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CHAPTER VII

Aspects of Partial Differential Equations

Abstract. This chapter provides an introduction to partial differential equations, particularly linear ones, beyond the material on separation of variables in Chapter I.

Sections 1–2 give an overview. Section 1 addresses the question of how many side conditions to impose in order to get local existence and uniqueness of solutions at the same time. The Cauchy–Kovalevskaya Theorem is stated precisely for first-order systems in standard form and for single equations of order greater than one. When the system or single equation is linear with constant coefficients and entire holomorphic data, the local holomorphic solutions extend to global holomorphic solutions. Section 2 comments on some tools that are used in the subject, particularly for linear equations, and it gives some definitions and establishes notation.

Section 3 establishes the basic theorem that a constant-coefficient linear partial differential equation $Lu = f$ has local solutions, the technique being multiple Fourier series.

Section 4 proves a maximum principle for solutions of second-order linear elliptic equations $Lu = 0$ with continuous real-valued coefficients under the assumption that $L(1) = 0$.

Section 5 proves that any linear elliptic equation $Lu = f$ with constant coefficients has a “parametrix,” and it shows how to deduce from the existence of the parametrix the fact that the solutions u are as regular as the data f . The section also deduces a global existence theorem when f is compactly supported; this result uses the existence of the parametrix and the constant-coefficient version of the Cauchy–Kovalevskaya Theorem.

Section 6 gives a brief introduction to pseudodifferential operators, concentrating on what is needed to obtain a parametrix for any linear elliptic equation with smooth variable coefficients.

1. Introduction via Cauchy Data

The subject of partial differential equations is a huge and diverse one, and a short introduction necessarily requires choices. The subject has its origins in physics and nowadays has applications that include physics, differential geometry, algebraic geometry, and probability theory. A small amount of complex-variable theory will be extremely helpful, and this will be taken as known for this chapter. We shall ultimately concentrate on single equations, as opposed to systems, and on partial differential equations that are linear. After the first two sections the topics of this chapter will largely be ones that can be approached through a combination of functional analysis and Fourier analysis.

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compact support into smooth functions. The operator Q that gives a two-sided inverse for Δ except for the smoothing term is called a **parametrix** for Δ .

The parametrix does not solve our equation for us, but it does supply useful information. As we shall see in Section 5, a parametrix will enable us to see that whenever u is a distribution solution of $\Delta u = f$ on an open set U , with f an arbitrary distribution on U , then u is smooth wherever f is smooth. In particular, any distribution solution of $\Delta u = 0$ is a smooth function. The argument will apply to any elliptic linear partial differential equation with constant coefficients. A first application of the method of pseudodifferential operators in Section 6 shows that the same conclusion is valid for any elliptic linear partial differential equation with smooth variable coefficients.

3. Local Solvability in the Constant-Coefficient Case

We come to the local existence of solutions to linear partial differential equations with constant coefficients.

Theorem 7.7. Let U be an open set in \mathbb{R}^N containing 0, and let f be in $C^\infty(U)$. If $P(D)$ is a linear differential operator with constant coefficients and with order ≥ 1 , then the equation $P(D)u = f$ has a smooth solution in a neighborhood of 0.

The proof will use multiple Fourier series as in Section III.7. Apart from that, all that we need will be some manipulations with polynomials in several variables and an integration. As in Section III.7, let us write \mathbb{Z}^N for the set of all integer N -tuples and $[-\pi, \pi]^N$ for the region of integration defining the Fourier series.

We shall give the idea of the proof, state a lemma, prove the theorem from the lemma, and then return to the proof of the lemma. The idea of the proof of Theorem 7.7 is as follows: We begin by multiplying f by a smooth function that is identically 1 near the origin and is identically 0 off some small ball containing the origin (existence of the smooth function by Proposition 3.5f), so that f is smooth of compact support, the support lying well inside $[-\pi, \pi]^N$. If we regard f as extended periodically to a smooth function, we can write $f(x) = \sum_{k \in \mathbb{Z}^N} d_k e^{ik \cdot x}$ by Proposition 3.30e. Let the unknown function u be given by $u(x) = \sum_{k \in \mathbb{Z}^N} c_k e^{ik \cdot x}$. Then $P(D)u(x)$ is given by

$$P(D)u(x) = \sum_{k \in \mathbb{Z}^N} c_k P(ik) e^{ik \cdot x},$$

and thus we want to take $c_k P(ik) = d_k$. We are done if $\frac{d_k}{P(ik)}$ decreases faster than any $|k|^{-n}$, by Proposition 3.30c and our computations. So we would like to prove that

$$|P(ik)|^{-1} \leq C(1 + |k|^2)^M \quad \text{for all } k \in \mathbb{Z}^N$$

and for some constants C and M , and then we would be done. Unfortunately this is not necessarily true; the polynomial $P(x) = |x|^2$ is a counterexample. What is true is the statement in the following lemma, and we can readily adjust the above idea to prove the theorem from this lemma.

Lemma 7.8. If $R(x)$ is any complex-valued polynomial not identically 0 on \mathbb{R}^N , then there exist $\alpha \in \mathbb{R}^N$ and constants C and M such that

$$|R(k + \alpha)|^{-1} \leq C(1 + |k|^2)^M \quad \text{for all } k \in \mathbb{Z}^N.$$

PROOF OF THEOREM 7.7. Apply the lemma to $R(x) = P(ix)$. Because of the preliminary step of multiplying f by something, we are assuming that f is smooth and has support near 0. Instead of extending f to be periodic, as suggested in the discussion before the lemma, we extend the function $f(x)e^{-i\alpha \cdot x}$ to be smooth and periodic. Thus write

$$f(x)e^{-i\alpha \cdot x} = \sum_{k \in \mathbb{Z}^N} d_k e^{ik \cdot x},$$

and put $c_k = \frac{d_k}{R(k + \alpha)}$. Since the $|d_k|$ decrease faster than $|k|^{-n}$ for any n , Lemma 7.8 and Proposition 3.30c together show that $\sum_{k \in \mathbb{Z}^N} c_k e^{ik \cdot x}$ is smooth and periodic. Define

$$u(x) = e^{i\alpha \cdot x} \sum_{k \in \mathbb{Z}^N} c_k e^{ik \cdot x} = \sum_{k \in \mathbb{Z}^N} c_k e^{i(k + \alpha) \cdot x}.$$

