1.1 MAT125 Midterm Exam Summer 2007.

Name:_____

Stony Brook ID Number: _____

Please note that calculators and written notes are not permitted. Cheating and other external assistance is forbidden and cheating will result in a report being made to the academic judiciary.

Any differentiation can be done using known rules, unless explicitly stated.

Questions 1-4 are out of 17, questions 5 and 6 are out of 16 and the bonus question is worth a maximum of 16 points. The exam is out of a total of 100.

Question 1.

The position of a particle in a physics experiment is observed to satisfy the equation

$$s(t) = 3t^3 - 2t^2 - 4$$

measured in meters from a fixed point, with t denoting time in seconds. At what time (or times) is the particle stationary?

Velocity is the instantaneous rate of change of position, so is calculated as $v(t) = \frac{ds}{dt}$. In this case, $v(t) = 9t^2 - 4t$. The times that this is equal to zero are $t = 0$ and $t = 4/9$.

Question 2.

(a) Simplify the following expressions:

(i)

$$\frac{\sqrt{x+3} - \sqrt{a+3}}{x-a}$$

(so that this does not resemble $0/0$ when $x = a$).

The important thing is to multiply the surd expression in the top, by its conjugate.

$$\begin{aligned}\frac{\sqrt{x+3} - \sqrt{a+3}}{x-a} &= \frac{(\sqrt{x+3} - \sqrt{a+3})(\sqrt{x+3} + \sqrt{a+3})}{(x-a)(\sqrt{x+3} + \sqrt{a+3})} \\ &= \frac{(x-a)}{(x-a)(\sqrt{x+3} + \sqrt{a+3})} \\ &= \frac{1}{\sqrt{x+3} + \sqrt{a+3}}\end{aligned}$$

(ii) $3\log_e(e^2) - \log_e(e) - 5$.

$3\log_e(e^2) - \log_e(e) - 5 = 6\log_e(e) - \log_e(e) - 5 = 6 - 1 - 5 = 0$.

(b) Calculate the inverse functions of

(i)

$$f(x) = \frac{x-2}{x+3},$$

$$\begin{aligned} x &= \frac{y-2}{y+3} \\ (y+3)x &= yx+3x = y-2 \\ y(x-1) &= = -3x-2 \\ y &= \frac{-3x-2}{x-1} \end{aligned}$$

which gives the inverse function of f.

(ii) $g(x) = \ln(\sqrt{x^2+1})$.

$$\begin{aligned} x &= \ln(\sqrt{y^2+1}) \\ y^2+1 &= e^{2x} \\ y &= \sqrt{e^{2x}-1} \end{aligned}$$

Question 3.

Differentiate the following functions by first principles (ie. by finding a limit). You may use the binomial formula $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$.

(a) $f(x) = x^3+2x$,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^3+2(x+h)-x^3-2x}{h} \\ &= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3+2x+2h-x^3-2x}{h} \\ &= \lim_{h\to 0} \frac{h(3x^2+3xh+h^2+2)}{h} \\ &= 3x^2+2. \end{aligned}$$

(b) $g(x) = \sqrt{x+3}$,

$$\begin{aligned} g'(a) &= \lim_{x\to a} \frac{\sqrt{x+3}-\sqrt{a+3}}{x-a} \\ &= \lim_{x\to a} \frac{1}{\sqrt{x+3}+\sqrt{a+3}} \\ &= \frac{1}{2\sqrt{a+3}} \end{aligned}$$

(c)

$$h(x) = \frac{1}{x+3}.$$

$$\begin{aligned} h'(x) &= \lim_{h\to 0} 1/h\Big(\frac{1}{x+h+3} - \frac{1}{x+3}\Big) \\ &= \lim_{h\to 0} \frac{x+3-(x+h+3)}{h(x+3)(x+h+3)} \\ &= \lim_{h\to 0} \frac{-1}{(x+3(x+3+h)} \\ &= \frac{-1}{(x+3)^2}. \end{aligned}$$

Question 4. Consider the function $f(x) = x + \frac{1}{x}$, defined on the interval $(0, \infty)$.

