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CHAPTER V

Distributions

Abstract. This chapter makes a detailed study of distributions, which are continuous linear functionals on vector spaces of smooth scalar-valued functions. The three spaces of smooth functions that are studied are the space $C_{\text{com}}^{\infty}(U)$ of smooth functions with compact support in an open set U , the space $C^{\infty}(U)$ of all smooth functions on U , and the space of Schwartz functions $S(\mathbb{R}^N)$ on \mathbb{R}^N . The corresponding spaces of continuous linear functionals are denoted by $\mathcal{D}'(U)$, $\mathcal{E}'(U)$, and $\mathcal{S}'(\mathbb{R}^N)$.

Section 1 examines the inclusions among the spaces of smooth functions and obtains the conclusion that the corresponding restriction mappings on distributions are one-one. It extends from $\mathcal{E}'(U)$ to $\mathcal{D}'(U)$ the definition given earlier for support, it shows that the only distributions of compact support in U are the ones that act continuously on $C^{\infty}(U)$, it gives a formula for these in terms of derivatives and compactly supported complex Borel measures, and it concludes with a discussion of operations on smooth functions.

Sections 2–3 introduce operations on distributions and study properties of these operations. Section 2 briefly discusses distributions given by functions, and it goes on to work with multiplications by smooth functions, iterated partial derivatives, linear partial differential operators with smooth coefficients, and the operation $(\cdot)^{\vee}$ corresponding to $x \mapsto -x$. Section 3 discusses convolution at length. Three techniques are used—the realization of distributions of compact support in terms of derivatives of complex measures, an interchange-of-limits result for differentiation in one variable and integration in another, and a device for localizing general distributions to distributions of compact support.

Section 4 reviews the operation of the Fourier transform on tempered distributions; this was introduced in Chapter III. The two main results are that the Fourier transform of a distribution of compact support is a smooth function whose derivatives have at most polynomial growth and that the convolution of a distribution of compact support and a tempered distribution is a tempered distribution whose Fourier transform is the product of the two Fourier transforms.

Section 5 establishes a fundamental solution for the Laplacian in \mathbb{R}^N for $N > 2$ and concludes with an existence theorem for distribution solutions to $\Delta u = f$ when f is any distribution of compact support.

1. Continuity on Spaces of Smooth Functions

Distributions are continuous linear functionals on vector spaces of smooth functions. Their properties are deceptively simple-looking and enormously helpful. Some of their power is hidden in various interchanges of limits that need to be

carried out to establish their basic properties. The result is a theory that is easy to implement and that yields results quickly. In the last section of this chapter, we shall see an example of this phenomenon when we show how it gives information about solutions of partial differential equations involving the Laplacian.

The three vector spaces of scalar-valued smooth functions that we shall consider in the text¹ of this chapter are $C^\infty(U)$, $\mathcal{S}(\mathbb{R}^N)$, and $C_{\text{com}}^\infty(U)$, where U is a nonempty open set in \mathbb{R}^N . Topologies for these spaces were introduced in Section IV.2, Section III.1, and Section IV.7, respectively. Let $\{K_p\}$ be an exhausting sequence of compact subsets of U , i.e., a sequence such that $K_p \subseteq K_{p+1}^o$ for all p and such that $U = \bigcup_{p=1}^\infty K_p$.

The vector space $C^\infty(U)$ of all smooth functions on U is given by a separating family of seminorms such that a countable subfamily suffices. The members of the subfamily may be taken to be $\|f\|_{p,\alpha} = \sup_{x \in K_p} |D^\alpha f(x)|$, where $1 \leq p < \infty$ and where α varies over all differentiation multi-indices.² The space of continuous linear functionals is denoted by $\mathcal{E}'(U)$, and the members of this space are called “distributions of compact support” for reasons that we recall in a moment.