This function is smooth but maybe is not periodic. Application of $P(D)$ gives

$$\begin{aligned} P(D)u(x) &= \sum_{k \in \mathbb{Z}^N} c_k P(i(k + \alpha)) e^{i(k + \alpha) \cdot x} \\ &= e^{i\alpha \cdot x} \sum_{k \in \mathbb{Z}^N} \frac{d_k}{R(k + \alpha)} P(i(k + \alpha)) e^{ik \cdot x} \\ &= e^{i\alpha \cdot x} \sum_{k \in \mathbb{Z}^N} d_k e^{ik \cdot x} = e^{i\alpha \cdot x} (f(x)e^{-i\alpha \cdot x}) = f(x), \end{aligned}$$

and hence u solves the equation for the original f in a neighborhood of the origin.

The proof of Lemma 7.8 requires two lemmas of its own.

Lemma 7.9. For each positive integer m and positive number $\delta < \frac{1}{m}$, there exists a constant C such that

$$\int_{-1}^1 |x - c_1|^{-\delta} \cdots |x - c_m|^{-\delta} dx \leq C$$

for any m complex numbers c_1, \dots, c_m .

PROOF. For $1 \leq j \leq m$, let E_j be the subset of $[-1, 1]$ where $|x - c_j|^{-\delta}$ is the largest of the m factors in the integrand. The integral in question is then

$$\begin{aligned} &\leq \sum_{j=1}^m \int_{E_j} |x - c_1|^{-\delta} \cdots |x - c_m|^{-\delta} dx \\ &\leq \sum_{j=1}^m \int_{E_j} |x - c_j|^{-m\delta} dx \leq \sum_{j=1}^m \int_{-1}^1 |x - c_j|^{-m\delta} dx \\ &\leq \sum_{j=1}^m \int_{-1}^1 |x - \operatorname{Re} c_j|^{-m\delta} dx \leq m \sup_{r \in \mathbb{R}} \int_{-1}^1 |x - r|^{-m\delta} dx. \end{aligned}$$

On the right side the integrand decreases pointwise with $|r|$ when $|r| \geq 1$, and hence the expression is equal to

$$\begin{aligned} &m \sup_{-1 \leq r \leq 1} \int_{-1}^1 |x - r|^{-m\delta} dx \\ &= m \sup_{-1 \leq r \leq 1} \left(\int_{-1}^r (r - x)^{-m\delta} dx + \int_r^1 (x - r)^{-m\delta} dx \right) \\ &= m(1 - m\delta)^{-1} \sup_{-1 \leq r \leq 1} \left((1 + r)^{1-m\delta} + (1 - r)^{1-m\delta} \right) \\ &\leq 2^{2-m\delta} m(1 - m\delta)^{-1}. \end{aligned}$$

Lemma 7.10. If $R(x)$ is any complex-valued polynomial on \mathbb{R}^N of degree $m > 0$, then $|R(x)|^{-\delta}$ is locally integrable whenever $\delta < \frac{1}{m}$.

PROOF. We first treat the special case that x_1^m has coefficient 1 in $R(x)$ and that integrability on the cube $[-1, 1]^N$ is to be checked. Write x' for (x_2, \dots, x_N) , so that $x = (x_1, x')$. Then $R(x) = x_1^m + \sum_{j=0}^{m-1} x_1^j p_j(x')$, where each p_j is a polynomial. For fixed x' , $R(x_1, x')$ is a monic polynomial of degree m in x_1 and factors as $(x_1 - c_1) \cdots (x_1 - c_m)$ for some complex numbers c_1, \dots, c_m depending on x' . Applying Lemma 7.9, we see that $\int_{-1}^1 |R(x_1, x')|^{-\delta} dx_1 \leq C$. Integration in the remaining $N - 1$ variables therefore gives $\int_{[-1, 1]^N} |R(x)|^{-\delta} dx \leq 2^{N-1} C$.

Turning to the general case, suppose that $R(x)$ and a point x_0 are given. We want to see that $F(x) = R(x + x_0)$ has the property that $|F(x)|^{-\delta}$ is integrable on some neighborhood of the origin in \mathbb{R}^N . The function F is still a polynomial of degree m . Let F_m be the sum of all its terms of total degree m . This cannot be identically 0 on the unit sphere since it is a nonzero homogeneous function,⁴ and thus $F_m(v_1) \neq 0$ for some unit vector v_1 . Extend $\{v_1\}$ to an orthonormal basis of \mathbb{R}^N , and define $G(y_1, \dots, y_N) = F_m(y_1 v_1 + \cdots + y_N v_N)$. The function G is a polynomial of degree m whose coefficient of y_1^m is $F_m(v_1)$ and hence is not 0, and it is obtained by applying an orthogonal transformation to the variables of F . Therefore $|G|^{-\delta}$ and $|F|^{-\delta}$ have the same integral over a ball centered at the origin. The special case shows that $|G|^{-\delta}$ is integrable over some such ball, and hence so is $|F|^{-\delta}$.

⁴A function F_m of several variables is **homogeneous of degree m** if $F_m(rx) = r^m F_m(x)$ for all $r > 0$ and all $x \neq 0$.

