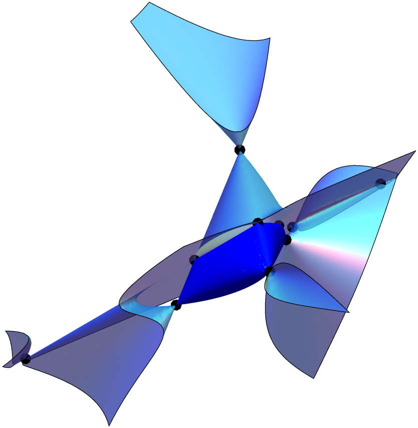
Quartic Symmetroids and Spectrahedra



Cynthia Vinzant,
University of Michigan

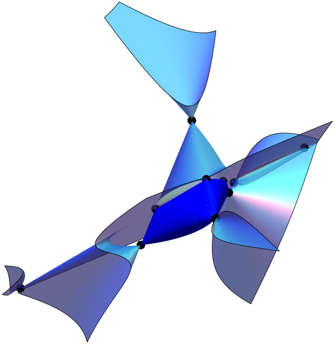
with
John Christian Ottem,
Kristian Ranestad, and
Bernd Sturmfels

Quartic Symmetroids

A quartic symmetroid is a surface $\mathcal{V}(f) \subset \mathbb{P}^3(\mathbb{C})$ given by

$$f \quad = \quad \det(A(x)) \quad = \quad \det(x_0A_0 + x_1A_1 + x_2A_2 + x_3A_3)$$

where A_0, A_1, A_2, A_3 are 4×4 symmetric matrices.



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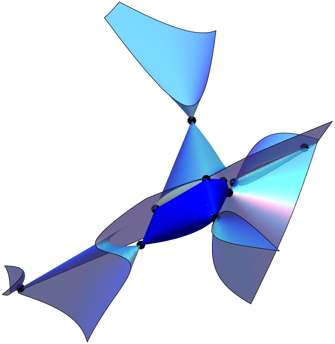
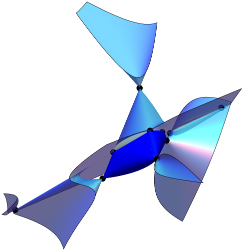


Fun facts:

- $\mathcal{V}(f)$ has 10 nodes (of rank 2)
- co-dimension 10 in $\mathbb{P}(\mathbb{C}[x_0, x_1, x_2, x_3]_4)$
- studied by Cayley in a set of memoirs 1869 - 1871

Here I'll talk about in surfaces $\mathcal{V}(\det(A(x)))$ where

- the matrices A_0, A_1, A_2, A_3 are real and
- their span contains a positive definite matrix.



Real Spectrahedral Symmetroids

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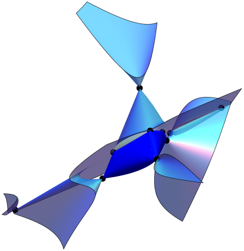
Motivation 1: The convex sets $\{x \in \mathbb{R}^4 \ : \ A(x) \succeq 0\}$ appear as feasible sets (spectrahedra) in semidefinite programming.



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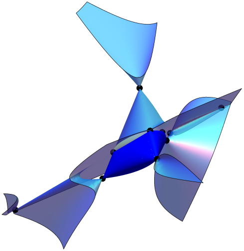

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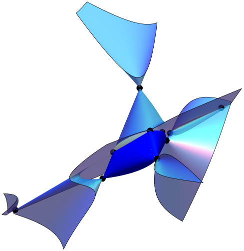
For example . . .

Friedland *et. al.* (1984) showed that in this case $\mathcal{V}(\det(A(x)))$ has a *real* node.



Linear spaces of matrices and spectrahedra

Let $A_0, A_1, \ldots, A_n$ be real symmetric $d \times d$ matrices and

$$A(x) = x_0 A_0 + x_1 A_1 + \ldots + x_n A_n.$$

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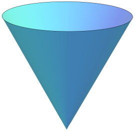
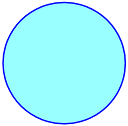
Example:

$$A(x) = \begin{pmatrix} x_0 + x_1 & x_2 \\ x_2 & x_0 - x_1 \end{pmatrix}$$





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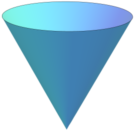
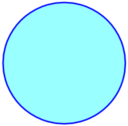


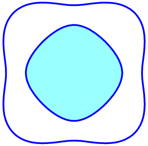


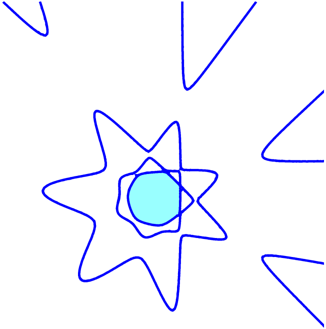
Goal: Understand the algebraic and topological properties of spectrahedra and their bounding polynomials.

