dimensional manifold that can be visualized in $\mathbb{R}^3$. We start from a rectangle of paper and start to bend it to be taped into the form of a cylinder, but before the cylinder is taped, we twist one end through half a turn. More precisely the Möbius band can be parametrized in $\mathbb{R}^3$ by two parameters s and t and the equations

$$\begin{aligned} x(s, t) &= (1 + \tfrac{t}{2}\cos\tfrac{s}{2})\cos s \\ y(s, t) &= (1 + \tfrac{t}{2}\cos\tfrac{s}{2})\sin s \\ z(s, t) &= \tfrac{t}{2}\sin\tfrac{s}{2}. \end{aligned}$$

Here s is to vary over a fixed half open interval $[c, c + 2\pi)$, and t is to vary over the open interval $(-1, 1)$. The equations are periodic in the t variable but with a twist:

$$(x(s + 2\pi, t), y(s + 2\pi, t), z(s + 2\pi, t)) = (x(s, -t), y(s, -t), z(s, -t)).$$

Problem 29 at the end of the chapter shows how to define a smooth manifold by means of two charts from this information, and Proposition 1.33 will lead from there to a proof that the manifold is not orientable. See Figure 1.2.

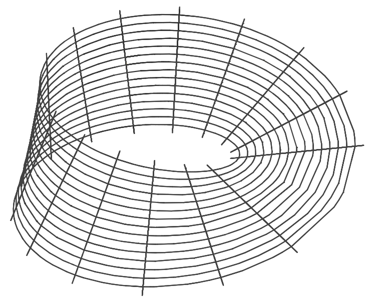

FIGURE 1.2. Möbius band.

EXAMPLE 4. The unit sphere $M = S^n$ in $\mathbb{R}^{n+1}$. This example was shown to be a smooth manifold in Section 1. It is orientable for $n \geq 1$, as will be deduced in Problem 15 at the end of the chapter. A general method applies for $n \geq 2$, and a special argument is needed for $n = 1$.

This is a good time to return to discuss orientation of a manifold M of dimension 0. In this case M is a discrete set of points, necessarily at most countable because our manifolds are assumed to be separable. The convention is that every smooth manifold of dimension 0 is orientable, being oriented by any atlas, and an **orientation** on it is the assignment of the scalar $+1$ or -1 to each of the points. This case is relevant in seeing how the general version of Stokes's Theorem reduces in one dimension to the Fundamental Theorem of Calculus, the boundary of a finite closed interval of the line being a two-point set. Vacuously every atlas exhibits a manifold of dimension 0 as oriented, and every chart is automatically positive.

Let us turn now to integration on smooth manifolds. In the special case that the manifold is a nonempty open subset U of Euclidean space $\mathbb{R}^m$, we introduce a notion of integration of smooth m forms. Any such form ω can be written as

$$\omega = F(x_1, \dots, x_m)\, dx_1 \wedge \cdots \wedge dx_m,$$

with $F(x_1, \dots, x_m)$ equal to some smooth real-valued function of the m variables on U. The integral of this m form, written as $\int_U F(x_1, \dots, x_m)\, dx_1 \wedge \cdots \wedge dx_m$, is defined simply to be the Lebesgue integral[21]

$$\int_U F(x_1, \dots, x_m)\, dx_1 \cdots dx_m$$

[21] Alternatively one can use the Riemann integral if the open set U has a sufficiently well behaved topological boundary. If the Lebesgue integral is used, there is no restriction on the topological boundary of the open set U.

with respect to Lebesgue measure. Notationally we just drop the signs $\wedge$. This integral raises some convergence questions, but we can avoid them either by assuming that ω has compact support in U or by working with the linear functional

$$f \mapsto \int_U f(x_1, \dots, x_m) F(x_1, \dots, x_m)\, dx_1 \cdots dx_m$$

defined for f in $C_{\text{com}}(U)$.

When $U = \mathbb{R}^m$, what happens to this definition of integration of m forms on $\mathbb{R}^m$ if the variables are written in a different order? For example, suppose that the positions of x_1 and x_2 are interchanged. The coefficient F of ω is unchanged, but the alternating tensor becomes $dx_2 \wedge dx_1 \wedge dx_3 \wedge \cdots \wedge dx_m$, which is the negative of $dx_1 \wedge dx_2 \wedge dx_3 \wedge \cdots \wedge dx_m$. Meanwhile the Lebesgue integral is unchanged if we replace $dx_1\, dx_2\, dx_3 \cdots dx_m$ by $dx_2\, dx_1\, dx_3 \cdots dx_m$. So we are off by a minus sign. The answer to this seeming contradiction is that orientation is playing a role in the definition of integration of an m form, a role that does not show up in the notation.[22]

Consider now any oriented smooth manifold M in the sense defined earlier in this section. The theorem below defines a notion of integration of top-degree differential forms that generalizes the one in open subsets of $\mathbb{R}^m$. After proving the theorem, we shall relate its statement to the Riesz Representation Theorem.[23]

Theorem 1.29. If ω is a smooth m form on the *oriented* smooth manifold M of dimension $m \geq 0$, then there exists a unique linear functional $f \mapsto \int_M f\omega$ on the space $C_{\text{com}}(M)$ of continuous functions of compact support on M with the property that whenever (M_α, α) is a *positive* compatible chart with local coordinates $\alpha = (x_1, \dots, x_m)$ and f is a member of $C_{\text{com}}(M)$ supported in M_α, then the value of the linear functional on any f that is compactly supported in M_α is

$$\int_M f\omega = \int_{\alpha(M_\alpha)} (f \circ \alpha^{-1})(x_1, \dots, x_m) F_\alpha(x_1, \dots, x_m)\, dx_1 \cdots dx_m, \qquad (*)$$

where α is given in local coordinates by $(x_1, \dots, x_m)$ and the local expression for ω in the local coordinates of $\alpha(M_\alpha)$ is

$$(\alpha^{-1})^*\omega = F_\alpha(x_1, \dots, x_m)\, dx_1 \wedge \cdots \wedge dx_m \qquad (**)$$

with $F_\alpha : \alpha(M_\alpha) \to \mathbb{R}$ smooth.[24] The integral on the right side of $(*)$ is understood to be an ordinary Lebesgue integral with respect to Lebesgue measure.

---

[22] We shall use notation like $\int_M f\omega$ in this text, but notation like $\int_{M,o} f\omega$ that indicates an orientation o along with M and the integrand $f\omega$, might serve as a better reminder that the orientation affects the value.

[23] Theorem 11.1 of *Basic Real Analysis*.

[24] The notation $(\alpha_i^{-1})^*\omega$ is no mystery. It refers to the pullback of ω under α^{-1}, i.e., the "push forward" of ω from its domain M_α to the open set $\alpha(M_\alpha)$ in Euclidean space. In other words, it is indeed the "local expression for ω in the local coordinates."

REMARKS.

(1) In other words the expression $\int_M f\omega$ is being defined by the right side of $(*)$. The content of the theorem is that the definition does not depend on the choice of local coordinates.

(2) Theorem 1.29 remains true if "manifold" in the statement is replaced by "manifold-with-boundary," which is a notion to be defined in Chapter II, or by "manifold-with-corners" or "Whitney manifold," which are notions to be defined in Chapter III. The proof requires no change other than an updating of the reference to the existence of a partition of unity.

(3) Once again: In the definition of $\int_M f\omega$, the notation "M" includes both M and its orientation. If the orientation is changed, then the value of the integral may change. The orientation enters the statement of the theorem in the requirement that (M_α, α) be a positive compatible chart.

(4) For our purposes the main role of having f present in the formula is to relate integration of differential m forms to Lebesgue integration in measure theory. We shall have more to say about this point after the end of the proof of the theorem. In the applications of this theorem after this section in this book, all the m forms that are involved in integration will have compact support within the set of integration, and then inclusion of f in the formula becomes a frill. Accordingly we shall tend to drop f in applications of this formula after this section.

(5) With f dropped, the formula of the theorem can be written briefly as

$$\int_M \omega = \int_{M_\alpha} \omega = \int_{\alpha(M_\alpha)} (\alpha^{-1})^*\omega$$

if ω is compactly supported in M_α. Orientations are implicit throughout the three members of this equation, the orientation on the right side being the standard orientation on Euclidean space.

PROOF. Let us first dispose of the case $m = 0$. Then ω is a 0 form, which is a real-valued function on the points of the discrete space. The integral $\int_M f\omega$ is to be interpreted as the sum over the points of the product of the value of f by the value of ω times the value of the orientation at the point, namely ± 1. This factor ± 1 is what by convention $F_\alpha(x_1, \ldots, x_m)\, dx_1 \wedge \cdots \wedge dx_m$ reduces to when $m = 0$.

For the remainder of the proof, assume that $m > 0$. Whenever f is compactly supported in M_α, then $f \circ \alpha^{-1}$ is compactly supported in $\alpha(M_\alpha)$ and the right side of $(*)$ is well defined. Thus let us define

$$\int_{M_\alpha} f\omega = \int_{\alpha(M_\alpha)} (f \circ \alpha^{-1})(x_1, \ldots, x_m) F_\alpha(x_1, \ldots, x_m)\, dx_1 \cdots dx_m.$$

This definition satisfies a certain consistency condition. To see this, suppose that f is compactly supported in an intersection $M_\alpha \cap M_\beta$. Then by our definition we have also

$$\int_{M_\beta} f\omega = \int_{\beta(M_\beta)} (f \circ \beta^{-1})(y_1, \ldots, y_m) F_\beta(y_1, \ldots, y_m)\, dy_1 \cdots dy_m. \qquad (\dagger)$$

To see that the right sides of $(*)$ and $(\dagger)$ are equal, we use the change of variables formula for multiple integrals.[25] The change of variables $y = \beta \circ \alpha^{-1}(x)$ in $(*)$ expresses $y_1, \ldots, y_m$ as functions of $x_1, \ldots, x_m$, and $(\dagger)$ therefore is

$$\begin{aligned} = \int_{\alpha(M_\alpha \cap M_\beta)} & f \circ \beta^{-1} \circ \beta \circ \alpha^{-1}(x_1, \ldots, x_m) \\ & \times F_\beta \circ \beta \circ \alpha^{-1}(x_1, \ldots, x_m) \Big| \det \Big\{ \frac{\partial y_i}{\partial x_j} \Big\}_{i,j=1,\ldots,m} \Big| \, dx_1 \cdots dx_m. \end{aligned}$$

The right side here will be equal to the right side of $(*)$ if it is shown that

$$F_\alpha \stackrel{?}{=} (F_\beta \circ \beta \circ \alpha^{-1}) \Big| \det \Big\{ \frac{\partial y_i}{\partial x_j} \Big\}_{i,j=1,\ldots,m} \Big|. \qquad (\dagger\dagger)$$

Now

$$\begin{aligned} F_\alpha \, dx_1 \wedge \cdots \wedge dx_m &= (\alpha^{-1})^* \omega && \text{from } (**) \\ &= (\beta \circ \alpha^{-1})^* (\beta^{-1})^* \omega \\ &= (\beta \circ \alpha^{-1})^* (F_\beta \, dy_1 \wedge \cdots \wedge dy_m) && \text{from } (**) \\ &= (F_\beta \circ \beta \circ \alpha^{-1}) \det \Big\{ \frac{\partial y_i}{\partial x_j} \Big\}_{i,j=1,\ldots,m} dx_1 \wedge \cdots \wedge dx_m \\ & && \text{by Proposition 1.17.} \end{aligned}$$

Thus

$$F_\alpha = (F_\beta \circ \beta \circ \alpha^{-1}) \det \Big\{ \frac{\partial y_i}{\partial x_j} \Big\}_{i,j=1,\ldots,m}. \qquad (\ddagger)$$

Since $\det \big\{ \frac{\partial y_i}{\partial x_j} \big\}_{i,j=1,\ldots,m}$ is everywhere positive, equality in $(\dagger\dagger)$ follows from $(\ddagger)$. Therefore

$$\int_{M_\alpha} f\omega = \int_{M_\beta} f\omega$$

whenever f is compactly supported in $M_\alpha \cap M_\beta$.

[25] Theorem 6.32 of *Basic Real Analysis*.

For future reference later in this section and also for use in the next chapter, we rewrite (‡) in terms of coordinates as

$$F_\alpha(y_1, \dots, y_m) = F_\beta(x_1, \dots, x_m) \det \left\{ \frac{\partial y_i}{\partial x_j} \right\}_{i,j=1,\dots,m}. \qquad (‡‡)$$

To define $\int_M f\omega$ for general f in $C_{\mathrm{com}}(M)$, we select finitely many open coordinate neighborhoods M_{α_i} that together cover the support of f, and we use Theorem 1.25 to form a smooth partition of unity $\{\psi_{\alpha_i}\}$ subordinate to the finite open cover $\{M_{\alpha_i}\}$ of the support of f. Then we can define

$$\int_M f\omega = \sum_i \int_{M_{\alpha_i}} (\psi_{\alpha_i} f)\omega. \qquad (\S)$$

Let us see that this definition is unchanged if the smooth partition of unity is changed. Indeed, suppose that $\{M_{\beta_j}\}$ is a second finite open cover of the support of f. Let $\{\phi_{\beta_j}\}$ be a smooth partition of unity subordinate to the finite open cover $\{M_{\beta_j}\}$ of the support of f. Linearity of the Lebesgue integral allows us to write the right side of (§) as

$$= \sum_i \sum_j \int_{M_{\alpha_i}} (f\psi_{\alpha_i}\phi_{\beta_j})\omega. \qquad (\S\S)$$

If $f\psi_{\alpha_i}\phi_{\beta_j}$ is not identically 0, it is supported in M_{α_i} and also in M_{β_j}. The fact that $(*)$ equals (†), which we proved above, means that we get the same result for $\int_M f\psi_{\alpha_i}\phi_{\beta_j}$ whether we treat f as a function supported in M_{α_i} or we treat it as a function supported in M_{β_j}, i.e.,

$$\int_{M_{\alpha_i}} (f\psi_{\alpha_i}\phi_{\beta_j})\omega = \int_{M_{\beta_j}} (f\psi_{\alpha_i}\phi_{\beta_j})\omega.$$

Thus (§§) is

$$= \sum_j \sum_i \int_{M_{\beta_j}} (f\psi_{\alpha_i}\phi_{\beta_j})\omega = \sum_j \int_{M_{\beta_j}} (f\phi_{\beta_j})\omega,$$

and this is the value of $\int_M f\omega$ we get by using the partition of unity $\{\phi_{\beta_j}\}$. □

When ω is fixed, it is apparent from (§) and the integral formula for $\int_{M_{\alpha_i}} (\psi_{\alpha_i} f)\omega$ that the map $f \mapsto \int_M f\omega$ is a linear functional on $C_{\mathrm{com}}(M)$. In dimension $m \geq 1$, we say that the m form ω is **everywhere positive** relative to the given atlas if each local expression $(**)$ has $F_\alpha(x_1, \dots, x_m)$ everywhere positive on $\alpha(U_\alpha)$. In dimension 0, a 0 form ω is interpreted as everywhere positive if the pointwise product of ω and the orientation is everywhere positive.

When ω is everywhere positive, the linear functional $f \mapsto \int_M f\omega$ is positive in the sense that $f \geq 0$ implies $\int_M f\omega \geq 0$. By the Riesz Representation Theorem,[26] there exists a unique (regular[27]) Borel measure $d\mu_\omega$ on M such that

$$\int_M f\omega = \int_M f(x)\,d\mu_\omega(x)$$

for all $f \in C_{\mathrm{com}}(M)$. The next two propositions tell how to create and recognize everywhere positive m forms ω.

Proposition 1.30. If an m dimensional manifold M with $m \geq 1$ admits a nowhere-vanishing m form ω, then M can be oriented so that ω is everywhere positive. Conversely if M is oriented, then M admits a nowhere-vanishing m form ω.

REMARKS. This proposition will allow us to classify the possible ways of orienting a smooth manifold m of dimension $m \geq 1$. An **orientation** of M is an equivalence class of nowhere-vanishing m forms on M, two such being equivalent if each is an everywhere positive function times the other. Indeed, the constructions in the proof below show that any nowhere-vanishing m form yields an atlas of compatible charts exhibiting M as oriented, that equivalent such forms lead to the same atlas, and that inequivalent such forms lead to distinct atlases. If a given orientation of M comes from a nowhere-vanishing m form ω_0, then the orientation that corresponds to $-\omega_0$ is called the **opposite orientation** to the given one. In Theorem 1.29, changing matters so that the oriented manifold M has the opposite orientation has the effect of multiplying $\int_M f\omega$ by -1.

PROOF. Suppose that M admits a nowhere-vanishing m form ω. Let $\{(M_\alpha, \alpha)\}$ be any atlas for M. The components of each M_α are open and cover M_α, and there is no loss of generality in assuming that each M_α is connected. For each M_α, let F_α be the function in $(**)$ of Theorem 1.29 in the local expression for ω in $\alpha(M_\alpha)$. Specifically

$$(\alpha^{-1})^*\omega = F_\alpha(x_1, \dots, x_m)\,dx_1 \wedge \cdots \wedge dx_m$$

with $F_\alpha : \alpha(M_\alpha) \to \mathbb{R}$ smooth. Since ω is nowhere vanishing and M_α is connected, F_α has constant sign on $\alpha(M_\alpha)$. If the sign is positive, we retain (M_α, α) in the atlas. If the sign is negative, we redefine[28] α by following it with the map $(x_1, x_2, \dots, x_m) \mapsto (-x_1, x_2, \dots, x_m)$, and then the redefined F_α is everywhere positive; in this case we instead include the redefined (M_α, α) in

[26] Theorem 11.1 of *Basic Real Analysis*.

[27] On any separable locally compact Hausdorff space, and in particular on any smooth manifold, all Borel measures are regular.

[28] This redefinition is possible since $m \geq 1$.

the atlas. In this way we can arrange that all F_α are everywhere positive on their domains. Referring to (‡‡) in the proof of Theorem 1.29, we see that each function $\det\left(\frac{\partial y_i}{\partial x_j}\right)$ is positive on its domain. Hence M is oriented. Since the F_α are all everywhere positive, ω is everywhere positive relative to this orientation.

Conversely suppose that M is oriented. Let $\{(M_\alpha, \alpha)\}$ be an atlas that exhibits M as oriented. For each α, define a smooth differential m form ω_α on M_α by

$$\omega_\alpha = \alpha^*(dx_1 \wedge \cdots \wedge dx_m).$$

If the intersection $M_\alpha \cap M_\beta$ is nonempty, then points in the intersection have also

$$\begin{aligned}
\omega_\beta &= \beta^*(dy_1 \wedge \cdots \wedge dy_m) \\
&= (\beta \circ \alpha^{-1} \circ \alpha)^*(dy_1 \wedge \cdots \wedge dy_m) \\
&= \alpha^*(\beta \circ \alpha^{-1})^*(dy_1 \wedge \cdots \wedge dy_m) && \text{by Proposition 1.18f} \\
&= \alpha^*(\beta \circ \alpha^{-1}) \det\left\{\frac{\partial y_i}{\partial x_j}\right\} dx_1 \wedge \cdots \wedge dx_m && \text{by Proposition 1.17} \\
&= \det\left\{\frac{\partial y_i}{\partial x_j}\right\} \alpha^*(dx_1 \wedge \cdots \wedge dx_m) \\
&= \det\left\{\frac{\partial y_i}{\partial x_j}\right\} \omega_\alpha.
\end{aligned}$$

In other words,

$$\omega_\beta(p) = \lambda_{\beta\alpha}(p)\omega_\alpha(p) \qquad (*)$$

for all points $p \in M_\alpha \cap M_\beta$ and some everywhere-positive function $\lambda_{\beta\alpha}$.

Let K be a compact subset of M to be specified. The various open sets M_α of charts cover K, we let $\{\alpha_1, \dots, \alpha_k\}$ be a finite subcover, and we use Theorem 1.25 to choose a smooth partition of unity $\{\psi_{\alpha_i}, 1 \le i \le k\}$ of M subordinate to the finite open cover $\{M_{\alpha_i}, 1 \le i \le k\}$ of K. Let $\omega = \sum_{i=1}^{k} \psi_{\alpha_i}\omega_{\alpha_i}$. The m form ω_{α_i} is nowhere-vanishing on M_{α_i}, being the pullback to M_{α_i} from $\alpha_i(M_{\alpha_i})$ of a nowhere-vanishing differential form on $\alpha_i(M_{\alpha_i})$. We can extend its domain to all of M by setting it equal to 0 off M_{α_i}, and then the product $\psi_{\alpha_i}\omega_{\alpha_i}$ is a smooth m form on M. Hence the sum ω is smooth on M.

Consider any point p in K. Since $\sum \psi_{\alpha_i} =$ is identically 1 and each ψ_{α_i} is ≥ 0, some $\psi_{\alpha_i}(p)$ is nonzero. Then also $\psi_{\alpha_i}\omega_{\alpha_i}(p)$ is nonzero. If any other index j has $\psi_{\alpha_j}(p) \ne 0$, then $(*)$ shows that $\psi_{\alpha_j}\omega_{\alpha_j}(p)$ is a positive multiple of $\psi_{\alpha_i}\omega_{\alpha_i}(p)$. Then it follows that $\omega(p)$ is not zero. In other words, the m form ω is nowhere vanishing on K.

If M were compact, we would be done at this point. In the general case we begin with the following observation: if we had specified in advance an open set U containing K, we could have arranged that ω vanishes at all points outside U simply by multiplying ω by a smooth function that is 1 on K and is 0 off U; Lemma 1.27 provides such a function. That being so, let $E_0 = \varnothing \subset E_1 \subset E_2 \subset \cdots$ be an exhausting sequence of compact subsets of M. Each set E_j is contained in the interior E_{j+1}^o of the next member of the sequence. For each $j \geq 0$, repeat the above procedure for the compact set $E_{j+1} - E_j^o$ in place of K, obtaining a smooth differential form ω_j, and arrange that the form ω_j vanishes off E_{j+2}^o. The form ω_j is nowhere vanishing on $E_{j+1} - E_j^o$, and the coefficients of the forms all have the same sign at all points where any of them is nonzero. Each point p has some j for which p is in $E_{j+1} - E_j$, and then a neighborhood of p lies in $E_{j+2}^o - E_j$. The points of that neighborhood are all outside $E_{k+1} - E_k^o$ for $k \geq j+2$ and $k+1 \leq j$. Thus that neighborhood meets at most the three sets $E_{j+2} - E_{j+1}^o$, $E_{j+1} - E_j^o$, and $E_j - E_{j-1}^o$ in the sequence. Consequently $\omega = \sum_{j=0}^{\infty} \omega_j$ is a well defined smooth m form. The form ω is nonvanishing at least at all points of $\bigcup_{j=0}^{\infty}(E_{j+1} - E_j^o)$; in other words, ω is nowhere vanishing. $\square$

Proposition 1.31. If a connected manifold M is oriented and if ω is a nowhere-vanishing smooth m form on M, then either ω is everywhere positive or $-\omega$ is everywhere positive.

REMARKS. The proposition says that the problem of finding nowhere-vanishing forms of the top degree m can be solved one connected component at a time: the manifold M is orientable if and only if each connected component is orientable, a connected component is orientable if and only if it has two equivalence classes of nowhere-vanishing m forms rather than just one, and nonvanishing m forms can be assembled for M one component at a time in arbitrary fashion.

PROOF. At each point p of M, all the functions F_α representing ω locally by means of a positive compatible chart as in $(**)$ of the statement of Theorem 1.29 have $F_\alpha(\alpha(p))$ nonzero of the same sign because of (‡‡), the nowhere-vanishing of ω, and the fact that M is oriented. Let S be the subset of M where this common sign is positive. Possibly replacing ω by $-\omega$, we may assume that S is nonempty. We show that S is open and closed. Let p be in S and let (M_{α_0}, α_0) be a positive compatible chart about p. Then $F_{\alpha_0}(\alpha_0(p)) > 0$ since p is in S, and hence $F_\alpha(\alpha(q))$ is positive at all points q in the neighborhood M_{α_0} of p for the one value α_0 of α. Since the sign is the same for the α's of all positive compatible charts, $F_\alpha(\alpha(q)) > 0$ for all α such that q is in $M_{\alpha_0} \cap M_\alpha$. Hence S is open. Let $\{p_n\}$ be a sequence in S converging to p in M, and let (M_{α_0}, α_0) be a positive compatible chart about p. Then $F_{\alpha_0}(\alpha_0(p_n)) > 0$ for large n, and hence $F_{\alpha_0}(\alpha_0(p)) \geq 0$ by continuity. Since ω is nowhere vanishing, $F_{\alpha_0}(\alpha_0(p)) > 0$.

Since the sign is the same for all positive compatible charts, $F_\alpha(\alpha(p))$ is > 0 for all α. Therefore p is in S, and S is closed. Since M is connected and S is nonempty open closed, $S = M$. $\square$

Propositions 1.30 and 1.31 together give us a better understanding of the notion of positive chart that was defined just before the four examples in this section. If M is connected and orientable, then there are exactly two possibilities for a nowhere-vanishing form of top degree m up to equivalence, and these are negatives of each other. If we fix the orientation, say in terms of the m form ω, then the positive compatible charts (M_α, α) are exactly the charts for which $(\alpha^{-1})^*\omega$ is a positive function times $dx_1 \wedge \cdots \wedge dx_m$. The set of such positive charts is an atlas.

Let us now examine the effect of mappings on orientation. Because orientation is determined by a differential m form η, we can check the effect of a mapping Φ by examining the pullback $\Phi^*\eta$. The situation is clearest in the case of a diffeomorphism.

Let M and N be oriented smooth manifolds of dimension m, and let $\Phi : M \to N$ be a diffeomorphism. If η is a nowhere-vanishing m form on N, then $\Phi^*\eta$ will be an m form on M, and Proposition 1.17 shows that it is nowhere vanishing. In fact, we can argue locally, writing η in local coordinates as the wedge product of m nowhere-vanishing 1 forms. Then Proposition 1.17 gives a local expression for $\Phi^*\eta$ as the wedge product of nowhere-vanishing 1 forms on M. Consequently the globally defined m form $\Phi^*\eta$ is nowhere vanishing.

We say that Φ is **orientation preserving** if whenever the nowhere-vanishing m form η is everywhere positive, then the nowhere-vanishing m form $\Phi^*\eta$ is everywhere positive. Similarly Φ is **orientation reversing** if whenever the nowhere-vanishing m form η is everywhere positive, then the nowhere-vanishing m form $\Phi^*\eta$ is everywhere negative. If Φ is orientation preserving, then for every positive chart (M_α, α) in the atlas for M, the chart $(\Phi(M_\alpha), \alpha \circ \Phi^{-1})$ is positive relative to the atlas for N. Consequently the atlas of compatible charts for N can be taken to be $\{(\Phi(M_\alpha), \alpha \circ \Phi^{-1})\}$. Then the change of variables formula for multiple integrals may be expressed using pullbacks as in the following proposition.

Proposition 1.32. Let M and N be oriented manifolds of dimension m, and let $\Phi : M \to N$ be an orientation-preserving diffeomorphism. If ω is any smooth m form on N, then

$$\int_N f\omega = \int_M (f \circ \Phi)\Phi^*\omega$$

for every $f \in C_{\text{com}}(N)$.

PROOF. Let the atlas for M be $\{(M_\alpha, \alpha)\}$, and take the atlas for N to be $\{(\Phi(M_\alpha), \alpha \circ \Phi^{-1})\}$. It is enough to prove the result for f compactly supported in a particular $\Phi(M_\alpha)$. For such f, Theorem 1.29 gives

$$\int_N f\omega = \int_{\alpha\circ\Phi^{-1}(\Phi(M_\alpha))} f\circ\Phi\circ\alpha^{-1}(x_1,\dots,x_m)F_\alpha(x_1,\dots,x_m)\,dx_1\cdots dx_m, \tag{$*$}$$

where F_α is the function with

$$((\alpha\circ\Phi^{-1})^{-1})^*\omega = F_\alpha(x_1,\dots,x_m)\,dx_1\wedge\cdots\wedge dx_m. \tag{$**$}$$

The function $f\circ\Phi$ is compactly supported in M_α, and Theorem 1.29 gives also

$$\int_M (f\circ\Phi)\Phi^*\omega = \int_{\alpha(M_\alpha)} f\circ\Phi\circ\alpha^{-1}(x_1,\dots,x_m)F_\alpha(x_1,\dots,x_m)\,dx_1\cdots dx_m \tag{$\dagger$}$$

since

$$(\alpha^{-1})^*\Phi^*\omega = ((\alpha\circ\Phi^{-1})^{-1})^*\omega = F_\alpha(x_1,\dots,x_m)\,dx_1\wedge\cdots\wedge dx_m$$

by $(**)$. The right sides of $(*)$ and $(\dagger)$ are equal, and hence so are the left sides. $\square$

The above discussion of diffeomorphisms and pullbacks extends to "immersions" between two smooth manifolds of the same dimension. If M and N are smooth manifolds, then an **immersion** Φ of M into N is a smooth function, not necessarily one-one, of M into N such that the derivative $D\Phi(p)$ is one-one from $T_p(M)$ into $T_{\Phi(p)}(N)$ for each point p in M. In this case when M and N have the same dimension, then the same argument as above shows for each nowhere-vanishing m form η on N that $\Phi^*\eta$ is a nowhere-vanishing m form on M. The next proposition is a consequence.

Proposition 1.33. The Möbius band of Example 3 in this section is not orientable.

PROOF. We assume that the Möbius band M has already been shown to be a manifold; this step is carried out in Problem 29 at the end of the chapter. To address orientability, we consider M as defined directly in terms of the parameters (s, t) in Example 3, rather than as a parametrically defined subset of $\mathbb{R}^3$. In the setup of the example, the subset $\mathbb{R}\times(-1, 1)$ gets mapped onto M in such a way that (s, t) maps to the same point of M as $(s+2\pi, -t)$, and hence also to the same point as $(s+4\pi, t)$. We carry out this process in two stages. In the first stage we pass from the manifold $\mathbb{R}\times(-1, 1)$ to the manifold $S^1\times(-1, 1)$ by taking the remainder modulo 4π in the s variable. Representatives of members of the image are the pairs (s, t) with $0\le s<4\pi$ and $-1<t<1$. In the second stage we identify any pair (s, t) with the pair $(s+2\pi, -t)$. This carries $S^1 \times (-1, 1)$ onto M and is a smooth 2-to-1 mapping that we call Φ; it is an immersion.

If we write h for the function that interchanges each pair (s, t) in $S^1 \times (-1, 1)$ with its mate $(s + 2\pi, -t)$, with $s + 2\pi$ understood to be adjusted by 4π if necessary so that it lies in $[0, 4\pi)$, then h is a diffeomorphism of $S^1 \times (-1, 1)$ onto itself that satisfies $h^*(ds \wedge dt) = -ds \wedge dt$. In other words, h is orientation reversing. Moreover, we have $\Phi = \Phi \circ h$. Arguing by contradiction, suppose that M is orientable. Then Proposition 1.30 supplies a nowhere-vanishing differential 2 form η on it. Passing to pullbacks from the equation $\Phi = \Phi \circ h$, we obtain

$$\Phi^*\eta = (\Phi h)^*\eta = h^*\Phi^*\eta.$$

The 2 form $\Phi^*\eta$, being nowhere vanishing, has to equal $F\,ds \wedge dt$ for some nowhere-vanishing function F on $S^1 \times (-1, 1)$. Then we are led to

$$F\,ds \wedge dt = h^*(F\,ds \wedge dt) = h^*(F)h^*(ds \wedge dt) = -(F \circ h)\,ds \wedge dt,$$

which is a contradiction since F has constant sign. □

7. Problems

1. Show that if $K_1 \subset K_2 \subset K_3 \subset \cdots$ is an exhausting sequence for a smooth manifold M and if C is a compact subset of M, then there is some j such that $C \subseteq K_j$.

2. The circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ was defined as a smooth manifold of dimension 1 in Section 1 by means of two charts (C_1, φ_1) and (C_2, φ_2), where

$$C_1 = S^1 - \{(0, +1)\} \quad \text{and} \quad \varphi_1(x, y) = \big(\tfrac{x}{1-y}\big),$$
$$C_2 = S^1 - \{(0, -1)\} \quad \text{and} \quad \varphi_2(x, y) = \big(\tfrac{x}{1+y}\big).$$

In Example 2 in Section 6, it was defined by means of two charts

$$M_1 = S^1 - \{(-1, 0)\} \quad \text{and} \quad \psi_1(\cos t, \sin t) = t \ \text{ for } \ -\pi < t < \pi,$$
$$M_2 = S^1 - \{(+1, 0)\} \quad \text{and} \quad \psi_2(\cos t, \sin t) = t \ \text{ for } \ 0 < t < 2\pi.$$

What steps need to be carried out to show that these smooth manifolds are the same? Carry out one such step.

3. Set-theoretically, the real n dimensional projective space $M = \mathbb{R}P^n$ can be defined as the result of identifying each member x of the unit sphere S^n in $\mathbb{R}^{n+1}$ with its antipodal point $-x$. Let $[x] \in \mathbb{R}P^n$ denote the class of $x \in S^n$. Do the following:
   (a) Show that $d([x],[y]) = \min\{|x-y|, |x+y|\}$ is well defined and makes $\mathbb{R}P^n$ into metric space such that the function $x \mapsto [x]$ is continuous and carries open sets to open sets.
   (b) For each j with $1 \le j \le n+1$, define
$$\alpha_j[(x_1,\ldots,x_{n+1})] = \Big(\frac{x_1}{x_j},\ldots,\frac{x_{j-1}}{x_j},\frac{x_{j+1}}{x_j},\ldots,\frac{x_{n+1}}{x_j}\Big)$$
on the domain $M_{\alpha_j} = \big\{[(x_1,\ldots,x_{n+1})] \bigm| x_j \ne 0\big\}$. Show that the system $\big\{(M_{\alpha_j},\alpha_j) \bigm| 1 \le j \le n+1\big\}$ is an atlas for $\mathbb{R}P^n$ and that the function $x \mapsto [x]$ from S^n to $\mathbb{R}P^n$ is smooth.

4. Prove that if p and q are two points in a connected smooth manifold, then there exists a diffeomorphism of the manifold mapping p to q.

5. The **product** of two manifolds M and N with respective atlases $\{(M_\alpha,\alpha)\}$ and $\{(N_\beta,\beta)\}$ is the set $M \times N$ with an atlas consisting of all charts $(M_\alpha \times N_\beta, \alpha\times\beta)$.
   (a) Show that $M\times N$ is a smooth manifold and that the projections $M \times N \to M$ and $M \times N \to N$ are smooth.
   (b) Show that if p is in M and q is in M, then the maps $i_p : N \to M \times N$ and $j_q : M \to M \times N$ given by $i_p(n) = (p,n)$ and $j_q(m) = (m,q)$ are smooth immersions.

6. Prove in $\mathbb{R}^3$ that if f is real-valued and F is vector-valued, then $\operatorname{div}\operatorname{curl} F = 0$ and that $\operatorname{curl}\operatorname{grad} f = 0$.

7. Prove by induction on the dimension that if ω is a smooth differential 1 form on $\mathbb{R}^n$ with $d\omega = 0$, then $\omega = df$ for some smooth real-valued function f defined on $\mathbb{R}^n$.

8. The proof in Problem 7 has to depend on special properties of $\mathbb{R}^n$ as the domain of ω because of the following example: Let ω be the 1 form on $\mathbb{R}^2 - \{(0,0)\}$ defined by
$$\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

Define a function θ on $\mathbb{R}^2 - \{(x,0) \mid x \geq 0\}$ by

$$\theta(x,y) = \begin{cases} \arctan y/x & \text{if } x > 0 \text{ and } y > 0 \\ \pi + \arctan y/x & \text{if } x < 0 \\ 2\pi + \arctan y/x & \text{if } x > 0 \text{ and } y < 0 \\ \pi/2 & \text{if } x = 0 \text{ and } y > 0 \\ 3\pi/2 & \text{if } x = 0 \text{ and } y < 0, \end{cases}$$

where arctan is the inverse function on $\mathbb{R}$ to tan on $(-\pi/2, \pi/2)$.

(a) Verify that $d\omega = 0$ on the domain of ω.

(b) Verify that if f is smooth on the domain of θ and if $\omega = df$ there, then f and θ have respective first partial derivatives equal on the domain of θ.

(c) Observe that a function f as in (b) has to be $f = \theta +$constant on the domain of θ and cannot extend continuously to $\mathbb{R}^2 - \{(0,0)\}$. Conclude that the equation $df = \omega$ has no smooth solution f on $\mathbb{R}^2 - \{(0,0)\}$.

9. If E and F are disjoint compact subsets of a smooth manifold M, prove that there exist functions $f \geq 0$ and $g \geq 0$ in $C^\infty_{\text{com}}(M)$ such that f is identically 1 on E and identifically 0 on F and such that g is identically 0 on E and identically 1 on F.

10. Let U be a nonempty connected open set in $\mathbb{R}^m$. Call a smooth k form ω on U **elementary** if it can be written as

$$\omega = d\varphi_1 \wedge \cdots \wedge d\varphi_k$$

for some set of k functions in $C^\infty(U)$.

(a) Prove that in this case, $\omega = d\eta$ for some smooth $k-1$ form η.

(b) Prove that any k form ω on U that can be written as

$$\omega = f_1(x_1) f_2(x_2) \cdots f_k(x_k)\, dx_1 \wedge \cdots \wedge dx_k$$

is elementary.

11. Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\Phi(r,s,t) = (r+s+t, rs+st+rt, rst)$. Compute $\Phi^*(dx \wedge dy)$ in terms of $dr \wedge ds$, $dr \wedge dt$, and $ds \wedge dt$.

Problems 12–18 introduce the notion of "contraction" (also called "interior multiplication") of a smooth differential form by a smooth vector field and use it to analyze the orientability of spheres. Let M be a smooth manifold and let X and $X_1, \ldots, X_{k-1}$ be smooth vector fields on M. If ω is a smooth differential k form on M, i.e., a member of $\Omega^k(M)$, then the **contraction** $c_X(\omega)$ of ω by X is defined pointwise on M by

$$c_X(\omega)_p\big((X_1)_p, \ldots, (X_{k-1})_p\big) = k\,\omega_p\big((X)_p, (X_1)_p, \ldots, (X_{k-1})_p\big).$$

12. Under the hypotheses above, expand $c_X(\omega)$ and the smooth vector fields within each chart by using the methods of Sections 1 and 3, and conclude that $c_X(\omega)$ is smooth and therefore c_X carries $\Omega^k(M)$ into $\Omega^{k-1}(M)$. Check also that $c_X(\omega)$ is $C^\infty(M)$ linear in the X variable.

13. (a) Prove for $k \geq 1$ and for all $\omega_1, \dots, \omega_k$ in $\Omega^1(M)$ that
$$c_X(\omega_1 \wedge \cdots \wedge \omega_k) = \sum_{i=1}^{k} (-1)^{i-1} \omega_i(X)(\omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_k).$$
(b) Deduce as a consequence that
$$c_X(\omega \wedge \eta) = c_X(\omega) \wedge \eta + (-1)^k \omega \wedge c_X(\eta)$$
if ω is in $\Omega^k(M)$ and η is in $\Omega^l(M)$.

14. Show that if $i : S \to M$ a one-one smooth immersion between manifolds and if ω is in $\Omega^k(M)$, then the member $i^*(\omega)$ of $\Omega^k(S)$ can be regarded as the restriction of ω to S. (This problem will be applied shortly to the immersion $i : S^n \to \mathbb{R}^{n+1}$.)

