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Along with a companion volume

Basic Real Analysis

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CHAPTER III

Topics in Euclidean Fourier Analysis

Abstract. This chapter takes up several independent topics in Euclidean Fourier analysis, all having some bearing on the subject of partial differential equations.

Section 1 elaborates on the relationship between the Fourier transform and the Schwartz space, the subspace of $L^1(\mathbb{R}^N)$ consisting of smooth functions with the property that the product of any iterated partial derivative of the function with any polynomial is bounded. It is possible to make the Schwartz space into a metric space, and then one can consider the space of continuous linear functionals; these continuous linear functionals are called “tempered distributions.” The Fourier transform carries the space of tempered distributions in one-one fashion onto itself.

Section 2 concerns weak derivatives, and the main result is Sobolev’s Theorem, which tells how to recover information about ordinary derivatives from information about weak derivatives. Weak derivatives are easy to manipulate, and Sobolev’s Theorem is therefore a helpful tool for handling derivatives without continually having to check the validity of interchanges of limits.

Sections 3–4 concern harmonic functions, those functions on open sets in Euclidean space that are annihilated by the Laplacian. The main results of Section 3 are a characterization of harmonic functions in terms of a mean-value property, a reflection principle that allows the extension to all of Euclidean space of any harmonic function in a half space that vanishes at the boundary, and a result of Liouville that the only bounded harmonic functions in all of Euclidean space are the constants. The main result of Section 4 is a converse to properties of Poisson integrals for half spaces, showing that harmonic functions in a half space are given as Poisson integrals of functions or of finite complex measures if their L^p norms over translates of the bounding Euclidean space are bounded.

Sections 5–6 concern the Calderón–Zygmund Theorem, a far-reaching generalization of the theorem concerning the boundedness of the Hilbert transform. Section 5 gives the statement and proof, and two applications are the subject of Section 6. One of the applications is to Riesz transforms, and the other is to the Beltrami equation, whose solutions are “quasiconformal mappings.”

Sections 7–8 concern multiple Fourier series for smooth periodic functions. The theory is established in Section 7, and an application to traces of integral operators is given in Section 8.

1. Tempered Distributions

We fix normalizations for the Euclidean Fourier transform as in *Basic*: For f in $L^1(\mathbb{R}^N)$, the definition is

$$\widehat{f}(y) = (\mathcal{F}f)(y) = \int_{\mathbb{R}^N} f(x)e^{-2\pi i x \cdot y} dx,$$

with $x \cdot y$ referring to the dot product and with the 2π in the exponent. The inversion formula is valid whenever \widehat{f} is in L^1 ; it says that f is recovered as

$$f(x) = (\mathcal{F}^{-1}\widehat{f})(x) = \int_{\mathbb{R}^N} \widehat{f}(y)e^{-2\pi i x \cdot y} dy$$

almost everywhere, including at all points of continuity of f . The operator \mathcal{F} carries $L^1 \cap L^2$ into L^2 and extends to a linear map \mathcal{F} of L^2 onto L^2 such that $\|\mathcal{F}f\|_2 = \|f\|_2$. This is the Plancherel formula.

The Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}^N)$ is the vector space of all functions f in $C^\infty(\mathbb{R}^N)$ such that the product of any polynomial by any iterated partial derivative of f is bounded. This is a vector subspace of $L^1 \cap L^2$, and it was shown in *Basic* that \mathcal{F} carries \mathcal{S} one-one onto itself. It will be handy sometimes to use a notation for partial derivatives and their iterates that is different from that in Chapter I.

Namely,¹ let $D_j = \frac{\partial}{\partial x_j}$. If $\alpha = (\alpha_1, \dots, \alpha_N)$ is an N -tuple of nonnegative integers, we write $|\alpha| = \sum_{j=1}^N \alpha_j$, $\alpha! = \alpha_1! \cdots \alpha_N!$, $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$, and $D^\alpha = D_1^{\alpha_1} \cdots D_N^{\alpha_N}$. Addition of such tuples α is defined component by component, and we say that $\alpha \leq \beta$ if $\alpha_j \leq \beta_j$ for $1 \leq j \leq N$. We write $|\alpha|$ for the total order $\alpha_1 + \cdots + \alpha_N$, and we call α a **multi-index**. If $Q(x) = \sum_{\alpha} a_{\alpha} x^{\alpha}$ is a complex-valued polynomial on \mathbb{R}^N , define $Q(D)$ to be the partial differential operator $\sum_{\alpha} a_{\alpha} D^{\alpha}$ with constant coefficients obtained by substituting, for each j with $1 \leq j \leq N$, the operator $D_j = \frac{\partial}{\partial x_j}$ for x_j . The Schwartz functions are then the smooth functions f on \mathbb{R}^N such that $P(x)Q(D)f$ is bounded for each pair of polynomials P and Q .

The Schwartz space is directly usable in connection with certain linear partial differential equations with constant coefficients. A really simple example concerns the Laplacian operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_N^2}$, which we can write as $\Delta = |D|^2$ in the new notation for differential operators. Specifically the equation

$$(1 - \Delta)u = f$$

has a unique solution u in \mathcal{S} for each f in \mathcal{S} . To see this, we take the Fourier transform of both sides, obtaining $\mathcal{F}u - \mathcal{F}(\Delta u) = \mathcal{F}f$ or $\mathcal{F}u - \mathcal{F}(|D|^2(u)) = \mathcal{F}f$. Using the formulas relating the Fourier transform, multiplication by polynomials, and differentiation,² we can rewrite this equation as $(1 + 4\pi^2|y|^2)\mathcal{F}(u) = \mathcal{F}(f)$. Problem 1 at the end of the chapter asks one to check that $(1 + 4\pi^2|y|^2)^{-1}g$ is in \mathcal{S} if

¹Some authors prefer to abbreviate $\frac{\partial}{\partial x_j}$ as ∂_j , reserving the notation D_j for the product of ∂_j and a certain imaginary scalar that depends on the definition of the Fourier transform.

²These, with hypotheses in place, appear as Proposition 8.1 of *Basic*.

g is in \mathcal{S} , and then existence of a solution in \mathcal{S} to the differential equation is proved by the formula $u = \mathcal{F}^{-1}((1 + 4\pi^2|y|^2)^{-1}\mathcal{F}(f))$. For uniqueness let u_1 and u_2 be two solutions in \mathcal{S} corresponding to the same f . Then $(1 - \Delta)(u_1 - u_2) = 0$, and hence $(1 + 4\pi^2|y|^2)\mathcal{F}(u_1 - u_2)(y) = 0$ for all y . Therefore $\mathcal{F}(u_1 - u_2)(y) = 0$ everywhere. Since \mathcal{F} is one-one on \mathcal{S} , we conclude that $u_1 = u_2$.

A deeper use of the Schwartz space in connection with linear partial differential equations comes about because of the relationship between the Schwartz space and the theory of “distributions.” Distributions are continuous linear functionals on vector spaces of smooth functions, i.e., continuous linear maps from such a space to the scalars, and they will be considered more extensively in Chapter V. For now, we shall be content with discussing “tempered distributions,” the distributions associated with the Schwartz space. In order to obtain a well-defined notion of continuity, we shall describe how to make $\mathcal{S}(\mathbb{R}^N)$ into a metric space.

For each pair of polynomials P and Q , we define

$$\|f\|_{P,Q} = \sup_{x \in \mathbb{R}^N} |P(x)(Q(D)f)(x)|.$$

Each function $\|\cdot\|_{P,Q}$ on \mathcal{S} is a **seminorm** on \mathcal{S} in the sense that³

- (i) $\|f\|_{P,Q} \geq 0$ for all f in \mathcal{S} ,
- (ii) $\|cf\|_{P,Q} = |c|\|f\|_{P,Q}$ for all f in \mathcal{S} and all scalars c ,
- (iii) $\|f + g\|_{P,Q} \leq \|f\|_{P,Q} + \|g\|_{P,Q}$ for all f and g in \mathcal{S} .

Collectively these seminorms have a property that goes in the converse direction to (i), namely

- (iv) $\|f\|_{P,Q} = 0$ for all P and Q implies $f = 0$.

In fact, f will already be 0 if the seminorm for $P = Q = 1$ is 0 on f .

Each seminorm gives rise to a pseudometric $d_{P,Q}(f, g) = \|f - g\|_{P,Q}$ in the usual way, and the topology on \mathcal{S} is the weakest topology making all the functions $d_{P,Q}(\cdot, g)$ continuous. That is, a base for the topology consists of all sets $U_{g,P,Q,n} = \{f \mid \|f - g\|_{P,Q} < 1/n\}$.

A feature of \mathcal{S} is that only countably many of the seminorms are relevant for obtaining the open sets, and a consequence is that the topology of \mathcal{S} is defined by a metric. The important seminorms are the ones in which P and Q are monomials, each with coefficient 1. In fact, if $P(x) = \sum_{\alpha} a_{\alpha}x^{\alpha}$ and $Q(x) = \sum_{\beta} b_{\beta}x^{\beta}$, then it is easy to check that $d_{P,Q}(f, g) \leq \sum_{\alpha,\beta} |a_{\alpha}b_{\beta}|d_{x^{\alpha},x^{\beta}}(f, g)$. Hence any open set that $d_{P,Q}$ defines is a union of finite intersections of the open sets defined by the finitely many $d_{x^{\alpha},x^{\beta}}$'s.

³The reader may notice that the definition of “seminorm” is the same as the definition of “pseudonorm” in *Basic*. The only distinction is that the word “seminorm” is often used in the context of a whole family of such objects, while the word “pseudonorm” is often used when there is only one such object under consideration.

Let us digress and consider the situation more abstractly because it will arise again later. Suppose we have a real or complex vector space V on which are defined countably many seminorms $\|\cdot\|_n$ satisfying (i), (ii), and (iii) above.

Each seminorm $\|\cdot\|_n$ gives rise to a pseudometric \tilde{d}_n on V and then to open sets defined relative to \tilde{d}_n . For any pseudometric $\tilde{\rho}$, the function $\rho = \min\{1, \tilde{\rho}\}$ is easily checked to be a pseudometric, and ρ defines the same open sets on V as $\tilde{\rho}$ does. We shall use the following abstract result about pseudometrics; this was proved as Proposition 10.28 of *Basic*, and we therefore omit the proof here.

Proposition 3.1. Suppose that V is a nonempty set and $\{d_n\}_{n \geq 1}$ is a sequence of pseudometrics on V such that $d_n(x, y) \leq 1$ for all n and for all x and y in V . Then $d(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n(x, y)$ is a pseudometric. If the open balls relative to d_n are denoted by $B_n(r; x)$ and the open balls relative to d are denoted by $B(r; x)$, then the B_n 's and B 's are related as follows:

- (a) whenever some $B_n(r_n; x)$ is given with $r_n > 0$, there exists some $B(r; x)$ with $r > 0$ such that $B(r; x) \subseteq B_n(r_n; x)$,
- (b) whenever $B(r; x)$ is given with $r > 0$, there exist finitely many $r_n > 0$, say for $n \leq K$, such that $\bigcap_{n=1}^K B_n(r_n; x) \subseteq B(r; x)$.

In the situation with countably many seminorms $\|\cdot\|_n$ for the vector space V , we see that we can introduce a pseudometric d such that three conditions hold:

- $d(x, y) = d(0, y - x)$ for all x and y ,
- whenever some x in V is given and an index n and corresponding number $r_n > 0$ are given, then there is a number $r > 0$ such that $d(x, y) < r$ implies $\|y - x\|_n < r_n$,
- whenever some x in V is given and some $r > 0$ is given, then there exist finitely many $r_n > 0$, say for $n \leq K$, such that any y with $\|y - x\|_n < r_n$ for $n \leq K$ implies $d(x, y) < r$.

If the seminorms collectively have the property that $\|x\|_n = 0$ for all n only for $x = 0$, then d is a metric, and we say that the family of seminorms is a **separating family**. The specific form of d is not important: in the case of \mathcal{S} , the metric d depended on the choice of the countable subfamily of pseudometrics and the order in which they were enumerated, and these choices do not affect any results about \mathcal{S} . The important thing about this construction is that it shows that the topology is given by *some* metric.

The three conditions marked with bullets enable us to detect continuity of linear functions with domain V and range another such space W by using the seminorms directly.

Proposition 3.2. Let $L : V \rightarrow W$ be a linear function between vector spaces that are both real or both complex. Suppose that V is topologized by means of

countably many seminorms $\|\cdot\|_{V,m}$ and W is topologized by means of countably many seminorms $\|\cdot\|_{W,n}$. Then L is continuous if and only if for each n , there is a finite set $F = F(n)$ of m 's and there are corresponding positive numbers δ_m such that $\|v\|_{V,m} \leq \delta_m$ for all $m \in F$ implies $\|L(v)\|_{W,n} \leq 1$.