Polynomials bounding spectrahedra

Spectrahedra are bounded by hyperbolic polynomials, $\det(A(x))$.

A polynomial f is hyperbolic with respect to a point p if every real line through p meets $\mathcal{V}(f)$ in only real points.





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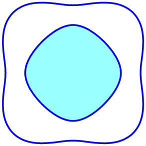
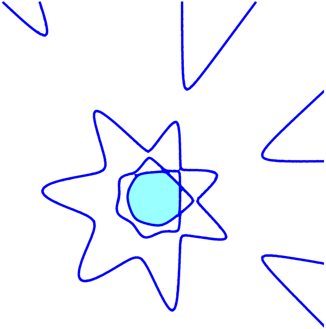
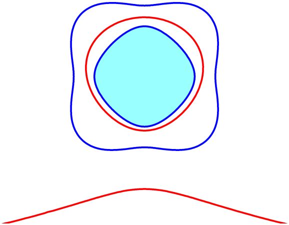


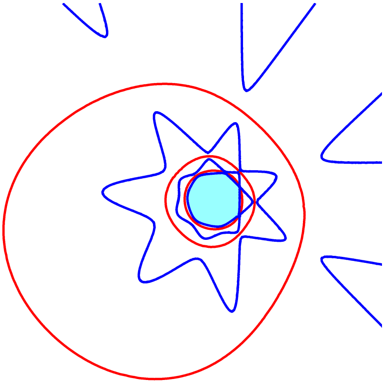


Theorem (Helton-Vinnikov 2007). A polynomial $f \in \mathbb{R}[x_0, x_1, x_2]_d$ bounds a spectrahedron if and only if f is hyperbolic.

Spectrahedra and interlacers

The diagonal $(d-1) \times (d-1)$ minors of $A(x)$ interlace the determinant $\det(A(x))$.





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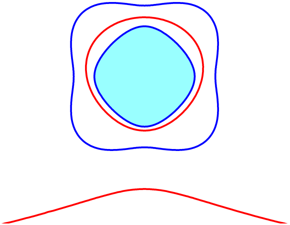
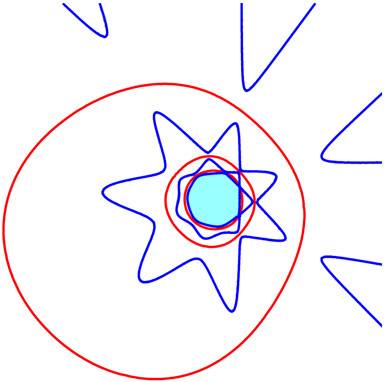


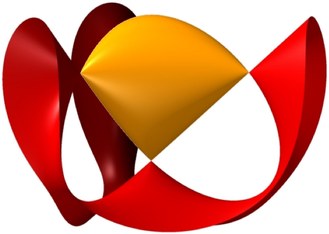


Theorem (Plaumann-V. 2013). The matrix $A(x)$ is definite at some point if and only if its minors interlace the determinant.

Determinantal surfaces and 3-dim'l spectrahedra ($n = 3$)

The variety of rank-$(d-2)$ matrices in $\mathbb{C}^{d\times d}_{sym}$ has

codimension 3 and degree $\binom{d+1}{3}$.



