The universal implosion and the multiplicative Horn problem

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The multiplicative Horn problem

Let K be a compact Lie group.

Let $\Delta = K/K$ denote the set of all conjugacy classes.

The *multiplicative Horn problem* asks:
For $A, B \in K$, how do the conjugacy classes of A, B constrain that of AB?

That is, what is the image of the natural map
$\{(A, B, C) \in K^3 \mid ABC = \mathrm{I}\} \to \Delta^3$?

The constraint is nontrivial: consider $A = \mathrm{I}$.

There are various generalizations.

The space of conjugacy classes

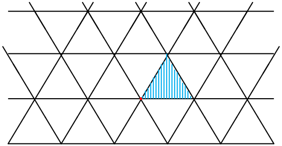
One quickly reduces to the case K simple, which we henceforth assume. The set Δ of conjugacy classes may then be identified with a simplex.

Indeed, let T be a maximal torus, $\mathfrak{t}$ its Lie algebra, $\Lambda = \ker \exp : \mathfrak{t} \to T$ the cocharacter lattice, $W = N_T/T$ the Weyl group.

Then $K/K = T/W = \mathfrak{t}/\tilde{W}$ where $\tilde{W} = W \ltimes \Lambda$ is the affine Weyl group.

A fundamental domain is the so-called Weyl alcove, a simplex in $\mathfrak{t}$.

E.g. for $K = SU_3$, here is the Weyl alcove shown in blue:



Conjugacy classes in SU_n

When $K = SU_n$, Δ parametrizes the logarithms of the eigenvalues:

$\Delta = \{\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq \alpha_1 - 1 \mid \sum_i \alpha_i = 0\}$

where $A = \mathrm{diag}(\exp 2\pi i \alpha_j)$.

Vertices correspond to central elements $\exp(2\pi ij/n)\mathrm{I}$.

Solution to the problem

The multiplicative Horn problem was solved by Belkale and Ressayre with contributions by Agnihotri, Kumar, Woodward, and others, following the solution of the additive problem by Klyachko and Knutson-Tao.

The image of $\{(A, B, C) \in K^3 \mid ABC = \mathrm{I}\} \to \Delta^3$ is a polyhedron $P \subset \Delta^3$.

This polyhedron will be our main focus, the protagonist of our drama.

For $K = SU_n$, its facets may be described as follows.

Each subset $I \subset \{1, \ldots, n\}$ of cardinality r determines a cohomology class $\sigma_I \in H^*(\mathrm{Gr}(r, n))$, the Poincaré dual of the corresponding Schubert cycle.

For every triple I, J, K of subsets of $\{1, \ldots, n\}$ of the same cardinality r, if the Gromov-Witten invariant of $\mathrm{Gr}(r, n)$ satisfies $\langle \sigma_I, \sigma_J, \sigma_K \rangle_d = 1$, then

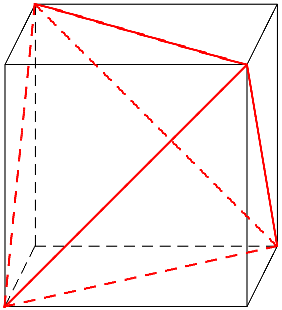
$$\sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j + \sum_{k \in K} \gamma_k \leq d$$

defines a facet of P.

Example: SU_2

Suppose $K = SU_2$.

Then Δ^3 is a cube, and P is a regular tetrahedron inscribed therein:



Explicit recursion for the Horn problem

Belkale has also given an explicit recursion in n for a finite set of inequalities defining P for $K = SU_n$.

These are redundant: they include the facets, but also hyperplanes touching P in higher codimension.

A dual question

Instead of asking for the *facets*, we explore another question:

What are the ***vertices*** *of the polyhedron P?*

Can one in any sense give a complete list?