15. Show that if a connected smooth manifold M has an atlas with just two charts and the charts have connected intersection, then M is orientable. Deduce that the unit sphere S^n in $\mathbb{R}^{n+1}$ is orientable if $n \geq 2$. A separate argument is needed to see that S^1 is orientable. The next three problems will produce an explicit nowhere-vanishing smooth n form on S^n, which has to exist by Proposition 1.30.

16. Let $i : S^n \to \mathbb{R}^{n+1}$ be the inclusion mapping, which is a one-one smooth immersion. For any point $p = (x_1, \dots, x_{n+1})$ in S^n and its image $i(p)$ in $\mathbb{R}^{n+1}$, check in two ways that $\mathbb{R}^{n+1}$ is the the direct sum of the tangent space to S^n at p and the 1 dimensional space $\mathbb{R}p$:
(a) First check via inner products and linear algebra, using the naive geometric interpretation of the tangent space as being geometrically tangent to the sphere at p.
(b) Second check via the definitions in this chapter of notions related to "tangent space." Specifically let $r = (r_1, \dots, r_{n+1})$ be any member of $\mathbb{R}^{n+1}$ such that the dot product $p \cdot r$ equals 0. Define a smooth curve γ_r in S^n for $|t| < \epsilon$ with $\epsilon > 0$ sufficiently small by
$$\gamma_r(t) = \frac{p + tr}{|p + tr|}.$$
Observe that $f \mapsto \frac{d}{dt} f(\gamma_r(t))|_{t=0}$ defines a derivation of the space of germs of smooth functions at p on S^n and therefore is a member X_r of $T_p(S^n)$. Show that the mapping $r \mapsto X_r$ is linear in r and is one-one, hence is onto $T_p(S^n)$. By dimensionality, conclude that $T_p(\mathbb{R}^{n+1}) = i(T_p(S^n)) \oplus \mathbb{R}X$, where $X = \{X_p\}$ is the vector field with $X_p = \sum_{j=1}^{n+1} x_j \frac{d}{dx_j}$ in $T_p(\mathbb{R}^{n+1})$.

17. With i, p, and X as in the previous problem, let $\omega = dx_1 \wedge \cdots \wedge dx_{n+1}$ on $\mathbb{R}^{n+1}$. Define a smooth n form η on S^n by $\eta = i^*(c_X(\omega))$. Using the results of Problems 14 and 16, prove that the n form η on S^n is nowhere vanishing.

18. With $M = \mathbb{R}^{n+1}$ and $\omega = dx_1 \wedge \cdots \wedge dx_{n+1}$, compute $c_X(\omega)$ for $X = \sum_{j=1}^{n+1} x_j(\partial/\partial x_j)$, showing that
$$c_X(\omega) = \sum_{j=1}^{n+1}(-1)^{j-1}x_j\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{n+1}.$$
(The differential form η of the previous problem involves also an application of i^*. Problem 14 observes that this application is just a matter of restricting domains, and it is customary not to incorporate it into the explicit notation.)

Problems 19–23 treat in a more general setting the orientation question that Proposition 1.33 settled for the Möbius band. Let M be a connected smooth manifold of dimension m, and let h be a diffeomorphism of M onto itself such that $h^2 = 1$ and such that $h(x) = x$ for no x.

19. (a) For x and y in M, define $x \sim y$ if $x = y$ or $y = h(x)$. Show that $\sim$ is an equivalence relation.
    (b) Write $[x]$ for the equivalence class of x, and let N denote the set of equivalence classes. If $d(x, y)$ is a metric for M such that $d(h(x), h(y)) = d(x, y)$, prove that the formula $d_0([x], [y]) = \min\{d(x, y), d(x, h(y))\}$ defines d_0 as a metric on N in such a way that the function $x \mapsto [x]$ of M onto N is continuous and open.
    (c) Show how to define charts that make the metric space N into a smooth manifold of dimension m for which that the quotient map $h(x) = [x]$ of M onto N is smooth and is an immersion.

20. Guided by the proof of Proposition 1.33, prove that if M is oriented and h is orientation reversing, then N is not orientable.

21. Using the charts constructed in Problem 19c, prove that if M is oriented and h is orientation preserving, then N is orientable.

22. The real projective space $\mathbb{R}P^n$ is defined in Problem 3 and also arises from Problem 19 when M is taken to be the sphere S^n and h is taken to be the antipodal map $h(x) = -x$.
    (a) Show that the smooth structures defined on $\mathbb{R}P^n$ by means of Problems 3 and 19c are the same.
    (b) The sphere S^n in $\mathbb{R}^{n+1}$ is orientable for $n \geq 1$ by Problem 15, and Problems 17–18 exhibited a nowhere-vanishing n form on it. Show that the antipodal map of $h : S^n \to S^n$ is orientation reversing if n is even and is orientation preserving if n is odd.

23. Conclude from Problems 20–22 that $\mathbb{R}P^n$ is orientable if n is odd and ≥ 1 and that it is not orientable if n is even and ≥ 2.

Problems 24–30 concern graphs, smooth immersions, "submanifolds," and "embeddings." A **submanifold** of a smooth manifold M is a subset S that has a smooth manifold structure of its own for which the inclusion $i : S \to M$ is a one-one immersion. A submanifold S of the manifold M is said to be **embedded** if the inclusion is a homeomorphism of S onto its image in M, i.e., if the manifold topology for S coincides with the subspace topology.

24. Let U be a nonempty open set in $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^k$ be a continuous function, not necessarily smooth. The **graph** of f, written $\operatorname{Graph}(f)$, is the subset of $\mathbb{R}^{n+k}$ of all points $(x, f(x))$ for x in U. Make $\operatorname{Graph}(f)$ into a smooth manifold with an atlas having just one chart, defined as (U, α) with $\alpha(x) = (x, f(x))$.
    (a) Verify that the mapping of U onto $\operatorname{Graph}(f)$ given by $\alpha(x) = (x, f(x))$ is a diffeomorphism of U onto Graph(f).
    (b) Let $I : \operatorname{Graph}(f) \to U \times \mathbb{R}^k$ be the inclusion mapping, and let $p : U \times \mathbb{R}^k \to \mathbb{R}^k$ be the projection to the second coordinate. Then the composition of the maps
    $$U \xrightarrow{\ \alpha\ } \operatorname{Graph}(f) \xrightarrow{\ I\ } U \times \mathbb{R}^k \xrightarrow{\ p\ } \mathbb{R}^k$$
    is $x \mapsto f(x)$, which need not be smooth. What is going on?

25. Let U be a nonempty open set in $\mathbb{R}^n$, and let $f : U \to \mathbb{R}$ be a smooth function. Define $\varphi : U \times \mathbb{R}^k \to U \times \mathbb{R}^k$ by $\varphi(x, y) = (x, y - f(x))$.
    (a) Verify that φ is a diffeomorphism.
    (b) Observe that $\varphi(\operatorname{Graph}(f)) = \{(u, v) \in U \times \mathbb{R}^k \mid v = 0\}$. In other words, $\operatorname{Graph}(f)$ is exhibited as the level set for level $v = 0$ in $\mathbb{R}^k$ of the smooth function φ.

26. Let $\gamma : (-\pi/2 \to 3\pi/2) \to \mathbb{R}^2$ be the function given by $\gamma(t) = (\sin 2t, \cos t)$. Its image looks something like the numeral 8 and is pictured in Figure 1.3. Show that γ is a one-one immersion, that its image is compact, and that it is not a smooth embedding.

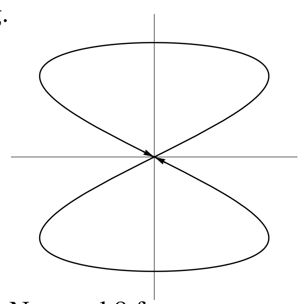

FIGURE 1.3. Numeral 8 from a one-one smooth immersion.

27. View S^1 as the set of elements in $\mathbb{C}$ of the form $e^{i\theta}$ for θ in $\mathbb{R}$, define the 2 dimensional torus T^2 to be the product $S^1 \times S^1$, and fix an irrational real number c. This problem observes that $\gamma(t) = (e^{2\pi i t}, e^{2\pi i c t})$ is a one-one immersion from $\mathbb{R}$ into T^2 but is not a smooth embedding. Its image is therefore a submanifold of T^2 but not an embedded submanifold.
    (a) Check that indeed $\gamma(t)$ is one-one and is an immersion.
    (b) Show for each $\epsilon > 0$ that some nonzero integer k has $|\gamma(k) - \gamma(0)| < \epsilon$.
    (c) Deduce that $\gamma(0)$ is a limit point of $\gamma(\mathbb{Z})$, and conclude that γ is not a homeomorphism with its image and therefore cannot be a smooth embedding.

28. This problem gives a mechanism for defining a manifold parametrically, i.e., as the image of a vector–valued function of several variables.
    (a) Let F be the smooth function from an interval of $\mathbb{R}$ into $\mathbb{R}^2$ given by $F(t) = \begin{pmatrix} x \\ y \end{pmatrix}$. Suppose that $x'(t_0) \neq 0$. Prove that the set of points $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ for t near t_0 is the embedded graph of a smooth function and is therefore an embedded submanifold in $\mathbb{R}^2$ of dimension 1.
    (b) Let U be a nonempty open subset of $\mathbb{R}^n$, let $F : U \to \mathbb{R}^k$ be a smooth function with $n < k$, and let $J(x)$ be the n-by-k Jacobian matrix of F at $x \in U$ with entries $\partial F_i / \partial x_j$. Suppose for each $x \in U$ that the rank of the matrix $J(x)$ is n, i.e., that $J(x)$ has n linearly independent columns. Use the Inverse Function Theorem to show for each x_0 in U that the set of points $F(x)$ in $\mathbb{R}^k$ for x near x_0 is an embedded submanifold in $\mathbb{R}^k$ of dimension n.

29. This problem constructs the Möbius band of Example 3 in Section 6 as a smooth 2 dimensional manifold in $\mathbb{R}^3$. (See Figure 1.2 for a picture.)
    (a) Example 3 of Section 6 explicitly defines three functions x, y, z as functions of the pair (s, t) for $-\infty < s < \infty$ and $-1 < t < 1$. Show that the Jacobian matrix of the function $(s, t) \mapsto (x, y, z)$ has rank two at every point (s, t), i.e., that the columns of the Jacobian matrix are linearly independent for each pair (s, t).
    (b) For fixed t, the functions x, y, z are periodic functions of period 4π in the variable s. Explain why this means that the function (x, y, z) of (s, t) descends to a smooth function into $\mathbb{R}^3$ with domain $M = \mathbb{R}/4\pi\mathbb{Z} \times (-1, 1)$.
    (c) Conclude that the image of the smooth function in (b) is a smooth manifold of dimension 2.

30. This problem gives a mechanism for defining a manifold implicitly, i.e., as the 0 locus of a vector-valued function of several variables.

(a) The unit circle in $\mathbb{R}^2$ is the set where $x^2 + y^2 = 1$. Define $F(x, y) = x^2 + y^2 - 1$, so that the circle is the set where $F(x, y) = 0$. The Jacobian matrix of F is

$$J(x, y) = \begin{pmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \end{pmatrix} = (2x \quad 2y).$$

Explain how the Implicit Function Theorem implies that near any point (x_0, y_0) on the circle for which $\frac{\partial F}{\partial x}(x_0, y_0) \neq 0$, the intersection of the circle with a suitable neighborhood of (x_0, y_0) is the graph of a smooth function $x = x(y)$. Why is this graph a smooth manifold?

(b) Repeat (a) for the unit sphere S^n in $\mathbb{R}^{n+1}$.

(c) More generally let U be a nonempty open subset of $\mathbb{R}^n$, let $F : U \to \mathbb{R}^n$ be a smooth function with $n > k$, and let $J(x)$ be the n-by-k Jacobian matrix of F at $x \in U$, with entries $\partial F_i / \partial x_j$. Suppose for each $x \in U$ that the rank of $J(x)$ is k, i.e., that $J(x)$ has k linearly independent columns. Use the Implicit Function Theorem to show that the subset of points $x \in U$ with $F(x) = 0$ is a smooth manifold of dimension $n - k$.

CHAPTER II

Manifolds-with-Boundary

Abstract. This chapter introduces oriented manifolds-with-boundary, obtains Stokes's Theorem for them, and shows that the classical theorems of Green, Gauss-Ostrogradsky, and Kelvin–Stokes fit into this framework.

Section 1 introduces the subject by working with ordinary oriented smooth manifolds, i.e., those oriented smooth manifolds without boundary. Stokes's Theorem for this situation reduces to a theorem about compactly supported differential forms in Euclidean space.

Section 2 introduces smooth manifolds-with-boundary of dimension m, charts being homeomorphisms from nonempty open subsets of the manifold-with-boundary onto relatively open subsets of the closed half space $\mathbb{H}^m$ of $\mathbb{R}^m$. One distinguishes manifold points and boundary points and observes that the set of manifold points yields a smooth manifold. The section defines smoothness of real-valued functions and associated objects, and for this setting, it goes through much of the same kind of development that was done for manifolds in Chapter I.

Section 3 defines orientability of a smooth manifold-with-boundary to mean orientability of the smooth manifold of manifold points. If a smooth manifold-with-boundary is orientable, then so is its boundary, and a particular choice of orientation of the boundary, known as the induced orientation, is defined so that the signs will eventually work out properly in Stokes's Theorem.

Section 4 states and proves Stokes's Theorem for oriented smooth manifolds-with-boundary, handling the case of dimension $m = 1$ separately from the case of dimension $m \geq 2$.

Section 5 examines the meaning of Stokes's Theorem in the settings that give rise to three classical integration theorems—Green's Theorem, the Divergence Theorem, and the Kelvin–Stokes Theorem—and in the setting of line integrals independent of the path.

1. Stokes's Theorem for Manifolds without Boundary

This section establishes Stokes's Theorem for oriented "manifolds without boundary," which is to say, for oriented manifolds in the sense of Chapter I.[1] It will always be assumed that the differential forms that appear in integrals have compact

[1]Terminology differs among mathematicians—whether manifolds are restricted to the kind that was defined in Chapter I or whether manifolds can have embellishments, such as some kind of attached boundary. For this section we stick to the kind that was defined in Chapter I. Starting in Section 2, we shall work with "manifolds-with-boundary," which are not necessarily manifolds in the sense of Chapter I. Instead they come with extra points satisfying some special conditions. The use of hyphens in the name "manifold-with-boundary" will be a continuing reminder that a manifold-with-boundary is not necessarily a manifold.

support, i.e., that they are 0 outside of some compact subset of the manifold. On a compact manifold this condition is automatically satisfied.

All forms of Stokes's Theorem are local theorems in the following sense: The heart of the matter is to prove the theorem in a "model space," the model space for manifolds of dimension m being $\mathbb{R}^m$. The validity of the theorem in the model space and the local nature of the result imply the validity of the theorem in any chart. Finally an orientation allows for the results for single charts to be added up with the help of a partition of unity.[2] It is as if the manifold in question is divided into pieces, and then the proof of the theorem proceeds one piece at a time and the results added. The virtue of using a partition of unity is that the borders between the pieces are smoothed out so as to avoid technical problems arising from discontinuities.[3]

Theorem 2.1. If M is a smooth oriented manifold of dimension m, then every smooth $m-1$ form ω with compact support on M has

$$\int_M d\omega = 0.$$

REMARKS. The prototype for this theorem with M noncompact is the case that $M = \mathbb{R}^1$. In this case, d is the usual differentiation operator on functions (regarded as 0 forms), and the statement comes down to the assertion that $\int_{\infty}^{\infty} f'(x)\,dx = 0$ for any f in $C^\infty_{\text{com}}(\mathbb{R}^1)$. This assertion is immediate from the Fundamental Theorem of Calculus. The prototype for this theorem with M compact is the case that M is the circle. We may then think in terms of smooth periodic functions of period 2π on the line, and the statement comes down to the assertion that any smooth f of period 2π on the line has $\int_{-\pi}^{\pi} f'(x)\,dx = 0$. Again the Fundamental Theorem of Calculus applies, giving $f(\pi) - f(-\pi) = 0$ as the value of the integral.

PROOF. We shall use the same approach in proving each version of Stokes's Theorem—the one here for manifolds, the one in Section II.4 for manifolds-with-boundary, the one in Section III.3 for manifolds-with-corners, and the one in Section III.6 for Whitney manifolds. The main step is to prove the theorem for the model space, which in this case is $\mathbb{R}^m$.

Thus we consider the special case $M = \mathbb{R}^m$ with the standard orientation. We may assume that ω is not 0. The support S of ω being compact, we choose real

[2]In other words, the only potential obstruction to extending Stokes's Theorem from a local result to a global result is the possible failure of the underlying manifold to be oriented.

[3]On the other hand, the shortcoming of using a partition of unity is that the method does not lend itself to actual computations.

numbers a_j and b_j for $1 \le j \le m$ such that all points $x = (x_1, \dots, x_m)$ of S have $a_j < x_j < b_j$ for all j. The smooth $m-1$ form ω necessarily has an expansion

$$\omega = \sum_{r=1}^{m} F_r(x_1, \dots, x_m)\, dx_1 \wedge \cdots \wedge \widehat{dx_r} \wedge \cdots \wedge dx_m,$$

the circumflex indicating a missing term. All the coefficient functions F_r are smooth and are equal to 0 off the compact set S. Then we have

$$\begin{aligned} d\omega &= \sum_{r=1}^{m} \sum_{s=1}^{m} \tfrac{\partial F_r}{\partial x_s}\, dx_s \wedge dx_1 \wedge \cdots \wedge \widehat{dx_r} \wedge \cdots \wedge dx_m \\ &= \sum_{r=1}^{m} (-1)^{r-1} \big(\tfrac{\partial F_r}{\partial x_r}\big)\, dx_1 \wedge \cdots \wedge dx_m. \end{aligned}$$

Hence the definition of integration of m forms on $\mathbb{R}^m$ gives

$$\int_{\mathbb{R}^m} d\omega = \sum_{r=1}^{m} (-1)^{r-1} \int_{\mathbb{R}^m} \big(\tfrac{\partial F_r}{\partial x_r}\big)\, dx_1 \cdots dx_m,$$

with the integral on the right side equal to an ordinary integral with respect to Lebesgue measure. Consider the r^{th} term of the sum on the right side. We can carry out the integration over $\mathbb{R}^m$ in any order, and we choose to do the x_r integration first. By the Fundamental Theorem of Calculus, that integral over x_r is

$$\begin{aligned} = \int_{\mathbb{R}^1} \big(\tfrac{\partial F_r}{\partial x_r}\big)\, dx_r &= \int_{a_r}^{b_r} \big(\tfrac{\partial F_r}{\partial x_r}\big)\, dx_r \\ &= F_r(x_1, \dots, b_r, \dots, x_m) - F_r(x_1, \dots, a_r, \dots, x_m). \end{aligned}$$

The right side is 0 because F_r vanishes off S. Therefore the r^{th} term is 0 for each r, and we conclude that

$$\int_{\mathbb{R}^m} d\omega = 0. \tag{$*$}$$

The proof is now complete for the model case $\mathbb{R}^m$.

To handle the general case, we proceed as follows: About each point p in M of the compact support S of ω, we choose a *positive* compatible chart (M_α, α). Since the sets M_{α_j} form an open cover of the compact set S, we can choose a finite subcover $\{M_{\alpha_1}, \dots, M_{\alpha_k}\}$. By Theorem 1.25 let $\{\psi_1, \dots, \psi_k\}$ be a smooth partition of unity of M subordinate to this finite open cover. For $1 \le i \le k$, the $m-1$ form $\psi_i\omega$ is compactly supported in M_{α_i}. Then we have

$$\begin{aligned} \int_M d(\psi_i\omega) = \int_{M_{\alpha_i}} d(\psi_i\omega) &= \int_{\alpha_i(M_{\alpha_i})} (\alpha^{-1})^* d(\psi_i\omega) && \text{by Theorem 1.29 and positivity} \\ &= \int_{\alpha_i(M_{\alpha_i})} d((\alpha^{-1})^*(\psi_i\omega)) && \text{by Proposition 1.24.} \end{aligned}$$

Since $\psi_i\omega$ is compactly supported on M_{α_i}, the $m-1$ form $(\alpha^{-1})^*(\psi_i\omega)$ is compactly supported in $\alpha_i(M_{\alpha_i}) \subseteq \mathbb{R}^m$. Extending this form to be 0 on the remainder of $\mathbb{R}^m$ and leaving its name unchanged, we obtain $\int_{\alpha_i(M_{\alpha_i})} d((\alpha^{-1})^*(\psi_i\omega)) = \int_{\mathbb{R}^m} d((\alpha^{-1})^*(\psi_i\omega))$. The right side is 0 by the result $(*)$ for the model case. In other words,

$$\int_M d(\psi_i\omega) = 0 \quad \text{for all } i.$$

Summing over i from 1 to k and using the fact that $\sum_i \psi_i$ is identically 1, we obtain

$$0 = \sum_i \int_M d(\psi_i\omega) = \int_M d\Big(\sum_i \psi_i\omega\Big) = \int_M d\omega,$$

and the proof of the general case is complete. $\square$

2. Elementary Properties and Examples

Smooth manifolds of dimension $m \geq 0$, as introduced in Chapter I, were defined as separable Hausdorff spaces that are locally modeled on open subsets of $\mathbb{R}^m$. In similar fashion the present section and the remainder of this chapter will work with smooth manifolds-with-boundary in dimension $m \geq 1$, which are separable Hausdorff spaces that are locally modeled on open subsets of the closed half space

$$\mathbb{H}^m = \{(x_1, \dots, x_{m-1}, x_m) \in \mathbb{R}^m \mid x_m \geq 0\}.$$

The **open subsets** of $\mathbb{H}^m$ are understood to be those subsets that are relatively open in the relative topology from $\mathbb{R}^m$. We write $\mathbb{H}^m_+$ for the interior of $\mathbb{H}^m$, namely the subset

$$\mathbb{H}^m_+ = \{(x_1, \dots, x_{m-1}, x_m) \in \mathbb{R}^m \mid x_m > 0\},$$

and we write $\partial\mathbb{H}^m$ for the boundary, namely the subset

$$\partial\mathbb{H}^m = \{(x_1, \dots, x_{m-1}, 0) \in \mathbb{R}^m\}.$$

Before coming to the formal definition of smooth manifold-with-boundary, we need to establish some definitions concerning smooth functions on $\mathbb{H}^m$. A real-valued function f defined on an open subset U of $\mathbb{H}^m$ will be said to be **smooth** if there is a smooth function F defined an open subset V of $\mathbb{R}^m$ such $U = V \cap \mathbb{H}^m$ and f is the restriction of F to U. The extending function F need not, of course, be unique.[4] With this definition of smoothness in place, we can define the space $\mathcal{C}_p(\mathbb{H}^m)$ of germs of smooth functions at points p of $\mathbb{H}^m$ and the tangent space $T_p(\mathbb{H}^m)$ at p. For $p \in \mathbb{R}^m_+$, these definitions are not new, but for $p \in \partial\mathbb{H}^m$, they are. We obtain facts about them in the same way as in Section I.1.

If U_1 and U_2 are two open subsets of $\mathbb{H}^m$, a **smooth map** $F : U_1 \to U_2$ is a continuous function whose m component functions are all smooth real-valued functions on U_1. The derivative $(DF)_p : T_p(U_1) \to T_{F(p)}(U_2)$ of the smooth map F at a point is defined just as in Section I.1. The smooth map F is a **diffeomorphism** if it is a homeomorphism with inverse $G : U_2 \to U_1$ such that the m component functions of each of F and G are smooth real-valued functions on U_1 and U_2, respectively. The composition of smooth maps is smooth, and the derivative of the composition is the composition of the derivatives. It follows that at each point the derivative of a diffeomorphism is an invertible linear function.

Although the components of a diffeomorphism F extend to be smooth functions on an open subset of $\mathbb{R}^m$ and similarly for G, no assertion is made about the extendability of the identities $F \circ G = 1$ and $G \circ F = 1$.

Let M be a separable Hausdorff topological space, and fix an integer $m \geq 1$. For purposes of working with manifolds-with-boundary, a **chart** (M_α, α) on M of **dimension** m is a homeomorphism α of a nonempty open subset M_α of M onto an open subset $\alpha(M_\alpha)$ of $\mathbb{H}^m$; the chart is said to be **about** a point p in M if p is in the domain M_α of α. When it is convenient to do so, we can restrict attention to charts (M_α, α) for which M_α is connected.

A smooth **manifold-with-boundary** of dimension m is a separable Hausdorff space M with a family $\mathcal{F}$ of charts (M_α, α) of dimension m such that

(i) any two charts (M_α, α) and (M_β, β) in $\mathcal{F}$ are (smoothly) **compatible** in the sense that $\beta \circ \alpha^{-1}$, as a mapping of the open subset $\alpha(M_\alpha \cap M_\beta)$ of $\mathbb{H}^m$ to the open subset $\beta(M_\alpha \cap M_\beta)$ of $\mathbb{H}^m$, is a diffeomorphism,

(ii) the family of compatible charts (M_α, α) is an **atlas** in the sense that the open sets M_α cover M, and

(iii) the family $\mathcal{F}$ is maximal among families of compatible charts on M.

In the presence of an understood atlas, a chart will be said to be **compatible** if it is compatible with all the members of the atlas.

As with smooth manifolds in the sense of Chapter I, any atlas of compatible charts for a smooth manifold-with-boundary can be extended in one and only one way to a maximal atlas of compatible charts. Also if U is any nonempty open subset of an m dimensional smooth manifold-with-boundary M, then U inherits the structure of a smooth manifold-with-boundary as follows: first define an atlas of U to consist of the intersection of U with all members of the atlas for M, using

[4]However, two extending functions F_1 and F_2 do have matching partial derivatives of all orders at every point of $U \cap \partial\mathbb{H}^m$, and we shall quietly make use of this fact.

the restrictions of the various functions α, and then discard occurrences of the empty set.

Later in this section we shall use charts to transfer our notions of tangent space, cotangent space, smooth function, smooth mapping, and derivative from $\mathbb{H}^m$ to general manifolds-with-boundary. But before we look at the details, let us underscore that manifolds-with-boundary are built from two distinct types of points.

The points of a smooth manifold-with-boundary divide into two distinct types—manifold points and boundary points. The **manifold points** are those points p for which there is a chart (M_α, α) about p with $\alpha(M_\alpha)$ contained in $\mathbb{H}^m_+$. The set of them will be denoted by M_+. The set M_+ is the union for all compatible charts (M_α, α) of the inverse image $\alpha^{-1}(\mathbb{H}^m_+)$, which is open in M by continuity of α. Thus M_+ is a nonempty open subset of M and is a smooth manifold of dimension m. The other points are called **boundary points**. One writes ∂M for the set of boundary points and calls ∂M the **boundary**.[5] As the complement of M_+ in M, it is a closed set.

Proposition 2.2. If M is a smooth manifold-with-boundary of dimension m, then

(a) each manifold point p has the property that every sufficiently small compatible chart (M_β, β) about p has $\beta(M_\beta)$ contained in $\mathbb{H}^m_+$,

(b) whenever (M_α, α) is a compatible chart for M, then its restriction to ∂M, namely $(M_\alpha \cap \partial M, \alpha\big|_{M_\alpha \cap \partial M})$, is a chart for ∂M of dimension $m - 1$ as long as $M_\alpha \cap \partial M$ is nonempty,

(c) whenever (M_α, α) and (M_β, β) are two compatible charts for M that meet ∂M, then the charts $(M_\alpha \cap \partial M, \alpha\big|_{M_\alpha \cap \partial M})$ and $(M_\beta \cap \partial M, \beta\big|_{M_\beta \cap \partial M})$ are compatible for ∂M,

(d) ∂M becomes a smooth manifold of dimension $m - 1$ if the atlas of charts is taken as the nonempty restrictions to ∂M of the charts in an atlas of compatible charts for M.

PROOF. For (a), suppose that (M_α, α) is a chart about p with $\alpha(M_\alpha) \subseteq \mathbb{H}^m_+$. If (M_β, β) is another chart about p, we are to show that $\beta(p)$ is in $\mathbb{H}^m_+$. Consider $\beta \circ \alpha^{-1}$ as a map from $\alpha(M_\alpha \cap M_\beta)$ to $\beta(M_\alpha \cap M_\beta)$. This map is smooth in the ordinary Euclidean sense with domain a Euclidean open set, it carries $\alpha(p)$ to $\beta(p)$, and its Jacobian determinant is nonzero at $\alpha(p)$. Therefore it carries a sufficiently small open set about $\alpha(p)$ onto a Euclidean open set about $\beta(p)$, by

[5] This notion of the boundary can differ from the set-theoretic notion, as Example 6 later in this section will show.

the Inverse Function Theorem.[6] The latter open set cannot be contained in $\mathbb{H}^m$ unless $\beta(p)$ lies in $\mathbb{H}^m_+$.

For (b), let (M_α, α) be a chart for M. In view of (a), α carries $M_\alpha \cap \partial M$ one-one onto $\alpha(M_\alpha) \cap \partial\mathbb{H}^m$. The set $M_\alpha \cap \partial M$ is relatively open in ∂M since M_α is open in M, and the set $\alpha(M_\alpha) \cap \partial\mathbb{H}^m$ is relatively open in $\partial\mathbb{H}^m$ since $\alpha(M_\alpha)$ is open in $\mathbb{H}^m$. The restrictions of α and α^{-1} are continuous. Thus $(M_\alpha \cap \partial M, \alpha\big|_{M_\alpha \cap \partial M})$ is a chart for ∂M; its dimension is $m - 1$ since the Euclidean space in question is $\partial\mathbb{H}^m$.

For (c), we are given (M_α, α) and (M_β, β) as in (b), we may assume that $M_\alpha \cap M_\beta$ is nonempty, and we are told that $\beta \circ \alpha^{-1} : \alpha(M_\alpha \cap M_\beta) \to \beta(M_\beta \cap M_\beta)$ and $\alpha \circ \beta^{-1} : \beta(M_\alpha \cap M_\beta) \to \alpha(M_\beta \cap M_\beta)$ are smooth. Put $\varphi = \beta \circ \alpha^{-1} = (\varphi_1, \dots, \varphi_m)$. The smoothness of φ means that each φ_j extends to a smooth real-valued function on an open neighborhood in $\mathbb{R}^m$ of its domain in $\mathbb{H}^m$. Then the restriction of φ_j to the intersection of that neighborhood with $\partial\mathbb{H}^m$ is certainly smooth, and hence $\beta \circ \alpha^{-1} : \alpha(M_\alpha \cap M_\beta \cap \partial M) \to \beta(M_\alpha \cap M_\beta \cap \partial M)$ is smooth. Similarly the restriction of $\alpha \circ \beta^{-1}$ is smooth. Thus the restrictions of the charts are compatible.

For (d), each nonempty restriction of a chart of dimension m for M is a chart of dimension $m - 1$ for ∂M by (b). These charts for ∂M are compatible with one another by (c), and they cover ∂M since the given charts cover M. Thus the charts for ∂M form an atlas. $\square$

EXAMPLES.

(1) Any smooth manifold of dimension ≥ 1 is a smooth manifold-with-boundary, the boundary being the empty set.

(2) In dimension 1, any interval of $\mathbb{R}$, whether open or closed or half open, is a manifold-with-boundary; the boundary consists of those endpoints that are present. The circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is a manifold of dimension 1 without boundary. The definitions allow no flexibility to declare that some of the points of the circle are boundary points and the rest are manifold points.

(3) The closed ball $B^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_1^2 + \cdots + x_m^2 \leq 1\}$ is a manifold-with-boundary of dimension m, the boundary being the sphere $S^{m-1} = \{(x_1, \dots, x_m) \mid x_1^2 + \cdots + x_m^2 = 1\}$.

(4) The closed unit square $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$ is not a manifold-with-boundary because of the presence of the corners. If the four corners are removed from the set, then the result is a manifold-with-boundary, the boundary consisting of the remaining points on the four edges.

(5) A closed figure 6 in $\mathbb{R}^2$ is not a 1 dimensional manifold-with-boundary because the point where the 6 closes on itself does not satisfy the definitions.

[6] Theorem 3.17 of *Basic Real Analysis*.

(6) If U is an open subset of $\mathbb{R}^m$ whose topological boundary $U^{\mathrm{cl}} - U$ is a smooth manifold of dimension $m-1$, then $M = (U^{\mathrm{cl}} - U) \cup U = U^{\mathrm{cl}}$ is a manifold-with-boundary of dimension m. However, if U is the union of the subsets where $|x| < 1$ and $1 < |x| < 2$ in $\mathbb{R}^2$, then the topological boundary of $U^{\mathrm{cl}} - U$, which is two circles, is different from the boundary ∂U^{cl} of the manifold, which is one circle.

(7) It is often possible to define regions in Euclidean space parametrically or implicitly and end up with a manifold-with-boundary. In $\mathbb{R}^2$, for example, the image of a curve $t \mapsto (x(t), y(t))$ in the plane is smooth if $x(t)$ and $y(t)$ are smooth and if the Implicit Function Theorem can be invoked around each point of the image to realize the set in question locally as the graph of a smooth function, i.e., if $x'(t)$ and $y'(t)$ are nowhere simultaneously vanishing. When such a curve is closed, in the sense of taking the same value at the two endpoints of the domain of definition, and when it is simple, in the sense of being one-one except for the equality of values at the endpoints, it bounds a region of the plane. The region and the curve together form a manifold-with-boundary of dimension 2.

(8) The same considerations apply in higher dimensions. It is also of interest to define smooth manifolds-with-boundary in higher dimensional spaces by using parametric equations and invoking the Implicit Function Theorem. The Möbius band, given as Example 3 in Section I.6, is a smooth manifold of dimension 2 defined parametrically in $\mathbb{R}^3$ by two parameters. As it was defined in that section, it is a noncompact smooth manifold without boundary. If the domain in the t variable is taken to be $-1 \le t \le 1$ instead of $-1 < t < 1$, then we obtain a compact smooth manifold-with-boundary of dimension 2. The boundary can be seen to be connected in this case; topologically it is a circle.

A **smooth real-valued function** $f : M \to \mathbb{R}$ on the smooth manifold-with-boundary M of dimension m is by definition a function such that for each $p \in M$ and each compatible chart (M_α, α) about p, the function $f \circ \alpha^{-1}$ is smooth as a function from the open subset $\alpha(M_\alpha)$ of $\mathbb{H}^m$ into $\mathbb{R}$. A smooth real-valued function is necessarily continuous.

To verify that a real-valued function f on the smooth manifold-with-boundary M is smooth, it is sufficient, for each point in M, to check smoothness within only one compatible chart about that point. The reason is the compatibility of the charts: if (M_α, α) and (M_β, β) are two compatible charts about p, then $f \circ \beta^{-1}$ is the composition of the smooth function $\alpha \circ \beta^{-1}$, which is smooth between open subsets of $\mathbb{H}^m$, followed by the smooth real-valued function $f \circ \alpha^{-1}$.

If E is a nonempty open subset of M, the space of smooth real-valued functions on E will be denoted by $C^\infty(E)$. The space $C^\infty(E)$ is an associative algebra over $\mathbb{R}$ under the pointwise operations, and it contains the constants. The **support** of a real-valued function is, as always, the closure of the set where the function is nonzero. We write $C^\infty_{\text{com}}(E)$ for the subset of $C^\infty(E)$ of functions whose support is a compact subset of M.

Transferring our notions of tangent space, cotangent space, smooth function, smooth mapping, and derivative from $\mathbb{H}^m$ to general manifolds-with-boundary can be done by suitably adjusting the definitions and proofs that we gave above for $\mathbb{H}^m$. Some care is appropriate, however: Although functions on $\mathbb{H}^m$ can be viewed as restrictions to $\mathbb{H}^m$ of functions on $\mathbb{R}^m$, we have no such global extended space to use with a general manifold-with-boundary. See Figure 2.1.

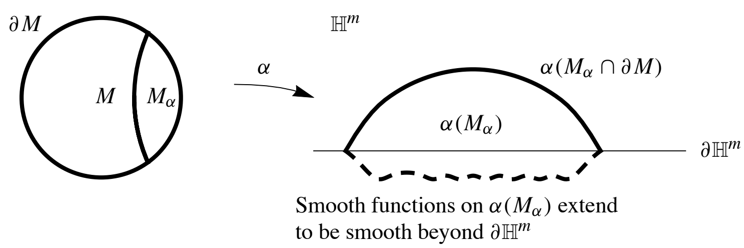

FIGURE 2.1. Nature of a chart about a boundary point.

If M is a smooth manifold-with-boundary of dimension m, we already have definitions of the tangent and cotangent spaces $T_p(M)$ and $T_p^*(M)$ at manifold points p, since M_+ is a smooth manifold. It is for boundary points p that we need to do something new. Thus let p be a boundary point. We define a **germ** at p to be an equivalence class of locally defined smooth real-valued functions in open neighborhoods of p. Arithmetic operations on germs mirror the corresponding operations on functions. The germs at p form an associative algebra $\mathcal{C}_p(M)$ over $\mathbb{R}$ with identity, just as in the manifold case. Derivations of $\mathcal{C}_p(M)$ are defined just as in the manifold case.

Observe, however, that in the case of a boundary point of $\mathbb{H}^m$, the open neighborhoods of boundary points are merely relatively open. They are somewhat one-sided and in particular are not open in $\mathbb{R}^m$.

The **tangent space** $T_p(M)$ at p is defined to be the real vector space of all derivations of $\mathcal{C}_p(M)$, just as it was in the manifold case in Section I.1. If a local coordinate system at p is given by means of a chart (M_α, α) with $\alpha = (x_1, \dots, x_m)$, then m examples of members of $T_p(M)$ are given by the derivations $\left[\frac{\partial}{\partial x_j}\right]_p$ defined by

$$\left[\frac{\partial f}{\partial x_j}\right]_p = \frac{\partial(f \circ \alpha^{-1})}{\partial u_j}\bigg|_{(u_1,\dots,u_m)=(x_1(p),\dots,x_m(p))} \qquad \text{for} \quad j = 1, \dots, m.$$

This is so even if p is a boundary point. In this case one or more of the partial derivatives may need to be interpreted as a one-sided partial derivative within

$\alpha(M_\alpha)$. Just as in the manifold case, the m derivations $\left[\frac{\partial}{\partial x_j}\right]_p$ form a vector-space basis of $T_p(M)$, regardless of whether p is a manifold point or a boundary point. Vector fields and smoothness of them are notions defined in the same way as in the manifold case.

The **derivative** DF of a smooth function $F : M \to N$ between manifolds-with-boundary is defined just as in the case of manifolds. If p is in M, then $(DF)(p)$ is a linear function from $T_p(M)$ to $T_{F(p)}(N)$. The **cotangent space** $T_p^*(M)$ is defined to be the dual of $T_p(M)$, just as in the manifold case. Differentials of smooth functions provide examples, the **differential** of f at p being defined by $(df)_p(L) = Lf$ for p in $T_p(M)$, just as in the manifold case. We can then go on to define differential 1 forms, differential k forms, and smoothness of differential forms. There are no surprises. The notion of **pullback** of a differential form is still meaningful.

The final preparatory step for working with manifolds-with-boundary is to make smooth partitions of unity be available. We begin with analogs of Lemmas 1.2 and 1.3.

Lemma 2.3. If U is a nonempty open subset of a smooth manifold-with-boundary M and if f is in $C^\infty_{\text{com}}(U)$, then the function F defined on M so as to equal f on U and to equal 0 off U is in $C^\infty_{\text{com}}(M)$ and has support contained in U.

REMARK. This is proved in the same way that Lemma 1.2 was proved for smooth manifolds. The argument makes use of the Hausdorff property of M.

Lemma 2.4. Suppose that p is a point in a smooth manifold-with-boundary M, that (M_α, α) is a compatible chart about p, and that K is a compact subset of M_α containing p. Then there is a smooth function $f : M \to \mathbb{R}$ with compact support contained in M_α such that f has values in [0, 1] and f is identically 1 on K.