PROOF OF LEMMA 7.8. Let R have degree m , which we may assume is positive without loss of generality. The function $S(x) = |x|^{2m} R\left(\frac{x}{|x|^2}\right)$ is then a polynomial of degree $\leq 2m$, and Lemma 7.10 shows that any number δ with $\delta < \frac{1}{2m}$ has the property that $|R|^{-\delta}$ and $|S|^{-\delta}$ are integrable for $|x| \leq 1$. Using spherical coordinates and making the change of variables $r \mapsto 1/r$ in the radial direction, we see that

$$\begin{aligned} \int_{|x| \geq 1} |R(x)|^{-\delta} |x|^{-2N} dx &= \int_{r=1}^{\infty} \int_{\omega \in S^{N-1}} |R(r\omega)|^{-\delta} r^{-2N} d\omega r^{N-1} dr \\ &= \int_{r=0}^1 \int_{\omega \in S^{N-1}} |R(r^{-1}\omega)|^{-\delta} d\omega r^{N-1} dr \\ &= \int_{|x| \leq 1} |R(x/|x|^2)|^{-\delta} dx \\ &= \int_{|x| \leq 1} |S(x)|^{-\delta} |x|^{2m\delta} dx \\ &\leq \int_{|x| \leq 1} |S(x)|^{-\delta} dx. \end{aligned}$$

The right side is finite. Since $(1 + |x|^2)^{-N} \leq 1 + |x|^{-2N}$, we see that

$$\int_{\mathbb{R}^N} |R(x)|^{-\delta} (1 + |x|^2)^{-N} dx < \infty.$$

Define $E = \{\alpha \in \mathbb{R}^N \mid 0 \leq \alpha_j < 1 \text{ for all } j\}$. By complete additivity, we can rewrite the above finiteness condition as

$$\int_{\alpha \in E} \left[\sum_{k \in \mathbb{Z}^N} |R(k + \alpha)|^{-\delta} (1 + |k + \alpha|^2)^{-N} \right] d\alpha < \infty.$$

Every pair (l, β) with $l \in \mathbb{Z}$ and $\beta \in [0, 1)$ has $(l + \beta)^2 \leq 2(1 + l^2)$. Summing N such inequalities gives $|k + \alpha|^2 \leq 2N + 2|k|^2 \leq 2N(1 + |k|^2)$. Thus we obtain $1 + |k + \alpha|^2 \leq 3N(1 + |k|^2)$, $(1 + |k + \alpha|^2)^{-N} \geq (3N)^{-N} (1 + |k|^2)^{-N}$, and

$$\int_{\alpha \in E} \left[\sum_{k \in \mathbb{Z}^N} |R(k + \alpha)|^{-\delta} (1 + |k|^2)^{-N} \right] d\alpha < \infty.$$

Therefore $\sum_{k \in \mathbb{Z}^N} |R(k + \alpha)|^{-\delta} (1 + |k|^2)^{-N}$ is finite almost everywhere $[d\alpha]$. Fix an α for which the sum is finite. If

$$\sum_{k \in \mathbb{Z}^N} |R(k + \alpha)|^{-\delta} (1 + |k|^2)^{-N} = K < \infty,$$

then $|R(k + \alpha)|^{-\delta} (1 + |k|^2)^{-N} \leq K$ for all $k \in \mathbb{Z}^N$ and hence $|R(k + \alpha)|^{-1} \leq K^{1/\delta} (1 + |k|^2)^{N/\delta}$ for all $k \in \mathbb{Z}^N$. This proves Lemma 7.8.

4. Maximum Principle in the Elliptic Second-Order Case

In this section we work with a second-order linear homogeneous elliptic equation $Lu = 0$ with continuous real-valued coefficients in a bounded connected open subset U of \mathbb{R}^N . It will be assumed that only derivatives of u , and not u itself, appear in the equation; in other words we assume that $L(1) = 0$. The conclusion will be that a real-valued C^2 solution u cannot have an absolute maximum or an absolute minimum inside U without being constant. This result was proved already in Corollary 3.20 for the special case that L is the Laplacian Δ .

Let us use notation for L of the kind in Proposition 7.5 and its proof. Then L is of the form

$$Lu = \sum_{i,j} b_{ij}(x) D_i D_j u + \sum_k c_k(x) D_k u$$

with the matrix $[b_{ij}(x)]$ real-valued and symmetric. Ellipticity of L at x means that $\sum_{i,j} b_{ij}(x) \xi_i \xi_j \neq 0$ for $\xi \neq 0$. Thus $|\sum_{i,j} b_{ij}(x) \xi_i \xi_j|$ has a positive minimum value $\mu(x)$ on the compact set where $|\xi| = 1$. By homogeneity of $|\sum_{i,j} b_{ij}(x) \xi_i \xi_j|$ and $|\xi|^2$, we conclude that

$$\left| \sum_{i,j} b_{ij}(x) \xi_i \xi_j \right| \geq \mu(x) |\xi|^2$$

for some $\mu(x) > 0$ and all ξ . The positive number $\mu(x)$ is called the **modulus of ellipticity** of L at x .

EXAMPLE. Let L be the sum of the Laplacian and first-order terms, i.e., $Lu = \Delta u + \sum_k c_k(x) D_k u$. Suppose that u is a real-valued C^2 function on U and that u attains a local maximum at x_0 in U . By calculus, $D_i u(x_0) = 0$ for each i and $D_i^2 u(x_0) \leq 0$, so that $Lu(x_0) \leq 0$. Therefore if we know that $Lu(x)$ is > 0 everywhere in U , then u can have no local maximum in U . To obtain a maximum principle, we want to relax two conditions and still get the same conclusion. One is that we want to allow more general second-order terms in L , and the other is that we want to get a conclusion from knowing only that $Lu(x)$ is ≥ 0 everywhere. The first step is carried out in Lemma 7.11 below, and the second step will be derived from the first essentially by perturbing the situation in a subtle way.

Lemma 7.11. Let $L = \sum_{i,j} b_{ij}(x) D_i D_j + \sum_k c_k(x) D_k$, with $[b_{ij}(x)]$ symmetric, be a second-order linear elliptic operator with real-valued coefficients in an open subset U of \mathbb{R}^N such that for every x in U , there is a number $\mu(x) > 0$ such that $\sum_{i,j} b_{ij}(x) \xi_i \xi_j \geq \mu(x) |\xi|^2$ for all $\xi \in \mathbb{R}^N$. If u is a real-valued C^2 function on U such that $Lu > 0$ everywhere in U , then u has no local maximum in U .

