Parabolic Limits of Renormalization

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Abstract of the Dissertation

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A unimodal map $f : [0,1] \to [0,1]$ is renormalizable if there is a sub-interval $I \subset [0,1]$ and an $n > 1$ such that $f^n|_I$ is unimodal. The renormalization of f is $f^n|_I$ rescaled to the unit interval.

We extend the well-known classification of limits of renormalization of unimodal maps with bounded combinatorics to a classification of the limits of renormalization of unimodal maps with essentially bounded combinatorics. Together with results of Lyubich on the limits of renormalization with essentially unbounded combinatorics, this completes the combinatorial description of limits of renormalization. The techniques are based on the towers of McMullen and on the local analysis around perturbed parabolic points. We define a parabolic tower to be a sequence of unimodal maps related by renormlization or *parabolic renormalization.* We state and prove the *combinatorial rigidity* of bi-infinite parabolic towers with complex bounds and essentially bounded combinatorics, which in turn implies the main theorem.

As an example we construct a natural unbounded analogue of the period-doubling fixed point of renormalization, called the *essentially period-tripling* fixed point.

To my parents

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$$\forall x \in \mathbb{R}\ \forall \epsilon > 0\ \exists \delta > 0\ \forall y \in \mathbb{R}\ |y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon$$

written on the board already. He claimed we would understand such nonsense within a few months and, miraculously, he was right.

I also thank all my friends and colleagues at Stony Brook. There are too many to lists here, but I particularly thank two people. I thank Janko for all the good times and for somehow surviving being my house-mate, apartment-mate, office-mate, beach-mate, basketball-mate, pool-mate, quake-mate, etc. for various years in various places. And last but definitely not least I thank Jenn for all of the hope, inspiration and love she has given me.

Chapter 1

Introduction

This thesis investigates renormalization of certain unimodal or quadratic-like maps (see §2 for definitions and background). For our purposes the goal is to answer the following

Main Question: *What, if any, are the limits of renormalization of an ∞-renormalizable unimodal or quadratic-like map?*

In this chapter we first give the context of our results. Then we state our results and discuss to what degree the main question has been answered.

1.1 Context

History begins, for us, in the 70's and early 80's when many people became interested in renormalizable maps with *period-doubling* combinatorics. One of the main results was the existence of a unique limit of renormalization in the period-doubling case: the *period-doubling fixed point of renormalization*. Let us be more specific. To fix notation, let $\mathcal{Q}_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{R}}$ denote the space of real-symmetric quadratic-like maps and germs, respectively. Let $\mathcal{R}f$ denote the renormalization of f. Let $p(f)$ denote the period of renormalization.

Theorem (Period Doubling Combinatorics, [Lan, E1, E2]). *There exists a unique $F \in \mathcal{G}_{\mathbb{R}}$ such that if $f \in \mathcal{Q}_{\mathbb{R}}$ is ∞-renormalizable with $p(\mathcal{R}^n f) = 2$ for all $n \geq 0$ then $\mathcal{R}^n f \to F$.*

Throughout this introduction we state theorems for real-symmetric quadratic-like maps. The generalizations for different classes of maps will be discussed in §1.3. Once the period-doubling fixed point was found, attention turned to the case of maps with *bounded combinatorics.*

Theorem (Bounded Combinatorics, [S, McM2]). *For $p > 1$ there exists a compact $\mathcal{R}$-invariant set $\mathcal{A} \subset \mathcal{G}_{\mathbb{R}}$ such that*

1. $\mathcal{R}|_\mathcal{A}$ *is conjugate to the full shift on* $\Omega(p)$*, the set of combinatorial types of period bounded above by* p*, and*

2. *if* $f \in \mathcal{Q}_\mathbb{R}$ *is* ∞*-renormalizable and* $\sup_n p(\mathcal{R}^n f) \leq p$ *then* $\mathcal{R}^n f \to \mathcal{A}$.

Since $\mathcal{R}$ is defined and continuous in a neighborhood of $\mathcal{A}$ it follows that the limits of renormalization of a map with bounded combinatorics are encoded by bi-infinite sequences of a finite number of symbols. However, this theorem still left unexplored the large set of maps with unbounded combinatorics. Lyubich answered the Main Question for the subset of maps with *essentially unbounded combinatorics*. Let $p_e(f)$ denote the *essential period* of a renormalizable $f \in \mathcal{Q}_\mathbb{R}$. The essential period roughly measures the period of renormalization minus the time spent near a "ghost parabolic point". See §2.3.3 for a precise definition.

Theorem (Essentially Unbounded Combinatorics, [L3]). *If* $f \in \mathcal{Q}_\mathbb{R}$ *is renormalizable then*

$$\operatorname{mod} \mathcal{R}f \geq \mu(p_e(f), \operatorname{mod} f) > 0$$

where $\mu \to \infty$ *when* $p_e(f) \to \infty$.

In this thesis we finish answering the Main Question by characterizing renormlization limits of maps with unbounded but essentially bounded combinatorics.

1.2 New Results

Our first result produces the essentially-bounded analog of the period-doubling fixed point of renormalization. In §3.1 we construct a countable collection of maximal tuned Mandelbrot copies $\{M_n^{(3)}\}_{n=1}^\infty$ that accumulate at $c = -1.75$, the root point of the real period-three Mandelbrot copy $M^{(3)}$. These copies have "essentially period tripling" combinatorics. They will play the roll of the period-doubling tuned Mandelbrot copy.

Theorem I (Essentially Period Tripling). *There is a unique* $F \in \mathcal{G}_\mathbb{R}$ *such that*

$$\mathcal{R}^n f \to F$$

for any ∞*-renormalizable* $f \in \mathcal{Q}_\mathbb{R}$ *with a tuning invariant*

$$\tau(f) = (M_{n_1}^{(3)}, M_{n_2}^{(3)}, \ldots, M_{n_k}^{(3)}, \ldots)$$

satisfying $n_k \to \infty$ *as* $k \to \infty$.

Our second result answers the Main Question for the essentially bounded combinatorics case.

Theorem II (Essentially Bounded Combinatorics). *For every* $p > 1$ *there exists a pre-compact* $\mathcal{R}$*-invariant set* $\mathcal{A} \subset \mathcal{G}_\mathbb{R}$ *such that*

1. *$\mathcal{R}|_{\mathcal{A}}$ is conjugate to the full shift on $\Omega_e(p)$, the set of combinatorial types with essential period bounded above by p, and*

2. *if $f \in \mathcal{Q}_{\mathbb{R}}$ is ∞-renormalizable and $\sup_n p_e(\mathcal{R}^n f) \leq p$ then $\mathcal{R}^n f \to \overline{\mathcal{A}}$.*

Let $h : \Pi_{-\infty}^{\infty}\Omega_e(p) \to \mathcal{A}$ denote the conjugacy. Then there is a compactification $\Omega_e^{cpt}(p)$ of $\Omega_e(p)$ such that h extends to a continuous map $h : \Pi_{-\infty}^{\infty}\Omega_e^{cpt}(p) \to \overline{\mathcal{A}}$. Moreover, suppose $f_1 \in \mathcal{Q}_{\mathbb{R}}$ has the same combinatorics as $f_2 \in \mathcal{A}$. Then for any sequence $n_k \to \infty$ and sequence $\bar{\sigma} \in \Pi_{-\infty}^{\infty}\Omega_e^{cpt}(p)$ such that $\mathcal{R}^{n_k} f_2 \to h(\bar{\sigma})$ one has $\mathcal{R}^{n_k} f_1 \to h(\bar{\sigma})$.

Note that there does not exists a neighborhood of $\overline{\mathcal{A}}$ on which $\mathcal{R}$ is continuous and, moreover, it is not know even if $\mathcal{R}|_{\mathcal{A}}$ extends continuously to $\overline{\mathcal{A}}$. If the map h were injective then we could conjugate the shift on $\Pi_{-\infty}^{\infty}\Omega_e^{cpt}(p)$ to a continuous extension of $\mathcal{R}$ on $\overline{\mathcal{A}}$. We expect h to be injective (see [Y2, Remark 3.1]). However, Theorem II states that any renormalization limit of a map with essentially bounded combinatorics has a combinatorial description. Let us discuss this theorem in more detail. Recall that the central objects of McMullen's argument [McM2] are *towers:* sequences of quadratic-like maps related by renormalization. A *forward tower* is a one-sided sequence and a *bi-infinite tower* is a two-sided infinite sequence. The convergence of renormalization is implied by the combinatorial rigidity of the corresponding limiting bi-infinite towers. That is, for the purposes of introduction, we can define the space Tow of McMullen towers to be the space of one-sided or two-sided sequences $\mathcal{T}$ of germs $f \in \mathcal{G}_{\mathbb{R}}$ such that $f_{n+1} = \mathcal{R} f_n$. If we embed the space X of germs of ∞-renormalizable maps with bounded combinatorics into Tow by the map $\imath : X \to Tow$ defined for all $n \geq 0$ by

$$\imath([f])(n) = \mathcal{R}^n[f]$$

then we have the following equality:

$$\mathcal{R}^n = \pi_0 \circ \theta^n \circ \imath$$

where θ is the shift operator and π_0 is the projection to the 0^{th}-coordinate. Since X embeds into a compact subspace $Tow(p)$ of Tow we can pass to the limit as $n \to \infty$ and obtain a limiting tower $\mathcal{T} \in \omega_\theta(\imath([f]))$. Since the tower $\mathcal{T}$ is *uniquely* determined (in $Tow(p)$) by its shuffle sequence, the theorem on convergence with bounded combinatorics follows.

We use a similar argument. However, X_e, the space of germs of ∞-renormalizable quadratic-like maps $f \in \mathcal{Q}_{\mathbb{R}}$ with $\sup_n p_e(\mathcal{R}^n f) \leq p$, does not embed into a pre-compact subspace of Tow. That is, for maps with essentially bounded combinatorics the limiting towers may contain parabolic maps and we lose the renormalization relation between levels. In this case a new relation appears: parabolic renormalization. That is, the maps in the limiting towers are related by either classical or parabolic renormalization. A tower which contains a parabolic renormalization is called a parabolic tower.

See §5.1 for the precise definitions of parabolic towers, and of the space of towers with essentially bounded combinatorics and complex bounds, denoted $Tow(m, B)$

where m is the modulus bound and B is the combinatorial bound. Our proof of the rigidity of bi-infinite parabolic towers with definite modulus and essentially bounded combinatorics consists of first analyzing forward towers and then analyzing bi-infinite towers.

Our analysis of forward parabolic towers was motivated by the work of A. Epstein [E], which considered general holomorphic dynamical systems (with maximal domains of definition) and their geometric limits. The phenomenon studied there was the renormalization (different from the sense used in this paper) of a parabolic orbit at the ends of its Ècalle-Voronin cylinders. The parabolic renormalization we study occurs away from the ends. As a result forward infinite parabolic towers share many properties with infinitely renormalizable real quadratic maps. In some sense many properties of ∞-renormalizable maps "pass to the limit".

The combinatorial rigidity of forward parabolic towers with polynomial base map follows from the theory of *quadratic-like families* and from the combinatorial rigidity of quadratic polynomials with complex bounds and real combinatorics (see Proposition 5.2.6). After analyzing the Julia set of a forward tower we prove any quasiconformal conjugacy of a forward infinite parabolic tower in $Tow(m, B)$ is a hybrid conjugacy (see §5.2.9).

Then following the arguments of McMullen we prove in §5.3 the rigidity of bi-infinite towers. That is, we first prove

Theorem 1.2.1 (Dynamical Hairiness) *The union of the Julia sets of the forward infinite sub-towers of a bi-infinite tower in $Tow(m, B)$ is dense in the plane.*

Then we prove

Theorem 1.2.2 (Quasi-conformal Rigidity) *Any quasiconformal equivalence of a bi-infinite tower in $Tow(m, B)$ is affine.*

and

Theorem 1.2.3 (Combinatorial Rigidity) *Any two normalized combinatorially equivalent bi-infinite towers in $Tow(m, B)$ are equivalent.*

1.3 Questions

The classification of renormalization limits for bounded combinatorics holds for a more general class of maps. For one, if f is an ∞-renormalizable real unimodal map that is C^{1+Z}-smooth with quadratic-like critical point and if $\sup_n p(\mathcal{R}^n f) \leq p$ then $\mathcal{R}^n f \to \mathcal{A}$ as well. One might hope to use the techniques from this thesis to prove the corresponding statement for unimodal maps with essentially bounded combinatorics. However, our result requires the existence of perturbed Fatou coordinates, which do not exist in the non-holomorphic class.

All of the theorems stated above are true for ∞-renormalizable maps $f \in \mathcal{Q}$ that only satisfy $I(f) \in \mathbb{R}$ where $I(f)$ is the inner-class of f. Theorems I and II are proved in this generality. One can also consider ∞-renormalizable quadratic-like maps that do not satisfy $I(f) \in \mathbb{R}$. Since the combinatorial rigidity of bi-infinite towers invokes real combinatorics in only a few places, we expect to be able to generalize Theorem II to certain classes of maps with complex combinatorics. However, the techniques used to prove Theorem II cannot succeed for all combinatorial types, since a vital ingredient in the proof is the existence of complex bounds, which is known to be false for certain ∞-renormalizable maps (see [M2, Theorem 6] and [McM2, Proposition 4.14]).

Let us mention a parallel with critical circle maps. The theory of renormalization of unimodal maps is closely related to renormalization theory of critical circle maps. The rotation number ρ, more specifically its continued fraction expansion, determines the combinatorics of a circle map. If the factors in its expansion are bounded then the map has bounded combinatorics and has unbounded combinatorics otherwise. If a circle map has unbounded combinatorics then the rotation numbers of the renormalizations contain rational limit points and the corresponding limit of renormalization contain parabolic periodic points. That is, the only kind of unbounded combinatorics in the theory of critical circle maps is the essentially bounded combinatorics. DeFaria [deF] analyzed the renormalization limits of critical circle maps with bounded combinatorics and Yampolsky [Y2] proves the analogue of Theorem II for critical circle maps with unbounded combinatorics.

1.4 Notation

- $\mathbb{H} \subset \mathbb{C}$ denotes the complex upper half-plane, $\widehat{\mathbb{C}}$ the Riemann sphere, $\mathbb{N} = \mathbb{N}_0$ the non-negative integers and $\mathbb{N}_+$ the positive integers.
- $[a, b]$ will also denote the interval $[b, a]$ if $b < a$.
- $\text{diam}(U)$ denotes the Euclidean diameter of $U \subset \mathbb{C}$ and $|I|$ the diameter of $I \subset \mathbb{R}$.
- $\text{cl}(X)$, $\text{int}(X)$ and ∂X denote the closure, interior and boundary of X in $\mathbb{R}$ if $X \subset \mathbb{R}$ and in $\mathbb{C}$ otherwise.
- $U \Subset V$ means U is compactly contained in V. Namely $\text{cl}(U)$ is compact and $\text{cl}(U) \subset V$.
- in a dynamical context f^n denotes f composed with itself n times.
- if V is a simply connected domain and $U \subset V$ then $\text{mod}\,(V, U) = \sup_A \text{mod}(A)$ where A is an annulus separating U from ∂V.
- $Dom(f)$ and $Range(f)$ denote the domain and range of f.
- $Im(f)$ denotes the image of f
- $\text{CC}(X)$ denotes the collection of connected components of X.

- $P_c(z) = z^2 + c$.
- an ϵ-scaled neighborhood of a domain U is an $\epsilon \cdot \operatorname{diam}(U)$ neighborhood of U

Chapter 2

Background

2.1 Quadratic-like Maps

We will use the notation $Dom(f)$ for the domain of a map f and $Range(f)$ for the image of f. A pair (U, x) consisting of a simply connected domain $U \subset \mathbb{C}$, $U \neq \mathbb{C}$, and a point $x \in U$ is called a *pointed disk*. We will call a simply connected domain that is not the entire plane a *topological disk* and we will call a topological disk whose boundary is a Jordan curve a *Jordan disk*. Given a pointed disk (U, x) let $R_{(U,x)} : \mathbb{D} \to U$ be the Riemann map normalized so that $R_{(U,x)}(0) = x$ and $R'_{(U,x)}(0) > 0$. Topologize the set $\mathcal{D}$ of pointed disks with the Caratheodory topology: the compact-open topology on the Riemann maps $R_{(U,x)}$.

Let $\mathcal{H}$ denote the space of holomorphic maps defined on topological disks with the Caratheodory topology. More specifically, define

$$\mathcal{H} = \{(f, U, x) \mid (U, x) \in \mathcal{D}, f : U \to \mathbb{C} \text{ holomorphic}\}$$

and impose on $\mathcal{H}$ the topology generated by neighborhoods of the form

$$\mathcal{N}(f, U, x, \epsilon, K, \mathcal{D}') = \{(g, V, y) \in \mathcal{H} \mid (V, y) \in \mathcal{D}',\ \|f - g\|_K < \epsilon\}$$

where $(f, U, x) \in \mathcal{H}$, $\epsilon > 0$, $K \subset U$ is compact and $\mathcal{D}'$ is a neighborhood of (U, x) in $\mathcal{D}$ such that $K \subset V$ for all $(V, y) \in \mathcal{D}'$. Let $\mathcal{H}_0 = \{(f, U, x) \in \mathcal{H} : x = 0\}$.

A map $f \in \mathcal{H}$ is *quadratic-like* if $Dom(f) \Subset Range(f)$ and f is a branched double cover of $Dom(f)$ onto $Range(f)$ (see [DH2]). An actual quadratic polynomial can be considered quadratic-like by restricting to the pre-image of $\{z : |z| < R\}$ for some large R. Unless otherwise indicated we will assume the critical point of a quadratic-like map is at the origin. A point $z \in Dom(f)$ is *non-escaping* if $f^n(z) \in Dom(f)$ for all $n \geq 0$. For a quadratic-like f define

$$\begin{aligned}
\text{modulus of } f &= \operatorname{mod} f &&= \operatorname{mod}(Range(f), Dom(f)) \\
\text{the filled Julia set of } f &= K(f) &&= \{z \in Dom(f) \mid z \text{ is non-escaping}\} \\
\text{the Julia set of } f &= J(f) &&= \partial K(f) \\
\text{the post-critical set of } f &= P(f) &&= \overline{\textstyle\bigcup_{n \geq 1} f^n(0)}.
\end{aligned}$$

Define the following subspaces of $\mathcal{H}_0$:

$$\begin{aligned}
\mathcal{Q} &= \{f \in \mathcal{H}_0 : f \text{ is quadratic-like}\} \\
\mathcal{Q}^n &= \{f \in \mathcal{Q} : f^k(0) \in Dom(f) \text{ for } k = 0, 1, \ldots, n\} \\
\mathcal{Q}^\infty &= \{f \in \mathcal{Q} : K(f) \text{ is connected}\} = \cap_{n \geq 0} \mathcal{Q}^n \\
\mathcal{Q}_\mathbb{R} &= \{f \in \mathcal{Q} : z \in Dom(f) \Rightarrow \bar{z} \in Dom(f) \text{ and } f(\bar{z}) = \overline{f(z)}\} \\
\mathcal{Q}(m) &= \{f \in \mathcal{Q} : \operatorname{mod} f \geq m\}.
\end{aligned}$$

Similarly define $\mathcal{Q}_\mathbb{R}^\infty$, $\mathcal{Q}_\mathbb{R}(m)$, $\mathcal{Q}^\infty(m)$, etc. as the appropriate intersections of the above subspaces.

Let $f \in \mathcal{Q}$. For a given $x \neq 0$ let $x' = f^{-1}(f(x)) \setminus \{x\}$. If $x = 0$ let $x' = 0$. There are, counted with multiplicity, two fixed points of f denoted α and β and, in the case $f \in \mathcal{Q}^\infty$, labeled so that either α is non-repelling or, if both fixed points are repelling, so that $J(f) \setminus \{\beta\}$ is connected. Note $\alpha = \beta$ iff $f'(\beta) = 1$. We say $f \in \mathcal{Q}^\infty$ is *normalized* if $\beta_f = 1$. We note the following folklore facts: (see, for example, [D], [McM1, Theorem 5.9], [McM2, Proposition 4.12], [McM1, Theorem 5.8])

Lemma 2.1.1 *The function* $\operatorname{diam} K(f)$ *is a continuous function on* $\mathcal{Q}$ *and the functions* α_f *and* β_f *are continuous on* $\mathcal{Q}^\infty$.

Lemma 2.1.2 *For any* $m > 0$ *and* $C < \infty$, *the set*

$$\{f \in \mathcal{Q}^\infty(m) : \operatorname{diam} K(f) = 1,\ d(0, \mathbb{C} \setminus Range(f)) \leq C\}$$

is compact.

2.1.1 Quadratic-like germs

Two holomorphic maps determine the same germ at z if they are equal on a neighborhood of z. That is, consider the equivalence relation $\sim_z$ defined on $\{f \in \mathcal{H} \mid z \in Dom(f)\}$ by

$$f \sim_z g \text{ iff there exists a nbhd } U \subset Dom(f) \cap Dom(g) \text{ of } z \text{ such that } f|_U = g|_U.$$

For a given $f \in \mathcal{H}$ with $z \in Dom(f)$ the *germ* of f, denoted $[f]_z$, is the equivalence class of f. The germ of a map $f \in \mathcal{H}_0$ at $z = 0$ will be denoted by $[f]$. We call a connected component F of $[f] \cap \mathcal{Q}$ a *quadratic-like germ* (see Fig. 2.1). This definition is slightly different from [L5, §3.1]. However, on $\mathcal{Q}^\infty$ they agree, as shown below.

Lemma 2.1.3 *Let* $f \in \mathcal{H}_0$ *and let* F *be a connected component of* $[f] \cap \mathcal{Q}$. *If* $f_1, f_2 \in F$ *then* $K(f_1) = K(f_2)$.

Proof: Let $X = \{g \in F : K(g) = K(f)\}$. We first claim X is closed: let $g_n \in X$ be sequence converging to $g \in F$. Let $W \subset Range(g)$ be a closed disk such that $g(0) \in \operatorname{int} W \supset Dom(g)$. Let $h = g$ restricted to $W_1 = g^{-1}(\operatorname{int} W)$. Then $K(h) = K(g)$.

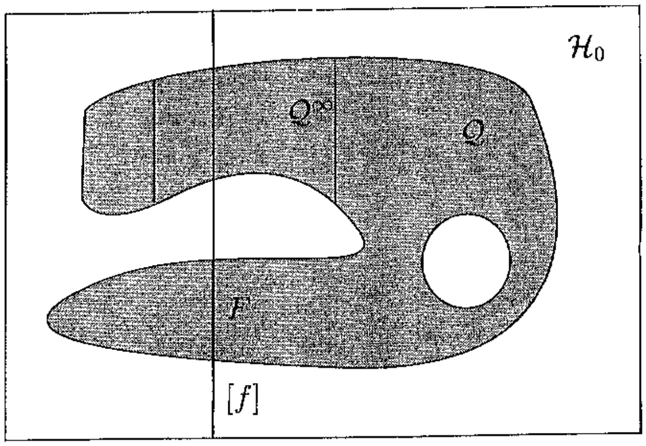

Figure 2.1: The space $\mathcal{H}_0$.

Also, since $W_1 \subset Dom(g_n)$ for large enough n, it follows from [McM1, Theorem 5.11] that $K(h) = K(g_n)$ for large enough n. Hence X is closed. Now we claim X is open. Let $g \in X$ and choose W as before. Choose a small $\epsilon > 0$ and let $U = F \cap \mathcal{N}(g, \epsilon, \overline{W_1})$, where recall $\mathcal{N}(f, \epsilon, K)$ generate the basis for the topology of $\mathcal{H}$. Again from [McM1, Theorem 5.11] it follows that $U \subset X$. □

Define the filled Julia set of F by $K(F) = K(f)$ for any $f \in F$. Similarly define the Julia set $J(F)$ of a marked germ F.

Lemma 2.1.4 *If $f \in \mathcal{Q}^\infty$ then $[f] \cap \mathcal{Q}^\infty$ is connected.*

Proof: Choose $g, h \in [f] \cap \mathcal{Q}^\infty$. From [McM2, Lemma 7.1] we have $K(g) = K(h) = K(f)$. Let U be the connected component of $Dom(g) \cap Dom(h)$ containing 0 and let $\phi = g = h$ restricted to U. Then $\phi \in [f] \cap \mathcal{Q}^\infty$ and $K(\phi) = K(f)$. Without loss of generality we may assume $Range(g)$ and $Range(h)$ are Jordan disks with piecewise smooth boundaries. We claim ϕ and g are in the same connected component of $[f]$. Foliate $\overline{Range(g) \setminus Dom(g)}$ with a continuous family of curves γ_θ for $\theta \in S^1$ such that for each θ, $\gamma_\theta(0) \in \partial Range(g)$ and $\gamma_\theta(1) \in \partial Dom(g)$. Pull back the foliation to be a foliation of $\overline{Range(g)} \setminus K(g)$, which we will still denote by γ_θ, though now the domain of γ_θ is $[0, \infty)$. That is, for all $t \geq 1$ and $\theta \in S^1$, γ_θ satisfies

$$g(\gamma_\theta(t)) = \gamma_{\theta'}(t-1)$$

where θ' satisfies $g(\gamma_\theta(1)) = \gamma_{\theta'}(0)$. Each curve γ_θ is called an *external ray* and for $t_0 \in [0, \infty]$ the level set $\cup_\theta \gamma_\theta(t_0)$ is called an *equipotential.* For each $x \in C = \partial Range(\phi)$ let $\theta(x)$ be the unique angle such that $x \in Im(\gamma_{\theta(x)})$ and let $t(x)$ be the unique time such that $x = \gamma_{\theta(x)}(t(x))$. For each $t \in [0, \infty)$ and $x \in C$ let

$$x_t = \gamma_{\theta(x)}(\max(0, t(x) - t)).$$

That is, x_t moves along the foliation until hitting $\partial Range(g)$ when it stops. Let V_t be the connected component of $\mathbb{C} \setminus \cup_{x \in C}\{x_t\}$ containing 0. Let $\phi_t = g$ restricted to $g^{-1}(V_t)$. Then ϕ_t is a path from $\phi = \phi_0$ to $g = \phi_T$ where $T = \max_{x \in C} t(x)$. □

We will often abuse notation and write $[f]$ to denote $[f] \cap \mathcal{Q}^\infty$, when defined.

Let $\mathcal{G}^\infty$ denote the set of quadratic-like germs with connected Julia set. We give $\mathcal{G}^\infty$ the following definition of convergence: $F_n \to F$ iff there are representatives $f_n \in F_n$ and $f \in F$ such that $f_n \to f$. See [L5, §4.1] for the definition of the underlying topology.

2.1.2 Straightening

We will assume the reader is familiar with the theory of quasiconformal maps and the Measurable Riemann Mapping Theorem (see [LV]). A *quasi-conformal equivalence* ϕ between quadratic-like maps f and g is a quasiconformal map from a neighborhood of $K(f)$ to a neighborhood of $K(g)$ such that $\phi \circ f = g \circ \phi$. A quasi-conformal equivalence is a *hybrid* equivalence if $\bar{\partial}\phi|_{K(f)} = 0$ as a distribution.