The vector space $\mathcal{S}(\mathbb{R}^N)$ of all Schwartz functions is another space given by a separating family of seminorms such that a countable subfamily suffices. The members of the subfamily may be taken to be $\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^N} |x^\alpha D^\beta f(x)|$, where α and β vary over all differentiation multi-indices.³ The space of continuous linear functionals is denoted by $\mathcal{S}'(U)$, and the members of this space are called “tempered distributions.”

The vector space $C_{\text{com}}^\infty(U)$ of all smooth functions of compact support on U is given by the inductive limit topology obtained from the vector subspaces $C_{K_p}^\infty$. The space $C_{K_p}^\infty$ consists of the smooth functions with support contained in K_p , the topology on $C_{K_p}^\infty$ being given by the countable family of seminorms $\|f\|_{p,\alpha} = \sup_{x \in K_p} |D^\alpha f(x)|$. The space of continuous linear functionals is traditionally⁴ written $\mathcal{D}'(U)$, and the members of this space are called simply “distributions.” Since the field of scalars is a locally convex topological vector space, Proposition 4.29 shows that the members of $\mathcal{D}'(U)$ may be viewed as arbitrary sequences of consistently defined continuous linear functionals on the spaces $C_{K_p}^\infty$.

¹A fourth space, the space of periodic smooth functions on \mathbb{R}^N , is considered in Problems 12–19 at the end of the chapter and again in the problems at the end of Chapter VII.

²The notation for the seminorms in Chapter IV was chosen for the entire separating subfamily and amounted to $\|f\|_{K_p, D^\alpha}$. The subscripts have been simplified to take into account the nature of the countable subfamily.

³The notation for the seminorms in Chapter III was chosen for the entire separating subfamily and amounted to $\|f\|_{x^\alpha, x^\beta}$. The subscripts have been simplified to take into account the nature of the countable subfamily.

⁴The tradition dates back to Laurent Schwartz’s work, in which $\mathcal{D}(U)$ was the notation for $C_{\text{com}}^\infty(U)$ and $\mathcal{D}'(U)$ denoted the space of continuous linear functionals.

For the spaces of smooth functions, there are continuous inclusions

$$\begin{aligned} C_{\text{com}}^{\infty}(U) &\subseteq C^{\infty}(U) && \text{for all } U, \\ C_{\text{com}}^{\infty}(\mathbb{R}^N) &\subseteq \mathcal{S}(\mathbb{R}^N) \subseteq C^{\infty}(\mathbb{R}^N) && \text{for } U = \mathbb{R}^N. \end{aligned}$$

We observed in Section IV.2 that $C_{\text{com}}^{\infty}(U) \subseteq C^{\infty}(U)$ has dense image. Proposition 4.12 showed that $C_{\text{com}}^{\infty}(\mathbb{R}^N) \subseteq \mathcal{S}(\mathbb{R}^N)$ has dense image, and it follows that $\mathcal{S}(\mathbb{R}^N) \subseteq C^{\infty}(\mathbb{R}^N)$ has dense image.

If $i : A \rightarrow B$ denotes one of these inclusions and T is a continuous linear functional on B , then $T \circ i$ is a continuous linear functional on A , and we can regard $T \circ i$ as the restriction of T to A . Since i has dense image, $T \circ i$ cannot be 0 unless T is 0. Thus each restriction map $T \mapsto T \circ i$ as above is one-one. We therefore have *one-one* restriction maps

$$\begin{aligned} \mathcal{E}'(U) &\rightarrow \mathcal{D}'(U) && \text{for all } U, \\ \mathcal{E}'(\mathbb{R}^N) &\rightarrow \mathcal{S}'(\mathbb{R}^N) \rightarrow \mathcal{D}'(\mathbb{R}^N) && \text{for } U = \mathbb{R}^N. \end{aligned}$$

This fact justifies using the term “distribution” for any member of \mathcal{D}' and for using the term “distribution” with an appropriate modifier for members of \mathcal{E}' and \mathcal{S}' .