PROOF. Suppose that u has a local maximum at x_0 . Applying Proposition 7.5, we can find a nonsingular matrix M such that the definition $D'_i = \sum_j M_{ij} D_j$ makes the second-order terms of L at x_0 take the form $\kappa_1 D_1'^2 + \cdots + \kappa_N D_N'^2$ with each κ_i equal to $+1$, -1 , or 0 . Examining the hypotheses of the lemma, we see that all κ_i must be $+1$. Hence the change of basis at x_0 via M converts the second-order terms of L into the form $D_1'^2 + \cdots + D_N'^2$. The argument in the example above is applicable at x_0 , and the lemma follows.

Theorem 7.12 (Hopf maximum principle). Let

$$L = \sum_{i,j} b_{ij}(x) D_i D_j + \sum_k c_k(x) D_k,$$

with $[b_{ij}(x)]$ symmetric, be a second-order linear elliptic operator with real-valued continuous coefficients in a connected open subset U of \mathbb{R}^N . If u is a real-valued C^2 function on U such that $Lu = 0$ everywhere in U , then u cannot attain its maximum or minimum values in U without being constant.

PROOF. First we normalize matters suitably. We have $|\sum_{i,j} b_{ij}(x) \xi_i \xi_j| \geq \mu(x) |\xi|^2$ with $\mu(x) > 0$ at every point. By continuity of the coefficients and connectedness of U , the expression within the absolute value signs on the left side is everywhere positive or everywhere negative. Possibly replacing L by $-L$, we shall assume that it is everywhere positive:

$$\sum_{i,j} b_{ij}(x) \xi_i \xi_j \geq \mu(x) |\xi|^2 \quad \text{for all } x \in U.$$

Because of the continuity of the coefficients of L , the coefficient functions are bounded on any compact subset of U and the function $\mu(x)$ is bounded below by a positive constant on any such compact set. Since u can always be replaced by $-u$, a result about absolute maxima is equivalent to a result about absolute minima. Thus we may suppose that u attains its absolute maximum value M at some x_1 in U , and we are to prove that u is constant in U . Arguing by contradiction, suppose that x_0 is a point in U with $u(x_0) < M$.

The idea of the proof is to use x_0 and x_1 to produce an open ball B with $B^{\text{cl}} \subseteq U$ and a point s in the boundary ∂B of B such that $u(s) = M$ and $u(x) < M$ for all x in $B^{\text{cl}} - \{s\}$. See Figure 7.1. For a suitably small open ball B_1 centered at s , we then produce a C^2 function w on \mathbb{R}^N such that $Lw > 0$ in B_1 and w attains a local maximum at the center s of B_1 . The existence of w contradicts Lemma 7.11, and thus the configuration with x_0 and x_1 could not have occurred.

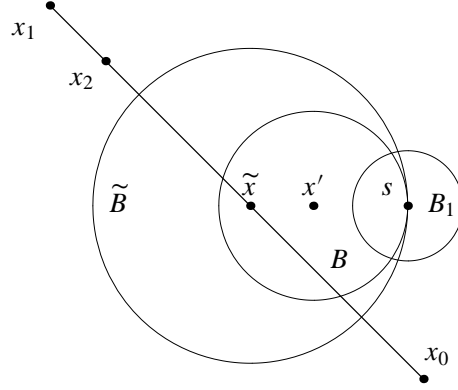


FIGURE 7.1. Construction in the proof of the Hopf maximum principle.

Since U is a connected open set in \mathbb{R}^N , it is pathwise connected. Let $p : [0, 1] \rightarrow U$ be a path with $p(0) = x_0$ and $p(1) = x_1$. Let τ be the first value of t such that $u(p(t)) = M$; necessarily $0 < \tau \leq 1$. Define $x_2 = p(\tau)$. Choose $d > 0$ such that $B(d; p(t))^{\text{cl}} \subseteq U$ for $0 \leq t \leq \tau$, and then fix a point $\tilde{x} = p(t)$ with $0 \leq t < \tau$ and with $|\tilde{x} - x_2| < \frac{1}{2}d$. By definition of d , $B(d; \tilde{x})^{\text{cl}} \subseteq U$. Let \tilde{B} be the largest open ball contained in U , centered at \tilde{x} , and having $u(x) < M$ for $x \in \tilde{B}$. Since $u(x_2) = M$ and $|\tilde{x} - x_2| < \frac{1}{2}d$, \tilde{B} has radius $< \frac{1}{2}d$. Thus $\tilde{B}^{\text{cl}} \subseteq B(d; \tilde{x})^{\text{cl}} \subseteq U$. The construction of \tilde{B} and the continuity of u force some point s of the boundary $\partial\tilde{B}$ to have $u(s) = M$. Let B be any open ball properly contained in \tilde{B} and internally tangent to \tilde{B} at s . Then $B^{\text{cl}} \subseteq \tilde{B} \cup \{s\}$, and hence $u(x) < M$ everywhere on B^{cl} except at s , where $u(s) = M$. Write $B = B(R; x')$.

To construct B_1 , fix $R_1 > 0$ with $R_1 < \frac{1}{2}R$, and let $B_1 = B(R_1; s)$. If x is in B_1^{cl} , then $|x - \tilde{x}| \leq |x - s| + |s - \tilde{x}| \leq R_1 + \frac{1}{2}d < \frac{1}{2}R + \frac{1}{2}d \leq d$, and hence $B_1^{\text{cl}} \subseteq B(d; \tilde{x})^{\text{cl}} \subseteq U$. Since B^{cl} and B_1^{cl} are compact subsets of U , the coefficients of L are bounded on $B^{\text{cl}} \cup B_1^{\text{cl}}$, and the ellipticity modulus is bounded below by a positive number. Let us say that

$$|b_{ij}(x)| \leq \beta, \quad |c_k(x)| \leq \gamma, \quad \mu(x) \geq \mu > 0 \quad \text{for } x \in B^{\text{cl}} \cup B_1^{\text{cl}}.$$