Proposition 2.1.5 ([DH2]) *Any quadratic-like map f is hybrid equivalent to a quadratic polynomial. If $K(f)$ is connected the polynomial is unique up to affine conjugacy. Moreover, if $f \in \mathcal{Q}^\infty(m)$ and $Dom(f)$ is a K-quasidisk with piecewise smooth boundary then the equivalence can be chosen to be a conjugacy on $Dom(f)$ with dilatation bounded above by $C(m, K) < \infty$.*

The *inner class* of a map $f \in \mathcal{Q}^\infty$, denoted $I(f)$, is the unique c value such that f is hybrid equivalent to P_c. Recall $P_c(z) = z^2 + c$. The inner class of a germ with connected Julia set is the inner class of any representative. Define the *Mandelbrot set*

$$\mathcal{M} = \{c \in \mathbb{C} : K(P_c) \text{ is connected}\}.$$

The *root* of $\mathcal{M}$ is the point $c = 1/4$, the unique parameter value so that $\alpha_{P_c} = \beta_{P_c}$. For a picture of $\mathcal{M}$ see Fig. 2.2. The inner classes parameterize the space of *hybrid classes*.

Proposition 2.1.6 ([DH2],[McM2, Proposition 4.7]) $I : \mathcal{Q}^\infty \to \mathcal{M}$ *is continuous.*

2.2 Renormalization

2.2.1 Complex renormalization

A parameter value $c \in \mathbb{C}$ is called *super-stable* if 0 is periodic under P_c. Fix a superstable $c_0 \neq 0$ and let p be the period of 0 under P_{c_0}. There exists a domain $U \ni 0$ such that

$$f_{c_0} = P_c^p|_U \in \mathcal{Q}^\infty.$$

For example, let U be a small neighborhood of the immediate basin B of 0 in the metric such that P_{c_0} is expanding on ∂B. The *Mandelbrot set tuned by c_0*, or, briefly, an *$\mathcal{M}$-copy* and denoted $\mathcal{M}^{c_0}$, is the set of parameter values c such that P_c^p has a

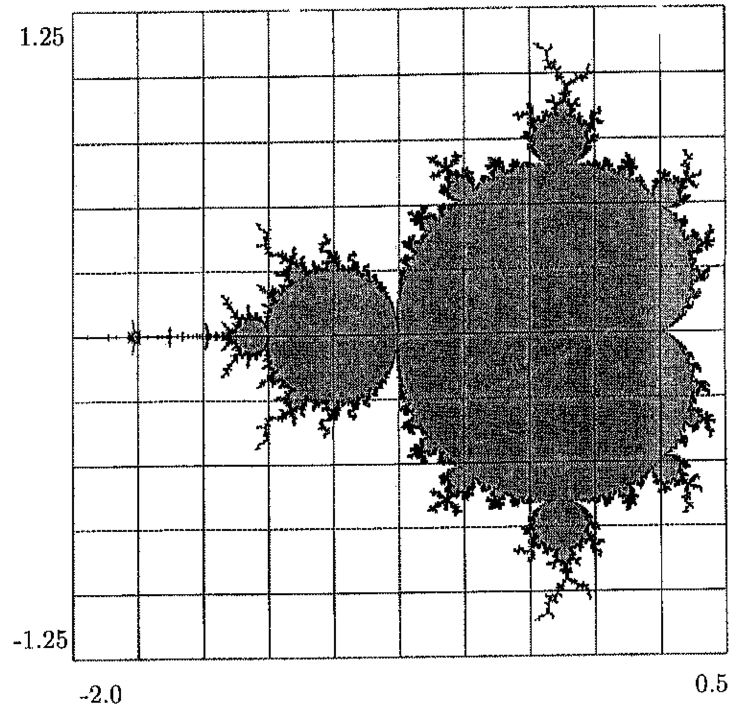

Figure 2.2: The Mandelbrot set.

quadratic-like restriction "near" f_{c_0} with a connected Julia set. More precisely, let $f_c = P_c^p$ and let $X \subset \mathcal{G}^\infty$ be the connected component of the set of germs $\cup_{c\in\mathbb{C}}[f_c] \cap \mathcal{Q}^\infty$ containing f_{c_0}. Define

$$\mathcal{M}^{c_0} = \{c \in \mathbb{C} : [f_c] \subset X\}.$$

Using the results summarized in §2.3.4 (the puzzle construction), one can show that $\mathcal{M}^c$ is homeomorphic to $\mathcal{M}$ or $\mathcal{M} \setminus \{1/4\}$. Note that this is not exactly the standard definition of an $\mathcal{M}$-copy, which would be the closure of what we call $\mathcal{M}^c$. The closure of $\mathcal{M}^c$ is always homeomorphic to $\mathcal{M}$. The *root* of $\mathcal{M}^c$ is the point corresponding to 1/4 and the *center* is the point c. An $\mathcal{M}$-copy that is not contained in any other $\mathcal{M}$-copy is called *maximal*. The *period* of an $\mathcal{M}$-copy $\mathcal{M}^c$ is the period of the origin under P_c.

Fix a maximal $\mathcal{M}$-copy M of period p and let $f \in I^{-1}(M)$. By definition there is a domain $U \ni 0$ such that $f^p|_U \in \mathcal{Q}^\infty$. The map $g = f^p|_U$ is called a (complex) *pre-renormalization* of f, and f is said to be *renormalizable* of period p. This pre-renormalization is always *simple*, meaning the iterates of $J(g)$ under f are either are disjoint or intersect only along the orbit of β_g. We say $\mathcal{M}^c$ is *real* if c is real. The only real maximal $\mathcal{M}$-copy for which the root point is not renormalizable is the period two copy $\mathcal{M}^{(2)}$. We will denote the real period three copy by $\mathcal{M}^{(3)}$. The root points of $\mathcal{M}^{(2)}$ and $\mathcal{M}^{(3)}$ are $c = -0.75$ and $c = -1.75$, respectively.

If $f^p|_U$ and $f^p|_{U'}$ are two pre-renormalizations then by Lemma 2.1.4 they represent the same quadratic-like germ. Hence we can define the *renormalization* $\mathcal{R}f$ as the quadratic-like germ $[f^p|_U]$ normalized so that $\beta = 1$. We define the renormalization of a germ to be the renormalization of a quadratic-like representative. A map $f \in \mathcal{Q}^\infty$

is *infinitely renormalizable* if $\mathcal{R}^n f$ is defined for all $n \geq 0$, or, equivalently, if $I(f)$ is contained in infinitely many $\mathcal{M}$-copies. We say an ∞-renormalizable f has *real combinatorics* if the all the maximal $\mathcal{M}$-copies containing $I(\mathcal{R}^n f)$ are real. See [M3, D2] for a more complete description of tuning.

2.2.2 Real renormalization

Let $I \subset \mathbb{R}$ be a closed interval. A continuous map $f : I \to I$ is *unimodal* if $f(\partial I) \subset \partial I$ and there is a unique extremum c of $f|_I$. For $f \in \mathcal{Q}_{\mathbb{R}}^{\infty}$ let $B_f = [\beta_f, \beta'_f]$, and $A_f = [\alpha_f, \alpha'_f] \subset B_f$. Note that $K(f) \cap \mathbb{R} = B_f$. The next lemma follows from Lemma 2.1.2 and the continuity of β_f and β'_f. We say A is C-*commensurable* to B if $C^{-1} \leq A/B \leq C$.

Lemma 2.2.1 *For $m > 0$, $|\beta_f|$ and $|B_f|$ are $C(m)$-commensurable to* $\operatorname{diam} K(f)$ *for any $f \in \mathcal{Q}_{\mathbb{R}}^{\infty}(m)$.*

Any $f \in \mathcal{Q}_{\mathbb{R}}^{\infty}$ is unimodal on B_f (and this interval is the only interval with this property). A unimodal map $f : I \to I$ is *real-renormalizable* if there is an interval $I' \ni c$ and an $n > 1$ such that $f^n|_{I'}$ is unimodal. Unlike complex renormalization, we can canonically define real-renormalization as acting on unimodal maps as follows. Define the real pre-renormalization g of a unimodal map f as $g = f^n|_{I'}$ where n is minimal and define the real-renormalization $\mathcal{R}f$ as g conjugated by $x \mapsto x/\beta_g$ where β_g is the boundary fixed point of g.

Suppose $f \in \mathcal{Q}_{\mathbb{R}}^{\infty}$ is real-renormalizable and positively oriented, where we say a unimodal map $f|_{[a,b]}$ with $a < b$ is *positively oriented* if $f(b) = b$. Note the quadratic family P_c is positively oriented. Let g be a pre-renormalization of f and let σ_f be the permutation induced on the orbit of B_g labeled from left to right. Any permutation that can be so realized is called a *unimodal non-renormalizable permutation*, or a *shuffle*. The permutation on two symbols we will denote by $\sigma^{(2)}$. If $\sigma_f = \sigma^{(2)}$ we say f is *immediately renormalizable*. The map $\mathcal{M}^c \mapsto \sigma_{P_c}$ from the set of real maximal $\mathcal{M}$-copies to the set of shuffles is a bijection. We will denote the shuffle corresponding to a real maximal $\mathcal{M}$-copy M by σ_M and the real maximal $\mathcal{M}$-copy corresponding to a shuffle σ by $\mathcal{M}^{\sigma}$. We will occasionally use the notation $\mathcal{R}_{\sigma}$ to denote the complex renormalization operator acting on $I^{-1}(\mathcal{M}^{\sigma})$ and on its germs. If $f \in I^{-1}(\mathcal{M}^{\sigma})$ then define $\sigma_f = \sigma$. For an infinitely renormalizable $f \in \mathcal{Q}^{\infty}$ with real combinatorics define $\bar{\sigma}_f$ to be the sequence of shuffles $\sigma_{\mathcal{R}^n f}$ for $n \geq 0$.

2.2.3 Complex bounds

An ∞-renormalizable map $f \in \mathcal{Q}^{\infty}$ has *complex bounds* if there is some $m > 0$ such that the domain U_k and range V_k of the k-th complex pre-renormalization f_k can be chosen to satisfy $\operatorname{mod}(V_k, U_k) \geq m$ for all $k \geq 1$. The following theorem establishes combinatorial rigidity of infinitely renormalizable maps with real combinatorics and complex bounds.

Theorem 2.2.2 ([L3]) *If P_c and $P_{c'}$ are two ∞-renormalizable quadratics with complex bounds and the same real combinatorics then $c = c'$.*

Complex bounds are proven to exist for real quadratics:

Theorem 2.2.3 ([LY, L2, S, LS]) *Real infinitely renormalizable quadratics have complex bounds. Moreover, U_k and V_k can be chosen to be K-quasidisks,*

$$\operatorname{diam}(V_k) \le C \cdot |B_{f_k}|,$$

and, if $\sigma_{\mathcal{R}^{k-1}f} \neq \sigma^{(2)}$ then the unbranched condition holds:

$$P(f) \cap V_k = P(f_k).$$

The values m, C and K are independent of f.

When we make an additional assumption on the combinatorics we obtain the unbranched condition on all levels.

Lemma 2.2.4 *Let $\epsilon > 0$. Suppose f is an infinitely renormalizable real quadratic with $I(\mathcal{R}^k f) \ge -2 + \epsilon$ for all $k \ge 0$. Then there is an $m > 0$ such that the domain U_k and range V_k of the k-th pre-renormalization can be chosen to satisfy*

- $\operatorname{mod}(V_k, U_k) \ge m$
- *U_k and V_k are K-quasidisks*
- $\operatorname{diam}(V_k) \le C \cdot |B_{f_k}|$
- $P(f) \cap V_k = P(f_k)$

for all $k \ge 1$. The constants m and K then depend on ϵ.

Proof: If $\sigma_{\mathcal{R}^{k-1}f} \neq \sigma^{(2)}$ then U_k and V_k can be chosen to be those given by Theorem 2.2.3. So assume $\sigma_{\mathcal{R}^{k-1}f} = \sigma^{(2)}$. Let $h : U'_{k-1} \to V'_{k-1}$ and $h_1 : U'_k \to V'_k$ be the $(k-1)$-st and k-th pre-normalization from Theorem 2.2.3 rescaled so that $\operatorname{diam} K(h) = 1$. Let $E = P(h) \setminus P(h_1)$. From the following lemma, Proposition 2.1.6 and the assumption $I(\mathcal{R}^k f) \ge -2 + \epsilon$ we obtain

$$\operatorname{dist}(E, B(h_1)) = |h^3(0) - \alpha_h| \ge C(\epsilon, m) > 0.$$

From a construction of Sands, V'_k can be chosen to be the union of a Euclidean disk centered at 0 of radius $|\beta_{h_1}|$ and two small Euclidean disks centered at $\pm\beta_{h_1}$ of radius $\epsilon' > 0$. The modulus $\operatorname{mod}(U'_k, V'_k)$ is bounded below by a function $m'(\epsilon') > 0$. Choose $\epsilon' < C(\epsilon, m)$. □

2.3 Generalized Quadratic-like Maps

Let $\mathcal{U}$ denote the subspace $\{(f, U, x) \in \mathcal{H} : f \text{ univalent and } U \Subset Range(f)\}$. Let

$$\mathcal{H}_n = (\mathcal{Q} \times \underbrace{\mathcal{U} \times \cdots \times \mathcal{U}}_{n-1})/S_{n-1}$$

where S_{n-1} is the permutation group on $n-1$ symbols acting on the last $n-1$ factors. Let $\mathcal{L}$ be the following subspace of $\sqcup_{n\geq 1}\mathcal{H}_n$:

$$\mathcal{L} = \left\{ (f_i, U_i, x_i)_{i=0}^{n-1} \in \mathcal{H}_n : \begin{array}{l} \forall 0 \leq i, j \leq n-1,\ U_i \cap U_j = \emptyset, \\ Range(f_i) = Range(f_j),\ x_i = f_i^{-1}(f_0(0)) \end{array} \right\}.$$

Since the domains U_i are disjoint we will identify the map $f = f_i|_{U_i}$ with the equivalence class of the tuple (f_i, U_i, x_i). A map $f \in \mathcal{L}$ is called *generalized quadratic-like.* The distinguished component $U_0 = U_0(f)$ is called the *central component* or the *critical piece.* The connected components $\cup_{j\neq 0}U_j = \cup_{j\neq 0}U_j(f)$ are called *off-critical pieces,* or *non-central pieces.* Choosing a representative (f_i, U_i, x_i) is simply the act of labeling the off-critical pieces, and we will use both phrases interchangeably. Define the *modulus* $\operatorname{mod} f$, the *filled Julia set,* $K(f)$, the *Julia set,* $J(f)$, and the *post-critical set,* $P(f)$, as for quadratic-like maps. Recall the modulus is the supremum of moduli of annuli in $Range(f)$ surrounding all the components in $Dom(f)$.

Define the following subspaces of $\mathcal{L}$:

$$\begin{aligned} \mathcal{L}^n &= \{f \in \mathcal{L} : f^k(0) \in Dom(f) \text{ for } k = 0, 1, \ldots, n\} \\ \mathcal{L}^\infty &= \cap_{n\geq 0}\mathcal{L}^n \\ \mathcal{L}_\mathbb{R} &= \{(f_i, U_i, x_i) \in \mathcal{L} : z \in U_i \Rightarrow \bar{z} \in U_i \text{ and } f_i(\bar{z}) = \overline{f_i(z)}\} \\ \mathcal{L}(m) &= \{f \in \mathcal{L} : \operatorname{mod} f \geq m\}. \end{aligned}$$

Similarly define $\mathcal{L}_\mathbb{R}$, $\mathcal{L}_\mathbb{R}(m)$, $\mathcal{L}^\infty(m)$, etc. as the appropriate intersections of the above subspaces.

Define the *geometry* of $(f_i, U_i, x_i) \in \mathcal{L}$ as

$$\operatorname{geo}(f) = \inf_i \frac{\operatorname{diam} K(f) \cap U_i}{\operatorname{diam} K(f)}.$$

The following lemma is a direct generalization of Lemma 2.1.2.

Lemma 2.3.1 *For a given $m > 0$ and $\lambda > 0$ the set*

$$\{f \in \mathcal{L}^\infty(m) : \operatorname{diam} K(f) = 1,\ n > 1,\ \operatorname{geo}(f) \geq \lambda\}$$

is compact.

Proof: Let us denote the set in question by $X(m, \lambda)$. Let $f = (f_i, U_i, x_i)$ and $K_i = K(f) \cap U_i$. Then since $f_i^{-1}(K(f)) = K_i$ we have

$$\operatorname{mod}(U_i, K_i) \geq \frac{\operatorname{mod}(Range(f), K(f))}{2} \geq \frac{\operatorname{mod} f}{2} \geq \frac{m}{2}.$$

Since $\operatorname{diam} K_i \geq \lambda$ it follows that U_i contains an $\epsilon(m, \lambda)$-neighborhood of K_i. Since $U_i \cap U_j = \emptyset$ for $i \neq j$ and $K(f) \subset \overline{\mathbb{D}}$ it follows that the number of connected components of $Dom(f)$ is bounded above in terms of m and λ. Moreover, since $x_i \in K(f)$ it follows that U_i contains an $\epsilon(m, \lambda)$-ball of x_i. Then $n > 1$ implies both U_i and its complement contain balls of radius $\epsilon(m, \lambda)$ with centers $x_i \in \overline{\mathbb{D}}$, which implies that each (U_i, x_i) ranges in a compact set of pointed disks. Hence each f_i is ranging in a compact subspace of $\mathcal{H}$ (see, for example, [McM1, Theorem 5.6]). Since $X(m, \lambda)$ is closed it is compact. □

2.3.1 First return maps and generalized renormalization

Fix an $f \in \mathcal{L}$ and let $U \subset \mathbb{C}$ be open. Define the open set D_0 by

$$D_0 = \{z : f^n(z) \in U \text{ for some } n \in \mathbb{N}_0\}.$$

For $z \in D_0$ define the *first landing time* $n_0 : D_0 \to \mathbb{N}_0$ by

$$n_0(z) = \min\{n \in \mathbb{N}_0 : f^n(z) \in U\}$$

and the *first landing map* $L_0 : D_0 \to U$ by

$$L_0(z) = f^{n_0(z)}(z).$$

We will also denote the first landing map by $L(f, U)$. Define D_+, n_+ and L_+ exactly as above except replacing $\mathbb{N}_0$ with $\mathbb{N}_+$. We call L_+ the *strict first landing map of f to U*. Define the *first return map of f to U*, denoted $R(f, U)$, as L_+ restricted to $D_+ \cap U$. If $P \subset Dom(R)$ is compact then we denote the restriction of R to the components of $Dom(R)$ containing P as $R(f, U|P)$. If 0 returns to $U_0(f)$ under f we define the *generalized renormalization* of f as $R(f, U_0(f)|P(f) \cup \{0\})$.

2.3.2 The return-type sequence

Fix a maximal, real $\mathcal{M}$-copy $M \neq \mathcal{M}^{\sigma^{(2)}}$ and let $f \in I^{-1}(M)$. This implies α_f is repelling. In real contexts we will assume $f \in \mathcal{Q}_{\mathbb{R}}^{\infty}$. Define the complex principal nest

$$V^0 \supset V^1 \supset V^2 \supset \dots$$

of f as follows. Choose a smooth closed embedded π_1-injective curve γ_0 in the fundamental annulus $Range(f) \setminus Dom(f)$. Curves of this type will be called *equipotentials.* Let D be the disk bounded by $f^{-1}(\gamma_0)$. Choose a straightening of $f|_D$ to a polynomial and pull back the external ray foliatation by the conjugacy. Cut D by the closure of the rays that land at α_f and at α'_f. The resulting set of connected components is called the *initial Yoccoz puzzle* Υ (see Fig. 2.3). Using the notation from [L3], these are the sets

$$\Upsilon = Y_1^{(1)} \cup Y^{(1)} \cup Z_1^{(1)}.$$

That is, $0 \in Y^{(1)}$, $\alpha_f \in \partial Y_1^{(1)}$ and $\alpha'_f \in \partial Z_1^{(1)}$. Let g be the first landing map $L(f, Z_1^{(1)})$ and let V^0 be the connected component of $f^{-2}(Z_1^{(1)})$ containing 0. We call

V^0 the *initial return domain w.r.t.* Υ. Note that $f^2(0) \in Z_1^{(1)}$ since $M \neq \mathcal{M}^{\sigma^{(2)}}$. For $m \geq 1$ define

$$g_m = R(f, V^{m-1}|P(f) \cup \{0\})$$

and define

$$V^m = U_0(g_m).$$

For $m \geq 0$ let $I^m = V^m \cap \mathbb{R}$. We will also denote the restriction to the real line $g_m : \cup_i I_i^m \to I^{m-1}$ by g_m. Number the intervals $\cup_i I_i^m$ (and domains V_i^m) from left to right and so that $0 \in I_0^m = I^m$.

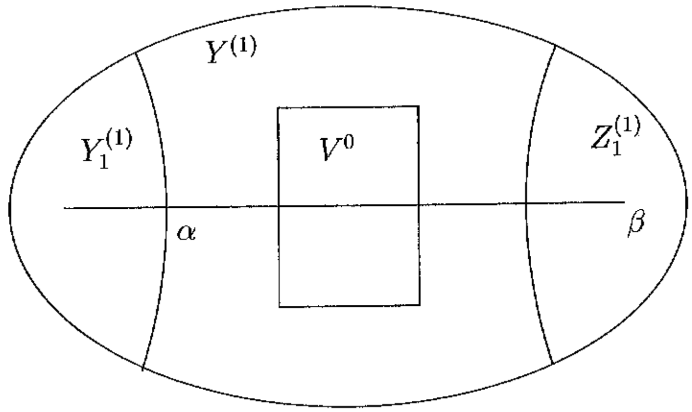

Figure 2.3: The initial Yoccoz puzzle.

The *return type* of g_m is defined as follows (see [L6] for details). Let $g \in \mathcal{L}_{\mathbb{R}}$ have finite type and let $\cup_i I_i = Dom(g) \cap \mathbb{R}$ numbered from left to right with $0 \in I_0$. Let (Γ, ϵ) be the free ordered signed semigroup generated by $\{I_i\}$ where $\epsilon : \{I_i\} \to \{\pm 1\}$ is the sign function defined for $i \neq 0$ by $\epsilon(I_i) = +1$ iff $g|_{I_i}$ is orientation preserving and for $i = 0$ by $\epsilon(I_0) = +1$ iff 0 is a local minimum of g. Let $h \in \mathcal{L}_{\mathbb{R}}$ be a restriction of the first return map $R(g, I_0)$ to finitely many components of its domain and let $\cup_j J_j = Dom(h) \cap \mathbb{R}$. Let (Γ', ϵ') be the corresponding signed semigroup for h. Let $\chi : (\Gamma', \epsilon') \to (\Gamma, \epsilon)$ be the homomorphism generated by assigning to each J_j the word $I_{i_1} I_{i_2} \cdots I_{i_n}$ where I_{i_k} is the interval containing $g^k(J_j)$ and n is the return time of J_j to I_0. The homomorphism $\chi : \Gamma' \to \Gamma$ is the *return type* of h.

A homomorphism $\chi : (\Gamma', \epsilon') \to (\Gamma, \epsilon)$ between free ordered signed semigroups is called *unimodal* if the image of every generator is a word ending with the central interval and if the map is strictly monotone on the intervals to the right and left of center and has an extremum at the center. We say a unimodal χ is *admissible* if

$$\epsilon'(I_j') = \operatorname{sgn}(j)\epsilon(\chi(I_j')) \text{ for } j \neq 0 \text{ and } \epsilon'(I_j') = \epsilon(\chi(I_j')) \text{ for } j = 0.$$

Let us describe the initial combinatorics of f. Let $\{I_{-2}, \ldots, I_2\}$ be the connected components of

$$(\text{int}(Y_1^{(1)}) \cap \mathbb{R}) \cup (\text{int}(Y^{(1)} \setminus V^0) \cap \mathbb{R}) \cup (\text{int}(V^0) \cap \mathbb{R}) \cup (\text{int}(Z_1^{(1)}) \cap \mathbb{R})$$

numbered from left to right on the real axis. Let (Γ_0, ϵ_0) be the signed semigroup generated by $-I_{-2}$, $+I_{-1}$, $+I_0$, $-I_1$ and $+I_2$. We say a homomorphism $\chi : (\Gamma, \epsilon) \to$

(Γ_0, ϵ_0) is *zero-admissible* if it can be realized as the return-type of the map g_1 for some unimodal map through the initial puzzle (Γ_0, ϵ_0). The initial combinatorics of f is described by the homomorphism assigning to each I_i^1 its itinerary by f through the intervals $\{I_i\}$. In general if h_1 is any restriction of the first return map to V^0 then the return type of h_1 is the homomorphism mapping to any interval in its domain its itinerary through the above intervals. Note that if f has negative orientation then repeat the construction with all signs reversed.

The combinatorics of f up to level m is described by the sequence S_m of admissible unimodal homomorphisms

$$\Gamma_m \overset{\chi_m}{\rightarrow} \Gamma_{m-1} \overset{\chi_{m-1}}{\rightarrow} \cdots \overset{\chi_2}{\rightarrow} \Gamma_1 \overset{\chi_1}{\rightarrow} \Gamma_0$$

where χ_m is the return type of g_m and χ_1 is zero-admissible. Each S_m is *irreducible*, meaning the orbit of the critical point enters every interval I_i^m. Since f is renormalizable there exists an m' such that Γ_m is the semigroup with one generator for all $m \geq m'$. Let $S(\sigma) = S_{m'}$ for the smallest such value of m'. Then the shuffle σ_f is uniquely specified by $S_{m'}$. Moreover, we have the following

Theorem 2.3.2 (**[L6]**) *Let S be an irreducible finite sequence of admissible unimodal homomorphisms:*

$$\Gamma_m \overset{\chi_m}{\rightarrow} \Gamma_{m-1} \overset{\chi_{m-1}}{\rightarrow} \cdots \overset{\chi_2}{\rightarrow} \Gamma_1 \overset{\chi_1}{\rightarrow} \Gamma_0.$$

Suppose Γ_m is the only semigroup with one generator, Γ_0 is as above and χ_1 is zero-admissible. Then there is a unique shuffle σ such that $S(\sigma) = S$.

2.3.3 Cascades and essential period

A level $m > 0$ is called *non-central* iff

$$g_m(0) \in V^{m-1} \setminus V^m.$$

Let $m(0) = 0$ and let $0 < m(1) < m(2) < \cdots < m(\kappa)$ enumerate the non-central levels, if any exist, and let $h_k \equiv g_{m(k)+1}$, $k = 0, \ldots, \kappa$.

The nest of intervals (or the corresponding nest of pieces V^m)

$$I^{m(k)+1} \supset I^{m(k)+2} \supset \ldots \supset I^{m(k+1)} \tag{2.1}$$

is called a *central cascade*. The *length* l_k of the cascade is defined as $m(k+1) - m(k)$. Note that a cascade of length 1 corresponds to a non-central return to level $m(k)$.

A cascade 2.1 is called *saddle-node* if $0 \notin h_k I^{m(k)+1}$. Otherwise it is called *Ulam-Neumann*. For a long saddle-node cascade the map h_k is combinatorially close to $z \mapsto z^2 + 1/4$. For a long Ulam-Neumann cascade it is close to $z \mapsto z^2 - 2$.

The next lemma shows that for a long saddle-node cascade, the map $h_k : I^{m(k)+1} \to I^{m(k)}$ is a small perturbation of a map with a parabolic fixed point.