PROOF. Let d_V and d_W be the distance functions in V and W . When n is given, the second item in the bulleted list shows that there is some $r > 0$ such that $d_W(0, w) \leq r$ implies $\|w\|_{W,n} \leq 1$. If L is continuous at 0, then there is a $\delta > 0$ such that $d_V(0, v) \leq \delta$ implies $d_W(0, L(v)) \leq r$. From the third item in the bulleted list, we know that there is a finite set F of indices m and there are corresponding numbers $\delta_m > 0$ such that $\|v\|_{V,m} \leq \delta_m$ implies $d_V(0, v) \leq \delta$. Then $\|v\|_{V,m} \leq \delta_m$ for all m in F implies $\|L(v)\|_{W,n} \leq 1$.

Conversely suppose for each n that there is a finite set F and there are numbers $\delta_m > 0$ for m in F such that the stated condition holds. To see that L is continuous at 0, let $\epsilon > 0$ be given. Choose K and numbers $\epsilon_n > 0$ for $n \leq K$ such that $\|w\|_{W,n} \leq \epsilon_n$ for $n \leq K$ implies $d_W(0, w) \leq \epsilon$. For each $n \leq K$, the given condition on L allows us to find a finite set F_n of indices m and numbers $\delta_m > 0$ such that $\|v\|_{V,m} \leq \delta_m$ implies $\|L(v)\|_{W,n} \leq 1$. If $\|v\|_{V,m} \leq \delta_m \epsilon_n$ for all m in $F = \bigcup_{n \leq K} F_n$, then $\|L(v)\|_{W,n} \leq \epsilon_n$ for all $n \leq K$ and hence $d_W(0, L(v)) \leq \epsilon$. We know that there is a number $\delta > 0$ such that $d_V(0, v) \leq \delta$ implies $\|v\|_{V,m} \leq \delta_m \epsilon_n$ for all m in F , and then $d_W(0, L(v)) \leq \epsilon$. Hence L is continuous at 0.

Once L is continuous at 0, it is continuous everywhere because of the translation invariance of d_V and d_W : $d_V(v_1, v_2) = d_V(0, v_2 - v_1)$ and $d_W(L(v_1), L(v_2)) = d_W(0, L(v_2) - L(v_1)) = d_W(0, L(v_2 - v_1))$.

Now we return to the Schwartz space \mathcal{S} to apply our construction and Proposition 3.2. The bulleted items above make it clear that it does not matter which countable set of generating seminorms we use nor what order we put them in; the open sets and the criterion for continuity are still the same. The following corollary is immediate from Proposition 3.2, the definition of \mathcal{S} , and the behavior of the Fourier transform under multiplication by polynomials and under differentiation.

Corollary 3.3. For the Schwartz space \mathcal{S} on \mathbb{R}^N ,

- (a) a linear functional ℓ is continuous if and only if there is a finite set F of pairs (P, Q) of polynomials and there are corresponding numbers $\delta_{P,Q} > 0$ such that $\|f\|_{P,Q} \leq \delta_{P,Q}$ for all (P, Q) in F implies $|\ell(f)| \leq 1$.
- (b) the Fourier transform mapping $\mathcal{F} : \mathcal{S} \rightarrow \mathcal{S}$ is continuous, and so is its inverse.

A continuous linear functional on the Schwartz space is called a **tempered distribution**, and the space of all tempered distributions is denoted by $\mathcal{S}' =$

$\mathcal{S}'(\mathbb{R}^N)$. It will be convenient to write $\langle T, \varphi \rangle$ for the value of the tempered distribution T on the Schwartz function φ . The space of tempered distributions is huge. A few examples will give an indication just how huge it is.

EXAMPLES.

(1) Any function f on \mathbb{R}^N with $|f(x)| \leq (1 + |x|^2)^n |g(x)|$ for some integer n and some integrable function g defines a tempered distribution T by integration: $\langle T, \varphi \rangle = \int_{\mathbb{R}^N} f(x)\varphi(x) dx$ when φ is in \mathcal{S} . In view of Corollary 3.3a, the continuity follows from the chain of inequalities

$$\begin{aligned} |\langle T, \varphi \rangle| &\leq \int_{\mathbb{R}^N} (|f(x)|(1 + |x|^2)^{-n})(1 + |x|^2)^n |\varphi(x)| dx \\ &\leq \left(\int_{\mathbb{R}^N} |g(x)| dx \right) \left(\sup_x \{(1 + |x|^2)^n |\varphi(x)|\} \right) \\ &= \|g\|_1 \|\varphi\|_{P,1} \quad \text{for } P(x) = (1 + |x|^2)^n. \end{aligned}$$

(2) Any function f with $|f(x)| \leq (1 + |x|^2)^n |g(x)|$ for some integer n and some function g in $L^\infty(\mathbb{R}^N)$ defines a tempered distribution T by integration: $\langle T, \varphi \rangle = \int_{\mathbb{R}^N} f(x)\varphi(x) dx$. In fact, $|f(x)| \leq (1 + |x|^2)^{n+N} (1 + |x|^2)^{-N} |g(x)|$, and $(1 + |x|^2)^{-N} |g(x)|$ is integrable; hence this example is an instance of Example 1.

(3) Any function f with $|f(x)| \leq (1 + |x|^2)^n |g(x)|$ for some integer n and some function g in $L^p(\mathbb{R}^N)$, where $1 \leq p \leq \infty$, defines a tempered distribution T by integration because such a distribution is the sum of one as in Example 1 and one as in Example 2.

(4) Suppose that f is as in Example 3 and that $Q(D)$ is a constant-coefficients partial differential operator. Then the formula $\langle T, \varphi \rangle = \int_{\mathbb{R}^N} f(x)(Q(D)\varphi)(x) dx$ defines a tempered distribution.

(5) In the above examples, Lebesgue measure dx may be replaced by any Borel measure $d\mu(x)$ on \mathbb{R}^N such that $\int_{\mathbb{R}^N} (1 + |x|^2)^{n_0} d\mu(x) < \infty$ for some n_0 . A particular case of interest is that $d\mu(x)$ is a point mass at a point x_0 ; in this case, the tempered distributions $\varphi \mapsto \langle T, \varphi \rangle$ that are obtained by combining the above constructions are the linear combinations of iterated partial derivatives of φ at the point x_0 .

(6) Any finite linear combination of tempered distributions as in Example 5 is again a tempered distribution.

Two especially useful operations on tempered distributions are multiplication by a Schwartz function and differentiation. Both of these definitions are arranged to be extensions of the corresponding operations on Schwartz functions. The definitions are $\langle \psi T, \varphi \rangle = \langle T, \psi \varphi \rangle$ and $\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle$; in the latter case the factor $(-1)^{|\alpha|}$ is included because integration by parts requires its presence when T is given by a Schwartz function.

A useful feature of distributions in connection with differential equations, as we shall see in more detail in later chapters, is that we can first look for solutions of a given differential equation that are distributions and then consider how close those distributions are to being functions. The special feature of *tempered* distributions is that the Fourier transform makes sense on them, as follows.

As with the operations of multiplication by a Schwartz function and differentiation, the definition of Fourier transform of a tempered distribution is arranged to be an extension of the definition of the Fourier transform of a member ψ of \mathcal{S} when we identify the function ψ with the distribution $\psi(x) dx$. If φ is in \mathcal{S} , then $\int \widehat{\psi} \varphi dx = \int \psi \widehat{\varphi} dx$ by the multiplication formula,⁴ which we reinterpret as $\langle \mathcal{F}(\psi dx), \varphi \rangle = \langle \psi dx, \widehat{\varphi} \rangle$. The definition is

$$\langle \mathcal{F}(T), \varphi \rangle = \langle T, \widehat{\varphi} \rangle$$

for $T \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$. To see that $\mathcal{F}(T)$ is in \mathcal{S}' , we have to check that $\mathcal{F}(T)$ is continuous. The definition is $\mathcal{F}(T) = T \circ \mathcal{F}$, and \mathcal{F} is continuous on \mathcal{S} by Corollary 3.3b. Thus the Fourier transform carries tempered distributions to tempered distributions.

Proposition 3.4. The Fourier transform \mathcal{F} is one-one from $\mathcal{S}'(\mathbb{R}^N)$ onto $\mathcal{S}'(\mathbb{R}^N)$.

PROOF. If T is in \mathcal{S}' and $\mathcal{F}(T) = 0$, then $\langle T, \mathcal{F}(\varphi) \rangle = 0$ for all φ in \mathcal{S} . Since \mathcal{F} carries \mathcal{S} onto \mathcal{S} , $\langle T, \psi \rangle = 0$ for all ψ in \mathcal{S} , and thus $T = 0$. Therefore \mathcal{F} is one-one on \mathcal{S}' .

If T' is given in \mathcal{S}' , put $T = T' \circ \mathcal{F}^{-1}$, where \mathcal{F}^{-1} is the inverse Fourier transform as a map of \mathcal{S} to itself. Then $T' = T \circ \mathcal{F}$ and $\mathcal{F}(T) = T \circ \mathcal{F} = T'$. Therefore \mathcal{F} is onto \mathcal{S}' .

2. Weak Derivatives and Sobolev Spaces

A careful study of a linear partial differential equation often requires attention to the domain of the operator, and it is helpful to be able to work with partial derivatives without investigating a problem of interchange of limits at each step. Sobolev spaces are one kind of space of functions that are used for this purpose, and their definition involves “weak derivatives.” At the end one wants to be able to deduce results about ordinary partial derivatives from results about weak derivatives, and Sobolev’s Theorem does exactly that.

We shall make extensive use in this book of techniques for regularizing functions that have been developed in *Basic*. Let us assemble a number of these in one place for convenient reference.

⁴Proposition 8.1e of *Basic*.

Proposition 3.5.

(a) (Theorems 6.20 and 9.13) Let φ be in $L^1(\mathbb{R}^N, dx)$, define $\varphi_\varepsilon(x) = \varepsilon^{-N} \varphi(\varepsilon^{-1}x)$ for $\varepsilon > 0$, and put $c = \int_{\mathbb{R}^N} \varphi(x) dx$.

(i) If f is in $L^p(\mathbb{R}^N, dx)$ with $1 \leq p < \infty$, then

$$\lim_{\varepsilon \downarrow 0} \|\varphi_\varepsilon * f - cf\|_p = 0.$$

(ii) If f is bounded on \mathbb{R}^N and is continuous at x , then $\lim_{\varepsilon \downarrow 0} (\varphi_\varepsilon * f)(x) = cf(x)$, and the convergence is uniform for any set E of x 's such that f is uniformly continuous at the points of E .

(b) (Proposition 9.9) If μ is a Borel measure on a nonempty open set U in \mathbb{R}^N and if $1 \leq p < \infty$, then $L^p(U, \mu)$ is separable, and $C_{\text{com}}(U)$ is dense in $L^p(U, \mu)$.

(c) (Corollary 6.19) Suppose that φ is a compactly supported function of class C^n on \mathbb{R}^N and that f is in $L^p(\mathbb{R}^N, dx)$ with $1 \leq p \leq \infty$. Then $\varphi * f$ is of class C^n , and $D^\alpha(\varphi * f) = (D^\alpha \varphi) * f$ for any iterated partial derivative D^α of order $\leq n$.

(d) (Lemma 8.11) If δ_1 and δ_2 are given positive numbers with $\delta_1 < \delta_2$, then there exists ψ in $C_{\text{com}}^\infty(\mathbb{R}^N)$ with values in $[0, 1]$ such that $\psi(x) = \psi_0(|x|)$, ψ_0 is nonincreasing, $\psi(x) = 1$ for $|x| \leq \delta_1$, and $\psi(x) = 0$ for $|x| \geq \delta_2$.

(e) (Consequence of (d)) If $\delta > 0$, then there exists $\varphi \geq 0$ in $C_{\text{com}}^\infty(\mathbb{R}^N)$ such that $\varphi(x) = \varphi_0(|x|)$ with φ_0 nonincreasing, $\varphi(x) = 0$ for $|x| \geq 1$, and $\int_{\mathbb{R}^N} \varphi(x) dx = 1$.

(f) (Proposition 8.12) If K and U are subsets of \mathbb{R}^N with K compact, U open, and $K \subseteq U$, then there exists $\varphi \in C_{\text{com}}^\infty(U)$ with values in $[0, 1]$ such that φ is identically 1 on K .