The next step is to construct an auxiliary function $z(x)$ on \mathbb{R}^N to be used in the definition of $w(x)$. Let α be a (large) positive number to be specified, and set

$$z(x) = e^{-\alpha|x-x'|^2} - e^{-\alpha R^2}.$$

The function $z(x)$ is > 0 on B , is 0 on ∂B , and is < 0 off B^{cl} . Let us see that we can choose α large enough to make $L(z)(x) > 0$ for x in B_1 . Performing the

differentiations explicitly, we obtain

$$\begin{aligned} L(z)(x) &= 2\alpha e^{-\alpha|x-x'|^2} \left(2\alpha \sum_{i,j} b_{ij}(x)(x_i - x'_i)(x_j - x'_j) \right. \\ &\quad \left. - \sum_k (b_{kk}(x) - c_k(x)(x_k - x'_k)) \right) \\ &\geq 2\alpha e^{-\alpha|x-x'|^2} (2\alpha\mu|x-x'|^2 - (\beta + \gamma|x-x'|)). \end{aligned}$$

All points x in B_1 have $\frac{1}{2}R < |x - x'| < \frac{3}{2}R$ and therefore satisfy

$$L(z)(x) \geq 2\alpha e^{-\alpha|x-x'|^2} (2\alpha\mu\frac{1}{4}R^2 - (\beta + \frac{3}{2}\gamma R)).$$

Consequently we can choose α large enough so that $L(z)(x) > 0$ for x in B_1 . Fix α with this property.

Let $\epsilon > 0$ be a (small) positive number to be specified, and define

$$w = u + \epsilon z.$$

For x in B_1 , we have $Lw = Lu + \epsilon Lz > 0$. Also,

$$w(s) = u(s) + \epsilon z(s) = u(s) = M \quad \text{since } s \text{ is in } \partial B.$$

Let us see that we can choose ϵ to make $w(x) < M$ everywhere on ∂B_1 . We consider ∂B_1 in two pieces. Let $C_0 = \partial B_1 \cap B^{\text{cl}}$. Since C_0 is a subset of $B^{\text{cl}} - \{s\}$, $u(x) < M$ at every point of C_0 . By compactness of C_0 and continuity of u , we must therefore have $u(x) \leq M - \delta$ on C_0 for some $\delta > 0$. Since the function $z(x)$ is everywhere $\leq 1 - e^{-\alpha R^2}$, any x in C_0 must have

$$w(x) = u(x) + \epsilon z(x) \leq M - \delta + \epsilon(1 - e^{-\alpha R^2}).$$

By taking ϵ small enough, we can arrange that $w(x) \leq M - \frac{1}{2}\delta$ on C_0 . Fix such an ϵ . The remaining part of ∂B_1 is $\partial B_1 - C_0$. Each x in this set has

$$w(x) = u(x) + \epsilon z(x) \leq M + \epsilon z(x) < M.$$

Thus $w(x) < M$ everywhere on ∂B_1 , as asserted.

Since $w(s) = M$ and $w(x) < M$ everywhere on ∂B_1 , w attains its maximum in B_1^{cl} somewhere in the open set B_1 . Since $Lw > 0$ on B_1 , we obtain a contradiction to Lemma 7.11. This completes the proof.

5. Parametrixes for Elliptic Equations with Constant Coefficients

In this section we use distribution theory to derive some results about an elliptic equation $P(D)u = f$ with constant coefficients. Initially we work on \mathbb{R}^N , yet in the end we will be able to work on any nonempty open set. We think of f as known and u as unknown. But we allow f to vary, so that we can see the effect on u of changing f . It will be important to be able to allow solutions that are not smooth functions, and thus u will be allowed to be some kind of distribution.

We begin by obtaining a parametrix, which at first will be a tempered distribution that approximately inverts $P(D)$ on $\mathcal{S}'(\mathbb{R}^N)$. More specifically it inverts $P(D)$ on $\mathcal{S}'(\mathbb{R}^N)$ up to an error term given by an operator equal to convolution with a Schwartz function.

At this point we can use the version Theorem 7.4 of the Cauchy–Kovalevskaya Theorem to obtain a **fundamental solution**, i.e., a member u of $\mathcal{D}'(\mathbb{R}^N)$ such that $P(D)u = \delta$. This step is carried out in Corollary 7.15 below. Convolution of $P(D)u = \delta$ with a member f of $\mathcal{E}'(\mathbb{R}^N)$ shows that Corollary 7.15 implies a global existence theorem: any elliptic equation $P(D)u = f$ with f in $\mathcal{E}'(\mathbb{R}^N)$ has a solution in $\mathcal{D}'(\mathbb{R}^N)$.

But it is not necessary, for purposes of examining regularity of solutions, to have an existence theorem. The next step is to modify the parametrix to have compact support. Once that has been done, the parametrix will invert $P(D)$ on $\mathcal{D}'(\mathbb{R}^N)$, up to a smoothing term, and we will deduce a regularity theorem about solutions saying that the singular support of u is contained in the singular support of f . In particular, solutions of $P(D)u = 0$ on \mathbb{R}^N are smooth. Finally we localize this result to see that the inclusion of singular supports persists even when the equation $P(D)u = f$ is being considered only on an open set U .

The starting point for our investigation is the following lemma.

Lemma 7.13. If $P(D)$ is an elliptic operator with constant coefficients, then the set of zeros of $P(2\pi i\xi)$ in \mathbb{R}^N is compact.

REMARK. The polynomial $P(2\pi i\xi)$ is the symbol of $P(D)$, as defined in Section 2. The important fact about the symbol is that the Fourier transform satisfies $\mathcal{F}(P(D)T) = P(2\pi i\xi)\mathcal{F}(T)$, which follows immediately from the formula $\mathcal{F}(D^\alpha T) = (2\pi i)^{|\alpha|}\xi^\alpha \mathcal{F}(T)$. This fact accounts for our studying the particular polynomial $P(2\pi i\xi)$.

