Refined Knot Invariants and Hilbert Schemes (joint with A. Negut)

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Outline

Reminder on Hilbert schemes

Hilbert scheme of points

The *symmetric power* $S^n\mathbb{C}^2$ is the moduli space of unordered n-tuples of points on $\mathbb{C}^2$.

The *Hilbert scheme* $\mathrm{Hilb}^n\,\mathbb{C}^2$ is the moduli space of codimension n ideals in $\mathbb{C}[x,y]$. Such an ideal is supported on a finite subset of n points in $\mathbb{C}^2$ (with multiplicities), this defines the Hilbert-Chow morphism:

$$HC : \mathrm{Hilb}^n\,\mathbb{C}^2 \to S^n\mathbb{C}^2.$$

Theorem (Fogarty)

$\mathrm{Hilb}^n\,\mathbb{C}^2$ *is a smooth manifold of dimension* $2n$.

Reminder on Hilbert schemes

Torus action

The natural scaling action of $(\mathbb{C}^*)^2$ lifts to an action on $S^n\mathbb{C}^2$ and on $\mathrm{Hilb}^n\,\mathbb{C}^2$. It has a finite number of fixed points corresponding to monomial ideals.

Example:

y^3	xy^3	x^2y^3	x^3y^3	x^4y^3	x^5y^3
y^2	xy^2	x^2y^2	x^3y^2	x^4y^2	x^5y^2
y	xy	x^2y	x^3y	x^4y	x^5y
1	x	x^2	x^3	x^4	x^5

The ideal is generated by y^3, xy^2, x^3y, x^4

Reminder on Hilbert schemes

Punctual Hilbert scheme

The *punctual Hilbert scheme* $\mathrm{Hilb}^n(\mathbb{C}^2, 0)$ is the scheme-theoretic fiber of the Hilbert-Chow morphism over $\{n \cdot 0\}$.

Theorem (Briançon,Haiman)

$\mathrm{Hilb}^n(\mathbb{C}^2, 0)$ *is reduced, irreducible and Cohen-Macaulay. Its dimension equals* $n - 1$*.*

Example

$\mathrm{Hilb}^2(\mathbb{C}^2, 0) = \mathbb{P}^1$; $\mathrm{Hilb}^3(\mathbb{C}^2, 0)$ is a (projective) cone over twisted cubic in $\mathbb{P}^3$. The vertex of the cone is the monomial ideal (x^2, xy, y^2).

Sheaves and operators

Tautological sheaf

We are interested in constructing various sheaves on $\mathrm{Hilb}^n \mathbb{C}^2$. The easiest one is the *tautological bundle* T of rank n, whose fiber over a point representing an ideal I equals $\mathbb{C}[x, y]/I$. One can consider its symmetric powers $S^n T$, exterior powers $\wedge^n T$. We will also need the formal classes $p_k(T)$ in the equivariant K-theory of $\mathrm{Hilb}^n \mathbb{C}^2$ defined by the equation

$$\sum_{k=1}^{\infty} p_k(T) t^{k-1} = \frac{d}{dt} \ln \left(\sum_{k=0}^{\infty} S^k T \cdot t^k \right).$$

We will need the operators

$$P_{0,k} : K_n \to K_n,\ [\mathcal{E}] \to [\mathcal{E} \otimes p_k(T)],$$

where $K_n = K_{(\mathbb{C}^*)^2} \mathrm{Hilb}^n \mathbb{C}^2$.

Sheaves and operators

Simple correspondences

Define

$$Hilb^{n,n+1} \subset \mathrm{Hilb}^n \times \mathrm{Hilb}^{n+1}, \ \ Hilb^{n,n+1} = \{(I,J) : I \subset J\}.$$

Theorem (Cheah, Tikhomirov, Ellingsrud...)

The space $Hilb^{n,n+1}$ is smooth of dimension $2n+2$.