As in Section III.1 it will turn out often to be useful to write the effect of a distribution T on a function φ as $\langle T, \varphi \rangle$, rather than as $T(\varphi)$, and we shall adhere to this convention systematically for the moment.⁵

We introduced in Section IV.2 the notion of “support” for any member of $\mathcal{E}'(U)$, and we now extend that discussion to $\mathcal{D}'(U)$. We saw in Proposition 4.10 that if T is an arbitrary linear functional on $C_{\text{com}}^{\infty}(U)$ and if U' is the union of all open subsets U_{γ} of U such that T vanishes on $C_{\text{com}}^{\infty}(U_{\gamma})$, then T vanishes on $C_{\text{com}}^{\infty}(U')$. We accordingly define the **support** of any distribution to be the complement in U of the union of all open sets U_{γ} such that T vanishes on $C_{\text{com}}^{\infty}(U_{\gamma})$. If T has empty support, then $T = 0$ because T vanishes on $C_{\text{com}}^{\infty}(U)$ and because $C_{\text{com}}^{\infty}(U)$ is dense in the domain of T . Proposition 4.11 showed that the members of $\mathcal{E}'(U)$ have compact support in this sense; we shall see in Theorem 5.1 that no other members of $\mathcal{D}'(U)$ have compact support.

An example of a member of $\mathcal{E}'(U)$ was given in Section IV.2: Take finitely many complex Borel measures ρ_{α} of compact support within U , the indexing being by multi-indices α with $|\alpha| \leq m$, and put $\langle T, \varphi \rangle = \sum_{|\alpha| \leq m} \int_U D^{\alpha} \varphi(x) d\rho_{\alpha}(x)$. Then T is in $\mathcal{E}'(U)$, and the support of T is contained in the union of the supports of the ρ_{α} 's. Theorem 5.1 below gives a converse, but it is necessary in general to allow the ρ_{α} 's to have support a little larger than the support of the given distribution T .

⁵A different convention is to write $\int_U \varphi(x) dT(x)$ in place of $\langle T, \varphi \rangle$. This notation emphasizes an analogy between distributions and measures and is especially useful when more than one \mathbb{R}^N variable is in play. This convention will provide helpful motivation in one spot in Section 3.

Theorem 5.1. If T is a member of $\mathcal{D}'(U)$ with support contained in a compact subset K of U , then T is in $\mathcal{E}'(U)$. Moreover, if K' is any compact subset of U whose interior contains K , then there exist a positive integer m and, for each multi-index α with $|\alpha| \leq m$, a complex Borel measure ρ_α supported in K' such that

$$\langle T, \varphi \rangle = \sum_{|\alpha| \leq m} \int_{K'} D^\alpha \varphi d\rho_\alpha \quad \text{for all } \varphi \in C^\infty(U).$$

REMARK. Problems 8–10 at the end of the chapter discuss the question of taking $K' = K$ under additional hypotheses.

PROOF. Let ψ be a member of $C_{\text{com}}^\infty(U)$ with values in $[0, 1]$ that is 1 on a neighborhood of K and is 0 on K'^c ; such a function exists by Proposition 3.5f. If φ is in $C_{\text{com}}^\infty(U)$, then we can write $\varphi = \psi\varphi + (1 - \psi)\varphi$ with $\psi\varphi$ in $C_{K'}^\infty$ and with $(1 - \psi)\varphi$ in $C_{\text{com}}^\infty(K^c)$. The assumption about the support of T makes $\langle T, (1 - \psi)\varphi \rangle = 0$, and therefore

$$\langle T, \varphi \rangle = \langle T, \psi\varphi \rangle + \langle T, (1 - \psi)\varphi \rangle = \langle T, \psi\varphi \rangle \quad \text{for all } \varphi \text{ in } C_{\text{com}}^\infty(U). \quad (*)$$