Lemma 2.3.3 (**[L2]**) *Let $h_k \in \mathcal{Q}_{\mathbb{R}}$ be a sequence of real-symmetric quadratic-like maps having saddle-node cascades of length $l_k \to \infty$. If $h_k \to f$ then $f \in I^{-1}(1/4)$.*

Proof: It takes l_k iterates for the critical point to escape $U_k = Dom(h_k)$ under iterates of h_k. Hence the critical point does not escape $Dom(f)$ under iterates of f. By the kneading theory [MT] f has on the real line topological type of $z^2 + c$ with $-2 \leq c \leq 1/4$. Since small perturbations of f have escaping critical point, the choice for c boils down to only two boundary parameter values, $1/4$ and -2. Since the cascades of h_k are of saddle-node type, $c = 1/4$. □

Let $x \in P(f) \cap (I^{m(k)} \setminus I^{m(k)+1})$ and let $h_k x \in I^j \setminus I^{j+1}$. Set

$$d(x) = \min\{j - m(k), m(k+1) - j\}.$$

This parameter shows how deep the orbit of x lands inside the cascade. Let us now define d_k as the maximum of $d(x)$ over all $x \in P(f) \cap (I^{m(k)} \setminus I^{m(k)+1})$. Given a saddle-node cascade (2.1), let us call all levels $m(k) + d_k < l < m(k+1) - d_k$ *neglectable.*

Let f be renormalizable and f_1 a pre-renormalization of f. Define the *essential period* $p_e = p_e(f)$ as follows. Let p be the period of the periodic interval $J = B(f_1)$, and set $J_k = f^k J$, for $0 \leq k \leq p-1$. Let us remove from the orbit $\{J_k\}_{k=0}^{p-1}$ all intervals whose first landing to some $I^{m(k)}$ belongs to a neglectable level, to obtain a sequence of intervals $\{J_{n_i}\}_{i=1}^{m}$. The essential period is the number of intervals which are left, $p_e(f) = m$. Note the essential period of a shuffle is well-defined and in this way we can define the essential period for any real maximal $\mathcal{M}$-copy M or $f \in I^{-1}(M)$.

Let us give some examples of combinatorial types involving long saddle-node cascades with neglectable levels. Let Γ, Γ', χ, χ' and χ_0 be from 3.1 and 3.2.

Example 2.3.1 (Goes Through Twice) *Let $\chi_2 : \Gamma \to \Gamma$ be the homomorphism generated by $I_0 \mapsto I_0$ and $I_{-1} \mapsto I_{-1}^2 I_0$. Then any sequence of the form*

$$\Gamma' \xrightarrow{\chi'} \Gamma \xrightarrow{\chi} \cdots \xrightarrow{\chi} \Gamma \xrightarrow{\chi_2} \Gamma \xrightarrow{\chi} \cdots \xrightarrow{\chi} \Gamma \xrightarrow{\chi_0} \Gamma_0$$

will correspond to a shuffle where the critical orbit moves up through the cascade until the top, returns to the level of χ_2, moves up through the cascade again and then returns to the renormalization interval. If the total number of levels in the sequence is m then the number of neglectable levels will be roughly $m - 2\min(d, m-d)$ where d is the level of χ_2.

Example 2.3.2 (Two Cascades) *As a second example imagine perturbing the right-hand picture in Fig. 3.4 so that the renormalization becomes hybrid equivalent to $z^2 + \frac{1}{4}$. Now any further perturbation will cause the parabolic orbit to bifurcate and we can create another long cascade. More specifically, let $\chi_3 : \Gamma \to \Gamma$ be the homomorphism generated by $I_0 \mapsto I_{-1} I_0$ and $I_{-1} \mapsto I_{-1}^2 I_0$ and consider a sequence of the form*

$$\Gamma' \xrightarrow{\chi'} \Gamma \xrightarrow{\chi} \cdots \xrightarrow{\chi} \Gamma \xrightarrow{\chi_3} \Gamma \xrightarrow{\chi} \cdots \xrightarrow{\chi} \Gamma \xrightarrow{\chi_0} \Gamma_0.$$

Since χ_3 has a non-central return the two long sequences of χ form two separate saddle-node cascades, each with a long sequence of neglectable levels.

2.3.4 Families of generalized quadratic-like maps

In this section we summarize the theory of holomorphic families of generalized quadratic-like maps. For further details see [L4]. Let $D \subset \mathbb{C}$ be a Jordan disk and fix $* \in D$. Let π_1 and π_2 be the coordinate projections of $\mathbb{C}^2$ to the first and second coordinates. Given a set $\mathbb{X} \subset \mathbb{C}^2$ let $X^\lambda = \pi_2(\mathbb{X} \cap \pi_1^{-1}(\lambda))$. An open set $\mathbb{X} \subset \mathbb{C}^2$ is a *Jordan bidisk* over D if $\pi_1(\mathbb{X}) = D$ and X^λ is a Jordan disk for all $\lambda \in D$. We say $\mathbb{X}$ admits an extension to the boundary if $\mathrm{cl}(\mathbb{X})$ is homeomorphic over $\mathrm{cl}(D)$ to $\mathrm{cl}(D) \times \mathrm{cl}(\mathbb{D})$. A section $\Psi : \mathrm{cl}(D) \to \mathrm{cl}(\mathbb{X})$ is a *trivial section* if $\Psi(\lambda) \in \mathrm{int}\, X^\lambda$ for all $\lambda \in \mathrm{cl}(D)$. Given a Jordan bidisk $\mathbb{X}$ which admits an extension to the boundary we define the *frame* $\delta\mathbb{X}$ as the torus $\cup_{\lambda \in \partial D} \cup_{z \in \partial X_\lambda} (\lambda, z)$. A holomorphic section $\Phi : D \to \mathbb{X}$ is *proper* if it admits a continuous extension to ∂D and $\Phi(\partial D) \subset \delta\mathbb{X}$. Let Φ be a proper section and let Ψ be a trivial section. Let $\phi = \pi_2 \circ \Phi$ and $\psi = \pi_2 \circ \Psi$. Define the *winding number* of Φ to be the winding number of the curve $(\phi - \psi)|_{\partial D}$ around the origin.

Lemma 2.3.4 (Argument Principle) *Let $\mathbb{X}$ be a Jordan bidisk over D that admits an extension to the boundary. Let $\Phi : D \to \mathbb{X}$ be a proper section and let $\Psi : \mathrm{cl}(D) \to \mathrm{cl}(\mathbb{X})$ be a continuous section, holomorphic on D. Let $\phi = \pi_2 \circ \Phi$ and $\psi = \pi_2 \circ \Psi$. Suppose there are no solutions to $\phi = \psi$ on ∂D. Then the number of solutions to $\phi = \psi$ counted with multiplicity is equal to the winding number of Φ.*

Let $\mathbb{U} = \cup_j \mathbb{U}_j$ be a pairwise disjoint collection of Jordan bidisks over D with $0 \in U_0^\lambda$. Let $\mathbb{V}$ be a Jordan bidisk over D such that each U_j^λ is compactly contained in V^λ. Let

$$\mathbf{f} : \mathbb{U} \to \mathbb{V}$$

be a fiber-preserving holomorphic map such that each fiber map $f_\lambda : U^\lambda \to V^\lambda$ is a generalized quadratic-like map with critical point at the origin and which on each branch $f_\lambda|_{U_j^\lambda}$ admits a holomorphic extension to a neighborhood of U_j^λ. Let $\mathbf{h}$ be a holomorphic motion

$$h_\lambda : (\partial V^*, \partial U^*) \to (\partial V^\lambda, \partial U^\lambda)$$

over D with basepoint $* \in D$ which respects the dynamics. We say $(\mathbf{f}, \mathbf{h})$ is a *holomorphic family of generalized quadratic-like maps over D*. When $\mathbb{U}$ consists of only one bidisk then the family is a *DH quadratic-like family*. A family is *proper* if

1. $\mathbb{V}$ admits an extension to the boundary
2. for each $z \in \partial U^*$ the section $\lambda \mapsto (\lambda, h_\lambda(z))$ extends continuously to ∂D and is a trivial section
3. the critical-value section $\Phi(\lambda) = (\lambda, f_\lambda(0))$ is proper.

The *winding number* of a proper family is the winding number of the critical value section.

Theorem 2.3.5 ([DH2]) *If $(\mathbf{f}, \mathbf{h})$ is a proper DH quadratic-like family over D with winding number 1 then*

$$M(\mathbf{f}, \mathbf{h}) = \{\lambda \in D : J(f_\lambda) \text{ is connected}\}$$

is homeomorphic to the standard Mandelbrot set M. The homeomorphism is given by the inner class map $\lambda \mapsto I(f_\lambda)$.

2.3.5 Generalized renormalization of families

Let $(\mathbf{f} : \cup_j \mathbb{U}_j \to \mathbb{V}, \mathbf{h})$ be a proper holomorphic family of generalized quadratic-like maps over D with winding number 1. If 0 returns under f_λ to U_0^λ let $\bar{i}_\lambda$ be the *return itinerary* of f_λ: the (possibly empty) sequence of indices of off-critical pieces $\{U_j^\lambda\}$ through which the critical point passes before returning to U_0^λ. For such an f_λ we can define a holomorphic motion $\mathbf{h}'$ of the boundaries of the domain and range of the return map to U_0^λ by pulling back the holomorphic motion $\mathbf{h}$ by f_λ. The motion $\mathbf{h}'$ has basepoint λ and is defined over the neighborhood of λ having the itinerary $\bar{i}_\lambda$.

Lemma 2.3.6 ([L4, Lemma 3.6]) *Let $(\mathbf{f} : \cup_j \mathbb{U}_j \to \mathbb{V}_j, \mathbf{h})$ be a proper generalized quadratic-like family over D with winding number 1. Let $* \in D$ be the basepoint and let g_* be the first return map $R(f_*, U_0^*)$ restricted to finitely many components. Suppose $g_* \in \mathcal{L}$. Then the set*

$$D' = \{\lambda \in D : \bar{i}_\lambda = \bar{i}_*\}$$

is a Jordan disk and the family of first return maps $(\mathbf{g}, \mathbf{h}')$ over D' is proper and has winding number 1.

2.4 Parabolic Periodic Points

The limits of maps with unbounded but essentially bounded combinatorics are maps with parabolic periodic points. This section reviews the local theory near parabolic orbits and their perturbations. The main results are the existence and continuity of Fatou coordinates. These results were proven in [DH1] and [La] for perturbations lying in an analytic family and later generalized in [Sh]. Our presentation is based on [Sh].

Throughout this section we give the space of holomorphic maps the "compact-open topology with domains". A basis for this topology is given by the sets

$$\mathcal{N}(f, K, \epsilon) = \{g : |g(z) - f(z)| < \epsilon \text{ for } z \in K\}$$

where $K \subset Dom(f)$ is compact and $\epsilon > 0$. If a sequence of maps $f_n \in \mathcal{H}$ converges in $\mathcal{H}$ then it also converges in this topology.

2.4.1 Unperturbed Fatou coordinates

Let $\mathcal{P}_0$ be the space of holomorphic maps f_0 with a fixed point ξ_0 that is *parabolic* and *non-degenerate*: $f_0'(\xi_0) = 1$ and $f_0''(\xi_0) \neq 0$. For example, choose any quadratic-like map f_0 hybrid equivalent to $z^2 + 1/4$. Choose a neighborhood $N \ni \xi_0$ so that $f_0|_N$ is a diffeomorphism and maps N onto a neighborhood $N' \ni \xi_0$.

Proposition 2.4.1 (Fatou coordinates) *Let $f_0 \in \mathcal{P}_0$ and choose N and N' as above. Then there exist topological disks $D_\pm \Subset N \cap N'$, whose union forms a punctured neighborhood of ξ_0 and which satisfy*

$$f_0^{\pm 1}(\overline{D_\pm}) \subset D_\pm \cup \{\xi_0\} \text{ and } \cap_{n \geq 0} f_0^{\pm n}(\overline{D_\pm}) = \{\xi_0\}.$$

Moreover, there exist univalent maps $\Phi_\pm : D_\pm \to \mathbb{C}$ such that

1. *$Range(\Phi_+)$ and $Range(\Phi_-)$ contain a right and left half-plane, respectively*
2. *$\Phi_\pm(f_0(z)) = \Phi_\pm(z) + 1$*

These maps are uniquely defined up to post-composing with a translation.

The disks $D_\pm$ are called *incoming* and *outgoing petals* and the maps $\Phi_\pm$ are called the *Fatou coordinates*. The Fatou coordinates induce conformal isomorphisms between the *Ècalle-Voronin cylinders* $\mathcal{C}_\pm = D_\pm / f_0$ and $\mathbb{C}/\mathbb{Z}$. Let $\pi_\pm$ denote the projection of $D_\pm$ to $\mathcal{C}_\pm$.

When $f_0 \in I^{-1}(1/4)$ we can extended the Fatou coordinates to be branched covers as follows. Let $B = \operatorname{int} K(f_0)$ be the attracting basin of ξ_0. Extend π_+ to B by $\pi_+(z) = \pi_+(f_0^n(z))$ for a large enough n. Lifting $\pi_+ : B \to \mathcal{C}_+$ to $\Phi_+ : B \to \mathbb{C}$ we obtain an extension of $\Phi_+ : D_+ \to \mathbb{C}$ to a branched covering map. The set of critical points is the backward orbit of 0 under f_0 and the set of critical values is a set of the form $\{\alpha - n\}_{n=0}^{\infty}$ for some $\alpha \in \mathbb{C}$ depending on the normalization of Φ_+ (see Fig. 2.4). We can also similarly extend Φ_-^{-1} to be a holomorphic map with range $Range(f_0)$. Note that Φ_-^{-1} will be a branched cover over $Range(f_0) \setminus \{\xi_0\}$. (see Fig. 2.5). In general, if f_0 is a branched cover then we can extend Φ_+ to the immediate attracting basin of ξ_0 and we can extend Φ_-^{-1} to a map with range $Range(f_0)$ which is a branched cover over range $Range(f_0) \setminus \{\xi_0\}$.

A *transit map* $g : \mathcal{C}_+ \to \mathcal{C}_-$ is a conformal isomorphism which respects the ends $\pm\infty$. A holomorphic map $h : U \to \mathbb{C}$ is a *local lift* of a transit map g if $U \subset D_+$, $Range(h) \subset D_-$, and

$$g \circ \pi_+ = \pi_- \circ h.$$

When written in Fatou coordinates, h is a translation T_a by a complex number a. The quantity $\bar{a} = a \mod \mathbb{Z}$, called the *phase*, depends only on g (and the normalization of Fatou coordinates) and uniquely specifies g. We will use the notation $g_{\bar{a}}$ to denote the transit map with phase $\bar{a}$.

To simplify future notation, let $\Phi = \Phi_+$ and $\phi = \Phi_-^{-1}$. Also, we shall freely use the notation Φ^n, Φ^f, $\mathcal{C}_\pm^n$, etc to indicate a dependence on an index n or map f.

2.4.2 Conformal dynamical systems

Given a collection $\{f_\alpha \in \mathcal{H}\}$ of holomorphic maps let $\langle f_\alpha \rangle$ denote the set of restrictions of all finite compositions of $\{f_\alpha\}$. Note that we consider a composition of zero maps to be the identity. A collection $\mathcal{F}$ of holomorphic maps closed under composition and restriction is called a *conformal dynamical system*. For a given $z \in \mathbb{C}$ we will let $\mathcal{F}_z$

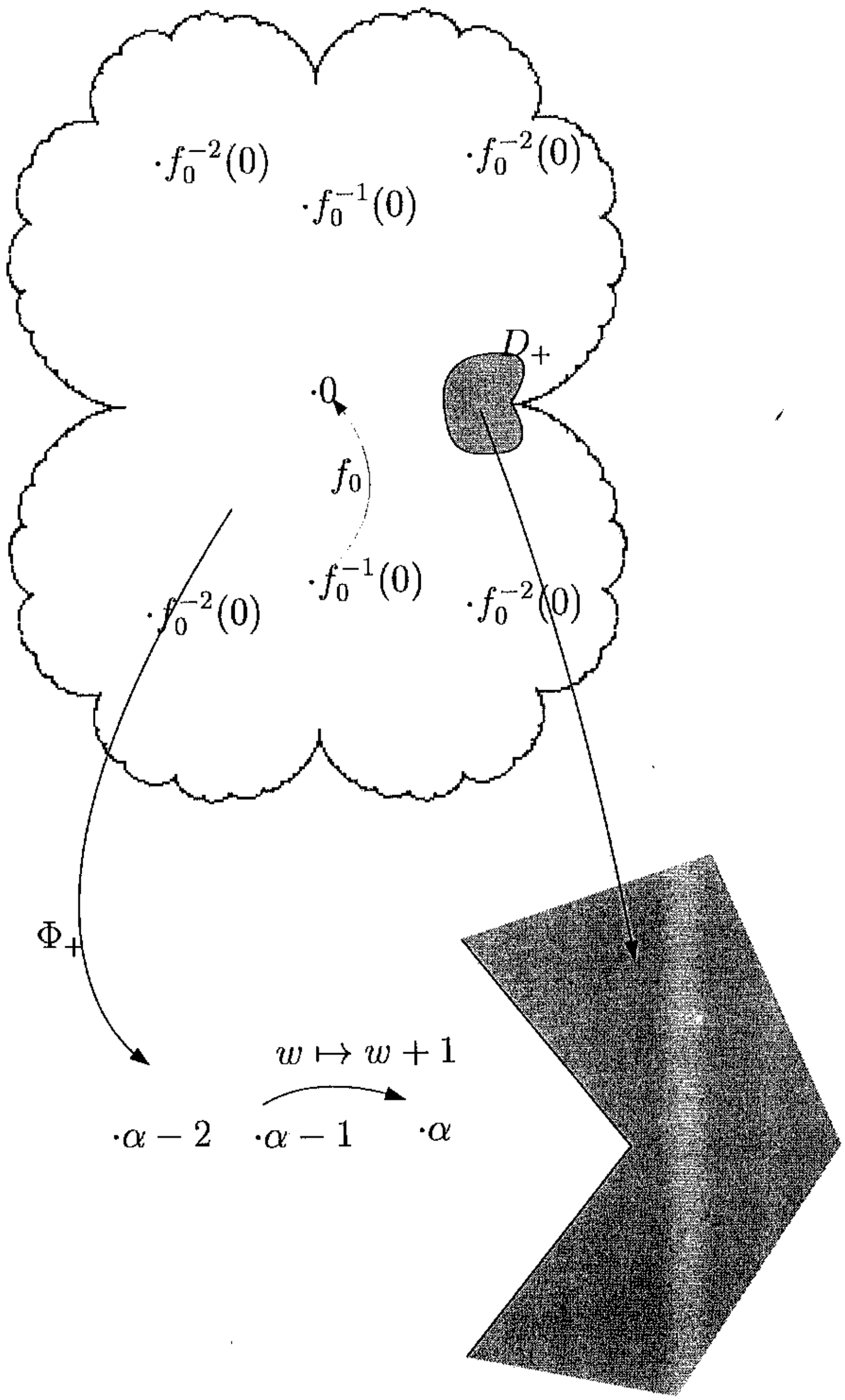

Figure 2.4: Extended attracting Fatou coordinates.

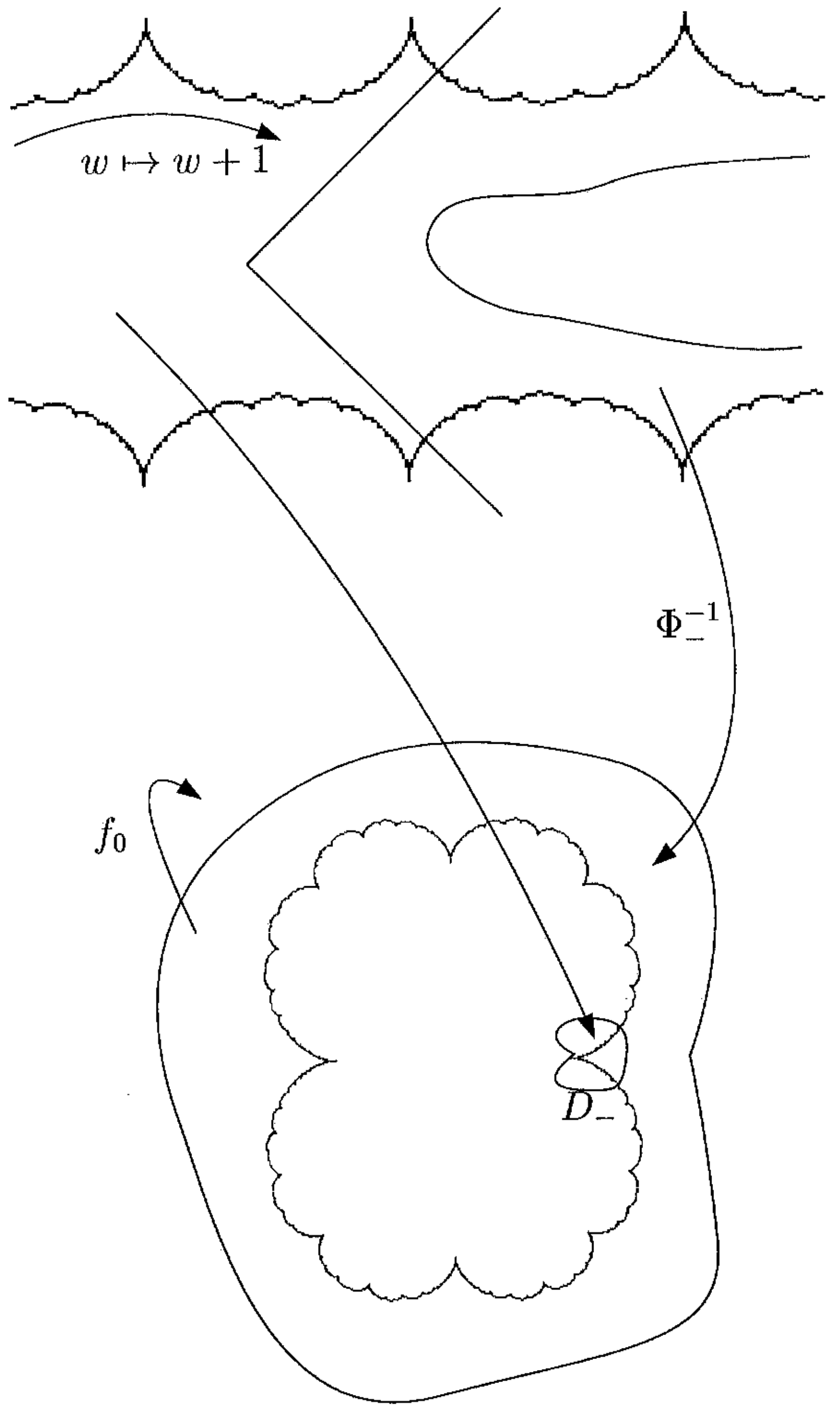

Figure 2.5: Extended repelling Fatou coordinates.

denote the set of germs $\{[h]_z\}$ of $h \in \mathcal{F}$ such that $z \in Dom(h)$. Define the orbit of a point $z \in \mathbb{C}$ as

$$\mathrm{orb}(z) = \mathrm{orb}(\mathcal{F}, z) = \bigcup_{h \in \mathcal{F}} h(z).$$

Let $f \in \mathcal{P}_0$ and let $g : \mathcal{C}_+ \to \mathcal{C}_-$ be a transit map. The conformal dynamical system

$$\mathcal{F}(f, g) = \langle f \cup \{\text{all local lifts of } g \text{ to } D_\pm\} \rangle$$

will be central to our study. Note that $\mathcal{F}(f, g)$ is independent of the choice of petals $D_\pm$. We define a total order on $\mathrm{orb}(z)$ as follows. First we order the germs $[h]_z$ of $h \in \mathcal{F}(f, g)$ at z. Fix $h, \widetilde{h} \in \mathcal{F}(f, g)$ such that $z \in Dom(h) \cap Dom(\widetilde{h})$. Write

$$\begin{aligned} h &= f^{n_k} \circ h_{k-1} \circ f^{n_{k-1}} \circ \cdots \circ h_1 \circ f^{n_1} \\ \widetilde{h} &= f^{m_l} \circ \widetilde{h}_{l-1} \circ f^{m_{l-1}} \circ \cdots \circ \widetilde{h}_1 \circ f^{m_1} \end{aligned}$$

for some $k, l \geq 1$ with appropriate lifts h_i and $\widetilde{h}_j$ of g and iterates of f. Since f commutes with local lifts of g, we can find a "common" local lift ϕ_1 so that

$$\begin{aligned} h_1 \circ f^{n_1} &= f^{s_2'} \circ \phi_1 \circ f^{s_1} \\ \widetilde{h}_1 \circ f^{m_1} &= f^{t_2'} \circ \phi_1 \circ f^{t_1} \end{aligned}$$

on a neighborhood of z for the appropriate choice of $s_1, s_2', t_1, t_2' \in \mathbb{N}$. We can continue rewriting local lifts until we have the expressions

$$\begin{aligned} h &= f^{s_k} \circ \phi_{k-1} \circ f^{s_{k-1}} \circ \cdots \circ f^{s_2} \circ \phi_1 \circ f^{s_1} \\ \widetilde{h} &= f^{t_l} \circ \phi_{l-1} \circ f^{t_{l-1}} \circ \cdots \circ f^{t_2} \circ \phi_1 \circ f^{t_1} \end{aligned}$$

on a neighborhood of z. Now define $[h]_z \leq [\widetilde{h}]_z$ iff

$$(s_1, s_2, \ldots, s_k) \leq (t_1, t_2, \ldots, t_l)$$

in the lexicographic order. Now order $\mathrm{orb}(z)$ by declaring $z_1 \leq z_2$ if the $[h_1]_z \leq [h_2]_z$ where $h_1(z) = z_1$, $h_2(z) = z_2$ and $[h_1]_z$ and $[h_2]_z$ are the minimizers with this property. We say $\mathcal{F}$ is contained in any *geometric limit* of a sequence $\mathcal{F}_n$ if for any $f \in \mathcal{F}$ there are $f_n \in \mathcal{F}_n$ such that $f_n \to f$ on compact sets.

We do not put a topology on the set of conformal dynamical systems. The systems we consider are generated by generalized quadratic-like maps and local lifts of transit maps, so instead we speak of the convergence of these generators.

2.4.3 Douady coordinates

We now consider perturbations of $f_0 \in \mathcal{P}_0$. Since ξ_0 is a non-degenerate parabolic fixed point the generic perturbation will cause it to bifurcate into two nearby fixed points ξ_f and ξ_f' with multipliers λ_f and λ_f', respectively. Let N be the neighborhood of ξ_0 chosen for Proposition 2.4.1 and let $\mathcal{P}$ be the space of holomorphic maps which are diffeomorphisms of N. Let $\mathcal{P}_1$ be the set of $f \in \mathcal{P}$ with exactly two fixed points ξ_f and ξ_f' in N satisfying

$$\arg(1 - \lambda_f), \arg(1 - \lambda_f') \in [\pi/4, 3\pi/4] \cup [-3\pi/4, -\pi/4]. \tag{2.2}$$

Theorem 2.4.2 (Douady coordinates) *Let $f_0 \in \mathcal{P}_0$. There is a neighborhood $\mathcal{N}$ of f_0 such that if $f \in (\mathcal{N} \cap \mathcal{P}_1)$ then there exist univalent maps $\Phi_f = \Phi_+^f$ and $\phi_f = (\Phi_-^f)^{-1}$, unique up to translation, and a constant $a_f \in \mathbb{C}$ satisfying*

1. *$\Phi_f(f(z)) = \Phi_f(z) + 1$ and $\phi_f(w+1) = f(\phi_f(w))$ where defined*
2. *$\mathcal{C}_+^f = Dom(\Phi_f)/f$ and $\mathcal{C}_-^f = Range(\phi_f)/f$ are conformally cylinders and one can choose fundamental domains $S_\pm^f$ to depend on $f \in \mathcal{P}_1$ continuously in the Hausdorff topology.*
3. *(see Fig. 2.6) for $z \in S_+^f$ there is an $n > 0$ such that $f^n(z) \in S_-^f$ and for n minimal*

$$f^n(z) = (\phi_f \circ T_{a_f+n} \circ \Phi_f)(z). \tag{2.3}$$

If we fix points $z_\pm \in D_\pm$ and normalize $\Phi_\pm^f$ by $\Phi_\pm^f(z_\pm) = 0$ then $\Phi_\pm^f$ depend continuously on $f \in \mathcal{N} \cap (\mathcal{P}_0 \cup \mathcal{P}_1)$.

