

MAT 203 Partial Solutions to Practice Midterm

- (1) Let $\vec{u} = \langle 1, 1, 1 \rangle$ and $\vec{v} = -3\vec{i} + 2\vec{j} + 4\vec{k}$. Compute the following:
- $\vec{u} \cdot \vec{v}$
 - $\vec{u} \times \vec{v}$
 - $\|\vec{u}\|$ and $\|\vec{v}\|$
 - Find the projection of \vec{u} onto \vec{v} .
 - Compute the angle between \vec{u} and \vec{v} .

Answer: For (a),(b),(c), just apply definitions. For (d),

$$Pro_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v},$$

for (e), let θ be the angle, then

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\| \|\vec{u}\|}.$$

- (2) How close does the line with direction vector $\vec{i} - \vec{j} + \vec{k}$ through the point $(4, 7, -2)$ come to the origin?

Answer: It is about finding a distance between a point and a line. The length of projection of $\langle 4, 7, -2 \rangle$ onto $\langle 1, -1, 1 \rangle$ is $a = \frac{\langle 1, -1, 1 \rangle \cdot \langle 4, 7, -2 \rangle}{\|\langle 1, -1, 1 \rangle\|}$, set $b = \|\langle 4, 7, -2 \rangle\|$, then the distance is $\sqrt{b^2 - a^2}$.

- (3) (a) Convert the point $(5/2, 4/3, -3/2)$ from rectangular to cylindrical and spherical coordinates.
 (b) Convert the point $(5, 3\pi/4, -5)$ from cylindrical to rectangular and spherical coordinates.
 (c) Convert the point $(4, 2\pi/3, \pi/4)$ from spherical to rectangular and cylindrical coordinates.

Answer: (a) From rectangular coordinates to spherical coordinates:

$$r_s = \sqrt{x^2 + y^2 + z^2}, \tan \theta = \frac{y}{x}, \cos \phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

where r_s is the distance between the point to the origin, which is also the first coordinate of spherical coordinates, ϕ is the angle with the positive z-axis, θ is angle with the positive x-axis. To cylindrical coordinates:

$$r_c = \sqrt{x^2 + y^2}, \tan \theta = \frac{y}{x}$$

keeping in mind that the last coordinate of cylindrical coordinate is the same as that of rectangular coordinates.

(b) From cylindrical to rectangular:

$$x = r_c \cos \theta, y = r_c \sin \theta.$$

where r_c is the first coordinate of cylindrical coordinates.
From cylindrical to spherical:

$$r_s = \sqrt{r_c^2 + z^2}, \cos \phi = \frac{z}{r_s}$$

keeping in mind their θ coordinates are the same.

(c) From spherical coordinates to rectangular coordinates, we can use the following relations

$$x = r_s \sin \phi \cos \theta, y = r_s \sin \phi \sin \theta, z = r_s \cos \phi.$$

From spherical coordinates to cylindrical:

$$r_c = \sqrt{r_s^2 - z^2}, z = r_s \cos \phi$$

(4) The lines given by

$$\vec{L}_1(t) = (t+1)\vec{i} + (t-1)\vec{j} + t\vec{k}$$

$$\vec{L}_2(t) = 2\vec{i} - 3t\vec{j} + (t+1)\vec{k}$$

lie in a plane. Write an equation for this plane.

Answer: Find the intersection point (x_0, y_0, z_0) of these two lines by $\vec{L}_1(t) = \vec{L}_2(t)$. Observe that $\langle 1, 1, 1 \rangle$ is the vector parallel to $\vec{L}_1(t)$, and that $\langle 0, -3, 1 \rangle$ is the one parallel to $\vec{L}_2(t)$, then $\langle 1, 1, 1 \rangle \times \langle 0, -3, 1 \rangle$ gives an orthogonal vector, say \vec{n} , to this two lines hence to the plane. Then the equation for this plane is given by

$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \vec{n} = 0$$

(5) Write the equation of a sphere with center $(4, -4, 12)$ and tangent to the plane $x = y + 1$.

Answer: Let $Q = (1, 0, 0)$ be a point on the plane, $P = (4, -4, 12)$, $\vec{n} = \langle 1, -1, 0 \rangle$ be the normal vector the plane, then the distance between the center and the plane is given by

$$r = \frac{P\vec{Q} \cdot \vec{n}}{\|\vec{n}\|}$$

hence the equation of the sphere is given by

$$(x - 4)^2 + (y + 4)^2 + (z - 12)^2 = r^2$$

(6) Omitted.

(7) Omitted.

(8) The surfaces given by

$$\begin{aligned}x^2 - 5y^2 + 2z^2 + 2y &= 3 \\ 3x^2 - 15y^2 + 6z^2 + 4z &= 7\end{aligned}$$

intersect at a curve. Does this curve lie in a plane? Which plane?

