1.1 Connections and Holonomy on Principal Bundles

These notes were produced as part of my preparation for my oral exam on Riemannian and complex algebraic geometry. No claim of originality is made and much of it is taken almost verbatim from my main references: Foundations of Differential Geometry by Kobayashi and Nomizu and Compact Manifolds of Special Holonomy. The first in particular is a gem.

Throughout this survey M will be a smooth manifold of dimension n and G will be a Lie group.

Definition 1.1. *Let M be a smooth manifold. A Principal G-bundle on M is a manifold P with a submersion $\pi : P \to M$ such that there is a free (right) G-action on P*

$$G \times P \to P$$
$$(g, p) \to pg$$

such that the fibre of π over $x \in M$ is pG for any $p \in \pi^{-1}(x)$.

This gives us the local triviality of the bundle. If $f : U \to P$ is a section over a small open set we can write

$$F : \pi^{-1}(U) \to U \times G$$
$$f(x) \cdot g \to (x, g)$$

This smoothly defines a trivialisation. We note here that each fibre is diffeomorphic to G but is not a group as there is no distinguished identity element.

Example The typical example (that I have seen) is as follows. If $E \to M$ is a vector bundle then set P to be the set of ordered bases of E_x for $x \in M$. If $\dim_{\mathbb{R}}(E_x) = k$ then P is a $GL(k, \mathbb{R})$-bundle over M.

If E has a Riemannian metric set F to be the set of ordered orthonormal bases of fibres. Then F is an $O(n)$-bundle.

If G is a Lie group and H is a closed subgroup then G/H is a smooth homogeneous space and $G \to G/H$ defines a principal H-bundle over G/H.

Definition 1.2. *A* connection *on a principal fibre bundle $P \to M$ is a distribution of n-planes on P. ie, $D_p \subset T_pP$ for all $p \in P$ is an n-dimensional subspace such that $\pi_* : D_p \to T_{\pi(p)}M$ is an isomorphism and $R_{g*}(D_p) = D_{pg} \subset T_{pg}P$. In other words, the distribrution projects to M by isomorphisms and is right invariant.*

$\pi : P \to M$ is a submersion so $\pi^{-1}(x) = pG$ is a submanifold of P. Also, the map $\xi_p : G \to pG$ by $p \to pg$ induces

$$\xi_{p*} : \mathfrak{g} \to T_pP. \tag{1.1}$$

The action is free so ξ_{p*} is injective. It gives an isomorphism between $\mathfrak{g}$ and $C_p = T_p\pi^{-1}(x)$ which is the tangent space of the fibre. We have the isomorphism, since $C_p = \ker(\pi_*)$ and since $\pi_*|_{D_p}$ is an isomorphism,

$$T_pP = C_p \oplus D_p \tag{1.2}$$

and we call C_p the vertical space and D_p the horizontal space. Define $\omega_p : C_p \to \mathfrak{g}$ at each point as

$$\omega_p(X) = \xi_{p*}^{-1}(X). \tag{1.3}$$

This can be extended to all of T_pP by setting it identically zero on D_p. Now we can compare ω_p for different p's in the same fibre. Take $X \in C_p$. Then $X = \frac{d}{dt}(pg_t)_{t=0}$ and $\omega_p(X) = \frac{dg_t}{dt}_{t=0}$. Then,

$$\begin{aligned} R_{g*}X &= \frac{d}{dt}(pg_tg)_{t=0} \\ &= \frac{d}{dt}(pg \cdot g^{-1}g_tg)_{t=0} \\ \text{so} \quad \omega_{pg}(R_{p*}X) &= \frac{d}{dt}(g^{-1}g_tg)_{t=0} \\ &= Ad_{g^{-1}}(\omega(X)). \end{aligned}$$