PROOF. Let M have dimension m. The set $\alpha(K)$ is a compact subset of the open subset $\alpha(M_\alpha)$ of $\mathbb{H}^m$. Let U be an open subset of $\mathbb{R}^m$ such that $U \cap \mathbb{H}^m = \alpha(M_\alpha)$. Lemma 1.1 produces a function g in $C^\infty_{\text{com}}(U)$ with values in [0, 1] that identically 1 on $\alpha(K)$. Let f be the pullback of g to M_α; that is, let $f = g \circ \alpha^{-1}$. Extending f to be 0 on the complement of M_α in M and applying Lemma 2.3, we see that the extended f has the desired properties. $\square$

The notion of a **smooth partition of unity** of a manifold-with-boundary M **subordinate** to the finite open cover $\{U_i\}$ of a compact subset K of M works just as in the case of smooth manifolds without boundary. The statement is as follows.

Proposition 2.5. Let M be a smooth manifold-with-boundary, let K be a nonempty compact subset, and let $\{U_i \mid 1 \le i \le r\}$ be a finite open cover of K.

Then there exist functions f_i in $C^\infty(M)$ for $1 \le i \le r$, taking values between 0 and 1 such that each f_i is identically 0 off a compact subset of U_i and $\sum_{i=1}^{r} f_i$ is identically 1 on K.

REMARKS. Except for changes in notation, the proof is the same as for Theorem 1.25. Specifically Lemmas 1.26 and 1.27 are unchanged except that "manifold" in each of their statements is to be replaced by "manifold-with-boundary." Lemma 1.28 is unchanged except that "manifold" in its statement is to be replaced by "manifold-with-boundary" and the citation of Lemma 1.3 is to be replaced by a citation of Lemma 2.4. Then the proof of Theorem 1.25 goes through without further change.

3. Induced Orientation on the Boundary

Let M be an m dimensional manifold-with-boundary with $m \ge 1$, let ∂M be its boundary, and let M_+ be its subset of manifold points. We shall say that M is **orientable** (or **oriented**) if M_+ is orientable (or oriented). This definition is meaningful because M_+ is a smooth manifold. The point of this section is to address the question of determining an orientation on ∂M from an orientation on M_+. We postpone the case where ∂M has dimension 0, namely the case $m = 1$, until the last example of this section. Thus for now, let $m \ge 2$.

The goal of the exercise is to be able to prove Stokes's Theorem, which gives the formula $\int_{\partial M} \omega = \int_M d\omega$ for any compactly supported smooth $m-1$ form on a manifold-with-boundary M of dimension m. In the formula, the integral over M is really an integral over the set M_+ of manifold points. Both M_+ and ∂M are smooth manifolds, and they are disjoint. To make sense of the two integrals, we need orientations for M_+ and ∂M, and they need to be correlated in some way. As in the special case of Theorem 2.1, Stokes's Theorem is really a local matter in the presence of an orientation. It is therefore necessary to understand what is happening in the model space $\mathbb{H}^m$.

EXAMPLE. $M = \mathbb{H}^m$ as a manifold-with-boundary. The manifold points are those in $\mathbb{H}^m_+$, and the boundary points are those in $\partial\mathbb{H}^m$. A single chart suffices for the whole manifold-with-boundary. The atlas for M consists of this one chart; its restriction to $\partial\mathbb{H}^m$ gives us a single chart for ∂M, hence an atlas for ∂M. The subset M_+ of manifold points is $\mathbb{H}^m_+$, which is an open subset of $\mathbb{R}^m$. As such, it can inherit the standard orientation from $\mathbb{R}^m$, which is the one determined by the m form $dx_1 \wedge \cdots \wedge dx_m$. To obtain an orientation for ∂M, we cannot simply let x_m tend to 0 in the latter m form. Instead, we can proceed by declaring some nowhere-vanishing $m-1$ form on the Euclidean space $\partial\mathbb{H}^m$ to be positive. For example, we could declare that the orientation for $\partial\mathbb{H}^m$ is determined by $dx_1 \wedge \cdots \wedge dx_{m-1}$ since the variable x_m is constantly equal to 0 on ∂M. Unfortunately this choice leads to the Stokes formula only up to a sign; specifically it leads to the formula modified by the inclusion of a factor of $(-1)^m$ on one of the two sides. Another approach is to renumber the variables so that the special variable that gets put equal to 0 on ∂M is the first variable. The m form on M_+ is still the same, and the temptation is to declare that the orientation for the Euclidean space where $x_1 = 0$ is determined by $dx_2 \wedge \cdots \wedge dx_m$. As is shown in Problem 10 at the end of the chapter, this choice leads to the Stokes formula modified by a single factor of (-1) on one of the two sides. Or one could try some other way of relating ∂M to M_+ notationally, and one may expect that we are always led to a parity question, namely whether we get the desired Stokes formula or we get that formula except for a minus sign. There is a traditional procedure for orienting ∂M so that the signs come out correctly, and we tell what that is momentarily. The point is that the choice of procedure we make is rather arbitrary. The motivation is to make the signs come out right at the end, and any geometric justification is secondary.

The traditional procedure is to take an outward-pointing tangent vector to $\partial M = \mathbb{R}^m_0$ into account, considering it to be primary. The outward-pointing vector in question can be $-\big[\frac{\partial}{\partial x_m}\big]$. We follow it with the standard basis of tangent vectors to $\partial\mathbb{H}^m$, including them in their standard order and obtaining

$$(-\partial/\partial x_m, \partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_{m-1}).$$

Then we use the nowhere-vanishing differential m form $dx_1 \wedge \cdots \wedge dx_m$ on $\mathbb{H}^m$ to determine the alternating $m-1$ linear form on $\partial\mathbb{H}^m$ given by

$$(v_1, \ldots, v_{m-1}) \mapsto (dx_1 \wedge \cdots \wedge dx_m)(-\partial/\partial x_m, v_1, \ldots, v_{m-1}).$$

The value of this expression is

$$= \frac{1}{m!} \det \begin{pmatrix} dx_1\big(-\frac{\partial}{\partial x_m}\big) & dx_1(v_1) & \cdots & dx_1(v_{m-1}) \\ & \vdots & & \\ dx_{m-1}\big(-\frac{\partial}{\partial x_m}\big) & dx_{m-1}(v_1) & \cdots & dx_{m-1}(v_{m-1}) \\ dx_m\big(-\frac{\partial}{\partial x_m}\big) & dx_m(v_1) & \cdots & dx_m(v_{m-1}) \end{pmatrix}$$

$$= \frac{1}{m!} \det \begin{pmatrix} 0 & dx_1(v_1) & \cdots & dx_1(v_{m-1}) \\ \vdots & \vdots & \ddots & \\ 0 & dx_{m-1}(v_1) & \cdots & dx_{m-1}(v_{m-1}) \\ -1 & dx_m(v_1) & \cdots & dx_m(v_{m-1}) \end{pmatrix}$$

$$= \frac{(-1)^m}{m!} \det \begin{pmatrix} dx_1(v_1) & \cdots & dx_1(v_{m-1}) \\ \vdots & \ddots & \\ dx_{m-1}(v_1) & \cdots & dx_{m-1}(v_{m-1}) \end{pmatrix}$$

$$= (-1)^m m^{-1} (dx_1 \wedge \cdots \wedge dx_{m-1})(v_1, \ldots, v_{m-1}),$$

and we see that the above form is nowhere vanishing on $\partial\mathbb{H}^m$. We take its equivalence class modulo everywhere positive functions to be the **induced orientation** on $\partial\mathbb{H}^m$. In other words, the induced orientation is determined by $(-1)^m(dx_1 \wedge \cdots \wedge dx_{m-1})$ up to a positive factor.

To work with this construction in the context of a general manifold-with-boundary M of dimension m, we shall make use of a particularly nice atlas of charts for M. This atlas consists of one compatible chart about each point of M. Distinct points are allowed to correspond to the same compatible chart.

For a manifold point p, we can use any chart about p that does not meet ∂M. For a boundary point p, we start from any compatible chart (M_α, α) about p such that M_α is connected. The charts are mutually compatible and cover M by construction. Thus the result is an atlas for M consistent with its manifold-with-boundary structure. The members of the atlas that do not meet ∂M are exactly the charts about boundary points.

The following proposition uses this constructed atlas on M to extend the notion of induced orientation from $\mathbb{H}^m$ and $\partial\mathbb{H}^m$ to M and ∂M.

Proposition 2.6. Let M be an m dimensional manifold-with-boundary, and suppose that $m \geq 2$ and that M_+ is oriented. Then the orientation on M_+ induces a nowhere-vanishing $m-1$ form η on ∂M with the property that on any connected positive chart (M_α, α) about a boundary point, η is the product of a positive function and the pullback $(-1)^m \alpha^*(dx_1 \wedge \cdots \wedge dx_{m-1})$.

REMARK. The orientation on ∂M obtained in this way, i.e., the equivalence class of η modulo everywhere-positive functions, is called the **induced orientation** for ∂M. It is uniquely determined by the orientation of M_+. A version of this proposition valid for $m = 1$ will be noted after the end of the proof.

PROOF. We start from the atlas for M constructed just before the statement of Proposition 2.6. It supplies one compatible chart about each point p of M. Applying Proposition 2.2 to this atlas, we obtain an atlas of compatible charts for ∂M by restriction, provided we discard those charts that do not meet ∂M. The ones that do not meet ∂M are all the charts about manifold points. Thus our construction has the property that the restrictions to ∂M of the charts about boundary points form an atlas of compatible charts for ∂M.

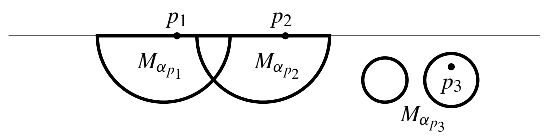

FIGURE 2.2. Some charts used in proving Proposition 2.6.

Since M_+ is orientable, Proposition 1.30 associates to each orientation of M_+ a nowhere-vanishing smooth m form on M. This m form is unique up to multiplication by a real-valued function that is everywhere positive. Fix such an ω for the given orientation of M_+. Before working with the atlas for ∂M, we shall make an adjustment to the charts about boundary points that we are including in the atlas for M.

For each p in ∂M, let $F_{\alpha_p} : \alpha_p(M_{\alpha_p}) \to \mathbb{R}$ be the smooth function such that

$$(\alpha_p^{-1})^*\omega = F_{\alpha_p}(x_1, \ldots, x_m)\, dx_1 \wedge \cdots \wedge dx_m.$$

The right side is the local expression for ω in the image in $\mathbb{H}^m$ of the chart (M_{α_p}, α_p). Since ω is nowhere vanishing and $\alpha_p(M_{\alpha_p})$ is assumed to be connected, F_{α_p} has constant sign on $\alpha_p(M_{\alpha_p})$. If the constant sign is positive, we retain (M_{α_p}, α_p) for the adjusted atlas. If the sign is negative, we take advantage of the fact that $m > 1$ to redefine α_p by following it with the linear map $T : \mathbb{R}^m \to \mathbb{R}^m$ given by $T(x_1, x_2, \ldots, x_m) = (-x_1, x_2, \ldots, x_m)$. In this case we instead include $(M_{\alpha_p}, T \circ \alpha_p)$ in the adjusted atlas. Since

$$((T \circ \alpha_p)^{-1})^*\omega = ((\alpha_p^{-1}) \circ T^{-1})^*\omega = (T^{-1})^*(\alpha_p^{-1})^*\omega$$

by Proposition 1.18f and since $T^{-1} = T$, we have

$$((T \circ \alpha_p)^{-1})^*\omega = -F_{\alpha_p}(-x_1, x_2, \ldots, x_m)\, dx_1 \wedge \cdots \wedge dx_m.$$

With this change the coefficient of $dx_1 \wedge \cdots \wedge dx_m$ is now everywhere positive on its domain $\alpha(M_{\alpha_p})$. This completes our adjustment to the charts about boundary points that we are including in our atlas.

Referring to (‡‡) in the proof of Theorem 1.29, we see that each function $\det \left\{ \dfrac{\partial y_i}{\partial x_j} \right\}_{i,j=1,\ldots,m}$ arising from a coordinate change between two of the charts about boundary points in the adjusted atlas for M is positive on its domain.

We now want to interpret this information for the atlas on ∂M. The members of the atlas for ∂M are obtained by restricting to ∂M the charts about boundary points. We want to see that this atlas for ∂M exhibits ∂M as oriented.[7] Thus

[7] It indeed exhibits ∂M as oriented, but sadly the resulting orientation on ∂M is not quite the one we seek.

suppose we have two such charts, say (M_α, α) and (M_β, β), with the property that $M_\alpha \cap M_\beta \cap \partial M$ is not empty. It is enough to consider

$$\beta \circ \alpha^{-1}\big|_{\partial \mathbb{H}^m} : \alpha(M_\alpha \cap M_\beta \cap \partial M) \to \beta(M_\alpha \cap M_\beta \cap \partial M).$$

This is the same as the full map $\beta \circ \alpha^{-1}$ but restricted to the set where $x_m = 0$, and we know from Proposition 2.2a that $y_m = 0$ for such points. Since the m^{th} coordinate function is 0, the Jacobian matrix has all entries equal to 0 in its m^{th} row except for the diagonal entry, which is $\frac{\partial y_m}{\partial x_m}$. If we write $J(x_1, \ldots, x_m)$ for the full Jacobian determinant and $J_{\partial M}(x_1, \ldots, x_{m-1})$ for the Jacobian determinant of the upper left $m-1$ by $m-1$ block, we obtain

$$J(x_1, \ldots, x_{m-1}, 0) = \frac{\partial y_m}{\partial x_m}(x_1, \ldots, x_{m-1}, 0)\, J_{\partial M}(x_1, \ldots, x_{m-1}).$$

We have seen that the left side is everywhere positive. If we can show that $\frac{\partial y_m}{\partial x_m}(x_1, \ldots, x_{m-1}, 0)$ is everywhere ≥ 0, then it will follow that every Jacobian determinant $J_{\partial M}(x_1, \ldots, x_{m-1})$ is everywhere positive, and we will have proved that the adjusted atlas exhibits ∂M as oriented. But this is just one-variable calculus: the m^{th} component of y_m is ≥ 0 for $x_m \geq 0$ and takes the value 0 at $x_m = 0$; its first derivative must then be ≥ 0 at $x_m = 0$.

Thus we have constructed an orientation on ∂M. It is not exactly the orientation we seek. We define the **induced orientation** on ∂M to be the constructed orientation if m is even and to be the opposite of the constructed orientation if m is odd. In symbols if η_1 is a nonvanishing $m-1$ form on ∂M defining the constructed orientation, we can use $\eta = (-1)^m \eta_1$ in every case as a nowhere-vanishing $m-1$ form on ∂M defining the induced orientation. □

The proof of Proposition 2.6 breaks down when $m = 1$. The smooth function $F_{\alpha_p} : \alpha(M_{\alpha_p}) \to \mathbb{R}$ of the third paragraph of the proof still makes sense. Since $m = 1$, it involves just one variable:

$$(\alpha_p^{-1})^* \omega = F_{\alpha_p}(x)\, dx.$$

It is still true that the function F_{α_p} necessarily has constant sign on $\alpha_p(M_{\alpha_p})$. But if that sign is negative, no variables are available to make use of the reflection function T. Thus we leave F_{α_p} as it is, positive or negative, and we make no adjustment to the charts about boundary points. Nevertheless, the restrictions of these charts to ∂M still exhibit ∂M as oriented. We just take the orientation of a point p to be the sign of $F_{\alpha_p}(0)$, and there is no contradiction. Following through for $m = 1$ on the sign convention in our definition above of the induced orientation for $m > 1$, we define the contribution of a point p to the value of an integral over ∂M of a 0 form in the **induced orientation** to be $-F_{\alpha_p}(0)$.

4. Stokes's Theorem for Manifolds-with-Boundary

Now we come to Stokes's Theorem, working with an oriented manifold-with-boundary M of dimension $m \geq 1$. Proposition 2.6 has shown how to obtain an induced orientation of ∂M when starting from a given orientation of M.

Theorem 2.7. Let M be an oriented manifold-with-boundary of dimension $m \geq 1$, and give its boundary ∂M the induced orientation. If ω is any smooth $m-1$ form on M of compact support, then

$$\int_{\partial M} \omega = \int_M d\omega.$$

PROOF. The model space is $\mathbb{H}^m$, and we first prove the theorem in this special case. The smooth $m-1$ form ω necessarily has an expansion

$$\omega = \sum_{r=1}^{m} F_r(x_1, \dots, x_r, \dots, x_m)\, dx_1 \wedge \cdots \wedge \widehat{dx_r} \wedge \cdots \wedge dx_m, \qquad (*)$$

the circumflex indicating a missing term. All the coefficient functions F_r are smooth and are equal to 0 off the compact support S of ω, and we have

$$\begin{aligned} d\omega &= \sum_{r=1}^{m} \sum_{s=1}^{m} \tfrac{\partial F_r}{\partial x_s}\, dx_s \wedge dx_1 \wedge \cdots \wedge \widehat{dx_r} \wedge \cdots \wedge dx_m \\ &= \sum_{r=1}^{m} (-1)^{r-1} \big(\tfrac{\partial F_r}{\partial x_r}\big)\, dx_1 \wedge \cdots \wedge dx_m. \qquad (**) \end{aligned}$$

The support of ω being compact, we choose real numbers a_j and b_j for $1 \leq j \leq m-1$ and a real number c such that all points $x = (x_1, \dots, x_m)$ of S have $a_j < x_j < b_j$ for $1 \leq j \leq m-1$ and $0 \leq x_m < c$.

On $\partial\mathbb{H}^m$, where x_m is identically 0 and dx_m is in effect 0, all the terms of $(*)$ drop out except for the term with $r = m$, and thus

$$\omega = F_m(x_1, \dots, x_{m-1}, 0)\, dx_1 \wedge \cdots \wedge dx_{m-1}.$$

We want to integrate ω over $\partial\mathbb{H}^m$, taking into account the orientation. Suppose for the moment that $m \geq 2$. Since $(-1)^m dx_1 \wedge \cdots \wedge dx_{m-1}$ is positively oriented on $\partial\mathbb{H}^m$ in the induced orientation, application of Theorem 1.29 gives

$$\begin{aligned} \int_{\partial\mathbb{H}^m} \omega &= (-1)^m \int_{\partial\mathbb{H}^m} F_m(x_1, \dots, x_{m-1}, 0)\, dx_1 \wedge \cdots \wedge dx_{m-1} \\ &= (-1)^m \int_{a_1}^{b_1} \cdots \int_{a_{m-1}}^{b_{m-1}} F_m(x_1, \dots, x_{m-1}, 0)\, dx_{m-1} \cdots dx_1. \qquad (\dagger) \end{aligned}$$

Special remarks are appropriate here when $m = 1$. Then $\partial\mathbb{H}^m$ reduces to a single point (0), and the differential form ω is the scalar $F_1(0)$ attached to that point. When it comes to integration, our convention about the induced orientation at the point 0 for this value of m, which was spelled out in the final paragraph of the previous section, is that we multiply $F_1(0)$ by -1. That is

$$\int_{\partial\mathbb{H}^1} \omega = -F_1(0),$$

and thus (†) still holds for $m = 1$. Therefore we take (†) as known for all $m \geq 1$.

Meanwhile, $d\omega$ is given on $\mathbb{H}^m$ for all $m \geq 1$ by (∗∗), and application of Theorem 1.29 and its Remark (2) yields

$$\begin{aligned}\int_{\mathbb{H}^m} d\omega &= \int_{\mathbb{H}^m} \sum_{r=1}^m (-1)^{r-1}\big(\tfrac{\partial F_r}{\partial x_r}\big)\, dx_1 \wedge \cdots \wedge dx_m \\ &= \sum_{r=1}^m (-1)^{r-1} \int_{\mathbb{H}^m} \big(\tfrac{\partial F_r}{\partial x_r}\big)\, dx_1 \cdots dx_m \end{aligned} \qquad (††)$$

with the integral on the right side equal to an ordinary integral with respect to Lebesgue measure. On the right side of (††) in the r^{th} term, the integration is taking place in m variables, and we choose to do the integration in the variable x_r first. Since the set of integration is a product set, the inside integral in the case that $r < m$ is

$$\int_{a_r}^{b_r} \big(\tfrac{\partial F_r}{\partial x_r}\big)\, dx_r.$$

The function F_r in its dependence on x_r is smooth and compactly supported in the open interval $a_r < x_r < b_r$. By the Fundamental Theorem of Calculus, the integral in the variable x_r is 0. For $r = m$, the inside integral on the right side of (††) is

$$\int_0^c \big(\tfrac{\partial F_m}{\partial x_m}\big)\, dx_m = F_m(x_1, \dots, x_{m-1}, c) - F_m(x_1, \dots, x_{m-1}, 0)$$

with $F_m(x_1, \dots, x_{m-1}, c) = 0$ by the support condition. Therefore the whole expression (††) boils down to

$$= (-1)^m \int_{a_1}^{b_1} \cdots \int_{a_{m-1}}^{b_{m-1}} F_m(x_1, \dots, x_{m-1}, 0)\, dx_{m-1} \cdots dx_1,$$

which exactly equals (†). We conclude that

$$\int_{\partial\mathbb{H}^m} \omega = \int_{\mathbb{H}^m} d\omega. \qquad (‡)$$

in the special case that $M = \partial\mathbb{H}^m$.

To handle the general case, we proceed in the same manner as in the proof of Theorem 2.1: About each point p in M of the compact support S of ω, we choose a *positive* compatible chart (M_α, α). Since the sets M_{α_j} form an open cover of the compact set S, we can choose a finite subcover $\{M_{\alpha_1}, \dots, M_{\alpha_k}\}$. By Proposition 2.5 (instead of Theorem 1.25), let $\{\psi_1, \dots, \psi_k\}$ be a smooth partition of unity of M subordinate to this finite open cover. For $1 \le i \le k$, the $m-1$ form $\psi_i\omega$ is compactly supported in M_{α_i}, and the $m-1$ form $(\alpha^{-1})^*(\psi_i\omega)$ is compactly supported in $\alpha_i(M_{\alpha_i}) \subseteq \mathbb{H}^m$. Let us extend it to all of $\mathbb{H}^m$ by setting it equal to 0 off $\alpha_i(M_{\alpha_i}) \subseteq \mathbb{H}^m$, leaving its name unchanged. Then

$$\begin{aligned}
\int_M d(\psi_i\omega) = \int_{M_{\alpha_i}} d(\psi_i\omega) &= \int_{\alpha_i(M_{\alpha_i})} (\alpha_i^{-1})^*(d(\psi_i\omega)) && \text{by Theorem 1.29}\\
&= \int_{\mathbb{H}^m} (\alpha_i^{-1})^*(d(\psi_i\omega)) && \text{after extension by 0}\\
&= \int_{\mathbb{H}^m} d((\alpha_i^{-1})^*(\psi_i\omega)) && \text{by Proposition 1.24}\\
&= \int_{\partial\mathbb{H}^m} (\alpha_i^{-1})^*(\psi_i\omega) && \text{by (‡)}\\
&= \int_{\partial M_{\alpha_i}} \psi_i\omega = \int_{\partial M} \psi_i\omega && \text{by Theorem 1.29.}
\end{aligned}$$

Summing over i from 1 to k and using the fact that $\sum_{i=1}^{k} \psi_i$ is identically 1, we obtain

$$\int_M d\omega = \sum_{i=1}^{k} \int_M d(\psi_i\omega) = \int_{\partial M} \Big(\sum_{i=1}^{k} \psi_i\omega\Big) = \int_{\partial M} \omega,$$

and the proof of the general case is complete. □

EXAMPLE. Suppose M is the closed bounded interval $[a, b]$ of $\mathbb{R}^1$. This manifold-with-boundary has $M_+ = (a, b)$ and $\partial M = \{a, b\}$. We can cover M with an atlas of two charts about boundary points, namely (M_α, α) and (M_β, β) with

$$\begin{aligned}
M_\alpha &= [a, b), &\qquad \alpha(x) &= x - a, &\qquad \alpha(M_\alpha) &= [0, b-a),\\
M_\beta &= (a, b], &\qquad \beta(x) &= b - x, &\qquad \beta(M_\beta) &= [0, b-a).
\end{aligned}$$

Proposition 2.6 does not modify this atlas before we restrict matters to ∂M. Fix a function ψ in $C^\infty_{\text{com}}([a, b))$ taking values in $[0, 1]$ and having $\psi(x) = 1$ near

$x = a$ and $\psi(x) = 0$ near $x = b$. Our partition of unity on $[a, b]$ can be taken as $\{\psi, 1 - \psi\}$.

The orientation on M_+ is given by the nowhere-vanishing 1 form $\eta = dx$; in other words, it is right to left as usual. The pullbacks of dx into the charts are given by $(\alpha^{-1})^*(dx) = dx$ and $(\beta^{-1})^*(dx) = -dx$. So the functions F_α and F_β in the proof of Proposition 2.6 are given by $F_\alpha = 1$ and $F_\beta = -1$. The induced orientation is obtained by multiplying these by -1 since m is odd. Thus we get a total contribution of -1 at the point a of ∂M and $+1$ at the point b.

Let ω be the 0 form $x \mapsto f(x)$, f being a C^∞ function on $[a, b]$. Then

$$\int_{\partial M} \omega = \int_{\partial M} \psi\omega + \int_{\partial M} (1 - \psi)\omega = -f(a) + f(b).$$

Meanwhile,

$$\int_M d\omega = \int_{(a,b)} \tfrac{d}{dx}(f(x))\, dx = \int_a^b f'(x)\, dx.$$

So the conclusion of the theorem, namely $\int_{\partial M} \omega = \int_M d\omega$, reduces to the Fundamental Theorem of Calculus, namely $f(b) - f(a) = \int_a^b f'(x)\, dx$.

5. Classical Vector Analysis

Vector analysis refers to the part of multivariable calculus in $\mathbb{R}^2$ and $\mathbb{R}^3$ that uses techniques of geometry and calculus to provide tools helpful in applications to science and engineering. These tools include

- vector notation for $\mathbb{R}^2$ and $\mathbb{R}^3$,
- dot product and vector product,
- various differentiation operators and notation for them,
- double and triple integrals,
- descriptions of curves and surfaces,
- tangent vectors and normal vectors,
- line integrals and surface integrals,
- arc length and surface area,
- Green's Theorem, the Divergence Theorem, and the Kelvin–Stokes Theorem.

Most of what is said in this section will be simply alternative notation for notions that are already known. Though the mathematics will not be new, it is important for good communication to be able to recognize this alternative notation and to be able to work with it.

The emphasis in this book has been and continues to be on the unified treatment of Green's Theorem, the Divergence Theorem, and the Kelvin–Stokes Theorem that was introduced by E. Cartan. The Cartan approach has led us to a certain amount of differential geometry (tangents vectors, tangent spaces, vector fields, differentials, derivatives, differential forms, pullbacks, integration of top-degree differential forms, and so on), and at the same time it has avoided making essential use of orthogonality anywhere. It avoided using orthogonality by being cast as a theory about general smooth manifolds with no additional structure. Some of the tools in vector analysis do make considerable use of orthogonality inherited from Euclidean space, and we shall touch on these tools only lightly.

In vector analysis one works with scalar-valued functions and vector-valued functions in two or three dimensions. To fix the ideas, let us work with dimension 3; dimension 2 van be handled by simply taking the third component to be 0. For a mathematician, $\mathbb{R}^3$ is often viewed as a space of column vectors of real numbers, such as $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Mathematicians allow themselves to write such a column vector horizontally with commas, as in (a, b, c), to save space, and the subject of vector analysis sometimes uses the same horizontal notation. Often in vector analysis, however, a different kind of abbreviation appears, in which one gives names to the three standard basis vectors, namely $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Then the vector (a, b, c) becomes $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Sometimes a vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = (a, b, c)$ is associated geometrically with an arrow that extends from the origin $(0, 0, 0)$ to the point (a, b, c). Vectors are often written with boldface symbols as in $\mathbf{v} = a\mathbf{i}+b\mathbf{j}+c\mathbf{k} = (a, b, c)$, or with symbols having arrows over them as in $\overrightarrow{v} = a\,\overrightarrow{i} + b\,\overrightarrow{j} + c\,\overrightarrow{k}$, but we shall usually not follow either of these conventions.

Functions into $\mathbb{R}^3$ with domain in $\mathbb{R}^1$ are called **curves** if they satisfy some additional properties, functions with domain in $\mathbb{R}^2$ are called **surfaces** if they satisfy some additional properties, and functions with domain in $\mathbb{R}^3$ are called "vector fields" in this language. The case of values in $\mathbb{R}^3$ requires some special comments. Such a function F, carrying part of $\mathbb{R}^3$ into $\mathbb{R}^3$, can be viewed conveniently as a system of arrows in $\mathbb{R}^3$, one such arrow having its tail is at the point (a, b, c) of the domain and having its tip is at $(a, b, c) + F(a, b, c)$. The arrows show how each point (a, b, c) moves under the function. Just as with vectors themselves, vector-valued functions are sometimes denoted by boldface symbols or by symbols with arrows over them, but we shall often not follow this convention.

Dot product in $\mathbb{R}^3$ is familiar to the reader from elementary linear algebra. The dot product of vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is written $u \cdot v$ in the language of vector analysis, and its value is $u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$. which is a scalar. As a function from $\mathbb{R}^3 \times \mathbb{R}^3$ into $\mathbb{R}$, dot product is linear in each variable, and it satisfies $u \cdot u \geq 0$ with equality if and only if $u = 0$. The length of a vector $u = (u_1, u_2, u_3)$, written $|u|$, is given by $|u| = \sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2 + u_3^2}$. Dot product has the geometric interpretation that $u \cdot v = |u||v| \cos\theta$, where θ is the angle that u and v make with the origin.

The cross product, also known as the vector product, of two vectors may be less well known. Cross product is defined reasonably only in $\mathbb{R}^3$ and does not generalize well to other dimensions. The **cross product** or **vector product** of vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ is the *vector* in $\mathbb{R}^3$ given by

$$u \times v = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

Fortunately there is a mnemonic for this definition, the formal expression being either

$$u \times v = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} \quad \text{or} \quad u \times v = \det \begin{pmatrix} \mathbf{i} & u_1 & v_1 \\ \mathbf{j} & u_2 & v_2 \\ \mathbf{k} & u_3 & v_3 \end{pmatrix},$$

whichever is more convenient. As a function from $\mathbb{R}^3 \times \mathbb{R}^3$ into $\mathbb{R}^3$, vector product is linear in each variable. It is 0 if and only if u and v are collinear.

Let $w = (w_1, w_2, w_3)$ be a third vector. The triple product $w \cdot (u \times v)$ is given by substituting into the mnemonic the coordinates of w for $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. From this fact it is clear that $u \times v$ is orthogonal to u and v. Moreover, a little computation shows that $|u \times v|^2 + (u \cdot v)^2 = |u|^2 |v|^2$. Therefore $|u \times v| = |u||v||\sin\theta|$, where θ is again the angle that u and v make with the origin. Consequently we know the magnitude of $u \times v$ (namely $|u||v||\sin\theta|$) and the direction up to sign, namely orthogonal to both u and v; that final sign can be determined from easy geometric considerations.[8] Finally if u, v, and w are three vectors, then $\det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$, up to sign, is the volume of the parallelepiped with sides u, v, and w. (See Problem 2 at the end of the chapter.)

Vector analysis makes use of three differential operators on functions on $\mathbb{R}^3$, known as gradient, divergence, and curl. They are defined by

$$\operatorname{grad} f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \qquad \text{for } f \text{ scalar-valued,}$$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \qquad \text{for } F = (F_1, F_2, F_3) \text{ vector-valued,}$$

and

$$\operatorname{curl} F = \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\mathbf{i} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\mathbf{j} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\mathbf{k}$$

for $F = (F_1, F_2, F_3)$ vector-valued. Observe that $\operatorname{grad} f$ and $\operatorname{curl} F$ are vector-valued, but $\operatorname{div} F$ is scalar-vaued. The symbolic vector $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$,

[8]That sign is determined by the right-hand rule or the left-hand rule, whichever applies to the valid formula $\mathbf{i} \times \mathbf{j} = \mathbf{k}$,

pronounced "nabla," allows us to write these definitions more economically as follows:

$$\operatorname{grad} f = \nabla f, \quad \operatorname{div} F = \nabla \cdot F, \quad \text{and} \quad \operatorname{curl} F = \nabla \times F.$$

These operators may be interpreted as special cases of the exterior derivative operator d, as was shown in Example 2 in Section I.4.

Double and triple integrals are familiar from Chapter III of *Basic Real Analysis*, and it is not necessary to say any more about them now except that $dx\,dy$ is sometimes abbreviated as dA in dimension 2 and $dx\,dy\,dz$ is sometimes abbreviated dV is dimension 3.

Curves and surfaces were discussed somewhat in problems at the end of Chapter I, and the reader may wish to refer to that material. In the present chapter we are interested only in smooth curves and smooth surfaces, which are often given either "parametrically" or "implicitly." Parametric curves are usually given by a function of 1 parameter into $\mathbb{R}^3$, while parametric surfaces are usually given by a function of 2 parameters into $\mathbb{R}^3$. In both cases the defining function is assumed to satisfy a certain nondegeneracy condition so that the Inverse Function Theorem can be applied. In the implicit case, curves and surfaces are usually given as the set of simultaneous solutions of some (nonlinear) equations, usually $n-1$ equations in n variables in the case of a curve or $n-2$ equations in n variables in the case of a surface. In addition, the defining equations are assumed to satisfy a certain nondegeneracy condition so that the Implicit Function Theorem can be applied. Nothing more needs to be added to these remarks at this time.

One way that a curve can arise in physics and engineering is as the trajectory of a particle in space. The position is often written as $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, in which case the velocity is $\mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$. The velocity vector is always tangent to the curve. The nondegeneracy condition that was mentioned in the previous paragraph is that the velocity vector is nowhere the 0 vector. This condition ensures that the curve is locally a smooth manifold of dimension 1.

A surface can arise, for example, as a membrane through which fluid is flowing, or as the two dimensional boundary of an open subset of $\mathbb{R}^3$. Say that the surface is given in terms of two parameters s and t by three functions $x(s,t)$, $y(s,t)$, $z(s,t)$. We write $\mathbf{r}(s,t) = x(s,t)\mathbf{i} + y(s,t)\mathbf{j} + z(s,t)\mathbf{k}$. The nondegeneracy condition that was mentioned above is that the surface has two linearly independent tangent vectors at each point, hence a genuine tangent plane at each point that varies nicely with the point. A vector orthogonal to this tangent plane is called a **normal vector**, and such a vector of length 1 is often denoted by $\mathbf{n}$. A unit normal vector at a particular point is determined up to sign. A smoothly embedded surface need not have a continuously varying unit normal vector; the Möbius band does not. A surface in $\mathbb{R}^3$ has a continuously varying unit normal vector if and only if it is orientable. Orientability is often disposed of quickly in physics and engineering applications, often by a phrase such as "with the region on the left" or "with an outward pointing unit normal vector." We shall see examples in the next section.

In the language of differential forms, line integrals are integrals of 1 forms over oriented manifolds of dimension 1, and surface integrals are integrals of 2 forms over oriented manifolds of dimension 2. In a sense, that fact remains true in the notation used in physics and engineering, but the differential forms are somewhat concealed. In the notation of physics and engineering, a line integral is an expression like

$$\int_C \mathbf{F}\cdot d\mathbf{s}.$$

where C is a curve. We are to think of $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ as a force field, assigning a quantity of force to a particle at any particular point of C or perhaps at any point of an open set containing C. Also we are to think of $d\mathbf{s}$ as $\mathbf{i}dx+\mathbf{j}dy+\mathbf{k}dz$. The integral represents the limit of a sum of infinitesimal displacements multiplied by values of $\mathbf{F}$, each summand of force times displacement representing a quantity of work (energy). A rigorous definition using a limiting process appears in Chapter III of *Basic Real Analysis*, but we need not be concerned with that point at present. Operationally we evaluate $\int_C \mathbf{F}\cdot d\mathbf{s}$ the same way we evaluate the integral of the 1 form $F_1\,dx + F_2\,dy + F_3\,dz$, namely by parametrizing the oriented curve with a parameter t, substituting for dx, dy, and dz in terms of dt, and evaluating an ordinary Riemann integral.

Similarly in the notation of physics and engineering, a surface integral is an expression like

$$\int_S \mathbf{F}\cdot d\mathbf{S},$$

where

$$d\mathbf{S} = \begin{pmatrix} dy\wedge dz \\ dz\wedge dx \\ dx\wedge dy \end{pmatrix}.$$

The integral $\int_S \mathbf{F}\cdot d\mathbf{S}$ is evaluated in the same way as the integral of the 2 form $F_1\,dy\wedge dz + F_2\,dz\wedge dx + F_3\,dx\wedge dy$. The interpretation of the integral is of the total "flux" crossing the surface, with $\mathbf{F}$ telling how much flux per unit area is crossing the surface at each point and with $d\mathbf{S}$ representing infinitesimal area of the surface. "Flux" is a term in physics whose exact meaning depends on the particular application. In hydrodynamics it is a quantity of fluid. The term is used also in electromagnetic theory. Since we know how to work with the integral of a smooth 2 form, we need not be concerned with incorporating a rigorous passage to the limit into our definition of surface integral.

Let us turn to arc length and surface area. Arc length was defined rigorously in Chapter III of *Basic Real Analysis* by a passage to the limit. For surface area, however, we found that an approach by taking a limit of areas of inscribed surfaces does not work, and consequently the surface area of a manifold of dimension 2 requires extra structure for a meaningful definition. It would be enough to have the surface smoothly embedded in $\mathbb{R}^3$, and then unit normal vectors to the surface are available. This concept takes us beyond the mathematics needed for Stokes's Theorem, and we shall not pursue it after this paragraph except to say that a unit normal vector to the surface in $\mathbb{R}^3$ defined parametrically by $\mathbf{r}(s, t)$ is

$$\mathbf{n} = \left| \frac{\partial \mathbf{r}(s,t)}{\partial s} \times \frac{\partial \mathbf{r}(s,t)}{\partial t} \right|^{-1} \left(\frac{\partial \mathbf{r}(s,t)}{\partial s} \times \frac{\partial \mathbf{r}(s,t)}{\partial t} \right)$$

and that the total surface area of a surface is given by integration of a scalar quantity dS over the surface, dS being related to the vector quantity $d\mathbf{S}$ by the formulas

$$\begin{aligned} d\mathbf{S} &= \frac{\partial \mathbf{r}(s,t)}{\partial s} \times \frac{\partial \mathbf{r}(s,t)}{\partial t}\, ds\, dt \\ dS &= \left| \frac{\partial \mathbf{r}(s,t)}{\partial s} \times \frac{\partial \mathbf{r}(s,t)}{\partial t} \right| ds\, dt \end{aligned}$$

Finally we want to see in some detail how Green's Theorem, the Divergence Theorem, and the Kelvin–Stokes Theorem arise as special cases in dimensions 2 and 3 of Stokes's Theorem. That is the topic for the next section.

6. Low Dimensional Cases of Stokes's Theorem

Let us examine the meaning of Stokes's Theorem for compact manifolds-with-boundary that are subsets of Euclidean spaces in dimensions 2 and 3. In every case we need to pay particular attention to orientations. In each case we shall state the classical result that comes from Theorem 2.7, explain the choices that are being made, and give a simple example. More complicated examples appear in the problems at the end of the chapter.