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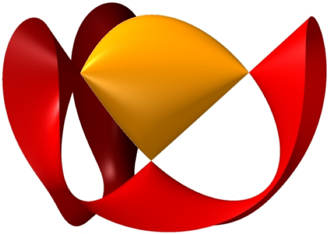

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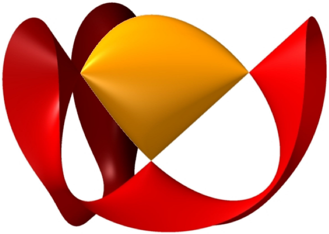
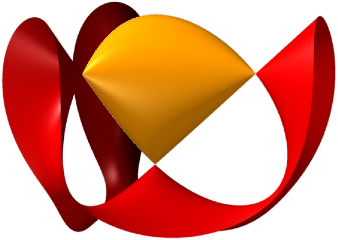
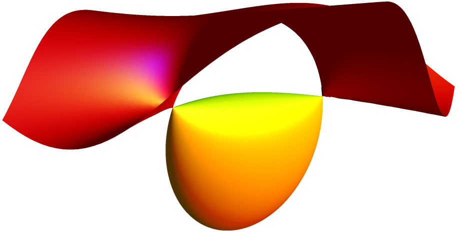
The complex surface $\mathcal{V}(\det(A(x))$ bounding a three-dimensional spectrahedron has $\binom{d+1}{3}$ nodes.



Three-dimensional spectrahedra bounded by cubics

Over $\mathbb{C}$ there are (generically) 4 nodes of rank one.

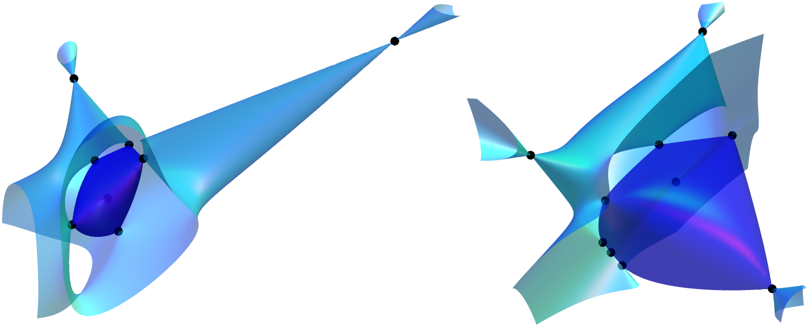




Either 2 or 4 of them are real and lie on the spectrahedron.

Three-dimensional spectrahedra bounded by quartics

Over $\mathbb{C}$ there are generically 10 nodes of rank two.

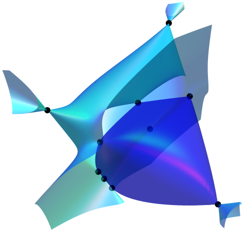


There are two flavors of real node (**on** or **off** the spectrahedron).
What configurations are possible?

Theorem (Degtyarev-Itenberg, 2011)

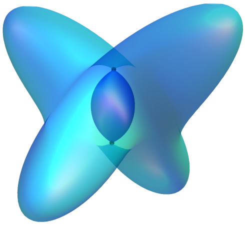
There is a (transversal) quartic spectrahedron with α nodes on its boundary and β nodes on its real surface if and only if

$$\alpha, \beta \text{ are even} \quad \text{and} \quad 2 \leq \alpha + \beta \leq 10.$$



$\alpha = 8$
$\beta = 2$

$\alpha = 0$
$\beta = 10$



$\alpha = 2$
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Back to the classics (Cayley's Symmetroids)

Idea of Cayley: Look at the projection of $\mathcal{V}(f)$ from a node p.

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This projection $\pi_p : \mathcal{V}(f) \to \mathbb{P}^2$ from a node p is a double cover of $\mathbb{P}^2$ whose branch locus is a sextic curve.

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Why? If $p = [1:0:0:0]$ then

$$f = a \cdot x_0^2 + b \cdot x_0 + c \quad \text{where} \quad a, b, c \in \mathbb{R}[x_1, x_2, x_3].$$

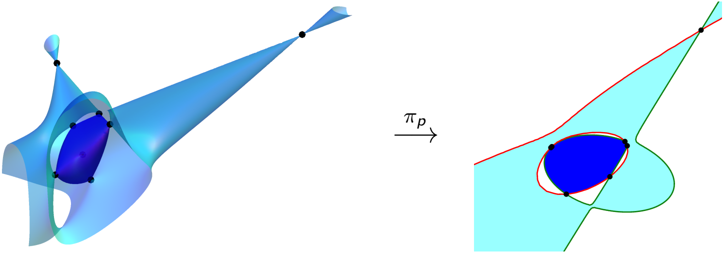
The branch locus of π_p is $\mathcal{V}(b^2 - 4ac)$.