Since the inclusion $C_{K'}^\infty \rightarrow C_{\text{com}}^\infty(U)$ is continuous, we can define a continuous linear functional T_1 on $C_{K'}^\infty$ by $T_1(\phi) = \langle T, \phi \rangle$ for ϕ in $C_{K'}^\infty$. For any φ in $C_{\text{com}}^\infty(U)$, $\phi = \psi\varphi$ is in $C_{K'}^\infty$, and (*) gives $\langle T, \varphi \rangle = \langle T, \psi\varphi \rangle = T_1(\psi\varphi)$. The continuity of T_1 on $C_{K'}^\infty$ means that there exist m and C such that

$$|T_1(\phi)| \leq C \sum_{|\alpha| \leq m} \sup_{x \in K'} |D^\alpha \phi(x)| \quad \text{for all } \phi \in C_{K'}^\infty. \quad (**)$$

Let M be the number of multi-indices α with $|\alpha| \leq m$.

We introduce the Banach space X of M -tuples of continuous complex-valued functions on K' , the norm for X being the largest of the norms of the components. The Banach-space dual of this space is the space of M -tuples of continuous linear functionals on the components, thus the space of M -tuples of complex Borel measures on K' .

We can embed $C_{K'}^\infty$ as a vector subspace of X by mapping ϕ to the M -tuple with components $D^\alpha \phi$ for $|\alpha| \leq m$. We transfer T_1 from $C_{K'}^\infty$ to its image subspace within X , and the result, which we still call T_1 , is a linear functional continuous relative to the norm on X as a consequence of (**). Applying the Hahn–Banach Theorem, we extend T_1 to a continuous linear functional \tilde{T}_1 on all of X without an increase in norm. Then \tilde{T}_1 is given on X by an M -tuple of complex Borel measures ρ'_α on K' , i.e., $\tilde{T}_1(\{f_\alpha\}_{|\alpha| \leq m}) = \sum_{|\alpha| \leq m} \int_{K'} f_\alpha d\rho'_\alpha$. Therefore any φ in $C_{\text{com}}^\infty(U)$ has

$$\langle T, \varphi \rangle = T_1(\psi\varphi) = \tilde{T}_1(\{D^\alpha(\psi\varphi)\}_{|\alpha| \leq m}) = \sum_{|\alpha| \leq m} \int_{K'} D^\alpha(\psi\varphi) d\rho'_\alpha. \quad (\dagger)$$

The right side of (\dagger) is continuous on $C^\infty(U)$, and therefore T extends to a member of $\mathcal{E}'(U)$. The formula in the theorem follows by expanding out each $D^\alpha(\psi\varphi)$ in (\dagger) by the Leibniz rule for differentiation of products, grouping the derivatives of ψ with the complex measures, and reassembling the expression with new complex measures ρ_α .

In Chapters VII and VIII we shall be interested also in a notion related to support, namely the notion of “singular support.” If f is a locally integrable function on the open set U , then f defines a member T_f of $\mathcal{D}'(U)$ by

$$\langle T_f, \varphi \rangle = \int_U f \varphi dx \quad \text{for } \varphi \in C_{\text{com}}^\infty(U).$$

If U' is an open subset of U and T is a distribution on U , we say that T **equals a locally integrable function** on U' if there is some locally integrable function f on U' such that $\langle T, \varphi \rangle = \langle T_f, \varphi \rangle$ for all φ in $C_{\text{com}}^\infty(U)$. We say that T **equals a smooth function** on U' if this condition is satisfied for some f in $C^\infty(U')$. In the latter case the member of $C^\infty(U')$ is certainly unique.