Suppose $f_0 \in \mathcal{P}_0$ and $f \in \mathcal{P}_1 \cap \mathcal{N}$ where $\mathcal{N}$ is from Theorem 2.4.2. The discontinuous map from S_+^f to S_-^f defined by equation 2.3 projects to a transit map $g_f : \mathcal{C}_+^f \to \mathcal{C}_-^f$ with phase $\bar{a}_f = a_f \mod \mathbb{Z}$. This map describes how a long orbit of f "passes though the gate" between ξ_f and ξ'_f. We call it the *transit map* of f.

The following lemma gives a simple condition under which perturbed Fatou coordinates exist.

Lemma 2.4.3 *Suppose f_n is a sequence of quadratic-like maps converging in the Carathéodory topology to a quadratic-like map $f \in \mathcal{P}_0$. Suppose the fixed points of f_n are repelling. Then $f_n \in \mathcal{P}_1$ for n large enough.*

Proof: Using the holomorphic index (see [M1]) one can prove that

$$\frac{1}{1-\lambda_{f_n}} + \frac{1}{1-\lambda'_{f_n}}$$

converges as $f_n \to f$. Since $\lambda, \lambda' \in \mathbb{C} \setminus \mathbb{D}$ it follows

$$|\arg(1-\lambda_{f_n})| \to \pi/2 \text{ and } |\arg(1-\lambda'_{f_n})| \to \pi/2$$

as $n \to \infty$. In particular, $f_n \in \mathcal{P}_1$ for n large. □

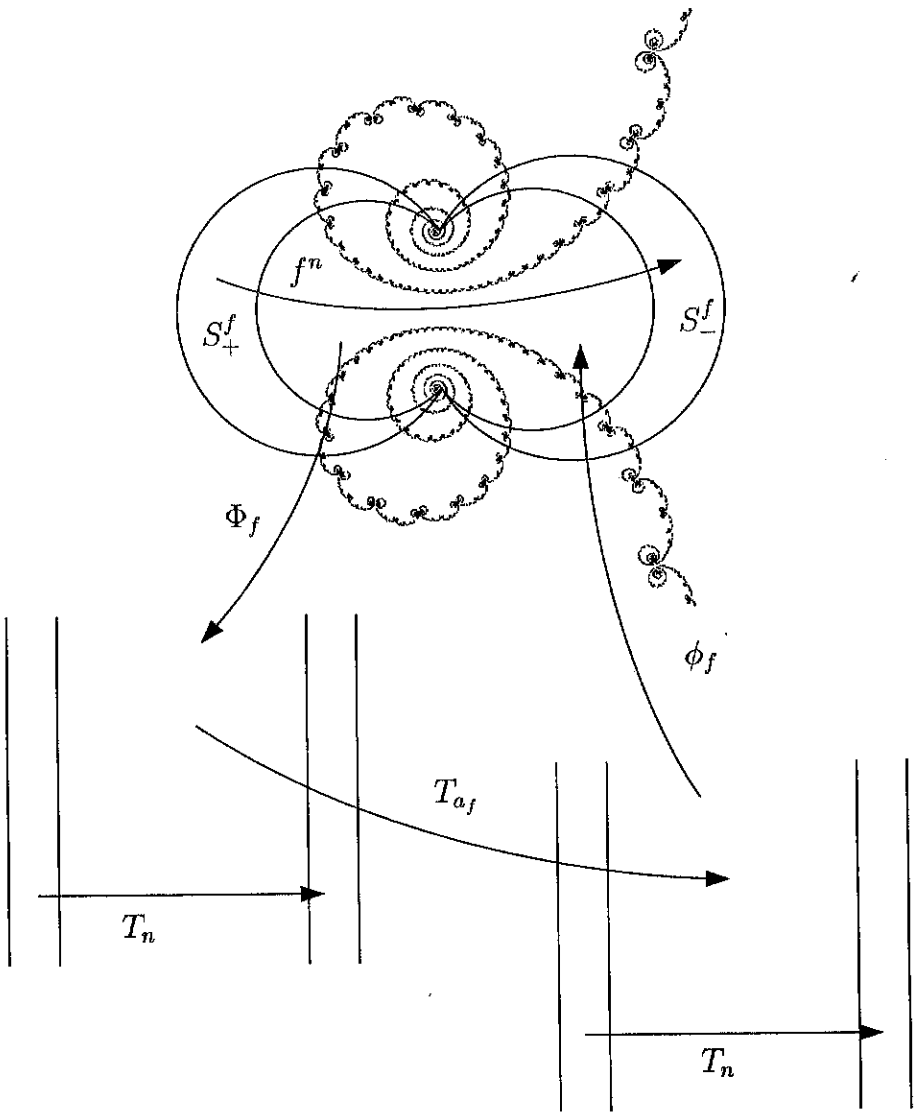

Figure 2.6: Perturbed Fatou coordinates.

Chapter 3

Statements of Main Theorems

3.1 Theorem I

Let us remind the reader of our first result:

Theorem I (Essentially Period Tripling). *There is a unique $F \in \mathcal{G}_{\mathbb{R}}$ such that*

$$\mathcal{R}^n f \to F$$

for any ∞-renormalizable $f \in \mathcal{Q}_{\mathbb{R}}$ with a tuning invariant

$$\tau(f) = (M^{(3)}_{n_1}, M^{(3)}_{n_2}, \ldots, M^{(3)}_{n_k}, \ldots)$$

satisfying $n_k \to \infty$ as $k \to \infty$.

In this chapter we define the $\mathcal{M}$-copies $\mathcal{M}^{(3)}_n$ and describe the limit operator of renormalization: parabolic renormalization. We will not use parabolic renormalization, as defined in this section, in the proofs. However, the definition and the properties of parabolic renormalization are easier to understand in the essentially period tripling case and so are worth discussing.

Let $f(x) = x^2 - 1.75$ and let ξ be the parabolic periodic orbit of period three. Recall $A = A_f = [\alpha_f, \alpha'_f]$. Let g be the first return map of f on A (see Fig. 3.1). Let I^0 and I^0_1 be the two indicated intervals satisfying $g|_{I^0} = f^3$ and $g|_{I^0_1} = f^2$.

Fix a small $\epsilon > 0$ and consider $c \in (-1.75, -1.75 + \epsilon)$. The periodic point ξ bifurcates and the orbit of the critical point under P^3_c now escapes the interval I^0. Let c_n be the parameter value (see Fig. 3.2) so that for $f = P_{c_n}$,

- $f^{3i}(0) \in I^0$ for $i = 1, \ldots, n-1$,
- $f^{3n}(0) \in I^0_1$,
- $f^{3n+2}(0) = 0$.

To justify the existence of c_n, consider the following signed semigroups generated by the specified intervals

$$\begin{aligned} \Gamma &= \langle +I_{-1}, -I_0 \rangle \\ \Gamma' &= \langle -I_0 \rangle \end{aligned} \tag{3.1}$$

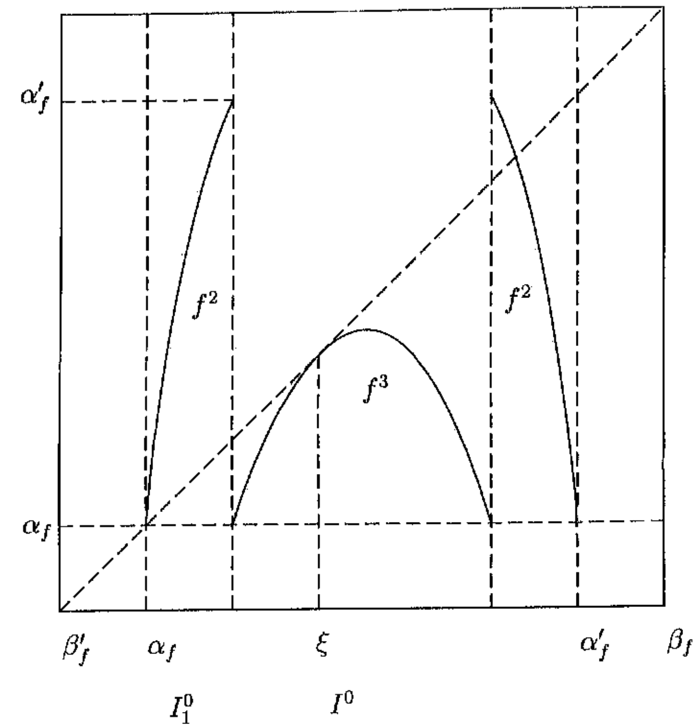

Figure 3.1: The first return map for $x^2 - 1.75$.

and consider the following homomorphisms

$$\begin{array}{lll} \chi_0 : \Gamma \to \Gamma_0 & \text{generated by} & I_{-1} \mapsto I_{-2}I_2I_{-1}I_{-2}I_0 \text{ and } I_0 \mapsto I_{-2}I_2I_0 \\ \chi : \Gamma \to \Gamma & \text{generated by} & I_{-1} \mapsto I_{-1}I_0 \text{ and } I_0 \mapsto I_0 \\ \chi' : \Gamma' \to \Gamma & \text{generated by} & I_0 \mapsto I_{-1}I_0. \end{array} \tag{3.2}$$

Then the sequence corresponding to $\mathcal{M}_n^{(3)} = \mathcal{M}^{c_n}$ is

$$\Gamma' \xrightarrow{\chi'} \Gamma \xrightarrow{\chi} \Gamma \xrightarrow{\chi} \cdots \xrightarrow{\chi} \Gamma \xrightarrow{\chi_0} \Gamma_0$$

where χ is repeated $n-2$ times. Let $\sigma_n^{(3)} = \sigma_{P_{c_n}}$.

In Fig. 3.3 we have drawn pictures of the Mandelbrot set zooming down to the $\mathcal{M}$-copy $\mathcal{M}_4^{(3)}$.

In Fig. 3.4 we have drawn the filled Julia sets for $z^2 - 1.75$ and for $z^2 - c_n$ for some c_n with n large, together with two blow-ups of the Julia set of $f = z^2 - c_n$. The "ghost" boundary of the basin of ξ is visible in the left picture and the pre-images of this ghost boundary nest down to $J(\mathcal{R}f)$ in the right picture.

3.1.1 Parabolic Renormalization

Let M be a maximal $\mathcal{M}$-copy with root c' and let $f \in I^{-1}(c')$. Let f_0 be a pre-renormalization of f and let $\xi = \beta_{f_0}$. Choose incoming and outgoing petals $D_\pm$ around the parabolic point ξ and let $\mathcal{C}_\pm$ denote the respective Ècalle-Voronin cylinders and $\pi'_\pm$

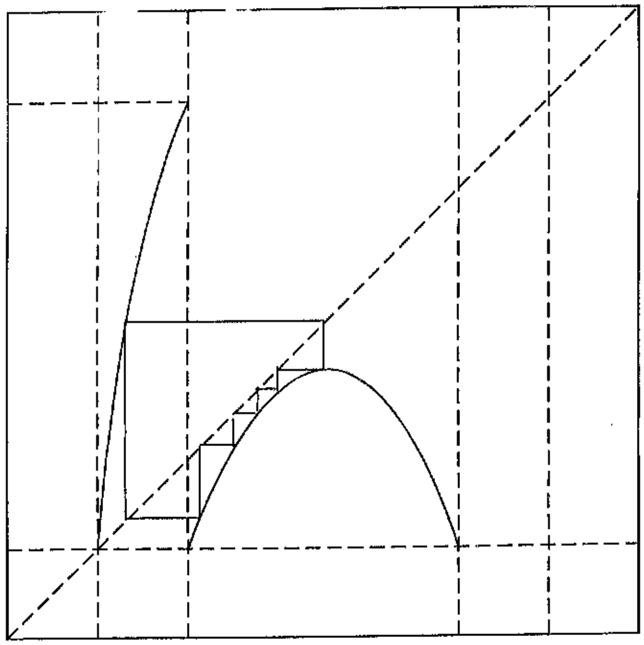

Figure 3.2: The first return map for $x^2 + c_5$ and the orbit of the origin.

the projections with π_+ extended to $B = \text{int}(K(f_0))$. Fix a transit map $g : \mathcal{C}_+ \to \mathcal{C}_-$ satisfying

$$g(\pi_+(0)) \notin \pi_-(K(f_0)).$$

We say the pair (f, g) is *parabolic renormalizable* if there is a neighborhood $U \ni 0$ and a map

$$h \in (\mathcal{F}(f, g) \setminus \langle f \rangle)$$

such that

$$h|_U \in \mathcal{Q}^\infty.$$

We call such an $h|_U$ a *parabolic pre-renormalization* of (f, g). From §2.4.2 there is a natural order on the germs in $\mathcal{F}(f, g)$ at 0. We call the normalized germ of a first pre-renormalization the *parabolic renormalization* of (f, g).

3.1.2 Construction of parabolic renormalizations

In this section we describe a construction from [DD] for finding a canonical representation of the parabolic renormalization in the essentially period tripling case. For simplicity we will state the construction for the quadratic map $P_{-1.75}$. However, it is clear how to generalize this construction to any map $f \in I^{-1}(-1.75)$.

Recall from §3.1 the sequence of maximal $\mathcal{M}$-copies $\mathcal{M}^{c_n}$ with essentially period tripling combinatorics accumulate at the root of the period three tuned copy, $\mathcal{M}^{-1.75}$. Let $f = P_{-1.75}$ and choose f_0 and $D_\pm$ as above. Let $B = \text{int}\, K(f_0)$ and let $f_n = P_{c_n}$. Choose n sufficiently large and let U_- be a connected component of $\text{int}\, K(f) \setminus B$ that

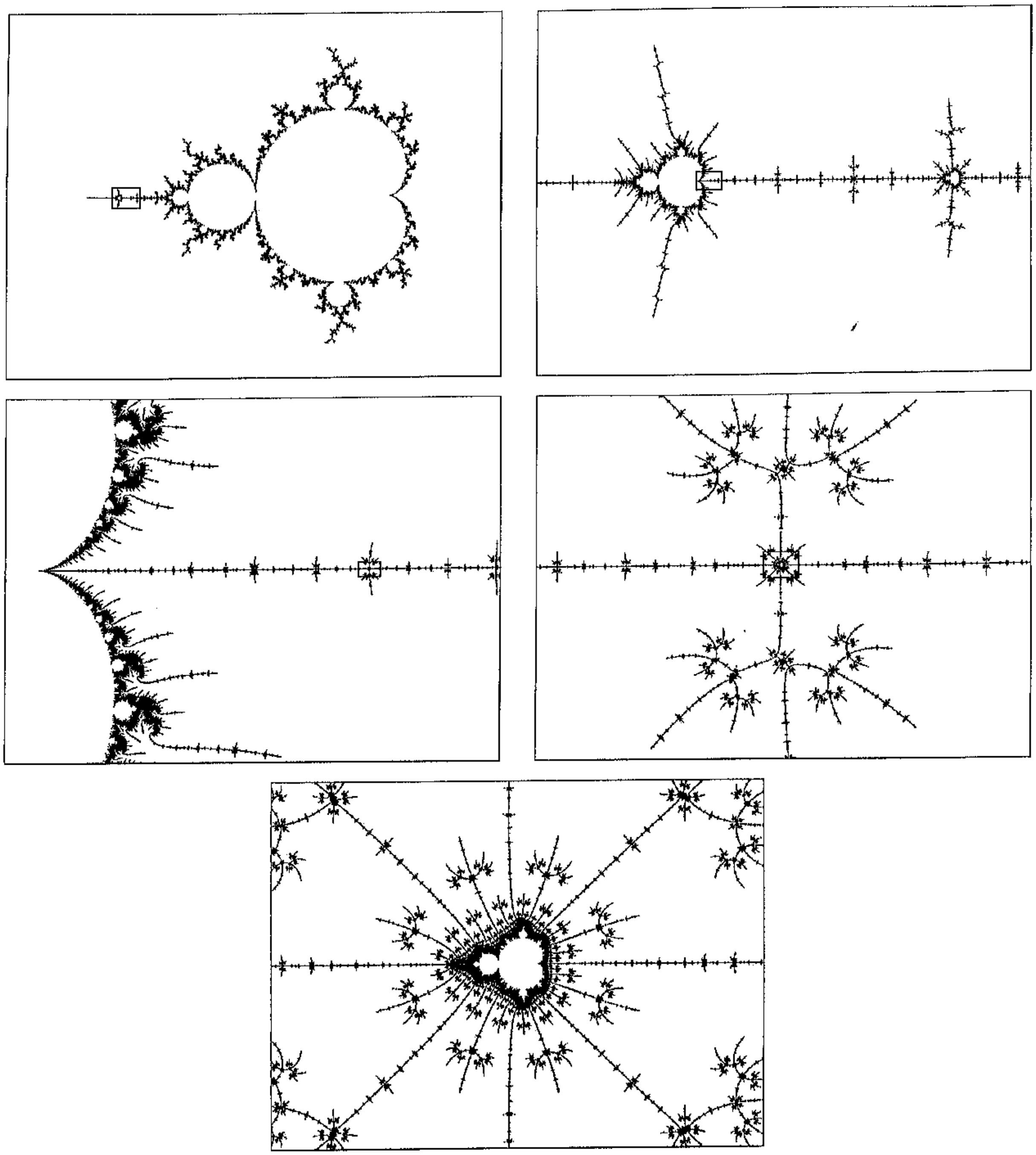

Figure 3.3: The Mandelbrot set zooming in to $\mathcal{M}_4^{(3)}$.

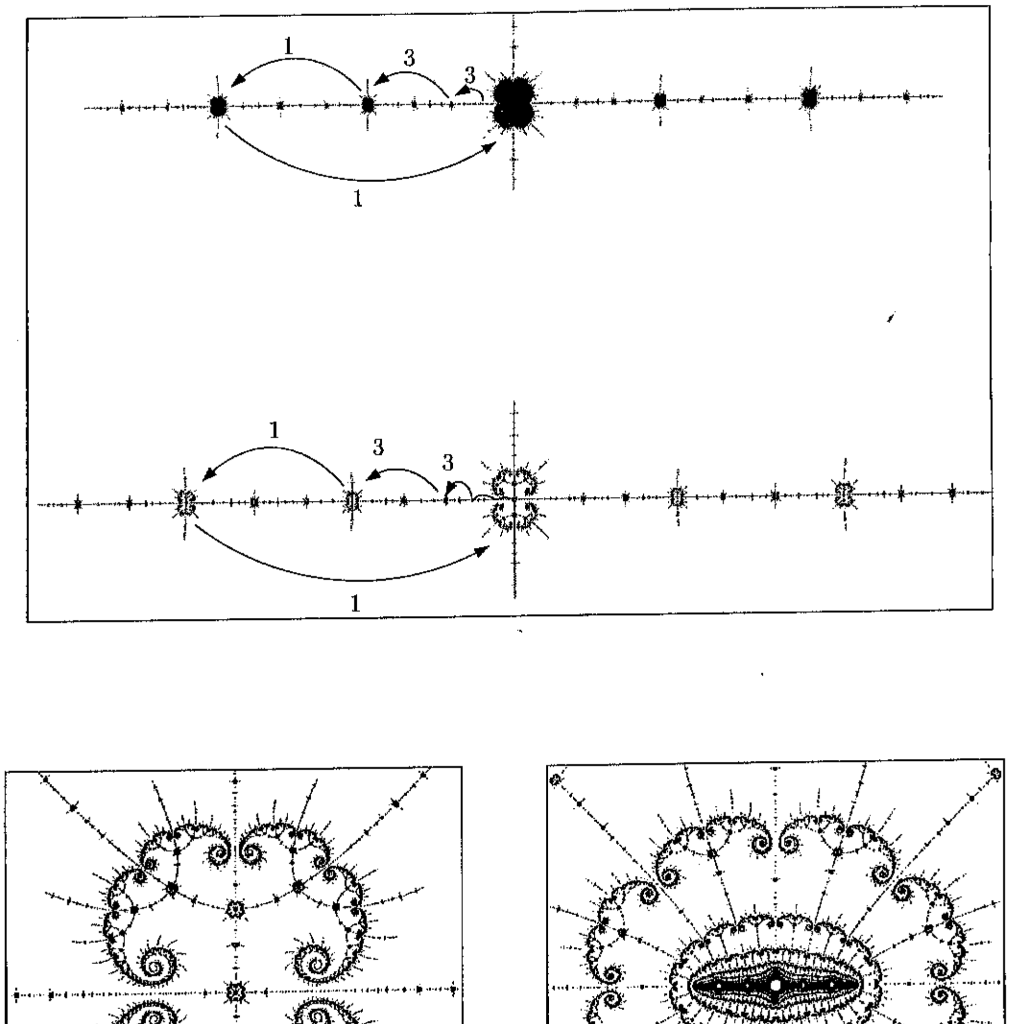

Figure 3.4: The Julia sets of $z^2 - 1.75$ and $z^2 + c_n$ for some large n and blow-ups.

intersects $P(f_n)$ and such that $U_- \Subset D_-$. Let t be the landing time of U_- to B under f.

Let

$$\mathcal{D}_f = \{g : \mathcal{C}_+ \to \mathcal{C}_- | g \text{ is a transit map and } g(\pi_+(0)) \in \pi_-(U_-)\}.$$

The phase map gives a conformal isomorphism of $\mathcal{D}_f$ to a disk $D_f \Subset \mathbb{C}/\mathbb{Z}$. Note that D_f is a Jordan domain. Choose a branch of π_-^{-1} so that the range contains U_-. For $g \in \mathcal{D}_f$ let W_g be the connected component of $(\pi_+^{-1} \circ g^{-1} \circ \pi_-)(U_-)$ containing 0. Since $\pi_-(U_-)$ is a topological disk, it follows the map $R_{\bar{a}} : W_g \to B$ given by

$$R_{\bar{a}} = f^t \circ \pi_-^{-1} \circ g_{\bar{a}} \circ \pi_+$$

is quadratic-like with possibly disconnected Julia set. If $J(R_{\bar{a}})$ is connected then we have constructed a parabolic pre-renormalization of (f, g).

Fix any $* \in D_f$. Define the holomorphic motion

$$h_{\bar{a}} : (\partial B, \partial W_{g_*}) \to (\partial B, \partial W_{g_{\bar{a}}})$$

on ∂B by the identity and locally on $\partial W_{g_{\bar{a}}}$ by pulling back under $R_{\bar{a}}$. Let $\mathbb{V} = \{(\bar{a}, z) : \bar{a} \in D_f, z \in B\}$ and $\mathbb{U} = \{(\bar{a}, z) : \bar{a} \in D_f, z \in W_{g_{\bar{a}}}\}$. Let $\mathbf{f} : \mathbb{U} \to \mathbb{V}$ be defined by

$$\mathbf{R}(\bar{a}, z) = (\bar{a}, R_{\bar{a}}(z)).$$

Lemma 3.1.1 *The family* $(\mathbf{R}, \mathbf{h})$ *is a proper DH quadratic-like family with winding number 1.*

Proof: The map f^t is a conformal isomorphism of a neighborhood of U_- onto a neighborhood of B. There is a branch of π_-^{-1} such that the map $(\pi_-^{-1} \circ g_{\bar{a}} \circ \pi_+)(0)$ is a conformal isomorphism of a neighborhood of D_f onto a neighborhood of U_-. The lemma follows. □

The following lemma states that the renormalization operators $\mathcal{R}_{\sigma_n^{(3)}}$ converge to essentially period tripling parabolic renormalization. In order to state the lemma we need some notation. Let $f \in I^{-1}(-1.75)$. Suppose $f_k \in I^{-1}(\mathcal{M}^{c_{n_k}})$ is a sequence of renormalizable maps with $f_k \to f$ and $n_k \to \infty$. From Lemma 2.4.3 there exist perturbed Fatou coordinates for f_k. Let $g_k : \mathcal{C}_{f_k,+} \to \mathcal{C}_{f_k,-}$ be the induced transit maps with phases $\bar{a}_k$.

Lemma 3.1.2 *With the above hypotheses,*

1. $\{\bar{a}_k\}$ *is pre-compact*

2. *if* $\bar{a}_{k_j} \to \bar{a}$ *is a convergent subsequence then* $J(R_{\bar{a}})$ *is connected and*

$$[h_{k_j}] \to [R_{\bar{a}}]$$

where h_k *is a pre-renormalization of* f_k

3. $\mathcal{F}(f, g_{\bar{a}})$ *is contained in any geometric limit of* $\langle f_{k_j} \rangle$

Proof: Let h_k be a pre-renormalization of f_k. Since $f_k \in I^{-1}(\mathcal{M}^{c_{n_k}})$ and $f_k \to f$ we can write

$$h_k = f_k^{N_1} \circ \widetilde{g}_k \circ f_k^{N_2} \tag{3.3}$$

on some neighborhood of the origin for some fixed N_1, N_2 and some choice of local lift $\widetilde{g}_k$ of the induced transit maps $g_k : \mathcal{C}_{f_k,+} \to \mathcal{C}_{f_k,-}$. The first claim is that h_k can be chosen in $\mathcal{Q}^\infty(m')$ for some $m' > 0$. Let V' be an ϵ-neighborhood of the central basin B of f for some small $\epsilon > 0$. Choose ϵ small enough and N_1 and N_2 large enough so that for large k the right-hand side of (3.3) can be used to define a pre-renormalization h_k with range V'. Let $U'_k = h_k^{-1}(V')$. By taking k larger still we can assume U'_k is contained in an $\epsilon/2$ neighborhood of B. It follows there is an $m' > 0$ so that $\operatorname{mod}(V', U'_k) \geq m'$. Moreover, $\operatorname{diam}(U'_k) \geq C > 0$ for some C independent of k. Hence (3.3) holds on a definite neighborhood of the origin.