Projection from a node

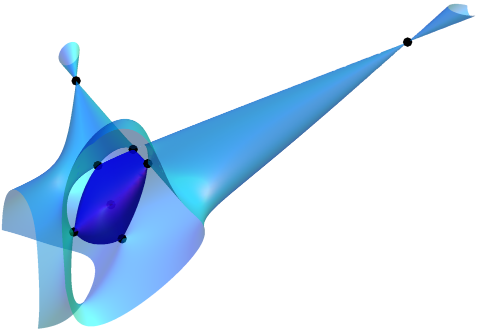
Theorem (Cayley 1869-71)

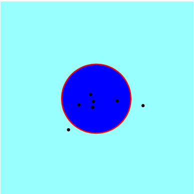
A quartic $f \in \mathbb{C}[x_0, x_1, x_2, x_3]_4$ with node p is a symmetroid if and only if the branch locus of π_p is the product of two cubics, $b_1 \cdot b_2$.

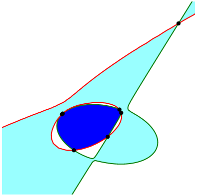
Moreover the images of the other 9 nodes are $\mathcal{V}(b_1) \cap \mathcal{V}(b_2)$.



The view from a node on or off the spectrahedron



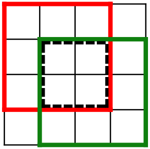




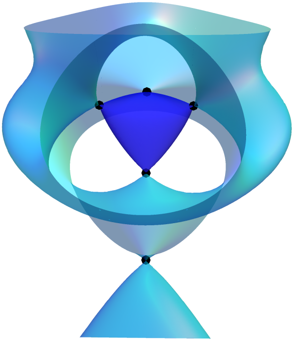
For $p \in$ *Spec*, $b_1 = \overline{b_2}$.
The image π_p(*Spec*) is the conic $\{a \geq 0\}$.

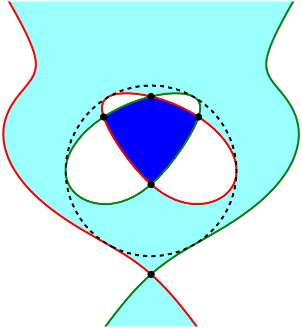
For $p \notin$ *Spec*, b_1, b_2 are real and hyperbolic.
The image π_p(*Spec*) is the intersection of cubic ovals.

The view from a node: interlacing branch locus

If $p = [1:0:0:0]$ and $A(x) \;=\; x_0 \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} +$ 

then the branch cubics b_1, b_2 are diagonal minors of $A(0, x_1, x_2, x_3)$.

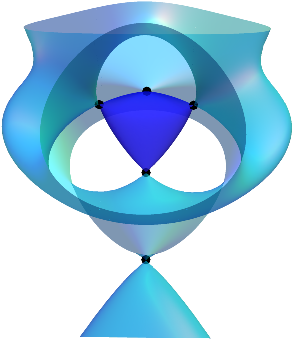


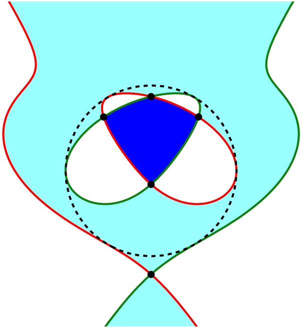


The view from a node: interlacing branch locus

The image of the spectrahedron is the intersection of cubic ovals.

$\rightarrow$ There are an even number of spectrahedral nodes.





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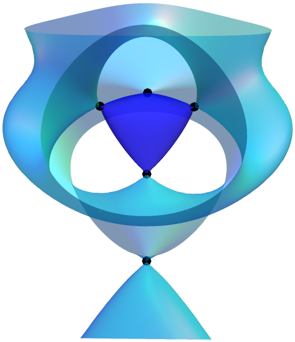
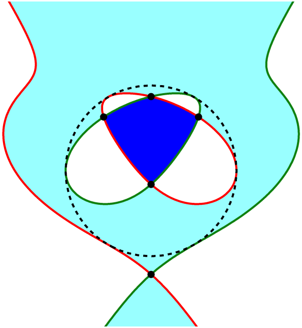
To understand the other direction of the Degtyarev-Itenberg Theorem . . .