In this section we work with a nonempty open subset U of \mathbb{R}^N , an index p satisfying $1 \leq p < \infty$, and the spaces $L^p(U) = L^p(U, dx)$, the underlying measure being understood to be Lebesgue measure. Let $p' = p/(p-1)$ be the dual index. For Sobolev's Theorem, we shall impose two additional conditions on U , namely boundedness for U and a certain regularity condition for the **boundary** $\partial U = U^{\text{cl}} - U$ of the open set U , but we do not impose those additional conditions yet.

Corollary 3.6. If U is a nonempty open subset of \mathbb{R}^N , then $C_{\text{com}}^\infty(U)$ is

- (a) uniformly dense in $C_{\text{com}}(U)$,
- (b) dense in $L^p(U)$ for every p with $1 \leq p < \infty$.

PROOF. Let f in $C_{\text{com}}(U)$ be given. Choose by Proposition 3.5e a function φ in $C_{\text{com}}^\infty(\mathbb{R}^N)$ that is ≥ 0 , vanishes outside the unit ball about the origin, and

has total integral 1. For $\varepsilon > 0$, define $\varphi_\varepsilon(x) = \varepsilon^{-N}\varphi(\varepsilon^{-1}x)$. The function $\varphi_\varepsilon * f$ is of class C^∞ by (c). If $U = \mathbb{R}^N$, let $\varepsilon_0 = 1$; otherwise let ε_0 be the distance from the support of f to the complement of U . For $\varepsilon < \varepsilon_0$, $\varphi_\varepsilon * f$ has compact support contained in U . As ε decreases to 0, Proposition 3.5a shows that $\|\varphi_\varepsilon * f - f\|_{\text{sup}}$ tends to 0 and so does $\|\varphi_\varepsilon * f - f\|_p$. This proves the first conclusion of the corollary and proves also that $C_{\text{com}}^\infty(U)$ is L^p dense in $C_{\text{com}}(U)$ if $1 \leq p < \infty$. Since Proposition 3.5b shows that $C_{\text{com}}(U)$ is dense in $L^p(U)$, the second conclusion of the corollary follows.

Suppose that f and g are two complex-valued functions that are **locally integrable** on U in the sense of being integrable on each compact subset of U . If α is a differentiation index, we say that $D^\alpha f = g$ in the sense of **weak derivatives** if

$$\int_U f(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_U g(x) \varphi(x) dx \quad \text{for all } \varphi \in C_{\text{com}}^\infty(U).$$

The definition is arranged so that g gives the result that one would expect for iterated partial differentiation of type α if the integrated or boundary term gives 0 at each stage. More precisely if f is in $C^{|\alpha|}(U)$, then the weak derivative of order α exists and is the pointwise derivative. To prove this, it is enough to handle a first-order partial derivative $D_j h$ for a function h in $C^1(U)$, showing that $\int_U h D_j \varphi dx = - \int_U (D_j h) \varphi dx$ for $\varphi \in C_{\text{com}}^\infty(U)$, i.e., that $\int_U D_j (h\varphi) dx = 0$. Because φ is compactly supported in U , $\psi = h\varphi$ makes sense as a compactly supported C^1 function on \mathbb{R}^N , and we are to prove that $\int_{\mathbb{R}^N} D_j \psi dx = 0$. The Fundamental Theorem of Calculus gives $\int_{-a}^a D_j \psi dx_j = [\psi]_{x_j=-a}^{x_j=a}$ for $a > 0$, and the compact support implies that this is 0 for a sufficiently large. Thus $\int_{\mathbb{R}} D_j \psi dx_j = 0$, and Fubini's Theorem gives $\int_{\mathbb{R}^N} D_j \psi dx = 0$.

The function g in the definition of weak derivative is unique up to sets of measure 0 if it exists. In fact, if g_1 and g_2 are both weak derivatives of order α , then $\int_U (g_1 - g_2) \varphi dx = 0$ for all φ in $C_{\text{com}}^\infty(U)$. Fix an open set V having compact closure contained in U . If f is in $C_{\text{com}}(V)$, then Corollary 3.6a produces a sequence of functions φ_n in $C_{\text{com}}^\infty(V)$ tending uniformly to f . Since $g_1 - g_2$ is integrable on V , the equalities $\int_V (g_1 - g_2) \varphi_n dx = 0$ for all n imply $\int_V (g_1 - g_2) f dx = 0$. By the uniqueness in the Riesz Representation Theorem, $g_1 = g_2$ a.e. on V . Since V is arbitrary, $g_1 = g_2$ a.e. on U .

EXAMPLE. In the open set $U = (-1, 1) \subseteq \mathbb{R}^1$, the function $e^{i/|x|}$ is locally integrable and is differentiable except at $x = 0$, but it does not have a weak derivative. In fact, if it had g as a weak derivative, we could use φ 's vanishing in neighborhoods of the origin to see that $g(x)$ has to be $-ix^{-2}(\text{sgn } x)e^{i/|x|}$ almost everywhere. But this function is not locally integrable on U .

If f has α^{th} weak derivative $D^\alpha f$ and $D^\alpha f$ has β^{th} weak derivative $D^\beta(D^\alpha f)$, then f has $(\beta + \alpha)^{\text{th}}$ weak derivative $D^{\beta+\alpha} f$ and $D^{\beta+\alpha} f = D^\beta(D^\alpha f)$. In fact, if φ is in $C_{\text{com}}^\infty(U)$, then this conclusion follows from the computation

$$\begin{aligned} \int_U f D^{\beta+\alpha} \varphi \, dx &= \int_U f D^\alpha(D^\beta \varphi) \, dx = (-1)^{|\alpha|} \int_U D^\alpha f D^\beta \varphi \, dx \\ &= (-1)^{|\alpha|+|\beta|} \int_U D^\beta(D^\alpha f) \varphi \, dx. \end{aligned}$$

If f has weak j^{th} partial derivative $D_j f$ and if ψ is in $C^\infty(U)$, then $f\psi$ has a weak j^{th} partial derivative, and it is given by $(D_j f)\psi + f(D_j \psi)$. In fact, this conclusion holds because $\int_U f \psi (D_j \varphi) \, dx = \int_U f D_j(\psi \varphi) \, dx - \int_U f (D_j \psi) \varphi \, dx = -\int_U (D_j f) \psi \varphi \, dx - \int_U f (D_j \psi) \varphi \, dx = -\int_U (f(D_j \psi) + (D_j f)\psi) \varphi \, dx$.

If f has β^{th} weak derivative $D^\beta f$ for every β with $\beta \leq \alpha$ and if ψ is in $C^\infty(U)$, then $f\psi$ has an α^{th} weak derivative. It is given by the **Leibniz rule**:

$$D^\alpha(f\psi) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} (D^\beta f)(D^{\alpha - \beta} \psi).$$

This formula follows by iterating the formula for $D_j(f\psi)$ in the previous paragraph.

Now we can give the definition of Sobolev spaces. Let $k \geq 0$ be an integer, and let $1 \leq p < \infty$. Define

$$L_k^p(U) = \{f \in L^p(U) \mid \text{all } D^\alpha f \text{ exist weakly for } |\alpha| \leq k \text{ and are in } L^p(U)\}.$$

Then $L_k^p(U)$ is a vector space, and we make it into a normed linear space by defining

$$\|f\|_{L_k^p} = \left(\sum_{|\alpha| \leq k} \int_U |D^\alpha f|^p \, dx \right)^{1/p}.$$

The normed linear spaces $L_k^p(U)$ are the **Sobolev spaces** for U . All the remaining results in this section concern these spaces.⁵

Proposition 3.7. If $k \geq 0$ is an integer and if $1 \leq p < \infty$, then the normed linear space $L_k^p(U)$ is complete.

⁵The subject of partial differential equations makes use of a number of families that generalize these spaces in various ways. Of particular importance is a family H^s such that $H^s = L_k^2$ when s is an integer $k \geq 0$ but s is a continuous real parameter with $-\infty < s < \infty$. The spaces $H^s(\mathbb{R}^N)$ are introduced in Problems 8–12 at the end of the chapter. For an open set U , the two spaces $H_{\text{com}}^s(U)$ and $H_{\text{loc}}^s(U)$ are introduced in Chapter VIII. All of these spaces are called **Sobolev spaces**.

PROOF. If $\{f_m\}$ is a Cauchy sequence in $L_k^p(U)$, then for each α with $|\alpha| \leq k$, the sequence $\{D^\alpha f_m\}$ is Cauchy in $L^p(U)$. Since $L^p(U)$ is complete, we can define $f^{(\alpha)}$ to be the $L^p(U)$ limit of $D^\alpha f_m$. For φ in $C_{\text{com}}^\infty(U)$, we then have

$$\int_U f^{(\alpha)} \varphi \, dx = \int_U (\lim_m D^\alpha f_m) \varphi \, dx = \lim_m \int_U (D^\alpha f_m) \varphi \, dx,$$

the second equality holding since φ is in the dual space $L^{p'}(U)$. In turn, this expression is equal to

$$(-1)^{|\alpha|} \lim_m \int_U (f_m)(D^\alpha \varphi) \, dx = (-1)^{|\alpha|} \int_U (f^{(0)})(D^\alpha \varphi) \, dx,$$

the second equality holding since $D^\alpha \varphi$ is in $L^{p'}(U)$. Therefore $f^{(\alpha)} = D^\alpha f^{(0)}$ and f_m tends to $f^{(0)}$ in $L_k^p(U)$.

Proposition 3.8. If $k \geq 0$ is an integer and if $1 \leq p < \infty$, then a function f is in $L_k^p(U)$ if f is in $L^p(U)$ and there exists a sequence $\{f_m\}$ in $C^k(U)$ such that

- (a) $\lim_m \|f - f_m\|_p = 0$,
- (b) for each α with $|\alpha| \leq k$, the iterated pointwise partial derivative $D^\alpha f_m$ is in $L^p(U)$ and converges in $L^p(U)$ as m tends to infinity.

PROOF. By (b), $\|D^\alpha (f_l - f_m)\|_p^p$ for each fixed α tends to 0 as l and m tend to infinity. Summing on α and taking the p^{th} root, we see that $\|f_l - f_m\|_{L_k^p}$ tends to 0. In other words, $\{f_m\}$ is Cauchy in $L_k^p(U)$. By Proposition 3.7, $\{f_m\}$ converges to some g in $L_m^p(U)$. The limit function g has to have the property that $\|f_m - g\|_p$ tends to 0, and (a) shows that we must have $g = f$. Therefore f is in $L_k^p(U)$.

The key theorem is the following converse to Proposition 3.8.

Theorem 3.9. If $k \geq 0$ is an integer and if $1 \leq p < \infty$, then $C^\infty(U) \cap L_k^p(U)$ is dense in $L_k^p(U)$.

On the other hand, despite Corollary 3.6b, it will be a consequence of Sobolev's Theorem that $C_{\text{com}}^\infty(U)$ is not dense in $L_k^p(U)$ if k is sufficiently large. The proof of the present theorem will be preceded by a lemma affirming that at least the members of $L_k^p(U)$ with compact support in U can be approximated by members of $C_{\text{com}}^\infty(U)$.