From the convergence of Fatou coordinates and the convergence of f_k it follows that $\{\bar{a}_k\}$ is pre-compact. Let $\bar{a}_{k_j} \to \bar{a}$ be a convergent subsequence. Then h_{k_j} converges on a definite neighborhood of the origin to the map $f^{N_1} \circ \widetilde{g}_{\bar{a}} \circ f^{N_2}$ for an appropriate local lift $\widetilde{g}_{\bar{a}}$ of $g_{\bar{a}}$. Since the origin is non-escaping under all h_k it follows $J(R_{\bar{a}})$ is connected. The last statement follows from the fact that $f_k \to f$ and from equation 2.3. $\square$

Moreover, the proof of the previous lemma can be modified to prove the following

Lemma 3.1.3 *Suppose* $f \in I^{-1}(-1.75)$ *and* $f_k \in I^{-1}(-1.75)$ *satisfy* $f_k \to f$. *Suppose* $g_k : \mathcal{C}_{f_k,+} \to \mathcal{C}_{f_k,-}$ *are transit maps with phases* $\bar{a}_k$ *such that* $R_k = R_{\bar{a}_k}$ *is defined. Then*

1. $\{\bar{a}_k\}$ *is pre-compact*
2. *if* $\bar{a}_{k_j} \to \bar{a}$ *is a convergent subsequence then*

$$R_{k_j} \to R_{\bar{a}}.$$

3. $\mathcal{F}(f, g_{\bar{a}})$ *is contained in any geometric limit of* $\mathcal{F}(f_{k_j}, g_{k_j})$

We finish this section with two useful properties of parabolic renormalization. The first property is that open sets intersecting the Julia set of the parabolic pre-renormalization iterate under $\mathcal{F}(f, g)$ to open sets intersecting $J(f)$.

Lemma 3.1.4 *Let* $f \in I^{-1}(-1.75)$ *and* $g : \mathcal{C}_+ \to \mathcal{C}_-$ *be a transit map with phase* $\bar{a}$ *such that* $J(R_{\bar{a}})$ *is connected. Suppose* U *is an open set satisfying*

$$U \cap J(R_{\bar{a}}) \neq \emptyset.$$

Then there is an $h \in \mathcal{F}(f, g)$ *such that* $U \cap Dom(h) \neq \emptyset$ *and*

$$h(U) \supset J(f).$$

Proof: From the construction of $R_{\bar{a}}$ it is clear that there is an $h \in \mathcal{F}(f,g)$ such that h is a quadratic-like extension of $R_{\bar{a}}$ to a small neighborhood of $B = Range(R_{\bar{a}})$. It follows that there an $m \geq 0$ such that $h^m(U) \cap \partial B \neq \emptyset$. But $\partial B \subset J(f)$. Iterating f further covers all of $J(f)$. $\square$

The second property is that no quadratic-like representative of $[R_{\bar{a}}]$ can have too large a domain.

Lemma 3.1.5 *Let $f \in I^{-1}(-1.75)$ and $g : \mathcal{C}_+ \to \mathcal{C}_-$ be a transit map with phase $\bar{a}$ such that $R_{\bar{a}}$ is defined. If $(\widetilde{f} : U \to V) \in \mathcal{Q}$ satisfies $[\widetilde{f}] = [R_{\bar{a}}]$ then*

$$U \subset Range(R_{\bar{a}}).$$

Proof: Let f_1 be a pre-renormalization of f and let $B = Range(R_{\bar{a}})$. Suppose $U \cap \partial B \neq \emptyset$ and let U' be the connected component of $U \cap B$ containing 0. Since f_1-preimages of 0 accumulate on $J(f_1) = \partial B$ there exists an $n > 0$ and $z_0 \in U'$ such that $f_1^n(z_0) = 0$. Since $[\widetilde{f}] = [R_{\bar{a}}]$ it follows $\widetilde{f}$ has a critical point at z_0, which is a contradiction. $\square$

3.2 Theorem II

Let us remind the reader of our second result:

Theorem II (Essentially Bounded Combinatorics). *For every $p > 1$ there exists a pre-compact $\mathcal{R}$-invariant set $\mathcal{A} \subset \mathcal{G}_{\mathbb{R}}$ such that*

1. *$\mathcal{R}|_{\mathcal{A}}$ is conjugate to the full shift on $\Omega_e(p)$, the set of combinatorial types with essential period bounded above by p, and*

2. *if $f \in \mathcal{Q}_{\mathbb{R}}$ is ∞-renormalizable and $\sup_n p_e(\mathcal{R}^n f) \leq p$ then $\mathcal{R}^n f \to \overline{\mathcal{A}}$.*

Let $h : \Pi_{-\infty}^{\infty}\Omega_e(p) \to \mathcal{A}$ denote the conjugacy. Then there is a compactification $\Omega_e^{cpt}(p)$ of $\Omega_e(p)$ such that h extends to a continuous map $h : \Pi_{-\infty}^{\infty}\Omega_e^{cpt}(p) \to \overline{\mathcal{A}}$. Moreover, suppose $f_1 \in \mathcal{Q}_{\mathbb{R}}$ has the same combinatorics as $f_2 \in \mathcal{A}$. Then for any sequence $n_k \to \infty$ and sequence $\bar{\sigma} \in \Pi_{-\infty}^{\infty}\Omega_e^{cpt}(p)$ such that $\mathcal{R}^{n_k} f_2 \to h(\bar{\sigma})$ one has $\mathcal{R}^{n_k} f_1 \to h(\bar{\sigma})$.

Let $\Omega_e(p)$ be the space of shuffles σ satisfying $p_e(\sigma) \leq p$. The sequence $\sigma_n^{(3)}$ from the previous section is a simple example of a sequence of shuffles with bounded essential period but unbounded period. In this section we construct a compactification $\Omega_e^{cpt}(p)$ of $\Omega_e(p)$ which will form the elements of our combinatorial description of renormalization limits.

Suppose $f \in \mathcal{Q}_{\mathbb{R}}^{\infty}$ is renormalizable and let

$$\Gamma_m \overset{\chi_m}{\to} \Gamma_{m-1} \overset{\chi_{m-1}}{\to} \cdots \overset{\chi_2}{\to} \Gamma_1 \overset{\chi_1}{\to} \Gamma_0$$

be its sequence of return types. Let l be a neglectable level and let $\chi_l : (\Gamma_l, \epsilon_l) \to (\Gamma_{l-1}, \epsilon_{l-1})$ be the return type of g_l. It is clear that if both level $l-1$ and $l+1$ are neglectable then (Γ_l, ϵ_l) and $(\Gamma_{l-1}, \epsilon_{l-1})$ are generated by configurations of the form

$$\pm I_{-p}, \pm I_{-p+1}, \ldots, \pm I_{-1}, -I_0$$

or by

$$+I_0, \pm I_1, \ldots, \pm I_{p-1}, \pm I_p$$

for some $p \geq 1$. We claim that $(\Gamma_1, \epsilon_l) \cong (\Gamma_{l-1}, \epsilon_{l-1})$ and that χ_l is defined by $I_i \mapsto I_i I_0$ for $i \neq 0$ and $I_0 \mapsto I_0$. First it is clear $I_0 \mapsto I_0$. Now if $\chi_l(I_i)$ contained more than one off-critical interval then l would not be a neglectable level. Since χ_l is unimodal it follows Γ_{l-1} contains at least as many intervals as Γ_l. Since the return type sequence is irreducible Γ_{l-1} contains exactly the same number of intervals as Γ_l. Hence $I_i \mapsto I_i I_0$. The claim that the signs agree follows from the condition that χ_l be admissible.

Hence we can "insert" another neglectable level into S before l to obtain another irreducible sequence S' of return types:

$$\Gamma_m \xrightarrow{\chi_m} \cdots \Gamma_l \xrightarrow{\chi_l} \Gamma_{l-1} \cong \Gamma_l \xrightarrow{\chi_l} \Gamma_{l-1} \cdots \xrightarrow{\chi_1} \Gamma_0.$$

From Theorem 2.3.2 there is a unique shuffle σ' such that $S(\sigma') = S'$.

We say two shuffles σ and σ' in $\Omega_e(p)$ are *essentially equivalent* if one can insert a finite number of neglectable levels into σ and σ' and obtain equal shuffles. Let Ξ be the partition of $\Omega_e(p)$ into essentially-equivalent equivalence classes. Let $U \in \Xi$ be a non-trivial equivalence class. Then there is an $n = n_U > 0$ such that for any $\sigma \in U$ the return type sequence $S(\sigma)$ has exactly n different cascades $S_1, S_2, \ldots, S_n$, canonically ordered, containing neglectable levels. Let l_k, $k = 1, \ldots, n$, denote the number of neglectable levels in the cascade S_k. The map $\theta_U : U \to \mathbb{N}_+^n$ given by $\sigma \mapsto (l_1, l_2, \ldots, l_n)$ is a homeomorphism. Let

$$\overline{\mathbb{N}}_+ = \mathbb{N}_+ \cup \{+\infty\}$$

be the one-point compactification of $\mathbb{N}$. Define $U^{cpt} \supset U$ as the unique space such that θ_U extends to a homeomorphism $\theta_U : U^{cpt} \to \overline{\mathbb{N}}_+^n$. Define $\Omega_e^{cpt}(p) \supset \Omega_e(p)$ as the union of the trivial classes of Ξ and of the spaces U^{cpt} for non-trivial $U \in \Xi$. An element of $\Omega_e^{cpt}(p) \setminus \Omega_e(p)$ is called an *end* and can be represented by a "sequence" of return types where infinitely long sequences of neglectable levels are allowed:

$$\Gamma_m \xrightarrow{\chi_m} \cdots \xrightarrow{\chi_{l+2}} \Gamma_{l+1} \xrightarrow{\chi_{l+1}} (\Gamma_l \xrightarrow{\chi_l} \Gamma_{l-1})^\infty \xrightarrow{\chi_{l-1}} \Gamma_{l-2} \xrightarrow{\chi_{l-3}} \cdots \xrightarrow{\chi_1} \Gamma_0.$$

The following lemma is evident from the definition of essential period and $\Omega_e^{cpt}(p)$.

Lemma 3.2.1 *For any $p > 1$ the space $\Omega_e^{cpt}(p)$ is metrizable and compact.*

Let

$$\{\mathcal{M}\}_p = \{\mathcal{M}^\sigma : \sigma \in \Omega_e(p)\}$$

be the collection of $\mathcal{M}$-copies corresponding to $\Omega_e(p)$ and let

$$\mathcal{C}^p = \{c : \mathcal{M}^c \in \{\mathcal{M}\}_p\}$$

be the corresponding collection of centers. We now describe the topology of $\mathcal{C}^p$ and how $\overline{\mathcal{C}^p}$ compares to $\Omega_e^{cpt}(p)$. For any $U \in \Xi$ with $n = n_U \geq 1$ let $\mathcal{C}_U \subset \mathcal{C}^p$ denote the collection of centers of $\{\mathcal{M}^\sigma : \sigma \in U\}$. Since Ξ is a finite partition it suffices to describe the topology of the sets $\mathcal{C}_U$. We claim for each non-trivial $U \in \Xi$ there is a homeomorphism of $\mathbb{R}$ which maps $\mathcal{C}_U$ to the image of the function $F : \mathbb{N}_+^n \to \mathbb{R}$ given by

$$F(x_1, x_2, \ldots, x_n) = 2^{-x_1} + 2^{-x_1 x_2 - 1} + \cdots + 2^{-x_1 x_2 \cdots x_n - n + 1}$$

where $n = n_U$ (see Fig. 3.5).

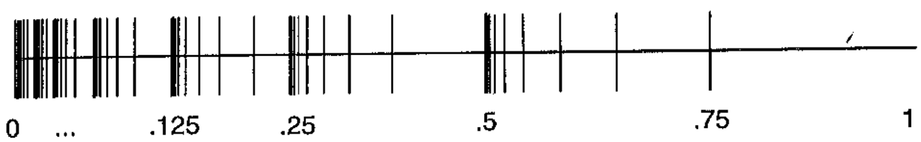

Figure 3.5: The image of F for $n = 2$.

To be more specific the limit points of $\mathcal{C}_U$ are root points of the $\mathcal{M}$-copies obtained by "truncating" the return type sequences of $\sigma \in U$ at the neglectable levels. Let us describe how to truncate a return type sequence

$$\Gamma_m \xrightarrow{\chi_m} \Gamma_{m-1} \xrightarrow{\chi_{m-1}} \cdots \xrightarrow{\chi_1} \Gamma_0$$

at a level l. Let (Γ_T, ϵ_T) be the semigroup generated by I_0 with $\epsilon_T(I_0) = \epsilon_l(I_0^l)$ and let χ_T be the homomorphism defined by $I_0 \mapsto \chi_l(I_0^l)$. Let S' be the sequence

$$\Gamma_T \xrightarrow{\chi_T} \Gamma_{l-1} \xrightarrow{\chi_{l-1}} \cdots \xrightarrow{\chi_2} \Gamma_1 \xrightarrow{\chi_1} \Gamma_0.$$

One can check that S' is a sequence of admissible unimodal return types. If S' is not irreducible then simply remove all intervals I_i^m not in the combinatorial orbit of the critical point and shorten the sequence if necessary. We obtain a unique shuffle $\sigma' = \lfloor \sigma \rfloor_l$, the shuffle σ truncated at level l.

Let $U \in \Xi$ satisfy $n = n_U \geq 1$. Any shuffle $\sigma \in U$ has n cascades with neglectable levels of lengths $x_1, \ldots, x_n$ respectively. As $x_1 \to \infty$, the corresponding centers accumulate at the root of the tuned $\mathcal{M}$-copy corresponding to any $\sigma \in U$ truncated at the first neglectable level. If we fix x_1 and let $x_2 \to \infty$ the corresponding centers accumulate at the root of the $\mathcal{M}$-copy corresponding to truncating at the second cascade of neglectable levels. In general if we fix the lengths of the first k sequences of neglectable levels and let the length of the $k+1$-st sequence grow the centers converge to the root of the $\mathcal{M}$-copy corresponding to truncating at the $(k+1)$-st neglectable sequence.

Given an end $\tau \in \Omega_e^{cpt}(p)$ let

$$c(\tau) = root(\lfloor \sigma \rfloor_l)$$

where $\sigma \in \Omega_e(p)$ is in a sufficiently small neighborhood of τ, l is a neglectable level of σ which belongs to the first infinitely long cascade of τ, and $root(\sigma)$ is the root of $\mathcal{M}^\sigma$. The map $c : \Omega_e^{cpt}(p) \to \mathbb{R}$ is continuous and its image is $\overline{C^p}$. This completes our description of the topology of $\{\mathcal{M}\}_p$ and how $\overline{C^p}$ compares to $\Omega_e^{cpt}(p)$.

We return to our examples from §2.3.3. Choose a large p so that the shuffles from Example 2.3.1 and Example 2.3.2 are contained in $\Omega_e(p)$.

First consider the essentially period tripling shuffles $\sigma_n^{(3)}$. Then $c(\sigma_n^{(3)}) \to root(\sigma^{(3)})$ where $\sigma^{(3)}$ is the period tripling shuffle. Moreover, $\sigma_n^{(3)}$ converges to an end $\tau_1 \in \Omega_e^{cpt}(p)$.

Now consider the shuffles $\sigma_{m,d}$ from Example 2.3.1 (Goes Through Twice). First fix $d > 1$ and let $m \to \infty$. Then $c(\sigma_{m,d}) \to root(\lfloor \sigma_{m,d} \rfloor_l) \neq root(\sigma^{(3)})$ where l is any neglectable level and, in much the same spirit as essential period tripling, $\sigma_{m,d}$ converges in $\Omega_e^{cpt}(p)$ to an end. Now fix $m - d > 1$ and let $m \to \infty$. Then $c(\sigma_{m,d}) \to root(\sigma^{(3)})$ and $\sigma_{m,d}$ converges to an end $\tau_2 \in \Omega_e^{cpt}(p)$.

Finally consider the shuffles σ_{l_1,l_2} from Example 2.3.2 (Two Cascades). Fix $l_1 > 1$ and let $l_2 \to \infty$. Then $c(\sigma_{l_1,l_2}) \to r_{l_1} = root(\lfloor \sigma_{l_1,l_2} \rfloor_l)$ where l is any neglectable level in the second cascade. The sequence $r_{l_1} \to root(\sigma^{(3)})$ as $l_1 \to \infty$. Moreover, for any sequence of l_2 if we let $l_1 \to \infty$ then $c(\sigma_{l_1,l_2}) \to root(\sigma^{(3)})$. Now consider the limits of σ_{l_1,l_2} in $\Omega_e^{cpt}(p)$. If we fix l_2 and let $l_1 \to \infty$ the shuffles will converge to an end τ_{∞,l_2}.

Chapter 4

Preliminary Constructions

In this section we describe a combinatorial framework that is more convenient for our proofs than shuffles and parabolic shuffles. The combinatorial objects, markings of generalized quadratic-like maps, are based on return-type homomorphisms. The maps, first through maps, are based on the Bernoulli schemes introduced in [L3] and they encode an entire cascade at once rather than as a sequence of first return maps. We then define the generalized parabolic renormalization of a generalized quadratic-like map with a saddle-node cascade.

4.1 Markings

Fix a map $f \in \mathcal{L}$. Let $A = \{\alpha_1, \ldots, \alpha_m\} \subset \partial Range(f)$, $m \geq 2$, be a finite number of marked points in $\partial Range(f)$. Let Γ be a collection of Jordan curves $\gamma : I = [0,1] \to \mathrm{cl}(Range(f) \setminus Dom(f))$ such that

$$\gamma(\partial I) \in \mathcal{A} = A \cup f^{-1}(A).$$

Let G be the graph with vertices $\mathcal{V} = \mathcal{A} \sqcup \mathrm{CC}(Dom(f))$ and with an edge between $x, y \in \mathcal{V}$ if either

- $x, y \in \mathcal{A}$ and there exists $\gamma \in \Gamma$ such that $\gamma(\partial I) = \{x, y\}$
- or $y \in \partial x$.

We say (A, Γ) is a *marking of* f iff G is a tree.

Example 4.1.1 *Any $f \in \mathcal{L}_{\mathbb{R}}$ with $Dom(f)$ and $Range(f)$ Jordan disks is naturally marked by choosing $A = Range(f) \cap \mathbb{R}$ numbered so that $\alpha_1 < \alpha_2$ and Γ such that $(Dom(f) \cap \mathbb{R}) \cup \bigcup_{\gamma \in \Gamma} Im(\gamma) = Range(f) \cap \mathbb{R}$.*

Two markings (A_1, Γ_1) and (A_2, Γ_2) are *homotopic* iff $A_1 = A_2$ and if there is a bijection $h : \Gamma_1 \to \Gamma_2$ such that each $\gamma \in \Gamma_1$ is homotopic rel $\mathcal{A}$ to $h(\gamma)$. A marking is *real* if

- $|A| = 2$

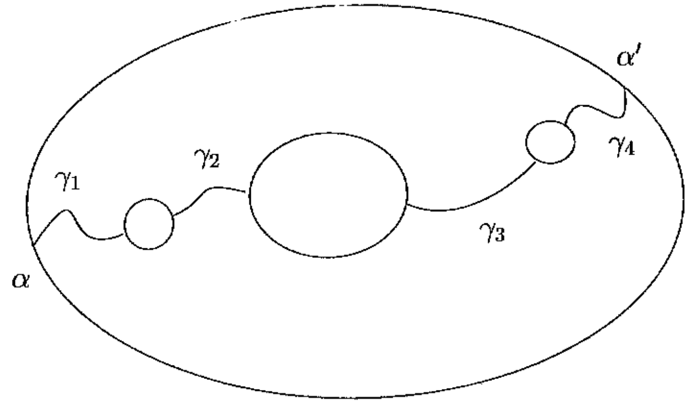

Figure 4.1: A marking $(\alpha, \alpha', \{\gamma_i\})$ of a generalized quadratic-like map.

- the span of $\mathrm{CC}(Dom(f))$ in G is isomorphic to a linear graph
- G is symmetric around the span of A.

A real marking induces an ordering on $\mathrm{CC}(Dom(f))$ by the rule

$$U_i \leq U_j \text{ iff } \begin{array}{l} \text{the shortest path from } \alpha_1 \text{ to } U_i \text{ in } G \\ \text{contains the shortest path from } \alpha_1 \text{ to } U_j \end{array}$$

and associates to f a signed ordered semigroup $\Lambda_{(f,A,\Gamma)}$ generated by $\mathrm{CC}(Dom(f))$ by defining the sign $\epsilon(U)$ of $U \in \mathrm{CC}(Dom(f))$ by the relative position of $f^{-1}(\alpha_1) \in \partial U$.

Let $\mathcal{ML}$ denote the space $(f, A, \Gamma)/\sim$ where $f \in \mathcal{L}$, $Dom(f)$ and $Range(f)$ are Jordan disks, (A, Γ) is a marking of f and

$$(f_1, A_1, \Gamma_1) \sim (f_2, A_2, \Gamma_2) \text{ iff } f_1 = f_2, A_1 = A_2 \text{ and } \Gamma_1 \text{ is homotopic to } \Gamma_2.$$

The topology on $\mathcal{ML}$ is the Caratheodory topology on f, the Hausdorff topology on A and the quotient the sup-norm topology on Γ. Let $\mathcal{ML}_{\mathbb{R}} \subset \mathcal{ML}$ denote the subspace where (A, Γ) is a real marking.

4.1.1 Induced Markings

Fix $(f, A, \Gamma) \in \mathcal{ML}$ and suppose 0 returns to $U_0(f)$ under f. Let g be a restriction of $R(f, U_0(f))$ to finitely many components of $Dom(R)$ so that $g \in \mathcal{L}$. Mark g by the following inductive construction.

First let $G_1 = f_0^{-1}(G)$ be the lift of the graph G to the central component $U_0(f)$ by the double cover f_0. Label the points $\{\alpha'_1, \alpha'_2, \ldots, \alpha'_{2m}\} = A' = f_0^{-1}(A)$ so that α'_1 and α_1 are in the same connected component of the graph $G \setminus U_0(f)$ and so that α'_2 is the dynamically symmetric point to α'_1:

$$\alpha'_2 = f_0^{-1}(f(\alpha'_1)) \setminus \{\alpha'_1\}.$$

Now suppose by induction that we have constructed G_i and suppose there is a connected component of $Dom(g)$ that is compactly contained in a component U of G_i.

Let G_U be the lift of G to U by the covering map f^n where n is the smallest integer such that $f^n(U) = Range(f)$. Define the graph G_{i+1} to be the graph G_i outside of U and to be G_U on U. Note that these graphs glue continuously across ∂U. Repeat this construction until all connected components of $Dom(g)$ are components of some G_N, $N \geq 1$. For each pair

$$(x, y) \in \mathrm{CC}(Dom(g)) \sqcup A'$$

there is a shortest path $p_{x,y}$ in G_N between x and y. For each component W of G_N on p choose a Jordan curve $\gamma_W \subset W$ between the neighbors of W in p. Since p is a finite path there is a well-defined concatenation $\gamma_{x,y}$ of $\{\gamma_W\}$ and of the curves of G_N in $p_{x,y}$ such that as x, y range over all allowed pairs we obtain a marking $(A', \{\gamma_{x,y}\})$ of g. The *induced marking of* g, denoted (A_g, Γ_g), is the smallest sub-marking of g from $(A', \{\gamma_{x,y}\})$ that contains $\{\alpha'_1, \alpha'_2\}$.

The induced marking is defined up to homotopy. In particular, let $(f, A, \Gamma) \in \mathcal{ML}_{\mathbb{R}}$ and suppose $g \in \mathcal{L}$ is a restriction of $R(f, U_0(f))$ to finitely many components such that $(g, A_g, \Gamma_g) \in \mathcal{ML}_{\mathbb{R}}$. Define the *return-type homomorphism* $\chi : \Lambda_{(g,A_g,\Gamma_g)} \to \Lambda_{(f,A,\Gamma)}$ of g as in §2.3.2.

4.1.2 Initial Markings

Suppose $f \in I^{-1}(\mathcal{M}^\sigma)$ for some shuffle $\sigma \neq \sigma^{(2)}$. Choose an initial Yoccoz partition Υ and let V^0 be the initial return domain w.r.t. Υ, as defined in §2.3.2. Let g be a restriction of $R(f, V^0)$ to finitely many components of $Dom(R)$ so that $g \in \mathcal{L}$. Mark g as follows.

Recall that V^0 is the connected component of $f^{-2}(Z_1^{(1)})$ containing 0. Let $A = \{\alpha_1, \alpha_2\} \subset \partial V^0$ be two pre-images of $f^{-2}(\alpha'_f)$, numbered so that the smallest positive angle of an external ray landing at α_1 is larger than the smallest positive angle of the ray landing at α_2. See Fig. 4.2 for a diagram of the situation.

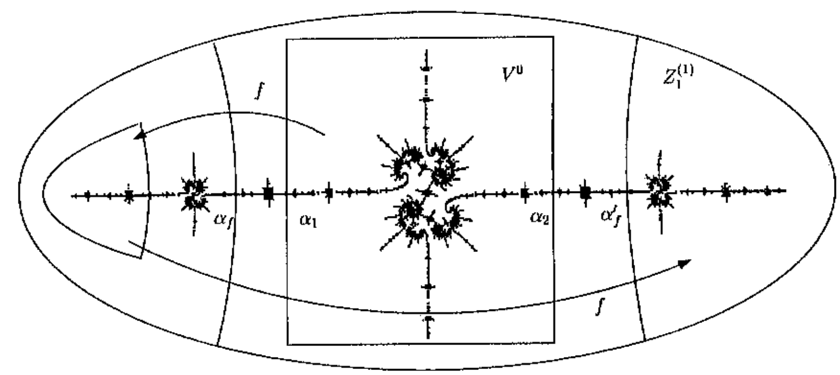

Figure 4.2: The marked points A in the initial marking.

Let D_{-1}, D_1 be the two connected components in $Y^{(1)}$ of $f^{-2}(Y^{(1)})$ and let $\cup_{n\geq1} Z_n$ be the connected components in $Z_1^{(1)}$ of the domain of the first landing map L to $Y^{(1)}$. Consider the map $\widetilde{f}$ which is f^2 on $D_{-1} \cup V^0 \cup D_1$ and L on $\cup_{n\geq1} Z_n$. Then apply the same inductive construction as for induced markings to the map $\tilde{f}$. That is, pull back the configuration $X_0 = \{D_{-1}, V^0, D_1, \cup_{n\geq 1} Z_n\}$ to V^0 and refine it as $X_1, X_2, \ldots$ until the components of $Dom(g)$ are not compactly contained in any component of X_N. Choose a collection of curves Γ with images in $\text{cl}(\cup X_N)$ so that (g, A, Γ) is a marking of g. We call $(g, A, \Gamma) \in \mathcal{ML}$ the *initial marking of* g.

If $(g, A, \Gamma) \in \mathcal{ML}_{\mathbb{R}}$ then the initial combinatorial type of g is given by the zero-admissible return-type homomorphism $\chi : \Lambda_{(g,A,\Gamma)} \to (\Gamma_0, \epsilon_0)$ as defined in §2.3.2.

4.2 First-through Maps

Fix an $f \in \mathcal{L}$. Let L be the first landing map of f to $\cup_{j\neq 0} U_j(f)$ and suppose $0 \in Dom(L)$. Define the *first through map*, $T(f, \cup_{j\neq 0} U_j(f))$, to be $f \circ L$ (see Fig. 4.3 and Fig. 4.4). Note that first through maps are not generalized quadratic-like maps, since if there exists a one branched point of T then all pre-images by L are branch points as well. However, first through maps are very similar to generalized quadratic-like in the following sense: instead of having a unique critical point, first through maps have a unique critical value. In any case we will construct restrictions of first through maps that are generalized quadratic-like.

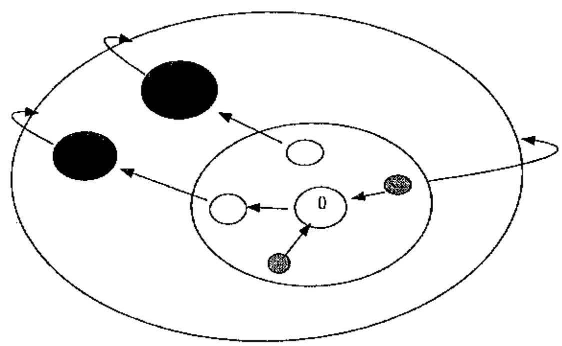

Figure 4.3: A first through map with off-critical (dark gray) and pre-critical (light gray) pieces.