$(A_0\ A_1\ A_2\ A_3)$ giving different types of spectrahedra

$$(\mathbf{2},\mathbf{2}) : \begin{bmatrix}3&4&1&-4\\4&14&-6&-10\\1&-6&9&2\\-4&-10&2&8\end{bmatrix} \begin{bmatrix}11&0&2&2\\0&6&-1&4\\2&-1&6&2\\2&4&2&4\end{bmatrix} \begin{bmatrix}17&-3&2&9\\-3&6&-4&1\\2&-4&13&10\\9&1&10&17\end{bmatrix} \begin{bmatrix}9&-3&9&3\\-3&10&6&-7\\9&6&18&-3\\3&-7&-3&5\end{bmatrix}$$

$$(\mathbf{4},\mathbf{4}) : \begin{bmatrix}18&3&9&6\\3&5&-1&-3\\9&-1&13&7\\6&-3&7&6\end{bmatrix} \begin{bmatrix}17&-10&4&3\\-10&14&-1&-3\\4&-1&5&-4\\3&-3&-4&6\end{bmatrix} \begin{bmatrix}8&6&10&10\\6&18&6&15\\10&6&14&9\\10&15&9&22\end{bmatrix} \begin{bmatrix}8&-4&8&0\\-4&10&-4&0\\8&-4&8&0\\0&0&0&0\end{bmatrix}$$

$$(\mathbf{6},\mathbf{6}) : \begin{bmatrix}10&8&2&6\\8&14&0&2\\2&0&5&7\\6&2&7&11\end{bmatrix} \begin{bmatrix}11&-6&10&9\\-6&10&-5&-5\\10&-5&14&11\\9&-5&11&9\end{bmatrix} \begin{bmatrix}6&2&6&-5\\2&9&2&0\\6&2&6&-5\\-5&0&-5&5\end{bmatrix} \begin{bmatrix}8&6&2&-2\\6&9&9&6\\2&9&13&12\\-2&6&12&13\end{bmatrix}$$

$$(\mathbf{8},\mathbf{8}) : \begin{bmatrix}5&3&-3&-4\\3&6&-3&-2\\-3&-3&6&4\\-4&-2&4&4\end{bmatrix} \begin{bmatrix}19&10&12&17\\10&14&10&7\\12&10&10&11\\17&7&11&17\end{bmatrix} \begin{bmatrix}5&1&3&-3\\1&5&-7&-1\\3&-7&22&7\\-3&-1&7&10\end{bmatrix} \begin{bmatrix}1&1&0&2\\1&1&0&2\\0&0&4&4\\2&2&4&8\end{bmatrix}$$

$$(\mathbf{10},\mathbf{10}) : \begin{bmatrix}18&6&6&-6\\6&2&2&-2\\6&2&2&-2\\-6&-2&-2&4\end{bmatrix} \begin{bmatrix}4&-6&6&4\\-6&13&-9&-8\\6&-9&9&6\\4&-8&6&5\end{bmatrix} \begin{bmatrix}1&0&-3&0\\0&4&0&6\\-3&0&9&0\\0&6&0&9\end{bmatrix} \begin{bmatrix}9&-3&0&0\\-3&10&9&-6\\0&9&9&-6\\0&-6&-6&4\end{bmatrix}$$

$$(\mathbf{2},\mathbf{0}) : \begin{bmatrix}20&6&-14&-4\\6&18&3&-12\\-14&3&17&-2\\-4&-12&-2&8\end{bmatrix} \begin{bmatrix}54&-27&16&12\\-27&18&-2&-15\\16&-2&20&-10\\12&-15&-10&21\end{bmatrix} \begin{bmatrix}42&-8&9&-3\\-8&10&5&-11\\9&5&29&7\\-3&-11&7&29\end{bmatrix} \begin{bmatrix}0&9&3&-3\\9&-9&-6&6\\3&-6&-3&3\\-3&6&3&-3\end{bmatrix}$$

$$(\mathbf{4},\mathbf{2}) : \begin{bmatrix}9&-4&1&1\\-4&5&-3&-2\\1&-3&3&1\\1&-2&1&1\end{bmatrix} \begin{bmatrix}6&1&3&4\\1&5&5&2\\3&5&6&2\\4&2&2&8\end{bmatrix} \begin{bmatrix}8&2&-6&4\\2&5&1&3\\-6&1&6&-2\\4&3&-2&3\end{bmatrix} \begin{bmatrix}-4&4&-2&2\\4&0&0&-2\\-2&0&0&1\\2&-2&1&-1\end{bmatrix}$$