In addition, the proof of the theorem will make use of an “exhausting sequence” and a smooth partition of unity based on it. Since U is locally compact and σ -compact, we can find a sequence $\{K_n\}_{n=1}^\infty$ of compact subsets of U with union U such that $K_n \subseteq K_{n+1}^o$ for all n . This sequence is called an **exhausting sequence**

for U . We construct the partition of unity $\{\psi_n\}_{n \geq 1}$ as follows. For $n \geq 1$, we use Proposition 3.5f to choose a C^∞ function φ_n with values in $[0, 1]$ such that

$$\varphi_1(x) = \begin{cases} 1 & \text{for } x \in K_3, \\ 0 & \text{for } x \in (K_4^o)^c, \end{cases}$$

and for $n \geq 2$,

$$\varphi_n(x) = \begin{cases} 1 & \text{for } x \in K_{n+2} - K_{n+1}^o, \\ 0 & \text{for } x \in (K_{n+3}^o)^c \cup K_n. \end{cases}$$

In the sum $\sum_{n=1}^{\infty} \varphi_n(x)$, each x has a neighborhood in which only finitely many terms are nonzero and some term is nonzero. Therefore $\varphi = \sum_{n=1}^{\infty} \varphi_n$ is a well-defined member of $C^\infty(U)$. If we put $\psi_n = \varphi_n / \varphi$, then ψ_n is in $C^\infty(U)$, $\sum_{n=1}^{\infty} \psi_n = 1$ on U , $\psi_1(x)$ is > 0 on K_3 and is $= 0$ on $(K_4^o)^c$, and for $n \geq 2$,

$$\psi_n(x) \begin{cases} > 0 & \text{for } x \in K_{n+2} - K_{n+1}^o, \\ = 0 & \text{for } x \in (K_{n+3}^o)^c \cup K_n. \end{cases}$$

Lemma 3.10. Let φ be a member of $C_{\text{com}}^\infty(\mathbb{R}^N)$ vanishing for $|x| \geq 1$ and having total integral 1, put $\varphi_\varepsilon(x) = \varepsilon^{-N} \varphi(\varepsilon^{-1}x)$ for $\varepsilon > 0$, and let f be a function in $L_k^p(U)$ whose support is a compact subset of U . For ε sufficiently small, $\varphi_\varepsilon * f$ is in $C_{\text{com}}^\infty(U)$, and

$$\lim_{\varepsilon \downarrow 0} \|\varphi_\varepsilon * f - f\|_{L_k^p} = 0.$$

PROOF. As in the proof of Corollary 3.6, $\varphi_\varepsilon * f$ has compact support contained in U if $\varepsilon < \varepsilon_0$, where ε_0 is 1 if $U = \mathbb{R}^N$ and ε_0 is the distance of the support of f to the complement of U if $U \neq \mathbb{R}^N$. Moreover, the function $\varphi_\varepsilon * f$ is in $C^\infty(\mathbb{R}^N)$ with $D^\alpha(\varphi_\varepsilon * f) = (D^\alpha \varphi_\varepsilon) * f$ for each α . Thus $\varphi_\varepsilon * f$ is in $C_{\text{com}}^\infty(U)$ if $\varepsilon < \varepsilon_0$. By the first remark after the definition of weak derivative, $\varphi_\varepsilon * f$ has weak derivatives of all orders for $\varepsilon < \varepsilon_0$, and they are given by the ordinary derivatives $D^\alpha(\varphi_\varepsilon * f)$. For $\varepsilon < \varepsilon_0$,

$$\begin{aligned} D^\alpha(\varphi_\varepsilon * f)(x) &= \int_U f(y) (D^\alpha \varphi_\varepsilon)(x - y) dy \\ &= (-1)^{|\alpha|} \int_U f(y) D^\alpha(y \mapsto \varphi_\varepsilon(x - y)) dy. \end{aligned}$$

Since f by assumption has weak derivatives through order k and since $y \mapsto \varphi_\varepsilon(x - y)$ has compact support in U , the right side is equal to

$$\int_U D^\alpha f(y) \varphi_\varepsilon(x - y) dy = (\varphi_\varepsilon * D^\alpha f)(x)$$

for $|\alpha| \leq k$. Therefore, for $\varepsilon < \varepsilon_0$ and $|\alpha| \leq k$, we have

$$\|D^\alpha(\varphi_\varepsilon * f - f)\|_p = \|\varphi_\varepsilon * (D^\alpha f) - D^\alpha f\|_p.$$

For these same α 's, Proposition 3.5a shows that the right side tends to 0 as ε tends to 0. Therefore $\varphi_\varepsilon * f - f$ tends to 0 in $L_k^p(U)$.

PROOF OF THEOREM 3.9. Let f be in $L_k^p(U)$. The idea is to break f into a countable sum of functions of compact support, apply the lemma to each piece, and add the results. The difficulty lies in arranging that each of the pieces of f have controlled weak derivatives through order k . Thus instead of using indicator functions to break up f , we shall use an exhausting sequence $\{K_n\}_{n \geq 1}$ and an associated partition of unity $\{\psi_n\}_{n \geq 1}$ of the kind described after the statement of the theorem. The discussion above concerning the Leibniz rule shows that each $\psi_n f$ has weak derivatives of all orders $\leq k$, and the construction shows that $\psi_n f$ has support in K_5^o for $n = 1$ and in $K_{n+4}^o - K_{n-1}$ for $n \geq 2$.

Let $\epsilon > 0$ be given, let φ be a member of $C_{\text{com}}^\infty(\mathbb{R}^N)$ vanishing for $|x| \geq 1$ and having total integral 1, and put $\varphi_\epsilon(x) = \epsilon^{-N} \varphi(\epsilon^{-1}x)$ for $\epsilon > 0$. Applying Lemma 3.10 to $\psi_n f$, choose $\epsilon_n > 0$ small enough so that the function $u_n = \varphi_{\epsilon_n} * (\psi_n f)$ has support in K_5^o for $n = 1$ and in $K_{n+4}^o - K_{n-1}$ for $n \geq 2$ and so that

$$\|u_n - \psi_n f\|_{L_k^p} < 2^{-n} \epsilon.$$

Put $u = \sum_{n=1}^\infty u_n$. Each x in U has a neighborhood on which only finitely many of the functions u_n are not identically 0, and therefore u is in $C^\infty(U)$. Also,

$$u = \sum_{n=1}^\infty (u_n - \psi_n f) + f \quad \text{since} \quad \sum_{n=1}^\infty \psi_n = 1.$$

Since for each compact subset of U , only finitely many $u_n - \psi_n f$ are not identically 0 on that set, the weak derivatives of order $\leq k$ satisfy $D^\alpha u = \sum_{n=1}^\infty D^\alpha (u_n - \psi_n f) + D^\alpha f$. Hence

$$D^\alpha (u - f) = \sum_{n=1}^\infty D^\alpha (u_n - \psi_n f).$$

Minkowski's inequality for integrals therefore gives

$$\|D^\alpha (u - f)\|_p \leq \sum_{n=1}^\infty \|D^\alpha (u_n - \psi_n f)\|_p \leq \sum_{n=1}^\infty \|u_n - \psi_n f\|_{L_k^p} \leq \sum_{n=1}^\infty \frac{\epsilon}{2^n} = \epsilon.$$

Finally we raise both sides to the p^{th} power, sum for α with $|\alpha| \leq k$, and extract the p^{th} root. If $m(k)$ denotes the number of such α 's, we obtain

$$\|u - f\|_{L_k^p} \leq m(k)^{1/p} \epsilon,$$

and the proof is complete.

Now we come to Sobolev's Theorem. For the remainder of the section, the open set U will be assumed bounded, and we shall impose a regularity condition on its boundary $\partial U = U^{\text{cl}} - U$. When we isolate one of the coordinates of points in \mathbb{R}^N , say the j^{th} , let us write y' for the other $N - 1$ coordinates, so that $y = (y_j, y')$. We say that U **satisfies the cone condition** if there exist positive constants c and h such that for each x in U , there are a sign \pm and an index j with $1 \leq j \leq N$ for which the closed truncated cone

$$\Gamma_x = x + \{y = (y_j, y') \mid \pm y_j \geq c|y'| \text{ and } |y| \leq h\}$$

lies in U for one choice of the sign \pm . See Figure 3.1. Problem 4 at the end of the chapter observes that if the bounded open set U has a C^1 boundary in a certain sense, then U satisfies the cone condition.

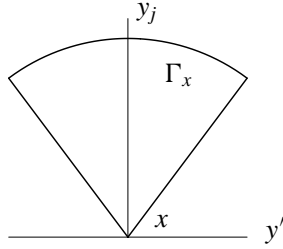


FIGURE 3.1. Cone condition for a bounded open set.

Theorem 3.11 (Sobolev's Theorem). Let U be a nonempty bounded open set in \mathbb{R}^N , and suppose that U satisfies the cone condition with constants c and h . If $1 \leq p < \infty$ and $k > N/p$, then there exists a constant $C = C(N, c, h, p, k)$ such that

$$\sup_{x \in U} |u(x)| \leq C \|u\|_{L_k^p}$$

for all u in $C^\infty(U) \cap L_k^p(U)$.

REMARK. Under the stated conditions on k and p , the theorem says that the inclusion of $C^\infty(U) \cap L_k^p(U)$ into the Banach space $C(U)$ of *bounded* continuous functions on U is a bounded linear operator relative to the norm of $L_k^p(U)$. Since $C^\infty(U) \cap L_k^p(U)$ is dense in $L_k^p(U)$ by Theorem 3.9 and since $C(U)$ is complete, the inclusion extends to a continuous map of $L_k^p(U)$ into $C(U)$. In other words, every member of $L_k^p(U)$ can be regarded as a bounded continuous function on U .

PROOF. Fix g in $C_{\text{com}}^\infty(\mathbb{R}^1)$ with $g(t)$ equal to 1 for $|t| \leq \frac{1}{2}$ and equal to 0 for $|t| \geq \frac{3}{4}$. Fix x in U and its associated sign \pm and index j . We introduce spherical

coordinates about x with the indices reordered so that j comes first, writing $x + y$ for a point near x with

$$\begin{aligned} y_j &= \pm r \cos \varphi, \\ y_1 &= r \sin \varphi \cos \theta_1, \\ &\vdots \quad (\text{with } y_j \text{ omitted}) \\ y_{N-1} &= r \sin \varphi \sin \theta_1 \cdots \sin \theta_{N-3} \cos \theta_{N-2}, \\ y_N &= r \sin \varphi \sin \theta_1 \cdots \sin \theta_{N-3} \sin \theta_{N-2}, \end{aligned}$$

when

$$\begin{aligned} 0 &\leq \varphi \leq \pi, \\ 0 &\leq \theta_i \leq \pi \text{ for } i < N-2, \\ 0 &\leq \theta_{N-2} \leq 2\pi. \end{aligned}$$

All the points $x + y$ with $0 \leq \varphi \leq \Phi(c)$, where $\Phi(c)$ is some positive number and $0 \leq r \leq h$, lie in the cone Γ_x at x . For such φ 's and for $0 \leq t \leq 1$, we define

$$F(t) = g\left(\frac{t}{h}\right)u\left(x + (\pm t \cos \varphi, t \sin \varphi \cos \theta_1, \dots)\right)$$

and expand F in a Taylor series through order $k - 1$ with remainder about the point $t = h$. Because of the behavior of g , F and all its derivatives vanish at $t = h$. Therefore $F(t)$ is given by the remainder term:

$$F(t) = \frac{1}{(k-1)!} \int_h^t (t-s)^{k-1} F^{(k)}(s) ds.$$

Putting $t = 0$, we obtain

$$\begin{aligned} u(x) &= \frac{1}{(k-1)!} \int_h^0 (-r)^{k-1} \frac{\partial^k}{\partial r^k} \left[g\left(\frac{r}{h}\right)u\left(x + (\dots)\right) \right] dr \\ &= \frac{(-1)^k}{(k-1)!} \int_0^h r^{k-N} \frac{\partial^k}{\partial r^k} \left[g\left(\frac{r}{h}\right)u\left(x + (\dots)\right) \right] r^{N-1} dr. \end{aligned}$$

We regard the integral on the right side as taking place over the radial part of the spherical coordinates that describe the set of y 's in Γ_x , and we want to extend the integration over all of Γ_x . To do so, we have to integrate over all values of $\theta_1, \dots, \theta_{N-2}$ and for $0 \leq \varphi \leq \Phi(c)$. We multiply by the spherical part of the Jacobian determinant for spherical coordinates and integrate both sides. The integrand on the left side is constant, being independent of y , and gives a *positive* multiple of $u(x)$. Dividing by that multiple, we get

$$u(x) = c_1 \int_{\Gamma_x - x} |y|^{k-N} \frac{\partial^k}{\partial r^k} \left[g\left(\frac{|y|}{h}\right)u(x+y) \right] dy.$$

Suppose temporarily that $p > 1$. With p' still denoting the index dual to p , application of Hölder's inequality gives

$$|u(x)| \leq c_1 \left(\int_{\Gamma_{x-x}} |y|^{(k-N)p'} dy \right)^{1/p'} \left(\int_{\Gamma_{x-x}} \left| \frac{\partial^k}{\partial r^k} \left[g\left(\frac{|y|}{h}\right) u(x+y) \right] \right|^p dy \right)^{1/p}.$$

The first integral on the right side is the critical one. The radius extends from 0 to h , and the integral is finite if and only if $(k - N)p' > -N > 0$, i.e., $k > N - N/p' = N/p$. This is the condition in the theorem.

The differentiation $\frac{\partial^k}{\partial r^k}$ in the second factor on the right can be expanded in terms of derivatives in Cartesian coordinates, and then the integration can be extended over all of U . The result is that the second factor is dominated by a multiple of $\|u\|_{L_k^p}$. This completes the proof when $p > 1$.