We have already handled the case of a closed interval in $\mathbb{R}^1$ as an example at the end of the previous section; it yields the Fundamental Theorem of Calculus on a closed bounded interval of $\mathbb{R}^1$. In dimensions 2 and 3, the situations to examine are those of Green's Theorem in $\mathbb{R}^2$, the Divergence Theorem in $\mathbb{R}^2$ and $\mathbb{R}^3$, the Kelvin–Stokes Theorem in $\mathbb{R}^3$, and integration of a differential along a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Before coming to the analysis of cases, let us summarize what we know about orientations from the previous section and Section I.6. Orientation of a smooth manifold M of dimension m amounts to a parity condition and is constant on connected components. One way of expressing it is as the sign of a nowhere-vanishing form of the top degree m. The standard orientation on $\mathbb{R}^m$ corresponds to the m form $dx_1 \wedge \cdots \wedge dx_m$. Permuting the variables corresponds to permuting an ordered basis, and the effect on orientation is given by the sign of the determinant of the change-of-basis matrix, which is the same as the sign of the permutation. An orientation on a smooth manifold-with-boundary is given by an orientation on the set of manifold points, and this induces an orientation on the boundary points. This process of inducing an orientation may or may not seem natural; primarily it is designed to make the signs come out right in Stokes's Theorem. One way of describing the process is the following: One works with the tangent space to M, starts with an outward pointing vector from the boundary, and extends that one-element set of vectors to an ordered basis of the tangent space by adjoining tangent vectors to the boundary. Then one takes that basis into account in parametrizing the boundary.

a. Green's Theorem. Rather than try to make the above general description more precise all at once, let us see how it is to work in increasingly complex examples. We begin with Green's Theorem, whose statement in the current setting is as follows.

Theorem 2.8 (GREEN'S THEOREM). Let M be a compact oriented smooth manifold-with-boundary of dimension 2 within $\mathbb{R}^2$. If P and Q are smooth functions on an open subset of $\mathbb{R}^2$ containing M, then

$$\int_{\partial M} P\,dx + Q\,dy = \int_M \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dx\,dy,$$

provided ∂M is given the induced orientation.

Here Theorem 2.7 is being applied on M to the 1 form $\omega = P\,dx + Q\,dy$. According to Example 1 in Section I.4, $d\omega$ equals $\big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\big)\,dx \wedge dy$. The manifold M_+ is understood to be given the standard orientation from $\mathbb{R}^2$, which is determined by the 2 form $dx \wedge dy$. Evaluation of the integral $\int_M d\omega$ can be done by using Theorem 1.29; since $dx \wedge dy$ corresponds to the standard orientation of $\mathbb{R}^2$, $dx \wedge dy$ is to be replaced by $dx\,dy$ in a double integral.

According to Theorem 2.8, ∂M is to be given the induced orientation. This means informally that a parametrization of the boundary curve ∂M is to trace out the curve "with the region on the left." More formally let the parametrization be $t \mapsto \big(\begin{smallmatrix} x(t) \\ y(t) \end{smallmatrix}\big)$. The derivative is $\big(\begin{smallmatrix} x'(t) \\ y'(t) \end{smallmatrix}\big)$; it is assumed that $x'(t)$ and $y'(t)$ never vanish simultaneously, so that at each point the Inverse Function Theorem applies either in x or in y and shows that locally one of the variables x and y is a smooth function of the other. At the point of the curve where $t = t_0$, the tangent space in $\mathbb{R}^2$ has an ordered basis $\big(v, \big(\begin{smallmatrix} x'(t_0) \\ y'(t_0) \end{smallmatrix}\big)\big)$, where v is a vector pointing outward from the boundary. This basis can be transformed into the standard basis of $\mathbb{R}^2$ by a linear map of nonzero determinant. In terms of orientations, the determinant is positive if and only if the parametrization of the curve at t_0 is consistent with the induced orientation of the boundary,[9] and it remains consistent for all t while the parametrization is in force.

EXAMPLE. Let M be the closed annulus $\{(x, y) \in \mathbb{R}^2 \mid 1 \le x^2 + y^2 \le 4\}$. The boundary consists of two circles, the outer circle being traversed counterclockwise (so that M_+ is on the immediate left) and the inner circle being traversed clockwise (so that M_+ is on the immediate left). Let $\omega = P\,dx + Q\,dy = y\,dx$. Then $d\omega = -dx \wedge dy$. So $\int d\omega = \int_M -1\,dx\,dy = -\text{Area}(M_+) = -3\pi$. We can parametrize the boundary by two circles, one being $t \mapsto (2\cos t, 2\sin t)$ for $0 \le t \le 2\pi$ and the other being $t \mapsto (\cos t, -\sin t)$ for $0 \le t \le 2\pi$. Then

$$\begin{aligned} \textstyle\int_{\partial M} x\,dx &= \textstyle\int_0^{2\pi} 2\sin t\,d(2\cos t) - \sin t\,d(\cos t) \\ &= \textstyle\int_0^{2\pi} (-4\sin^2 t + \sin^2 t)\,dt = -3\pi. \end{aligned}$$

b. Divergence Theorem. The Divergence Theorem works for manifolds-with-boundary in any number of dimensions $m \ge 2$. Let us begin with dimension 3. We state the theorem in that case, remark about orientation, and give an example. Then we make remarks about the case of general dimension m and say how the result in dimension 2 compares with Green's Theorem. Finally we restate the Divergence Theorem in dimension 3 in the notation of the previous section that is often used in physics and engineering.

Theorem 2.9 (DIVERGENCE THEOREM). Let M be a compact oriented smooth manifold-with-boundary of dimension 3 within $\mathbb{R}^3$. If F_1, F_2, F_3 are smooth real-valued functions on an open subset of $\mathbb{R}^3$ containing M, then

$$\int_{\partial M} F_1\,dy \wedge dz + F_2\,dz \wedge dx + F_3\,dx \wedge dy = \int_M \Big(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\Big)\,dx\,dy\,dz,$$

provided ∂M is given the induced orientation.

Theorem 2.9 is the special case of Theorem 2.7 applied to the 2 form

$$F_1\,dy \wedge dz + F_2\,dz \wedge dx + F_3\,dx \wedge dy.$$

According to Example 2 in Section I.4, $d\omega = \big(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial x}\big)\,dx \wedge dy \wedge dz$. M is oriented by the standard orientation of $\mathbb{R}^3$, the one determined by $dx \wedge dy \wedge dz$. To sort out the meaning of the induced orientation, we start with a

[9] Positive determinant means that the tangent vector to the curve points to the left of the outward-pointing vector v; thus the inward-pointing vector $-v$ points to the left of the tangent vector, and the region is on the left of the curve.

parametrization of the surface ∂M, say $(s,t) \mapsto \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix}$. This parametrization does not have to work everywhere on M at once. Consistent local parametrizations will be good enough because of the assumed orientability. The derivative matrix is the 3-by-2 matrix $\begin{pmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y \partial t \\ \partial z/\partial s & \partial z/\partial t \end{pmatrix}$. The assumption on the parametrization to make it locally invertible is that this derivative matrix has rank 2 everywhere. Then about every point of the surface, one can in principle solve for one of the variables x, y, z in terms of the other two, according to the Inverse Function Theorem. At the point of the surface where $(s,t) = (s_0, t_0)$, the tangent space in $\mathbb{R}^3$ has an ordered basis $\left(v, \begin{pmatrix} \partial x/\partial s \\ \partial y/\partial s \\ \partial z/\partial s \end{pmatrix}_{(s_0,t_0)}, \begin{pmatrix} \partial x/\partial t \\ \partial y/\partial t \\ \partial z/\partial t \end{pmatrix}_{(s_0,t_0)}\right)$, where v is a vector pointing outward from the boundary. This basis can be transformed into the standard basis of $\mathbb{R}^3$ by a linear map of nonzero determinant. If the determinant is positive, then the parametrization of the surface near (s_0, t_0) is consistent with the induced orientation of the boundary. Conversely if the determinant is negative, then the parametrization of the surface is consistent with the opposite of the induced orientation.

EXAMPLE. Let M be the closed unit ball

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \le 1\}.$$

The boundary is the unit sphere $\partial M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$, and it is to be given the induced orientation. Let $\omega = z\,dx \wedge dy$. Then $d\omega = dx \wedge dy \wedge dz$. So $\int_M d\omega = \int_M dx\,dy\,dz = \text{Volume}(M_+) = 4\pi/3$.

To evaluate $\int_{\partial M} z\,dx \wedge dy$ directly and check Theorem 2.7 in this case, we need to parametrize the sphere. We can use ordinary spherical coordinates (φ, θ) near most points for this purpose:

$$\begin{pmatrix} x(\varphi,\theta) \\ y(\varphi,\theta) \\ z(\varphi,\theta) \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \end{pmatrix}$$

for $0 < \varphi < \pi$ and $-\pi < \theta < \pi$. The derivative matrix is

$$\begin{pmatrix} -\sin\varphi & 0 \\ \cos\varphi\cos\theta & -\sin\varphi\sin\theta \\ \cos\varphi\sin\theta & \sin\varphi\cos\theta \end{pmatrix}.$$

This derivative matrix is convenient to examine at $(\varphi, \theta) = (\pi/2, 0)$, which corresponds to $(x, y, z) = (0, 1, 0)$. At this point, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an outward pointing vector from the closed unit ball. The derivative matrix at this point is

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

If the outward pointing vector is adjoined to this matrix as its first column, the determinant of the resulting 3-by-3 matrix is $+1$, positive. Thus our parametrization gives us the induced orientation, not its opposite. To evaluate $\int_{\partial M} z\,dx \wedge dy$, we compute

$$\begin{aligned} dx \wedge dy &= \left(\tfrac{\partial(x,y)}{\partial(\varphi,\theta)}\right) d\varphi \wedge d\theta \\ &= \det \begin{pmatrix} -\sin\varphi & 0 \\ \cos\varphi\cos\theta & -\sin\varphi\sin\theta \end{pmatrix} d\varphi \wedge d\theta = \sin^2\varphi \sin\theta \, d\varphi \wedge d\theta. \end{aligned}$$

Then

$$\int_{\partial M} z\,dx \wedge dy = \int_{\varphi=0}^{\pi} \int_{\theta=-\pi}^{\pi} \sin\varphi \sin\theta \sin^2\varphi \sin\theta \, d\theta \, d\varphi = \pi \int_{\varphi=0}^{\pi} \sin^3\varphi \, d\varphi.$$

One readily checks that the φ integral equals $4/3$, and thus the surface integral equals $4\pi/3$, in agreement with the statement of the Divergence Theorem.

REMARK. It may at first appear that Theorem 2.9 applies to many familiar regions of $\mathbb{R}^3$. But one has to remember that the hypotheses require the closure of the region to be a smooth manifold-with-boundary. In particular the boundary has to be smooth. A solid hemisphere does not fit the hypotheses. Often the region between two smooth surfaces suffers the same drawback, having "corners" where the two surfaces meet. The relevant setting to handle this situation is that of a "smooth manifold-with-corners." Such objects will be discussed in Chapter III.

In general dimension m, Theorem 2.7 gives

$$\int_{\partial M} \sum_{i=1}^{m} (-1)^{i-1} F_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_m = \int_M \sum_{i=1}^{m} \frac{\partial F_i}{\partial x_i} \, dx_1 \cdots dx_m,$$

where the circumflex indicates a missing factor. In dimension 2, the formula reduces to

$$\int_{\partial M} F_1 dy - F_2\,dx = \int_M \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dx\,dy.$$

This matches the formula of Green's Theorem if we put $F_2 = -P$ and $F_1 = Q$.

In the notation of the previous section that often arises in physics and engineering, the integral formula in the Divergence Theorem can be written more briefly as

$$\int_{\partial M} F \cdot d\mathbf{S} = \int_M (\operatorname{div} F)\,dV$$

or as $\int_{\partial M} F \cdot d\mathbf{S} = \int_M (\nabla \cdot F)\,dV$, where dV is shorthand for the volume element. The manifold-with-boundary M has dimension 3 and is assumed to lie in $\mathbb{R}^3$. It follows that its set M_+ of manifold points is an open subset of $\mathbb{R}^3$. Then M_+ inherits the standard orientation from $\mathbb{R}^3$, and it is understood that ∂M gets the induced orientation. Thus nothing explicit needs to be said about orientations or normal vectors.

c. Kelvin–Stokes Theorem. The classical form of Stokes's Theorem, also known as the Kelvin–Stokes Theorem, applies to a manifold-with-boundary of dimension 2 realized in $\mathbb{R}^3$. First we state the theorem and relate it to Theorem 2.7 for differential forms, and we make a few general comments about orientations in the Kelvin–Stokes Theorem. Second we work through a simple example, paying particular attention to orientations. Third we look at the example in the light of the notation in the previous section.

Theorem 2.10 (KELVIN–STOKES THEOREM). Let M be a compact oriented smooth manifold-with-boundary of dimension 2 within $\mathbb{R}^3$. If P, Q, R are smooth real-valued functions on an open subset of $\mathbb{R}^3$ containing M, then

$$\iint_S \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\Big)\, dy \wedge dz + \Big(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\Big)\, dz \wedge dx + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\, dx \wedge dy$$
$$= \int_\gamma P\, dx + Q\, dy + R\, dz.$$

provided ∂M is given the induced orientation.

Here Theorem 2.7 is being applied to the 1 form

$$\omega = P\, dx + Q\, dy + R\, dz.$$

According to Example 2 in Section I.4,

$$d\omega = \big(\tfrac{\partial R}{\partial y} - \tfrac{\partial Q}{\partial z}\big)\, dy \wedge dz + \big(\tfrac{\partial P}{\partial z} - \tfrac{\partial R}{\partial x}\big)\, dz \wedge dx + \big(\tfrac{\partial Q}{\partial x} - \tfrac{\partial P}{\partial y}\big)\, dx \wedge dy.$$

About orientations for this setting, M_+ is not an open subset of the Euclidean space $\mathbb{R}^3$ is which it lives; thus it does not automatically inherit an orientation from $\mathbb{R}^3$. By assumption, M_+ is orientable, and we must actually choose an orientation. One way to do so is to make use of a local parametrization, since a local parametrization allows us to identify part of M_+ with an open subset of the Euclidean space of parameters and transfer the standard orientation from that Euclidean space to M_+. Once that step is done, then the orientation of M_+ can be pieced together, ∂M acquires the induced orientation, and we can proceed. Observe that an orientation of $\mathbb{R}^3$ plays no role in this construction.

EXAMPLE. Let M be the cylinder in Figure 2.3 given by

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ \ z^2 \le 1\}.$$

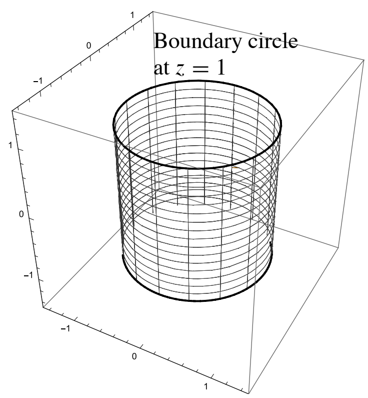

FIGURE 2.3. M and ∂M in an example for the Kelvin–Stokes Theorem.

The boundary ∂M consists of two circles:

$$\partial M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \text{ and } z = \pm 1\},$$

and it is to be given the induced orientation from M_+, whatever the orientation of M_+ might mean. To get at such an orientation locally, we can parametrize the cylinder locally with a pair (r, θ) of parameters, r for the z component and θ for the angle made with (x, y). One parametrization of M_+ is

$$\alpha(r, \theta) = \begin{pmatrix} x(r,\theta) \\ y(r,\theta) \\ z(r,\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ r \end{pmatrix},$$

valid for $-1 < r < 1$ and $-\pi < \theta < \pi$, let us say. The space of all parameters (r, θ), being an open subset of $\mathbb{R}^2$, contains a standard orientation given by $dr \wedge d\theta$, and we move this over to M_+ by the pullback of α^{-1}. Thus we obtain a nowhere-vanishing 2 form on an open set of M_+. We can argue similarly with different parameters for a second open subset of M_+, and the two open sets together cover M_+. The assumption that M_+ is orientable implies that these nowhere-vanishing 2 forms can be chosen consistently from the one open set to the other, and then we have realized our orientation of M_+ more or less concretely. To use this information, we form the derivative matrix of α, which is

$$D\alpha(r, \theta) = \begin{pmatrix} 0 & -\sin\theta \\ 0 & \cos\theta \\ 1 & 0 \end{pmatrix}.$$

Its columns span the tangent space of M_+ at the point of M corresponding to (r, θ).

We can parametrize the boundary circles one at a time, the one at $z = 1$ being given by

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ 1 \end{pmatrix}, \quad \text{with derivative} \quad \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix}.$$

To orient this component of ∂M, we seek a tangent vector to M that points outward from ∂M. Consider a single point of ∂M, say $(1, 0, 1)$, which arises when $r = 1$ and $\theta = 0$. The tangent space at this point has basis $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$. An example of an outward pointing tangent vector at this point is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, since the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is in the span of the two columns of the derivative matrix and is not a multiple of the second vector in the basis. Working with the induced orientation of ∂M means that when ∂M is parametrized, the derivative vector of the parametrization points in a direction that is a positive multiple of the second column. In other words the above parametrization of the circle at $z = 1$ is consistent with the induced orientation on ∂M.

The candidates for such a vector are $\pm$ any vector that is in the span of the two columns of the full derivative matrix but is not a multiple of the second column, and $(0, 0, 1)$ will do fine. To have the vector point outward, we can use $(0, 0, 1)$ at $z = 1$. In our ordering of basis vectors yielding an orientation for M, this vector is to precede a tangent vector to ∂M, and that situation is already the case with the columns of the derivative matrix as is. Thus the above parametrization of the circle at $z = 1$ is consistent with the induced orientation.

Now let us orient the boundary circle[10] at $z = -1$. We select a single point of this part of the boundary to examine. The point $(0, 1, -1)$, which arises when $\theta = \pi/2$, will do fine. An outward pointing tangent vector can be taken to be $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$. If we write v for the second vector in a basis purporting to give the induced orientation on ∂M, then the linear map that carries the tangent space to itself, sends $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and sends v to $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ must have positive determinant. This means that v is a negative multiple of $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. In other words the parametrization of ∂M as $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ -1 \end{pmatrix}$ is inconsistent with the induced orientation on ∂M. So we

[10] An observant reader will say that we are merely ensuring that the circle with $z = -1$ is traversed with the region on its left, just as the circle with $z = 1$ was. Resorting to familiar geometric intuition is all very well in this case, but the method being discussed here works even in higher dimensional cases when ∂M need not be 1 dimensional.

should use its opposite as in Figure 2.4, parametrizing the circle by

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \\ -1 \end{pmatrix} \quad \text{with derivative} \quad \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix}.$$

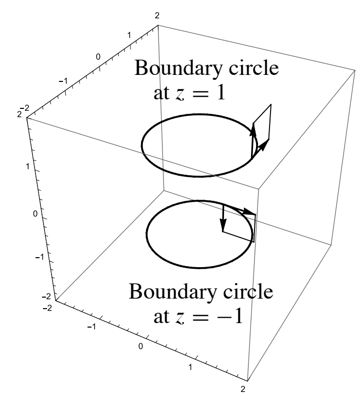

FIGURE 2.4. Tangent planes at points on the boundary circles in the example. The indicated planes are the respective tangents at $(1, 0, 1)$ and $(0, 1, -1)$.

Now let $\omega = yz\,dx$. Having parametrized both components of ∂M consistently with the induced orientation, we shall evaluate $\int_{\partial M} \omega = \int_{\partial M} yz\,dx$ in the two ways that Theorem 2.10 says should give the same answer. The signs will be crucial. One way is directly as the sum of two line integrals, namely as

$$\begin{aligned} &= \int_{z=1} yz\,dx + \int_{z=-1} yz\,dx \\ &= \int_{-\pi}^{\pi} (\sin t)(+1)(-\sin t)\,dt + \int_{-\pi}^{\pi} (\sin t)(-1)(\sin t)\,dt = -2\pi. \end{aligned}$$

The other way is as

$$\begin{aligned} \int_M d\omega &= \int_M d(yz) \wedge dx \\ &= \int_M z\,dy \wedge dx + \int_M y\,dz \wedge dx \\ &= -\int_M z\,dx \wedge dy + \int_M y\,dz \wedge dx. \end{aligned}$$

Referring to the derivative matrix $D\alpha(r, \theta)$, we have

$$dx \wedge dy = \tfrac{\partial(x,y)}{\partial(r,\theta)} = \det \begin{pmatrix} 0 & -\sin\theta \\ 0 & \cos\theta \end{pmatrix} = 0$$

and

$$dz \wedge dx = \tfrac{\partial(z,x)}{\partial(r,\theta)} = \det \begin{pmatrix} 1 & 0 \\ 0 & -\sin\theta \end{pmatrix} = -\sin\theta.$$

Therefore

$$\int_M d\omega = \int_M y\,dz \wedge dx = \int_0^{2\pi}\int_{-1}^{1} \sin\theta(-\sin\theta)\,dr\,d\theta$$
$$= -2\int_0^{2\pi} \sin^2\theta\,d\theta = -2\pi.$$

Thus indeed the two computations give the same answer.

Finally let us review the example in the light of the other systems of notation. The vector-valued function that we have been using is $F = (yz, 0, 0)$ or $F = yz\mathbf{i}$ or $F = \begin{pmatrix} yz \\ 0 \\ 0 \end{pmatrix}$, and

$$\operatorname{curl} F = \det\begin{pmatrix} \mathbf{i} & \frac{\partial}{\partial x} & yz \\ \mathbf{j} & \frac{\partial}{\partial y} & 0 \\ \mathbf{k} & \frac{\partial}{\partial z} & 0 \end{pmatrix} = -\mathbf{k}\big(\tfrac{\partial(yz)}{\partial y}\big) + \mathbf{j}\big(\tfrac{\partial(yz)}{\partial z}\big) = y\mathbf{j} - z\mathbf{k}.$$

Then

$$\operatorname{curl} F \cdot d\mathbf{S} = \begin{pmatrix} 0 \\ y \\ -z \end{pmatrix} \cdot \begin{pmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{pmatrix} = y\,dz \wedge dx - z\,dx \wedge dy.$$

d. Integration of a differential along a curve. In many applications of Stokes's Theorem, we are given $\int_{\partial M} \omega$ and we want to compute $\int_M d\omega$. Occasionally an application arises in which one wants to go in the other direction. In this case we are evaluating an integral $\int_M \eta$ for some m form η on M, where m is the dimension of M, and we recognize η as d of something, say $\eta = d\omega$. Then we can use the equality $\int_M \eta = \int_M d\omega = \int_{\partial M} \omega$.

This is what happens in the last low dimensional instance of Stokes's Theorem mentioned at the beginning os this section, namely the integration of a differential along a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$. We are to compute a line integral $\int_C \eta$, where η is a 1 form and C is a smooth curve with endpoints A and B in $\mathbb{R}^2$ or $\mathbb{R}^3$. A smooth curve with endpoints present is an example of a 1 dimensional manifold-with-boundary, and the above theory can apply. The only case in which Stokes's Theorem applies in straightforward fashion, however, is the case that the 1 form η is d of something, specifically d of a smooth function f. Thus suppose that the 1 form η that we are integrating is equal to a differential df. Then we have

$$\int_C \eta = \int_C df = f(B) - f(A).$$

This formula is an instance of Stokes's Theorem, but it is really easier than that. If the curve C is parametrized as $\gamma(t)$ for $a \le t \le b$ with $\gamma(a) = A$ and $\gamma(b) = B$, then an application of the Fundamental Theorem of Calculus to the composition $f \circ \gamma$ immediately gives

$$\int_C df = \int_a^b f(\gamma(t))\,dt = f(\gamma(b)) - f(\gamma(a)) = f(B) - f(A).$$

e. Final remarks. In many authors' formulations of versions of Stokes's Theorem, inner products and normal vectors play a role in the statements of the theorems and in the proofs. This is so in the formulations of the classical theorems of the Introduction, for example. In the text we have systemically avoided this extra layer of structure. Stokes's Theorem is really something about the exterior derivative and integration of differential forms, not about orthogonality, and the text has sought to emphasize this point. The cost has been small. We have had to work with "outward pointing tangent vectors" from the boundary of a manifold-with-boundary rather than outward "normal vectors." The inner product focuses attention on one good choice of an outward vector, but it does not help otherwise in the theory.

7. Problems

1. In $\mathbb{R}^3$, show that $|u \times v|^2 + (u \cdot v)^2 = |u|^2|v|^2$.
2. If $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, and $w = (w_1, w_2, w_3)$ are vectors in $\mathbb{R}^3$, show that $\det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$, up to sign, is the volume of the parallelepiped with sides u, v, and w.
3. If $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, and $w = (w_1, w_2, w_3)$ are vectors in $\mathbb{R}^3$, which of the six expressions $u \cdot (v \times w)$, $u \cdot (w \times v)$, $v \cdot (u \times w)$, $v \cdot (w \times u)$, $w \cdot (u \times v)$, and $w \cdot (v \times u)$ are equal to the first one. What is the relationship of the first one to the others?
4. (a) Compute div F and curl F for $F = x^2y\mathbf{i} - (z^3 - 3x)\mathbf{j} + 4y^2\mathbf{k}$.
   (b) Compute div F and curl F for $F = (3x + 2z^2)\mathbf{i} + x^3y^2\mathbf{j} - (z - 7x)\mathbf{k}$.
5. Let M be a smooth compact orientable manifold without boundary of dimension m. Proposition 1.30 showed that M has a nowhere-vanishing smooth m form ω. Use Stokes's Theorem to show that ω cannot be obtained as $d\eta$ for a smooth $m-1$ form η.
6. (a) Exhibit a smooth differential 2 form ω on $\mathbb{R}^4$ such that $\omega \wedge \omega \neq 0$.
   (b) Suppose that M is a compact orientable smooth manifold of dimension $2n$ without boundary. Suppose that α is a smooth differential 1 form on M, so that $d\alpha$ is a 2 form. Can the n fold wedge product $\omega = d\alpha \wedge \cdots \wedge d\alpha$ be nowhere vanishing? If so, exhibit such an ω for some M. If not, prove that such an ω can never be nowhere vanishing.

7. (a) Show that
$$\omega = \frac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$$
has $d\omega = 0$ in $\mathbb{R}^3 - \{0\}$.
(b) Let T be the torus in $\mathbb{R}^3$ given by rotating the unit circle in the x-z plane about the line where $x = 2$ and $y = 0$. It is the locus where
$$\left(\sqrt{(x-2)^2+y^2} - 2\right)^2 + z^2 = 1.$$
Evaluate the integral $\int_T \omega$, where ω is as in (a) and where T is oriented so that the unbounded component of $\mathbb{R}^3 - T$ is "outside" the torus.

8. Let M be the subset of $\mathbb{R}^3$ lying between the sphere S_1 of radius 1 and the sphere S_a of positive radius a with $a < \frac{1}{2}$. Regard M as a manifold-with-boundary that inherits its orientation from the standard orientation of $\mathbb{R}^3$, and give its boundary $S = S_1 \cup S_a$ the induced orientation. Let F be the vector-valued function $F(x) = |x|^{-3}x$.
(a) Show that div $F = 0$ on M.
(b) Why is $\int_{S_1} F\cdot d\mathbf{S} = \int_{S_a} F\cdot d\mathbf{S}$?

9. Generalize the formula in (a) of the Problem 7a by finding a smooth $n-1$ form $\omega = f(x_1,\dots,x_n)^{-1}\eta$ on $\mathbb{R}^n - \{0\}$ such that $d\eta = dx_1\wedge\cdots\wedge dx_n$ and $d\omega = 0$.

10. By examining the example of $\mathbb{H}^m$ in Sections 3 and 4, show for every $m \geq 1$ that if $\partial\mathbb{H}^m$ is made to correspond to $x_1 = 0$ and if $\partial\mathbb{H}^m$ gets its orientation from $dx_2\wedge\cdots\wedge dx_m$, then one is led to Stokes formula for $\mathbb{H}^m$ with a single minus sign (rather than $(-1)^m$) on one of the two sides of the formula.

Problems 11–15 concern surface integrals and the Kelvin–Stokes Theorem in $\mathbb{R}^3$.

11. Evaluate the surface integral $\int_S x\mathbf{i}\cdot d\mathbf{S}$, where S is the surface in $\mathbb{R}^3$ given by $z = x^2 + y^2$ for $z \leq 4$ and S is oriented by an outward/downward pointing normal vector.

12. Use the Kelvin–Stokes Theorem to compute $\int_S \operatorname{curl} F\cdot d\mathbf{S}$, where $F(x,y,z) = yz\mathbf{i} + xy\mathbf{k}$ and S is the part of sphere $x^2+y^2+z^2 = 4$ that lies inside the cylinder $x^2+y^2 = 1$ and above the x-y plane. The surface is oriented by an outward pointing normal vector.

13. Let F be the vector-valued function $F = (-yz, 4y+1, xy+e^z)$, and let C be the oriented curve $\mathbf{s}(t) = (3\cos t, 4, 3\sin t)$. This is the circle of radius 3 given by $x^2+y^2 = 9$ and $y = 4$. With the help of the Kelvin–Stokes Theorem, evaluate the line integral $\int_C F\cdot d\mathbf{s}$.

14. Use the Kelvin–Stokes Theorem to evaluate $\int_S \operatorname{curl} F \cdot d\mathbf{S}$ if $F = (y, -x, yx^3)$ and S is the portion of the sphere of radius 4 about the origin having $z \geq 0$ and the upward orientation.

15. Evaluate $\int_C F \cdot d\mathbf{s}$, where $F(x, y, z) = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$, and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. The curve C is to be oriented counterclockwise when the x-y plane is viewed as horizontal and the curve is viewed from above. Do this in two ways, as follows:
   (a) directly by parametrizing the curve by the angle θ in the x-y plane,
   (b) by using the Kelvin–Stokes Theorem, taking C to be the boundary of the filled ellipse in the plane where $y + z = 2$.

Problems 16–20 establish the Brouwer Fixed-Point Theorem, which says that whenever a continuous function f carries the closed unit ball $\overline{B} = \{x \in \mathbb{R}^n \bigm| |x \leq 1\}$ of $\mathbb{R}^n$ into itself, then there is some x in the ball with $f(x) = x$. Let

$$B = \{x \in \mathbb{R}^n \bigm| |x < 1\} \quad \text{and} \quad \partial B = \{x \in \mathbb{R}^n \bigm| |x = 1\}.$$

A **retraction** of $\overline{B}$ into ∂B is a continuous function $r : \overline{B} \to \partial B$ such that r is the identity on ∂B. The line of proof will be to show that there is no smooth retraction, that the fixed-point theorem follows in the smooth case from the nonexistence of a smooth retraction, and that the fixed-point theorem in the smooth case implies the fixed-point theorem in the general case.

16. This problem and the next show that there is no smooth retraction of $\overline{B}$ onto ∂B. In fact, suppose that a smooth retraction $r : \overline{B} \to \partial B$ exists. Let ω be a nowhere-vanishing $n-1$ form on ∂B; this has to exist on ∂B by Proposition 1.30 because Problem 15 at the end of Chapter I showed that all spheres are oriented. Justify the following steps in a computation for the smooth manifold-with-boundary $\overline{B}$:

$$0 < \int_{\partial B} \omega = \int_{\partial B} r^*(\omega) = \int_B dr^*(\omega) = \int_B r^*(d\omega).$$

17. Explain why the right side is 0 in the displayed line of the previous problem and why the retraction r cannot exist.

18. Show that if $f : \overline{B} \to \overline{B}$ is a smooth function such that $f(x) \neq x$ for all x in $\overline{B}$, then one can construct from f a smooth retraction r of $\overline{B}$ onto ∂B. Since the previous two problems have shown that there is no such smooth retraction, every smooth $f : \overline{B} \to \overline{B}$ has a fixed point.

19. If $f : \overline{B} \to \overline{B}$ is a continuous function, show that there exists a sequence $\{f_k\}$ of smooth functions from $\overline{B}$ into $\overline{B}$ that converges uniformly to f on $\overline{B}$.

20. If $f : \overline{B} \to \overline{B}$ is a continuous function, choose by the previous problem a sequence $\{f_k\}$ of smooth functions carrying $\overline{B} \to \overline{B}$ and converging uniformly to f on $\overline{B}$. Using Problem 18, let x_k be a point in $\overline{B}$ with $f_k(x_k) = x_k$. If x_0 is a limit point of $\{x_k\}$ in $\overline{B}$, show that $f(x_0) = x_0$. Consequently f has a fixed point in $\overline{B}$.

CHAPTER III

Whitney's Setting for Stokes's Theorem

Abstract. This chapter looks for a single setting in which Stokes's Theorem applies at once to all situations of practical interest. It begins by developing the theory in the setting of manifolds-with-corners and continues with a theory in a more general setting studied by H. Whitney.

Section 1 introduces the model space $\mathbb{Q}^m$ for $m \geq 2$, in terms of which manifolds-with-corners are defined. The section contains one result that is relatively hard to prove: the index of a point of $\mathbb{Q}^m$ is taken to be the number of coordinates that are equal to 0, and it is shown that any diffeomorphism between open sets in $\mathbb{Q}^m$ maps points of one index into points of the same index. Consequently the notion of index is well defined for the points of a manifold-with-corners. Other definitions concerning manifolds translate easily into corresponding definitions for manifolds-with-corners. These include smooth real-valued function, support, germ, tangent space, cotangent space, smooth differential forms, pullbacks of differential forms, and the derivative of a smooth map between manifolds-with-corners.

Section 2 introduces strata, the stratum $S_k(M)$ consisting of all points of index k in a manifold-with-corners M. Strata have a number of useful properties, one of which is that the strata of index 0 and 1 combine to yield a manifold-with-boundary.

Section 3 gives a version of the Stokes's Theorem for manifolds-with-corners, saying $\int_{\partial M} \omega = \int_M d\omega$ as usual. In this equality the integral on the left is over the stratum of all points of index 1, and the integral on the right is over the stratum of all points of index 0. Simple examples show that this theorem is not a trivial consequence of the theorem about manifolds-with-boundary when applied to the manifold-with-boundary consisting of all points of index 0 and 1 in M.

Section 4 establishes a version of the Divergence Theorem due to Whitney that applies to any bounded region of $\mathbb{R}^m$ for $m \geq 2$ when most of the topological boundary behaves as it does for a manifold-with-boundary and when the set of exceptional points of the topological boundary is small in a specific technical sense. Such a region will be called a Whitney domain. If the set of exceptional points is finite, then it is small in the technical sense.

Section 5 examines in some detail the technical condition in Section 4. That condition becomes: the set of exceptional points is compact and either is empty or has $m-1$ dimensional Minkowski content 0. It is shown that the condition that a compact set has ℓ dimensional Minkowski content 0 is intrinsic to the set as a subset of a Euclidean space and does not depend on its embedding. Furthermore any function from one Euclidean space to another that satisfies a Lipschitz condition always carries compact subsets of ℓ dimensional Minkowski content 0 to compact sets of ℓ dimensional Minkowski content 0. Consequently the notion "ℓ dimensional Minkowski content 0" is well defined for compact subsets of smooth manifolds and is preserved under smooth mappings into Euclidean spaces. The section concludes with examples of Whitney domains constructed from the zero loci of polynomials.

Section 6 extends the scope of Stokes's Theorem to Whitney manifolds, a class of spaces that includes all manifolds-with-corners and that allows all Whitney domains as additional model cases. The result is that the Stokes formula applies in what seems to be the full set of practical situations of interest to mathematicians, physicists, and engineers.

1. Definition and Examples of Manifolds-with-Corners

Smooth manifolds of dimension $m \geq 0$, as introduced in Chapter I, were defined as separable Hausdorff spaces that are locally modeled on open subsets of $\mathbb{R}^m$. In similar fashion smooth manifolds-with-boundary of dimension $m \geq 1$, as introduced in Chapter II, were defined as separable Hausdorff spaces that are locally modeled on open subsets of the closed half space

$$\mathbb{H}^m = \{(x_1, \dots, x_{m-1}, x_m) \in \mathbb{R}^m \mid x_m \geq 0\}.$$

In the first part of this chapter, we work with smooth manifolds-with-corners of dimension $m \geq 2$ as separable Hausdorff spaces that are locally modeled on open subsets of the closed generalized quadrant

$$\mathbb{Q}^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_j \geq 0 \text{ for } 1 \leq j \leq m\}.$$

The **open subsets** of $\mathbb{Q}^m$ are understood to be those subsets that are relatively open in the relative topology from $\mathbb{R}^m$. The goal of the first three sections of this chapter is to prove an extension of Stokes's Theorem to manifolds-with-corners. At the least such a theorem will simultaneously handle balls and rectangular solids in $\mathbb{R}^m$. The failure of the theorems of Chapters I and II to handle balls and rectangular solids at the same time was a weakness of the earlier theory that we shall now be able to remedy. We can do much better, and we begin the development of an improved theory in Section 4.

Before coming to the formal definition of smooth manifold-with-corners, we need to establish some definitions concerning smooth functions on $\mathbb{Q}^m$, just as we did with $\mathbb{H}^m$ in Section II.2. A real-valued function f defined on an open subset U of $\mathbb{Q}^m$ will be said to be **smooth** if there is a smooth function F defined an open subset V of $\mathbb{R}^m$ such $U = V \cap \mathbb{Q}^m$ and f is the restriction of F to U. The extending function F need not, of course, be unique. With this definition of smoothness in place, we can define the space $\mathcal{C}_p(\mathbb{Q}^m)$ of germs of smooth functions at points p of $\mathbb{Q}^m$ and the tangent space $T_p(\mathbb{Q}^m)$ at p.

We write $\mathbb{Q}^m_+$ for the interior of $\mathbb{Q}^m$, namely the subset

$$\mathbb{Q}^m_+ = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_j > 0 \text{ for } 1 \leq j \leq m\}.$$

and we write $\partial\mathbb{Q}^m$ for the topological boundary, namely the subset

$$\partial\mathbb{Q}^m = \{(x_1, \dots, x_m) \in \mathbb{Q}^m \mid x_j = 0 \text{ for at least one } j \text{ with } 1 \leq j \leq m\}.$$

The definitions of $\mathcal{C}_p$ and T_p are not new for p in $\mathbb{Q}^m_+$, but for p in $\partial\mathbb{Q}^m$, they are. We obtain facts about $\mathcal{C}_p$ and T_p in the same way as in Section I.1.

If U_1 and U_2 are two open subsets of $\mathbb{Q}^m$, a **smooth map** $F : U_1 \to U_2$ is function whose m component functions are all smooth real-valued functions on

U_1. The derivative $(DF)_p : T_p(U_1) \to T_{F(p)}(U_2)$ of the smooth map F at a point is defined just as in Section I.1. The smooth map F is a **diffeomorphism** if it is a homeomorphism with inverse $G : U_2 \to U_1$ such that the m component functions of each of F and G are smooth real-valued functions on U_1 and U_2, respectively. The composition of smooth maps is smooth, and the derivative of the composition is the composition of the derivatives. It follows that at each point the derivative of a diffeomorphism is an invertible linear function.

In the study of manifolds-with-boundary, we distinguished two kinds of points, manifold points and boundary points, and the distinction was straightforward. In a corresponding but more subtle fashion for manifolds-with-corners, we define[1] the **index** of a point $(x_1, \dots, x_m)$ in $\mathbb{Q}^m$ to be the number of indices j for which $x_j = 0$. The points in $\mathbb{Q}^m_+$ have index 0 and the points in $\partial\mathbb{Q}^m$ have index ≥ 1. For this notion to be usable with a general manifold-with-corners, we need Proposition 3.1 below, whose proof will make use of a lemma.

Proposition 3.1. If $F : U \to V$ is a diffeomorphism of one nonempty open subset of $\mathbb{Q}^m$ onto another, then every $p \in U$ has the property that the index of p equals the index of $F(p)$.