PROOF. Let P have order m , and let Z be the set of zeros of $P(2\pi i\xi)$ in \mathbb{R}^N . Since $P(D)$ is elliptic, the principal symbol $P_m(2\pi i\xi)$ is nowhere 0 on the unit sphere of \mathbb{R}^N . By compactness of the sphere, $|P_m(2\pi i\xi)| \geq c > 0$ there, for some constant c . Taking into account the homogeneity of P_m , we see that $|P_m(2\pi i\xi)| \geq c|\xi|^m$ for all ξ in \mathbb{R}^N . If we write $P(2\pi i\xi) = P_m(2\pi i\xi) + Q(2\pi i\xi)$, then

$Q(2\pi i\xi)| \leq C|\xi|^{m-1}$ for $|\xi| \geq 1$ and for some constant C . If ξ is in Z and $|\xi| \geq 1$, then we have $c|\xi|^m \leq P_m(2\pi i\xi) = |Q(2\pi i\xi)| \leq C|\xi|^{m-1}$, and we conclude that $|\xi| \leq C/c$. This proves the lemma.

Fix an elliptic operator $P(D)$, and choose $R > 0$ by the lemma such that all the zeros in \mathbb{R}^N of $P(2\pi i\xi)$ lie in the closed ball of radius R centered at the origin. Fix numbers R' and R'' with $R' > R'' > R$. Let χ be a smooth function on \mathbb{R}^N with values in $[0, 1]$ such that $\chi(\xi)$ is 0 when $|\xi| \leq R''$ and is 1 when $|\xi| \geq R'$. The formal computation is as follows: if we define v in terms of f by

$$v(x) = \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} \frac{\mathcal{F}(f)(\xi)}{P(2\pi i\xi)} \chi(\xi) d\xi,$$

then Fourier inversion gives

$$\begin{aligned} (P(D)v)(x) &= \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} \mathcal{F}(f)(\xi) \chi(\xi) d\xi \\ &= f(x) + \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} (\chi(\xi) - 1) \mathcal{F}(f)(\xi) d\xi, \end{aligned}$$

and the second term on the right side will be seen to be a smoothing term. Let us now state a precise result and use properties of distributions to make this computation rigorous.

Theorem 7.14. Let $P(D)$ be an elliptic operator on \mathbb{R}^N with constant coefficients. Then there exist a distribution $k \in \mathcal{S}'(\mathbb{R}^N)$ and a Schwartz function $h \in \mathcal{F}^{-1}(C_{\text{com}}^\infty(\mathbb{R}^N))$ such that

$$P(D)k = \delta + T_h,$$

as an equality in $\mathcal{S}'(\mathbb{R}^N)$. Here δ is the Dirac distribution $\langle \delta, \varphi \rangle = \varphi(0)$. Consequently whenever f is in $\mathcal{E}'(\mathbb{R}^N)$, then the distribution $v = k * f$ is tempered and satisfies $P(D)v = f + (h * f)$.

REMARKS. The convolution operator $f \mapsto k * f$ is called a **parametrix** for $P(D)$ on $\mathcal{E}'(\mathbb{R}^N)$. More precisely it is a right parametrix, and a left parametrix can be defined similarly. The operator $f \mapsto h * f$ is called a **smoothing operator** because $h * f$ is in $C^\infty(\mathbb{R}^N)$ whenever f is in $\mathcal{E}'(\mathbb{R}^N)$. To see the smoothing property, we observe that h , as a Schwartz function, is identified with a tempered distribution when we pass to T_h . Theorem 5.21 shows that $T_h * f$ is a tempered distribution with Fourier transform $\mathcal{F}(h)\mathcal{F}(f)$. Both factors $\mathcal{F}(h)$ and $\mathcal{F}(f)$ are smooth functions, and $\mathcal{F}(h)$ has compact support. Therefore $\mathcal{F}(h * f)$ is smooth of compact support, and $h * f$ is a Schwartz function.

PROOF. The function $\sigma(\xi) = \chi(\xi)/P(2\pi i\xi)$ is smooth and is bounded on \mathbb{R}^N because, in the notation used in the proof of Lemma 7.13, $|P(2\pi i\xi)| \geq |P_m(2\pi i\xi)| - |Q(2\pi i\xi)| \geq (c|\xi| - C)|\xi|^{m-1}$ and because $(c|\xi| - C)|\xi|^{m-1} \geq 1$ as soon as $|\xi|$ is large enough. Since σ is bounded, integration of the product of σ and any Schwartz function is meaningful, and T_σ is therefore in $\mathcal{S}'(\mathbb{R}^N)$. Define $k = \mathcal{F}^{-1}(T_\sigma)$. This is in $\mathcal{S}'(\mathbb{R}^N)$ and has $\mathcal{F}(k) = T_\sigma$. Define $h = \mathcal{F}^{-1}(\chi - 1)$. Since $\chi - 1$ is in $C_{\text{com}}^\infty(\mathbb{R}^N)$, h is in $\mathcal{S}(\mathbb{R}^N)$.

Now let f in $\mathcal{E}'(\mathbb{R}^N)$ be given, and define $v = k * f$. Theorem 5.21 shows that v is in $\mathcal{S}'(\mathbb{R}^N)$ and that $\mathcal{F}(v) = \mathcal{F}(k)\mathcal{F}(f) = \sigma\mathcal{F}(f)$. Then

$$\begin{aligned} \mathcal{F}(P(D)v) &= P(2\pi i\xi)\mathcal{F}(v) = P(2\pi i\xi)\sigma(\xi)\mathcal{F}(f) \\ &= \chi(\xi)\mathcal{F}(f) = \mathcal{F}(f) + (\chi(\xi) - 1)\mathcal{F}(f) = \mathcal{F}(f) + \mathcal{F}(h)\mathcal{F}(f). \end{aligned}$$

Taking the inverse Fourier transform of both sides yields $P(D)v = f + h * f$ as asserted. For the special case $f = \delta$, we have $v = k * \delta = k$, and then $P(D)k = \delta + T_h$. This completes the proof.