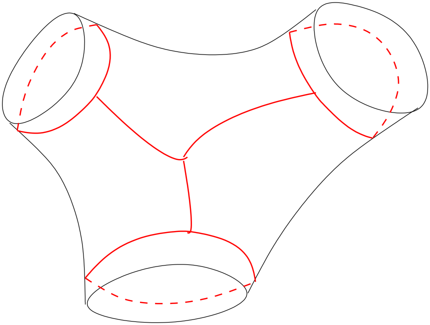
Linear programming provides algorithms determining vertices from facets, but this leads to no general formula.

Last October, the `polymake` software package determined the vertices for SU_6 in a few minutes.

It has been thinking about SU_7 ever since.

Geometric interpretation

The trinion Σ has $\pi_1(\Sigma) = \dfrac{\langle a, b, c \rangle}{\langle abc = 1 \rangle}$.



So a triple $ABC = \mathrm{I} \in K$ determines a flat K-bundle on the trinion.

Moduli spaces of flat bundles

If the conjugacy classes α, β, γ of A, B, C are fixed, the moduli space $M_{\alpha,\beta,\gamma}$ of such flat bundles is a symplectic stratified space.
Indeed it is a complex projective variety.

Our polyhedron P is the set of conjugacy classes for which $M_{\alpha,\beta,\gamma} \neq \varnothing$.

In symplectic geometry, polyhedra arise as images of moment maps of torus actions.

Theorem (Atiyah, Guillemin-Sternberg):
If a torus T acts on compact symplectic M with moment map $\mu : M \to \mathfrak{t}^*$, then $\mu(M)$ is the convex hull of the finite set $\mu(M^T)$.

Can we find a master space: a symplectic M with T^3-action such that $M_{\alpha,\beta,\gamma} = \mu^{-1}(\alpha, \beta, \gamma)/T^3$?

Such an M is provided by the work of Hurtubise-Jeffrey-Sjamaar.

It is a quasi-Hamiltonian quotient, a stratified symplectic space.
Perhaps it is even a complex projective variety.

Group-valued symplectic quotients

Alekseev-Malkin-Meinrenken introduced the notion of a *quasi-Hamiltonian group action* of G on M, having an invariant $\omega \in \Omega^2(M)$ and a *group-valued moment map* $\mu : M \to G$, satisfying certain conditions.

They defined the *quasi-Hamiltonian quotient* $M /\!\!/ G$ to be $\mu^{-1}(\mathrm{I})/G$.

If G is a torus, then ω is symplectic and $\log \mu$ is the classical moment map.

If the action of $G \times H$ on M is quasi-Hamiltonian, then so is the residual action of H on $M /\!\!/ G$.

Hence H a torus (or 1) implies $M /\!\!/ G$ symplectic.

The universal implosion

This construction applies to the *group-valued universal implosion Q*.

A section s of $K \to \Delta \subset \mathfrak{t}$ is provided by $s(A) = \exp(2\pi i A)$.

The centralizer of $s(A)$ depends only on the face F containing A.
Let K_F be the semisimple part of this centralizer.

Define $Q = \bigsqcup_{F \subset \Delta} F \times K/K_F$.

Give it the quotient topology from the natural surjection $\Delta \times K \to Q$.

It is a quasi-Hamiltonian stratified space.

The natural action of $K \times T$ has group-valued moment map
$\mu(A, kK_F) = (k \exp(2\pi i A) k^{-1}, \exp(2\pi i A)) \in K \times T$.

Construction of the master space

Let $M = Q^3 /\!\!/ K$ with the residual action of T^3.

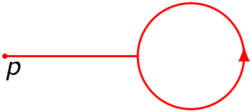
That $M_{\alpha,\beta,\gamma} = \mu^{-1}(\alpha, \beta, \gamma)/T^3$ will become clear presently.

For example, when $K = SU_2$, we have $Q = S^4$, the suspension of S^3, and $M = (S^4)^3 /\!\!/ SU_2 = CP^3$.

The smoothness in the SU_2 case is accidental.