Now suppose that $p = 1$. Then the above result from applying Hölder's inequality is replaced by the inequality

$$|u(x)| \leq c_1 \| |y|^{k-N} \|_{\infty, \Gamma_{x-x}} \int_{\Gamma_{x-x}} \left| \frac{\partial^k}{\partial r^k} \left[g\left(\frac{|y|}{h}\right) u(x+y) \right] \right| dy.$$

The first factor is finite if $k \geq N$, and the second factor is handled as before. This completes the proof if $p = 1$.

Corollary 3.12. Suppose that U is a nonempty bounded open subset of \mathbb{R}^N satisfying the cone condition, and suppose that $1 < p < \infty$ and that m and k are integers ≥ 0 such that $k > m + N/p$. If f is in $L_k^p(U)$, then f can be redefined on a set of measure 0 so as to be in $C^m(U)$.

PROOF. Choose by Theorem 3.9 a sequence $\{f_i\}$ in $C^\infty(U) \cap L_k^p(U)$ such that $\lim f_i = f$ in $L_k^p(U)$. For $|\alpha| \leq m$, we apply Theorem 3.11 to see that

$$\sup_U |D^\alpha f_i - D^\alpha f_j|$$

tends to 0 as i and j tend to infinity. Thus all the $D^\alpha f_i$ converge uniformly. It follows that the uniform-limit function $\tilde{f} = \lim f_i$ is in $C^m(U)$. Since $f_i \rightarrow f$ in $L^p(U)$ and $f_i \rightarrow \tilde{f}$ uniformly, we conclude that $\tilde{f} = f$ almost everywhere. Thus \tilde{f} tells how to redefine f on a set of measure 0 so as to be in $C^m(U)$.

3. Harmonic Functions

Let U be an open set in \mathbb{R}^N . The discussion will not be very interesting for $N = 1$, and we exclude that case. A function u in $C^2(U)$ is **harmonic** in U if $\Delta u = 0$ identically in U . Harmonic functions were introduced already in Chapter I and investigated in connection with certain boundary-value problems. In the present

section we examine properties of harmonic functions more generally. Harmonic functions in a half space, through their boundary values and the Poisson integral formula, become a tool in analysis for working with functions on the Euclidean boundary, and the behavior of harmonic functions on general open sets becomes a prototype for the behavior of solutions of further “elliptic” second-order partial differential equations.

Harmonic functions will be characterized shortly in terms of a certain mean-value property. To get at this characterization and its ramifications, we need the N -dimensional “Divergence Theorem” of Gauss for two special cases—a ball and a half space. The result for a ball will be formulated as in Lemma 3.13 below; we give a proof since this theorem was not treated in *Basic*. The argument for a half space is quite simple, and we will incorporate what we need into the proof of Proposition 3.15 below. For the case of a ball, recall⁶ that the change-of-variables formula $x = r\omega$, with $r \geq 0$ and $|\omega| = 1$, for transforming integrals in Cartesian coordinates for \mathbb{R}^N into spherical coordinates involves substituting $dx = r^{N-1} dr d\omega$, where $d\omega$ is a certain rotation-invariant measure on the unit sphere S^{N-1} that can be expressed in terms of $N - 1$ angular variables. The open ball of radius x_0 and radius r is denoted by $B(r; x_0)$, and its boundary is $\partial B(r; x_0)$.

Lemma 3.13. If F is a C^1 function in an open set on \mathbb{R}^N containing the closed ball $B(r; 0)^{\text{cl}}$ and if $1 \leq j \leq N$, then

$$\int_{x \in B(r; 0)} \frac{\partial F}{\partial x_j}(x_0 + x) dx = \int_{r\omega \in \partial B(r; 0)} x_j F(x_0 + r\omega) r^{N-2} d\omega.$$

REMARKS. The usual formula of the **Divergence Theorem** is $\int_U \operatorname{div} \mathbf{F} dx = \int_{\partial U} (\mathbf{F} \cdot \mathbf{n}) dS$, where U is a suitable bounded open set, $\partial U = U^{\text{cl}} - U$ is its boundary, \mathbf{n} is the outward-pointing unit normal, \mathbf{F} is a vector-valued C^1 function, and dS is surface area. In Lemma 3.13, U is specialized to the ball $B(r; 0)$, dS is the $(N - 1)$ -dimensional area measure $r^{N-1} d\omega$ on the surface $\partial B(r; 0)$ of the ball, \mathbf{F} is taken to be the product of F by the j^{th} standard basis vector e_j , and $e_j \cdot \mathbf{n}$ is $r^{-1}x_j$.

PROOF. Without loss of generality, we may take $j = 1$ and $x_0 = 0$. Write $x = (x_1, x')$, where $x' = (x_2, \dots, x_N)$, and write $\omega = (\omega_1, \omega')$ similarly. The left side in the displayed formula is equal to

$$\begin{aligned} \int_{|x'| \leq r} \int_{x_1 = -\sqrt{r^2 - |x'|^2}}^{\sqrt{r^2 - |x'|^2}} \frac{\partial F}{\partial x_1}(x_1, x') dx_1 dx' \\ = \int_{|x'| \leq r} [F(\sqrt{r^2 - |x'|^2}, x') - F(-\sqrt{r^2 - |x'|^2}, x')] dx'. \end{aligned}$$

⁶From Section VI.5 of *Basic*.

Thus the lemma will follow if it is proved that

$$\int_{|x'| \leq r} F(\sqrt{r^2 - |x'|^2}, x') dx' = \int_{|\omega|=1, \omega_1 \geq 0} x_1 F(r\omega) r^{N-2} d\omega \quad (*)$$

and

$$- \int_{|x'| \leq r} F(-\sqrt{r^2 - |x'|^2}, x') dx' = \int_{|\omega|=1, \omega_1 \leq 0} x_1 F(r\omega) r^{N-2} d\omega. \quad (**)$$

Let us use ordinary spherical coordinates for ω , with

$$\begin{pmatrix} r\omega_1 \\ \vdots \\ r\omega_N \end{pmatrix} = \begin{pmatrix} r \cos \theta_1 \\ r \sin \theta_1 \cos \theta_2 \\ \vdots \\ r \sin \theta_1 \cdots \sin \theta_{N-2} \cos \theta_{N-1} \\ r \sin \theta_1 \cdots \sin \theta_{N-2} \sin \theta_{N-1} \end{pmatrix}$$

and

$$d\omega = \sin^{N-2} \theta_1 \sin^{N-3} \theta_2 \cdots \sin \theta_{N-2} d\theta_1 \cdots d\theta_{N-1}.$$

The right side of (*) is equal to

$$\begin{aligned} & \int_{|\omega|=1, \omega_1 \geq 0} F(r\omega) \omega_1 r^{N-2} d\omega \\ &= \int_{\substack{0 \leq \theta_1 \leq \pi/2, \\ 0 \leq \theta_j \leq \pi \text{ for } 1 < j < N-1, \\ 0 \leq \theta_{N-1} \leq 2\pi}} F(r\omega) r^{N-1} \cos \theta_1 \sin^{N-2} \theta_1 \sin^{N-3} \theta_2 \cdots \sin \theta_{N-2} d\theta_1 \cdots d\theta_{N-1}, \end{aligned}$$

and we show that it equals the left side of (*) by carrying out for the left side of (*) the change of variables $x' \leftrightarrow (\theta_1, \dots, \theta_{N-1})$ given with r constant by

$$x' = \begin{pmatrix} x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} r \sin \theta_1 \cos \theta_2 \\ \vdots \\ r \sin \theta_1 \cdots \sin \theta_{N-2} \cos \theta_{N-1} \\ r \sin \theta_1 \cdots \sin \theta_{N-2} \sin \theta_{N-1} \end{pmatrix}.$$

The Jacobian matrix is the same as for the change to spherical coordinates $(r, \theta_2, \dots, \theta_{N-1})$ except that the first column has a factor $r \cos \theta_1$ instead of 1 and the other columns have an extra factor of $\sin \theta_1$. Consequently

$$dx' = r^{N-1} (|\cos \theta_1| \sin^{N-2} \theta_1) (\sin^{N-3} \theta_2 \cdots \sin \theta_{N-2}) d\theta_1 \cdots d\theta_{N-1}.$$

Therefore the measures match in the two transformed sides, the sets of integration for $(\theta_1, \dots, \theta_{N-1})$ are the same, and the integrands are the same because $\cos \theta_1 = |\cos \theta_1|$. This proves (*). For (**) we make the same computation but the interval of integration for θ_1 is $\pi/2 \leq \theta_1 \leq \pi$. To get a match, the minus sign is necessary because $\cos \theta_1 = -|\cos \theta_1|$.

Proposition 3.14 (Green's formula⁷ for a ball). Let B be an open ball in \mathbb{R}^N , let ∂B be its surface, and let $d\sigma$ be the surface-area measure of ∂B . If u and v are C^2 functions in an open set containing B^{cl} , then

$$\int_B (u \Delta v - v \Delta u) dx = \int_{\partial B} \left(u \frac{\partial v}{\partial \mathbf{n}} - v \frac{\partial u}{\partial \mathbf{n}} \right) d\sigma,$$

where $\mathbf{n} : \partial B \rightarrow \mathbb{R}^N$ is the outward-pointing unit normal vector.

PROOF. Apply Lemma 3.13 to $F = u \frac{\partial v}{\partial x_j}$ and then to $F = v \frac{\partial u}{\partial x_j}$, and subtract the results. Then sum on j .

Let Ω_{N-1} be the surface area $\int_{S^{N-1}} d\omega$ of the unit sphere in \mathbb{R}^N . A continuous function u on an open subset U of \mathbb{R}^N is said to have the **mean-value property** in R if the value of u at each point x in U equals the average value of u over each sphere centered at x and lying in U , i.e., if

$$u(x) = \frac{1}{\Omega_{N-1}} \int_{\omega \in S^{N-1}} u(x + t\omega) d\omega$$

for every x in U and for every positive t less than the distance from x to U^c .

The mean-value property over spheres implies a corresponding average-value property over balls. In fact, the volume $|B(t_0; 0)|$ of the ball $B(t_0; 0)$ is given by $\int_0^{t_0} \int_{S^{N-1}} t^{N-1} d\omega dt = N^{-1} t_0^N \int_{S^{N-1}} d\omega = N^{-1} t_0^N \Omega_{N-1}$. When the mean-value property over spheres is satisfied and t_0 is less than the distance from x to U^c , we can apply the operation $N t_0^{-N} \int_0^{t_0} (\text{---}) dt$ to both sides of the mean-value formula and obtain

$$u(x) = \frac{N t_0^{-N}}{\Omega_{N-1}} \int_0^{t_0} \int_{\omega \in S^{N-1}} u(x + t\omega) t^{N-1} d\omega dt = \frac{1}{|B(t_0; 0)|} \int_{B(t_0; 0)} u(x + y) dy.$$

Proposition 3.15 (Green's formula for a half space). Let R^+ be the subset of $\mathbb{R}^N = \{(x', x_n) \mid x' \in \mathbb{R}^{N-1} \text{ and } x_n \in \mathbb{R}\}$ where $x_n > 0$. Denote its boundary by $\partial R^+ = \mathbb{R}^{N-1}$, and suppose that u and v are C^2 functions on an open subset of \mathbb{R}^{N-1} containing $(R^+)^{\text{cl}}$ and that at least one of u and v is compactly supported. Then

$$\int_{x \in R^+} (u \Delta v - v \Delta u) dx = \int_{x' \in \mathbb{R}^{N-1}} \left(v \frac{\partial u}{\partial x_n} - u \frac{\partial v}{\partial x_n} \right) dx'.$$

PROOF. Suppose F is a C^1 function compactly supported on an open subset of \mathbb{R}^{N-1} containing $(R^+)^{\text{cl}}$. If $1 \leq j \leq N-1$, then $\int_{R^+} \frac{\partial F}{\partial x_j} dx = 0$ since the integral

⁷This formula is related to but distinct from the formula with the same name at the beginning of Section I.3.

with respect to dx_j is the difference between two values of F and since these are 0 by the compactness of the support. For $j = N$, however, one of the boundary terms may fail to be 0, and the result is that $\int_{\mathbb{R}^+} \frac{\partial F}{\partial x_N} dx = - \int_{\mathbb{R}^{N-1}} F(x') dx'$.

Apply the j^{th} of these formulas first to $F = u \frac{\partial v}{\partial x_j}$ and then to $F = v \frac{\partial u}{\partial x_j}$, sum the results on j , and subtract the two sums. The result is the formula of the proposition.