The function h is the inverse Fourier transform of a member of $C_{\text{com}}^\infty(\mathbb{R}^N)$, specifically $h(x) = \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} (\chi(\xi) - 1) d\xi$. Since the integration is really taking place on a compact set, we see that we can replace x by a complex variable z and obtain a holomorphic function in all of \mathbb{C}^N . In other words, h extends to a holomorphic function on \mathbb{C}^N . If we single out any variable, say x_1 , then the ellipticity of $P(D)$ implies that $D_{x_1}^m$ has nonzero coefficient in $P(D)$, and $P(D)w = h$ is therefore an equation to which the global Cauchy–Kovalevskaya Theorem applies in the form of Theorem 7.4. The theorem says that the equation $P(D)w = h$, in the presence of globally holomorphic Cauchy data, has not just a local holomorphic solution but a global holomorphic one. Therefore $P(D)w = h$ has an entire holomorphic solution w . Let us regard w and h as yielding distributions T_w and T_h on $C_{\text{com}}^\infty(\mathbb{R}^N)$, so that the equation reads $P(D)T_w = T_h$. Subtracting this from $P(D)k = \delta + T_h$ yields $P(D)(k - T_w) = \delta$. In summary we have the following corollary.

Corollary 7.15. If $P(D)$ is an elliptic operator on \mathbb{R}^N with constant coefficients, then there exists e in $\mathcal{D}'(\mathbb{R}^N)$ with $P(D)e = \delta$.

The distribution e is called a **fundamental solution** for $P(D)$ in $\mathcal{D}'(\mathbb{R}^N)$. A consequence of the existence of e is that $P(D)u = f$ has a solution u in $\mathcal{D}'(\mathbb{R}^N)$ for each f in $\mathcal{E}'(\mathbb{R}^N)$. This represents an improvement in the conclusion (fundamental solution vs. parametrix) of Theorem 7.14.

Think of Corollary 7.15 as being an existence theorem. We now turn to a discussion of the regularity of solutions. For this we do not need the existence result, and thus we shall proceed without making further use of Corollary 7.15.

Proposition 7.16. Let $P(D)$ be an elliptic operator on \mathbb{R}^N with constant coefficients. Then the tempered distribution $k = \mathcal{F}^{-1}(T_\sigma)$, where $\sigma(\xi) = \chi(\xi)/P(2\pi i\xi)$, is a smooth function on $\mathbb{R}^N - \{0\}$. Therefore, for any neighborhood of 0, the elliptic operator $P(D)$ has a parametrix $k_0 \in \mathcal{E}'(\mathbb{R}^N)$ with compact support in that neighborhood. In particular, there is a smooth function h_1 with support in that neighborhood such that whenever f is in $\mathcal{E}'(\mathbb{R}^N)$, then the distribution $v = k_0 * f$ is in $\mathcal{E}'(\mathbb{R}^N)$ and satisfies $P(D)v = f + (h_1 * f)$.

SKETCH OF PROOF. One checks that

$$D^\beta(\xi^\alpha k) = (2\pi i)^{|\beta|}(-2\pi i)^{-|\alpha|}\mathcal{F}^{-1}(T_{\xi^\beta D^\alpha \sigma}).$$

Here $\xi^\beta D^\alpha \sigma$ is a C^∞ function, and we are interested in its integrability. It is enough to consider what happens for $|\xi| \geq R'$, where $\sigma(\xi) = 1/P(2\pi i\xi)$. The function $1/P(2\pi i\xi)$ is bounded above by a multiple of $|\xi|^{-m}$, and an inductive argument on the order of the derivative shows that $|\xi^\beta D^\alpha \sigma| \leq C|\xi|^{|\beta|-|\alpha|-m}$ for $|\xi| \geq R'$, for a constant C independent of ξ .

Take $\beta = 0$. If $|\alpha|$ is large enough, we see that $D^\alpha \sigma$ is in $L^1(\mathbb{R}^N)$. Then $\mathcal{F}^{-1}(D^\alpha \sigma) = (2\pi i)^{|\alpha|}\xi^\alpha k$ is given by the usual integral formula for \mathcal{F} , but with $e^{-2\pi i x \cdot \xi}$ replaced by $e^{2\pi i x \cdot \xi}$. Therefore $\xi^\alpha k$ is a bounded continuous function when $|\alpha|$ is large enough. Applying this observation to $(\sum_{j=1}^n |\xi_j|^{2l})k$ for large enough l , we find that k is a continuous function on $\mathbb{R}^N - \{0\}$.

Next take $|\beta| = 1$ and increase l by 1, writing α' for the new α . Then $\xi^\beta D^{\alpha'} \sigma$ is integrable, and it follows⁵ that $\xi^{\alpha'} k$ has a pointwise partial derivative of type β and is continuous. Thus the same thing is true of k on $\mathbb{R}^N - \{0\}$.

Iterating this argument by adding 1 to one of the entries of β to obtain β' , we find for each β that we consider, that the functions $D^\beta(\sum_{j=1}^n |\xi_j|^{2l})k$ and $D^{\beta'}(\sum_{j=1}^n |\xi_j|^{2l'})k$ are integrable for l' sufficiently large, and we deduce that $D^\beta k$ has all first partial derivatives continuous. Since β' is arbitrary, k equals a smooth function on $\mathbb{R}^N - \{0\}$.

To finish the argument, let k and h be as in Theorem 7.14, and let ψ in $C_{\text{com}}^\infty(\mathbb{R}^N)$ be identically 1 near 0 and have support in whatever neighborhood of 0 has been specified. If we write $k = \psi k + (1 - \psi)k$, then $k_0 = \psi k$ has support in that same neighborhood, and $T = (1 - \psi)k$ is of the form T_{h_0} for some smooth function h_0 , by what we have shown. Substituting $k = k_0 + T_{h_0}$ into $P(D)k = \delta + T_h$, we find that $P(D)k_0 = \delta + T_h - T_{P(D)h_0}$. The function $h_1 = h - P(D)h_0$ is smooth, and it must have compact support since $P(D)k_0$ and δ have compact support.

Corollary 7.17. If u is in $\mathcal{D}'(\mathbb{R}^N)$ and $P(D)$ is elliptic, then $\text{sing supp } u \subseteq \text{sing supp } P(D)u$, where “sing supp” denotes singular support.

⁵The precise result to use is Proposition 8.1f of *Basic*.

REMARK. At first glance it might seem that the rough spots of $P(D)u$ are surely at least as bad as the rough spots of u for any D . But consider a function on \mathbb{R}^2 of the form $u(x, y) = g(y)$ and apply $P(D) = \partial/\partial x$. The result is 0, and thus $\text{sing supp } u$ can properly contain $\text{sing supp } P(D)u$ for $P(D) = \partial/\partial x$. The corollary says that this kind of thing does not happen if $P(D)$ is elliptic.