Modular interpretation of the universal implosion

Let S be a lollipop:



A flat K-bundle on S, framed at p, is determined by its holonomy $A \in K$.

Its structure group canonically reduces to the centralizer K_A. Let $K_A \to T_A$ be the quotient by the commutator subgroup.

Then Q is the moduli space of flat K-bundles on a lollipop, framed at p, and with a (global) trivialization of the associated T_A-bundle.

When $K = U_n$, it is the moduli space of flat unitary vector bundles, framed at p, and with a trivialization of the determinant line of every eigenspace of holonomy.

When $K = SU_n$, same thing, but compatible with the overall trivialization of determinant.

Modular interpretation of the master space

It is now clear that $M = Q^3 /\!\!/ K$ is the moduli space of flat K-bundles on a trinion, with trivializations of the associated T_A-bundle on each boundary circle.

A fixed point (A, B, C) of the T^3-action is one where there exist global automorphisms of the flat K-bundle canceling out any reframings on the boundary circles. That is, there is a torus in $Z_K(A, B, C) = Z_K(A) \cap Z_K(B)$ surjecting onto $T_A \times T_B \times T_C$.

One possibility of course is that $T_A = T_B = T_C = 1$, that is, $A, B, C \in Z(K)$. For $K = SU_n$, these are vertices of Δ^3, hence definitely vertices of P.

In fact the finite group $\Gamma = \{(A, B, C) \in Z(K) \,|\, ABC = \mathrm{I}\}$ acts on Δ^3 preserving P. These vertices are $\Gamma \cdot (\mathrm{I}, \mathrm{I}, \mathrm{I})$. Call them the *central vertices*. Indeed, really $S_3 \ltimes \Gamma$ acts.

The hunt for vertices: SU_2, SU_3, SU_4

For $K = SU_2$, we know that the central vertices are the only vertices.
For $K = SU_3$ this again proves true.
For $K = SU_4$ it is false!

$A = \mathrm{diag}(1, \quad 1, -1, -1)$
$B = \mathrm{diag}(1, -1, \quad 1, -1)$
$C = \mathrm{diag}(1, -1, -1, \quad 1)$

The torus
$S_A = \mathrm{diag}(t, t, t^{-1}, t^{-1})$
acts nontrivially on the eigenspaces of A but trivially on those of B and C.
Similarly for B and C.

The needed surjection is then provided by $S_A \times S_B \times S_C \to T_A \times T_B \times T_C$.

The vertices of P fall into two Γ-orbits: the central orbits and this orbit.

The hunt for vertices: SU_5

For $K = SU_5$ it is again false.

$A = \mathrm{diag}(1, 1, \quad 1, -1, -1)$
$B = \mathrm{diag}(1, 1, -1, \quad 1, -1)$
$C = \mathrm{diag}(1, 1, -1, -1, \quad 1)$

Same as with SU_4 but with the first simultaneous eigenspace *dilated*.

Still, at least all fixed points of T^3 on M map to vertices of P, and all parametrize commuting triples (A, B, C).

Indeed, as with SU_4, the fixed points fall into two Γ-orbits, that of $(\mathrm{I}, \mathrm{I}, \mathrm{I})$ and that of (A, B, C) above.

The hunt for vertices: SU_6

When $K = SU_6$, the vertices of P fall into 4 $S_3 \ltimes \Gamma$-orbits:

$A = B = C = \mathrm{I}$
the central vertices

$A = \mathrm{diag}(1, 1, 1, \quad 1, -1, -1)$
$B = \mathrm{diag}(1, 1, 1, -1, \quad 1, -1)$
$C = \mathrm{diag}(1, 1, 1, -1, -1, \quad 1)$
dilated from SU_5

$A = \mathrm{diag}(\ \ 1, \ \ 1, \ \ 1, \ \ 1, -1, -1)$
$B = \mathrm{diag}(\ \ i, \ \ i, -i, -i, \ \ i, -i)$
$C = \mathrm{diag}(-i, -i, \ \ i, \ \ i, \ \ i, -i)$
also dilated from SU_5