Theorem 3.16. Let U be an open set in \mathbb{R}^N , and let u be a continuous scalar-valued function on U . If u is harmonic on U , then u has the mean-value property on U . Conversely if u has the mean-value property on U , then u is in $C^\infty(U)$ and is harmonic on U .

PROOF. Suppose that u is harmonic on U . We prove that u has the mean-value property. It is enough to treat $x = 0$. Green's formula, as in Proposition 3.14, directly extends from balls to the difference of two balls.⁸ Thus we have

$$\int_E (u \Delta v - v \Delta u) dx = \int_{\partial E} \left(u \frac{\partial v}{\partial \mathbf{n}} - v \frac{\partial u}{\partial \mathbf{n}} \right) d\sigma \quad (*)$$

whenever E is a closed ball B_t of radius t contained in U or is the difference $B_t - (B_\epsilon)^o$ of two concentric balls with $\epsilon < t$. Taking $E = B_t$ and $v = 1$ in (*), we obtain

$$\int_{\partial B_t} \frac{\partial u}{\partial \mathbf{n}} d\sigma = 0. \quad (**)$$

Routine computation shows that the function given by

$$v(x) = \begin{cases} |x|^{-(N-2)} & \text{for } N > 2, \\ \log |x| & \text{for } N = 2, \end{cases}$$

is harmonic for $x \neq 0$ and has $\frac{\partial v}{\partial r}$ equal to a nonzero multiple of $|x|^{-(N-1)}$, r being the spherical coordinate radius $|x|$. If we apply (*) to this v and our harmonic u when $E = B_t - (B_\epsilon)^o$, we obtain

$$\int_{\partial(B_t - (B_\epsilon)^o)} \left(u \frac{\partial v}{\partial \mathbf{n}} - v \frac{\partial u}{\partial \mathbf{n}} \right) d\sigma = 0.$$

Since v depends only on $|x|$, (**) shows that the second term of the integrand yields 0. Thus this formula becomes

$$\int_{\partial(B_t - (B_\epsilon)^o)} u \frac{\partial v}{\partial \mathbf{n}} d\sigma = 0.$$

⁸For the extended result, suppose that the balls have radii $r_1 < r_2$. Then u and v are defined from radius $r_1 - \epsilon$ to $r_2 + \epsilon$ for some $\epsilon > 0$. We can adjust u and v by multiplying by a suitable smooth function that is identically 1 for radius $\geq r_1 - \frac{1}{3}\epsilon$ and identically 0 for radius $\leq r_1 - \frac{2}{3}\epsilon$, and then u and v will extend as smooth functions for radius $< r_2 + \epsilon$. Consequently Proposition 3.14 will apply on each ball to the adjusted functions, and subtraction of the results gives the desired version of Green's formula.

The normal vector for the inner sphere points toward the center. Hence we can rewrite our equality as

$$\int_{|x|=\epsilon} u \frac{\partial v}{\partial r} d\sigma = \int_{|x|=t} u \frac{\partial v}{\partial r} d\sigma.$$

Since $\frac{\partial v}{\partial r} = c|x|^{-(N-1)}$ with $c \neq 0$, we obtain

$$\epsilon^{-(N-1)} \int_{|x|=\epsilon} u d\sigma = t^{-(N-1)} \int_{|x|=t} u d\sigma.$$

On the left side, $d\sigma = \epsilon^{N-1} d\omega$, while on the right side, $d\sigma = t^{N-1} d\omega$. Therefore

$$\int_{|\omega|=1} u(\epsilon\omega) d\omega = \int_{|\omega|=1} u(t\omega) d\omega$$

whenever $0 < \epsilon < t$ and B_t is contained in U . Dividing by Ω_{N-1} , letting ϵ decrease to 0, and using the continuity of u , we see that $u(0) = \int_{\omega \in S^{N-1}} u(t\omega) d\omega$. Thus u has the mean-value property.

For the converse direction suppose initially that u is in $C^2(U)$. Define

$$m_t(u)(x) = \Omega_{N-1}^{-1} \int_{|\omega|=1} u(x + t\omega) d\omega$$

whenever x is in U and t is a positive number less than the distance of x to U^c . With x fixed, the function $m_t(u)(x)$ has two continuous derivatives. We shall show that

$$\frac{d^2}{dt^2} m_t(u)(x) \Big|_{t=0} = N^{-1} \Delta u(x), \quad (\dagger)$$

the derivatives being understood to be one-sided derivatives as t decreases to 0. If u is assumed to have the mean-value property, $m_t(u)(x)$ is constant in t , and we can conclude from (\dagger) that $\Delta u(x) = 0$. The computation of $\frac{d^2}{dt^2} m_t(u)(x)$ is

$$\begin{aligned} m_t(u)(x) &= \Omega_{N-1}^{-1} \int_{|\omega|=1} u(x_1 + t\omega_1, \dots, x_N + t\omega_N) d\omega, \\ \frac{d}{dt} m_t(u)(x) &= \Omega_{N-1}^{-1} \int_{|\omega|=1} \sum_{j=1}^N \omega_j D_j u(x + t\omega) d\omega, \\ \frac{d^2}{dt^2} m_t(u)(x) &= \Omega_{N-1}^{-1} \int_{|\omega|=1} \sum_{j,k=1}^N \omega_j \omega_k D_j D_k u(x + t\omega) d\omega. \end{aligned}$$

Letting t decrease to 0, we obtain

$$\frac{d^2}{dt^2} m_t(u)(x) \Big|_{t=0} = \Omega_{N-1}^{-1} \sum_{j,k=1}^N D_j D_k u(x) \int_{|\omega|=1} \omega_j \omega_k d\omega.$$

If $j \neq k$, then $\int_{|\omega|=1} \omega_j \omega_k d\omega = 0$ since the integrand is an odd function of the j^{th} variable taken over a set symmetric about 0. The integral $\int_{|\omega|=1} \omega_j^2 d\omega$ is

independent of j and has the property that N times it is equal to $\int_{|\omega|=1} |\omega|^2 d\omega = \int_{|\omega|=1} d\omega = \Omega_{N-1}$. Thus $\int_{|\omega|=1} \omega_j^2 d\omega = N^{-1} \Omega_{N-1}$, and

$$\frac{d^2}{dt^2} m_t(u)(x) \Big|_{t=0} = N^{-1} \sum_{j=1}^N D_j^2 u(x) = N^{-1} \Delta u(x).$$

This proves (†) and completes the argument that a C^2 function in U with the mean-value property is harmonic.

Finally suppose that u has the mean-value property and is assumed to be merely continuous. Proposition 3.5e allows us to choose a function $\varphi \geq 0$ in $C_{\text{com}}^\infty(\mathbb{R}^N)$ with $\varphi(x) = \varphi_0(|x|)$, $\int_{\mathbb{R}^N} \varphi(x) dx = 1$, and $\varphi(x) = 0$ for $|x| \geq 1$. Put $\varphi_\varepsilon(x) = \varepsilon^{-N} \varphi(\varepsilon^{-1}x)$, and define $u_\varepsilon(x) = \int_{\mathbb{R}^N} u(x-y) \varphi_\varepsilon(y) dy$ in the open set $U_\varepsilon = \{x \in U \mid D(x, U^c) > \varepsilon\}$. Proposition 3.5c shows that u_ε is in $C^\infty(U_\varepsilon)$, and the mean-value property of u , in combination with the radial nature of φ_ε as expressed by the equality $\varphi_\varepsilon(t\omega) = \varphi_\varepsilon(te_1)$, forces $u_\varepsilon(x) = u(x)$ for all x in U_ε :

$$\begin{aligned} u_\varepsilon(x) &= \int_{t=0}^\varepsilon \int_{|\omega|=1} u(x-t\omega) \varphi_\varepsilon(t\omega) t^{N-1} d\omega dt \\ &= \int_{t=0}^\varepsilon \Omega_{N-1} u(x) \varphi_\varepsilon(te_1) t^{N-1} dt \\ &= u(x) \int_{\mathbb{R}^N} \varphi_\varepsilon(y) dy = u(x). \end{aligned}$$

Since ε is arbitrary, u is in $C^\infty(U)$. The function u has now been shown to be in $C^2(U)$, and it is assumed to have the mean-value property. Therefore the previous case shows that it is harmonic.

Corollary 3.17. If u is harmonic on an open subset U of \mathbb{R}^N , then u is in $C^\infty(U)$.

PROOF. This follows by using both directions of Theorem 3.16.

A sequence of functions $\{u_n\}$ on a locally compact Hausdorff space X is said to converge **uniformly on compact subsets** of X if $\lim u_n = u$ pointwise on X and if for each compact subset K of X , the convergence is uniform on K . For example the sequence $\{x^n\}$ converges to the 0 function on $(0, 1)$ uniformly on compact subsets.

Corollary 3.18. If $\{u_n\}$ is a sequence of harmonic functions on an open subset U of \mathbb{R}^N and if $\{u_n\}$ converges uniformly on compact subsets to u , then u is harmonic on U .

PROOF. About any point of U is a compact neighborhood lying in U , and the convergence is uniform on that neighborhood. Therefore u is continuous. Each integration needed for the mean-value property occurs on a compact subset

of U , and the uniform convergence allows us to interchange limit and integral. Therefore the mean-value property for each u_n , valid because of one direction of Theorem 3.16, implies the mean-value property for u . Hence u is harmonic by the converse direction of Theorem 3.16.

Suppose that U is open in \mathbb{R}^N and that u is harmonic on U . If B is an open ball in U , then $\int_U u \Delta \psi \, dx = 0$ for all $\psi \in C_{\text{com}}^\infty(B)$ by Green's formula (Proposition 3.14), since ψ and $\frac{\partial \psi}{\partial \mathbf{n}}$ are both identically 0 on the boundary of B . We shall use a smooth partition of unity to show that $\int_U u \Delta \psi \, dx$ is therefore 0 for all $\psi \in C_{\text{com}}^\infty(U)$. Corollary 3.19 below provides a converse; we shall use the converse in a crucial way in Corollary 3.23 below.

The argument to construct the partition of unity goes as follows. To each point of $K = \text{support}(\psi)$, we can associate an open ball centered at that point whose closure is contained in U . As the point varies, these open balls cover K , and we extract a finite subcover $\{U_1, \dots, U_k\}$. Lemma 3.15b of *Basic* constructs an open cover $\{W_1, \dots, W_k\}$ of K such that W_i^{cl} is a compact subset of U_i for each i . Now we argue as in the proof of Proposition 3.14 of *Basic*. A second application of Lemma 3.15b of *Basic* gives an open cover $\{V_1, \dots, V_k\}$ of K such that V_i^{cl} is compact and $V_i^{\text{cl}} \subseteq W_i$ for each i . Proposition 3.5f constructs a smooth function $g_i \geq 0$ that is 1 on V_i^{cl} and is 0 off W_i . Then $g = \sum_{i=1}^k g_i$ is smooth and ≥ 0 on \mathbb{R}^N and is > 0 everywhere on K . A second application of Proposition 3.5f produces a smooth function $h \geq 0$ on \mathbb{R}^N that is 1 on the set where g is 0 and is 0 on K . Then $g+h$ is everywhere positive on \mathbb{R}^N , and the functions $\varphi_i = g_i/(g+h)$ form the smooth partition of unity that we shall use.

To apply the partition of unity, we write $\psi = \sum_i \varphi_i \psi$. Then each term $\varphi_i \psi$ is smooth and compactly supported in an open ball whose closure is contained in U . Consequently we have $\int_U u \Delta(\varphi_i \psi) \, dx = 0$ for each i . Summing on i , we obtain $\int_U u \Delta \psi \, dx = 0$, which was what was being asserted.

Corollary 3.19. Suppose that U is open in \mathbb{R}^N , that u is continuous on U , and that $\int_U u \Delta \psi \, dx = 0$ for all $\psi \in C_{\text{com}}^\infty(U)$. Then u is harmonic on U .