Lemma 3.2. Let $A = (a_{ij})_{i,j=1}^m$ be a square matrix with real entries. If there is an integer k with $1 \leq k \leq m$ such that $a_{ij} = 0$ whenever $i \leq k$ and $j \geq k$, then $\det A = 0$.

PROOF OF LEMMA 3.2. The proof is by induction on k simultaneously for all m. The base case of the induction is $k = 1$. In this case, $a_{ij} = 0$ for $i = 1$ and all j. In other words, the first row of A is 0. Hence $\det A = 0$.

Suppose that the lemma has been proved for the integer $k - 1 \geq 1$ and that we are to consider a matrix A for the integer k. We expand $\det A$ in cofactors about the first row, obtaining an alternating sum of terms with a coefficient a_{1j} that multiplies the determinant of a matrix of size $m - 1$. The upper left entry of that matrix is a_{22} for the first term and is a_{21} for the subsequent terms. Since the coefficient a_{1j} is 0 for $j \geq k$, we need only consider the first $k - 1$ terms in the expansion. Each of those terms corresponds to a matrix of the form in the lemma but with k replaced by $k - 1$. By inductive hypothesis, each such determinant is 0. Therefore $\det A = 0$, and the induction is complete. □

PROOF OF PROPOSITION 3.1. Possibly replacing F by F^{-1}, we see that it is enough to prove that the index I of $F(p)$ is $\leq$ the index J of p. It will simplify the ideas if we think of U and V as lying in distinct copies of $\mathbb{Q}^m$, so that the order of the variables in U does not affect the order of the variables in V. Let us write p

[1] Some authors use the term "depth" in place of "index."

as $(x_1, \dots, x_m)$ and F as $(F_1, \dots, F_m)$, and let us concentrate on a single point p of $\mathbb{Q}^m$, say $p = p_0 = (x_{1,0}, \dots, x_{m,0})$. The function F being a diffeomorphism, the m-by-m Jacobian matrix A of the derivative $DF(p_0)$ is invertible.

We reorder the variables of U so that the first J of the entries of p_0 are 0 and the others are > 0. Then we reorder the variables of V so that the first I of the entries of $F(p_0)$ are 0 and the others are > 0. Consider the restriction of $F_1, \dots, F_I$ to points $(x_{1,0}, \dots, x_{J,0}, x_{J+1}, \dots, x_m)$ as a function of several variables $(x_{J+1}, \dots, x_m)$. This function is ≥ 0 everywhere in a Euclidean neighborhood of $(x_{J+1,0}, \dots, x_{m,0})$ and takes on its minimum value 0 at $(x_{J+1,0}, \dots, x_{m,0})$. Thus the first partial derivatives of $F_1, \dots, F_I$ with respect to $(x_{J+1}, \dots, x_m)$ must be 0 at any point where the minimum value is attained. In symbols,

$$\Big(\frac{\partial F_i}{\partial x_j}\Big)(p_0) = 0 \quad \text{for } i \leq I \text{ and } j \geq J+1. \tag{$*$}$$

Arguing by contradiction, suppose that $I > J$. If $i \leq I$ and $j \geq I$, then we have $j \geq I > J$ and hence $j \geq J + 1$. In view of $(*)$, the Jacobian matrix A of $DF(p_0)$, whose $(i, j)^{\text{th}}$ entry is $a_{ij} = \big(\partial F_i/\partial x_j\big)(p_0)$, has $a_{ij} = 0$ for $i \leq I$ and $j \geq I$. Taking $k = I$ in Lemma 3.2, we see that the matrix A has $\det A = 0$, in contradiction to the fact that A is invertible. This contradiction shows that we must after all have had $I \leq J$. $\square$

Now we can introduce manifolds-with-corners. Let M be a separable Hausdorff topological space, and fix an integer $m \geq 2$. For purposes of working with manifolds-with-corners, a **chart** (M_α, α) on M of **dimension** m is a homeomorphism α of a nonempty open subset M_α of M onto an open subset $\alpha(M_\alpha)$ of $\mathbb{Q}^m$; the chart is said to be **about** a point p in M if p is in the domain M_α of α. When it is convenient to do so, we can restrict attention to charts (M_α, α) for which M_α is connected.

A smooth **manifold-with-corners** of dimension m is a separable Hausdorff space M with a family $\mathcal{F}$ of charts (M_α, α) of dimension m such that

(i) any two charts (M_α, α) and (M_β, β) in $\mathcal{F}$ are (smoothly) **compatible** in the sense that $\beta \circ \alpha^{-1}$, as a mapping of the open subset $\alpha(M_\alpha \cap M_\beta)$ of $\mathbb{Q}^m$ to the open subset $\beta(M_\alpha \cap M_\beta)$ of $\mathbb{Q}^m$, is a diffeomorphism,

(ii) the family of compatible charts (M_α, α) is an **atlas** in the sense that the open sets M_α cover M, and

(iii) the family $\mathcal{F}$ is maximal among families of compatible charts on M.

In the presence of an understood atlas, a chart will be said to be **compatible** if it is compatible with all the members of the atlas.

Because of Proposition 3.1, we can unambiguously transfer the definition of "index" from $\mathbb{Q}^m$ to any smooth manifold-with-corners M: if (M_α, α) is a chart about a point p in M, then the **index** of p in M is defined to be the index of $\alpha(p)$ in $\mathbb{Q}^m$. The points of index 0 are the **manifold points**, the points of index ≥ 1 are sometime called boundary points, and the points of index ≥ 2 are sometimes called corner points.

As with smooth manifolds in the sense of Chapter I and with smooth manifolds-with-boundary in Chapter II, any atlas of compatible charts for a smooth manifold-with-corners can be extended in one and only one way to a maximal atlas of compatible charts. Also if U is any nonempty open subset of an m dimensional smooth manifold-with-corners M, then U inherits the structure of a smooth manifold-with-corners as follows: first define an atlas of U to consist of the intersection of U with all members of the atlas for M, using the restrictions of the various functions α, and then discard occurrences of the empty set.

We turn to examples. Some of these will be examples of (smooth) manifolds-with-corners, and some will be nearly-but-not-quite examples of manifolds-with-corners. For some of the latter, there will be a simple way of subdividing or triangulating the given space that exhibits it as a finite union of manifolds-with-corners. In any case the theorem in Section 3 is going to be that the Stokes formula, $\int_{\partial M} \omega = \int_M d\omega$, holds for all manifolds-with-corners. In this equality the integral on the left side is carried on the points of index 1, and the integral on the right side is carried on the points of index 0. Our decompositions of some of the near manifolds-with-corners as finite unions of genuine manifolds-with-corners will have the Stokes formula holding on each piece, and we shall be able to add these formulas for the pieces to obtain the Stokes formula for the union.

EXAMPLES.

(1) Any smooth manifold-with-boundary of dimension ≥ 2 is a smooth manifold-with-corners, there being no corner points. Any filled compact convex polygon in dimension 2 is a manifold-with-corners, as a consequence of the definition.

(2) No manifold-with-corners has any phantom corner points, in which a boundary point can be interpreted either as a corner or not. In dimension 2, for example, the boundary changes direction at each corner point, and there are no angles of 0 or 360 degrees. This is a consequence of Proposition 3.1 and the fact that index is well defined. See Figure 3.1.

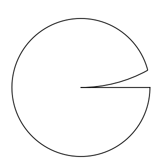

FIGURE 3.1. Manifolds-with-corners have no angles of 0 or 360 degrees.

(3) A manifold-with-corners has a certain local convexity to it. In dimension 2, for example, no interior angle of more than π can occur at a boundary point. This is a consequence of the fact that $\mathbb{Q}^m$ has this property at its boundary points.

(4) In dimension 2, a space that looks like the filled space in Figure 3.2, although not itself a manifold-with-corners (according to Example 3), can be subdivided into two adjacent pieces that are manifolds-with-corners by inserting an auxiliary line that becomes part of the boundary of each piece. (The auxiliary line is dashed in the figure.) An orientation on the set of manifold points yields by restriction an orientation on each of the two adjacent components and then yields an induced orientation on the boundaries of each piece. Since a single reflection is involved in passing from the induced orientation for the one component to the induced orientation for the other component, the two orientations on the auxiliary line will cancel in the computation of integrals over the boundary. Thus the validity of the Stokes formula $\int_{\partial M} \omega = \int_M d\omega$ for each of the constituents will imply the validity of the Stokes formula for the whole space.

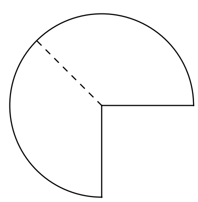

FIGURE 3.2. Triangulation available for angles greater than π.

(5) A filled closed cube in $\mathbb{R}^3$ is a smooth manifold-with-corners. The interior points have index 0, the points on the interiors of the six faces have index 1, the points on the interiors of the eight edges have index 2, and the eight vertices have index 3. The subset consisting of the faces, edges, and vertices is not a manifold-with-corners because no open neighborhood of a vertex is diffeomorphic to an open subset of any $\mathbb{Q}^m$.

(6) A filled closed tetrahedron in $\mathbb{R}^3$ is a manifold-with-corners, but a filled closed square pyramid in $\mathbb{R}^3$ is not. In the latter case, the pyramid can be subdivided into two adjacent pieces that are manifolds-with-corners (tetrahedra actually) by inserting an auxiliary triangle whose base is a diagonal of the square

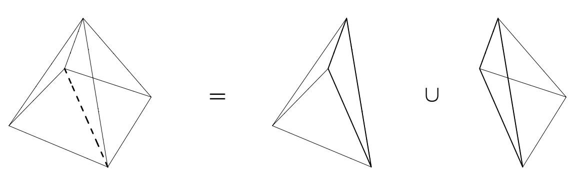

FIGURE 3.3. Square pyramid subdivided into two tetrahedra. The vertex of the pyramid appears at the top of each solid.

base of the pyramid and whose vertex is the vertex of the pyramid. See Figure 3.3. The relevant diagonal of the base is shown dashed. The auxiliary triangle becomes part of the boundary of each piece. As in Example 4, the two induced orientations on the added triangle are opposite and then cancel when computing integrals over boundaries. Thus the validity of the Stokes formula $\int_{\partial M} \omega = \int_M d\omega$ for each of the constituents will imply the validity of the Stokes formula for the whole space.

(7) A solid cylinder in $\mathbb{R}^3$ is a manifold-with-corners. Its surface, which consists of two closed disks for the ends and the product of a circle and a closed interval for the side of the cylinder, is a manifold-with-boundary. A solid cone in $\mathbb{R}^3$ such as $z^2 \leq x^2 + y^2 \leq 25$ is not a manifold-with-corners because the cone point at the origin has no open neighborhood diffeomorphic to an open subset of $\mathbb{Q}^3$; no simple way is evident for decomposing this solid cone into the union of nonoverlapping manifolds-with-corners.

(8) Whenever M is a smooth manifold-with-corners, then the points of index 0 form a smooth manifold, the points of index 0 or 1 form a smooth manifold-with-boundary, and the points of index 0 through 2 form a smooth manifold-with-corners. This example will be amplified in the next section when we introduce "strata" for smooth manifolds-with-boundary.

(9) The numeral 8 in Figure 3.4, once it has been filled, is not a manifold-with-corners because it has no neighborhood of the crossing point that is diffeomorphic to an open subset of $\mathbb{Q}^2$. However, the top half of the filled numeral is a manifold-with-corners, there being just the one corner at the crossing point. Similarly the bottom half is a manifold-with-corners. The whole space is thus the union of two manifolds-with-corners whose intersection is simply the crossing point. Accordingly the Stokes formula applies to each half. Since the crossing point has index 2 in both cases, it plays no role in integrations. Thus the validity of the Stokes formula for each piece will imply the validity of the Stokes formula for the whole filled numeral.

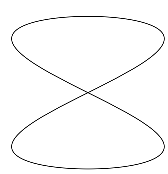

FIGURE 3.4. Numeral 8 centered at the origin, to be regarded as filled.

Finally we are in a position to introduce the notion of a smooth function and various related constructs for smooth manifolds-with-corners. A **smooth real-valued function** $f : M \to \mathbb{R}$ on the smooth manifold-with-corners of dimension m is by definition a function such that for each $p \in M$ and each compatible chart (M_α, α) about p, the function $f \circ \alpha^{-1}$ is smooth as a function from the open subset $\alpha(M_\alpha)$ of $\mathbb{Q}^m$ into $\mathbb{R}$. This is the expected definition, and there are no surprises. A smooth real-valued function is necessarily continuous.

If E is a nonempty open subset of M, the space of smooth real-valued functions on E will be denoted by $C^\infty(E)$. The space $C^\infty(E)$ is an associative algebra over $\mathbb{R}$ under the pointwise operations, and it contains the constants. The **support** of a real-valued function is, as always, the closure of the set where the function is nonzero. We write $C^\infty_{\text{com}}(E)$ for the subset of $C^\infty(E)$ of functions whose support is a compact subset of M.

Relative to a point p of the manifold-with-corners M, we define a **germ** at p, the **tangent space** $T_p(M)$ at p, and the **cotangent space** at p in the same way as in the manifold case. For the manifold points of M, the definition is completely unchanged. The only difference occurs in the case of boundary points: when matters are referred back to the model space $\mathbb{Q}^m$, the open sets of $\mathbb{Q}^m$ do not need to be open in the underlying Euclidean space $\mathbb{R}^m$. The space $\mathcal{C}_p(M)$ of germs at p is an associative algebra over $\mathbb{R}$ with identity.

The nature of $T_p(M)$ and that of $T^*_p(M)$ are unchanged from the manifold case. If (M_α, α) is a chart about p, and if $\alpha = (x_1, \dots, x_m)$, then a basis of $T_p(M)$ consists of the m first partials $\big[\partial/\partial x_j\big]$ evaluated at p. If p has j^{th} coordinate 0 in $\mathbb{Q}^m$, then $\big[\partial/\partial x_j\big]_p$ can be computed as a one-sided partial derivative. Examples of members of $T^*_p(M)$ are the differentials of smooth functions at p, the differential of f at p being defined by $(df)_p(L) = Lf$ for L in $T_p(M)$, just as in the manifold case.

We can then go on to define differential 1 forms, differential k forms, and smoothness of differential forms. There are no surprises. The notion of **pullback** of a differential form is still meaningful.

The derivative DF of a smooth function between manifolds-with-corners is defined just as in the case of manifolds. Let $F : M \to N$ be a smooth function from a smooth manifold-with-corners M of dimension m into a smooth manifold-with-corners N of dimension n. For any $p \in M$, the function F allows any germ $g \in \mathcal{C}_{F(p)}(N)$ to be pulled back to a germ $g \circ F$ in $\mathcal{C}_p(M)$. Then any tangent vector L in $T_p(M)$ is carried into a tangent vector $(DF)_p(L)$ in $T_{F(p)}(N)$ by the formula $(DF)_p(L)(g) = L(g \circ F)$. The result is a linear function $(DF)_p : T_p(M) \to T_{F(p)}(N)$ called the **derivative** of F at p.

The final preparatory step for working with manifolds-with-corners is to make smooth partitions of unity be available. We proceed exactly as at the end of Section II.2, beginning with analogs of Lemma 2.3 and 2.4.

Lemma 3.3. If U is a nonempty open subset of a smooth manifold-with-corners M and if f is in $C^\infty_{\text{com}}(U)$, then the function F defined on M so as to equal f on U and to equal 0 off U is in $C^\infty_{\text{com}}(M)$ and has support contained in U.

REMARK. This is proved in the same way that Lemma 1.2 was proved for smooth manifolds. The argument makes use of the Hausdorff property of M.

Lemma 3.4. Suppose that p is a point in a smooth manifold-with-corners M, that (M_α, α) is a compatible chart about p, and that K is a compact subset of M_α

containing p. Then there is a smooth function $f : M \to \mathbb{R}$ with compact support contained in M_α such that f has values in $[0, 1]$ and f is identically 1 on K.

REMARK. Except for changes in notation, this is proved in the same way as Lemma 2.4.

The notion of a **smooth partition of unity** of a manifold-with-corners M **subordinate** to the finite open cover $\{U_i\}$ of a compact subset K of M works just as in the case of smooth manifolds-with-boundary. The statement is as follows.

Proposition 3.5. Let M be a smooth manifold-with-corners, let K be a nonempty compact subset, and let $\{U_i \mid 1 \le i \le r\}$ be a finite open cover of K. Then there exist functions f_i in $C^\infty(M)$ for $1 \le i \le r$, taking values between 0 and 1 such that each f_i is identically 0 off a compact subset of U_i and $\sum_{i=1}^{r} f_i$ is identically 1 on K.

REMARK. Except for changes in notation, this is proved in the same way as Proposition 2.5.

2. Index and Strata

Let M be a smooth manifold-with-corners of dimension m. If p is in M and (M_α, α) is a chart about p, we have defined the index of p to be the number of integers k for which the member $\alpha(p)$ of $\mathbb{Q}^m$ has k^{th} coordinate 0. Proposition 3.1 showed that this number is independent of the chart, hence depends only on M and p. It is denoted by $\operatorname{index}_M(p)$. It satisfies $0 \le \operatorname{index}_M(p) \le m$.

The set M_+ of all points p of M with $\operatorname{index}_M(p) = 0$ is a smooth manifold of dimension m, and we have defined those points to be **manifold points**. The remaining points, those with $\operatorname{index}_M(p) \ge 1$, are sometimes called boundary points, and those with $\operatorname{index}_M(p) \ge 2$ are sometimes called corner points.

We define $S_k(M) = \{p \in M \mid \operatorname{index}_M(p) = k\}$ for $0 \le k \le m$, calling it the **stratum** of points in M of index k. It is plain that M is the disjoint union of its strata. Strata satisfy the additional conditions listed in the following proposition.

Proposition 3.6. If M is a smooth manifold-with-corners of dimension m, then its strata are such that

(a) each nonempty stratum $S_k(M)$ has the structure of smooth manifold of dimension $m - k$,
(b) for each k for which $S_k(M)$ is nonempty, the union of all strata $S_l(M)$ for $l \le k$ is a manifold-with-corners of dimension m,
(c) the closure of $S_k(M)$ is the union of all strata $S_l(M)$ for $l \ge k$,
(d) M is a smooth manifold if and only $S_k(M)$ is empty for all $k > 0$, and

(e) M is a smooth manifold-with-boundary if and only $S_k(M)$ is empty for all $k > 1$, and in this case the boundary is $S_1(M)$.

REMARKS. An example to bear in mind is that of a solid cube M in $\mathbb{R}^3$. The stratum $S_0(M)$ is the interior, $S_1(M)$ is the union of the six faces but without edges and vertices, $S_2(M)$ is the union of the eight edges but without the vertices, and $S_3(M)$ is the set of eight vertices. It might seem unfortunate that $S_1(M) \cup S_2(M) \cup S_3(M)$ and $S_2(M) \cup S_3(M)$ are not manifolds-with-corners, but such features will not affect us because our concern is only with Stokes's Theorem.

WARNING. Although those unfortunate features do not concern us, they do affect some authors who have different goals. Often those authors will change one or another definition in the theory to achieve their purposes. For example, counting each vertex twice allows one to make $S_2(M) \cup S_3(M)$ into the disjoint union of four closed intervals; in this way $S_2(M) \cup S_3(M)$ becomes a manifold-with-boundary. It is therefore necessary always to be alert to an author's definitions of manifold-with-corners and related concepts.

PROOF. In (a) for the case that $M = \mathbb{Q}^m$, $S_k(\mathbb{Q}^m)$ is the set of points that lie on exactly k hyperplanes $\{x_i = 0\}$. This is a smooth manifold, being diffeomorphic to the disjoint union of Euclidean spaces of dimension $m - k$. For general M, if p is a point in $S_k(M)$ and (M_α, α) is a compatible chart of dimension m about $p \in M$, then the set $S_k(M) \cap M_\alpha$ and the restriction of α form a chart of dimension $m - k$ about $p \in S_k(M)$. These charts in $S_k(M)$ are compatible and provide an atlas for $S_k(M)$.

Conclusions (b), (d), and (e) follow directly from the definitions.

In (c), the result is clear for the case that M is $\mathbb{Q}^m$ or is a nonempty open subset of $\mathbb{Q}^m$. Hence if (M_α, α) is a compatible chart for M, then the closure of $S_k(M_\alpha)$ in M_α is the union of all strata $S_l(M_\alpha)$ for $l \geq k$. Consequently the closure of $S_k(M)$ in M contains the union of all strata $S_l(M_\alpha)$ for $l \geq k$. This being so for all α, the closure of $S_k(M)$ contains the union of all strata $S_l(M)$ for $l \geq k$. Arguing by contradiction, suppose that the closure contains a point p that is not in the union. This p must be a limit point of $S_k(M)$. Choose a chart (M_α, α) about p. Since the complement of M_α in M is closed, p must be a limit point of $S_k(M_\alpha)$. By what we have already proved, p must be in some $S_l(M_\alpha)$ for $l \geq k$. Then also p lies in the larger set $S_l(M)$, in contradiction to the assumption that p is not in the union of the $S_l(M)$ for $l \geq k$. $\square$

3. Stokes's Theorem for Manifolds-with-Corners

A version of Stokes's Theorem is valid for manifolds-with-corners, the formula being $\int_{\partial M} \omega = \int_M d\omega$ as usual. Proposition 3.1b, which says that $M_+ \cup S_1(M)$ is a manifold-with-boundary, gives us the proper framework. The integration over M is really to be an integral over the manifold M_+, and the integration over ∂M is to be an integration over $S_1(M)$. Just as M_+ is a dense open manifold in M, so too $S_1(M)$ is a dense open manifold of the topological boundary ∂M of M, according to Propositions 3.6c and 3.6a.

Accordingly it is a reasonable question to ask why Stokes's Theorem for manifolds-with-corners is not just a special case of Theorem 2.7. The answer is that Theorem 2.7 assumes that the given $m-1$ form ω has compact support in the manifold-with-boundary. An example will illustrate. Let M be the closed filled unit square in $\mathbb{R}^2$. This is a compact manifold-with-corners, but the associated manifold-with boundary omits the four corners, each of which has index 2. Theorem 2.7 thus asks that the given ω have compact support in the space consisting of the square with the four corners deleted. Running through the usual argument would thus show us that the formula of Green's Theorem, namely,

$$\int_{\partial M} P\,dx + Q\,dy = \int_M \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dx\,dy,$$

is valid whenever P and Q are smooth functions on the square that vanish in a neighborhood of each of the corners. Attempting to derive the theorem for general smooth P and Q on the square from this special case requires a passage to the limit that is more difficult to justify than the complete proof of Stokes's Theorem for manifolds-with-corners that we give later in this section. We shall not abandon the thought of handling matters by a passage to the limit, however, but shall merely postpone consideration of it until Section 4. A close look at the passage to the limit lies behind the theory of Whitney's that we develop starting in Section 4.

Let M be an m dimensional manifold-with-corners with $m \geq 2$, let ∂M be its boundary, and let M_+ be its subset of manifold points. We shall say that M is **orientable** (or **oriented**) if M_+ is orientable (or oriented). This definition is meaningful because M_+ is a smooth manifold. Then $S_1(M)$ acquires an **induced orientation** as in Section II.3, since $M_+ \cup S_1(M)$ is a manifold-with-boundary.

Theorem 3.7. Let M be an oriented manifold-with-corners of dimension $m \geq 2$, regard its boundary ∂M as $S_1(M)$, and give the boundary the induced orientation. If ω is any smooth $m-1$ form on M of compact support, then

$$\int_{\partial M} \omega = \int_M d\omega.$$

PROOF. The model space is $\mathbb{Q}^m$, and we first prove the theorem in this special case. The smooth $m-1$ form ω necessarily has an expansion

$$\omega = \sum_{j=1}^m F_j\,dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_m,$$

the circumflex pointing to a missing term. All the coefficient functions F_j are smooth, and we have

$$d\omega = \sum_{j=1}^{m} \frac{\partial F_j}{\partial x_j}\, dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_m = \sum_{j=1}^{m} (-1)^{j-1} \frac{\partial F_j}{\partial x_j}\, dx_1 \wedge \cdots \wedge dx_m.$$

Since the support of ω is compact, we may assume that each F_j vanishes outside $[0, R]^m$ for some number R. Theorem 1.29 gives

$$\int_{\mathbb{Q}^m} d\omega = (-1)^{j-1} \sum_{j=1}^{m} \int_{[0,R]^m} \frac{\partial F_j}{\partial x_j}(x_1, \dots, x_m)\, dx_1 \cdots dx_m$$

because $\mathbb{Q}^m$ has the standard orientation for $\mathbb{R}^m$. We can evaluate the j^{th} integration by the Fundamental Theorem of Calculus, obtaining

$$\begin{aligned}\int_0^R \frac{\partial F_j}{\partial x_j}(x_1, \dots, x_m)\, dx_j &= F_j(x_1, \dots, R, \dots, x_m) - F_j(x_1, \dots, 0, \dots, x_m) \\ &= -F_j(x_1, \dots, 0, \dots, x_m).\end{aligned}$$

Therefore

$$\int_{\mathbb{Q}^m} d\omega = (-1)^{j-1} \sum_{j=1}^{m} \int_{[0,R]^{m-1}} (-F_j)(x_1, \dots, 0, \dots, x_m)\, dx_1 \cdots \widehat{dx_j} \cdots dx_m. \tag{$*$}$$

To compute $\int_{\partial M} \omega = \int_{S_1(\mathbb{Q}^m)} \omega$, we have to sort out the orientation of each component of $S_1(\mathbb{Q}^m)$. There are m components, the j^{th} one being given by

$$Z_j = \{(x_1, \dots, x_m) \mid x_j = 0 \text{ and all other } x_i > 0\}.$$

To orient Z_1, for example, we take an outward pointing vector like $(-1, 0, \dots, 0)$, follow it by the standard basis for the subspace where $z_1 = 0$, and see what is needed to transform it into the standard basis of the whole space. The change requires one sign change and the identity permutation, and hence Z_1 has the opposite orientation from the standard one. For Z_j, we argue similarly, and its orientation is $(-1)^j$ times the standard one. Meanwhile, dx_j equals 0 on Z_j, and only one term of ω survives in the integration. Thus Theorem 1.29 gives

$$\begin{aligned}\int_{S_1(\mathbb{Q}^m)} \omega &= \sum_{j=1}^{m} \int_{Z_j} \omega \\ &= (-1)^j \sum_{j=1}^{m} \int_{[0,R]^{m-1}} F_j(x_1, \dots, 0, \dots, x_m)\, dx_1 \cdots \widehat{dx_j} \cdots dx_m.\end{aligned} \tag{$**$}$$

From $(*)$ and $(**)$, we conclude that

$$\int_{\mathbb{Q}^m} d\omega = \int_{S_1(\mathbb{Q}^m)} \omega, \tag{$\dagger$}$$

and the proof of the theorem is complete when $M = \mathbb{Q}^m$.

To handle the general case, we proceed in the same manner as in the proof of Theorem 2.7: About each point p in M of the compact support S of ω, we choose a *positive* compatible chart (M_α, α). Since the sets M_{α_j} form an open cover of the compact set S, we can choose a finite subcover $\{M_{\alpha_1}, \dots, M_{\alpha_k}\}$. By Proposition 3.5 (instead of Proposition 2.5), let $\{\psi_1, \dots, \psi_k\}$ be a smooth partition of unity of M subordinate to this finite open cover. For $1 \le i \le k$, the $m-1$ form $\psi_i\omega$ is compactly supported in M_{α_i}, and the $m-1$ form $(\alpha^{-1})^*(\psi_i\omega)$ is compactly supported in $\alpha_i(M_{\alpha_i}) \subseteq \mathbb{Q}^m$. Let us extend it to all of $\mathbb{Q}^m$ by setting it equal to 0 off $\alpha_i(M_{\alpha_i}) \subseteq \mathbb{Q}^m$, leaving its name unchanged. Then

$$\begin{aligned}
\int_M d(\psi_i\omega) = \int_{M_{\alpha_i}} d(\psi_i\omega) &= \int_{\alpha_i(M_{\alpha_i})} (\alpha_i^{-1})^*(d(\psi_i\omega)) && \text{by Theorem 1.29} \\
&= \int_{\mathbb{Q}^m} (\alpha_i^{-1})^*(d(\psi_i\omega)) && \text{after extension by 0} \\
&= \int_{\mathbb{Q}^m} d((\alpha_i^{-1})^*(\psi_i\omega)), && \text{by Proposition 1.24} \\
&= \int_{\partial\mathbb{Q}^m} (\alpha_i^{-1})^*(\psi_i\omega) && \text{by } (\dagger) \\
&= \int_{\partial M_{\alpha_i}} \psi_i\omega = \int_{\partial M} \psi_i\omega && \text{by Theorem 1.29.}
\end{aligned}$$

Summing over i from 1 to k and using the fact that $\sum_{i=1}^k \psi_i$ is identically 1, we obtain

$$\int_M d\omega = \sum_{i=1}^k \int_M d(\psi_i\omega) = \int_{\partial M} \Big(\sum_{i=1}^k \psi_i\omega\Big) = \int_{\partial M} \omega,$$

and the proof of the general case is complete. $\square$

4. Whitney's Generalization of the Divergence Theorem

Although Theorem 3.7 handles many situations of practical interest for Stokes's Theorem, it by no means handles all. In Section 1 we saw at least five examples of spaces of geometric interest that could almost be handled by Theorem 3.7 but did not fit the hypotheses completely. In four of those examples, we identified ad hoc techniques that reduced those examples to ones that could be handled directly. Those techniques all essentially amounted to introducing a specific triangulation to subdivide the space into simpler spaces for which Theorem 3.6 could apply directly.[2] The sum of the Stokes formulas for the simpler spaces yielded the Stokes formula for the given space.

Working with triangulations is hard and asks for more of a geometric grasp of the space globally than we often have. In addition, we had no technique at all for handling the circular cone in Example 7. So we need a new device. The new device will come down to justifying the kind of passage to the limit that we tried to avoid early in the previous section. The main theorem that will incorporate that passage to the limit is Hassler Whitney's form of Stokes's Theorem.

In this section let us concentrate on situations where the underlying manifold or generalized manifold is a subset of $\mathbb{R}^m$ of full dimension m. This is the core of the problem. Effectively we are thus working on generalizing the m dimensional Divergence Theorem, which handled this case for manifolds-with-boundary and manifolds-with-corners when the space in question can be realized as a subset of $\mathbb{R}^m$. We shall see how one theorem of Whitney's handles all situations in $\mathbb{R}^m$ without further effort. We postpone to Section 6 a general theorem about cases of Stokes's Theorem that are not embedded in $\mathbb{R}^m$.

An example to keep in mind is the one in Example 7 that we could not handle, namely that of a filled ice-cream cone. So that we can concentrate on the vertex, let us think of the cone as infinite in size. The thought that suggests itself is that we might be able to handle the cone as a manifold-with-corners if we were to remove some of it near the vertex, and perhaps then we could pass to the limit.

Thus we return to the question we set aside early in Section III.3. If we have an exceptional set E on the boundary that we do not have tools to handle, can we discard the exceptional set so as to obtain a noncompact manifold-with-boundary, apply Theorem 2.7 to any compactly supported $m-1$ form ω on the manifold-with-boundary, and then pass to the limit to eliminate the support restriction on ω? Whitney's answer is yes as long as the exceptional set is not too large in a technical sense.

To fix the ideas, let U be a nonempty bounded open set in $\mathbb{R}^m$ with (compact) topological boundary B, and let E be a compact subset of B that we think of as small and exceptional. We shall impose conditions on B so that $(B-E)\cup U$ is a noncompact manifold-with-boundary. We are to be given a smooth $m-1$ form ω on $B\cup U$, with smoothness meaning as usual that in an open neighborhood of each point of $B\cup U$, ω extends to a smooth $m-1$ form on the open neighborhood.

[2]We have not sought techniques for handling general roughness of the differential forms that are involved. We work only with smooth forms and regard rough ones as not of practical interest.

We want to prove the Stokes formula $\int_{(B-E)} \omega = \int_U d\omega$ without making any assumption about the support of ω.

Write $D(x, E)$ for the distance from a point x in $\mathbb{R}^m$ to the compact set E. The key to quantifying the smallness of E is the order of magnitude of the Lebesgue measure of the open set where $D(x, E) < \delta$ when $\delta > 0$ is small; we may think of this open set as a thickened version of E. For example if $m = 2$ and E is a one-point set, then the set where $D(x, E) < \delta$ is a disk of radius δ, whose measure is $\pi\delta^2$. Still in $\mathbb{R}^2$ if E instead is a line segment of length 1, then the set where $D(x, E) < \delta$ has the shape of a filled racetrack, and its measure is $2\delta + \pi\delta^2$. In other words a one-point set leads us to the order of magnitude of δ^2, whereas a line segment leads us to the order of magnitude of δ. This distinction is what will allow us to handle each missing corner of a square in Green's Theorem, but we would not be able to handle a whole missing side.

More generally let $|A|$ be the Lebesgue measure of a Borel subset A of $\mathbb{R}^m$. Whitney's generalization of the Divergence Theorem in dimension m, given as Theorem 3.8 below,[3] will say that the condition

$$\lim_{\delta \downarrow 0} \delta^{-1} |\{x \in \mathbb{R}^2 \mid D(x, E) < \delta\}| = 0$$

is just the right hypothesis to allow us to ignore the exceptional set E and treat the whole generalized manifold as an ordinary manifold-with-boundary. We shall investigate sets E with this property in the next section.

In the meantime let us observe that a one-point set E in dimension m always satisfies this condition because $\delta^{-1}|\{x \in \mathbb{R}^2 \mid D(x, E) < \delta\}|$ is approximately a constant times δ^{m-1} for small δ. We already saw a number of cases in Section 1 where E consists of just a single point, and we shall recall them after proving the theorem. They will furnish our first examples where the theorem applies.

Theorem 3.8. Let U be a nonempty bounded open set in $\mathbb{R}^m$ with $m \geq 2$, let B be its topological boundary, and let E be a closed subset of B. Suppose further that $(B - E) \cup U$ is a smooth manifold-with-boundary of dimension m in the following sense:

> to each point p of $B - E$, there exists a unit vector $v(p)$ such that if axes in $\mathbb{R}^m$ are chosen with $v(p)$ in the x_1 direction, then the set of points of $B - E$ in some neighborhood of p is given by a smooth function $x_1 = h(x_2, \ldots, x_n)$ and the set of points of U in this neighborhood is given by the inequality $x_1 < h(x_2, \ldots, x_m)$.

[3]This is Theorem 14A in Whitney's book listed in the Selected References. The theorem here is what Whitney's published theorem says in case the differential form ω has no smoothness problems up to and including the boundary. The published theorem allows the differential form to have a certain amount of roughness.

Suppose further that E has the property that either E is empty or

$$\lim_{\delta \downarrow 0} \delta^{-1} |\{x \in \mathbb{R}^m \mid D(x, E) < \delta\}| = 0. \tag{$*$}$$

Let U be given the standard orientation from $\mathbb{R}^m$, and let $B - E$ be given the induced orientation. If ω is a compactly supported smooth $m-1$ form on $B \cup U$, then the Stokes formula holds in the sense that

$$\int_{B-E} \omega = \int_U d\omega. \tag{$**$}$$

REMARKS.

(1) Let us refer to the triple (U, B, E) in the theorem as a **Whitney domain** in $\mathbb{R}^m$. In his book Whitney himself referred to such triples with U connected as "standard domains." In our case, on the one hand, we want to treat certain examples with U disconnected, such as a filled numeral 8, as Whitney domains, and on the other hand, connectedness plays no role in the proof of Theorem 3.8. Thus we have deviated from Whitney's treatment and dropped the hypothesis of connectedness.

(2) The inset condition in the theorem describes certain charts about points of $B - E$ and tells how to orient them relative to the orientation of the underlying space $\mathbb{R}^m$. It really amounts to the condition that $(B - E) \cup U$ is a smooth manifold-with-boundary, saying in addition that all the charts are positively oriented with the induced orientation.

(3) If E is empty, condition $(*)$ is to be ignored, and the theorem still applies. In this case it amounts to the m dimensional Divergence Theorem for the compact smooth manifold-with-boundary $M = U \cup B$ and is a special case of Theorem 2.7.

(4) The condition that ω is smooth is to be understood to mean that about each point of $U \cup B$, the differential form ω extends to a smooth differential form in an open set of $\mathbb{R}^m$. Concretely this means that in a neighborhood of the point, ω has an expansion $\sum_{j=1}^{m} F_j(x, \dots, x_m)\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_m$ with each F_j smooth in a neighborhood of the point.

PROOF. Fix a closed ball X in $\mathbb{R}^m$ large enough to contain all the points of interest. We shall approximate ω by smooth forms ω_k that have compact support in $(B - E) \cup U$, apply Theorem 2.7 to each ω_k, and then pass to the limit.

Let I_k be the indicator function of the subset of $x \in X$ where $D(x, E) \geq 2^{-k}$, i.e., let $I_k(x)$ be 1 on that subset and 0 off the subset. Let J_k be the indicator function of the set of x where $D(x, E) < 2^{-k}$. Then $I_k(x) = 1 - J_k(x)$ for $x \in X$. Fix a smooth function $\varphi \geq 0$ on $\mathbb{R}^m$ that is supported on the closed unit ball and has $\int_{\mathbb{R}^m} \varphi\, dx = 1$, and let $\varphi_{k+1} = 2^{(k+1)m} \varphi(2^{(k+1)} x)$. The function φ_{k+1} is ≥ 0, has total integral 1, and is supported on the ball where $|x| \leq 2^{-(k+1)}$.

The function $I_k * \varphi_{k+1}$ is smooth and vanishes off the set where $D(x, E) \geq 2^{-(k+1)}$. The differential form $\omega_k = (I_k * \varphi_{k+1})\omega$ is smooth on $B \cup U$ and vanishes off the set where $D(x, E) \geq 2^{-(k+1)}$. Consequently it is a compactly supported form on the manifold-with-boundary $(B - E) \cup U$, and Theorem 2.7 applies to it. The theorem gives

$$\int_{B-E} \omega_k = \int_U d\omega_k$$

for all k. We shall prove that

$$\lim_k \int_{B-E} \omega_k = \int_{B-E} \omega \qquad \text{and} \qquad \lim_k \int_U d\omega_k = \int_U d\omega, \tag{†}$$

and then we will have proved $(**)$ and the theorem.

Let us examine the difference

$$\begin{aligned} \omega - \omega_k &= \omega(1 - (I_k * \varphi_{k+1})) \\ &= \omega(1 - (1 - J_k) * \varphi_{k+1}) \\ &= \omega(J_k * \varphi_{k+1}). \end{aligned} \tag{††}$$

The function $J_k * \varphi_{k+1}$ vanishes off the set where $D(x, E) > 2^{-(k-1)}$ and is ≤ 1 everywhere. Thus $\lim \omega_k = \omega$ pointwise in the complement of E, and dominated convergence applies to yield the first formula of (†).

Toward the second formula of (†), let us use Proposition 1.23a to write

$$d\omega - d\omega_k = d(J_k * \varphi_{k+1}) \wedge \omega + (J_k * \varphi_{k+1})\, d\omega$$

and

$$\Big| \int_U d\omega - \int_U d\omega_k \Big| \leq \Big| \int_U d(J_k * \varphi_{k+1}) \wedge \omega \Big| + \Big| \int_U (J_k * \varphi_{k+1})\, d\omega \Big|. \tag{‡}$$

The easy term to handle in (‡) is the second term. In it the form $d\omega$, being smooth on $B \cup U$, is integrable on U, and we saw in the previous paragraph that $J_k * \varphi_{k+1}$ tends to 0 pointwise off E, always being ≤ 1. Thus

$$\lim_k \textstyle\int_U (J_k * \varphi_{k+1}) d\omega = 0$$

by dominated convergence, i.e., the second term of (‡) tends to 0.