PROOF. Let $E = (\text{sing supp } P(D)u)^c$. By definition the restriction of $P(D)u$ to $C_{\text{com}}^\infty(E)$ is of the form T_ψ with ψ in $C^\infty(E)$. Let U be any nonempty open set with U^{cl} compact and with $U^{\text{cl}} \subseteq E$. It is enough to exhibit a smooth function η equal to u on U . Choose an open set V with V^{cl} compact such that $U^{\text{cl}} \subseteq V \subseteq V^{\text{cl}} \subseteq E$. Multiply ψ by a smooth function of compact support in E that equals 1 on V^{cl} , obtaining a function $\psi_0 \in C_{\text{com}}^\infty(E)$ such that $\psi_0 = \psi$ on V .

Choose an open neighborhood W of 0 such that $W = -W$ and such that the set of sums $U^{\text{cl}} + W^{\text{cl}}$ is contained in V . Applying Proposition 7.16, we can write $P(D)k_0 = \delta + h'$ with $k_0 \in \mathcal{E}'(\mathbb{R}^N)$ and $h' \in C_{\text{com}}^\infty(\mathbb{R}^N)$. The proposition allows us to insist that the support of k_0^\vee be contained in W . Then also h' has support contained in W .

We are to produce $\eta \in C^\infty(U)$ with $\langle T_\eta, \varphi \rangle = \langle u, \varphi \rangle$ for all $\varphi \in C_{\text{com}}^\infty(U)$. Our choice of W forces $k_0^\vee * \varphi$ to have support in V . Hence

$$\langle k_0 * P(D)u, \varphi \rangle = \langle P(D)u, k_0^\vee * \varphi \rangle = \langle T_\psi, k_0^\vee * \varphi \rangle = \langle T_{\psi_0}, k_0^\vee * \varphi \rangle = \langle k_0 * \psi_0, \varphi \rangle.$$

On the other hand, application of Corollary 5.14 gives

$$\langle k_0 * P(D)u, \varphi \rangle = \langle P(D)k_0 * u, \varphi \rangle = \langle (\delta + h') * u, \varphi \rangle = \langle u, \varphi \rangle + \langle h' * u, \varphi \rangle.$$

Combining the two computations, we see that $\langle u, \varphi \rangle = \langle k_0 * \psi_0 - h' * u, \varphi \rangle$, and the proof is complete if we take η to be $k_0 * \psi_0 - h' * u$.

The final step is to localize the result of Corollary 7.17.

Corollary 7.18. If $P(D)$ is elliptic with constant coefficients, if U is nonempty and open in \mathbb{R}^N , and if u and f are members of $\mathcal{D}'(U)$ with $P(D)u = f$, then $\text{sing supp } u \subseteq \text{sing supp } f$. Consequently if f is a smooth function on U , then so is u .

REMARKS. For the Laplacian this result gives something beyond the results in Chapter III: Part of the statement is that *any* distribution solution u of $\Delta u = 0$ on an open set U equals a smooth function on U . Previously the best result of this kind that we had was Corollary 3.17, which says that any distribution solution equal to a C^2 function is a smooth function.

PROOF. It is enough to prove that $E \cap \text{sing supp } u \subseteq E \cap \text{sing supp } f$ for each open set E with E^{cl} compact and $E^{\text{cl}} \subseteq U$. Choose ψ in $C_{\text{com}}^\infty(U)$ with ψ equal to 1 on E^{cl} . The equality $\langle \psi u, \varphi \rangle = \langle u, \psi \varphi \rangle = \langle u, \varphi \rangle$ for all $\varphi \in C_{\text{com}}^\infty(E)$ shows that $E \cap \text{sing supp } u = E \cap \text{sing supp } \psi u$. Regard ψu as in $\mathcal{E}'(\mathbb{R}^N)$, and define $g = P(D)(\psi u)$. Both ψu and g are in $\mathcal{E}'(\mathbb{R}^N)$, and every $\varphi \in C_{\text{com}}^\infty(E)$ satisfies

$$\begin{aligned} \langle g, \varphi \rangle &= \langle P(D)(\psi u), \varphi \rangle = \langle \psi u, P(D)^{\text{tr}} \varphi \rangle \\ &= \langle u, P(D)^{\text{tr}} \varphi \rangle = \langle P(D)u, \varphi \rangle = \langle f, \varphi \rangle. \end{aligned}$$

Hence $E \cap \text{sing supp } g = E \cap \text{sing supp } f$. Application of Corollary 7.17 therefore gives

$$E \cap \text{sing supp } u = E \cap \text{sing supp } \psi u \subseteq E \cap \text{sing supp } g = E \cap \text{sing supp } f,$$

and the result follows.

6. Method of Pseudodifferential Operators

Linear elliptic equations with variable coefficients were already well understood by the end of the 1950s. The methods to analyze them combined compactness arguments for operators between Banach spaces with the use of Sobolev spaces and similar spaces of functions. Those methods were of limited utility for other kinds of linear partial equations, but some isolated methods had been developed to handle certain cases of special interest. In the 1960s a general theory of pseudodifferential operators was introduced to include all these methods under a single umbrella, and it and its generalizations are now a standard device for studying linear partial differential equations. They provide a tool for taking advantage of point-by-point knowledge of the zero locus of the principal symbol.

As with distributions, pseudodifferential operators make certain kinds of calculations quite natural, and many verifications lie behind their use. We shall omit most of this detail and concentrate on some of the ideas behind extending the theory of the previous section to variable-coefficient operators.

We start with a nonempty open subset U of \mathbb{R}^N and a linear differential operator $P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ whose coefficients $a_\alpha(x)$ are in $C^\infty(U)$. If u is in $C_{\text{com}}^\infty(U)$, we can regard u as in $C_{\text{com}}^\infty(\mathbb{R}^N)$. The function u is then a Schwartz function, and the Fourier inversion formula holds:

$$u(x) = \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} \widehat{u}(\xi) d\xi,$$