And, both sides are zero if $X \in D_p$ so ω is a G-equivariant 1-form on P with values in $\mathfrak{g}$. ω is called the connection 1-form.
Let P be a pricipal G-bundle on M. And let $\rho : G \to \mathrm{Aut}(V)$ be a representation of G. Then we can define $E = P \times_G V := \frac{P \times V}{\sim}$ where we have the eqivalence relation

$$(p, v) \sim (pg, \rho(g)^{-1}v). \tag{1.4}$$

The set E is a vector bundle on M. We now make the claim that there is an equivalence between G-equivariant functions$f : P :\to V$ and sections $\sigma : M \to E = P \times_G V$. Suppose $f : P \to V$ satisfies $f(pg) = \rho(g)^{-1}f(p)$. Then define

$$\sigma : M \to E \quad \sigma(x) = [p, f(p)] \quad \text{for } p \in \pi^{-1}(x)$$

$$[p, f(p)] = [pg, \rho(g)^{-1}f(p)] = [pg, f(pg)]$$

so the section σ is well defined. Similarly, a section $\sigma : M \to E$ defines a function.

In a more common formulation, connections arise as methods for differentiating sections of a vector bundle. This is seen in this case by considering them as equivariant functions on P. Let $f : P \to V$ be such that $f(pg) = \rho(g)^{-1}f(p)$. Then if $X \in C_p$,

$$X_p = \frac{d}{dt}(pg_t)_{t=0} \tag{1.5}$$

$$
\begin{aligned}
df(X) &= \frac{d}{dt}(f(pg_t))|_{t=0} \\
&= \frac{d}{dt}(\rho(g_t)^{-1} \cdot f(p)) \\
&= -\frac{d}{dt}(\rho(g_t)|_{t=0}) \cdot f(p) \\
&= -\rho_*(\omega_p(X) \cdot f(p).
\end{aligned}
$$

where $\rho : G \to \mathrm{Aut}(V)$ is the representation and $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$ is the induced Lie algebra homomorphism. I'll neglect the ρ and say that $df(X) + \omega(X)f = 0$ for X vertical. We can extend this expression to all of T_pP by setting

$$\nabla f(X) = (df + \omega(\cdot)f)(X)$$

for $X \in T_pP$. f is equivariant so

$$
\begin{aligned}
df(R_{g*}X) &= \rho(g)^{-1} df(X) \\
\text{and} \quad \omega_{pg}(R_{g*}X) \cdot f(pg) &= \rho(g)^{-1}\omega_p(X)\rho(g)\rho(g)^{-1}f(p) \\
&= \rho(g)^{-1}\omega_p(X)f(p)
\end{aligned}
$$

$$\text{so} \quad \nabla f(R_{g*}X) = \rho(g)^{-1}\nabla f(X). \tag{1.6}$$

For any $X \in \Gamma(TM)$, a vector field on M, we can lift X to X', a horizontal and right invariant vector field on P. If we define

$$\nabla_X f = df(X') + \omega(X')f \tag{1.7}$$

the previous discussion shows that we obtain a G-equivariant function $P \to V$. Hence, ∇ gives

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E). \tag{1.8}$$

This is tensorial in the first argument and is a derivation in the second. This is consistent with our previous experience with connections on vector bundles.

Now, given these definitions, we want to define the curvature of a connection.

Definition 1.3. *The curvature of a connection D with connection form ω is given as the $\mathfrak{g}$-valued 2-form on P,*

$$\Omega(X, Y) = d\omega(X, Y) + \frac{1}{2}[\omega(X), \omega(Y)]. \tag{1.9}$$

Ω can also be defined as

$$\Omega(X, Y) = d\omega(hX, hY) \tag{1.10}$$

where h is the projection of X, Y onto D_p, the horizontal subspace. The second of these two definitions indicates that rather than being the second derivative obtained using a connection, the curvature is the derivative of the connection (in some sense). The equivalence of these statements is more or less clear when X, Y are either both vertical or both horizontal. If $X \in D_p$ is horizontal and $Y \in C_p$ vertical we can extend X to be horizontal in a neighbourhood of p and we can extend Y as

$$Y_q = \frac{d}{dt}(qg_t)|_{t=0} \quad \text{for all } q \in P. \tag{1.11}$$

Then $\omega_q(Y) = \frac{d}{dt}(g_t)|_{t=0}$ is constant in q. Then

$$\begin{aligned} d\omega(X,Y) &= \frac{1}{2}(X\omega(Y) - Y\omega(X) - \omega([X,Y])) \\ &= \frac{-1}{2}\omega([X,Y]) \end{aligned}$$