$$(\mathbf{6},\mathbf{4}) : \begin{bmatrix}6&-1&5&5\\-1&2&1&-3\\5&1&6&2\\5&-3&2&9\end{bmatrix} \begin{bmatrix}5&-5&5&-3\\-5&6&-5&5\\5&-5&5&-3\\-3&5&-3&9\end{bmatrix} \begin{bmatrix}6&-3&5&2\\-3&5&-3&2\\5&-3&9&-4\\2&2&-4&9\end{bmatrix} \begin{bmatrix}0&-2&-2&0\\-2&1&2&1\\-2&2&3&1\\0&1&1&0\end{bmatrix}$$

$$(\mathbf{8},\mathbf{6}) : \begin{bmatrix}4&0&4&-2\\0&5&-2&5\\4&-2&8&-4\\-2&5&-4&6\end{bmatrix} \begin{bmatrix}2&3&-1&-1\\3&6&-1&-4\\-1&-1&6&-3\\-1&-4&-3&6\end{bmatrix} \begin{bmatrix}6&2&0&1\\2&8&4&-2\\0&4&8&-2\\1&-2&-2&1\end{bmatrix} \begin{bmatrix}2&-3&0&1\\-3&5&0&0\\0&0&0&0\\1&0&0&5\end{bmatrix}$$

$$(\mathbf{10},\mathbf{8}) : \begin{bmatrix}5&-1&-1&4\\-1&6&-3&5\\-1&-3&2&-4\\4&5&-4&9\end{bmatrix} \begin{bmatrix}8&0&0&-4\\0&1&0&-1\\0&0&2&0\\-4&-1&0&3\end{bmatrix} \begin{bmatrix}6&5&1&-2\\5&9&-3&-4\\1&-3&6&4\\-2&-4&4&4\end{bmatrix} \begin{bmatrix}8&0&0&-4\\0&8&4&4\\0&4&2&2\\-4&4&2&4\end{bmatrix}$$

Combinatorial types of quartic spectrahedra (11-20)

$$(\mathbf{4},\mathbf{0}):\quad \begin{bmatrix}21&10&1&-6\\10&10&0&-1\\1&0&2&-3\\-6&-1&-3&6\end{bmatrix}\ \begin{bmatrix}0&6&-6&2\\6&3&0&-4\\-6&0&-3&5\\2&-4&5&-3\end{bmatrix}\ \begin{bmatrix}0&0&0&2\\0&0&0&-1\\0&0&0&-1\\2&-1&-1&5\end{bmatrix}\ \begin{bmatrix}0&3&-1&1\\3&-3&8&-5\\-1&8&-5&4\\1&-5&4&-3\end{bmatrix}$$

$$(\mathbf{6},\mathbf{2}):\quad \begin{bmatrix}7&-1&5&2\\-1&5&-1&5\\5&-1&4&1\\2&5&1&7\end{bmatrix}\ \begin{bmatrix}-1&-2&1&-2\\-2&-3&2&-6\\1&2&-1&2\\-2&-6&2&0\end{bmatrix}\ \begin{bmatrix}4&4&2&-2\\4&0&4&-2\\2&4&0&-1\\-2&-2&-1&1\end{bmatrix}\ \begin{bmatrix}-1&1&2&1\\1&-1&-2&-1\\2&-2&-3&-1\\1&-1&-1&0\end{bmatrix}$$

$$(\mathbf{8},\mathbf{4}):\quad \begin{bmatrix}16&-4&-16&10\\-4&18&0&-13\\-16&0&20&-9\\10&-13&-9&19\end{bmatrix}\ \begin{bmatrix}0&1&-1&0\\1&-5&6&1\\-1&6&-7&-1\\0&1&-1&0\end{bmatrix}\ \begin{bmatrix}0&-16&0&-8\\-16&0&16&-16\\0&16&0&8\\-8&-16&8&-16\end{bmatrix}\ \begin{bmatrix}7&9&16&3\\9&-9&-12&9\\16&-12&-15&15\\3&9&15&0\end{bmatrix}$$