The **singular support** of a member T of $\mathcal{D}'(U)$ is the complement of the union of all open subsets U' of U such that T equals a smooth function on U' . The uniqueness of the smooth function on such a subset implies that if T equals the smooth function f_1 on U'_1 and equals the smooth function f_2 on U'_2 , then $f_1(x) = f_2(x)$ for x in $U'_1 \cap U'_2$. In fact, T equals the smooth function $f_1|_{U'_1 \cap U'_2}$ on $U'_1 \cap U'_2$ and also equals the smooth function $f_2|_{U'_1 \cap U'_2}$ there. The uniqueness forces $f_1|_{U'_1 \cap U'_2} = f_2|_{U'_1 \cap U'_2}$. Taking the union of all the open subsets on which T equals a smooth function, we see that T is a smooth function on the complement of its singular support.

EXAMPLE. Take $U = \mathbb{R}^1$, and define

$$\langle T, \varphi \rangle = \lim_{\varepsilon \downarrow 0} \int_{|x| \geq \varepsilon} \frac{\varphi(x) dx}{x} \quad \text{for } \varphi \in C_{\text{com}}^\infty(\mathbb{R}^1).$$

To see that this is well defined, we choose η in $C_{\text{com}}^\infty(\mathbb{R}^1)$ with η identically 1 on the support of φ and with $\eta(x) = \eta(-x)$ for all x . Taylor's Theorem gives $\varphi(x) = \varphi(0) + xR(x)$ with R in $C^\infty(\mathbb{R}^1)$. Multiplying by $\eta(x)$ and integrating for $|x| \geq \varepsilon$, we obtain

$$\int_{|x| \geq \varepsilon} \frac{\varphi(x) dx}{x} = \varphi(0) \int_{|x| \geq \varepsilon} \frac{\eta(x) dx}{x} + \int_{|x| \geq \varepsilon} R(x) \eta(x) dx.$$

The first term on the right side is 0 for every ε , and therefore

$$\langle T, \varphi \rangle = \int_{\mathbb{R}^1} R(x) \eta(x) dx.$$

It follows that T is in $\mathcal{D}'(\mathbb{R}^1)$. On any function compactly supported in $\mathbb{R}^1 - \{0\}$, the original integral defining T is convergent. Thus T equals the function $1/x$ on $\mathbb{R}^1 - \{0\}$. Since $1/x$ is nowhere zero on $\mathbb{R}^1 - \{0\}$, the (ordinary) support of T has to be a closed subset of \mathbb{R}^1 containing $\mathbb{R}^1 - \{0\}$. Therefore T has support \mathbb{R}^1 . On the other hand, T does not equal a function on all of \mathbb{R}^1 , and T has $\{0\}$ as its singular support.

Starting in Section 2, we shall examine various operations on distributions. Operations on distributions will be defined by duality from corresponding operations on smooth functions. For that reason it is helpful to know about continuity of various operations on spaces of smooth functions. These we study now.

We begin with multiplication by smooth functions and with differentiation. If ψ is in $C^\infty(U)$, then multiplication $\varphi \mapsto \psi\varphi$ carries $C_{\text{com}}^\infty(U)$ into itself and also $C^\infty(U)$ into itself. The same is true of any iterated partial derivative operator $\varphi \mapsto D^\alpha\varphi$. We shall show that these operations are continuous. A multiplication $\varphi \mapsto \psi\varphi$ need not carry $\mathcal{S}(\mathbb{R}^N)$ into itself, and we put aside $\mathcal{S}(\mathbb{R}^N)$ for further consideration later.

The kind of continuity result for $C^\infty(U)$ that we are studying tends to follow from an easy computation with seminorms, and it is often true that the same argument can be used to handle also $C_{\text{com}}^\infty(U)$. Here is the general fact.

Lemma 5.2. Suppose that $L : C^\infty(U) \rightarrow C^\infty(U)$ is a continuous linear map that carries $C_{\text{com}}^\infty(U)$ into $C_{\text{com}}^\infty(U)$ in such a way that for each compact $K \subseteq U$, C_K^∞ is carried into $C_{K'}^\infty$ for some compact $K' \supseteq K$. Then L is continuous as a linear map from $C_{\text{com}}^\infty(U)$ into $C_{\text{com}}^\infty(U)$.