The first term of (‡) involves a sum of terms $(\partial/\partial x_j)(J_k * \varphi_{k+1})(dx_j \wedge \omega)$. Since $dx_j \wedge \omega$ is smooth on the compact set $B \cup U$, integration with it operates as the product of a bounded function by Lebesgue measure. Thus to show that the first term of (‡) tends to 0, it is enough to show that the integral of the coefficient function $(\partial/\partial x_j)(J_k * \varphi_{k+1})$ with respect to Lebesgue measure tends to 0. Let us abbreviate $\partial/\partial x_j$ as ∇_j and consider the coefficient $\nabla_j(J_k * \varphi_{k+1}) = J_k * \nabla_j \varphi_{k+1}$.

By the chain rule, $\nabla_j \varphi_{k+1}(x) = 2^{(k+1)} 2^{m(k+1)} (\nabla_j \varphi)(2^{(k+1)x})$, and we can write this as $\nabla_j \varphi_{k+1}(x) = 2^{k+1}(\nabla_j \varphi)_{k+1}$ if we continue to use subscript notation for dilations by powers of 2. With $\|\cdot\|_1$ denoting the L^1 norm with respect to Lebesgue measure, we have

$$\nabla_j(J_k * \varphi_{k+1}) = J_k * \nabla_j \varphi_{k+1} = J_k * 2^{k+1} (\nabla_j \varphi)_{k+1}$$

and
$$\begin{aligned} \|\nabla_j(J_k * \varphi_{k+1})\|_1 &\le 2^{k+1} \|J_k\|_1 \|(\nabla_j \varphi)_{k+1}\|_1 \\ &= 2^{k+1} \|\nabla_j \varphi\|_1 \big|\{x \in \mathbb{R}^m \mid D(x, E) < 2^{-k}\}\big|. \end{aligned}$$

The right side is a multiple of $\delta^{-1}\big|\{x \in \mathbb{R}^m \mid D(x, E) < \delta\}\big|$ for $\delta = 2^{-k}$, and it tends to 0 by hypothesis $(*)$. Thus the first term of (‡) tends to 0, and this completes the proof of the second formula of (†). $\square$

EXAMPLES. We have observed that the exceptional set certainly satisfies condition $(*)$ if it consists of just finitely many points, provided $m \ge 2$. The following were potential examples in dimension 2 in this situation that were mentioned in Section 1. We now see that they are all Whitney domains and that Theorem 3.8 is therefore applicable:

(1) any manifold-with-corners of dimension $m = 2$ embedded in $\mathbb{R}^2$, and in particular any filled compact convex polygon in dimension 2,

(2) any filled simple polygon in dimension 2, convex or not,

(3) any filled simple region in dimension 2 with finitely many curved sides even if those curved sides make angles of 0, 180, or 360 degrees with one another,

(4) a filled numeral 8 in $\mathbb{R}^2$.

In the next section we shall examine condition $(*)$ more closely, and we shall be led to examples with more complicated exceptional sets.

5. Sets with ℓ Dimensional Minkowski Content Zero

Let us examine more closely the condition $(*)$ on the exceptional set E so that Theorem 3.8 applies, namely that

$$\lim_{\delta \downarrow 0} \delta^{-1} |\{x \in \mathbb{R}^m \mid D(x, E) < \delta\}| = 0.$$

For $0 \le \ell \le m$ and $m \ge 1$, we define the ℓ **dimensional Minkowski content** of a nonempty compact set E in $\mathbb{R}^m$ to be

$$\mathcal{M}^\ell(E) = \lim_{\delta \downarrow 0} \big|\{x \in \mathbb{R}^m \mid D(x, E) < \delta\}\big| \big/ (\alpha_{m-\ell} \delta^{m-\ell})$$

if this limit exists. Here $\alpha_{m-\ell}$ is the $m-\ell$ dimensional volume of the unit ball in $\mathbb{R}^{m-\ell}$ if $\ell < m$, and we take $\alpha_{m-\ell}$ to be 1 if $\ell = m$. If the limit does not exist, then one refers to the lim sup and lim inf as the "upper ℓ dimensional Minkowski content" and "lower ℓ dimensional Minkowski content" of E, respectively. If $\ell = m$, the m dimensional Minkowski content of a compact set exists and equals the Lebesgue measure of the set.

In the setting of Theorem 3.8, ℓ equals $m-1$, and the assumption $(*)$ in the theorem is that the limit exists and equals 0. *Thus the assumption* $(*)$ *is that the* $m-1$ *dimensional Minkowski content of* E *is* 0.

In what follows it will simplify statements to adopt the convention that the ℓ dimensional Minkowski content of the empty set is 0.

It is useful to keep in mind the following example. With $\ell \le m$, suppose that E is an ℓ dimensional cube of side 1 positioned in m dimensional space as the product $[0,1]^\ell \times \{0\}^{m-\ell}$. To compute the volume of the δ neighborhood of E, we can integrate 1 over that neighborhhood. The integration then extends in each of the first ℓ variables over an interval of length between $1+\delta$ and $1+2\delta$, while in the last $m-\ell$ variables it extends over the ball of radius δ centered at the origin in $\mathbb{R}^{m-\ell}$, whose volume is $\alpha_{m-\ell}$. The result of the integration thus has to be something between $\alpha_{m-\ell}\delta^{m-\ell}(1+\delta)^\ell$ and $\alpha_{m-\ell}\delta^{m-\ell}(1+2\delta)^\ell$. Dividing by $\alpha_{m-\ell}\delta^{m-\ell}$ and letting δ tend to 0, we obtain $\mathcal{M}^\ell(E) = 1$. Thus the ℓ dimensional cube E in $\mathbb{R}^m$ has ℓ dimensional Minkowski content 1; the Minkowski content of that cube is 0 in dimensions larger than ℓ and is infinite in dimensions smaller than ℓ.

The set function $\mathcal{M}^\ell(E)$ is not asserted to be defined on all compact subsets of $\mathbb{R}^m$, but when it is defined, it is anyway nonnegative, and it has the property that if A and B are compact sets, then

$$\mathcal{M}^\ell(E \cup F) \le \mathcal{M}^\ell(E) + \mathcal{M}^\ell(F)$$

with equality if E and F are disjoint. In fact, the containment

$$\{x \in \mathbb{R}^m \mid D(x, E \cup F) < \delta\} \subseteq \{x \in \mathbb{R}^m \mid D(x,E) < \delta\} \cup \{x \in \mathbb{R}^m \mid D(x,F) < \delta\}$$

is valid for all δ; if E and F are disjoint and nonempty, then they are at a positive distance δ_0 from one another and the above containment is an equality for $\delta \le \delta_0$.

Because of condition $(*)$ in Theorem 3.8, our main interest is in what happens when the ℓ dimensional Minkowski content exists and equals 0 for a compact subset of $\mathbb{R}^m$ when $\ell \le m$. Let us record three easy facts about that situation:

(1) If E_1 and E_2 are compact in $\mathbb{R}^m$ with $E_2 \subseteq E_1$ and if E_1 has ℓ dimensional Minkowski content 0, then so does E_2.
(2) If E_1 and E_2 are compact in $\mathbb{R}^m$ with E_1 and E_2 having ℓ dimensional Minkowski content 0, then the same thing is true of $E_1 \cup E_2$.
(3) If the compact set E in $\mathbb{R}^m$ has finite ℓ dimensional Minkowski content, then E has k dimensional Minkowski content 0 for every k with $\ell < k \le m$, as follows by comparing the definitions of $\mathcal{M}^k(E)$ and $\mathcal{M}^\ell(E)$.

Let us pause and assess what this little theory tells us for Theorem 3.8. A Whitney domain in $\mathbb{R}^m$ was defined in effect as the closure of a nonempty bounded open set U in $\mathbb{R}^m$ such that the topological boundary B can be written as $B = (B - E) \cup E$, where $U \cup (B - E)$ is an m dimensional manifold-with-boundary and E is a compact subset of B of $m - 1$ dimensional Minkowski content 0.

However E is defined as a compact subset of B, the hope is that E has dimension $m-2$ or less and that consequently it has $m-1$ dimensional Minkowski content 0. The reality is that E is often hard to deal with. Accordingly we shall introduce some tools for working with the notion of ℓ dimensional Minkowski content 0.

To begin with, the definition of Minkowski content of a nonempty compact set E supplies a value that depends on external information about E. We shall establish an equivalent definition that depends only on internal information about E. Define

$$E^\delta = \{x \in \mathbb{R}^m \mid D(x, E) > \delta\} \qquad \text{and} \qquad B^\delta = \{x \in \mathbb{R}^m \mid |x| < \delta\}.$$

Since E is compact, only finitely many open balls of radius $< \delta$ are needed to cover E. Let

$$N(E, \delta) = \left\{\begin{matrix}\text{minimum number of open balls of} \\ \text{diameter} < \delta \text{ needed to cover } E\end{matrix}\right\}$$

and

$$N_{\text{sep}}(E, \delta) = \left\{\begin{matrix}\text{maximum number of points of } E \\ \text{at distance} \geq \delta \text{ from one another}\end{matrix}\right\}.$$

Lemma 3.9. For E compact and nonempty in $\mathbb{R}^m$,

(a) $N(E, \delta) \leq N_{\text{sep}}(E, \delta)$.
(b) $N_{\text{sep}}(E, \delta) \leq N(E, \delta/2)$,
(c) $|E^\delta| \leq N(E, \delta)|B_\delta|$, and
(d) $|E^\delta| \geq N_{\text{sep}}(E, \delta)|B_{\delta/2}|$.

PROOF. Write $B_r(x)$ for the open ball of all points y in $\mathbb{R}^m$ with $|y - x| < r$.

For (a), if $k = N_{\text{sep}}(E, \delta)$, choose a set $S = \{x_1, \dots, x_k\}$ of points of E such that $|x_i - x_j| \geq \delta$ for all $i \neq j$. The balls $B_{2\delta}(x_1), \dots, B_{2\delta}(x_k)$ must cover E because otherwise some point y of E has $|x_i - y| \geq 2\delta$ for all i and $S \cup \{y\}$ is a set of $k + 1$ points of E at distance $\geq \delta$ from one another. Thus some system of k open balls of radius 2δ covers E. Shrinking each of these balls a sufficiently small amount still leaves them covering E but having radius $< 2\delta$, therefore diameter $< \delta$. The number $N(E, \delta)$ is by definition $\leq$ this number k, and therefore $N(E, \delta) \leq N_{\text{sep}}(E, \delta)$.

For (b), if $k = N_{\text{sep}}(E, \delta)$, choose a set $S = \{x_1, \dots, x_k\}$ of points of E with $|x_i - x_j| \geq \delta$ for all $i \neq j$. If $\mathcal{C}$ is a collection of balls $B_{r_1}(y_1), \dots, B_{r_n}(y_n)$ of radius $< \delta/4$ that cover E, then we can associate to each index j of the members of S some index $i = i(j)$ of the members of $\mathcal{C}$ such that $B_{r_i}(y_i)$ contains x_j. No two members of S can be in any single $B_{r_i}(y_i)$ because the diameter of $B_{r_i}(u_i)$ is less than $2r_i$, which is less than $\delta/2$. Thus the function $j \mapsto i(j)$ is one-one from S into $\mathcal{C}$. This proves that the number of balls is $\geq$ the number of points in S. Hence the minimum possible number of balls is $\geq N_{\text{sep}}(E, \delta)$.

For (c), if $k = N(E, \delta)$, let $\mathcal{C} = \{B_{r_1}(x_1), \dots, B_{r_k}(x_k)\}$ be a collection of k open balls of radius $< \delta/2$ in $\mathbb{R}^m$ that cover E. If x is in E^δ, the compactness of E implies that there is a point y in E with $|x - y| = \delta$. The point y must lie in some $B_{r_j}(x_j)$, and thus $|x - x_j| \leq |x - y| + |y - x_j| \leq \delta/2 + r_j < \delta$. Thus the collection of balls $B_\delta(x_1), \dots B_\delta(x_k)$ covers E^δ, and we must have $|E^\delta| \leq k|B_\delta|$, as asserted.

For (d), if $k = N_{\text{sep}}(E, \delta)$, choose a set $S = \{x_1, \dots, x_k\}$ of points of E such that $|x_i - x_j| \geq \delta$ for all $i \neq j$. The balls $B_{\delta/2}(x_j)$ are pairwise disjoint and lie completely in E^δ. Thus $|E^\delta| \geq |B_{\delta/2}(x_1)| + \cdots + |B_{\delta/2}(x_k)| = k|B_{\delta/2}| = N_{\text{sep}}(E, \delta)|B_{\delta/2}|$. $\square$

Proposition 3.10. If $\ell \leq m$, a nonempty compact set E in $\mathbb{R}^m$ has ℓ dimensional Minkowski content equal to 0 if and only if

$$\lim_{\delta \downarrow 0} \delta^\ell N(E, \delta) = 0,$$

where

$$N(E, \delta) = \begin{Bmatrix} \text{minimum number of open balls of} \\ \text{diameter} < \delta \text{ needed to cover } E \end{Bmatrix}.$$

REMARK. In view of parts (a) and (b) of Lemma 3.9, it would be equivalent to write the condition as $\lim_{\delta \downarrow 0} \delta^\ell N_{\text{sep}}(E, \delta) = 0$. This equality depends only on E as a metric space and does not make use of any embedding. However, we will find the formulation of the condition as $\lim_{\delta \downarrow 0} \delta^\ell N(E, \delta) = 0$ to be more useful.

PROOF. Applying (a), (d), and (c) of Lemma 3.9 in turn, we obtain

$$N(E, \delta)|B_{\delta/2}| \leq N_{\text{sep}}(E, \delta)|B_{\delta/2}| \leq |E^\delta| \leq N(E, \delta)|B_\delta| = 2^m N(E, \delta)|B_{\delta/2}|,$$

and thus

$$2^{-m}\delta^m N(E, \delta)|B_1| \leq |E^\delta| \leq \delta^m N(E, \delta)|B_1|.$$

The proposition follows. $\square$.

A function F from a nonempty subset of $\mathbb{R}^a$ into $\mathbb{R}^b$ is said to satisfy a **Lipschitz condition** on a set E with constant C if $|F(x) - F(y)| \leq C|x - y|$ for all x and y in E. It follows from Taylor's Theorem with integral remainder[4] that any smooth function from an open convex set in $\mathbb{R}^a$ into $\mathbb{R}^b$ satisfies a Lipschitz condition when restricted to any compact subset of the domain.

[4]Theorem 3.11 of *Basic Real Analysis*.

Proposition 3.11. Let F be a function from a compact subset E of $\mathbb{R}^a$ into $\mathbb{R}^b$ that satisfies a Lipschitz condition, and suppose that $\ell \geq 0$ is an integer. If E has ℓ dimensional Minkowski content equal to 0 in $\mathbb{R}^a$, then $F(E)$ has ℓ dimensional Minkowski content equal to 0 in $\mathbb{R}^b$.

REMARK. No relationship between a and b is assumed.

PROOF. Decomposing F as the composition of a dilation followed by a function satisfying a Lipschitz condition with Lipschitz constant 1, we see that it is enough to prove the corollary in the case that the Lipschitz constant is 1. Under this assumption let E be a compact subset of $\mathbb{R}^a$ that has ℓ dimensional Minkowski content 0. We may assume that E is nonempty. In view of Proposition 3.10, we are assuming that $\lim_\delta \delta^\ell N(E, \delta) = 0$, and we want to prove that $\lim_\delta \delta^\ell N(F(E), \delta) = 0$.

Let E be covered by N open balls of diameter $< \delta$, say

$$E \subseteq B_{r_1}(x_1) \cup \cdots \cup B_{r_k}(x_k).$$

Then

$$F(E) \subseteq F(B_{r_1}(x_1)) \cup \cdots \cup F(B_{r_k}(x_k)),$$

and the right side is

$$\subseteq B_{r_1}(F(x_1)) \cup \cdots \cup B_{r_k}(F(x_k))$$

because F satisfies a Lipschitz condition with Lipschitz constant 1. This shows that

$$N(F(E), \delta)) \leq N(E, \delta),$$

and Proposition 3.11 follows from Proposition 3.10. $\square$

Proposition 3.11 allows us to introduce a well defined notion of ℓ dimensional Minkowski dimension 0 for compact subsets of any smooth manifold of dimension $\geq \ell$ and to show that smooth mappings of these manifolds into any Euclidean space of dimension $\geq \ell$ carry these sets into compact sets of ℓ dimensional Minkowski content 0 in the Euclidean space. The details are as follows.

Corollary 3.12. Let M be a smooth manifold of dimension m, let $\ell \geq 0$ be an integer, and let E be a nonempty compact subset of M. Suppose that $\{M_\alpha, \alpha)\}$ is an atlas for M such that some finite open cover $\{M_{\alpha_1}, \ldots, M_{\alpha_r}\}$ of E has the property that for each j with $1 \leq j \leq r$, each compact subset S of $M_{\alpha_j} \cap E$ has $\alpha_j(S)$ of ℓ dimensional Minkowski content 0 in $\mathbb{R}^n$. Then for every (M_β, β) in the atlas, each compact subset T of $M_\beta \cap E$ has $\beta(T)$ of ℓ dimensional Minkowski content 0 in $\mathbb{R}^n$.

REMARKS.

(1) When a finite open cover of E exists as in the lemma, we say that the compact subset E of M has ℓ **dimensional Minkowski content 0**.

(2) With this definition the finite union of compact subsets of ℓ dimensional Minkowski content 0 in a smooth manifold of dimension $n \geq \ell$ has ℓ dimensional Minkowski content 0. This is a consequence of the corresponding fact about compact subsets of Euclidean space.

(3) With only cosmetic changes in the proof, this corollary remains valid if "smooth manifold" in the statement is replaced by "smooth manifold-with-boundary" or "smooth manifold-with-corners."

PROOF. Fix the open cover $\{M_{\alpha_1}, \dots, M_{\alpha_r}\}$ of E, and choose by Lemma 1.26b an open subcover $\{P_{\alpha_1}, \dots, P_{\alpha_r}\}$ of E such that $P^{\text{cl}}_{\alpha_j} \subseteq M_{\alpha_j}$ for each j. Suppose that M_β is any member of the atlas and that T is a compact subset of $M_\beta \cap E$. Then $T = (P^{\text{cl}}_{\alpha_1} \cap T) \cup \cdots \cup (P^{\text{cl}}_{\alpha_r} \cap T)$ exhibits T as the union of respective compact subsets $P^{\text{cl}}_{\alpha_j} \cap T$ of $M_{\alpha_j} \cap E$. The set $\alpha_j(P^{\text{cl}}_{\alpha_j} \cap T)$ is a compact subset of $\alpha_j(M_{\alpha_j} \cap E)$ and by hypothesis has ℓ dimensional Minkowski content 0 in $\mathbb{R}^m$.

Let us apply Proposition 3.11 to the smooth mapping $F = \beta \circ \alpha_j^{-1}$, which is a diffeomorphism from the open set $\alpha_j(M_{\alpha_j} \cap M_\beta)$ onto the open set $\beta(M_{\alpha_j} \cap M_\beta)$. Since $\alpha_j(P^{\text{cl}}_{\alpha_j} \cap T)$ is a compact subset of $\alpha_j(M_{\alpha_j} \cap M_\beta)$ of ℓ dimensional Minkowski content 0, its image $\beta(P^{\text{cl}}_{\alpha_j} \cap T)$ under F is a compact subset of $\beta(M_{\alpha_j} \cap M_\beta) \subseteq \mathbb{R}^m$ of ℓ dimensional Minkowski content 0. Taking the union over j, we see that $\beta(T)$ has ℓ dimensional Minkowski content 0 in $\mathbb{R}^m$. $\square$

It is now easy to extend certain results about ℓ dimensional Minkowski content 0 from Euclidean space to smooth manifolds. Some extensions of this kind appear in the problems at the end of the chapter.

GEOMETRIC EXAMPLES.

(1) The above results allow us to see that various polyhedral sets meet condition $(*)$ for exceptional sets in Theorem 3.8. A filled square pyramid in $\mathbb{R}^3$ has four vertices, eight edges, five faces, and the solid part. The Stokes formula involves the solid part and the faces. All other potential contributions are compact of dimension ≤ 1, which is two less than the ambient dimension, and there are only finitely many of them. Corollary 3.14 says that each of them has 2 dimensional Minkowski content 0, and the finite union of compact sets of 2 dimensional Minkowski content 0 has Minkowski content 0. Therefore condition $(*)$ in Theorem 3.8 is satisfied, and the Stokes formula holds for a solid square pyramid.

(2) More generally any closed convex **polytope** in $\mathbb{R}^m$, i.e., the generalization to dimension m of a closed convex polyhedron in $\mathbb{R}^3$, fits this description. Aside from the solid and the faces, all other potential contributions can be taken to

be compact of dimension $\leq m-2$, and there are only finitely many of them. Corollary 3.14 says that each of them has $m-1$ dimensional Minkowski content 0, and the finite union of compact sets of $m-1$ dimensional Minkowski content 0 has Minkowski content 0. Thus again condition $(*)$ is satisfied, and theorem 3.8 applies.

(3) In any manifold-with-corners of dimension m that is embedded in Euclidean space $\mathbb{R}^m$, the exceptional set that arises in Theorem 3.8 consists of all points of index ≥ 2, i.e., of all corner points. The subset of corner points that lies within the support of a given smooth $m-1$ form is compact, and Corollary 3.14 says that this subset satisfies condition $(*)$. Thus Theorem 3.7 is a special case of Theorem 3.8 if the given manifold-with-corners embeds in $\mathbb{R}^m$. A filled ice cream cone in $\mathbb{R}^3$ is an example. The full version of Stokes's Theorem that we give in the next section will apply to all Whitney manifolds and in particular will apply to all smooth manifolds-with-corners, whether embedded in $\mathbb{R}^m$ or not.

A further class of examples for Theorem 3.8 is of an algebraic nature and arises from zero loci of real polynomials in several real variables $x_1, \ldots, x_m$. We shall assume that a given polynomial F is a function of m variables and is irreducible over $\mathbb{R}$. Guided by Theorem 3.8, we consider the region in $\mathbb{R}^m$ where $F < 0$. The statement of that theorem gives us a clue what to expect with the boundary. At a point on the topological boundary, if $\partial F/\partial X_j$ is nonzero for some j, then the Implicit Function Theorem allows us to solve locally for x_j in terms of the other variables, obtaining a smooth function f of $m-1$ variables, and locally a part of the boundary of the region will be the graph of f with the part of $\mathbb{R}^m$ below the graph corresponding to the interior of the region under study. The subset of the boundary for which this condition holds is thus part of the boundary of a manifold-with-boundary in the familiar sense. The subset of the boundary for which the condition fails is called the **singular set** of F and is taken as the exceptional set E in the theorem.

When we are applying Theorem 3.8, it is helpful for our regions in $\mathbb{R}^m$ to be bounded, so that integrals are well defined, and we think of intersecting our set of interest with a large closed ball $\{x \mid |x| \leq C\}$ for some C. Since the goal is to have a theorem for differential forms of compact support, we take always take C large enough so that every point of the support has $|x| \leq C$, and the part of the boundary where $|x| = C$ does not enter the Stokes formula. The adjustment of requiring $|x| \leq C$ results in temporarily enlarging the boundary so that some points with $|x| = C$ are included. These new boundary points are uninteresting for our current purposes, since they play no role at the end.[5]

[5]This description is not quite good enough. To avoid problems from the sharp edge of the region where $|x| = C$, we actually work with the region where $F < 0$ and a specific smooth auxiliary function in $C^\infty_{\text{com}}(\mathbb{R}^m)$ is > 0. The auxiliary function can be taken to be $\varphi(C^{-1}x)$, where φ is a function ≥ 0 in $C^\infty_{\text{com}}(\mathbb{R}^m)$ that is identically 1 for $|x| \leq \frac{1}{2}$ and is identically 0 for $|x| \geq 1$. This auxiliary function is smooth and equals 0 for $|x| \geq C$.

ALGEBRAIC EXAMPLES.

(1) With two variables, let $F(x, y) = x^4 + y^4 - 1$. Our region becomes the set where $x^4 + y^4 < 1$. This is a bounded region of $\mathbb{R}^2$. The respective first partial derivatives of F are $4x^3$ and $4y^3$, and they do not simultaneously vanish at any point of our locus. Thus the exceptional set E is empty, and Theorem 3.8 for this case reduces to the ordinary Divergence Theorem for $\mathbb{R}^2$, hence to Green's Theorem in the plane if we adjust notation suitably. The business with introducing a large closed ball is unnecessary since our region is already bounded.

(2) Let the underlying space be $\mathbb{R}^4$, which we can identify with the space of 2-by-2 real matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ if we want. We take F to be the determinant function $ad - bc$, and we consider the set of all matrices x for which $\det x \le 0$. To a first approximation the open set U in Theorem 3.8 will be the set of all matrices x for which $\det x < 0$, and the topological boundary B will be the set where $\det x = 0$. However, we are not interested in effects from considering large matrices, and we therefore consider only those matrices x for which $|x| \le C$ for some positive constant C, where $|x|^2$ is the sum of the squares of the entries. Thus the actual U is the set of x with $\det x < 0$ and $|x| < C$. The actual topological boundary B consists of an interesting part where $\det x = 0$ and $|x| < C$ and an uninteresting part where $|x| = C$. The first partial derivatives of det are d, $-b$, $-c$, and a, respectively, and they vanish simultaneously only when $x = 0$. The point with $x = 0$ happens to be one of the points on the locus $\det x = 0$. Thus the singular set consists of $x = 0$ alone.

Thus the interesting part of the boundary B consists of the all points where $\det x = 0$. Points x in its nonsingular part have $x \ne 0$, and the exceptional set E consists of 0 alone.[6] Since a one-point set satisfies condition $(*)$ of Theorem 3.8, (U, B, E) is a Whitney domain, and the Stokes formula is applicable in this situation.

(3) Let the underlying Euclidean space be $\mathbb{R}^9$ realized as the space of all 3-by-3 real matrices. We study the set where $\det x \le 0$. Again we want to know where $\det x = 0$, and we want to identify the singular set. Each matrix entry function $x \mapsto x_{ij}$ is a coordinate function, and we want to examine the first partial derivative $\partial(\det x)/\partial x_{ij}$. Thus let e_{ij} be the matrix for which $x_{ij}(e_{ij}) = \delta_{ij}$. By definition,

$$\frac{\partial(\det x)}{\partial x_{ij}} = \frac{d}{dt}\,\det(x + t x_{ij})\Big|_{t=0} = \lim_{t \to 0} t^{-1}(\det(x + t e_{ij}) - \det x).$$

Since det is an alternating multilinear function of its columns, the expression within the outer parentheses on the right equals the determinant of a matrix

[6] The uninteresting part of B consists of all points with $|x| = C$. No point with $|x| = C$ has all four first partial derivatives equal to 0, and therefore the singular set for this example is completely contained in the interesting part of the boundary.

that equals x in all but the j^{th} column and there equals the j^{th} column of te_{ij}. Expanding the determinant by cofactors, we see that the limit collapses to $(-1)^{i+j}$ times the $(i, j)^{\text{th}}$ minor[7] of x. So the partial derivative that we seek is just a 2-by-2 minor of x. The set where all first partial derivatives vanish is exactly the set where all 2-by-2 minors are 0, which is the set of all matrices of rank at most 1. The condition on the minors implies that $\det x = 0$, and consequently the singular set of the locus where $\det x = 0$ is the set of matrices of rank ≤ 1. Let E be this set.

We shall want to apply Theorem 3.8. The open set U has dimension 9, and the nonsingular part $B - E$ of the boundary has dimension 8. What we might expect is that somehow the singular set E has dimension at most 7, and then condition $(*)$ ought to be satisfied in the theorem. The set E is not a manifold, however, and some care is needed. What we really want is for the compact set E to have 8 dimensional Minkowski content 0. To see this, we shall write E as the union of 9 compact subsets of 5 dimensional vector subspaces of $\mathbb{R}^9$, and each of these compact subsets will have 8 dimensional Minkowski content 0; then Remark 2 after Corollary 3.12 will allow us to conclude that E has 8 dimensional Minkowski content 0. For the moment fix attention on the first row and column of matrices, and consider a member x of E with $x_{11} \neq 0$. Since each x in E has rank ≤ 1, the second and third columns of this x must be multiples of the first column. The set of matrices for which the second and third columns are multiples of the first is a linear subspace of $\mathbb{R}^9$ of dimension 5, and x lies in this subspace. (The first column contributes 3 to the dimension, and each multiple contributes one more.)

We can argue similarly with each of the nine pairs of indices (i, j), not just $(1, 1)$. If a member x of E has $x_{ij} \neq 0$, then x lies in a certain (different) 5 dimensional vector subspace of $\mathbb{R}^9$. The member 0 of E lies in all of these subspaces. The conclusion is that E lies in the union of nine specific subspaces of $\mathbb{R}^9$ of dimension 5. The intersection of E with each subspace is closed, hence compact, and thus E is exhibited as the finite union of compact sets lying in 5 dimensional subspaces. We have seen that any compact subset of $\mathbb{R}^k$ has $m - 1$ dimensional Minkowski content 0 if $k < m - 1$. Here we have $k = 5$ and $m = 9$, and the conclusion is that E has 8 dimensional Minkowski content 0.

Therefore condition $(*)$ is met, and Theorem 3.8 applies. Once again we are skipping lightly over the uninteresting part of the boundary where $|x| = C$. We may do so because we are interested only in differential forms of compact support.[8] Anyway the Stokes formula applies to differential forms of degree 9 with U as the set of 3-by-3 matrices of negative determinant, with $B - E$ as the

[7] The $(i, j)^{\text{th}}$ **minor** of an n-by-n matrix is the determinant of the matrix of size $n - 1$ obtained by deleting the i^{th} row and j^{th} column.

[8] In this example, there are matrices in the set E that lie on the sphere $|x| = C$, but they can be ignored because of the smoothing technique mentioned in an earlier footnote.

set of 3-by-3 matrices x of rank 2, and with E as the set of 3-by-3 matrices of rank ≤ 1.

6. Whitney's Global Form of Stokes's Theorem

For the final stage in our work with Stokes's Theorem, we shall expand our repertory of model cases. Then we can piece together local results to get the global form of Stokes's Theorem that we seek. The setting will be a "Whitney manifold" of dimension m, an object that we define below. In the end we will allow three types of model cases: $\mathbb{R}^m$, $\mathbb{H}^m$, and Whitney domains in $\mathbb{R}^m$, Whitney domains having been defined in the remarks with Theorem 3.8. It would be enough to use Whitney domains themselves as the sole kind of model, but it will help us to include $\mathbb{R}^m$ and $\mathbb{H}^m$ so that we can easily handle manifold points and well behaved boundary points with models that do not involves Whitney domains.

There is one subtle qualitative difference between the settings of manifolds-with-boundary and manifolds-with-corners vs. the setting of Whitney manifolds. In the earlier settings, there were different kinds of points: manifold points and boundary points in the case of manifolds-with-boundary, and points of different index in the case of manifolds-with-corners. Telling one kind of point apart from another was a question intrinsic to the point. With a Whitney domain (U, B, E) and therefore also with Whitney manifolds, the distinction between different kinds of points is no longer intrinsic. Indeed, we shall still have manifold points corresponding to U, ordinary boundary points corresponding to $B - E$, and exceptional boundary points corresponding to E, but it is always possible to change the label of one boundary point in B from ordinary to exceptional without affecting the validity of Theorem 3.8. Thus identifying exceptional points depends at least partly on how we label them. In order to have a theory that parallels the theories of manifolds-with-boundary and manifolds-with-corners, it will be necessary to carry along this information about labels in some of our definitions. As we make the definitions, it will be helpful to keep one nontrivial example in mind.

EXAMPLE. The surface S of an ice-cream cone in $\mathbb{R}^3$. The curved part of the surface can be realized as

$$\{(x, y, z) \mid x^2 + y^2 = z^2 \text{ and } 0 \leq z \leq 1\},$$

let us say. The points $\{(x, y, z) \mid x^2 + y^2 = 1 \text{ and } z = 1\}$ can be taken to be ordinary points of the boundary, and the point $(0, 0, 0)$ is an exceptional point of the boundary. This example is not a smooth manifold-with-boundary because of the behavior near the origin, and it is not covered by Theorem 3.8 because the surface is not a subset of dimension 3 in $\mathbb{R}^3$. Thus at this stage we do not know whether the Stokes formula is valid for S or not. Theorem 3.12 will affirm that it is indeed valid. In the statement of the formula, the integration over the boundary turns out to be limited to the 1 dimensional part of the boundary; the point $(0, 0, 0)$ plays no role.

Fix an integer $m \geq 2$. A **local Whitney domain** in $\mathbb{R}^m$ is the intersection of a Whitney domain $W = (U, B, E)$ in $\mathbb{R}^m$ with an open set O of $\mathbb{R}^m$ under the assumption that $U \cap O$ is nonempty. The subset of $\mathbb{R}^m$ of interest is then $(U \cup B) \cap O$, and the triple is $W \cap O = (U \cap O, B \cap O, E \cap O)$. The set $B \cap O$ is relatively closed in the closure of U, and $E \cap O$ is relatively closed in $B \cap O$. Observe that the set $E \cap O$ need not be compact.

We shall define "Whitney manifolds" M of dimension m. Let M be a locally compact separable metric space, let ∂M be a closed subset of M, and let $\partial_0 M$ be a closed subset of ∂M. The points of ∂M will be called the **boundary points** of M, and the points of $\partial_0 M$ will be called the **exceptional points**. Either of ∂M or ∂M_0 is allowed to be empty. For purposes of defining M as a Whitney manifold, a **Whitney chart** (M_α, α) on M of **dimension** m is a homeomorphism α of a nonempty open subset M_α of M onto some local Whitney domain $W_\alpha \cap O_\alpha$ in $\mathbb{R}^m$, say with $W_\alpha = (U_\alpha, B_\alpha, E_\alpha)$, such that the restriction of α to $M_\alpha \cap \partial M$ is a homeomorphism onto $B_\alpha \cap O_\alpha$ and the restriction of α to $M \cap \partial_0 M$ is a homeomorphism onto $E_\alpha \cap O_\alpha$. The image of α is understood to be $(U_\alpha \cup B_\alpha) \cap O_\alpha$. When the local Whitney domain has no exceptional points, i.e., when $E_\alpha \cap O_\alpha$ is empty, a Whitney chart is just an ordinary chart.

The Whitney chart (M_α, α) is said to be **about** a point p in M if p is in the domain M_α of α.

On such a space M, two charts (M_α, α) and (M_β, β) for which $M_\alpha \cap M_\beta$ is nonempty will be said to be smoothly **compatible** if $\beta \circ \alpha^{-1}$, as a mapping of the subset $\alpha(M_\alpha \cap M_\beta)$ of $\mathbb{R}^m$ to the subset $\beta(M_\alpha \cap M_\beta)$ of $\mathbb{R}^m$, is smooth and its inverse $\alpha \circ \beta^{-1}$ is smooth. As usual, smoothness at a boundary point means that the function extends to a smooth function in a neighborhood of the boundary point.

The locally compact separable metric space M is said to be a **Whitney manifold** of dimension m if a system $\mathcal{F}$ of Whitney charts $(M_\alpha, \varphi_\alpha)$ on M of dimension m is specified such that

(i) any two charts (M_α, α) and (M_β, β) in $\mathcal{F}$ are smoothly **compatible**,
(ii) the system of compatible charts (M_α, α) is an **atlas** in the sense that the sets M_α together cover M, and
(iii) $\mathcal{F}$ is maximal among families of compatible charts on M.

The next step is to review for Whitney manifolds all the constructions of smooth functions, tangent spaces, differential forms, etc. that we did for smooth manifolds, then for smooth manifolds-with-boundary, and finally for smooth manifolds-with-corners and check that the whole theory goes through with no surprises. This step is repetitious, and we omit it.

Let M be a Whitney manifold of dimension m. We say that M is **oriented** if the smooth manifold $M - \partial M$ is oriented. In this case, $\partial M - \partial_0 M$ is the finite or countably infinite union of its open components, each of which is a connected smooth manifold of dimension $m - 1$. We give each component the orientation induced from $M - \partial M$, and the result is that $\partial M - \partial_0 M$ becomes an oriented smooth manifold of dimension $m - 1$.

Theorem 3.12. Suppose that M is an oriented Whitney manifold of dimension m with boundary ∂M and exceptional set $\partial_0 M$, and suppose further that $\partial M - \partial_0 M$ is given the induced orientation. If ω is a compactly supported smooth differential $m - 1$ form on M, then the Stokes formula holds for M in the sense that

$$\int_{\partial M - \partial_0 M} \omega = \int_{M - \partial M} d\omega.$$

REMARKS. This theorem is based on Theorem 18A of Whitney's book in the Selected References. What we have stated here is mostly formalism, the deep result being Theorem 3.8 above. However, Theorem 3.12 is not a tautology, since as we shall see, it does say something new about the surface of an ice-cream cone in $\mathbb{R}^3$.

The notion of a **smooth partition of unity** of a Whitney manifold M **subordinate** to the finite open cover $\{U_i\}$ of a compact subset K of M works just as in the cases of smooth manifolds, smooth manifolds-with-boundary, and smooth manifolds-with-corners. This step too requires a little checking, and we omit it. The statement is as follows.

Lemma 3.13. Let M be an Whitney manifold, let K be a nonempty compact subset, and let $\{U_i \mid 1 \leq i \leq r\}$ be a finite open cover of K. Then there exist functions f_i in $C^\infty(M)$ for $1 \leq i \leq r$, taking values between 0 and 1, such that each f_i is identically 0 off a compact subset of U_i and $\sum_{i=1}^{r} f_i$ is identically 1 on K.

PROOF OF THEOREM 3.12. About each point p in M of the compact support S of ω, we choose a *positive* compatible Whitney chart (M_α, α). This is possible since the positive compatible charts form an atlas, M being oriented. Since the sets M_{α_j} form an open cover of the compact set S, we can choose a finite subcover $\{M_{\alpha_1}, \dots, M_{\alpha_k}\}$. By Lemma 3.13 let $\{\psi_1, \dots, \psi_k\}$ be a smooth partition of unity of M subordinate to this finite open cover.

For each i with $1 \leq i \leq k$, $\alpha_i(M_{\alpha_i})$ is open in one of the model spaces $\mathbb{R}^m$, $\mathbb{H}^m$, or a Whitney domain (U, B, E), and $\psi_i\omega$ is compactly supported within that open subset of the model space. Since the model space is Hausdorff, the extension of $\psi_i\omega$ by 0 on the complement of $\alpha_i(M_{\alpha_i})$ is compactly supported and smooth on the whole model space.

For $1 \leq i \leq k$, the $m-1$ form $\psi_i\omega$ is compactly supported in M_{α_i}, and the $m-1$ form $(\alpha^{-1})^*(\psi_i\omega)$ is compactly supported in the open subset $\alpha_i(M_{\alpha_i})$ of one of the model spaces. Let us extend it to the whole model space by setting it equal to 0 off $\alpha_i(M_{\alpha_i})$, leaving its name unchanged. The computation is then the same in all cases, but the notation has to be interpreted a little differently when the model space is a Whitney domain. When the model space is $\mathbb{R}^m$ or $\mathbb{H}^m$, the computation is

$$\begin{aligned}
\int_M d(\psi_i\omega) = \int_{M_{\alpha_i}} d(\psi_i\omega) &= \int_{\alpha_i(M_{\alpha_i})} (\alpha_i^{-1})^*(d(\psi_i\omega)) && \text{by Theorem 1.29} \\
&= \int_{\text{model}} (\alpha_i^{-1})^*(d(\psi_i\omega)) && \text{after extension by 0} \\
&= \int_{\text{model}} d((\alpha_i^{-1})^*(\psi_i\omega)), && \text{by Proposition 1.24} \\
&= \int_{\partial(\text{model})} (\alpha_i^{-1})^*(\psi_i\omega) && \text{by Stokes for model} \\
&= \int_{\partial M_{\alpha_i}} \psi_i\omega = \int_{\partial M} \psi_i\omega && \text{by Theorem 1.29.}
\end{aligned}$$

In the above computation the first five integrations are understood to extend over the set of manifold points, not the full space indicated, and with that understanding we get the desired equality $\int_M d(\psi_i\omega) = \int_{\partial M} \psi_i\omega$. The expression "Stokes for model" refers to Theorem 2.1 or 2.7.