since X is horizontal and $\omega(Y)$ is constant. Now, $Y_q = \frac{d}{dt}(qg_t)|_{t=0}$ so $\varphi_t(q) = qg_t = R_{g_t}q$ is the local flow of Y. Hence,

$$[Y,X]_p = \lim_{t\to 0} 1/t[\varphi_{t*}(X_{-t}) - X_p] \tag{1.12}$$

If $X_{\varphi_{-t}(p)} \in D_{pg_t^{-1}}$ then $R_{g_t*}X_{-t} \in D_p$ so $1/t[\varphi_{t*}X_{-t} - X_p] \in D_p$ for all t so $[X,Y]_p \in D_p$ so $\omega([X,Y]) = 0$. This means that the first definition for Ω vanishes if one of the arguments is vertical. From the definition the second must also vanish in this case. The other thing we must note at this stage is that the form Ω is also equivariant under the adjoint action of G. That is, for $X, Y \in T_pP$,

$$\begin{aligned} (R_g^*\Omega)_q(X,Y) &= \Omega_{qg}(R_{g*}X, R_{g*}Y) \\ &= g^{-1}\Omega_q(X,Y)g \\ \text{so } R_g^*\Omega &= Ad_{g^{-1}}\Omega. \end{aligned}$$

These statements come from the fact that the connection form ω and the bracket on $\mathfrak{g}$ are Ad-equivariant. What this amounts to is that the curvature defines a section (cf. equivariant function) of the vector bundle $Ad(P)$. This is the vector bundle $P \times_G \mathfrak{g}$ with typical fibre $\mathfrak{g}$ where G has the adjoint representation.

We can also see that this is consistent with our previous concept of curvature of connections on vector bundles. Take $X, Y \in \Gamma(TM)$. We can lift them to vector fields $X', Y' \in \Gamma(TP)$. Also, since $\pi_*X' = X$ and $\pi_*Y' = Y$, $\pi_*[X',Y'] = [X,Y]$ so $[X,Y]$ lifts to $[X',Y']$. If $\sigma \to E = P \times_G V$ we can obtain a G-equivariant function $f : P \to V$. And,

$$R(X,Y)\sigma = \nabla_X\nabla_Y\sigma - \nabla_Y\nabla_x\sigma - \nabla_{[X,Y]}\sigma. \tag{1.13}$$

Also, $\nabla_Y\sigma = Y'f + \omega(Y')f$ and

$$\nabla_X\nabla_Y\sigma = X'Y'f + X'(\omega(y')f) + \omega(X')Y'f + \omega(X')\omega(Y')f \tag{1.14}$$

so with a little cancellation we obtain

$$\begin{aligned} R(X,Y)\sigma &= (X'Y'-Y'X')f + (X'\omega(Y') - Y'\omega(X'))f + (\omega(X')\omega(Y') \\ &\quad -\omega(Y')\omega(X'))f - [X',Y']f - \omega([X',Y'])f \\ &= 2(d\omega(X',Y') + 1/2[\omega(X'),\omega(Y')])f \end{aligned}$$

so $R(X,Y)(p)$ is the endomorphism $2\Omega_p(X'_p, Y'_p)$ so the two concepts of curvature coincide up to a fixed constant.

Definition 1.4. *A piecewise smooth curve* $\gamma : [0,1] \to P$ *is called horizontal if* $\gamma'(t) \in D_{\gamma(t)}$ *ie., is a horizontal vector for all* t.

Proposition 1.1. *For any piecewise smooth curve* $\gamma : [0,1] \to P$ *and* $p \in M$ *with* $\pi(p) = \gamma(0)$ *there is a horizontal curve* $\tilde{\gamma} : [0,1] \to P$ *with* $\pi \circ \tilde{\gamma} = \gamma$ *and* $\tilde{\gamma}(0) = p$.