PROOF. Proposition 4.29b shows that it is enough to prove for each K that the composition of $L : C_K^\infty \rightarrow C_{K'}^\infty$ followed by the inclusion of $C_{K'}^\infty$ into $C_{\text{com}}^\infty(U)$ is continuous, and we know that the inclusion is continuous. Fix K , choose K_p in the exhausting sequence containing the corresponding K' , and let α be a multi-index. By the continuity of $L : C^\infty(U) \rightarrow C^\infty(U)$, there exist a constant C , some integer q with $q \geq p$, and finitely many multi-indices β_i such that $\|L(\varphi)\|_{p,\alpha} \leq C \sum_i \|\varphi\|_{q,\beta_i}$. Since $L(\varphi)$ has support in $K' \subseteq K_p$ and φ has support in $K \subseteq K' \subseteq K_p \subseteq K_q$, this inequality shows that $\sup_{x \in K'} |D^\alpha(L(\varphi))(x)| \leq C \sum_i \sup_{x \in K} |D^{\beta_i}\varphi(x)|$. Hence $L : C_K^\infty \rightarrow C_{K'}^\infty$ is continuous, and the lemma follows.

Proposition 5.3. If ψ is in $C^\infty(U)$, then $\varphi \mapsto \psi\varphi$ is continuous from $C^\infty(U)$ to $C^\infty(U)$ and from $C_{\text{com}}^\infty(U)$ to $C_{\text{com}}^\infty(U)$. If α is any differentiation multi-index, then $\varphi \mapsto D^\alpha\varphi$ is continuous from $C^\infty(U)$ to $C^\infty(U)$ and from $C_{\text{com}}^\infty(U)$ to $C_{\text{com}}^\infty(U)$.

PROOF. The Leibniz rule for differentiation of products gives $D^\alpha(\psi\varphi) = \sum_{\beta \leq \alpha} c_\beta (D^{\beta-\alpha}\psi)(D^\beta\varphi)$ for certain integers c_β . Then

$$\|\psi\varphi\|_{p,\alpha} \leq \sum_{\beta \leq \alpha} c_\beta m_\beta \|\varphi\|_{p,\beta},$$

where $m_\beta = \sup_{x \in K_p} |D^{\beta-\alpha}\psi(x)|$, and it follows that $\varphi \mapsto \psi\varphi$ is continuous from $C^\infty(U)$ into itself. Taking $K' = K$ in Lemma 5.2, we see that $\varphi \mapsto \psi\varphi$ is continuous from $C_{\text{com}}^\infty(U)$ into itself.

Since $\|D^\alpha\varphi\|_{p,\beta} = \|\varphi\|_{p,\alpha+\beta}$, the function $\varphi \mapsto D^\alpha\varphi$ is continuous from $C^\infty(U)$ into itself, and Lemma 5.2 with $K' = K$ shows that $\varphi \mapsto D^\alpha\varphi$ is continuous from $C_{\text{com}}^\infty(U)$ into itself.

We can combine these two operations into the operation of a **linear partial differential operator**

$$P(x, D) = \sum_{|\alpha| \leq m} c_\alpha(x) D^\alpha \quad \text{with all } c_\alpha \text{ in } C^\infty(U)$$

by means of the formula $P(x, D)\varphi = \sum_{|\alpha| \leq m} c_\alpha(x) D^\alpha\varphi$. It is to be understood that the operator has smooth coefficients. It is immediate from Proposition 5.3 that $P(x, D)$ is continuous from $C^\infty(U)$ into itself and from $C_{\text{com}}^\infty(U)$ into itself.

An operator $P(x, D)$ as above is said to be of **order** m if some $c_\alpha(x)$ with $|\alpha| = m$ has c_α not identically 0. The operator reduces to an operator of the form $P(D)$ if the coefficient functions c_α are all constant functions.