Let h be a restriction of $T(f, \cup_{j\neq 0} U_j(f))$ to finitely many components of $Dom(T)$ and suppose $h \in \mathcal{L}$. Let $f_0 = f|_{U_0(f)}$. Suppose f is marked by (A, Γ). As before let G_1 be the lift of G by f_0. Refine the graph G_i inductively until all components of $Dom(h)$ are components of some G_N, $N \geq 1$ and define the *induced marking* of h in the same way as for return maps.

See Fig. 4.5 for the first few pull-backs of $\cup_{j\neq 0} U_j(f)$ and Γ.

Let us define the combinatorial type of a real-marked through map. Let $(f, A, \Gamma) \in \mathcal{ML}_{\mathbb{R}}$ and suppose $g \in \mathcal{L}$ is a restriction of $T(f, \cup_{j\neq 0} U_j(f))$ to finitely many components such that $(g, A_g, \Gamma_g) \in \mathcal{ML}_{\mathbb{R}}$. For each $U \in \text{CC}(Dom(g))$, define $d_{U,0} = n_0(0) - n_0(U)$ and $d_{U,A} = n_0(U)$ Define the *depth* $d(U)$ of $U \in \text{CC}(Dom(g))$ as

$$d(U) = \begin{cases} -d_{U,0} & d_{U,0} < d_{U,A} \\ d_{U,A} & \text{otherwise} \end{cases}$$

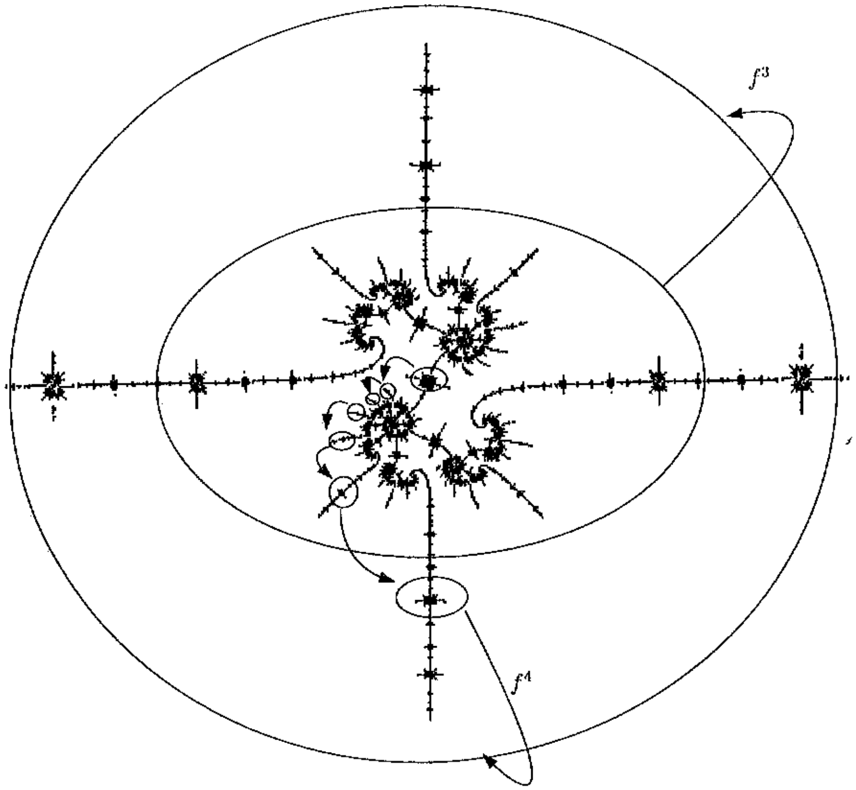

Figure 4.4: A restriction of a first through map.

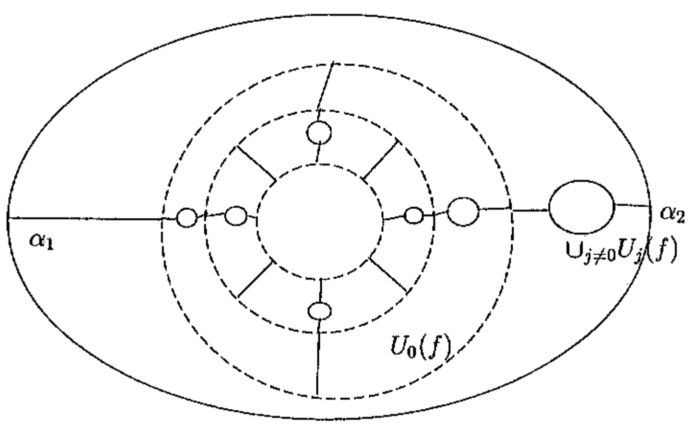

Figure 4.5: A few curves of a marked first through map.

Define the *through-type homomorphism* $\chi : \Lambda_{(g,A_g,\Gamma_g)} \to \Lambda_{(f,A,\Gamma)} \times \mathbb{Z} \times \mathbb{N}_0$ to be the homomorphism generated by

$$\chi(U) = (L(U), d(U), n_0(0)).$$

4.2.1 Saddle-node cascades

We say $(f, A, \Gamma) \in \mathcal{ML}_{\mathbb{R}}$ has a *saddle-node cascade* if

- $(T(f, \cup_{j\neq 0} U_j(f)|P(f)), A_T, \Gamma_T) \in \mathcal{ML}_{\mathbb{R}}$
- f has a *low return*: let $a \in A_T \cap \partial U_0(T)$ be in the same connected component of $G_T \setminus U_0(T)$ as α_1. Then f has a low return if the component $U \ni f(0)$, $U \in \mathrm{CC}(Dom(T))$ is in the same connected component of $G_T \setminus U_0(T)$ as a and $T(a)$.

The *length* $\ell(f)$ of a saddle-node cascade is the first landing time of 0 in $\cup_{j\neq 0} U_j(f)$ under f.

The following lemma is the complex analogue of Lemma 2.3.3:

Lemma 4.2.1 *Suppose $(f_n, A_n, \Gamma_n) \in \mathcal{ML}_{\mathbb{R}}$ converges to $(f, A, \Gamma) \in \mathcal{ML}_{\mathbb{R}}$ and suppose each f_n has a saddle-node cascade with $\ell(f_n) \to \infty$. Then $f|_{U_0(f)} \in I^{-1}(1/4)$.*

Proof: Apply the pull-back argument. Let $\widetilde{f}_n = f_n|_{U_0(f_n)}$ and let $W_n = Range(f_n) \setminus Dom(\widetilde{f}_n)$. We can assume that W_n is bounded by piecewise smooth curves. Since f_n has a saddle-node cascade there is a unique $N_n \to \infty$ so that $c_n = \widetilde{f}_n^{N_n}(0) \in W_n$. Let B be the disk of radius 4 around the origin. Let $g_n = z^2 + 1/4 + \epsilon_n$ where $\epsilon_n > 0$ is chosen so that $g_n^{N_n}(0) \in B$ but $g_n^{N_n+1}(0) \notin \overline{B}$. For n large enough g_n is quadratic-like. Let $W_n' = B \setminus g_n^{-1}(B)$ be the fundamental annulus of g_n. Now there is a $C > 0$ such that $\mathrm{mod}\, W_n \geq C$ and $\mathrm{mod}\, W_n' \geq C$. Since (f_n, A_n, Γ_n) converges to (f, A, Γ) it follows that there is a $K > 0$ so that for n large enough there is a K-quasiconformal map $h_n : W_n \to W_n'$ mapping (A_n, Γ_n), or a homotopic smooth marking, to the real axis, mapping c_n to $g_n^{N_n}(0)$ and conjugating $\widetilde{f}_n$ and g_n on the inner boundary of W_n and W_n', respectively. Pull back h_n to get a K-quasiconformal conjugacy $\widetilde{h}_n$ from $\widetilde{f}_n$ to g_n. This pull back is well defined since the marking determines which branch of g_n^{-1} to choose. Note that the limiting pull-backs extend over $K(\widetilde{f}_n)$ as K-quasiconformal maps and they glue across the pre-images of ∂W_n as K-quasiconformal maps. Choose a convergent subsequence $\widetilde{h}_{n_k}$ so that $\widetilde{h}_{n_k} \to h$. Let $\widetilde{f} = f|_{U_0(f)}$. Then h is a K-quasiconformal equivalence of $\widetilde{f}$ and $z^2 + 1/4$. From rigidity of parabolic parameter values, $\widetilde{f}$ is hybrid equivalent to $z^2 + 1/4$. □

Suppose (f, A, Γ) has a saddle-node cascade. Let h be the first through map $T(f, \cup_{j\neq 0} U_j(f))$ restricted to finitely many components such that $(h, A_h, \Gamma_h) \in \mathcal{ML}_{\mathbb{R}}$. The combinatorics of h is said to be *essentially bounded by B* if

- number of connected components of $Dom(f) \leq B$ and
- $d(U) \leq B$ for all $U \in \mathrm{CC}(Dom(h))$.

4.3 Generalized Parabolic Renormalization

In this section we modify the construction of parabolic renormalization to act on generalized quadratic-like maps. The idea is to consider the first through map of $\mathcal{F}(f,g)$.

Fix an $f \in \mathcal{L}$ that satisfies

$$f_0 = f|_{U_0(f)} \in I^{-1}(1/4).$$

Let $\xi = \beta_{f_0}$. Choose incoming and outgoing petals $D_\pm$ around the parabolic point ξ and let $\mathcal{C}_\pm$ denote the respective Ècalle-Voronin cylinders and $\pi_\pm$ the projections with π_+ extended to $B = \text{int}(K(f_0))$.

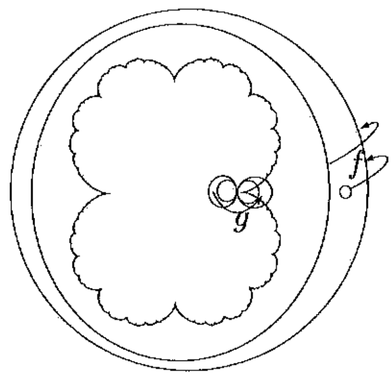

Figure 4.6: The generators f and g of $\mathcal{F}(f,g)$.

For a given $g : \mathcal{C}_+ \to \mathcal{C}_-$ let L be the first landing map under $\mathcal{F}(f_0,g)$ to $\cup_{j\neq 0}U_j(f)$. This map is well-defined because the germs of $\mathcal{F}(f_0,g)$ at z are well ordered (see §2.4.2).

Note that if C is a connected component of $Dom(L)$ then there is an $h \in \mathcal{F}(f_0,g)$ extending L to a branched cover of $Range(f)$. (see §2.4). Let $T(f,g,\cup_{j\neq 0}U_j(f))$ be the *first through map*

$$T = f \circ L. \tag{4.1}$$

4.3.1 Marking

Now suppose $(f, A, \Gamma) \in \mathcal{ML}_\mathbb{R}$ and let $h \in \mathcal{L}$ be a restriction of T to finitely many components of $Dom(T)$. Mark h in the following way.

First consider the collection of curves

$$\Gamma_1 = \cup_{n\geq 0}\{f_0^{-n}(\gamma)\}.$$

Since $0 \in K(f_0)$ each lift is well-defined for any $n \geq 0$. Let $T_1 = T(f,\cup_{j\neq 0}U_j(f))$ and let $\mathcal{A}_1 = A \cup T_1^{-1}(A)$. Let G be the graph with vertices

$$\mathcal{V}_1 = \mathcal{A}_1 \sqcup \text{CC}(Dom(T_1))$$

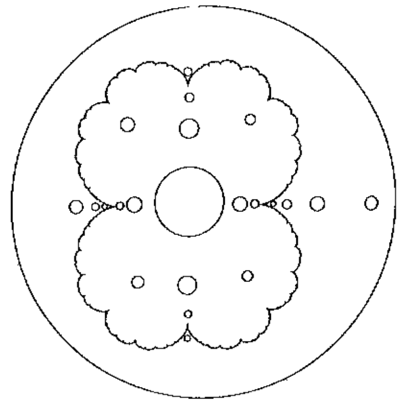

Figure 4.7: The first through map of $\mathcal{F}(f,g)$.

and with an edge between $x, y \in \mathcal{V}_1$ if either $x, y \in \mathcal{A}_1$ and there exists a chain $\mathcal{C} = \{\gamma_k\}$ with $\gamma_k \in \Gamma_1$ such that $\gamma_\mathcal{C}(\partial I) = \{x, y\}$ or $x \in \partial y$. Each connected component of G has a well-defined limit point on $J(f_0)$, and, using the Caratheodory loop γ around $J(f_0)$ we can assign an angle θ to each connected component G^θ of G so that $\gamma(\theta) = \mathrm{cl}(G^\theta) \setminus G^\theta$. Since $\mathcal{V}_1 \cup Im(\Gamma_1)$ is f_0-invariant, we can project G to the repelling Écalle cylinder $\mathcal{C}_-$. Take the pre-image by g to obtain a graph on the attracting cylinder $\mathcal{C}_+$. Now lift by π_+ to obtain a graph G_+ in the basin B.

Now we make two simplifying assumptions. First, that $g(\pi_+(0))$ lies in the connected component G^0, the component of G with zero angle. This connected component limits onto the fixed point ξ. If this assumption is satisfied then we say the transit map is *combinatorially real.* Second, we assume all the components of $Dom(h)$ lie on

$$G^0 \sqcup G^\pi \sqcup G^0_+.$$

One could define the induced marking for maps that do not satisfy these assumptions. But for simplicity of exposition we keep the assumptions.

Now using the graphs G^0, G^π and G^0_+, it is clear how to modify the constructions from §4.1 to construct a marking (A_h, Γ_h) of h. We omit the details.

Let us define the combinatorial type of a real-marked through map. Let $(f, A, \Gamma) \in \mathcal{ML}_\mathbb{R}$ and let $g : \mathcal{C}_+ \to \mathcal{C}_-$ be combinatorially real. Suppose $h \in \mathcal{L}$ is a restriction of $T(f, g, \cup_{j\neq 0} U_j(f))$ to finitely many components such that $(h, A_h, \Gamma_h) \in \mathcal{ML}_\mathbb{R}$. Define the *depth* of a component $U \in \mathrm{CC}(Dom(h))$ as follows. For each $U \in \mathrm{CC}(Dom(T_1) \cap Dom(h))$, define $d(U = n_0(U)$ where n_0 is the first landing time of U to $\cup_{j\neq 0} U_j(f)$. For every other $U \in \mathrm{CC}(Dom(h))$ define $d(U)$ as the first landing time to $\cup_{j\neq 0} U_j(f)$ relative to the origin.

Define the *through-type homomorphism* $\chi : \Lambda_{(h, A_h, \Gamma_h)} \to \Lambda_{(f, A, \Gamma)} \times \mathbb{Z}$ to be the homomorphism generated by

$$\chi(U) = (L(U), d(U)).$$

Chapter 5

Towers

5.1 Definition of a tower

A tower is a forward infinite or bi-infinite sequence of maps. First let us define the "product" space $\Pi_*^\infty X$ of a given topological space X to topologize the forward infinite or bi-infinite sequences of elements of X. Let $\mathcal{S} = [-\infty, 0]$ denote the one-point compactification of $-\mathbb{N}_0$. Given an $m \in \mathcal{S}$ let $S_m = \{n \in \mathbb{Z} : n \geq m\}$. Define

$$\Pi_*^\infty X = \{f : S_m \to X \mid m \in \mathcal{S}\}$$

with the following topology. Fix an $f \in \Pi_*^\infty X$ with index set $S(f) = S_{m(f)}$. Let A be a neighborhood of $m(f)$ in $\mathcal{S}$ and let $m' = \sup A$. Fix $N, M \in [m', \infty)$ with $N \leq M$ and for each $n \in [N, M]$ let X_n be a neighborhood of $f(n)$ in X. A basis for the topology of $\Pi_*^\infty X$ is generated by neighborhoods of the form

$$\{g \in \Pi_*^\infty X : m(g) \in A,\ g(n) \in X_n \text{ for } n \in [N, M]\}.$$

In words, $\Pi_*^\infty X$ is the space of one-sided sequences $(x_m, x_{m+1}, \dots)$ with $m \in -\mathbb{N}_0$ and two-sided sequences $(\dots, x_{-1}, x_0, x_1, \dots)$ with the topology of pointwise convergence.

The space of towers will be a subspace of a product space $\Pi_*^\infty \mathcal{L}'$. We now define the space $\mathcal{L}'$. Let

$$\mathcal{L}_\mathcal{P} = \{(f, g) : f \in \mathcal{L},\ f|_{U_0(f)} \in I^{-1}(1/4),\ g : \mathcal{C}_+ \to \mathcal{C}_- \text{ a transit map}\}$$

and let $\mathcal{L}' = \mathcal{L} \sqcup \mathcal{L}_\mathcal{P}$ with the topology generated by the following subbasis. A neighborhood in $\mathcal{L}'$ of an $f \in \mathcal{L}$ is a neighborhood $\mathcal{N} \subset \mathcal{L}$ of f. Now fix an $(f, g) \in \mathcal{L}_\mathcal{P}$ and fix a neighborhood N of the parabolic fixed point of $f|_{U_0(f)}$ on which f is a diffeomorphism. Let $\mathcal{N}$ be the neighborhood of f from Lemma 2.4.2. Let $\mathcal{N}' \subset \mathbb{C}/\mathbb{Z}$ be a neighborhood of the phase $\bar{a}$ of g. Then a neighborhood of (f, g) is the union of

1. $(f', g') \in \mathcal{L}_\mathcal{P}$ satisfying $f' \in \mathcal{N} \cap \mathcal{P}_0$ and $\bar{a}' \in \mathcal{N}'$ where $\bar{a}'$ is the phase of g'.
2. $f' \in \mathcal{L}$ satisfying $f' \in \mathcal{N} \cap \mathcal{P}_1$ and $\bar{a}' \in \mathcal{N}'$ where $\bar{a}'$ is the phase of the induced transit map $g_{f'}$.

Let us introduce some notation for a given $\mathcal{T} \in \Pi_*^\infty \mathcal{L}'$. The index set of $\mathcal{T}$ will be denoted by $S = S(\mathcal{T})$. We will often write $\mathcal{T} = \{f_n, g_n\}$ where it is implicit that g_n is not defined for all $n \in S$. Also, let $U^n = Dom(f_n)$ and $V^n = Range(f_n)$. Let U_0^n be the central component of f_n and let $\cup U_{j\neq 0}^n = \cup_{j\neq 0} U_j(f_n)$ be the off-critical pieces of f_n. Let

$$S_{\mathcal{P}} = \{n \in S : f_n|_{U_0(f_n)} \in I^{-1}(1/4)\}$$

and let

$$S_{\mathcal{Q}} = \{n \in S : f_n \in \mathcal{Q}^\infty\}.$$

We will often identify a transit map g_n with the set of local lifts of g_n to some choice of petals. Let

$$\mathcal{T}_m = \{f_m\} \cup \{g_n : n \geq m\}$$

and let $\mathcal{F}_m = \langle \mathcal{T}_m \rangle$ and define the post-critical set of $\mathcal{T}_m$ as

$$P^m = \overline{\mathrm{orb}(\mathcal{F}_m, 0)}.$$

We will often also denote $\mathcal{F}_{\min(S)}$ simply by $\mathcal{F}$.

The space of towers Tow is the set of $\mathcal{T} \in \Pi_*^\infty \mathcal{L}'$ such that $\sup S_{\mathcal{Q}} = \infty$, for each level $n \in S$ one of the following conditions hold:

- T1: $n \in S_{\mathcal{Q}}$, f_n is immediately renormalizable and $[f_{n+1}] = [h]$ where h is a pre-renormalization of f_n of minimal period
- T2: $n \in S_{\mathcal{Q}}$, f_n is not immediately renormalizable and $f_{n+1} = R(f_n, V^{n+1}|P^n)$ where V^{n+1} is the initial return domain w.r.t. some initial Yoccoz partition Υ_n of f_n
- T3: $n \notin (S_{\mathcal{Q}} \cup S_{\mathcal{P}})$ and $f_{n+1} = R(f_n, U_0^n|P^n)$
- T4: $n \notin (S_{\mathcal{Q}} \cup S_{\mathcal{P}})$ and $f_{n+1} = T(f_n, \cup U_{j\neq 0}^n|P^n)$
- T5: $n \in S_{\mathcal{P}}$ and $f_{n+1} = T(f_n, g_n, \cup U_{j\neq 0}^n|P^n)$

unless $n+1 \in S_{\mathcal{Q}}$, in which case one of the above conditions holds for the germ $[f_{n+1}]$.

If $S_{\mathcal{P}} \neq \emptyset$ then $\mathcal{T}$ is a *parabolic tower*. If $S = \mathbb{Z}$ then $\mathcal{T}$ is a *bi-infinite tower*. If $S \neq \mathbb{Z}$ and $f_{\min(S)} \in \mathcal{Q}^\infty$ then $\mathcal{T}$ is a *forward tower*. The map $f_{\min(S)}$ in a forward tower is called the *base map*. A tower with $f_0 \in \mathcal{Q}^\infty$ is *normalized* if $\beta_{f_0} = 1$. Let

$$Tow_0 = \{\mathcal{T} \in Tow : S(\mathcal{T}) = \mathbb{N}_0\}.$$

For any $n \in S_{\mathcal{Q}}$ we denote the next $m \in S_{\mathcal{Q}}$ by $\mathrm{succ}(n)$. That is,

$$\mathrm{succ}(n) = \min\{m \in S_{\mathcal{Q}} : m > n\}.$$

A tower $\mathcal{T}$ has *complex bounds* if there exists $m > 0$ such that $\mathrm{mod}\, f_n \geq m$ for all $n \in S(\mathcal{T})$. For a given $m > 0$ let

$$Tow(m) = \{\mathcal{T} \in Tow : \mathrm{mod}\, f_n \geq m \text{ for all } n \in S\}.$$

We say a tower $\mathcal{T}$ is *unbranched* iff

$$P^n = P^0 \cap V^n$$

for all $n \in S$.

5.1.1 Combinatorics

A tower $\mathcal{T} = \{f_n, g_n\} \in Tow_0$ is naturally marked as follows. First mark all levels n satisfying condition T2 with the initial marking (A_n, Γ_n). Then inductively mark levels $k = n+1, n+2, \dots, \mathrm{succ}(n)-1, \mathrm{succ}(n)$ with the marking (A_k, Γ_k) induced from level $k-1$. Levels satisfying T1 are not marked. Clearly the markings $\{(A_n, \Gamma_n)\}_{n \in S}$ depend continuously on $\mathcal{T}$. That is, for each level n (not satisfying T1) the injective map $f_n \mapsto (f_n, A_n, \Gamma_n) \in \mathcal{ML}$ is continuous. We say a tower has *real combinatorics* iff every $(f_n, A_n, \Gamma_n) \in \mathcal{ML}_{\mathbb{R}}$ and every g_n is combinatorially real.

Suppose now that $\mathcal{T}$ has real combinatorics. Let Λ_n denote the signed ordered semigroup $\Lambda_{(f_n, A_n, \Gamma_n)}$ and define a sequence of homomorphisms χ_n as follows. If level n satisfies T2 then define $\chi_n : \Lambda_n \to (\Gamma_0, \epsilon_0)$ to be the homomorphism generated by mapping each generator of Λ_n to its itinerary through the initial Yoccoz partition Υ_n. If level n satisfies T3 then define $\chi_n : \Lambda_n \to \Lambda_{n-1}$ to be the homomorphism generated by mapping each generator of Λ_n to its itinerary through Λ_{n-1}' until the first return to V^n. If level n satisfies T4 then define $\chi_n : \Lambda_n \to \Lambda_{n-1} \times \mathbb{Z} \times \mathbb{N}_0$ to be the homomorphism generated by $\chi_n(U) = (L(U), d(U), n_0(0))$ where L is the first landing map of f_{n-1} to the off-critical pieces, d is the depth function and n_0 is the first landing time function. If level n satisfies T5 then define $\chi_n : \Lambda_n \to \Lambda_{n-1} \times \mathbb{Z}$ to be the homomorphism generated by $\chi_n(U) = (L(U), d(U))$ where again L is the first landing map of $\mathcal{F}(f_{n-1}, g_{n-1})$ to the off-critical pieces and d is the depth function.

The combinatorics of $\mathcal{T}$ is encoded by the sequence $\{\chi_n\}_{n \in S}$. We say $\mathcal{T}$ has essential combinatorics bounded by B if the following conditions hold

- if $n \in S_Q$ then $|n - \mathrm{succ}(n)| < B$
- the return times on T2 and T3 levels are bounded above by B
- the depths of all components on T4 and T5 levels are bounded above by B
- number of connected components of $U^n \leq B$ for all $n \in S$.

Let us describe how to associate to the sequence χ_n a sequence of (parabolic) shuffles

$$\bar{\sigma}_{\mathcal{T}} = \{\sigma_n\}_{n \in S_Q}.$$

To each level n satisfying T1 let $\sigma_n = \sigma^{(2)}$. Now suppose level n satisfies T2. We will assign a parabolic shuffle to the group of levels $E = \{n, n+1, \dots, \mathrm{succ}(n) - 1\}$. First, if $E \cap S_{\mathcal{P}} = \emptyset$ then $f_n \in \mathcal{M}^{\sigma}$ for some shuffle σ and $[f_{n+1}] = \mathcal{R} f_n$. Hence the combinatorics of the levels in E are uniquely specified by the shuffle $\sigma_n = \sigma$.

So suppose $D = E \cap S_{\mathcal{P}} \neq \emptyset$. For each $m \in D$ let $\chi_m^{(k)} : \Lambda_n \to \Lambda_{n-1} \times \mathbb{Z} \times \mathbb{N}_0$ be the homomorphism generated by $\chi_m^{(k)}(U) = (\chi_m(U), k)$. Now for each k there is a unique shuffle $\sigma^{(k)}$ with the return-type and through-type sequence

$$\chi_n, \chi_{n+1}, \dots, \chi_{m-1}, \chi_m^{(k)}, \chi_{m+1}, \dots, \chi_{\mathrm{succ}(n)-1}.$$

Moreover, there is a $p < \infty$ such that $\sigma^{(k)} \in \Omega_e(p)$ for all k. Let

$$\sigma_n = \lim_{k \to \infty} \sigma^{(k)}.$$

It is apparent from the definition of $\Omega_e^{cpt}(p)$ that the limit is well-defined. The combinatorial objects σ_n and $\{\chi_n\}_{n\in E}$ are comparable. That is, if the essential combinatorics of $\mathcal{T}$ restricted to E are bounded by B then then $\sigma_n \in \Omega_e^{cpt}(p)$ for some p, depending on B. Vice versa, if $\sigma_n \in \Omega_e^{cpt}(p)$ then the essential combinatorics of $\mathcal{T}$ are bounded by B depending only on p. The bounds p and B tend to infinity together.

Two towers $\mathcal{T}$ and $\mathcal{T}'$ with real combinatorics are *combinatorially equivalent* iff

$$S = S', S_Q = S'_Q, \text{ and } \bar{\sigma} = \bar{\sigma}'.$$

For a given $m > 0$ and $B \in \mathbb{N}$ let $Tow(m, B)$ be the space of towers $\mathcal{T} \in Tow(m)$ with essential combinatorics bounded by B.