There is a natural bundle $\mathcal{L} := J/I$ on $Hilb^{n,n+1}$, and one can define operators

$$P_{1,k} : K_n \to K_{n+1}, P_{1,k}(\mathcal{E}) := p_{(n+1)*}\left(\mathcal{L}^k \otimes p_n^* \mathcal{E}\right),$$

where $p_n : Hilb^{n,n+1} \to \mathrm{Hilb}^n$ and $p_{n+1} : Hilb^{n,n+1} \to \mathrm{Hilb}^{n+1}$ denote the natural projections.

Sheaves and operators

Algebra of correspondences

Theorem (Schiffmann-Vasserot,Feigin-Tsymbaliuk)

The operators $P_{0,k}$ and $P_{1,k}$ generate (a half of) an algebra $\mathcal{A}$, which is known as:

- *Elliptic Hall algebra*
- *Double affine Hecke algebra of GL_∞*
- *Shuffle algebra*

In particular, there is an action of $SL(2,\mathbb{Z})$ on the algebra $\mathcal{A}$. For a pair of integers (n,m) with $GCD(m,n)=d$ one can choose a matrix $\gamma \in SL(2,\mathbb{Z})$ such that $\gamma(d,0)=(n,m)$; we define an operator

$$P_{n,m} = \gamma(P_{d,0}).$$

Sheaves and operators

Flag Hilbert schemes

Consider the moduli space of flags

$$\mathrm{Hilb}^{k,k+1,\ldots,k+n} := \{J_k \supset J_{k+1} \supset J_{k+2} \supset \ldots \supset J_{k+n}\},$$

where J_i is an ideal in $\mathbb{C}[x,y]$ of codimension i and all quotients J_i/J_{i+1} are supported at the origin. There are two projections:

$$p_k : \mathrm{Hilb}^{k,k+1,\ldots,k+n} \to \mathrm{Hilb}^k, \;\; p_{n+k} : \mathrm{Hilb}^{k,k+1,\ldots,k+n} \to \mathrm{Hilb}^{k+n},$$

and n line bundles $\mathcal{L}_i := J_i/J_{i+1}$ on $Hilb^{k,k+1,\ldots,k+n}$.

Example

$\mathrm{Hilb}^{0,1,2} = \mathrm{Hilb}^2 = \mathbb{P}^1$; $\mathrm{Hilb}^{0,1,2,3}$ is isomorphic to the Hirzebruch surface $\mathbb{P}(\mathcal{O} + \mathcal{O}(-3)) \to \mathbb{P}^1$. It is a blowup of the singular cone $\mathrm{Hilb}^3(\mathbb{C}^2, 0)$.

Sheaves and operators

Flag Hilbert schemes: operators

In general, flag Hilbert scheme is singular (and reducible). There is a way to define a virtual tangent bundle to it, so that it is a virtual local complete intersection.

Theorem (Negut)

Suppose that $GCD(n, m) = 1$, then the operator $P_{n,m}$ is defined by the equation:

$$P_{n,m} : K_r \to K_{r+n}, \; P_{n,m}(\mathcal{E}) := p_{(r+n)*}\left(\prod_i \mathcal{L}_{r+i}^{S_i} \otimes p_r^* \mathcal{E}\right),$$

where

$$S_i = \left\lfloor \frac{mi}{n} \right\rfloor - \left\lfloor \frac{m(i-1)}{n} \right\rfloor.$$

Sheaves and operators

Flag Hilbert schemes: localization

The natural action of $(\mathbb{C}^*)^2$ on the Hilbert scheme lifts to an action on the flag Hilbert scheme.
A torus fixed point on $\mathrm{Hilb}^{k,k+1,\dots,k+n}$ is a tuple of Young diagrams $\lambda_k \subset \lambda_{k+1} \subset \dots \subset \lambda_{k+n}$ such that $|\lambda_i| = i$.
These are in one-to-one correspondence with standard Young tableaux (SYT) of skew shape $\lambda_{k+n} \setminus \lambda_k$:

3			
1	5		
		2	4

$\longleftarrow$ A fixed point on $\mathrm{Hilb}^{2,3,4,5,6,7}$

Sheaves and operators

Localization and Macdonald polynomials

One can use localization to match geometry with known and new representation-theoretic constructions:

- The space K_n is identified with the space of degree n symmetric polynomials
- The space $K := \bigoplus_{n=0}^{\infty} K_n$ is identified with the space of symmetric polynomials in infinitely many variables.
- The fixed point basis in K_n is identified with the Haiman's *modified Macdonald basis* in K.
- The operators $P_{n,0}$ are identified with the multiplication operators by p_n
- The operators $P_{0,n}$ are identified with the Macdonald operators (as they diagonalize in Macdonald basis)

The localization also provide formulae for the matrix elements of the operators $P_{n,m}$ as sums over standard Young tableaux.

Sheaves and operators

Example: Pieri rule

Let $\widetilde{H}_\lambda$ denote the modified Macdonald polynomial, then

$$P_{1,0}\left(\widetilde{H}_\lambda\right) = p_1 \widetilde{H}_\lambda = \sum_{\mu=\lambda+\square} d_{\lambda\mu} \widetilde{H}_\mu,$$

where $d_{\lambda\mu}$ is a certain explicit coefficient.
For example,

$$p_1 \cdot \widetilde{H}_{\square\square} = \frac{1-t}{q^2-t}\widetilde{H}_{\square\square\square} + \frac{1-q^2}{t-q^2}\widetilde{H}_{\substack{\square\\ \square\square}}$$

Geometrically, there is exactly one fixed point on $\mathrm{Hilb}^{k,k+1}$ which projects to λ on Hilb^k and to μ on Hilb^{k+1}.

Sheaves and operators

Example: q, t-Catalan numbers

Consider the line bundle $\mathcal{O}(1) = \wedge^n T$ on $\mathrm{Hilb}^n(\mathbb{C}^2, 0)$.

Theorem (Haiman)

a) $H^i(\mathrm{Hilb}^n(\mathbb{C}^2, 0), \mathcal{O}(1)) = 0$ *for* $i > 0$*;*
b) $\dim H^0(\mathrm{Hilb}^n(\mathbb{C}^2, 0), \mathcal{O}(1)) = \frac{1}{n+1}\binom{2n}{n}$.

The bigraded character of $H^0(\mathrm{Hilb}^n(\mathbb{C}^2, 0), \mathcal{O}(1))$ is called the q, t-Catalan number, it has many interesting combinatorial properties.

Theorem (G., Negut)

The following identity hold in K_n*:*

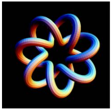
$$[\mathcal{O}(1) \otimes \mathcal{O}_{\mathrm{Hilb}^n(\mathbb{C}^2,0)}] = P_{n,n+1} \cdot 1.$$

Knot invariants

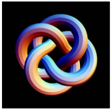
Torus knots

T(3,2)

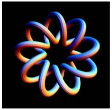
T(5,2)



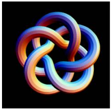
T(7,2)



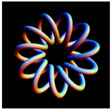
T(4,3)



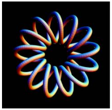
T(9,2)



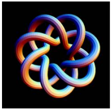
T(5,3)



T(11,2)

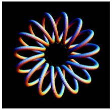


T(13,2)

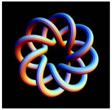


T(7,3)

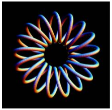
T(5,4)



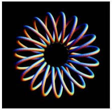
T(15,2)



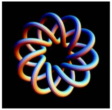
T(8,3)



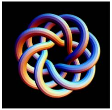
T(17,2)



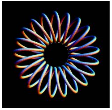
T(19,2)



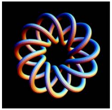
T(10,3)



T(7,4)



T(21,2)



T(11,3)