We introduce the **transpose operator** $P(x, D)^{\text{tr}}$ by the formula

$$P(x, D)^{\text{tr}}\varphi(x) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha(c_\alpha(x)\varphi(x)).$$

Expanding out the terms $D^\alpha(c_\alpha(x)\varphi(x))$ by means of the Leibniz rule, we see that $P(x, D)^{\text{tr}}$ is some linear partial differential operator of the form $Q(x, D)$. The next proposition gives the crucial property of the transpose operator.

Proposition 5.4. Suppose that $P(x, D)$ is a linear partial differential operator on U . If u and v are in $C^\infty(U)$ and at least one of them is in $C_{\text{com}}^\infty(U)$, then

$$\int_U (P(x, D)^{\text{tr}}u(x))v(x) dx = \int_U u(x)(P(x, D)v(x)) dx.$$

PROOF. It is enough to prove that the partial derivative operator D_j with respect to x_j satisfies $\int_U (D_j u)v dx = -\int_U u(D_j v) dx$ since iteration of this formula gives the result of the proposition. Moving everything to one side of the equation

and putting $w = uv$, we see that it is enough to prove that $\int_{\mathbb{R}^N} I_U D_j w \, dx = 0$ if w is in $C_{\text{com}}^\infty(U)$, where I_U is the indicator function of U . We can drop the I_U from the integration since $D_j w$ is 0 off U , and thus it is enough to prove that $\int_{\mathbb{R}^N} D_j w \, dx = 0$ for w in $C_{\text{com}}^\infty(\mathbb{R}^N)$. By Fubini's Theorem the integral may be computed as an iterated integral. The integral on the inside extends over the set where x_j is arbitrary in \mathbb{R} and the other variables take on particular values, say $x_i = c_i$ for $i \neq j$. The integral on the outside extends over all choices of the c_i for $i \neq j$. The inside integral is already 0, because for suitable a and b , it is of the form $\int_a^b D_j w \, dx_j = [w]_{x_j=a}^{x_j=b} = 0 - 0 = 0$.

Next let us consider convolution, taking $U = \mathbb{R}^N$. We shall be interested in the function $\psi * \varphi$ given by

$$\psi * \varphi(x) = \int_{\mathbb{R}^N} \psi(x-y)\varphi(y) \, dy = \int_{\mathbb{R}^N} \psi(y)\varphi(x-y) \, dy,$$

under the assumption that ψ and φ are in $C^\infty(\mathbb{R}^N)$ and that one of them has compact support.

A simple device of localization helps with the analysis of this function: If K is the support of ψ , then the values of $\psi * \varphi(x)$ for x in a bounded open set S depend only on the value of φ on the bounded open set of differences $S - K$. Consequently we can replace φ by $\eta\varphi$, where η is a member of $C_{\text{com}}^\infty(\mathbb{R}^N)$ that is 1 on $S - K$, and the values of $\psi * \varphi(x)$ will match those of $\psi * (\eta\varphi)(x)$ for x in S . The latter function is the convolution of two smooth functions of compact support and is smooth by Proposition 3.5c. Therefore $\psi * \varphi$ is always in $C^\infty(\mathbb{R}^N)$ if ψ is in $C_{\text{com}}^\infty(\mathbb{R}^N)$ and φ is in $C^\infty(\mathbb{R}^N)$. We shall use this same device later in treating convolution of distributions.

Proposition 5.5. If ψ is in $C_{\text{com}}^\infty(\mathbb{R}^N)$ and φ is in $C^\infty(\mathbb{R}^N)$, then

- (a) $D^\alpha(\psi * \varphi) = (D^\alpha\psi) * \varphi = \psi * (D^\alpha\varphi)$,
- (b) convolution of three functions in $C^\infty(\mathbb{R}^N)$ is associative when at least two of the three functions have compact support,
- (c) convolution with ψ is continuous from $C^\infty(\mathbb{R}^N)$ into itself and from $C_{\text{com}}^\infty(\mathbb{R}^N)$ into itself,
- (d) convolution with φ is continuous from $C_{\text{com}}^\infty(\mathbb{R}^N)$ into $C^\infty(\mathbb{R}^N)$.