(a) On what interval(s) is $f'(x) \geq 0$?

$f'(x) = 1 - 1/x^2$ which is non-negative for $x \geq 1$.

(b) What is the value of $f(x)$ at the point at which $f'(x) = 0$?

$f(1) = 2$.

(c) On what interval is $f''(x) \geq 0$?

$f''(x) = 2/x^3$ which is positive on $(0, \infty)$.

(d) Calculate $\lim_{x\to 0} f(x), \quad \lim_{x\to\infty} f'(x)$.

$\lim_{x\to 0} f(x) = +\infty$, and $\lim_{x\to\infty} f'(x) = 1$.

(e) Use the above informtion to graph $y = x + \frac{1}{x}$.

Question 5.

(a) Complete the statements by refering to the higher derivatives of f.

(i) f is *concave up* at a point x if $\underline{f''(x) \geq 0}$

(ii) f is *concave down* at a point x if $\underline{f''(x) \leq 0}$

(b) Sketch the approximate graph of a function that satisfies
(i) $f'(x) \leq 0$ on $(-\infty, -2)$, $f'(x) \geq 0$ on $(-2, \infty)$,
(ii) $f''(x) \geq 0$ on $(-4, 0)$ and $f''(x) \leq 0$ on $(0, \infty)$,
(iii) $f(0) = 0$, $\lim_{x \to -\infty} f(x) = 3$, and $\lim_{x \to \infty} f(x) = 2$.

Question 6.
(a) Find the slope of the line tangent to the curve $y = e^x$ at the point $(0, 1)$.

The derivative of e^x is e^x so the slope of the tangent line at $(0, 1)$ is $e^0 = 1$.

(b) Estimate the slope if the line tangent to the curve $y = e^x + e^{-x}$ at the point $(0, 2)$. (Hint: draw the graph of this function by considering the graphs of e^x and e^{-x} separately).

The graph of e^{-x} is obtained from that of e^x by reflecting across the y-axis. The tangent to the graph of e^{-x} at $(0, 1)$ is therefore the reflection of the tangent to e^x at $(0, 1)$. This reflection muliplies the slope by -1. Ie, e^x has derivative 1 at $x = 0$ and e^{-x} has derivative -1 at $x = 0$ so $e^x + e^{-x}$ has slope 0 at $x = 0$.

(c) Using (a) and (b), estimate the slope of the tangent line to the graph $y = e^x - e^{-x}$ at the point $(0, 0)$.

By the same reasoning as above, $e^x - e^{-x}$ has derivative $1 - (-1) = 2$ at $x = 0$.

Question 7. (Bonus Question) In this question we will attempt to differentiate the

function $f(x) = \sqrt{x^2+3}$ by first principles. The later parts of this question will use the earlier ones. It may be useful to recall the eequalities, $(a+b)^2 = a^2 + 2ab + b^2$ and $a^2 - b^2 = (a-b)(a+b)$.

(a) By considering the right hand side, show the following equation.

$$h = \frac{(x+h)^2 + 3 - (x^2 + 3 + h^2)}{2x}.$$

This question is hard, and there is an error in how it is stated, so I won't answer it here. Contact me if you'd like a solution.

(b) What is the slope of the secant line for the graph of f, between the points $(x, f(x))$ and $(x+h, f(x+h))$? Show, using part (a), that this is equal to

$$2x\frac{\sqrt{(x+h)^2+3} - \sqrt{x^2+3}}{((x+h)^2+3) - (x^2+3+h^2)}.$$

(c) Show that this expression can be simplified to equal

$$2x\frac{1}{\sqrt{(x+h)^2+3} + \sqrt{x^2+3+h^2}}.$$

(d) Calculate the derivative of $f(x) = \sqrt{x^2 + 3}$. That is, calculate the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$