$$(\mathbf{10},\mathbf{6}):\quad \begin{bmatrix}18&-13&15&1\\-13&22&2&-16\\15&2&30&-20\\1&-16&-20&30\end{bmatrix}\ \begin{bmatrix}-15&7&8&5\\7&-3&-4&-3\\8&-4&-4&-2\\5&-3&-2&0\end{bmatrix}\ \begin{bmatrix}1&0&1&-3\\0&0&0&0\\1&0&-8&-15\\-3&0&-15&-7\end{bmatrix}\ \begin{bmatrix}-15&0&-6&2\\0&15&6&8\\-6&6&0&4\\2&8&4&4\end{bmatrix}$$

$$(\mathbf{6},\mathbf{0}):\quad \begin{bmatrix}3&6&-4&-4\\6&13&-5&-5\\-4&-5&19&20\\-4&-5&20&23\end{bmatrix}\ \begin{bmatrix}0&-1&-3&0\\-1&3&6&0\\-3&6&9&0\\0&0&0&0\end{bmatrix}\ \begin{bmatrix}8&2&-2&2\\2&-4&-2&2\\-2&-2&0&0\\2&2&0&0\end{bmatrix}\ \begin{bmatrix}1&-2&1&3\\-2&-5&-11&-15\\1&-11&-8&-6\\3&-15&-6&0\end{bmatrix}$$

$$(\mathbf{8},\mathbf{2}):\quad \begin{bmatrix}3&-3&3&-1\\-3&4&-3&2\\3&-3&5&0\\-1&2&0&2\end{bmatrix}\ \begin{bmatrix}-1&1&-1&-2\\1&0&0&0\\-1&0&0&0\\-2&0&0&0\end{bmatrix}\ \begin{bmatrix}0&0&-1&-2\\0&0&0&0\\-1&0&1&0\\-2&0&0&-4\end{bmatrix}\ \begin{bmatrix}-1&1&1&0\\1&3&-1&2\\1&-1&-1&0\\0&2&0&1\end{bmatrix}$$

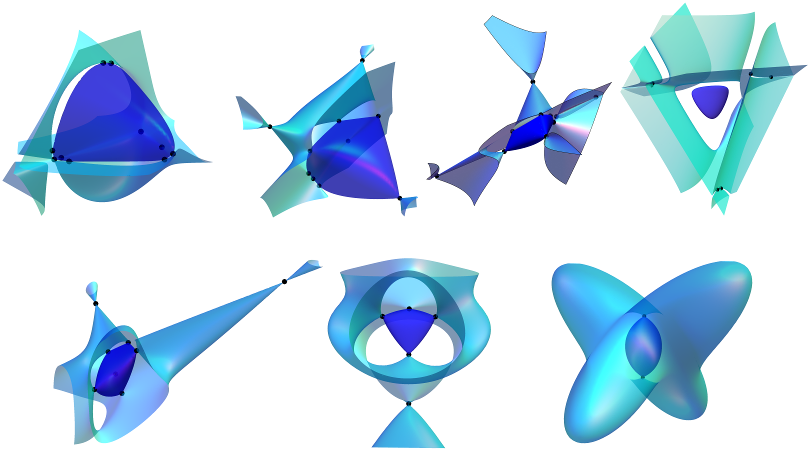
$$(\mathbf{10},\mathbf{4}):\quad \begin{bmatrix}5&-1&-3&1\\-1&2&2&0\\-3&2&4&-1\\1&0&-1&3\end{bmatrix}\ \begin{bmatrix}0&0&0&0\\0&-4&-4&-2\\0&-4&-4&-2\\0&-2&-2&0\end{bmatrix}\ \begin{bmatrix}0&4&-4&-6\\4&0&2&1\\-4&2&-4&-4\\-6&1&-4&-3\end{bmatrix}\ \begin{bmatrix}-3&0&-1&-2\\0&0&0&0\\-1&0&0&-1\\-2&0&-1&-1\end{bmatrix}$$

$$(\mathbf{8},\mathbf{0}):\quad \begin{bmatrix}9&0&-7&-10\\0&5&0&2\\-7&0&15&5\\-10&2&5&13\end{bmatrix}\ \begin{bmatrix}8&6&5&8\\6&-8&-5&-4\\5&-5&-3&-2\\8&-4&-2&0\end{bmatrix}\ \begin{bmatrix}8&4&11&4\\4&0&10&0\\11&10&5&10\\4&0&10&0\end{bmatrix}\ \begin{bmatrix}-4&-4&2&4\\-4&-4&2&4\\2&2&0&0\\4&4&0&0\end{bmatrix}$$