$A = \mathrm{diag}(\xi^5, \xi^1, \xi^5, \xi^1, \xi^5, \xi^1)$
$B = \mathrm{diag}(\xi^0, \xi^0, \xi^2, \xi^2, \xi^4, \xi^4)$
$C = \mathrm{diag}(\xi^1, \xi^5, \xi^5, \xi^3, \xi^3, \xi^1)$
where $\xi = e^{2\pi i/6}$, new.

The abelian vertices

If a fixed point of the T^3-action on M corresponds to commuting A, B, C, call it an *abelian* fixed point.
For $K = SU_n$, this means that it splits as a sum of flat line bundles.

If, moreover, its image in P is a vertex, call it an *abelian* vertex.
All of the vertices we have seen thus far are abelian.

The space of flat T-bundles on Σ is of course
$\{(A, B, C) \in T^3 \,|\, ABC = \mathrm{I}\} \cong T \times T$.

The natural map $L : \mathfrak{t} \times \mathfrak{t} \to T \times T \to P$ is piecewise linear.

For $K = SU_2$ its image is the *boundary* of the tetrahedron we saw earlier.
For $K = SU_3$ it plunges into the interior of P but also hits all the vertices.
For general K, the abelian vertices are the images of vertices of the polyhedra where L is linear.

The abelian vertices of P all lie on $\partial\Delta^3$,
indeed on the r-skeleton of Δ^3, where $r = \dim T$.
This is more clear from the following perspective.

Periodic chamber decompositions

Define a *periodic chamber decomposition* of a finite-dimensional vector space V to be a locally finite collection of hyperplanes, periodic with respect to some lattice Λ of full rank.

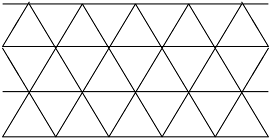
A *vertex* of such a thing is a vertex of one of its polyhedral chambers.

There are finitely many vertices modulo Λ.

A union of finitely many PCD's is a PCD.

So is the inverse image of a PCD under a linear surjection.

The key example is $\mathfrak{t}$ with the hyperplanes fixed by elements of the affine Weyl group $\tilde{W}$. Call this C. Here it is again for $K = SU_3$:



Now, on $\mathfrak{t} \oplus \mathfrak{t}$, consider the PCD $\pi_1^{-1}(C) \cup \pi_2^{-1}(C) \cup A^{-1}(C)$, where $A : \mathfrak{t} \oplus \mathfrak{t} \to \mathfrak{t}$ is addition. How does one classify its vertices modulo $\Lambda \oplus \Lambda$?

I wish I knew. Does anyone here?

Two infinite families

There are two infinite families of abelian fixed points.

In SU_{n^2},
$A = \mathrm{diag}(\zeta^1, \zeta^2, \ldots, \zeta^{n^2})$ where $\zeta = e^{2\pi i/n}$
$B = \mathrm{diag}(\zeta^1 \mathrm{I}_n, \zeta^2 \mathrm{I}_n, \ldots, \zeta^n \mathrm{I}_n)$
$C = (AB)^{-1}$

In SU_{2n},
$A = \mathrm{diag}(\xi^{-1}, \xi, \xi^{-1}, \xi, \ldots, \xi^{-1}, \xi)$ where $\xi = e^{2\pi i/2n}$
$B = \mathrm{diag}(\xi^2, \xi^2, \xi^4, \xi^4, \ldots, \xi^{2n}, \xi^{2n})$
$C = \mathrm{diag}(\xi^{-1}, \xi^{-3}, \xi^{-3}, \xi^{-5}, \ldots, \xi^{-2n-1})$

We have already seen them in SU_4 and SU_6.

They are not always vertices of P.

Are there more?