PROOF. For (a), let K be the support of ψ . Concentrating on x 's lying in a bounded open set S , choose a function η in $C_{\text{com}}^\infty(\mathbb{R}^N)$ that is 1 on $S - K$, and then $\psi * \varphi(x) = \psi * (\eta\varphi)(x)$ for x in S . Proposition 3.5c says that

$$D^\alpha(\psi * (\eta\varphi))(x) = (D^\alpha\psi) * (\eta\varphi)(x) = \psi * D^\alpha(\eta\varphi)(x)$$

for all x in \mathbb{R}^N , and consequently

$$D^\alpha(\psi * \varphi)(x) = (D^\alpha\psi) * \varphi(x) = \psi * D^\alpha\varphi(x)$$

for all x in S . Since S is arbitrary, (a) follows. The proof of (b) is similar.

For (c), again let K be the support of ψ , and apply (a). Then

$$\begin{aligned} \|\psi * \varphi\|_{p,\alpha} &= \sup_{x \in K_p} |D^\alpha(\psi * \varphi)(x)| = \sup_{x \in K_p} |\psi * (D^\alpha \varphi)(x)| \\ &\leq \sup_{x \in K_p} \int_K |\psi(y)| |D^\alpha \varphi(x - y)| dy \leq \|\psi\|_1 \sup_{z \in K_p - K} |D^\alpha \varphi(z)|, \end{aligned}$$

and the right side is $\leq \|\psi\|_1 \|\varphi\|_{q,\alpha}$ if q is large enough so that $K_p - K \subseteq K_q$. This proves the continuity on $C^\infty(\mathbb{R}^N)$, and the continuity on $C_{\text{com}}^\infty(\mathbb{R}^N)$ then follows from Lemma 5.2.

For (d), Proposition 4.29b shows that it is enough to prove that $\psi \mapsto \psi * \varphi$ is continuous from C_K^∞ into $C^\infty(\mathbb{R}^N)$ for each compact set K . The same estimate as for (c) gives

$$\|\psi * \varphi\|_{p,\alpha} \leq \|\psi\|_1 \|\varphi\|_{q,\alpha} \leq |K| \|\varphi\|_{q,\alpha} (\sup_{x \in K} |\psi(x)|)$$

if q is large enough so that $K_p - K \subseteq K_q$. The result follows.

2. Elementary Operations on Distributions

In this section we take up operations on distributions. If f is a locally integrable function on the open set U , we defined the member T_f of $\mathcal{D}'(U)$ by

$$\langle T_f, \varphi \rangle = \int_U f \varphi dx$$

for φ in $C_{\text{com}}^\infty(U)$. If f vanishes outside a compact subset of U , then T_f is in $\mathcal{E}'(U)$, extending to operate on all of $C^\infty(U)$ by the same formula.

Starting from certain continuous operations L on smooth functions, we want to extend these operations to operations on distributions. So that we can regard L as an extension from smooth functions to distributions, we insist on having $L(T_f) = T_{L(f)}$ if f is smooth. To tie the definition of L on distributions T_f to the definition on general distributions T , we insist that L be the “transpose” of *some* continuous operation M on functions, i.e., that $\langle L(T), \varphi \rangle = \langle T, M(\varphi) \rangle$. Taking $T = T_f$ in this equation, we see that we must have $\int_U L(f) \varphi dx = \int_U f M(\varphi) dx$. On the other hand, once we have found a continuous M on smooth functions with $\int_U L(f) \varphi dx = \int_U f M(\varphi) dx$, then we can make the definition $\langle L(T), \varphi \rangle = \langle T, M(\varphi) \rangle$ for the effect of L on distributions. In particular the operator M on smooth functions is unique if it exists. We write $L^{\text{tr}} = M$ for it. In summary, our