PROOF. Let B be an open ball of radius r with closure contained in U , fix $\varepsilon > 0$ so as to be $< r$, and let B_ε be the open ball of radius $r - \varepsilon$ with the same center as B . Construct φ_ε as in the proof of Theorem 3.16, and let $u_\varepsilon = u * \varphi_\varepsilon$. Suppose that ψ is in $C_{\text{com}}^\infty(B_\varepsilon)$. For t and x in \mathbb{R}^N with $|t| \leq \varepsilon$, define $\psi_t(x) = \psi(t+x)$. Since ψ is supported in B_ε , ψ_t is supported in B , and therefore

$$\int_B u(x-t) \Delta \psi_t(x) \, dx = \int_B u(x) \Delta \psi(x+t) \, dx = \int_B u \Delta \psi_t \, dx = 0,$$

the last equality holding by the hypothesis. Multiplying by $\varphi_\varepsilon(t)$, integrating for $|t| \leq \varepsilon$, and interchanging integrals, we obtain

$$0 = \int_B \int_{\mathbb{R}^N} u(x-t) \varphi_\varepsilon(t) \Delta \psi_t(x) \, dt \, dx = \int_B u_\varepsilon(x) \Delta \psi(x) \, dx.$$

Since ψ vanishes identically near the boundary of B , this identity and Green's formula (Proposition 3.14) together yield $\int_B \psi(x) \Delta u_\varepsilon(x) dx = 0$ for all ψ in $C_{\text{com}}^\infty(B_\varepsilon)$. Application of Corollary 3.6a allows us to extend this conclusion to all ψ in $C_{\text{com}}(B_\varepsilon)$, and then the uniqueness in the Riesz Representation Theorem shows that we must have $\Delta u_\varepsilon(x) = 0$ for all x in B_ε . As ε decreases to 0, u_ε tends to u uniformly on compact sets. By Corollary 3.18, u is harmonic in B . Since the ball B is arbitrary in U , u is harmonic in U .

Corollary 3.20. Let U be a connected open set in \mathbb{R}^N . If u is harmonic in U and $|u|$ attains a maximum somewhere in U , then u is constant in U .

PROOF. Suppose that $|u|$ attains a maximum at x_0 . Multiplying u by a suitable constant $e^{i\theta}$, we may assume that $u(x_0) = M > 0$. The subset E of U where $u(x)$ equals M is closed and nonempty. It is enough to prove that E is open. Let x_1 be in E , and choose an open ball B centered at x_1 , say of some radius $r > 0$, that lies in U . We show that B lies in E . For $0 < t < r$, Theorem 3.16 says that u has the mean-value property

$$\Omega_{N-1}^{-1} \int_{S^{N-1}} u(x_1 + t\omega) d\omega = u(x_1) = M.$$

Arguing by contradiction, suppose that $u(x_1 + t_0\omega_0) \neq u(x_1)$ for some $t_0\omega_0$ with $0 < t_0 < r$. Then $\text{Re } u(x_1 + t_0\omega_0) < M - \epsilon$ for some $\epsilon > 0$, and continuity produces a nonempty open set S in the sphere S^{N-1} such that $\text{Re } u(x_1 + t_0\omega) < M - \epsilon$ for ω in S . If σ is the name of the measure on S^{N-1} , then we have

$$\begin{aligned} M\Omega_{N-1} &= \text{Re} \left(\int_{S^{N-1}} u(x_1 + t\omega) d\omega \right) \\ &= \int_S \text{Re } u(x_1 + t\omega) d\omega + \int_{S^{N-1}-S} \text{Re } u(x_1 + t\omega) d\omega \\ &\leq \int_S (M - \epsilon) d\omega + \int_{S^{N-1}-S} M d\omega \\ &= (M - \epsilon)\sigma(S) + M\sigma(S^{N-1} - S) \\ &= M\Omega_{N-1} - \epsilon\sigma(S), \end{aligned}$$

and we have arrived at a contradiction since $\sigma(S) > 0$.

Corollary 3.21. Let U be a bounded open subset of \mathbb{R}^N , and let ∂U be its boundary. If u is harmonic in U and u is continuous on U^{cl} , then $\sup_{x \in U} |u(x)| = \max_{x \in \partial U} |u(x)|$.

PROOF. Since u is continuous and U^{cl} is compact, $|u|$ assumes its maximum M somewhere on U^{cl} . If $|u(x_0)| = M$ for some x_0 in U , then Corollary 3.20 shows that u is constant on the component of U to which x_0 belongs. The closure of that component cannot equal that component since \mathbb{R}^N is connected. Thus the closure of that component contains a point of ∂U , and $|u|$ must equal M at that point of ∂U . Consequently $\sup_{x \in U} |u(x)| \leq \max_{x \in \partial U} |u(x)|$. Since every point of ∂U is the limit of a sequence of points in U , the reverse inequality is valid as well, and the corollary follows.

Corollary 3.22 (Liouville). Any bounded harmonic function on \mathbb{R}^N is constant.

REMARKS. The best-known result of Liouville of this kind is one from complex analysis—that a bounded function analytic on all of \mathbb{C} is constant. This complex-analysis result is actually a consequence of Corollary 3.22 because the real and imaginary parts of a bounded analytic function on \mathbb{C} are bounded harmonic functions on \mathbb{R}^2 .

PROOF. Suppose that u is harmonic on \mathbb{R}^N with $|u(x)| \leq M$. Let x_1 and x_2 be distinct points of \mathbb{R}^N , and let $R > 0$. Since u has the mean-value property over spheres by Theorem 3.16, u equals its average value over balls. Hence $u(x_1) = |B(R; 0)|^{-1} \int_{B(R; x_1)} u(x) dx$ and $u(x_2) = |B(R; 0)|^{-1} \int_{B(R; x_2)} u(x) dx$. Subtraction gives

$$\begin{aligned} u(x_1) - u(x_2) &= |B(R; 0)|^{-1} \left(\int_{B(R; x_1)} u(x) dx - \int_{B(R; x_2)} u(x) dx \right) \\ &= |B(R; 0)|^{-1} \left(\int_{B(R; x_1) - B(R; x_2)} u(x) dx - \int_{B(R; x_2) - B(R; x_1)} u(x) dx \right). \end{aligned}$$

Therefore

$$|u(x_1) - u(x_2)| \leq |B(R; 0)|^{-1} \int_{B(R; x_1) \Delta B(R; x_2)} |u(x)| dx,$$

where $B(R; x_1) \Delta B(R; x_2)$ is the symmetric difference $(B(R; x_1) - B(R; x_2)) \cup (B(R; x_2) - B(R; x_1))$. Hence

$$|u(x_1) - u(x_2)| \leq \frac{M |B(R; x_1) \Delta B(R; x_2)|}{|B(R; 0)|} = \frac{MR^N |B(1; x_1/R) \Delta B(1; x_2/R)|}{R^N |B(1; 0)|}.$$

The right side is $|B(1; x_1/R) \Delta B(1; x_2/R)|$, apart from a constant factor, and the sets $B(1; x_1/R) \Delta B(1; x_2/R)$ decrease and have empty intersection as R tends to infinity. By complete additivity of Lebesgue measure, the measure of the symmetric difference tends to 0. We conclude that $u(x_1) = u(x_2)$. Therefore u is constant.

In the final two corollaries let \mathbb{R}_+^{N+1} be the open half space of points (x, t) in \mathbb{R}^{N+1} such that x is in \mathbb{R}^N and $t > 0$.

Corollary 3.23 (Schwarz Reflection Principle). Suppose that $u(x, t)$ is harmonic in \mathbb{R}_+^{N+1} , that u is continuous on $(\mathbb{R}_+^{N+1})^{\text{cl}}$, and that $u(x, 0) = 0$ for all x . Then the definition $u(x, -t) = -u(x, t)$ for $t > 0$ extends u to a harmonic function on all of \mathbb{R}^{N+1} .

PROOF. Define

$$w(x, t) = \begin{cases} u(x, t) & \text{for } t \geq 0, \\ -u(x, -t) & \text{for } t \leq 0. \end{cases}$$

The function w is continuous. We shall show that $\int_{\mathbb{R}^N} w \Delta \psi \, dx = 0$ for all $\psi \in C_{\text{com}}^\infty(\mathbb{R}^{N+1})$, and then Corollary 3.19 shows that w is harmonic. Write ψ as the sum of functions even and odd in the variable t . Since w is odd in t , the contribution to $\int_{\mathbb{R}^N} w \Delta \psi \, dx$ from the even part of ψ is 0. We may thus assume that ψ is odd in t .

For $\varepsilon > 0$, let $R_\varepsilon = \{(x, t) \mid t > \varepsilon\}$. It is enough to show that $\int_{R_\varepsilon} u \Delta \psi \, dx \, dt$ has limit 0 as ε decreases to 0 since $\int_{\mathbb{R}^{N+1}} w \Delta \psi \, dx \, dt$ is twice this limit. We apply Green's formula for a half space (Proposition 3.15) with $v = \psi$ on the set $R_\varepsilon \subseteq \mathbb{R}^{N+1}$ except for one detail: to get the hypothesis of compact support to be satisfied, we temporarily multiply ψ by a smooth function that is identically 1 for $t \geq \varepsilon$ and is identically 0 for $t \leq \frac{1}{2}\varepsilon$. Since u is harmonic in R_ε , the result is that

$$-\int_{R_\varepsilon} u \Delta \psi \, dx \, dt = \int_{R_\varepsilon} (\psi \Delta u - u \Delta \psi) \, dx \, dt = \int_{\{(x,t) \mid t=\varepsilon\}} \left(u \frac{\partial \psi}{\partial t} - \psi \frac{\partial u}{\partial t} \right) dx.$$

On the right side, $\lim_{\varepsilon \downarrow 0} \int_{\{(x,t) \mid t=\varepsilon\}} u \frac{\partial \psi}{\partial t} \, dx = 0$ since $u(\cdot, \varepsilon)$ tends uniformly to 0 on the relevant compact set of x 's in \mathbb{R}^N .

Thus it is enough to prove that $\lim_{\varepsilon \downarrow 0} \int_{\{(x,t) \mid t=\varepsilon\}} \psi \frac{\partial u}{\partial t} \, dx = 0$. Since $\psi(x, t)$ is of class C^2 , is odd in x , and is compactly supported, we have $|\psi(x, t)| \leq Ct$ uniformly in x for small positive t . Thus it is enough to prove that

$$\lim_{t \downarrow 0} \left| t \frac{\partial u}{\partial t}(x, t) \right| = 0 \quad (*)$$

uniformly on compact subsets of \mathbb{R}^N .

To prove (*), let φ be a function as in Proposition 3.5e, and let $\varphi_\varepsilon(x, t) = \varepsilon^{-(N+1)} \varphi(\varepsilon^{-1}(x, t))$. Fix x_0 in \mathbb{R}^N , and define $X_0 = (x_0, t_0)$ and $X = (x, t)$. If $|X - X_0| < \frac{1}{3}t_0$, then the mean-value property of u in \mathbb{R}_+^{N+1} gives $u(X) = (u * \varphi_{\frac{1}{3}t_0})(X)$. Hence we have

$$\begin{aligned} \frac{\partial u}{\partial t}(X) &= \frac{\partial}{\partial t} \int_{\mathbb{R}^{N+1}} \varphi_{\frac{1}{3}t_0}(X - Y) u(Y) \, dY \\ &= \int_{\mathbb{R}^{N+1}} \frac{\partial}{\partial t} \left[\left(\frac{1}{3}t_0\right)^{-(N+1)} \varphi\left(\left(\frac{1}{3}t_0\right)^{-1}(X - Y)\right) \right] u(Y) \, dY. \end{aligned}$$

In the computation of the partial derivative on the right side, the variable t appears as the last coordinate of X . Therefore this expression is equal to

$$\left(\frac{1}{3}t_0\right)^{-1} \int_{\mathbb{R}^{N+1}} \left(\frac{1}{3}t_0\right)^{-(N+1)} \frac{\partial \varphi}{\partial t} \left(\left(\frac{1}{3}t_0\right)^{-1}(X - Y) \right) u(Y) \, dY.$$

Changing variables in the integration by a dilation in Y shows that this expression is equal also to

$$\left(\frac{1}{3}t_0\right)^{-1} \int_{\mathbb{R}^{N+1}} \frac{\partial \varphi}{\partial t} \left(\left(\frac{1}{3}t_0\right)^{-1} X - Y\right) u\left(\frac{1}{3}t_0 Y\right) dY.$$

If we write $Y = (y, s)$ and take absolute values, we obtain

$$\left| \frac{\partial u}{\partial t}(x_0, t) \right| \leq 3t_0^{-1} \left\| \frac{\partial \varphi}{\partial t} \right\|_1 \sup_{\substack{|s-t_0| < 2t_0/3, \\ Y \text{ near } X_0}} |u(Y)|.$$

The required behavior of $t \frac{\partial u}{\partial t}$ follows from this estimate.

Corollary 3.24. Suppose that $u(x, t)$ is harmonic in \mathbb{R}_+^{N+1} , that u is continuous on $(\mathbb{R}_+^{N+1})^{\text{cl}}$, and that $u(x, 0) = 0$ for all x . If u is bounded, then u is identically 0.

REMARK. Without the assumption of boundedness, the function $u(x, t) = t$ is a counterexample.