Conjugacy classes of order 2

Suppose that $ABC = \mathrm{I} \in SU_n$ all have order 2:
$A^2 = B^2 = C^2 = \mathrm{I}$.

Then $BA = B^{-1}A^{-1} = (AB)^{-1} = AB$,
so A, B, C may be simultaneously diagonalized.

Their conjugacy classes α, β, γ must then satisfy $M_{\alpha,\beta,\gamma} =$ point,

which implies $(\alpha, \beta, \gamma) \in \partial P$.

The dimensions i, j, k of their -1-eigenspaces must satisfy the Clebsch-Gordan rules: $|i - j| \leq k \leq i + j$, etc.

Does this generalize in any way?

Constraints on the vertices

Continue seeking abelian vertices of P when $K = SU_n$.

Recall that $T \supset S_A \times S_B \times S_C \to T_A \times T_B \times T_C$ is surjective.

Hence, if A, B, C have n_1, n_2, n_3 eigenspaces respectively, then

$$n - 1 \geq (n_1 - 1) + (n_2 - 1) + (n_3 - 1).$$

Not too many eigenspaces!

On the other hand, if $n_1 = 1$, then $A, B, C \in Z_{SU_n}$, so vertex is central.

If $n_1 = 2$, must have the infinite family in SU_{2n} or its dilations.

Also, if vertex is not a dilation, then n is number of simultaneous eigenspaces of A, B, C, so

$$n \leq \min(n_1 n_2, n_1 n_3, n_2 n_3).$$

Not too few eigenspaces!

Combinatorial constraints

The eigenspaces of three diagonal matrices A, B, C are the coordinate subspaces corresponding to a triple of partitions

$$\{1, \ldots, n\} = \bigcup_{i \in I} U_i = \bigcup_{j \in J} V_j = \bigcup_{k \in K} W_k.$$

They must satisfy: for all i, j with $U_i \cap V_j \neq \varnothing$, there exists k with

$$U_i \cap V_j = U_i \cap W_k = V_j \cap W_k.$$

And similarly with i, j, k permuted.

If any $U_i \subset V_j$, then $U_i = V_j \cap W_k$ and dividing by U_i leads to a smaller triple with $V_j \cap W_k = \varnothing$.

So wlog we may assume that no $U_i \subset V_j$, and similarly.

This leads to a graphical calculus.

E.g. for SU_7 the only remaining case is $(n_1, n_2, n_3) = (3, 3, 3)$.

A nonabelian vertex

For some time I thought that all vertices were abelian. Not so!

Let $\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$

Then for $K = SU_8$,

$A = \operatorname{diag}(\sigma_1, i, -i, i, -i, -1, -1)$
$B = \operatorname{diag}(\sigma_2, i, -i, -1, -1, i, -i)$
$C = \operatorname{diag}(\sigma_3, -1, -1, i, -i, i, -i)$

is a nonabelian vertex.

Again, this lies on the r-skeleton, indeed the $(r-1)$-skeleton, of Δ^3.

Broader perspectives

A more sophisticated approach to the multiplicative Horn problem would address two broader topics:

(1) **Symplectic volumes.**
Duistermaat-Heckman: volume of $M_{\alpha,\beta,\gamma}$ is piecewise polynomial.
One should determine this piecewise polynomial function supported on P.
Work in this direction by Meinrenken.

(2) **Parabolic bundles.**
Mehta-Seshadri: flat SU_n-bundles on Σ correspond to algebraic vector bundles on CP^1, semistable with respect to a *parabolic structure*.
Thus everything turns into algebraic geometry.
Vertices should correspond to parabolic bundles with some rigidity.
Complicated, as on the "far wall" of P the correspondence is actually with *parahoric bundles*.

In conclusion

As the meeting is now ending, one thing remains to be said:

Can anyone offer me a ride to Ronkonkoma?

But also, many thanks to the conference organizers for putting on such a delightful event! And thank you for listening.