Proof. We can take the tangent vector field to γ and lift the vectors to be horizontal vectors in each distribution space above γ. Extend to a vector field in a neighbourhood and then observe that the integral curve to this vector field, passing through p must be a horizontal curve that projects onto γ. □

Then, let $\gamma : [0,1] \to M$ be a path with $\gamma(0) = \gamma(1) = x$. γ lifts to $\tilde{\gamma}$ into P with $\tilde{\gamma}(0) = p \in \pi^{-1}(x)$. $\tilde{\gamma}(1) = q$ is also in $\pi^{-1}(x)$ so $q = pg$ for some $g \in G$. This allows us to define the holonomy group of a connection.

Definition 1.5. *Let* $\pi : P \to M$ *be a principal* G*-bundle over* M *and* $D = \{D_p\}$ *a connection on* P. *Let* $p \in P$. *Then,*

$$Hol_p(P,D) = \{g \in G; p \textit{ can be joined to } pg \textit{ by a horizontal curve in } P\}.$$

Take $g_1, g_2 \in Hol_p(P,D)$. Then there exist γ_1, γ_2 such that $\gamma_1(0) = \gamma_2(0) = p$ and $\gamma_i(1) = pg_i$. Then define,

$$\gamma(t) = \begin{cases} \gamma_2(1-2t)g_2^{-1} & t \in [0,\frac{1}{2}] \\ \gamma_1(2t-1)g_2^{-1} & t \in [\frac{1}{2},1] \end{cases} \tag{1.15}$$

Since the γ_i are horizontal and the distribution is G-invariant, the curve γ is horizontal, starts at p and ends at $pg_1g_2^{-1}$ so $g_1g_2^{-1} \in Hol_p(P,D)$. It is thus a group.

This can be brought to be consistent with our previous experience. A typical example is of a vector bundle $E \to M$ with Riemannian metric, ie an inner product on each fibre. Set F to be the set of ordered orthonormal bases of the fibres of E. This defines a principal $O(n)$-bundle over M. Let ∇ be a connection consistent with the metric. Then parallel transport around a closed loop in M sends one orthonormal basis,call it e, for E to another. This defines an orthogonal matrix (since it is an orthogonal transformation that sends a specific basis to another). The set of such transformations is the group $Hol_e(F,\nabla) \subset O(n)$.

Definition 1.6. *The restricted holonomy group at p is given as*

$$Hol_p^0(P,D) = \{g \in G; p \text{ can be joined to } pg \text{ by a horizontal curve } \gamma \text{ in } P \text{ satisfying } \pi \circ \gamma : [0,1] \to M \text{ is null-homotopic.}\}$$

This is the group of holonomy transformations arising from the lifts of contractable closed paths in M. This too is a group.

A path $\gamma : [0,1] \to M$ based at $x = \pi(p)$ defines $P_\gamma \in Hol_p(P,D)$ and a homotopy $F : [0,1] \times [0,1] \to M$ of γ to a constant path lifts to P_{γ_s}, a path in $Hol_p(P,D)$ linking $P_\gamma = P_{\gamma_0}$ to $P_{\gamma_1} = Id$. Thus $Hol_p^0(P,D)$ is path connected. We can quote a theorem of Yamabe: A path-connected subgroup of a Lie group is a connected Lie subgroup. A Lie subgroup in this context means the image of an injective immersion. It may not be embedded or a closed submanifold. Hence, $Hol_p^0(P,D)$ is a connected Lie subgroup of G, although it may not be closed in G if it is not an actual submanifold.

From this point I will assume that $Hol_p^0(P,D)$ is a closed subgroup of G. I will refine the definition I made earlier. Define an equivalence relation $\sim$ on P. Say that $p \sim q$ if and only if p and q can be joined by a piecewise smooth horizontal curve. p and q are not necessarily in the same fibre over M. According to this definition,

$$H = Hol_p(P,D) = \{g \in G; p \sim pg\}.$$

If $E \to M$ is a Riemannian vector bundle with metric connection ∇ and F is the frame bundle of E, parallel translation of an orthonormal basis at $x \in M$ gives another orthonormal basis, perhaps of a different vector space. However, it may not be the case that every two orthonormal bases can be connected in this way.