PROOF. Corollary 3.23 shows that u extends to a bounded harmonic function on all of \mathbb{R}^{N+1} , and Corollary 3.22 shows that the extended function is constant, hence identically 0.

4. \mathcal{H}^p Theory

As was said at the beginning of Section 3, harmonic functions in a half space, through their boundary values and the Poisson integral formula, become a tool in analysis for working with functions on the Euclidean boundary. The Poisson integral formula, which was introduced in Chapters VIII and IX of *Basic*, generates harmonic functions from boundary values.

The details are as follows. Let \mathbb{R}_+^{N+1} be the open half space of pairs (x, t) in \mathbb{R}^{N+1} with $x \in \mathbb{R}^N$ and with $t > 0$ in \mathbb{R}^1 . We view the boundary $\{(x, 0) \mid x \in \mathbb{R}^N\}$ as \mathbb{R}^N . The function

$$P(x, t) = P_t(x) = \frac{c_N t}{(t^2 + |x|^2)^{\frac{1}{2}(N+1)}},$$

for $t > 0$, with $c_N = \pi^{-\frac{1}{2}(N+1)} \Gamma\left(\frac{N+1}{2}\right)$, is called the **Poisson kernel** for \mathbb{R}_+^{N+1} . The **Poisson integral formula** for \mathbb{R}_+^{N+1} is $u(x, t) = (P_t * f)(x)$, where f is any given function in $L^p(\mathbb{R}^N)$ and $1 \leq p \leq \infty$, and the function u is called the **Poisson integral** of f .

If f is in L^p , then u is harmonic on \mathbb{R}_+^{N+1} , $u(\cdot, t)$ is in L^p for each $t > 0$, and $\|u(\cdot, t)\|_p \leq \|f\|_p$. For $1 \leq p < \infty$, $\lim_{t \downarrow 0} u(\cdot, t) = f$ in the norm topology of L^p , while for $p = \infty$, $\lim_{t \downarrow 0} u(\cdot, t) = f$ in the weak-star topology of L^∞ against L^1 . In both cases, $\lim_{t \downarrow 0} \|u(\cdot, t)\|_p = \|f\|_p$, and $\lim_{t \downarrow 0} u(x, t) = f(x)$ a.e.; this latter result is known as **Fatou's Theorem**. When $p = \infty$, the a.e. convergence occurs at any point where f is continuous, and the pointwise convergence is uniform on any subset of \mathbb{R}^N where f is uniformly continuous.

The L^p theory for $p = 1$ extends from integrable functions to the Banach space $M(\mathbb{R}^N)$ of finite complex Borel measures. Specifically if ν is a finite complex Borel measure on \mathbb{R}^N , then the Poisson integral of ν is defined to be the function $u(x, t) = (P_t * \mu)(x) = \int_{\mathbb{R}^N} P_t(x - y) d\nu(y)$. Then u is harmonic on \mathbb{R}_+^{N+1} , $\|u(\cdot, t)\|_1 \leq \|\nu\|$ for each $t > 0$, $\lim_{t \downarrow 0} u(\cdot, t) = \nu$ in the weak-star topology of $M(\mathbb{R}^N)$ against $C_{\text{com}}(\mathbb{R}^N)$, and $\lim_{t \downarrow 0} \|u(\cdot, t)\|_1 = \|\mu\|$.

The new topic for this section is a converse to the above considerations. For $1 \leq p \leq \infty$, we define $\mathcal{H}^p(\mathbb{R}_+^{N+1})$ to be the vector space of functions $u(x, t)$ on \mathbb{R}_+^{N+1} such that

- (i) $u(x, t)$ is harmonic on \mathbb{R}_+^{N+1} ,
- (ii) $\sup_{t>0} \|u(\cdot, t)\|_p < \infty$.

With $\|u\|_{\mathcal{H}^p}$ defined as $\sup_{t>0} \|u(\cdot, t)\|_p$, the vector space $\mathcal{H}^p(\mathbb{R}_+^{N+1})$ is a normed linear space. If f is in $L^p(\mathbb{R}^N)$, then the facts about the Poisson integral formula show that the Poisson integral of f is in $\mathcal{H}^p(\mathbb{R}_+^{N+1})$ and its $\mathcal{H}^p(\mathbb{R}_+^{N+1})$ norm matches the $L^p(\mathbb{R}^N)$ norm of f . For $p = 1$, we readily produce further examples. Specifically if ν is any member of $M(\mathbb{R}^N)$, then the Poisson integral of ν is in $\mathcal{H}^1(\mathbb{R}_+^{N+1})$, with the $\mathcal{H}^1(\mathbb{R}_+^{N+1})$ norm matching the $M(\mathbb{R}^N)$ norm. The theorem of this section will say that there are no other examples.

The members of $\mathcal{H}^\infty(\mathbb{R}_+^{N+1})$ are exactly the bounded harmonic functions in the half space \mathbb{R}_+^{N+1} , and the tool for obtaining an L^∞ function on \mathbb{R}^N from this harmonic function is the preliminary form of Alaoglu's Theorem proved in *Basic*:⁹ any norm-bounded sequence in the dual of a separable normed linear space has a weak-star convergent subsequence.¹⁰ We shall use Corollary 3.24 to see that the harmonic function has to be the Poisson integral of this L^∞ function.

Theorem 3.25. If $1 < p \leq \infty$, then any harmonic function in $\mathcal{H}^p(\mathbb{R}_+^{N+1})$ is the Poisson integral of a function in $L^p(\mathbb{R}^N)$. For $p = 1$, any harmonic function in $\mathcal{H}^1(\mathbb{R}_+^{N+1})$ is the Poisson integral of a finite complex measure in $M(\mathbb{R}^N)$.

PROOF. We begin by proving that $u(x, t)$ is bounded for $t \geq t_0$. For this step we may assume that $p < \infty$. Theorem 3.16 shows that u has the mean-value

⁹Theorem 5.58 of *Basic*.

¹⁰The full-fledged version of Alaoglu's Theorem will be stated and proved in Chapter IV.

property. We know as a consequence that if B denotes the ball with center (x, t) and radius $\frac{1}{2}t_0$, then the value of u at (x, t) equals the average value over B :

$$u(x, t) = \frac{1}{|B|} \int_B u(y, s) dy ds.$$

Since the measure $|B|^{-1} dy ds$ on B has total mass 1, Hölder's inequality gives

$$\begin{aligned} |u(x, t)|^p &\leq \frac{1}{|B|} \int_B |u(y, s)|^p dy ds \\ &\leq \frac{1}{|B|} \int_{|s-t| \leq \frac{1}{2}t_0} \int_{y \in \mathbb{R}^N} |u(y, s)|^p dy ds \\ &\leq [(\frac{1}{2}t_0)^{N+1} \Omega_N]^{-1} (N+1)t_0 \|u\|_{\mathcal{H}^p}^p, \end{aligned}$$

and the boundedness is proved.

For each positive integer k , define $f_k(x) = u(x, 1/k)$ and $w(x, t) = (P_t * f_k)(x)$. Then the function $w_k(x, t) - u(x, t + 1/k)$ is

- (i) harmonic in (x, t) for $t > 0$ since w_k and any translate of u are harmonic,
- (ii) bounded as a function of (x, t) for $t \geq 0$ since $u(x, t + 1/k)$ is bounded for $t \geq 0$, according to the previous paragraph, and since w_k is the Poisson integral of the bounded function f_k ,
- (iii) continuous in (x, t) for $t \geq 0$ since $u(x, t + 1/k)$ and $w_k(x, t)$ both have this property, the latter because f_k is continuous and bounded.

By Corollary 3.24, $w_k(x, t) - u(x, t + 1/k) = 0$. That is,

$$u(x, t + 1/k) = \int_{\mathbb{R}^N} P_t(x - y) f_k(y) dy.$$

Now suppose $p > 1$, so that L^p is the dual space to $L^{p'}$ if $p^{-1} + p'^{-1} = 1$. Since u is in \mathcal{H}^p , $\|f_k\|_p \leq M$ for the constant $M = \|u\|_{\mathcal{H}^p}$. By the preliminary form of Alaoglu's Theorem, there exists a subsequence $\{f_{k_j}\}$ of $\{f_k\}$ that is weak-star convergent to some function f in $L^{p'}$. Since for each fixed t , P_t is in $L^1 \cap L^\infty$ and hence is in $L^{p'}$, each (x, t) has the property that

$$u(x, t + 1/k_j) = \int_{\mathbb{R}^N} P_t(x - y) f_{k_j}(y) dy \rightarrow \int_{\mathbb{R}^N} P_t(x - y) f(y) dy.$$

But $u(x, t + 1/k_j) \rightarrow u(x, t)$ by continuity of u . We conclude that $u(x, t) = \int_{\mathbb{R}^N} P_t(x - y) f(y) dy$.

This proves the theorem for $p > 1$. If $p = 1$, the above argument falls short of constructing a function f in L^1 since L^1 is not the dual of L^∞ . Instead, we treat f_k as a complex measure $f_k(x) dx$. The norm of $f_k(x) dx$ in $M(\mathbb{R}^N)$ equals $\|f_k\|_1$, and thus the norms of the complex measures $f_k(x) dx$ are bounded. The space $M(\mathbb{R}^N)$ is the dual of $C_{\text{com}}(\mathbb{R}^N)$ and hence also of its uniform closure, which is the Banach space $C_0(\mathbb{R}^N)$ of continuous functions on \mathbb{R}^N vanishing at infinity. Let $\{f_{k_j}(x) dx\}$ be a weak-star convergent subsequence of $\{f_k(x) dx\}$, with limit ν in $M(\mathbb{R}^N)$. Since each function $y \mapsto P_t(x - y)$ is in $C_0(\mathbb{R}^N)$, we have $\lim_k \int_{\mathbb{R}^N} P_t(x - y) f_{k_j}(y) dy = \int_{\mathbb{R}^N} P_t(x - y) d\nu(y)$. This completes the proof.

For $N = 1$, every analytic function in the upper half plane \mathbb{R}_+^2 is automatically harmonic, and one can ask for a characterization of the subspace of analytic members of $\mathcal{H}^p(\mathbb{R}_+^2)$. Aspects of the corresponding theory are discussed in Problems 13–20 at the end of the chapter.

5. Calderón–Zygmund Theorem

The Calderón–Zygmund Theorem asserts the boundedness of certain kinds of important operators on $L^p(\mathbb{R}^N)$ for $1 < p < \infty$. It is an N -dimensional generalization of the theorem giving the boundedness of the Hilbert transform, which was proved in Chapters VIII and IX of *Basic*. We state and prove the Calderón–Zygmund Theorem in this section, and we give some applications to partial differential equations in the next section.

Theorem 3.26 (Calderón–Zygmund Theorem). Let $K(x)$ be a C^1 function on $\mathbb{R}^N - \{0\}$ homogeneous¹¹ of degree 0 with mean value 0 over the unit sphere, i.e., with

$$\int_{S^{N-1}} K(\omega) d\omega = 0.$$

For each $\varepsilon > 0$, define

$$T_\varepsilon f(x) = \int_{|t| \geq \varepsilon} \frac{K(t)}{|t|^N} f(x - t) dt$$

whenever $1 < p < \infty$ and f is in $L^p(\mathbb{R}^N)$. Then

- (a) $\|T_\varepsilon f\|_p \leq A_p \|f\|_p$ for a constant A_p independent of ε and f ,
- (b) $\lim_{\varepsilon \downarrow 0} T_\varepsilon f = Tf$ exists as an L^p limit,
- (c) $\|Tf\|_p \leq A_p \|f\|_p$ for a constant A_p independent of f .

REMARKS. If $1 \leq p < \infty$ and if p' is the dual index to p , then the function equal to $K(t)/|t|^N$ for $|t| \geq \varepsilon$ and equal to 0 for $|t| < \varepsilon$ is in $L^{p'}$. Therefore, for each such p , $T_\varepsilon f$ is the convolution of an $L^{p'}$ function and an L^p function and is a well-defined bounded uniformly continuous function. In proving the theorem, we shall use less about $K(x)$ than the assumed C^1 condition on $\mathbb{R}^N - \{0\}$ but more than continuity. The precise condition that we shall use is that $|K(x) - K(y)| \leq \psi(|x - y|)$ on S^{N-1} for a nondecreasing function $\psi(\delta)$ of one variable that satisfies $\int_0^1 \frac{\psi(\delta)}{\delta} d\delta < \infty$.

¹¹A function F of several variables is **homogeneous of degree m** if $F(rx) = r^m F(x)$ for all $r > 0$ and all $x \neq 0$.