5.2 Forward Towers

In this section we analyze forward towers. For convenience we will assume all towers in this chapter are indexed by $S = \mathbb{N}_0$.

Let $\mathcal{T}$ be a forward tower. Let us define the (filled) Julia set. We say orb(z) *escapes* if orb$(z) \cap (V^0 \setminus U^0) \neq \emptyset$. Define the *filled Julia set*, $K(\mathcal{T})$, the *Julia set*, $J(\mathcal{T})$, as for quadratic-like maps.

Two forward towers $\mathcal{T}$ and $\mathcal{T}'$ with $S(\mathcal{T}) = S(\mathcal{T}')$ are *quasi-conformally equivalent* if there is a quasi-conformal map ϕ such that

1. ϕ is a quasi-conformal conjugacy of f_n and f'_n on a neighborhood of $K(f_n)$ to a neighborhood of $K(f'_n)$ for all $n \in S$,
2. ϕ induces a quasi-conformal conjugacy of the transit maps g_n and g'_n for $n \in S_{\mathcal{P}}$.

A quasi-conformal equivalence ϕ between two forward towers is a *hybrid* equivalence if $\bar{\partial}\phi|_{K(\mathcal{T})} \equiv 0$ and is a *holomorphic* equivalence if ϕ is holomorphic. A forward towers $\mathcal{T}$ is *equal on* $K(\mathcal{T})$ to a forward tower $\mathcal{T}'$ if

1. $K(\mathcal{T}) = K(\mathcal{T}')$
2. $\mathcal{F}_z = \mathcal{F}'_z$ for every $z \in K(\mathcal{T})$.

Here $\mathcal{F}$ is the dynamical system generated by $\mathcal{T}$, $\mathcal{F}'$ the one generated by $\mathcal{T}'$, and $\mathcal{F}_z$ is the set of germs at z of the dynamical system $\mathcal{F}$.

Fix a $z \in U^0$. We note without proof the following two facts:

1. the germs $\mathcal{F}_z$ are well-ordered by an extension of the well-ordering defined in §2.4.2
2. any $h \in \mathcal{F}$ with $z \in Dom(h)$ can be extended to a holomorphic map $\widetilde{h} \in \mathcal{F}$ with range V^0 by extending local lifts of transit maps using the extended Fatou coordinates of §2.4. A restriction of $\widetilde{h}$ is a cover map of $V^0 \setminus P^0$. In particular if $V^n \subset V^0$ then $f_n \in \mathcal{F}$.
3. given a tower $\mathcal{T}$ and a level $n \in S_Q$, a different choice of initial Yoccoz partition Υ_n yields a tower equal to $\mathcal{T}$ on $K(\mathcal{T})$.

5.2.1 Straightening

Proposition 5.2.1 (Straightening) *Any forward tower $\mathcal{T}$ is hybrid equivalent to a tower with a quadratic base map.*

Proof: Let f_0 be the base map of $\mathcal{T}$. From Proposition 2.1.5 there is a hybrid equivalence ϕ between f_0 and a unique polynomial of the form $z^2 + c$. Let $u(z)$ be the complex dilatation of ϕ and let $\mu = u(z)d\bar{z}/dz$ be the corresponding Beltrami differential. Since ϕ is quasi-conformal there is a $k < 1$ such that $\|u(z)\|_\infty \leq k$. Let $U \supset K(f_0)$ be the domain on which ϕ is a conjugacy.

Define the Beltrami differential μ' by

$$\mu'|_{K(\mathcal{T})} \equiv 0$$

and if $z \in (U \setminus K(\mathcal{T}))$ by

$$\mu'|_{U'} = h^*(\mu)$$

where $h \in \mathcal{F}(\mathcal{T})$ and $U' \ni z$ satisfy $h(U') \subset (U \setminus K(f_m))$. The well-ordering of the germs $\mathcal{F}_z$ implies μ' is well-defined.

First we claim there are restrictions f_n' of f_n such that $Range(f_n') \subset Range(f_n)$ and $\mathcal{T}' = \{f_n', g_n\}$ is a forward tower equal to $\mathcal{T}$ on $K(\mathcal{T})$ and μ' is invariant under $\mathcal{T}'$. Let $f_n' = f_n$ for $n = 0, 1, \ldots, m$ where $m \in S_Q$ is the first level where $Range(f_m) \not\subset Range(f_0)$, if it exists. Let $N \in \mathbb{N}$ be large enough so that $V_m^N = f_m^{-N}(Range(f_m)) \subset Range(f_0)$. Let $f_m' = f_m$ restricted to V_m^{N+1}. Choose a new initial Yoccoz partition Υ for f_n' and construct the restrictions f_n' of f_n as required for an equal tower for levels $n = m+1, m+2, \ldots, m_2$ until the level $m_2 \in S_Q$ and repeat. The resulting sequence of maps $\{f_n', g_n\}$ will be the desired tower.

Write $\mu'(z) = u'(z)d\bar{z}/dz$. Since all maps in $\mathcal{T}'$ are holomorphic $\|u'(z)\|_\infty \leq k < 1$. Let ϕ_1 be the solution to the Beltrami equation

$$\bar{\partial}\phi_1 = u' \cdot \partial\phi_1$$

and let

$$\mathcal{T}'' = \{\phi_1 \circ h \circ \phi_1^{-1} : h \in \mathcal{T}'\}.$$

We claim $\mathcal{T}''$ is again a forward tower and that ϕ_1 is a hybrid equivalence between $\mathcal{T}$ and $\mathcal{T}''$. Let $n \in S_P$ and $g_n \in \mathcal{T}'$. Let $g_n'' = \phi_1 \circ g_n \circ \phi_1^{-1}$ and $f_n'' = \phi_1 \circ f_n' \circ \phi_1^{-1}$. Since ϕ_1 conjugates forward and backward orbits of f_n to orbits of f_n'', it follows that g_n'' is a map on the Ècalle-Voronin cylinders of f_n''. Since ϕ_1 is a homeomorphism, it is evident that g_n'' is a homeomorphism. Moreover, μ' is invariant under g_n, and so g_n'' is conformal. That is, the conjugate of a transit map in $\mathcal{T}$ is a transit map in $\mathcal{T}''$. The other properties of a tower are clear.

The base map of $\mathcal{T}''$ is holomorphically equivalent to a polynomial. Hence $\mathcal{T}''$ is holomorphically equivalent to a tower with a polynomial base map. □

5.2.2 Expansion of the Hyperbolic Metric

One of the central ideas in McMullen's arguments is that maps in a tower expand the hyperbolic metric on the complement of the post-critical set. In this section we prove similar propositions.

Lemma 5.2.2 *There are continuous increasing functions $C_1(s)$ and $C_2(s)$ such that if $f : X \hookrightarrow Y$ is an inclusion between two hyperbolic Riemann surfaces and $x \in X$ then, letting $s = d(x, Y \setminus X)$,*

$$0 < C_1(s) \leq \|Df(x)\| \leq C_2(s) < 1.$$

Moreover, $C_2(s) \to 0$ as $s \to 0$.

Proof: The inequality $\|Df(x)\| \leq C_2(s) < 1$ and the properties of $C_2(s)$ are found in [McM2]. Lift f to the universal cover $\pi : \mathbb{D} \to Y$ and normalize so that $x = f(x) = 0$. The inclusion $B_s \equiv \{z : d_{\mathbb{D}}(0, z) < s\} \hookrightarrow \mathbb{D}$ factors through f and so $\|Df(0)\| \geq 1/r(s)$ where $r(s)$ is the radius of B_s measured in the Euclidean metric. □

The following Proposition states when maps in a forward tower $\mathcal{T} \in Tow_0$ expand the hyperbolic metric on $V^0 \setminus P^0$ and gives an estimate on the amount of expansion and the variation of expansion.

We will use the notation ρ, $\|\cdot\|$, $d(\cdot,\cdot)$ and $\ell(\cdot)$ to denote the hyperbolic metric, norm, distance and length on $V^0 \setminus P^0$.

Proposition 5.2.3 *Let $\mathcal{T} \in Tow_0$ be a forward tower with base map f_0. Let $h \in \mathcal{F}(\mathcal{T})$ and $Q_h = h^{-1}(P^0)$. Then*

$$\|Dh(z)\| > 1$$

for any $z \in (Dom(h) \setminus Q_h)$. Moreover, if $(Q_h \setminus P^0) \neq \emptyset$ then

$$C_2^{-1}(s_2) \leq \|Dh(z)\| \leq C_1^{-1}(s_1)$$

where $s_1 = d(z, Q_h \cup \partial Dom(h))$ and $s_2 = d(z, Q_h)$. Finally, if γ is a path in $Dom(h) \setminus Q_h$ with endpoints z_1 and z_2, then

$$\|Dh(z_2)\|^{1/\alpha} \leq \|Dh(z_1)\| \leq \|Dh(z_2)\|^{\alpha}$$

where $\alpha = exp(M\ell(h(\gamma)))$ for a universal $M > 0$.

Proof: Let $h \in \mathcal{F}(\mathcal{T})$, $Q_h = h^{-1}(P^0)$, $U = (Dom(h) \setminus Q_h)$ and $z \in U$. There exists an $H \in \mathcal{F}$ such that $H|_U = h$ and H is a covering map onto $V^0 \setminus P^0$. Since the inclusion

$$\imath : Dom(H) \hookrightarrow (V^0 \setminus P^0)$$

is a contraction by the Schwarz Lemma, we see H expands ρ. Since $H|_U = h$ we have $\|Dh(z)\| > 1$. More precisely, the Schwarz Lemma states that if $f : U \to V$ is a holomorphic map between hyperbolic Riemann surfaces $U, V \subset \mathbb{C}$ then, with ρ_U and ρ_V denoting the hyperbolic metrics on U and V,

$$\frac{|f'(z)|\rho_V(f(z))}{\rho_U(z)} \leq 1$$

with equality iff f is a covering map. With $U = Dom(H)$, $V = V^0 \setminus P^0$ and $f = \imath$ we see that $\rho_V(z) \leq \rho_U(z)$ and hence

$$\|DH(z)\| = \frac{|H'(z)|\rho_V(h(z))}{\rho_V(z)} \geq \frac{|H'(z)|\rho_V(h(z))}{\rho_U(z)} = 1,$$

where the last equality comes from the Scharz Lemma applied to the covering map $H : U \to V$. We rule out equality since $V \neq U$.

Now we estimate how much h expands ρ. Apply Lemma 5.2.2 to the inclusion $\imath : U \hookrightarrow V$ to get the inequalities

$$C_2^{-1}(s) \leq \|DH(z)\| \leq C_1^{-1}(s)$$

where $s = d(z, V \setminus U)$. Assume $Q_h \setminus P^0 \neq \emptyset$. Then since C_1 and C_2 are increasing,

$$C_2^{-1}(s_2) \leq \|DH(z)\| \leq C_1^{-1}(s_1)$$

where $s_1 = d(z, Q_h \cup \partial Dom(h)) \leq s$ and $s_2 = d(z, Q_h) \geq s$.

To conclude let us prove the last statement about the variation of expansion. From [McM1, Cor 2.27] the variation in $\|DH(z)\|$ is controlled by the distance between z_1 and z_2 measured in the hyperbolic metric on U. Since H is a covering map, this distance is bounded above by the length $\ell(H(\gamma))$ of $H(\gamma)$ measure on V. But $h(\gamma) = H(\gamma)$. □

5.2.3 Repelling Cycles in J

Fix a forward tower $\mathcal{T} = \{f_n, g_n\} \in Tow_0$. For a point $z \in U^0$ we say a (possibly finite) sequence $(z_0, z_1, z_2, ...)$ is a *sub-orbit* of z (in $\mathcal{T}$) if the following conditions are satisfied:

- $z_0 = z$
- if $z_i \in V^0 \setminus U^0$ then z_{i+1} is not defined
- if $z_i = 0$ then $z_{i+1} = 0$
- if $z_i \in U^h$ then $z_{i+1} = h(z_i)$ for some local lift $h \in \mathcal{T}$ of g_n
- otherwise $z_{i+1} = f_0(z_i)$

Note any sub-orbit of z is a subset of $\text{orb}(z)$ and $\text{orb}(z)$ escapes iff there exists a sub-orbit $z_0, \ldots, z_N$ such that $z_N \in V^0 \setminus U^0$.

A point $z \in U^0$ is called *periodic* (in $\mathcal{T}$) if there exists $h \in \mathcal{F}(\mathcal{T})$, $h \neq id$, such that $h(z) = z$. Equivalently, $z \neq 0$ is periodic iff there is an $x \in \text{orb}(z)$ such that $z \in \text{orb}(x)$ and a sub-orbit $x_0, x_1 = h_1(x), \ldots, x_N = h_N(x)$ of x such that $x_0 = x_N$ and $x_0 \neq x_i$ for $0 < i < N$. The *multiplier*, λ, of the periodic orbit through z is defined to be $Dh_N(x)$. The multiplier does not depend on the sub-orbit. A periodic orbit is called *superattracting, attracting, repelling, neutral* if λ satisfies $\lambda = 0, |\lambda| < 1, |\lambda| > 1, |\lambda| = 1$, respectively.

Lemma 5.2.4 *Let $\mathcal{T} \in Tow_0$. The only non-repelling periodic orbits in $\mathcal{T}$ are the orbits through the parabolic points of f_n for $n \in S_{\mathcal{P}}$.*

Proof: Let $z_0, \ldots, z_N$ be the periodic orbit. Since the only non-repelling periodic orbits in $P(\mathcal{T})$ are the orbits through the parabolic points, we can assume the orbit is disjoint from $P(\mathcal{T})$. By Proposition 5.2.3,

$$\|Dh_N(z)\| > 1$$

But then

$$|\lambda| = |Dh_N(z)| > 1$$

in the Euclidean metric as well. □

For a given level $n \in S_{\mathcal{P}}$ let B_n be the *central basin* of level n. A connected compact set $K \subset U^0$ is *iterable* if $K \cap \partial B_n = \emptyset$ for all central basins B_n. Mimicking the definition of sub-orbits of points, we say a (possibly finite) sequence of compact sets $(K_0, K_1, K_2, \ldots)$ is a *sub-orbit* of K (in $\mathcal{T}$) if the following conditions are satisfied:

- $K_0 = K$
- all K_i are iterable except possibly the last one, if it exists
- if $K_i \subset Dom(h)$ then $K_{i+1} = h(K_i)$ for some local lift $h \in \mathcal{T}$ of g_n
- otherwise $K_{i+1} = f_0(K_i)$.

Now that we have said what it means to iterate an iterable compact set, we can prove the following

Proposition 5.2.5 *Suppose $\mathcal{T} \in Tow_0$ and let $y \in J(\mathcal{T})$. The following are two equivalent definitions of the Julia set:*

1. $J(\mathcal{T}) = \text{cl}\{z \in U^0 : z \textit{ is a repelling periodic point}\}$
2. $J(\mathcal{T}) = \text{cl}\{z \in U^0 : z \textit{ is a pre-image of } y\}$

Proof: The proposition is well-known when $S_{\mathcal{P}} = \emptyset$ (for example [M1, Theorem 11.1]). So assume $S_{\mathcal{P}} \neq \emptyset$. Let us first prove that

$$J(\mathcal{T}) \subset \mathrm{cl}\{z \in U^0 : z \text{ is a repelling periodic point}\}.$$

Let $z \in \partial K(\mathcal{T})$ and let W be a connected neighborhood of z. Our claim is that there is a repelling periodic orbit of $\mathcal{T}$ in W. Since such orbits are dense in $J(f_0)$ we can assume $W \subset \mathrm{int}(K(f_0))$. Let $K = \overline{W}$. We can form the suborbit $K_i = h_i(K)$ from K until the first moment when K_i is not iterable. Such a moment must exist since the orbit of $z \in K$ never escapes but the orbit of some other point in K does escape, by the definition of $K(\mathcal{T})$.

Case 1: Suppose $\mathrm{int}(K_i) \cap \partial B_n \neq \emptyset$ for some $n \in S$. We follow the proof in [M1] of Theorem 11.1. By arguing as in Lemma 3.1.4 there is a open set $W' \subset K_i$ and composition $h \in \mathcal{F}$ defined on W' such that $h(W') \cap J(f_0) \neq \emptyset$. There is then an open set $W'' \subset h(W')$ and an $N \geq 0$ such that $\beta_{f_0} \in f_0^N(W'')$. Let $z' \in W$ be the pre-image of β_{f_0}:

$$z = (f_0^N \circ h \circ h_i)^{-1}(\beta_{f_0}).$$

Let W_0 be a neighborhood of β_{f_0} on which f_0 is univalent. There exists an $M > 0$ such that $z \in f_0^M(W_0)$. Let $z_1 \in W_0$ be a pre-image of z by f_0^M:

$$z_1 = (f_0^M)^{-1}(z).$$

Since $\beta_{f_0} \notin P^0$ we can assume $\mathrm{orb}\, z_1 \cap P^0 = \emptyset$. Then a small enough neighborhood W_1 of z_1 will map univalently under

$$f_0^N \circ h \circ h_i \circ f_0^M$$

to a neighborhood V of β_{f_0}. Pull W_1 back by $f_0|_{W_0}$ until it is contained in V. Say the resulting domain is $V_1 = f_0^{-N_1}(W_1)$. Then we have found a univalent map

$$f_0^N \circ h \circ h_i \circ f_0^{N_1+M}$$

from V_1 to $V \supset V_1$. Moreover $f_0^{N_1+M}(V_1) \subset W$. Hence there is a repelling periodic orbit passing through W.

Case 2: If K_i is not iterable because $K_i \cap (V_0 \setminus U_0) \neq \emptyset$, then by perhaps choosing a smaller neighborhood W and iterating f_0 more, we can assume that the moment when K_i is not iterable is because $\mathrm{int}(K_i) \cap \partial B_n \neq \emptyset$ for $n = \min S_{\mathcal{P}}$ and we can argue as in case 1.

Case 3: Suppose $\mathrm{int}(K_i) \cap \partial B_n = \emptyset$ for some $n \in S_{\mathcal{P}}$ but that $\partial K_i \cap \partial B_n \neq \emptyset$. Then by choosing a slightly smaller neighborhood W we can assume K_i is iterable and continue iterating the sub-orbit. We claim this case can only happen a finite number of times. For otherwise every time K_i is not iterable K_i falls into this case. Then by choosing the slightly smaller neighborhoods so that they all contain some definite neighborhood W' of z we see that the orbit of W' is defined for all iterates. But this is impossible since then W' never escapes, contradicting the fact that $z \in \partial K(\mathcal{T})$.

Thus after a finite number of restrictions, the non-iterable set K_i must fall into the cases considered above. This finishes the claim that

$$J(\mathcal{T}) \subset \mathrm{cl}\{z \in U^0 : z \text{ is a repelling periodic point}\}.$$

Now let us prove the other containment. Let $z \in K(\mathcal{T})$ and let W be a connected neighborhood of z. Suppose W contains a repelling periodic point z_0. Again let $K = \mathrm{cl}(W)$ and start forming the sub-orbit $K_i = h_i(K)$ through K. Claim there is a moment when K_i is not iterable. For otherwise the maps h_i form a normal family on W and that contradicts the fact that W contains a repelling periodic point. Thus there is a non-iterable iterate K_i.

Case 1: Just as case 1 above, there is a open set $W' \subset K_i$ and composition $h \in \mathcal{F}(\mathcal{T})$ defined on W' such that $h(W') \cap J(f_0) \neq \emptyset$. But then there is a point in $h(W')$ that escapes and thus there is a point in W that escapes as well.

Case 2: If K_i is not iterable because $K_i \cap (V^0 \setminus U^0) \neq \emptyset$, then we have found a point in W that escapes.

Case 3: Suppose $\mathrm{int}(K_i) \cap \partial B_n = \emptyset$ but that $\partial K_i \cap \partial B_n \neq \emptyset$. Then by choosing a slightly smaller neighborhood W that still contains the repelling periodic point z_0, we can assume K_i is iterable and continue iterating the sub-orbit. We claim this case can only happen a finite number of times. For otherwise every time K_i is not iterable K_i falls into this case. Then by choosing the slightly smaller neighborhoods so that they all contain some definite neighborhood W' containing z_0 we see that the orbit of W' is defined for all iterates. But this is impossible since the iterates of W' cannot form a normal family. Thus after a finite number of restrictions, the non-iterable set K_i must fall into the cases considered above. Thus we have proven the first statement of the proposition.

To prove the second statement, notice that the argument proving the first also proves that if $y \in J(\mathcal{T})$ then any point in U_0 has a pre-image arbitrarily close to y. That is,

$$J(\mathcal{T}) \subset \mathrm{cl}\{z \in U^0 : \text{ there is an } h \text{ such that } h(z) = y\}.$$

The reverse inclusion follows from the fact that $J(\mathcal{T})$ is closed and backward invariant and that $y \in J(\mathcal{T})$. □

5.2.4 Forward Rigidity

In this section we prove the combinatorial rigidity of forward towers. That is, the germ of the base map and the sequence χ_n determine the germs of the quadratic-like maps and transit maps uniquely:

Proposition 5.2.6 *Let $\mathcal{T} = \{f_n, g_n\}$ and $\mathcal{T}' = \{f'_n, g_n\}$ be forward towers with real combinatorics and complex bounds. Suppose $S(\mathcal{T}) = S(\mathcal{T}') = \mathbb{N}$, $\mathcal{T}$ is combinatorially equivalent to $\mathcal{T}'$ and $[f_0] = [f'_0]$. Then $\mathcal{T}$ is equal to $\mathcal{T}'$ on $K(\mathcal{T})$.*

The proof involves constructing families of generalized quadratic-like maps, which we discuss first.

Families of first through maps

Let $(\mathbf{f} : \cup_j \mathbb{U}_j \to \mathbb{V}, \mathbf{h})$ be a proper holomorphic family of generalized quadratic-like maps over $\lambda \in D$ with winding number 1. Assume $\mathbf{f}$ is not a DH-family and let $T_\lambda = T(f_\lambda, \cup_{j\neq 0} U_j(f_\lambda))$. Let $* \in D$ be the base point of the family and suppose $0 \in Dom(T_*)$. Let $D' \subset D$ be the connected component of $X_* = \{\lambda : 0 \in Dom(T_\lambda)\}$ containing $*$. Let g_* be a restriction of $T(f_*, \cup_{j\neq 0} U_j(f_*))$ so that $g_* \in \mathcal{L}$. One can pull back the motion h to a holomorphic motion $\mathbf{h}'$ of $\partial Dom(g_*)$ defined over D'.

Lemma 5.2.7 *Let $(\mathbf{f} : \cup_j \mathbb{U}_j \to \mathbb{V}_j, \mathbf{h})$ be a proper generalized quadratic-like family over D with winding number 1. Let $* \in D$ be the basepoint and let g_* be the first through map $T(f_*, \cup_{j\neq 0} U_j(f_*))$ restricted to finitely many components. Suppose $g_* \in \mathcal{L}$. Then the connected component D' of X_* containing $*$ is a Jordan disk and the family of first through maps $(\mathbf{g}, \mathbf{h}')$ over D' is proper and has winding number 1.*

Note that when $(A_\lambda, \Gamma_\lambda)$ is a continuous family of markings of f then D' is uniquely specified by the induced marking on g_* and the through-type homomorphism of g_*.

Families of generalized parabolic renormalizations

Now suppose $f \in \mathcal{L}$ satisfies

$$f_0 = f|_{U_0(f)} \in I^{-1}(1/4).$$

We keep the setup from §4.3 as follows. Let $\xi = \beta_{f_0}$. Choose incoming and outgoing petals $D_\pm$ around the parabolic point ξ and let $\mathcal{C}_\pm$ denote the respective Ècalle-Voronin cylinders and $\pi_\pm$ the projections with π_+ extended to $B = \text{int}(K(f_0))$. For a given $g : \mathcal{C}_+ \to \mathcal{C}_-$ let L be the first landing map under $\mathcal{F}(f_0, g)$ to $\cup_{j\neq 0} U_j(f)$.

The goal is to construct a family in $\mathcal{L}$ by varying the transit map $g : \mathcal{C}_+ \to \mathcal{C}_-$. Let $X \subset \mathbb{C}/\mathbb{Z}$ be the set of phases $\bar{a}$ such that for $g = g_{\bar{a}}$,

$$0 \in Dom(L).$$

It is clear that X is a countable pairwise disjoint collection of Jordan disks. Choose a connected component D of X and fix $* \in D$ and a restriction T_* of $T(f, g_*, \cup_{j\neq 0} U_j(f))$ to finitely many components of $Dom(T)$ such that $T_* \in \mathcal{L}$. The construction of the holomorphic motion $\mathbf{h}$ of $(\partial Dom(T_*), \partial Range(T_*))$ described before Lemma 3.1.1 carries over unchanged to this situation. Moreover, one can modify the proof of Lemma 3.1.1 to prove

Lemma 5.2.8 *For any connected component D of X, the family $(\mathbf{T}, \mathbf{h})$ over D is a proper generalized quadratic-like family with winding number 1.*

We are now able to prove Proposition 5.2.6:

Proof: If $S_\mathcal{P} = \emptyset$ then the result is a restatement of Theorem 2.2.2. So suppose $S_\mathcal{P} \neq \emptyset$ and let $n = \min S_\mathcal{P}$. Then since $[f_0] = [f'_0]$ it follows that

$$[f_n|_{U_0(f_n)}] = [f'_n|_{U_0(f'_n)}].$$

Hence the Écalle-Voronin cylinders of level n are equal: $\mathcal{C}_\pm = \mathcal{C}'_\pm$. We claim that $g_n = g'_n$. In what follows we have implicitly fixed a pair of Fatou coordinates for level n. Also, by intersecting initial Yoccoz partitions and passing to towers equal on $K(\mathcal{T})$ and $K(\mathcal{T}')$ we can assume

$$f_n = f'_n.$$

For each $\bar{a} \in \mathbb{C}/\mathbb{Z}$ let $L_{\bar{a}}$ be the first landing map of $\mathcal{F}(f_n, g_{\bar{a}})$ to $\cup_{j\neq 0} U_j(f_n)$ and let

$$X = \{\bar{a} : 0 \in Dom(L_{\bar{a}})\}.$$

Let $L = L(f_n, g_n)$ and let $U_0 = U_0(L)$. Similarly let $L' = L(f_n, g'_n)$ and let $U'_0 = U_0(L')$. Let $\bar{a}_0$ be the phase of g_n and let $\bar{a}'_0$ be the phase of g'_n. Let $D \subset X$ and $D' \subset X$ be the connected components of X containing $\bar{a}_0$ and $\bar{a}'_0$, respectively. First we claim $D = D'$.

Since $\mathcal{T}$ and $\mathcal{T}'$ are combinatorially equivalent the through-type homomorphisms are equal: $\chi_n = \chi'_n$. Let $\chi_n(U_0) = U_1$ and let $\chi'_n(U'_0) = U'_1$. Also, since $[f_0] = [f'_0]$ we know $K(f_0) = K(f'_0)$. Let $K = K(f_0)$. Since $\chi_k = \chi'_k$ for $k = 0, \ldots, n$ we know

$$K \cap U_1 = K \cap U'_1.$$

Let $\widetilde{K} = K \cap U_1$. Since $L(0) \in \widetilde{K}$ and $L'(0) \in \widetilde{K}$ it follows that $\bar{a}_0$ and $\bar{a}'_0$ are in the same connected component of X. Hence $D = D'$.