If M is path connected, the restriction $\pi : Q \to M$ is surjective. Also, if $g \in H$ and $q \in Q$ we must have that $p \sim pg$ and $p \sim q$ so $pg \sim qg$ and $p \sim qg$ by transitivity. In other words, Q admits an action of H coming from the restriction of the action of G on P. Furthermore, if $q_1, q_2 \in Q$ lie in the same fibre of the projection π, $q_1 = q_2 g$ for $g \in G$. We will see that g is in fact in H. There must exist horizontal curves γ_1, γ_2 linking the q_i to p. Then define,

$$\gamma(t) = \begin{cases} \gamma_2(2t) & t \in [0, \frac{1}{2}] \\ \gamma_1(2-2t)g^{-1} & t \in [\frac{1}{2}, 1]. \end{cases}$$

This is a piecewise smooth horizontal path in P connecting p and pg^{-1}. We must have that g^{-1}, and so g also, lies in H. This means that a fibre of the projection $\pi|_Q$ is precisely an orbit of H on Q. If we assume that H is closed in G, Q is a submanifold of P. This, with a little more work, shows that Q is a principal fibre bundle with fibre H. Q is called the holonomy bundle of P, D (based at the point p).

We now consider the connection distribution again. Take $V \in D_q \subseteq T_qP$. Let $\gamma : [0,1] \to P$ be such that $\frac{d\gamma}{dt}|_{t=0} = V$. Assume that γ is horizontal. Then, necessarily, $\gamma(t) \in Q$ for all t. so $V \in T_qQ$. In other words, $D_q \subseteq T_qQ$. The distribution is H-invariant and $\pi_{Q*}D_q = T_{\pi(p)}M$ so D defines a connection on the holonomy bundle Q. Now, the fibre $\pi^{-1}(x) \cong H$ and so for $q \in \pi^{-1}(x)$

$$T_q(\pi^{-1}(x)) \cong \mathfrak{hol}_p(P,D) \subseteq \mathfrak{g}.$$

Where $\mathfrak{h} = \mathfrak{hol}_p(P,D)$ is the Lie algebra of H. The connection defines the connection 1-form ω given earlier. $\omega : T_qP \to \mathfrak{g}$ restricts to $\omega : T_qQ \to \mathfrak{h}$. We have thus proven:

Theorem 1.2. *Let P be a principal G-bundle over M and $D = \{D_p\}$ a connection on P. Let $p \in P$ and let $Q = Q_p$ be the holonomy bundle of D through p. Then for any $q \in Q$, the curvature 2-form*

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]$$

takes values in $\mathfrak{hol}_p(P,D) \otimes \bigwedge^2 T_qQ$

This needs to be stated delicately because if p is chosen differently, the group $H = Hol_p(P,D)$ and H-bundle Q can be affected (not very much but a little bit). There is at least a partial converse to this statement, in that the holonomy group consists of no more than elements of this form.

Theorem 1.3 (Ambrose-Singer). *Let P be a principal G-bundle over M. Let D be a connection on P, Ω the curvature 2-form. For $p \in P$ let $H = Hol_p(P,D)$ be the holonomy group of P and D, based at p and let Q be the holonomy bundle with structure group H. Then, $\mathfrak{hol}_p(P,D)$ is the subspace of $\mathfrak{g}$ spanned by elements $\Omega_q(X,Y)$ for $q \in Q$ and X and Y horizontal vectors in D_q.*

Proof. This will be proven in esentially four parts. In the first we will define a subspace $\mathfrak{g}'$ of $\mathfrak{hol}_p(P,D)$ and show that it is an ideal. We will then define a distribution on the manifold Q and show that it is smooth. In the third part we will show that the distribution is integrable and finally consider the maximal integral submanifolds of the foliation to show that $\mathfrak{g}' = \mathfrak{h}$.