Answer: To find the intersecting curve, we set the first equation by (1), and second by (2), then $(2)-3(1)=4z-6y=-2$, which gives a relation between z and y , namely, $z = 3/2y - 1/2$. Now replace z in the equation (1) by $3/2y - 1/2$, you can get a equation only in x and y , namely,

$$2x^2 - y^2 - 2y = 5$$

which is a curve lying in $z = 3/2y - 1/2$ plane.

(9) A piece of driftwood floats in the ocean. It is carried by a current flowing due north at a speed of 1.8km/hr . Just as the driftwood passes by a stationary buoy, a strong breeze begins blowing northwest accelerating the driftwood at a rate of 0.25m/s^2 . Find a vector valued function describing the location of the driftwood t seconds after it passes the buoy.

Answer: The location of the driftwood consists of two coordinates $x(t)$ and $y(t)$. To find these two coordinates, we need to consider the displacement of each direction, namely north and west. We know that there is an acceleration on northwest, it can be decomposed into two direction, with $\sqrt{2}/2 \times 0.25$ for each direction. So

$$x(t) = \int_0^t -\sqrt{2}/2 \times 0.25t dt = -\sqrt{2}/16t^2,$$

$$y(t) = \int_0^t \sqrt{2}/2 \times 0.25t dt - 1800/3600t = \sqrt{2}/16t^2 + 1800/3600t$$

$x(t)$ is negative since we consider east and north as positive. Hence the location of the driftwood at t is

$$r(t) = (-\sqrt{2}/16t^2, \sqrt{2}/16t^2 + 1800/3600t)$$

(10) Find the unit tangent \vec{T} and unit normal \vec{N} at $t = 0$ for the curve defined by the vector valued function $\vec{r}(t) = 5t\vec{i} + e^t\vec{j} + e^{-t}\vec{k}$.

Answer: $\vec{r}'(t) = 5\vec{i} + e^t\vec{j} - e^{-t}\vec{k}$, $\|\vec{r}'(t)\| = \sqrt{25 + e^{2t} + e^{-2t}}$.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

$$\vec{T} = \vec{T}(0) = \frac{\langle 5, 1, -1 \rangle}{3\sqrt{3}}$$

$$\vec{N} = \frac{\vec{T}'(0)}{\|\vec{T}'(0)\|}.$$

- (11) Find the arc length of the curve given by $\vec{r}(t) = t^3\vec{i} + \cos(6t^2)\vec{j} + \sin(6t^2)\vec{k}$ between $\vec{r}(0)$ and $\vec{r}(5)$.

Answer: $x(t) = t^3, y(t) = \cos(6t^2), z(t) = \sin(6t^2)$, then

$$\begin{aligned} L &= \int_0^5 \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt \\ &= \int_0^5 \sqrt{9t^4 + 144t^2} dt \\ &= \int_0^5 3t\sqrt{t^2 + 16} dt \\ &= \int_0^{25} 3/2\sqrt{u + 16} du \\ &= 41^{3/2} - 64 \end{aligned}$$

Hence the arc length of the curve is $(41^{3/2} - 64)$.

- (12) Find and plot the domain of the function

$$f(x, y) = \frac{\sqrt{x-3}}{x^2 - y}$$

Answer: The domain of this function consists of two parts:

$$x \leq 3, x^2 \neq y$$

hence is $\{x \leq 3, y \neq 9\}$.

- (13) Consider the following functions

- (a) $f(x, y) = \sqrt{x^2 + y^2}$
- (b) $f(x, y) = x^2 + y^2 - 3$
- (c) $f(x, y) = \arctan x$
- (d) $f(x, y) = -y^2 e^{-x}$

Graph them and compute

$$f_x(x, y), f_y(x, y), f_{xy}(x, y)$$

. Where are these functions differentiable?

Answer: (a)

$$f_x = \frac{x}{\sqrt{x^2 + y^2}}, f_y = \frac{y}{\sqrt{x^2 + y^2}}, f_{xy} = \frac{1 + x \cdot y}{\sqrt{x^2 + y^2}},$$

these functions are differential everywhere except the origin $(0, 0)$, since only f_x, f_y, f_{xy} are continuous everywhere except the origin.

(b) Just take derivatives, and all these functions are differentiable everywhere.

(c)

$$f_x = \frac{1}{\sqrt{x^2 + 1}}, f_y = f_{xy} = 0.$$

All are differentiable everywhere.

(d)

$$f_x = y^2 e^{-x}, f_y = -2y e^{-x}, f_{xy} = 2y e^{-x},$$

all these functions are differentiable everywhere.