When the model space is a Whitney domain, the expression "Stokes for model" refers to Theorem 3.8. The first five lines of the above display again extend over the set of manifold points, and that is the way that Theorem 3.12 writes them. The integrations over the boundary extend only over the ordinary points of the boundary, according to Theorem 3.8, and an adjustment to the above notation needs to be made to take this fact into account.

In short we obtain the formula

$$\int_{M-\partial M} d(\psi_i\omega) = \int_{\partial M - \partial_0 M} \psi_i\omega$$

in every case. Summing over i from 1 to k and using the fact that $\sum_{i=1}^{k} \psi_i$ is identically 1, we obtain

$$\int_{M-\partial M} d\omega = \sum_{i=1}^{k} \int_{M-\partial M} d(\psi_i\omega) = \int_{\partial M-\partial_0 M} \big(\sum_{i=1}^{k} \psi_i\omega\big) = \int_{\partial M - \partial_0 M} \omega,$$

and the proof of the theorem is complete. $\square$

EXAMPLE. The surface S of an ice-cream cone in $\mathbb{R}^3$, continued. Let us see that the surface is a Whitney manifold of dimension 2 with just one Whitney chart. The image of the chart is the Whitney domain in $\mathbb{R}^2$ given by the punctured unit disk, with the puncture considered as an exceptional point of the boundary. Thus $U = \{(a,b) \in \mathbb{R}^2 \mid 0 < a^2 + b^2 < 1\}$, $B = \{(0,0)\} \cup \{(a,b) \mid a^2 + b^2 = 1\}$, and $E = \{(0,0)\}$. The chart is (S, φ) with the mapping $\varphi : S \to U \cup B$ given by

$$(a,b) = \varphi(x,y,z) = (x,y) \quad \text{for} \quad 0 \le x^2 + y^2 \le 1.$$

Since E consists of a single point, we have seen E satisfies the key hypothesis $(*)$ in Theorem 3.8, and therefore (U, B, E) is a Whitney domain.[9] The function φ is a homeomorphism of S onto the closed unit disk $U \cup B$. Since there is just one chart, no compatibility of charts needs to be proved. Theorem 3.12 applies. According to the theory, computations proceed just as with Green's Theorem for the unit disk; the exceptional point $(0,0)$ plays no role in the integrations.

7. Problems

1. (a) A **compact convex polyhedron** in $\mathbb{R}^3$ is a compact set that does not lie in a single plane and that is the intersection of finitely many closed half planes. It has a number F of 2 dimensional faces, a number E of 1 dimensional edges, and a number V of 0 dimensional vertices. According to a formula due to Euler, these numbers are related by $F + V = E + 2$. Assume that the polyhedron is nondegenerate in the sense that no three vertices are collinear, and for simplicity assume that it is in "general position," which means that no four vertices are coplanar. Prove that the polyhedron can be triangulated, i.e., that it can be be written as the union of tetrahedra in such a way that each vertex of a tetrahedron is a vertex of the original polyhedron and that any two tetrahedra either are disjoint or intersect in a single face.

   (b) Deduce Stokes's Theorem for compact convex polyhedra in $\mathbb{R}^3$ from the result for tetrahedra, which is an instance of Theorem 3.7. Handle the necessary cancellation in the boundary integral in the same way as in Example 4 of Section 1.

2. Show that a compact manifold-with-corners of dimension m that is embedded in $\mathbb{R}^m$ is an example of a Whitney domain of dimension m, the exceptional set consisting of all points of index ≥ 2.

[9] It is possible to verify $(*)$ using the more sophisticated theory of Section 5 rather than the direct computation that appeared in Section 4. In the terminology of Section 5, E has 1 dimensional Minkowski content equal to 0 because, for example, it has finite nonzero 0 dimensional Minkowski content.

3. Guided by the third algebraic example in Section 5, show that a bounded portion of the subset of the space of 4-by-4 real matrices where $\det x \leq 0$ can be made into a Whitney domain for which the exceptional set E is the set of all matrices of rank ≤ 2.

4. For which of the following functions and vector spaces of matrices does the procedure of the algebraic examples of Section 5 lead to a Whitney domain (U, B, E)? Describe B and E in each case.
   (a) $F(x, y, z) = z(z - xy)$ and the space $\mathbb{R}^4$,
   (b) $F(x) = \mathrm{Re}(\det(x))$ and the space of all 2-by-2 complex matrices.
   (c) $F(x) = \det(x)$ and the space of all skew-symmetric 4-by-4 real matrices,

Problems 5–9 concern the Divergence Theorem.

5. Let V be the solid in $\mathbb{R}^3$ given by

$$\{(x, y, z) \mid x^2 + y^2 + (z-2)^2 \leq 4 \quad \text{and} \quad x^2 + y^2 + (z+1)^2 \leq 1\}.$$

   (a) Check that V is a manifold-with-corners.
   (b) If S is the surface of V, evaluate $\int_S x^2\, dy \wedge dz$, where S is oriented via an outward pointing vector.

6. Evaluate $\int_S F \cdot d\mathbf{S}$, where $F = 3y\mathbf{i} + 2x\mathbf{j} + (z-8)\mathbf{k}$ and S is the surface of the solid in $\mathbb{R}^3$ bounded by the coordinate planes $x = 0$, $y = 0$, and $z = 0$, and by the plane $4x + 2y + z = 8$. Again S is oriented by an outward pointing vector.

7. Let S be the surface in $\mathbb{R}^3$ defined by

$$x^4 + y^4 + z^4 = a^4,$$

   where $a > 0$ is chosen so that the region V enclosed by S has volume 7. Let $\omega = x\, dy \wedge dz + y dz \wedge dx + z dx \wedge dy$, and let S be oriented toward the outside. Evaluate the integral $\int_S \omega$.

8. Let $F(x, y, z)$ be the vector field

$$F(x, y, z) = z^2 \log(1 + y^2)\mathbf{i} + (5y + 2x^2)\mathbf{j} + (\cos^4 x + 3y)\mathbf{k}.$$

   If S is the surface of the half ball where $x^2 + y^2 + z^2 \leq 4$ and $z \geq 0$, compute $\int_S F \cdot d\mathbf{S}$ if S is oriented with an outward pointing vector.

9. Let M be a compact manifold-with-boundary embedded in $\mathbb{R}^2$, and suppose that $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ are smooth functions such that $f < g$ everywhere.
   (a) Show that the subset

$$V = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in M \text{ and } f(x, y) \leq z \leq g(x, y)\}$$

   is a manifold-with-corners.
   (b) Identify subsets U, B, and E of $\mathbb{R}^3$ so that V can be viewed as the Whitney domain (U, B, E).

Problems 10-12 concern integration over piecewise C^∞ curves and other geometric objects that lend themselves to a canonical decomposition into pieces.

10. Let f be a (continuous) **piecewise smooth function** from a closed interval $I = [a, b]$ into $\mathbb{R}$. Specifically there is to be a partition, say

$$a = t_0 < t_1 < \cdots < t_k = b \quad \text{with } k \geq 1$$

such that $f(t)$ is a continuous function on $[a, b]$ and is of class C^∞ on each of $I_j = [t_{j-1}, t_j]$ for $1 \leq j \leq k$. Put $f_j = f|_{[t_{j-1},t_j]}$ for $1 \leq j \leq k$.

(a) Taking into account all the assumptions on f, verify that $\int_{I_j} f'(t)\,dt = f(t_j) - f(t_{j-1})$ for $1 \leq j \leq k$ and conclude that $\int_I f'(t)\,dt = f(b) - f(a)$.

(b) Interpret the results of (a) via Section II.6d as saying that Stokes's Theorem holds for the 0 form $\omega = f$ on $[a, b]$ and the 1 form $d\omega = f'(t)\,dt$ even though ω is only piecewise smooth. (Educational note: In other words, Stokes's Theorem readily extends in $\mathbb{R}^1$ from smooth 0 forms to piecewise smooth 0 forms.)

(c) Relate the cancellation that occurred in (a) to a question about orientations, and say what abstract hypothesis on orientations to impose in order to ensure this cancellation.

11. Proceeding similarly with objects in one higher dimension, introduce a notion of a piecewise smooth function on the faces and edges of a tetrahedron, and derive a version of Stokes's Theorem for the surface of a tetrahedron, the boundary integral being an integral of a 1 form on the union of the edges, all consistently oriented.

12. If the same procedure is followed with a square pyramid, is there any substantial difference in what happens?

Problems 13–19 primarily concern the notion of ℓ dimensional Minkowski content $\mathcal{M}^\ell(E)$.

13. Let $\ell \geq 0$ be an integer, and let $F : M \to N$ be a smooth mapping between smooth manifolds of dimension $\geq \ell$. Prove that if E is a compact subset of ℓ dimensional Minkowski content 0 in M, then $F(E)$ is a compact subset of ℓ dimensional content 0 in N. (The notion of ℓ dimensional Minkowski content 0 in the setting of a smooth manifold is defined in Corollary 3.12 and its remarks.)

14. Let M be a smooth manifold of dimension $m \geq 2$. Prove that the smooth image in M of any compact subset of a smooth manifold of dimension $\leq m - 2$ has $m - 1$ dimensional Minkowski content 0.

15. Show that any compact m dimensional manifold-with-corners, not necessarily embedded in $\mathbb{R}^m$, is an example of a Whitney manifold of dimension m.

16. In his book Whitney defined a set E in $\mathbb{R}^m$ to be of **zero ℓ extent** if the following is true: For each $\epsilon > 0$, there is some $\zeta_0 > 0$ such that for any $\zeta \le \zeta_0$ there are balls $B_1, \dots, B_k$ for some k such that
$$E \subseteq B_1 \cup \cdots \cup B_k, \qquad \operatorname{diam}(B_i) \le \zeta \ \text{ for all } i, \qquad k\zeta^\ell < \epsilon.$$
In his formulation of the result given as Theorem 3.8 here, he required that the exceptional set E be of zero $m-1$ extent. Prove that a nonempty compact set of $\mathbb{R}^m$ is of zero ℓ extent if and only if it has ℓ dimensional Minkowski content 0.

17. If E_1 and E_2 are nonempty compact subsets of $\mathbb{R}^{a_1}$ and $\mathbb{R}^{a_2}$, respectively, so that $E_1 \times E_2$ is a subset of $\mathbb{R}^{a_1+a_2}$, prove that
$$N(E_1 \times E_2, \delta) \le N(E_1, \delta/2)N(E_2, \delta/2),$$
where $N(E, \delta)$ is as in Section 5.

18. If E is a nonempty compact subset of $\mathbb{R}^a$ and $N(E, \delta)$ is as in Section 5, prove that $\limsup_{\delta \downarrow 0} \delta^a N(E, \delta)$ is finite.

19. Suppose that E_1 and E_2 are compact subsets of $\mathbb{R}^{a_1}$ and $\mathbb{R}^{a_2}$, respectively, and suppose further that E_1 has ℓ_1 dimensional Minkowski content 0, where $\ell_1 \le a_1$.
    (a) Prove that if E_2 is a compact subset of $\mathbb{R}^{a_2}$ with ℓ_2 dimensional Minkowski content 0, where $\ell_2 \le a_2$, then $E_1 \times E_2$ is a compact subset of $\mathbb{R}^{a_1+a_2}$ of $\ell_1 + \ell_2$ dimensional Minkowski content 0.
    (b) Prove that if E_2 is a compact subset of $\mathbb{R}^{a_2}$, then $E_1 \times E_2$ is a compact subset of $\mathbb{R}^{a_1+a_2}$ of $\ell_1 + a_2$ dimensional Minkowski content 0.

20. Let (U_1, B_1, E_1) be a Whitney domain in $\mathbb{R}^{m_1}$, and let M be a compact smooth manifold-with-boundary of dimension m_2 in $\mathbb{R}^{m_2}$. Write M_+ for the set of manifold points in M and ∂M for the boundary.
    (a) Under the special assumption that (U_1, B_1, E_1) arises as in the geometric examples of Section 5 from a bounded portion of the subset of $\mathbb{R}^m$ where a real-valued polynomial F of m variables is < 0, prove that the product $(U, B, E) = (U_1, B_1, E_1) \times M$ has the natural structure of a Whitney domain in $\mathbb{R}^{m_1+m_2}$ if one defines
    $$U = U_1 \times M_+,$$
    $$B = (U_1 \times \partial M) \cup (B_1 \times M_+),$$
    and
    $$E = \big(E_1 \times M\big) \cup \big(B_1 \times \partial M\big).$$
    (b) Does the conclusion of (a) still hold without the special assumption that (U_1, B_1, E_1) arises from a bounded portion of the subset of $\mathbb{R}^m$ where a real-valued polynomial takes on negative values?

HINTS FOR SOLUTIONS OF PROBLEMS

Chapter I

1. The interior of K_{j+1} contains K_j for all j, and the union of the K_j equals M. The interiors of the sets K_{j+1} therefore form an open cover of C. A finite subcover suffices by compactness of C, and a single K_{j+1} suffices because the sets are nested.

2. The smooth manifolds will be the same if it is shown that their maximal atlases coincide, and this will happen if it is shown that the charts C_1 and C_2 are smoothly compatible with the atlas $\{M_1, M_2\}$ and that the charts M_1 and M_2 are smoothly compatible with the atlas $\{C_1, C_2\}$. One step in the verification is to check that $\varphi_1 \circ \psi_1^{-1}$ is smooth from $\psi_1(M_1 \cap C_1)$ to $\varphi_1(M_1 \cap C_1)$. The function $\varphi_1 \circ \psi_1^{-1}$ carries t to $(\cos t, \sin t)$ and then to $(\cos t)/(1 - \sin t))$ for $-\pi < t < -\pi$ and $t \neq \pi/2$, and the result is a smooth function.

3. For (a), the triangle inequality needs to be checked. Thus we are to show that

$$\min\{|x-y|, |x+y|\} \leq \min\{|x-z|, |x+z|\} + \min\{|z-y|, |z+y|\}.$$

Since

$$|x-y| \leq |x-z| + |z-y| \quad \text{and} \quad |x+y| \leq |x-z| + |z+y|,$$

we have

$$\min\{|x-y|, |x+y|\} \leq |x-z| + \min\{|z-y|, |z+y|\}.$$

Replacing z by $-z$ yields

$$\min\{|x-y|, |x+y|\} \leq |x+z| + \min\{|z-y|, |z+y|\}.$$

Then it follows that

$$\min\{|x-y|, |x+y|\} \leq \min\{|x-z|, |x+z|\} + \min\{|z-y|, |z+y|\},$$

as required. The continuity of $x \mapsto [x]$ is immediate from the inequality $d([x],[y]) \leq |x-y|$. If x is given, then the image of the set of y such that $|x-y| < \varepsilon$ is the set of $[y]$ with $d([x],[y]) < \varepsilon$, and thus open sets map to open sets.

For (b), the checking of the compatibility of the charts is similar to that in Section 1 for the sphere. The continuity of $x \mapsto [x]$ was proved in (a), and the smoothness is straightforward.

4. Let the manifold be M. Fix a point p_0 in M and consider the set of all points p in M for which there is a diffeomorphism of M carrying p_0 to p. This set is nonempty since it contains p_0, and we prove it is open and closed. Matters come down to considering an open neighborhood of a single point p, which may assume in local coordinates is a cube centered at the origin. It is then enough to produce a diffeomorphism of the open unit cube that is the identity near the boundary and carries the origin to any other point. We give the construction in $\mathbb{R}^1$, and then the general case follows by using a product of the functions of one variable. Thus we are to produce a smooth monotone function carrying $(-1, 1)$ onto itself, fixing all points near -1 and 1, and carrying 0 to some specified point p_0 in $(-1, 1)$. Subtracting the function $g(x) = x$, we see that it is enough to produce a smooth function f of compact support in $(-1, 1)$ such that $-1 < f'(x) < 1$ everywhere and such that $f(0) = p_0$. The assumption about p_0 is that p_0 is in the interval $(-1, 1)$. Constructing such a function out of standard smooth functions of compact support is easy.

5. This is elementary.

6. These are special cases of the formula $d^2 = 0$ of Proposition 1.23b. See Example 2 in Section 4.

7. This problem was addressed in *Basic Real Analysis* in another guise. Let $\omega = \sum_j P_j\, dx_j$. The condition that $d\omega = 0$ is the condition that $\partial P_j/\partial x_i = \partial P_i/\partial x_j$ for all i and j. In the language of Section III.12 of *Basic Real Analysis*, the function $F = (P_1, \ldots, P_m)$ is a conservative vector field, and Proposition 3.48 of that book shows that F is the gradient of a function f, proceeding by induction on the dimension. This f is the required function.

8. Part (a) comes down to observing that $\frac{\partial}{\partial x}\big(x/(x^2 + y^2)\big) = -\frac{\partial}{\partial y}\big(y/(x^2 + y^2)\big)$ away from $(0, 0)$. Part (b) is a routine computation with several cases. The domain of θ is the complement in $\mathbb{R}^2$ of the nonnegative x-axis. For (c), it has been shown that f and θ have matching first partial derivatives on the complement of the nonegative real axis. This set is connected, and therefore f and θ differ by a constant there. Since this set is dense in $\mathbb{R}^2 - \{(0, 0)\}$, the existence of a smooth f on $\mathbb{R}^2 - \{(0, 0)\}$ of this type would imply that θ has a continuous extension to $\mathbb{R}^2 - \{(0, 0)\}$. There is no continuous extension, and therefore no smooth solution f to $df = \omega$ exists.

9. Choose disjoint open sets A and B such that $E \subseteq A$ and $F \subseteq B$. Next choose by Theorem 1.25 a smooth partition of unity $\{f, g\}$ subordinate to the open cover $\{A, B\}$ of $E \cup F$. Then f and g take values in $[0, 1]$, f equals 0 off a compact subset of A, g equals 0 off a compact subset of B, and $f + g = 1$ on $E \cup F$. Hence f and g have the required properties.

10. For (a), take $\eta = \varphi_1\, d\varphi_2 \wedge \cdots \wedge d\varphi_k$, for example. In (b), for each j with $1 \le j \le k$, the function f_j is a smooth function of one variable defined on the subset of $x_j \in \mathbb{R}^1$ such that $(x_1, \ldots, x_m)$ is in U for some value of the variables other than x_j. This subset is a union of open sets in $\mathbb{R}^1$ and is therefore open. For such an open set in $\mathbb{R}^1$, we define a function F_j component by component so that $F_j' = f_j$ on each

component. Then the expansion $\omega = dF_1 \wedge \cdots \wedge dF_k$ exhibits ω as elementary.

11. We refer to Examples 2 and 3 in Section 3 and find that $\varphi^*(dx) = d\Phi_1 = d(r+s+t) = dr + ds + dt$ and $\varphi^*(dy) = d\Phi_2 = d(rs+st+rt) = r\,ds + s\,dr + s\,dt + t\,ds + r\,dt + t\,dr$. Thus $\varphi^*(dx \wedge dy)$ equals $\varphi^*(dx) \wedge \varphi^*(dy)$, which is

$$\begin{aligned}
&= (dr + ds + dt) \wedge (r\,ds + s\,dr + s\,dt + t\,ds + r\,dt + t\,dr)\\
&= (r + t - s - t)(dr \wedge ds) + (r + s - r - t)(ds \wedge dt) + (s + r - s - t)(dr \wedge dt)\\
&= (r - s)(dr \wedge ds) + (s - t)(ds \wedge dt) + (r - t)(dr \wedge dt).
\end{aligned}$$

12. This is straightforward.

13. For (a), the left side on $(X_2, \ldots, X_k)$ equals $k(\omega_1 \wedge \cdots \wedge \omega_k)(X, X_2, \ldots, X_k)$, which by Corollary 1.16 equals

$$\frac{k}{k!} \det \begin{pmatrix} \omega_1(X) & \omega_2(X) & \cdots & \omega_k(X) \\ \omega_1(X_2) & \omega_2(X_2) & \cdots & \omega_k(X_2) \\ \vdots & & & \\ \omega_1(X_k) & \omega_2(X_k) & \cdots & \omega_k(X_k) \end{pmatrix}.$$

When this determinant is expanded in cofactors about the first row and account is taken of the coefficient, the i^{th} term of the expansion is exactly

$$(-1)^{i-1}\omega_i(X)(\omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_k)(X_2, \ldots, X_k).$$

The result follows.

For (b), we may assume without loss of generality that $\omega = \omega_1 \wedge \cdots \wedge \omega_k$ and that $\eta = \omega_{k+1} \wedge \cdots \wedge \omega_{k+l}$. Applying (a) to each yields

$$c_X(\omega) = \sum_{i=1}^{k} (-1)^{i-1}\omega_i(X)(\omega_1 \wedge \cdots \widehat{\omega_i} \wedge \cdots \wedge \omega_k)$$

and

$$\begin{aligned}
c_X(\eta) &= \sum_{j=1}^{l} (-1)^{j-1}\omega_{k+j}(X)(\omega_{k+1} \wedge \cdots \widehat{\omega_{k+j}} \wedge \cdots \wedge \omega_{k+l})\\
&= \sum_{m=k+1}^{k+l} (-1)^{m-k-1}\omega_m(X)(\omega_{k+1} \wedge \cdots \widehat{\omega_m} \wedge \cdots \wedge \omega_{k+l}).
\end{aligned}$$

Therefore

$$\begin{aligned}
c_X(\omega) \wedge \eta + (-1)^k(\omega \wedge c_X(\eta)) &= \sum_{i=1}^{k+l} (-1)^{i-1}\omega_i(X)(\omega_1 \wedge \cdots \widehat{\omega_i} \wedge \cdots \wedge \omega_{k+l})\\
&= c_X(\omega \wedge \eta).
\end{aligned}$$

14. The expanded formula for $i^*(\omega)$ is

$$i^*(\omega)_p((X_1)_p, \ldots, (X_k)_p) = \omega_{i(p)}\big((Di)_p X_1)_{i(p)}, \ldots, (Di)_p X_k)_{i(p)}\big),$$

where $(X_1)_p, \ldots, (X_k)_p$ are in $T_p(S)$, $(Di)_p$ is the derivative of i at p, ω is an alternating k multilinear form on M, and $i^*(\omega)$ is the pullback alternating k multilinear form on S. The derivative $(Di)_p$ may be regarded as an inclusion of $T_p(S)$ into $T_{i(p)}(M)$, and the arguments of $\omega_{i(p)}$ within $T_{i(p)}(M)$ are obtained by taking the arguments of $i^*(\omega)_p$ and regarding them as included in $T_{i(p)}(M)$. Inclusions and restrictions are the same thing from a different point of view.

15. We go back to the definition of "orientable" near the beginning of Section 6. Let the two charts be (M_1, φ_1) and $M_2, \varphi_2)$. The condition of orientability is that $\det(\varphi_2 \circ \varphi_1^{-1})$ and $\det(\varphi_1 \circ \varphi_2^{-1})$ are both positive. The second determinant is the reciprocal of the first. If they are positive, we are done. If they are negative, then we redefine φ_1 by following φ_1 with the map $(x_1, x_2, \ldots, x_m) \mapsto (-x_1, x_2, \ldots, x_m)$; use of the composition changes the determinants from negative to positive.

16. For (a), a point in S^n may be identified with a vector in $\mathbb{R}^{n+1}$. As a vector, $p = (x_1, \ldots, x_{n+1})$ is orthogonal to the tangent space at the point of tangency p on the sphere. Thus the tangent space consists of all $p + x \in \mathbb{R}^{n+1}$ with $x \cdot p = 0$. Viewed as through the origin, the tangent space is simply $\{x \in \mathbb{R}^{n+1} \mid x \cdot p = 0\}$, i.e., the orthogonal complement $(\mathbb{R}p)^\perp$ of the 1 dimensional space $\mathbb{R}p$. Any subspace of a finite dimensional inner product vector space is the direct sum of itself and its orthogonal complement. With these identifications, $\mathbb{R}^{n+1} = (\mathbb{R}p)^\perp \oplus T_p(S^n)$.

In (b), the derivation property of $f \mapsto \frac{d}{dt} f(\gamma_r(t))\big|_{t=0}$ is immediate from the one-variable rule for differentiating products. Write $\gamma_r(t)$ in coordinates as $\gamma_r(t) = \big((x_1(t), \ldots, x_{n+1}(t)\big)$, and expand the derivative in question as

$$\frac{d}{dt}(\gamma_r(t))\Big|_{t=0} = \frac{\partial f}{\partial x_1}(p)\,\frac{dx_1}{dt}(0) + \cdots + \frac{\partial f}{\partial x_{n+1}}(p)\,\frac{dx_{n+1}}{dt}(0).$$

To compute this, we write

$$\gamma_r(t) = \frac{p + tr}{|p + tr|} = \frac{p + tr}{\sqrt{(p + tr) \cdot (p + tr)}}.$$

Since $p \cdot p = 1$ and $p \cdot r = 0$, $\gamma_r(t)$ simplifies to $\frac{p+tr}{\sqrt{1+t^2|r|^2}}$, whose derivative at $t = 0$ is r since there are no first-order terms in t in the denominator. The result follows.

17. For $(X_1)_p, \ldots, (X_n)_p$ in $T_p(S^n)$, we have

$$i^*(c_X(\omega))_p((X_1)_p, \ldots, (X_n)_p) = \omega_{i(p)}(X, i(X_1)_p, \ldots, i(X_n)_p),$$

where $i(X_j)_p$ means the effect of the derivative Di on $(X_j)_p$, namely $(Di)_p(X_j)_p$. Take $\{(X_1)_p, \ldots, (X_n)_p\}$ at p to be a basis of $T_p(S^n)$. Then $\{i(X_1)_p, \ldots, i(X_n)_p\}$ is a basis of the vector space $iT_p(S^n)$, which we know from Problem 16 to equal $(\mathbb{R}p)^\perp$. Since $X_p = p$, $\{X_p, i(X_1)_p, \dots, i(X_n)_p\}$ at p is a basis of $iT_p(S^n) \oplus \mathbb{R}p = \mathbb{R}^{n+1}$. Since the given ω is nonzero at p, its value at p does not vanish on any basis of $\mathbb{R}^{n+1}$. Therefore $i^*(c_X(\omega))_p((X_1)_p, \dots, (X_n)_p) \neq 0$.

18. Corollary 1.16 yields

$$\begin{aligned} c_X(\omega) &= (n+1)(dx_1 \wedge \cdots \wedge dx_{n+1})(X, X_1, \dots, X_n) \\ &= \frac{n+1}{(n+1)!} \det \begin{pmatrix} dx_1(X) & dx_1(X_1) & \cdots & dx_1(X_n) \\ dx_2(X) & dx_2(X_1) & \cdots & dx_2(X_n) \\ \cdots & & & \\ dx_{n+1}(X) & dx_{n+1}(X_1) & \cdots & dx_{n+1}(X_n) \end{pmatrix}. \end{aligned}$$

We can evaluate the entries in the first column as follows. For the i^{th} entry we have $dx_i(X) = \sum\limits_k x_k \, dx_i\big(\frac{\partial}{\partial x_k}\big) = x_i$. Then we expand the whole determinant by cofactors about the first column. With the coefficient $(n!)^{-1}$ in place, the expansion gives a sum over i of an alternating sign $(-1)^{i-1}$ times the coefficient x_i, times the complementary determinant, which is

$$(dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{n+1})(X_1, \dots, X_n).$$

Thus $c_X(\omega) = \sum_i (-1)^{i-1} \, (dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{n+1})$ as required.

19. In (a), symmetry of $\sim$ follows from the fact that $h^2 = 1$. For the transitive property, we observe that if $y = h(x)$ and $z = h(y)$, then $z = h^2(x) = x$ and hence $z \sim x$. In (b), the argument is similar to that for Problem 3, which deals with a special case. In (c) to define a chart about x in M, use the open ball about x of each radius less than half the distance from x to $h(x)$,

20. With the proof of Proposition 1.33 as a guide, this is easy.

21. With the proof of Proposition 1.33 as a guide, this is easy.

22. For (b), a nowhere vanishing n form for S^n can be taken to be a restriction of

$$\sum_{j=1}^{n+1} x_j \, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{n+1}.$$

The anitpodal map has the effect of sending each x_i into its negative and each dx_j into its negative. Thus it has the effect of introducing $n+1$ minus signs in each term, thus of multiplying the whole expression by $(-1)^{n+1}$. Consequently the n form is preserved by the antipodal map if n is odd and is reversed if n is even. The n form gives the orientation, up to an everywhere positive factor, and so the orientation is preserved if n is odd and is reversed if n is even.

23. This is immediate.

24. In (a), the mapping α and its inverse are continuous because f is continuous. For smoothness of σ and its inverse, we are to compose before and after with the chart mappings, and we end up with the identity, which is smooth. In (b), the mappings α and p are smooth, and so is the composition $\alpha \circ I \circ p$; thus the inclusion map I must not be smooth.

25. In (a), φ is smooth, and its inverse is $\varphi^{-1}(u, v) = (u, v + f(u))$, which is smooth. Then (b) is an observation.

26. The derivative is $(2\cos 2t, -\sin t)$. For this to be $(0, 0)$, $\sin t$ must be 0, which means that t is a multiple of π. Then $2t$ is a multiple of 2π, and $\cos 2t = 1$. Thus both entries cannot be 0 for the same t, and γ is an immersion. It is easy to check that γ is one-one over an interval of length 2π. Finally its image is compact, being closed and bounded. Specifically it contains all its limit points, since the only point that needs checking is $(0, 0)$, which is $\gamma(\pi/2) = (0, 0)$ and is therefore already in the image. The topology of the domain of γ is that of an open interval, which is not compact, and the topology of the image is compact. Thus the two topologies do not coincide, and the immersion is not an embedding.

27. In (a), $\gamma'(t) = (2\pi i e^{2\pi i t}, 2\pi i c e^{2\pi i c t})$, and neither coordinate is ever 0. So $\gamma'(t)$ nowhere vanishing, and γ is an immersion. If $\gamma(t_1) = \gamma(t_2)$, then $e^{2\pi i t_1} = e^{2\pi i t_2}$ and $e^{2\pi i c t_1} = e^{2\pi i c t_2}$. Hence $t_1 - t_2$ is an integer, and so is $c(t_1 - t_2)$. Since c is irrational, this is possible only if $t_1 - t_2 = 0$. Hence γ is one-one.

For (b), it follows from (a) that $\{\gamma(k) \mid k \in \mathbb{Z}\}$ is an infinite set. Thus it has a limit point in $\mathbb{C}$, say z. Choose a sequence $\{k_n\}$ such that $\lim_n \gamma(k_n) = z$. Given $\varepsilon > 0$, choose two distinct integers r and s in the sequence such that $|\gamma(r) - \gamma(s)| < \varepsilon$. Then $k = r - s$ is a nonzero integer with $|\gamma(k) - \gamma(0)| < \varepsilon$.

For (c), repeating this construction for a sequence of values of ε tending to 0 shows that there is a sequence of points in $\gamma(\mathbb{Z})$ tending to 1 but not equal to 1. Hence $\gamma(Z)$ does not have the discrete topology, and γ is not an embedding.

28. In (a), since the function $x(t)$ is smooth near t_0 and its derivative is nonzero there, the one-variable Inverse Function Theorem says that near the point t_0, $x(t)$ can in principle be inverted to give a unique smooth inverse function $t = t(x)$. This result can be substituted into the expression $y(t)$ to yield $y(t) = y(t(x))$ as a function of x near $x(t_0)$. More specifically put $x(t_0) = x_0$. Then the set of points $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ in a suitably small rectangle in $\mathbb{R}^2$ about $\begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix}$ is the embedded graph of the smooth function $g(f(t))$.

In (b), fix x_0, and suppose that n of the columns of $J(x_0)$ are linearly independent. Possibly by permuting the variables, we may assume that the first n columns are linearly independent. Write $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$, so that the n-by-n square matrix $\left\{\left(\frac{\partial (F_1)_i}{\partial x_j}\right)\right\}$ is invertible at $x = x_0$. By the Inverse Function Theorem, we can in principle solve uniquely in a neighborhood of $(x_0, F(x_0))$ to write x as a smooth function $x = x(F_1)$ there. Then the set of points in a suitably small rectangular neighborhood of $\begin{pmatrix} x_0 \\ F(x_0) \end{pmatrix}$

in $\mathbb{R}^{n+k}$ is the embedded graph of the smooth function $F_2(x(F_1))$.

29. For (a), the Jacobian matrix of (x, y, z) with respect to (s, t) is

$$\begin{pmatrix} -\sin s - \frac{1}{2}t\cos(s/2)\sin s - \frac{1}{4}t\sin(s/2)\cos s & \frac{1}{2}\cos(s/2)\cos s \\ \cos s + \frac{1}{2}t\cos(s/2)\cos s - \frac{1}{4}t\sin(s/2)\sin s & \frac{1}{2}\cos(s/2)\sin s \\ \frac{1}{4}t\cos(s/2) & \frac{1}{2}\sin(s/2) \end{pmatrix}.$$

The 2-by-2 determinant from the first two rows is

$$\begin{aligned} &= -\tfrac{1}{2}\cos(s/2)\sin^2 s - \tfrac{1}{4}t\cos^2(s/2)\sin^2 s - \tfrac{1}{8}t\sin(s/2)\cos(s/2)\sin s\cos s \\ &\quad - \tfrac{1}{2}\cos(s/2)\sin^2 s - \tfrac{1}{4}t\cos^2(s/2)\cos^2 s + \tfrac{1}{8}t\sin(s/2) \\ &= -\tfrac{1}{2}\cos(s/2) - \tfrac{1}{4}t\cos^2(s/2) \\ &= -\tfrac{1}{2}\cos(s/2)\big(1 - \tfrac{1}{2}t\cos(s/2)\big), \end{aligned}$$

and this has the same sign as $-\frac{1}{2}\cos(s/2)$. When $\cos(s/2) = 0$, the Jacobian matrix simplifies to

$$\begin{pmatrix} -\sin s - \frac{1}{4}t\sin(s/2)\cos s & 0 \\ \cos s - \frac{1}{4}t\sin(s/2)\sin s & 0 \\ 0 & \frac{1}{2}\sin(s/2) \end{pmatrix}.$$

When $\cos(s/2) = 0$, we see that $\sin(s/2)$ is ± 1, $\sin s$ is 0, and $\cos s$ is ± 1. Thus the determinant from the first and third rows equals $(\pm\frac{1}{2})(\pm\frac{1}{4}t)$, which is nonzero unless $t = 0$. When $\cos(s/2) = 0$ and $t = 0$, then the determinant from the second and third rows equals $(\pm\frac{1}{2})\cos s$, which is not zero. Thus the Jacobian matrix has rank two for every pair (s, t) under consideration.

Part (b) is clear. For (c), the image of the smooth function is locally a smooth function, by the Inverse Function Theorem. Since the function is only two-to-one, it is locally invertible. Hence the image is a smooth manifold.

30. In (a), the function $F(x, y) = x^2 + y^2 - 1$ is smooth near the point (x_0, y_0), which has $F(x_0, y_0) = 0$, and the assumption is that $\frac{\partial F}{\partial x}(x_0, y_0) \neq 0$. That is, the 1-by-1 matrix with entry $\frac{\partial F}{\partial x}(x_0, y_0) \neq 0$ is invertible. The theorem says that in a suitable rectangular neighborhood $I \times J$ of (x_0, y_0) with $I \subseteq \mathbb{R}^1$ and $J \subseteq \mathbb{R}^1$, each y value yields a unique x value with $F(x, y) = 0$ and the resulting function $x = f(y)$ for $x \in I$ is smooth and satisfies $F(f(y), y) = 0$ for all y in J. Then the open subset $I \times J$ of $\mathbb{R}^2$ contains the embedded graph of a smooth function, as in Problem 25.

In (b), the same procedure is to be applied to the function $F(x_1, \ldots, x_{n+1}) - 1$ and the point $((x_1)_0), \ldots, (x_{n+1})_0)$ on S^n under the assumption that

$$\Big(\frac{\partial F}{\partial x_1}\Big)((x_1)_0, \ldots, (x_{n+1})_0),$$

namely $2(x_1)_0$, is nonzero. The Implicit Function Theorem yields a rectangular open neighborhood $I \times J$ of $((x_1)_0, \ldots, (x_{n+1})_0)$ with $I \subseteq \mathbb{R}^1$ and $J \subseteq \mathbb{R}^n$ such that each value of $(x_2, \dots, x_{n+1})$ in J yields a unique x_1 in I with $F(x_1, \dots, x_{n+1}) = 0$ and the resulting function $x_1 = f(x_2, \dots, x_{n+1})$ is smooth and satisfies

$$F(f(x_2, \dots, x_{n+1}), x_2, \dots, x_{n+1}) = 0 \quad \text{for all } (x_2, \dots, x_{n+1}) \in J.$$

Then the open subset $I \times J$ of $\mathbb{R}^{n+1}$ contains the embedded graph of a smooth function, as in Problem 25.

In (c), fix x_0, and suppose that k of the columns of $J(x_0)$ are linearly independent. Possibly by permuting the variables, we may assume that the first k of the columns are linearly independent. Regard F as a function of n variables whose entries $(F_1, \dots, F_k)$ are members of $\mathbb{R}^k$. The assumption is that the matrix $\big\{\frac{\partial F_i}{\partial x_j}(x_0)\big\}$ is nonsingular. The Implicit Function Theorem yields a rectangular set $I \times J \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}$ centered at $((x_1)_0, \dots (x_n)_0)$ and a smooth function $f(x_{k+1}, \dots, x_n)$ defined in J such that for each $(x_{k+1}, \dots, x_n)$ in J, there is a unique $(x_1, \dots, x_k)$ in I with $F(x_1, \dots, x_n) = 0$ and the resulting function $(x_1, \dots, x_k) = f(x_{k+1}, \dots, x_n)$ is smooth and satisfies $F(f(x_{k+1}, \dots, x_n), x_{k+1}, \dots, x_n) = 0$ for all $(x_2, \dots, x_{n+1})$ in J. Then the open subset $I \times J$ of $\mathbb{R}^n$ contains the embedded graph of a smooth function, as in Problem 25.

Chapter II

1. Straightforward calculation.

2. Two ways of proving this result that generalize to all dimensions are to make use of Corollary 1.16 of the present text and to proceed via row reduction of matrices as outlined in Section III.10 of *Basic Real Analysis.*

For dimension 3 an argument is available that makes use of cross product, as follows: We compute the volume of the parallelepiped spanned by u, v, and w as the area of the base spanned by u and v, times the height. The area of the base we know to be $|u \times v| = |u||v||\sin\theta|$. The height is the magnitude of the projection of w in the direction perpendicular to the base, i.e., in the direction of $u \times v$. Thus the height is $\left|\dfrac{w \cdot (u \times v)}{|u \times v|^2}\,(u \times v)\right| = \dfrac{|w \cdot (u \times v)|}{|u \times v|}$. Then the product of the base and height is $|w \cdot (u \times v)|$, which is the determinant in question.