We can suppose that we are already on the H-bundle Q. Let $\mathfrak{g}'$ be the subspace of $\mathfrak{h} = \mathfrak{hol}_p(P,D)$ spanned by elements $\Omega_q(X,Y)$ as above. We have seen that the curvature form satisfies

$$\begin{aligned}(R_g^*\Omega)_q(X,Y) &= \Omega_{qg}(R_{g*}X, R_{g*}Y) \\ &= Ad_{g^{-1}}(\Omega_q(X,Y))\end{aligned}$$

for $X, Y \in D_q$, which is equivalent to saying that if X, Y are horizontal lifts of vector fields on M, $\Omega(X, Y)$ defines an Ad-equivariant $\mathfrak{h}$-valued function on P. Take $X = \frac{dg_t}{dt}|_{t=0} \in \mathfrak{h}$ and take $Y = \Omega_q(V, W) \in \mathfrak{g}'$. Then,

$$
\begin{aligned}
[X, Y] &= \lim_{t \to 0} \frac{1}{t}[Ad_{g^{-1}} Y - Y] \\
\text{so } [X, \Omega_q(V, W)] &= \lim_{t \to 0} \frac{1}{t}[Ad_{g^{-1}}(\Omega_q(V, W)) - \Omega_q(V, W)] \\
&= \lim_{t \to 0} \frac{1}{t}[\Omega_{qg}(R_{g*}V, R_{g*}W)) - \Omega_q(V, W)]
\end{aligned}
$$

so since $\Omega_{qg}(R_{g*}V, R_{g*}W),\ \Omega_q(V, W) \in \mathfrak{g}'$ for all t, the final expression is in $\mathfrak{g}'$. Hence, $\mathfrak{g}'$ is an ideal in $\mathfrak{h}$.

Recall that the group H acts on Q. Then for any $A = \frac{dg_t}{dt}|_{t=0}$ we can define the vector field A^* by

$$A^*_q = \frac{d(qg_t)}{dt}|_{t=0}$$

for all q. The vector field is defined as $A \mapsto \{q \mapsto \xi_{q*} A\}$ as we mentioned earlier. This defines an isomorphism between $\mathfrak{h}$ and the vertical subspace. This isomorphism is inverted by $\omega : T_q(\pi^{-1}(x)) \to \mathfrak{h}$. We can now define the distribution S on Q. At $q \in Q$ set $S_q \subseteq T_q Q$ to be the subspace spanned by all horizontal vectors X, and the vertical vectors A^*_q for $A \in \mathfrak{g}' \subseteq \mathfrak{h}$. Suppose that $A_1, \ldots, A_r$ span $\mathfrak{g}'$ and $X_1, \ldots, X_n$ span the horizontal spaces locally. Then the A^*_i, X_j provide a smooth basis for S at each point so S is smooth. To show that it is integrable it is sufficient to show that $[A^*_i, A^*_j], [A_i, X_j]$ and $[X_i, X_j]$ lie in S_q. In the first case it is clear since $\mathfrak{g}'$ is an ideal of $\mathfrak{h}$ and $[A^*_i, A^*_j] = [A_i, A_j]^*$. In the second case we refer to equation 1.12 where we saw that for such vertical and horizontal fields the braket $[A^*_i, X_j]$ is horizontal since the distribution is H-invariant and the flow of the field A^*_i is just right translation by a one parameter subgroup of H.

In the last case we only consider the vertical component of $[X_i, X_j]$ since we obviously know that the horizontal component is in S_q. For a pair of horizontal vector fields we have $\Omega(X_i, X_j) = -\frac{1}{2}\omega([X_i, X_j]) = \omega([X_i.X_j]^v) \in \mathfrak{g}'$ so necessarily, $[X_i, X_j]^v \in S_q$. Hence, the distribution is integrable.

Now we consider the maximal interal submanifolds of the foliation passing through p. Every point in the holonomy bundle Q can be connected to p by a horizontal path. The distribution contains all the horizontal subspaces so any horizontal path much be contained in a maximal submanifold. This means that every point in Q is contained in the submanifold containing p. The submanifold must equal Q so in particular have the same dimension. Hence the vertical part of $S_q \cong \mathfrak{g}'$ equals $T_q(\pi^{-1}(x)) \cong \mathfrak{hol}_p(P, D)$. Ie, $\mathfrak{g}' \subseteq \mathfrak{hol}_p(P, D)$ have the same dimension so must be equal. □