Knot invariants

Polynomial invariants

Various invariants of knots has been developed. The ones most relevant for this talk are the colored Reshetikhin-Turaev invariants $P_{\lambda,N}(q)$. They are parametrized by an integer N and a Young diagram λ ("color"), and their specializations include:

- Jones polynomial ($N = 2, \lambda = \Box$)
- Colored Jones polynomial ($N = 2$)
- $sl(N)$ skein invariants ($\lambda = \Box$)

The invariants for various N can be unified by the colored HOMFLY polynomials $P_\lambda(a, q)$ such that

$$P_{\lambda,N}(q) = P_\lambda(a = q^N, q).$$

Knot invariants

Recent developments

Khovanov and Rozansky developed a knot homology theory, which assigns a collection of homology groups to each knot. The Euler characteristic of this homology coincides with the HOMFLY polynomial.

It is known that Khovanov-Rozansky homology carry nontrivial geometric information: for example, they can be used for genus bounds. However, their definition uses knot diagram, and explicit computations are very hard.

Based on physical ideas, Aganagic and Shakirov defined *refined Chern-Simons invariants* for torus knots using Macdonald polynomials. In all examples, Aganagic-Shakirov and Khovanov-Rozansky invariants agree.

Knot invariants

Main theorem

Theorem (G., Negut)

For $\lambda = \square$ the Aganagic-Shakirov invariant is given by the formula:

$$\mathcal{P}_{\square}(T(m,n)) = \sum_{T} \frac{\prod_i \chi_i^{S_i}(1 - a\chi_i^{-1})}{\prod_{i=2}^{n}(1-\chi_i)(1 - qt\chi_{i-1}/\chi_i)} \prod_{i<j} \omega\left(\frac{\chi_i}{\chi_j}\right),$$

where the summation is over standard Young tableaux T of size n, χ_i denote q,t-contents of boxes in T,

$$S_i = \left\lfloor \frac{mi}{n} \right\rfloor - \left\lfloor \frac{m(i-1)}{n} \right\rfloor$$

and

$$\omega(x) = \frac{(1-x)(1-qtx)}{(1-qx)(1-tx)}.$$

Knot invariants

Idea of proof

The construction of Aganagic and Shakirov is motivated by topological quantum field theory and runs as follows:

- To the two-dimensional torus they associate a vector space $Z(T^2)$ with a distinguished 'vacuum vector' $\mathbf{1}$
- To a torus knot $T(n, m)$ they associate an operator $W_{n,m}$ on $Z(T^2)$ and a vector $W_{n,m} \cdot \mathbf{1}$
- There is an action of $SL(2, \mathbb{Z})$ on the algebra generated by $W_{n,m}$ such that $W_{n,m} = \gamma(W_{1,0})$ for appropriate γ and coprime m, n; this action is defined using the work of Etingof and Kirillov on Macdonald polynomials
- The sphere S^3 is glued from two solid tori. One of them contains $T(m, n)$, the other is empty and generates a vector $v(a)$ in $Z(T^2)$.
- Finally, the knot invariant is computed as $(W_{n,m} \cdot \mathbf{1}, v(a))$.

Knot invariants

Idea of proof cont'd

We match this construction to the Hilbert scheme picture:

- The space $Z(T^2)$ is identified with K
- Using the results of Cherednik, Schiffmann and Vasserot on Macdonald polynomials and DAHA, the operators $W_{n,m}$ can be matched with $P_{n,m}$
- The vacuum vector $\mathbf{1}$ represents the class of $\mathrm{Hilb}^0 = pt$
- The vector $v(a)$ is identified with $\sum_i (-a)^i \Lambda^i T^*$
- Finally, the knot invariant equals

$$(W_{n,m} \cdot \mathbf{1}, v(a)) = \sum_i (-a)^i \int_{\mathrm{Hilb}^n \mathbb{C}^2} \Lambda^i T^* \otimes (P_{n,m} \cdot \mathbf{1})$$

The theorem then computes this invariant by localization in fixed points.

Thank you