3. The first, fourth, and fifth are equal. The second, third, and sixth are the negative of these.

4. For (a), div $F = 2xy$ and curl $F = (8y - 3z^2)\mathbf{i} - (x^2 + 3x)\mathbf{k}$.
For (b), div $F = 2 + 2x^3y$ and curl $F = (4z - 7)\mathbf{j} + (3x^2y^2)\mathbf{k}$.

5. Without loss of generality we may assume that M is connected. If a smooth $m-1$ form η exists with $d\eta = \omega$, then Stokes's Theorem says that $\int_{\partial M} \eta = \int_M d\eta = \int_M \omega$. In a connected compatible chart $\alpha = (x_1, \dots, x_m)$, $\alpha^*(\omega)$ can be written as $F_\alpha\, dx_1 \wedge \cdots \wedge dx_m$ for some nowhere-vanishing smooth function F_α. Then F_α does not change sign, and $\int_M \omega$ is not zero. Consequently $\int_{\partial M} \eta \neq 0$. But this contradicts Theorem 2.1 since ∂M is a smooth manifold without boundary.

6. For (a), we can take $\omega = (dx_1 \wedge dx_2) + (dx_3 \wedge dx_4)$.

For (b), we have $\omega = d\eta$ with $\eta = \alpha \wedge d\alpha \wedge \cdots \wedge d\alpha$ since $d^2 = 0$. Then the previous problem shows that ω has to vanish somewhere.

7. In (a), the value of $d\omega$ is the sum of $(\partial/\partial x)\big(x(x^2+t^2+z^2)^{-3/2}\big)\,(dx \wedge dy \wedge dz)$ and two similar terms. The coefficient of $dx \wedge dy \wedge dz$ is

$$\begin{aligned}(x^2+y^2+z^2)^{-3/2} &+ x(-3/2)(x^2+y^2+z^2)^{-5/2}(2x)\\ &= (x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2-3x^2)(x^2+y^2+z^2)^{-1}.\end{aligned}$$

The contributions from the other two terms are similar except that x is to be replaced by y and then z. The sum of the three terms is then

$$(x^2+y^2+z^2)^{-5/2}(3(x^2+y^2+z^2)-3x^2-3y^2-3z^2) = 0.$$

In (b), let M be the "inside" of T. We can apply Stokes's Theorem (Theorem 4.7) to T since ω is smooth everywhere inside and on T. Then we have $\int_T \omega = \int_M d\omega = 0$.

8. Part (a) is a restatement of Problem 7a.

In (b), the Divergence Theorem gives $\int_S F \cdot d\mathbf{S} = 0$ since $\operatorname{div} F = 0$. The orientation on S is given by an outward normal from M, which is then outward on S_1 and in toward the origin on S_a. Hence $0 = \int_S F \cdot d\mathbf{S} = \int_{S_1} F \cdot d\mathbf{S} - \int_{S_a} F \cdot d\mathbf{S}$.

9. Take $\omega = \frac{1}{n}\sum_{j=1}^{n}(-1)^{j-1}dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$ and $f(x_1,\dots,x_n) = (x_1^2+\cdots+x_n^2)^{n/2}$.

10. In Section 3 the paragraph beginning "The traditional procedure" is irrelevant and can be omitted. In the statement of Proposition 2.6, $(-1)^m\alpha^*(dx_1 \wedge\cdots\wedge dx_{m-1})$ is to be replaced by $-\alpha^*(dx_1\wedge\cdots\wedge dx_m)$. (Note the sign!) The proof of Proposition 2.6 is unchanged down to the paragraph beginning "Thus we have constructed." For the case $m = 1$, we still have F_{α_p} as it is, positive or negative. The orientation at p is still the sign of $F_{\alpha_p}(0)$.

In Section 4, formulas $(*)$ and $(**)$ are unchanged. In the paragraph beginning "On $\partial\mathbb{H}^m$," some changes are needed. We have

$$\omega = F_1(0, x_2, \dots, x_m)\, dx_2 \wedge \cdots \wedge dx_m.$$

For the case $m \geq 2$ the proof becomes, "Since $-dx_2\wedge\cdots\wedge dx_m$ is positively oriented in the orientation of the boundary that we are using, application of Theorem 1.29 gives

$$\begin{aligned}\int_{\partial\mathbb{H}^m}\omega &= -\int_{\partial\mathbb{H}^m} F_1(0,x_2,\dots,x_m)\,dx_m\cdots dx_2\\ &= -\int_{a_2}^{b_2}\cdots\int_{a_m}^{b_m} F_1(0,x_2,\dots,x_m)\,dx_m\cdots dx_2.\end{aligned}\qquad (\dagger)$$

For $m = 1$, we get $\int_{\partial\mathbb{H}^m}\omega = -F_1(0)$. So $(\dagger)$ holds for all $m \geq 1$.

Formula (††) is still valid, and we still do the integration in the variable x_r first. For $r \geq 1$, we get 0 from the inside integral. For $r = 1$, the inside integral is

$$\int_0^c \left(\frac{\partial F_1}{\partial x_1}\right) dx_1 = F_1(c, x_2, \ldots, x_m) - F_1(0, x_2, \ldots, x_m)$$

with $F(c, x_2, \ldots, x_m) = 0$ by the support condition. Therefore (††) boils down to

$$-\int_{a_2}^{b_2} \cdots \int_{a_m}^{b_m} F_1(0, x_2, \ldots, x_m)\, dx_m \cdots dx_2,$$

which equals (†). Thus we get (‡), and the remainder of the proof is unchanged.

11. We can parametrize the surface by using s and t as parameters, with s standing for x and t standing for y. Then the parametrization is $(s,t) \mapsto \begin{pmatrix} s \\ t \\ s^2+t^2 \end{pmatrix}$ with derivative $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2s & 2t \end{pmatrix}$. Then we have $\dfrac{\partial(x,y)}{\partial(s,t)} = 1$, $\dfrac{\partial(y,z)}{\partial(s,t)} = -2s$, and $\dfrac{\partial(z,x)}{\partial(s,t)} = -2t$. The integrand $F = x\mathbf{i} \cdot d\mathbf{S}$ is $x\,dy \wedge dz = x(-2s)\,ds\,dt = -2s^2\,ds\,dt$. There is no natural orientation on the surface, but we are told to orient the surface by using an outward/downward vector. That is, we are to consider the basis of the tangent space at a point of the surface, include an outward/downward vector before it (a vector with third component negative), and see whether our parametrization is consistent with this basis of $\mathbb{R}^3$. To fix the ideas, take $(s,t) = (0,0)$. Then the basis we choose of $\mathbb{R}^3$ can be $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. The matrix formed from these basis vectors has determinant -1, and our parametrization is the opposite of what we need. Let us therefore start over, using $(s,t) \mapsto \begin{pmatrix} t \\ s \\ s^2+t^2 \end{pmatrix}$ with derivative $\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2s & 2t \end{pmatrix}$ as parametrization. Then $\dfrac{\partial(y,z)}{\partial(s,t)} = 2t$. With this parametrization the integrand becomes $F = x\mathbf{i} \cdot d\mathbf{S} = x\,dy \wedge dz = x(2t)\,ds\,dt = 2t^2\,ds\,dt$. The integration extends over the set where $s^2 + t^2 \leq 4$. Switching to polar coordinates in the s-t plane shows that the integral is $\int_0^2 \int_0^{2\pi} 2r^2(\sin^2\theta)\, r\,dr\,d\theta = \pi \int_0^2 2r^3\,dr = \boxed{8\pi}$.

As it should, this orientation gives minus the answer we would get with the opposite orientation. Had we not taken the orientation into account properly, we would have integrated $-2s^2\,ds\,dt$ over the set where $s^2 + t^2 \leq 4$ and gotten -8π as the answer.

12. The boundary curve of S is given by the subset of points (x,y,z) that satisfy both conditions, namely $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$, and have $z \geq 0$. Substitution gives $z^2 = 3$. Thus the intersection is the circle with $z = \sqrt{3}$ and $x^2 + y^2 = 1$. Stokes's Theorem says that the integral is equal to $\int_C F \cdot d\mathbf{s}$, but we have to orient C properly. Since the orientation of S is upward, this situation is like looking at the ordinary unit circle in the x-y plane. The circle is therefore to be traversed with S on the left, and the parametrization can be taken as $t \mapsto (\sqrt{3}\cos t, \sqrt{3}\sin t, \sqrt{3})$.

The derivative is $(-\sqrt{3}\,\sin t, \sqrt{3}\,\cos t, 0)$. On the circle the value of F in terms of the parameter t is $(yz, 0, xy) = (3\sin t, 0, \cos t \sin t)$ Thus the integral is

$$\begin{aligned} &= \textstyle\int_0^{2\pi} (3\sin t, 0, 3\cos t \sin t)\cdot(-\sqrt{3}\,\sin t, \sqrt{3}\,\cos t, 0)\,dt \\ &= \textstyle\int_0^{2\pi} -3\sqrt{3}\sin^2 t\,dt = \boxed{-3\pi\sqrt{3}} \end{aligned}$$

13. A direct attack on the line integral leads to an unpleasant term $e^{3\sin t}$ because of the presence of e^z of F. In preparation for using the Kelvin–Stokes Theorem, direct computation gives $\operatorname{curl} F = (x, -2y, y)$ with the e^z gone. By the Kelvin–Stokes Theorem the integral equals $\int_S (\operatorname{curl} F)\cdot d\mathbf{S}$ when S is any oriented smooth surface with boundary curve C, provided the orientations match properly. An example of such a surface is the disk given by $x^2 + y^2 \le 9$ and $y = 4$ with a suitable orientation. By the same token the surface integral equals the line integral $\int_C G\cdot d\mathbf{s}$, where $G = (-yz, 0, xy)$, since $\operatorname{curl} G = \operatorname{curl} F$. (In changing F into G, we can drop pure x terms from the first entry, pure y terms from the second entry, and pure z terms from the third entry without changing the curl.) Since $\mathbf{s}'(t) = (-3\sin t, 0, 3\cos t)$, the given line integral is

$$\begin{aligned} &= \textstyle\int_0^{2\pi} \big[(-4(3\sin t))(-3\sin t) + 4(3\cos t)(3\cos t)\big]\,dt \\ &= \textstyle\int_0^{2\pi} (36\sin^2 t + 36\cos^2 t)\,dt = \int_0^{2\pi} 36\,dt = \boxed{72\pi} \end{aligned}$$

14. The boundary is the circle C in the plane $z = 0$ with $x^2 + y^2 = 16$. Since S is oriented upward, the induced orientation on C is clockwise (with the hemisphere on the left). Thus C can be parametrized as $t \mapsto (4\cos t, 4\sin t, 0)$ with derivative $-4\sin t, 4\cos t, 0)$. The given integral is therefore

$$= \textstyle\int f\cdot d\mathbf{s} = \int_0^{2\pi} \big(y(-4\sin t) - x(4\cos t) + 0\big)\,dt = \int_0^{2\pi} (-16\sin^2 t - 16\cos^2 t)\,dt,$$

which equals $\boxed{-32\pi}$

15. In (a), the circle can be parametrized as $\theta \mapsto \begin{pmatrix} \cos\theta \\ \sin\theta \\ 2-\sin\theta \end{pmatrix}$ for $0 \le \theta \le 2\pi$. We are given $F(x, y, z) = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$, and we have

$$d\mathbf{s} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} + (-\cos\theta)\,\mathbf{k}.$$

Then

$$\textstyle\int_C F\cdot d\mathbf{s} = \int_0^{2\pi} (-\sin^2\theta)(-\sin\theta) + (\cos\theta)(\cos\theta) + (2-\sin\theta)^2(-\cos\theta)\,d\theta = \pi.$$

In (b), the filled ellipse is to be oriented upward. We can parametrize it as $(r, \theta) \mapsto \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 2-r\sin\theta \end{pmatrix}$ with derivative $\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ -\sin\theta & -r\cos\theta \end{pmatrix}$. Then

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r, \qquad \frac{\partial(x, z)}{\partial(r, \theta)} = r(\sin^2\theta - \cos^2\theta), \qquad \frac{\partial(y, z)}{\partial(r, \theta)} = 0.$$

Direct calculation gives curl $F = (1+2y)\mathbf{k}$, and then curl $F \cdot d\mathbf{S} = (1+2y)\frac{\partial(x,y)}{\partial(r,\theta)} = r(1+2r\sin\theta) = r + 2r^2\sin\theta$. Thus the integral is $\int_0^1 \int_0^{2\pi} (r + 2r^2 \sin\theta)\, d\theta\, dr = 2\pi \int_0^1 r\, dr = \pi$.

16. The leftmost inequality sign follows from the fact that ω is nowhere vanishing, the argument being like the one for Problem 5. The first of the three equalities follows because the fact that r is a retraction shows up on the level of pullbacks as meaning that r^* is the identity on forms located where r is the identity, i.e., on ∂B. The second equality is by Stokes's Theorem, Theorem 4.7. The third equality is by Proposition 1.24, which says that exterior derivative commutes with pullback.

17. In the previous problem, there is some virtue in making explicit the role of the inclusion $i : \partial B \to \overline{B}$ is the computation. The fact that r is a retraction means that $f \circ i = 1_{\partial B}$, and this translates into the identity $i^* f^* = 1$ on forms of each degree. The computation is less ambiguous if it is written as

$$0 < \int_{\partial B} \omega = \int_{\partial B} i^* r^*(\omega) = \int_B dr^*(\omega) = \int_B r^*(d\omega).$$

Remembering that pullbacks preserve degree and that r^* therefore carries $\Omega^k(\partial B)$ into $\Omega^k(B)$ for each k, we can track down the degrees of the various forms in the computation. The ω on the left is in $\Omega^{n-1}(\partial B)$, $i^*r^*(\omega)$ is in $\Omega^{n-1}(\partial B)$, $dr^*(\omega)$ is in $\Omega^n(B)$ by Stokes's Theorem, and $r^*(d\omega)$ is in $\Omega^n(B)$. Since $r^*(d\omega)$ ends up in $\Omega^n(B)$, r^* must have been acting on something in $\Omega^n(\partial B)$. This space is 0 since ∂B has dimension $n-1$, and thus $r^*(d\omega) = r^*(0) = 0$.

18. For any point p in B, the fact that $f(p) \neq p$ implies that there is a unique line passing through p and $f(p)$. This line meets the sphere ∂B in two points, and we define $r(p)$ to be the point that is closer to $f(p)$. (To complete the definition, we define r to be the identity on points of ∂B.) Let us write the definition of r is symbols, and then we can see that f is smooth. The parametrically defined line $t \mapsto (1-t)p + tf(p)$ passes through p when $t = 0$ and passes through $f(p)$ when $t = 1$. From the geometry it is evident that it meets ∂B twice, once for some negative value of t and once for some value of t greater than 1. We seek an expression for the value of t greater than 1. Thus we set $|(1-t)p + tf(p)|^2 = 1$ and solve the resulting quadratic equation for t. The coefficient of t^2 is

$$|p|^2 - 2p \cdot f(p) + |f(p)|^2 = |p - f(p)|^2,$$

and this is positive since $f(p) \neq p$. The constant term is $|p|^2 - 1$, which is negative since p is in B. Thus the two roots t have opposite sign, and our desired root t is the one with the plus sign in the quadratic formula. Consequently we can obtain an explicit formula for $r(p)$, and its dependence on p is smooth if f is smooth. The function r is a smooth retraction, which the previous problem shows cannot exist. Therefore f must have a fixed point.

19. Regard f as extended to $\mathbb{R}^n$ by extending it as 0 outside $\overline{B}$. Choose a member $\varphi \geq 0$ of $C^\infty_{\text{com}}(\mathbb{R}^n)$ of total integral 1, let $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$ for $\varepsilon > 0$, and convolve the scalar-valued function φ_ε with each entry of f. Then $\varphi_\varepsilon * f$ converges uniformly to f on $\mathbb{R}^n$ as ε tends to 0. Thus the sequence $\{f_k\}$ may be taken to be the sequence of restrictions to $\overline{B}$ of the functions $\varphi_{1/k} * f$.

20. We are assuming that $\{f_k\}$ is a sequence of smooth functions carrying $\overline{B}$ to itself such that $f_k(x_k) = x_k$ for all k and such that $\{f_k\}$ converges uniformly to f on $\overline{B}$. The Bolzano-Weierstrass Theorem produces a limit point x_0 in $\overline{B}$ for the sequence $\{x_k\}$. Passing to a subsequence and renumbering, we may assume that $\lim_k x_k = x_0$. Then we have

$$|f(x_0) - x_0| \leq |f(x_0) - f(x_k)| + |f(x_k) - f_k(x_k)| + |f_k(x_k) - x_k| + |x_k - x_0|.$$

On the right side, the first term tends to 0 by continuity of f, the second term tends to 0 by the uniformity of the convergence, the third term is 0 because x_k is a fixed point of f_k, and the fourth term tends to 0 since $\lim_k x_k = x_0$. Since the left side is independent of k, it must be 0.

Chapter III

1. For (a), let S be the set of vertices. We proceed by induction on the cardinality V of S, the base case of the induction being the case $V = 4$ of a tetrahedron. For a tetrahedron the assertion is clear. Let a polyhedron be given with $n \geq 5$ faces, and assume that a triangulation exists whenever a compact convex polyhedron has $\leq n - 1$ faces. We shall attempt to introduce a plane that divides S into two proper but overlapping subsets; if we can do this, then by induction we can do the triangulation for the polyhedron associated to each subset of vertices, and the union of the triangulations will be a triangulation of the given polyhedron. We fix attention on any three vertices and consider the unique plane that contains them. Let this plane be the set where some linear functional L is 0. One subset of S will consist of those vertices for which $L \geq 0$, and the other subset will consist of those vertices for which $L \leq 0$. We have seen that we are done if both these subsets are proper.

Thus suppose that one or the other of the subsets is all of S. Then the plane that passes through our three vertices is completely on one side of our polyhedron and those three vertices must span a face. In other words, we have associated a unique face to to each triple of vertices. On the other hand, if a face is given, then the vertices of that (triangular) face are a triple of vertices. We conclude the F equals the number of triples of vertices, which is $\frac{1}{6}V(V-1)(V-2)$.

Meanwhile to each edge we can associate two vertices, and distinct edges yield distinct pairs of vertices. Thus $E \leq \frac{1}{2}V(V-1)$. Substituting into Euler's formula, we obtain $\frac{1}{6}V(V-1)(V-2) + V = F + V = E + 2 \leq \frac{1}{2}V(V-1) + 2$, and we are led to the inequality $V^3 - 6V^2 + 11V - 12 \leq 0$. The derivative of the polynomial $P(V)$ on the left is $3V^2 - 12V + 11$, whose larger root is $\frac{1}{6}(12 + \sqrt{23})$, which is less than 4. Thus $P(V)$ is an increasing function for $V \geq 4$. Computation gives $P(4) = 0$. Therefore $P(V) > 0$ for $V > 4$, and we cannot have our required inequality $P(V) \leq 0$ for $V > 4$. Tracing back, we see we are forced to conclude that when $V \geq 5$, it is possible to divide S into two proper subsets by some plane and thereby to complete the induction.

2. For each p in M, let (M_p, α_p) be a compatible chart about p for the manifold-with-corners M of dimension m; here M_p is an open neighborhood of p, and $\alpha_p(M_p)$ is open in $\mathbb{Q}^m$. We may assume that no point of M_p has larger index than p does. Now let $F : M \to \mathbb{R}^m$ be an embedding. Since F is continuous and M is compact, $F(M)$ is bounded. Since F is an embedding, F is a homeomorphism of M_+ onto its image in $\mathbb{R}^m$. Define U to be the open set $F(M_+)$, let $B = U^{\text{cl}} - U$, and let E be the image under F of all points in M of index ≥ 2. We are to see that $U \cup (B - E)$ is a smooth manifold-with-boundary, that E is compact, and that E has $m-1$ dimensional Minkowski content 0 in $\mathbb{R}^m$. Proposition 3.6c shows that the set of points of index ≥ 2 is closed in M. Hence it is compact, and its image E in $\mathbb{R}^m$ is compact. We have arranged that $F(M_+) = U$, and hence F carries the set of points of index 1 onto $B - E$.

For each point p in M of index 0 or 1, the open subset M_p of M consists completely of points of index 0 or 1. Hence F carries M_p into $U \cup (B - E)$. The pairs $(F(M_p), \alpha_p \circ F^{-1})$ form an atlas for $U \cup (B - E)$ and exhibit $U - (B - E)$ as a manifold-with-boundary.

Finally we are to see that E has $m - 1$ dimensional Minkowski content 0 in $\mathbb{R}^m$. This step follows from Corollary 3.12.

3. This is a routine adaptation of the argument for Example 3. The condition for the first partial derivative of the $(i, j)^{\text{th}}$ entry of x to vanish is that the $(i, j)^{\text{th}}$ minor of x should vanish. The set where all these 3-by-3 minors vanish is the set of matrices of rank ≤ 2. Call this set E.

We shall exhibit E as the union of 16 compact subsets of vector subspaces of $\mathbb{R}^{16}$ of dimension 12. Each of these will have 15 dimensional Minkowski content 0; then we can conclude that E has 15 dimensional Minkowski content 0, and the set where $\det x \leq 0$ is a Whitney domain.

Consider a matrix x for which the upper left 2-by-2 determinant is nonzero. The vector subspace of $\mathbb{R}^{16}$ corresponding to this choice of entries has dimension $8 + 2 + 2 = 12$, and x lies in this subspace. Thus all matrices of rank 2 lie in the union of 16 vector subspaces of $\mathbb{R}^{16}$ of dimension 12. The matrices of rank < 2 lie in this same finite union, and we see that E has 15 dimensional Mnkowski content 0. Thus the set where $\det x \leq 0$ is a Whitney domain in $\mathbb{R}^{16}$.

4. All of them. In (a), the respective first partial derivatives are $-yz$, $-xz$, and $2z - xy$. If these are simultaneously all 0, then $z = 0$ and also $x = 0$ or $y = 0$; also the converse is true. Thus U is the set where $z(z - xy) < 0$, i.e., the set where z and $z - xy$ are nonzero quantities of the same sign. Also B is the set where $z = 0$ or

$z = xy$, and E is the set where $z = 0$ and $xy = 0$.

In (b), write $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that $\det(x) = ad - bc$. The set U in question is where $\mathrm{Re}\,(ad - bc) < 0$. There are eight variables, namely the real and imaginary parts of a, b, c, d. If all eight first partial derivatives are 0, we are led to $a = b = c = d = 0$. Thus b is the set where $\mathrm{Re}\,(ad - bc) = 0$, and $E = \{0\}$.

In (c), we proceed somewhat as in Algebraic Example 3 in Section 5. We study the set of 4-by-4 skew-symmetric matrices with $\det x \le 0$. We want to know where $\det x = 0$, and we want to identify the singular set. We can use each entry function above the diagonal as coordinates. The partial derivative in question with respect to the variable in the first row and second column is

$$\frac{d}{dt} \det \begin{pmatrix} 0 & x_{12}+t & x_{13} & x_{14} \\ -x_{12}-t & 0 & -x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & a_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{pmatrix} \Bigg|_{t=0}$$

In this expression will appear constant terms, terms with t, and terms with t^2. We use the multilinearity of the determinant to isolate the cofficient of t and find that it equals the sum of two 3-by-3 determinants. Some of the terms cancel, and we find that the derivative at $t = 0$ is the determinant of the 2-by-2 matrix in positions 3 and 4. At any singular point all such derivatives at $t = 0$ have to be 0. The bottom line is that the only singular point is $x = 0$. So again $E = \{0\}$.

5. For (a), V is the intersection of two closed balls. Each of them is a manifold-with-boundary. Then for each point where one or both of the inequalities are strict has an open neighborhood of the kind in a manifold-with-boundary. Each point where both equalities hold has an open neighborhood diffeomorphic to an open neighborhood of $(1, 0, 0)$ in $\mathbb{Q}^3$, and thus we have a manifold-with-corners.

For (b), we are working with $F = x^2\mathbf{i}$, for which $\operatorname{div} F = 2x$. The Divergence Theorem (Theorem 3.7) gives $\int_S x^2 y\, dy \wedge dx = \int_V 2x\, dx\, dy\, dz$. Since V is symmetric about 0 in the x variable and the integrand is odd in the x variable, the integral is 0.

6. For $F = 3y\mathbf{i} + 2x\mathbf{j} + (z - 8)\mathbf{k}$, $\operatorname{div} F = 1$. Thus the given surface integral equals the volume of the tetrahedron that is decribed. The maximum values of x, y, and z subject to $4x + 2y + z = 8$ with all variables ≥ 0 are $x = 2$, $y = 4$, and $z = 8$. The volume in question is 1/6 of the volume of a parallelepiped with sides 2, 4, and 8. It is therefore $64/6 = \boxed{32/3}$.

7. Here $F = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\operatorname{div} F = 3$. Thus the integral equals $3 \cdot 7 = \boxed{21}$.

8. For this F, $\operatorname{div} F = 5$. Since the volume of a closed half ball of radius 2 is $\frac{2}{3}\pi 2^3$, the integral equals $5 \cdot (\frac{2}{3}\pi 2^3) = \boxed{80\pi/3}$.

9. In (b),

$$
\begin{aligned}
U &= \{(x,y,z) \,\big|\, (x,y) \in M_+ \text{ and } f(x,y) < z < g(x,y)\}, \\
B &= \{(x,y,z) \,\big|\, (x,y) \in \partial M \text{ and } f(x,y) \le z \le g(x,y)\} \\
&\qquad \cup \{(x,y,z) \,\big|\, (x,y) \in M_+ \text{ and } z = f(x,y)\} \\
&\qquad \cup \{(x,y,z) \,\big|\, (x,y) \in M_+ \text{ and } z = g(x,y)\}, \\
E &= \{(x,y,z) \,\big|\, (x,y) \in \partial M \text{ and } z = f(x,y)\} \\
&\qquad \cup \{(x,y,z) \,\big|\, (x,y) \in \partial M \text{ and } z = g(x,y)\}.
\end{aligned}
$$

10. In (a), we apply the Fundamental Theorem of Calculus on each subinterval I_j, completely ignoring the other subintervals, and there is no problem. Then we add the results and obtain $\int_a^b f'(t)\,dt = \sum_{j=1}^k \int_I f'(t)\,dt = \sum_{j=1}^k \int_{I_j} f_j'(t)\,dt = \sum_{j=1}^k (f_j(t_j) - f_j(t_{j-1})) = f_k(t_k) - f_0(t_0) = f(b) - f(a)$ because the (finite) series before the next-to-last equality sign telescopes.

In (b) and (c), we can indeed interpret the j^{th} equality as saying the 0 form f_j and the 1 form $df_j = f_j'(t)\,dt$ together satisfy $\int_{\{a_j,b_j\}} f_j = \int_{I_j} df_j$ under a certain orientation. Combining these equalities into a single equality for f requires a certain consistency for the orientations, so that the series in (a) can be seen to telescope at the last step. The orientations on the two-point sets $\{a_j, b_j\}$ are the induced orientations from the various intervals $[a_j, b_j]$, and these are arranged so that each intermediate point $a_1, \dots, a_{k-1}$ occurs with opposite orientations the two times it occurs.

When this framework is applied to a closed triangle—that is, when the t interval is regarded as parametrizing the edge of the triangle—consistent orientations are obtained by orienting the triangle and giving each edge the induced orientation. In this case the expression $f(b) - f(a)$ on the right is 0, since $a = b$. Thus the theorem is that the integral of the derivative is 0; in other words, the result is a version of Theorem 2.1.

11. The definition of a piecewise smooth 1 form on the (closed) faces and edges of a tetrahedron can be taken to be that it is continuous function from the union of the faces and edges of the tetrahedron whose restriction to each face is a smooth 1 form on the closed face. Stokes's Theorem applies to each face as a manifold-with-corners, and we obtain the usual formula $\int_{\text{edges}} \omega = \int_{\text{face}} d\omega$. The 3 dimensional part of the tetrahedron is not present, but if it were and if we were to orient it, then we could use the induced orientation on each face. With this choice when we take all the faces into account, we again have cancellation in pairs for the contributions from the lower dimensional integrals, and the conclusion is that $\sum \int_{\text{faces}} d\omega = 0$.

12. No.

13. Since E is compact and F is continuous, $F(E)$ is compact. Choose compatible charts $(M_{\alpha_1}, \alpha_1), \dots, (M_{\alpha_r}, \alpha_r)$ in M such that $E \subseteq M_{\alpha_1} \cup \cdots \cup M_{\alpha_r}$, and choose by Lemma 1.26b an open cover $\{P_{\alpha_1}, \dots, P_{\alpha_r}\}$ of E such that $P_{\alpha_i}^{\text{cl}} \subseteq M_{\alpha_i}$ for each i.

Then $E = (P^{\mathrm{cl}}_{\alpha_1} \cap E) \cup \cdots \cup (P^{\mathrm{cl}}_{\alpha_r} \cap E)$ exhibits E as the union of the respective compact subsets $P^{\mathrm{cl}}_{\alpha_i} \cap E$ of $M_{\alpha_i} \cap E$. The set $\alpha_i(P^{\mathrm{cl}}_{\alpha_i} \cap E)$ is a compact subset of $\alpha_i(M_{\alpha_i} \cap E)$ and by hypothesis has ℓ dimenional Minkowski content 0 in $\mathbb{R}^m$. It is enough to show that the compact set $F(P^{\mathrm{cl}}_{\alpha_i} \cap E) = (F \circ \alpha_i^{-1})(\alpha_i(M_{\alpha_i} \cap E))$ has ℓ dimensional Minkowski content 0 in N.

In other words we may assume from the outset that M is an open subset of $\mathbb{R}^m$, that we are given a compact subset E of M of ℓ dimensional Minkowski content 0, and that we are to show that $F(E)$ has ℓ dimensional Minkowski content 0 in N.

Choose charts $(N_{\beta_1}, \beta_1), \dots, (N_{\beta_s}, \beta_s)$ in N so that $F(E) \subseteq N_{\beta_1} \cup \cdots \cup N_{\beta_s}$, and then choose by Lemma 1.26b an open cover $\{Q_{\beta_1}, \dots, Q_{\beta_s}\}$ of $F(E)$ such that each Q^{cl}_j is compact and $Q^{\mathrm{cl}}_{\beta_j} \subseteq N_{\beta_j}$ for each j.

For each p in E, choose an open neighborhood M_p of p such that $F(M_p)$ is contained in a single Q_{β_j}. These open neighborhoods cover E, and finitely many of them, say $M_{p_1}, \dots, M_{p_t}$, suffice to cover E. Choose by Lemma 1.26b an open cover $\{R_{p_1}, \dots, R_{p_t}\}$ of E such that $R^{\mathrm{cl}}_{p_k} \subseteq M_{p_k}$ for each k.

The restriction $F\big|_{M_{p_k}}$ of F is smooth from M_{p_k} into some Q_{β_j}, say $Q_{\beta_{j(k)}}$. When it is followed by $\beta_{j(k)}$, the result is a smooth function from an open subset of $\mathbb{R}^m$ into a Eucldiean space. Proposition 3.11 applies to this function and shows that it carries compact sets of ℓ dimensional Minkowski content 0 into compact sets of ℓ dimensional Minkowski content 0. From the inclusions

$$(\beta_{j(k)} \circ F)(R_{p_k}) \subseteq (\beta_{j(k)} \circ F)(R^{\mathrm{cl}}_{p_k}) \subseteq \beta_{j(k)}(Q_{\beta_{j(k)}}),$$

we see that $(\beta_{j(k)} \circ F)(R^{\mathrm{cl}}_{p_k})$ has ℓ dimensional Minkowski content 0 in Euclidean space. Thus $F(R^{\mathrm{cl}}_{p_k})$ has ℓ dimensional Minkowski content 0 in N. We combine this fact with the chain of inclusions

$$F(E) \subseteq F(R_{p_1} \cup \cdots \cup R_{p_t}) = F(R_{p_1}) \cup \cdots \cup F(R_{p_t}) \subseteq F(R^{\mathrm{cl}}_{p_1}) \cup \cdots \cup F(R^{\mathrm{cl}}_{p_t}),$$

and we conclude that $F(E)$ has ℓ dimensional Minkowski content 0 in N.

14. Arguing as in Problem 13, we see that it is enough to see that the smooth image in N of any compact subset E of a Euclidean space $\mathbb{R}^d$ of dimension $d \leq n-2$ has $n-1$ dimensional Minkowski content 0. A compact subset E of $\mathbb{R}^d$ has d dimensional Minkowski content equal to its Lebesgue measure, and then E has $d+1$ dimensional Minkowski content equal to 0. Since $d+1 \leq n-1$, E has $n-1$ dimensional Minkowski content 0. Problem 13 then allows us to conclude that that the smooth image of E in any smooth manifold of dimension $\geq n-1$ has $n-1$ dimensional Minkowski content 0.

15. This is similar to Problem 2. The relevance of the assumption of compactness is in proving that the (closed) set of points of index ≥ 2 is compact.

16. This equivalence is essentially the content of Proposition 3.10. In one direction suppose that E has ℓ dimensional Minkowski content 0 and therefore that

$\lim_{\delta \downarrow 0} \delta^\ell N(E,\delta) = 0$. Then for any $\epsilon > 0$, there is a δ_0 such that $\delta < \delta_0$ implies $\delta^\ell N(E,\delta) < \epsilon$. Take $k = N(E,\delta)$, and let $B_1, \dots, B_k$ have $\operatorname{diam}(B_i) < \delta$. Then $k\delta^\ell < \epsilon$, and E is of zero ℓ extent. In the converse direction suppose E is of zero ℓ extent. Let $\epsilon > 0$ be given, and choose ζ_0 according to that condition. Whenever $\zeta < \zeta_0$ is given, choose k so that $E \subseteq B_1 \cup \cdots \cup B_k$, $\operatorname{diam}(B_i) \le \zeta$, and $k\zeta^\ell < \epsilon$. Then $\zeta^\ell N(E,\zeta_0) < \epsilon$, and $\zeta_0^\ell N(E,\zeta_0) \le \epsilon$. In other words, $\lim_{\zeta_0 \downarrow 0} \zeta_0^\ell N(E,\zeta_0) = 0$, and then E has ℓ dimensional Minkowski content 0 by Proposition 3.10.

17. As in Section 5, let

$$N_{\text{sep}}(E,\delta) = \begin{Bmatrix} \text{maximum number of points of } E \\ \text{at distance} \ge \delta \text{ from one another} \end{Bmatrix}.$$

Suppose we have a configuration of N_1 points x_1 of E_1 that are at distance $\ge \delta$ from one another, and suppose also that we have a configuration of N_2 points x_2 of E_2 that are at distance $\ge \delta$ from one another. Then the corresponding set of points (x_1, x_2) in $E_1 \times E_2$ has the property that any two distinct members of the product set have

$$|(x_1, x_2) - (x_1', x_2')| \ge \max\{|x_1 - x_1'|, |x_2 - x_2'|\} \ge \delta.$$

Therefore there exist $N_1 N_2$ points of $E_1 \times E_2$ at distance $\ge \delta$ from one another, and the definition of N_{sep} gives

$$N_{\text{sep}}(E_1 \times E_2, \delta) \le N_1 N_2.$$

Taking the minimum over all such configurations allows us to conclude that

$$N_{\text{sep}}(E_1 \times E_2, \delta) \le N_{\text{sep}}(E_1, \delta) N_{\text{sep}}(E_2, \delta).$$

Combining ths inequality with the first two conclusions of Lemma 3.9 yields

$$\begin{aligned} N(E_1 \times E_2, \delta) &\le N_{\text{sep}}(E_1 \times E_2, \delta) \\ &\le N_{\text{sep}}(E_1, \delta) \\ &\le N_{\text{sep}}(E_2, \delta) \\ &\le N(E_1, \delta/2) N(E_2, \delta/2), \end{aligned}$$

and the result follows.

18. We know that a dimensional Minkowski content coincides with Lebesgue measure for compact subsets of $\mathbb{R}^a$. Also Lemma 3.9 shows that $|E^\delta|$ is comparable in size to $\delta^a N(E,\delta)$. As δ tends to 0, $|E^\delta|$ tends to $|E|$ by complete additivity of Lebesgue measure, and this limit is finite since E is compact. Thus $\delta^a N(E,\delta)$ is bounded as δ tends to 0.

19. In both parts of the problem, Problem 17 gives

$$N(E_1 \times E_2, \delta) \le N(E_1, \delta)N(E_2, \delta). \tag{$*$}$$

We multiply through by $\delta^{\ell_1+\ell_2}$ for (a) and by $\delta^{\ell_1+m_2}$ for (b). Then we let δ tend to 0.

In (a), Proposition 3.10 shows that $\delta^{\ell_1}N(E_1, \delta)$ and $\delta^{\ell_2}N(E_2, \delta)$ tend to 0. By $(*)$, $\delta^{\ell_1+\ell_2}N(E_1 \times E_2, \delta)$ tends to 0. Thus the converse direction of Proposition 3.10 shows that $E_1 \times E_2$ has $\ell_1 + \ell_2$ dimensional Minkowski content 0.

For (b), we argue in the same way except that we use Problem 18 to see that $\delta^{a_2}N(E_2, \delta)$ is bounded as δ tends to 0. This bounded quantity is multiplied by $\delta^{\ell_1}N(E_1, \delta)$, which tends to 0, and the product thus tends to 0. We conclude that $\delta^{\ell_1+m_2}N(E_1 \times E_2, \delta)$ tends to 0, and it follows that $E_1 \times E_2$ has $\ell_1 + m_2$ dimensional Minkowski content 0.

20. For (a), we are to show that (U, B, E) has the properties of a Whitney domain in $\mathbb{R}^{m_1+m_2}$. The set U is open in $\mathbb{R}^{m_1+m_2}$ because its factors are open in $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, and B is closed and is the boundary of U because B equals $U^{\text{cl}} - U$. The set U is bounded in $\mathbb{R}^{m_1+m_2}$ because U_1 is bounded in $\mathbb{R}^{m_1}$ and M is compact in $\mathbb{R}^{m_2}$. The set E is compact as the product of two compact sets. What needs to be shown is that E has $m_1 + m_2 - 1$ dimensional Minkowski content 0.

It is enough to prove that each of $E_1 \times M$ and $B_1 \times \partial M$ has $m_1 + m_2 - 1$ dimensional Minkowski content 0. Consider $E_1 \times M$. Since E_1 has $m_1 - 1$ dimensional Minkowski content 0 and M is compact in $\mathbb{R}^{m_2}$, Problem 19b shows that $E_1 \times M$ has $m_1 + m_2 - 1$ dimensional Minkowski content 0.

Consider $B_1 \times \partial M$. The subset B_1 of $\mathbb{R}^{m_1}$ by assumption is a closed bounded portion of the set in $\mathbb{R}^{m_1}$ where a nonzero real-valued polynomial in m_1 variables equals 0. Problem 10 of Chapter VI of *Basic Real Analysis* shows that the compact set B_1 has m_1 dimensional Lebesgue measure 0. It therefore has m_1 dimensional Minkowski content 0. The set ∂M is a compact manifold of dimension $m_2 - 1$. Using the style of argument in Problems 13 and 14 and applying Problem 19, we see that $B_1 \times \partial M$ is a compact set of $m_1 + m_2 - 1$ dimensional Minkowski content 0.

At this writing, the author does not know the answer to (b).

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INDEX OF NOTATION

This list indexes recurring symbols introduced in Chapters I through III (pages 1–125). In the list below, each piece of notation is regarded as having a key symbol. The first group consists of those items for which the key symbol is a fixed Latin letter, and the items are arranged roughly alphabetically by that key symbol. The next group consists of those items for which the key symbol is a Greek letter. The final group consists of those items for which the key symbol is a variable or a nonletter, and these are arranged by type. To locate an item below, first proceed on the assumption that the key symbol is a Latin or Greek letter; if the item does not appear to be in the list, then treat it as if its key symbol is a variable or a nonletter.

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