$$(\mathbf{10},\mathbf{2}):\quad \begin{bmatrix}29&-22&4&-4\\-22&26&-7&5\\4&-7&25&-6\\-4&5&-6&5\end{bmatrix}\ \begin{bmatrix}-1&-4&-1&-4\\-4&-12&-4&-14\\-1&-4&-1&-4\\-4&-14&-4&-15\end{bmatrix}\ \begin{bmatrix}-5&9&6&7\\9&8&-2&5\\6&-2&-4&-2\\7&5&-2&3\end{bmatrix}\ \begin{bmatrix}-5&16&-1&-10\\16&-12&20&4\\-1&20&7&-14\\-10&4&-14&0\end{bmatrix}$$

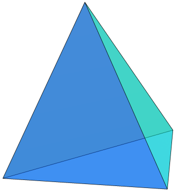
$$(\mathbf{10},\mathbf{0}):\quad \begin{bmatrix}51&-34&5&60\\-34&147&30&-37\\5&30&99&40\\60&-37&40&135\end{bmatrix}\ \begin{bmatrix}15&97&64&36\\97&-13&-50&76\\64&-50&-63&40\\36&76&40&48\end{bmatrix}\ \begin{bmatrix}-27&45&-27&51\\45&0&-30&10\\-27&-30&48&-44\\51&10&-44&24\end{bmatrix}\ \begin{bmatrix}-60&30&10&-52\\30&45&-55&-2\\10&-55&40&32\\-52&-2&32&-32\end{bmatrix}$$

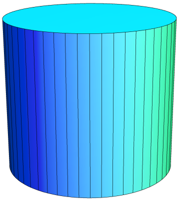
Many flavors of quartic spectrahedra

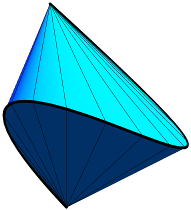


Special Quartic Spectrahedra

Non-generically, the span of A_0, A_1, A_2, A_3 might contain a curve of rank-two matrices.

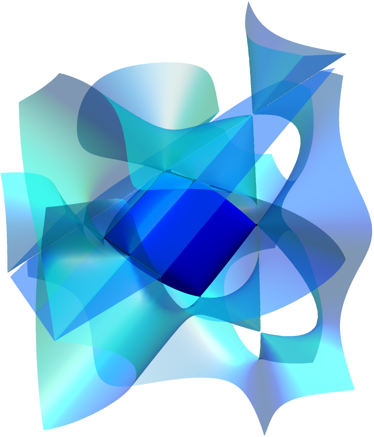






$$\begin{pmatrix} x_0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_3 \end{pmatrix} \quad \begin{pmatrix} x_0 + x_1 & x_2 & 0 & 0 \\ x_2 & x_0 - x_1 & 0 & 0 \\ 0 & 0 & x_0 + x_3 & 0 \\ 0 & 0 & 0 & x_0 - x_3 \end{pmatrix} \quad \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_0 & x_1 & x_2 \\ x_2 & x_1 & x_0 & x_1 \\ x_3 & x_2 & x_1 & x_0 \end{pmatrix}$$

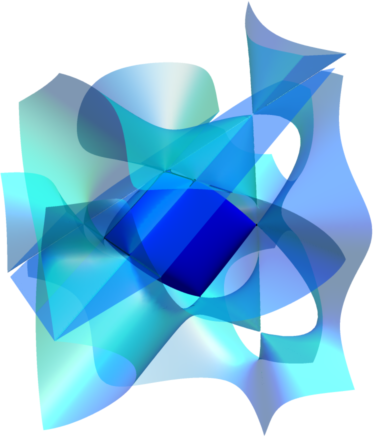
Conclusions



Spectrahedra can be understood using beautiful and classical algebraic geometry.

There is still lots to understand. What are the combinatorial types of spectrahedra of higher dimensions and degrees?

Conclusions



Spectrahedra can be understood using beautiful and classical algebraic geometry.

There is still lots to understand. What are the combinatorial types of spectrahedra of higher dimensions and degrees?

Thanks for your attention!