Let $(\mathbf{T}, \mathbf{h})$ be the holomorphic family over D constructed in Lemma 5.2.8 such that

$$T_{\bar{a}_0} = f_{n+1}$$
$$T_{\bar{a}'_0} = f'_{n+1}.$$

Note that $(\mathbf{T}, \mathbf{h})$ is a proper generalized quadratic-like family over the Jordan disk D with winding number 1. The base point can be chosen as either $\bar{a}_0$ or $\bar{a}'_0$. Let $(\mathbf{f}_{n+1}, \mathbf{h}_{n+1}) = (\mathbf{T}, \mathbf{h})$ and suppose $n+1 \notin S_{\mathcal{Q}} \cup Q_{\mathcal{P}}$. That is, suppose level $n+1$ satisfies condition T3 or T4 in the definition of towers. If level $n+1$ satisfies T3 then let $(\mathbf{f}_{n+2}, \mathbf{h}_{n+2})$ be the family of first return maps from Lemma 2.3.6 such that $f_{n+2,\bar{a}_0} = f_{n+2}$. This family is a proper family with winding number 1. If level $n+1$ satisfies T4 then let $(\mathbf{f}_{n+2}, \mathbf{h}_{n+2})$ be the family of first through maps from Lemma 5.2.7 such that $f_{n+2,\bar{a}_0} = f_{n+2}$. Again, this family is a proper family with winding number 1. In either case, let D_{n+2} be the parameter domain for $(\mathbf{f}_{n+2}, \mathbf{h}_{n+2})$. Inductively construct families $(\mathbf{f}_k, \mathbf{h}_k)$ for levels $k = n+2, n+3, \ldots, m$ where

$$m = \inf\{m' \in S_{\mathcal{P}} \cup S_{\mathcal{Q}} : m' > n\}.$$

Since $\chi_k = \chi'_k$ for $k = n+2, n+3, \ldots, m$ we see $(\mathbf{f}_k, \mathbf{h}_k)$ are also the families of first return and first through maps for f'_k with base point $\bar{a}'_0$.

Now consider the DH quadratic-like families $(\mathbf{f}_m, \mathbf{h}_m)$ restricted to the central domain, denoted by $\mathbf{F}$. From Theorem 2.3.5 the set

$$\mathbf{M} = \{\bar{a} \in D_m : J(F_{\bar{a}}) \text{ is connected}\}$$

is homeomorphic to the Mandelbrot set M. It follows that $\bar{a}_0, \bar{a}'_0 \in \mathbf{M}$. Now we claim that $\bar{a}_0 = \bar{a}'_0$.

If $F_{\bar{a}_0}$ has a parabolic orbit then clearly $\bar{a}_0 = \bar{a}'_0$ from the rigidity of parabolic parameter values. Otherwise $F_{\bar{a}_0}$ is infinitely renormalizable with real combinatorics and complex bounds and so by Theorem 2.2.2 we see $\bar{a}_0 = \bar{a}'_0$.

Now by induction on $n \in S_{\mathcal{P}}$ we see $g_n = g'_n$ and $[f_m] = [f'_m]$ for $m = \mathrm{succ}(n)$. Hence $K(\mathcal{T}) = K(\mathcal{T}')$ and $\mathcal{T}$ is equal to $\mathcal{T}'$. □

In fact by examining the above proof we see the only use of complex bounds was for Theorem 2.2.2. Thus the above proof also yields the slightly stronger result:

Proposition 5.2.9 *Let $\mathcal{T} = \{f_n, g_n\}$ and $\mathcal{T}' = \{f'_n, g_n\}$ be forward towers with real combinatorics. Suppose $S(\mathcal{T}) = S(\mathcal{T}') = \mathbb{N}$, $\mathcal{T}$ is combinatorially equivalent to $\mathcal{T}'$ and $[f_0] = [f'_0]$. Suppose in addition that $\sup S_{\mathcal{P}} = \infty$. Then $\mathcal{T}$ is equal to $\mathcal{T}'$ on $K(\mathcal{T})$.*

Combining rigidity with straightening we have the following

Corollary 5.2.10 *Any two combinatorially equivalent forward towers $\mathcal{T}, \mathcal{T}' \in Tow_0$ with real combinatorics and complex bounds are hybrid equivalent.*

Proof: Straighten $\mathcal{T}$ and $\mathcal{T}'$ to towers $\mathcal{T}_1$ and $\mathcal{T}_2$ with quadratic base maps. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are combinatorially equivalent it follows from Theorem 2.2.2 and the uniqueness of root points that the base maps have the same germ at 0. Hence by Proposition 5.2.6 $\mathcal{T}_1$ and $\mathcal{T}_2$ are hybrid equivalent. □

5.2.5 Compactness

The goal in this section is to prove Proposition 5.2.16. To do so we will need several lemmas about the geometry of first return and first through maps.

Bounded geometry

First we control the geometry of levels satisfying condition T2 of a tower:

Lemma 5.2.11 ([L3]) *Let $m > 0$. Let $f \in I^{-1}(M) \cap \mathcal{Q}(m)$ for some maximal, real $\mathcal{M}$-copy $M \neq \mathcal{M}^{\sigma(2)}$ and let V^0 be the initial return domain w.r.t. some initial Yoccoz partition. Let g be a restriction of $R(f, V^0)$ to finitely many components of $Dom(R)$ so that $g \in \mathcal{L}$. Suppose there is $n > 0$ so that the return time of any $z \in Dom(g)$ to V^0 is bounded above by n. Then there are functions $C_1 > 0$ and $C_2 > 0$, depending only on m and n, such that $\mathrm{geo}(g) \geq C_1$ and $(\mathrm{diam}\, K(g))/(\mathrm{diam}\, K(f)) \geq C_2$.*

Next we control the geometry of levels satisfying condition T3:

Lemma 5.2.12 *Let $r \in \mathbb{N}$, $m > 0$, $\lambda > 0$. Suppose $f \in \mathcal{L}(m)$ and $\mathrm{geo}(f) \geq \lambda$. Consider the first return map $R = R(f, U_0(f))$ and suppose $0 \in Dom(R)$. Let h be R restricted to finitely many components so that $h \in \mathcal{L}$. Suppose*

$$\sup_{z \in Dom(h)} n_+(z) \leq r.$$

Then there exists $C_1 > 0$ and $C_2 > 0$, depending only on m, λ and r such that $\mathrm{geo}(h) \geq C_1$ and $(\mathrm{diam}\, K(h))/(\mathrm{diam}\, K(f)) \geq C_2$.

Proof: Assume $\operatorname{diam} K(f) = 1$. If $Dom(f)$ has one connected component then $K(h) = K(f)$. So assume $Dom(f)$ has more that one connected component. Then from Lemma 2.3.1 we know f is ranging in a compact set $X(m, \lambda)$. Let $X(m, \lambda, r)$ be the subset of $f \in X(m, \lambda)$ such that if $R = R(f, U_0(f))$ then $0 \in Dom(R)$ and $n_+(0) \leq r$. Since $X(m, \lambda, r)$ is closed it is also compact.

Let $Y \subset \mathcal{L}$ denote the set h of restrictions of $R(f, U_0(f))$, $f \in X(m, \lambda, r)$ such that

$$\sup_{z \in Dom(h)} n_+(z) \leq r.$$

Then Y is compact since $X(m, \lambda, r)$ is compact and since Y consists of finitely many connected components and within each component h depends continuously on f. Since the geometry function $\operatorname{geo}(h)$ is continuous it follows that $\operatorname{geo}(h) \geq C_1$ where C_1 depends only on m, λ and r.

The second statement similarly follows from the continuity of the function $\operatorname{diam} K(h)$. □

The following lemma controls the geometry of restrictions of first through maps where the landing time is bounded above. The proof is a minor modification to the proof of the lemma above, and we omit it.

Lemma 5.2.13 *Let $r \in \mathbb{N}$, $m > 0$, $\lambda > 0$. Suppose $f \in \mathcal{L}(m)$ and $\operatorname{geo}(f) \geq \lambda$. Consider the first through map $T = T(f, \cup_{j \neq 0} U_j(f))$ and suppose $0 \in Dom(T)$. Let h be T restricted to finitely many components so that $h \in \mathcal{L}$. Suppose*

$$\sup_{z \in Dom(h)} n_0(z) \leq r.$$

Then there exists $C_1 > 0$ and $C_2 > 0$, depending only on m, λ and r such that $\operatorname{geo}(h) \geq C_1$ and $(\operatorname{diam} K(h))/(\operatorname{diam} K(f)) \geq C_2$.

Let us now turn to those levels satisfying T4 with long saddle-node cascades. Let $m > 0$ and let

$$X = \{f \in \mathcal{Q}(m) : f \in I^{-1}(1/4) \text{ and } \operatorname{diam} K(f) = 1\}.$$

From Lemma 2.1.2, X is compact (in the Carathéodory topology). For each $f \in X$ choose a neighborhood $N \ni \beta_f$ on which f is a diffeomorphism and let $\mathcal{N}_1, \ldots, \mathcal{N}_k$ be a finite cover of X by the neighborhoods from Theorem 2.4.2. By rescaling we can extend the neighborhoods $\mathcal{N}_i$ to be a finite cover of $\{f \in \mathcal{Q}(m) : f \in I^{-1}(1/4)\}$. Note the coordinates do not necessarily agree on the overlaps $\mathcal{N}_i \cap \mathcal{N}_j$. Now if $(f, A, \Gamma) \in \mathcal{ML}_{\mathbb{R}}(m)$ has a long enough saddle-node cascade then by Lemma 4.2.1 it follows that $f \in (\mathcal{N}_i \cap \mathcal{P}_1)$ for some $1 \leq i \leq k$. That is, there is a function ℓ_m such that if $\ell(f) \geq \ell_m$ then $f \in (\mathcal{N}_i \cap \mathcal{P}_1)$ for some i. We can use the perturbed Fatou coordinates to control the geometry of a first through map:

Lemma 5.2.14 *Let $\lambda > 0$, $m > 0$ and $B \in \mathbb{N}$. Suppose $(f, A, \Gamma) \in \mathcal{ML}_{\mathbb{R}}(m)$ satisfies $\operatorname{geo}(f) \geq \lambda$ and $\ell(f) \geq \ell_m$. Let $(g, A_g, \Gamma_g) \in \mathcal{ML}_{\mathbb{R}}$ be the first through map T of f restricted to finitely many components such that the combinatorics of g is essentially bounded by B. Then there exists $C_1 > 0$ and $C_2 > 0$, depending only on λ, m, and B, such that $\operatorname{geo}(g) \geq C_1$ and $(\operatorname{diam} K(g))/(\operatorname{diam} K(f)) \geq C_2$.*

Proof: Let us prove the first statement. Suppose $f \in \mathcal{N}_i$. All statements that depend on Fatou coordinates implicitly use the coordinates from the neighborhood $\mathcal{N}_i$. Let $c_1 = f^{r_1}(0)$ be the first moment when the orbit of 0 lands in $S_{f,+}$. We can assume r_1 is uniform over the neighborhood $\mathcal{N}_i$. Then c_1 lies in a compact subset of $\mathcal{C}^f_+$. Let g_f be the induced transit map of f and let $c_2 = g_f(c_1)$. Since $f^n(c_2) \in \cup_{j \neq 0} U_j$ for some $n \leq B$, it follows c_2 lies in a compact subset of $\mathcal{C}^f_-$. Hence the phase $\bar{a}_f$, measured in the coordinates from $\mathcal{N}_i$, lies in a pre-compact subset of $\mathbb{C}/\mathbb{Z}$. Now it is easy to modify the proof of Lemma 5.2.12 to finish the proof: (f, g) is ranging in a pre-compact subset and so the set Y of restrictions with combinatorics essentially bounded by B is pre-compact. □

Finally we bound the geometry on levels that satisfy T5. The proof is a minor modification of the above lemma.

Lemma 5.2.15 *Let $\lambda > 0$, $m > 0$ and $B \in \mathbb{N}$. Suppose $(f, A, \Gamma) \in \mathcal{ML}_{\mathbb{R}}(m)$ satisfies* $\mathrm{geo}(f) \geq \lambda$ *and*

$$f_0 = f|_{U_0(f)} \in I^{-1}(1/4).$$

Let $g : \mathcal{C}_+ \to \mathcal{C}_-$ be a combinatorially real transit map of f_0. Let $(h, A_h, \Gamma_h) \in \mathcal{ML}_{\mathbb{R}}$ be the first through map $T(f, g, \cup_{j \neq 0} U_j(f))$ restricted to finitely many components such that the combinatorics of h is essentially bounded by B. Then there exists $C_1 > 0$ and $C_2 > 0$, depending only on λ, m, and B, such that $\mathrm{geo}(h) \geq C_1$ *and* $(\mathrm{diam}\, K(h))/(\mathrm{diam}\, K(f)) \geq C_2$.

Compactness

We are now able to prove the main proposition of this section.

Proposition 5.2.16 *For any $m > 0$ and $B \in \mathbb{N}$ the set of normalized towers $\mathcal{T} \in Tow_0(m, B)$ is compact.*

Proof: Let $\mathcal{T} = \{f_n, g_n\}$ be a tower in $Tow_0(m, B)$. First we claim the geometry of f_n is uniformly bounded. Indeed, the following table gives the appropriate lemma for each type of level:

Level	Lemma
T1	[McM2, Proposition 4.13]
T2	Lemma 5.2.11
T3	Lemma 5.2.12
T4	Lemma 5.2.13 and Lemma 5.2.14
T5	Lemma 5.2.15

That is, there exist $C_1 > 0$ and $C_2 > 0$, depending only on m and B such that

$$\mathrm{geo}(f_n) \geq C_1$$

and

$$\frac{\operatorname{diam} K(f_n)}{\operatorname{diam} K(f_{n-1})} \geq C_2.$$

Let $\mathcal{T}_k = \{f_{k,n}, g_{k,n}\}$ be a sequence in $Tow_0(m, B)$. From the geometry bounds above and from Lemma 2.3.1 and Lemma 2.1.2 (the extra condition

$$d(0, \mathbb{C} \setminus Range(f)) \leq C$$

is satisfied since $\beta_{f_n} \notin Range(f_{\mathrm{succ}(n)})$) we can select a subsequence $\mathcal{T}_{m_k}$ so that $f_{m_k,n}$ converges on all levels $n \in S$ to some generalized quadratic-like maps f_n. Let $S_{\mathcal{P}} \subset S$ be the levels with $f_n|_{U_0^n} \in I^{-1}(1/4)$. As in the proof of Lemma 5.2.14 we can choose a further subsequence so that the transit maps on each level $n \in S_{\mathcal{P}}$ converge. It is clear the limiting collection of maps will form a tower. □

5.2.6 Continuity of P

Lemma 5.2.17 *Let $m > 0$, $B \in \mathbb{N}$ and $\mathcal{T} \in Tow_0(m, B)$. Then* $\operatorname{diam} K(f_n) \to 0$ *as $n \to \infty$.*

Proof: We can assume there are an infinite number of levels $n \to \infty$ where f_n is not immediately renormalizable, for otherwise $\mathcal{T}$ is eventually a McMullen tower with period-doubling combinatorics and the result follows. Choose a subsequence of levels $n_k \to \infty$ so that f_{n_k} has at least one off-critical piece.

Suppose by contradiction that $\operatorname{diam} K(f_{n_k}) \geq \epsilon > 0$. Let $\cup_j U_{k,j} = Dom(f_{n_k})$ and $K_{k,j} = K(f_{n_k}) \cap U_{k,j}$. We may assume $K_{k+1,j} \subset K_{k,0}$ by selecting levels of first return.

Then since $\operatorname{geo}(f_{n_k}) \geq C(m, B) > 0$ and $\operatorname{mod}(K_{k,j}, U_{k,j}) \geq m$ it follows that $U_{k,j}$ contains a definite neighborhood of $K_{k,j}$. Hence there is eventually some $j_1, j_2 \neq 0$ and $k_2 > k_1$ with $K_{k_2,j_2} \cap U_{k_1,j_1} \neq \emptyset$. But this is a contradiction since $K_{k_2,j_2} \subset K_{k_1,0}$ and $K_{k_1,0} \cap U_{k_1,j_1} = \emptyset$. □

Proposition 5.2.18 ([McM2, Corollary 5.12]) *Let $m > 0$ and $B \in \mathbb{N}$. Then the postcritical set $P(\mathcal{T})$ varies continuously with $\mathcal{T} \in Tow_0(m, B)$.*

Proof: Let $\mathcal{T}_m$ be a sequence of towers in $Tow_0(m, B)$ converging to a tower $\mathcal{T}$. If $z \in \operatorname{orb}(\mathcal{T}, 0)$ then $d(z, P(\mathcal{T}_m)) \to 0$ as $m \to \infty$ since $\mathcal{F}(\mathcal{T})$ is contained in any geometric limit of $\mathcal{T}_m$. Hence $P(\mathcal{T}) \subset \liminf_m P(\mathcal{T}_m)$. We must show $\limsup_m P(\mathcal{T}_m) \subset P(\mathcal{T})$.

For $n \in S_Q$ let $K_n(0) = K(f_n)$ and let $K_n(i)$ enumerate the orbit of $K(f_n)$ by $\mathcal{T}$. That is,

$$\cup_i K_n(i) = \{h(z) : z \in K(f_n), h \in \mathcal{F}(\mathcal{T})\}.$$

Let $\delta_n = \sup_i \operatorname{diam} K_n(i)$. The arguments proving $\operatorname{diam} K_n(0) \to 0$ can be adapted to prove $\delta_n \to 0$. Let $\epsilon > 0$ and let N be large enough so that $\delta_N < \epsilon$. Let

$$\cup_i K_{m,n}(i) = \{h(z) : z \in K(f_{m,n}), h \in \mathcal{F}(\mathcal{T}_m)\}.$$

Since $\mathcal{T}_m \to \mathcal{T}$ it follows that for $m > N$ large enough $\cup_i K_{m,n}(i)$ is contained in an ϵ-neighborhood of $\cup_i K_n(i)$. Hence $P(\mathcal{T}_m)$ is contained in a 2ϵ-neighborhood of $P(\mathcal{T})$. □

5.2.7 Definite Expansion

The following corollary can be used to control the expansion of the hyperbolic metric on one level with bounds from a deeper level. For a given tower $\mathcal{T}$ and level $n \in S_Q$ we will use the notation ρ_n, $\|\cdot\|_n$, $d_n(\cdot,\cdot)$ and $\ell_n(\cdot)$ to denote the hyperbolic metric, norm, distance and length on $V^n \setminus P^n$, where recall P^n is the post-critical set of $\mathcal{T}$ restricted to levels $m \geq n$. That is, for any $m \in S$ recall $\mathcal{F}_m = \langle \{f_m\} \cup \{g_n : n \geq m\}\rangle$ and $P^m = \overline{\text{orb}(\mathcal{F}_m, 0)}$. Recall we say $\mathcal{T}$ is *unbranched* iff

$$P^n = P^0 \cap V^n$$

for all $n \in S$.

Corollary 5.2.19 *Let $m > 0$ and $B \in \mathbb{N}$. Let $\mathcal{T} \in Tow_0(m, B)$ be an unbranched forward tower. Suppose $n \in S_Q$ is a level such that $V^n \subset V^0$. Let $h \in \mathcal{F}_n$ and let $Q_h^n = h^{-1}(P^n)$. Then if $(Q_h^n \setminus P^n) \neq \emptyset$ and $z \in Dom(h) \setminus Q_h^n$,*

$$C_2^{-1}(s_2) \leq \|Dh(z)\|$$

where $s_2 = d_n(z, Q_h^n)$.

Proof: Since $V^n \subset V^0$ and $P^n = P^0 \cap V^n$ we see

$$(V^n \setminus P^n) \subset (V^0 \setminus P^0)$$

and so

$$d_0(z, Q_h^n) \leq d_n(z, Q_h^n).$$

Since the function C_2 in Proposition 5.2.3 is increasing,

$$C_2^{-1}(d_n(z, Q_h^n)) \leq C_2^{-1}(d_0(z, Q_h^n)). \tag{5.1}$$

Then from $Range(h) \subset V^n$ and $V^n \cap P^0 = P^n$ we see

$$h^{-1}(P^n) = h^{-1}(P^0).$$

Since $h \in \mathcal{F}(\mathcal{T})$ it follows from Proposition 5.2.3 that

$$C_2^{-1}(d_0(z, Q_h^n)) \leq \|Dh(z)\|_0. \tag{5.2}$$

The lemma follows from equations 5.1 and 5.2. ☐

In order to apply this corollary we need to get a bound on $s_2 = d_n(z, Q_h^n)$. This is done by compactness. For any $m > 0$, $C < \infty$, $K < \infty$ and $B \in \mathbb{N}$ let $Tow_0(m, B, C, K)$ be the set of towers $\mathcal{T} = \{f_n, g_n\}$ in $Tow_0(m, B, C)$ where all $U \in \text{CC}(U^n)$ and V^n are K-quasidisks for all $n \in S$.

Lemma 5.2.20 *Let $m > 0$, $C < \infty$, $K < \infty$ and $B \in \mathbb{N}$. Let $\mathcal{T} \in Tow_0(m, B, C, K)$, $n \in S_Q$ and $z \in f_n^{-1}(V^n \setminus U^n)$. Then there exists a $C_1 < \infty$ depending only on m, C, K and B such that $d_n(z, Q_{f_n}^n) \leq C_1$.*

Proof: By shifting level n to level 0 and restricting to a subtower we can assume $n = 0$. Since V^0 and U^0 are K-quasidisks, the set

$$W_0 = \overline{f_0^{-1}(V^0 \setminus U^0)}$$

varies continuously with $\mathcal{T} \in Tow_0(m, B, C, K)$. Since $P(\mathcal{T})$ varies continously the hyperbolic metric ρ and the set $Q^0_{f_0}$ vary continuously. Therefore the function F on $Tow_0(m, B, C, K)$ given by

$$F(\mathcal{T}) = \sup_{z \in W_0} d(z, Q^0_{f_0})$$

is continuous. Since $Tow_0(m, B, C, K)$ is compact by Lemma 5.2.16, there is a C_1 such that $F(\mathcal{T}) \leq C_1$. □

5.2.8 The Interior of K

An infinitely renormalizable quadratic-like map $f \in \mathcal{Q}^\infty_\mathbb{R}$ has a filled Julia set with empty interior. The same statement holds for forward towers:

Proposition 5.2.21 *For any unbranched* $\mathcal{T} \in Tow_0(m, B, C, K)$,

$$\text{int}(K(\mathcal{T})) = \emptyset.$$

The proof of Proposition 5.2.21 is broken into propositions Proposition 5.2.25 and Proposition 5.2.26 and will occupy the rest of this section.

Fix an unbranched tower $\mathcal{T} \in Tow_0(m, B, C, K)$. Suppose by contradiction that

$$\mathcal{O} = \text{CC}(\text{int}(K(\mathcal{T})))$$

is non-empty. Let $U \in \mathcal{O}$ and $z \in U$. Let $K \subset U$ be a compact and connected neighborhood of z. Recall B_n are the central basins of $\mathcal{T}$. Since $\partial B_n \subset J(\mathcal{T})$ for all $n \in S_\mathcal{P}$ it follows that K is iterable. Since $J(\mathcal{T})$ is backward invariant we see that all the iterates of K are iterable as well. Thus the orbit of K is well defined and contains the orbit of z and so, letting K range over larger and larger compact subset of U, we can define the orbit of U, orb(U), to be components containing the orbit of K.

Periodic Components

A component $U \in \mathcal{O}$ is called *periodic* if $U' \in \text{orb}(U)$ implies $U \in \text{orb}(U')$. A component $U \in \mathcal{O}$ is called *pre-periodic* if U is not itself periodic but there is a periodic component in orb(U).

The classification of periodic components is based on the following classical propositions (see, for example, [L1, M1]):

Proposition 5.2.22 *Let $h : U \to U$ be an analytic transform of a hyperbolic Riemann surface U. The we have one of the following possibilities:*

1. *h has an attracting or superattracting fixed point in U to which all orbits converge*

2. *all orbits tend to infinity*

3. *h is conformally conjugate to an irrational rotation of the disk, the punctured disk or an annulus*

4. *h is a conformal homeomorphism of finite order*

Proposition 5.2.23 *Let U be a hyperbolic domain on the sphere, and $h : U \to U$ an analytic transform continuous up to the boundary. Suppose that the set of fixed points of h on ∂U is totally disconnected. Then in case 2) of Proposition 5.2.22 there is a fixed point $\alpha \in \partial U$ such that $h_m(z) \to \alpha$ for every $z \in U$.*

The following lemma is useful for controlling the dynamics near the ends of the Ècalle-Voronin cylinders. Let $f \in I^{-1}(1/4)$ and choose petals $D_{\pm}$. For any $z \in D_+ \cap D_-$ define the Ècalle-Voronin transformation $\mathcal{E}$ by

$$\mathcal{E}(\pi_-(z)) = \pi_+(z).$$

One can show that $\mathcal{E}$ extends holomorphically to the two ends of $\mathcal{C}_-$ by using the Fatou coordinates and the standard isomorphism $\pi(z) = \exp(2\pi i z)$ of $\mathbb{C}/\mathbb{Z}$ to $\mathbb{C} \setminus 0$.

Lemma 5.2.24 *Suppose $f_0 \in I^{-1}(1/4)$ and $g : \mathcal{C}_+ \to \mathcal{C}_-$ is a transit map such that the critical point of f_0 escapes $K(f_0)$ under iterates of f_0 and local lifts of g. Then*

$$|(g \circ \mathcal{E})'(\pm\infty)| > 1.$$

Proof: We will prove the lemma with the critical point escaping after just one iterate of a local lift of g. Assume the critical point of f is at the origin. Let $R = g \circ \mathcal{E}$ and $J_- = \pi_-(J(f_0))$. Let $V_{\pm\infty}$ denote the connected components of $(\mathcal{C}_- \setminus J_-) \cup \{\pm\infty\}$ containing $\pm\infty$ and let $U_{\pm\infty} = g^{-1}(V_{\pm\infty})$ (see Fig. 5.1).

Note that $\mathcal{E}$ can be extended to $V_{\pm\infty}$ as a branched cover. The set of critical points is the backward orbit of 0 and the only critical value is $\pi_+(0)$.

Since $\pi_+(0) \notin U_{\pm\infty}$ and each $U_{\pm\infty}$ is simply connected there is a branch of $\mathcal{E}^{-1}$ defined on $U_{\pm\infty}$ preserving $\pm\infty$. Composing $\mathcal{E}^{-1} \circ g^{-1}$ we have constructed a branch of R^{-1} which maps each $V_{\pm\infty}$ inside itself and fixes $\pm\infty$. In fact, since $\mathcal{E}^{-1}(\overline{U_{\pm\infty}}) \subset \pi_-(\operatorname{int} K(f_0))$ it follows that R^{-1} maps $V_{\pm\infty}$ strictly inside itself. The lemma follows from the Schwarz lemma. □

We can now prove the following

Proposition 5.2.25 *No $U \in \mathcal{O}$ is periodic or pre-periodic.*

Proof: Suppose $U \in \mathcal{O}$ is a periodic component. Suppose $\mathrm{cl}(U)$ is iterable and all iterates of $\mathrm{cl}(U)$ are iterable. Then since U is periodic there exists a univalent map $h \in \mathcal{F}(\mathcal{T})$ defined on a neighborhood of $\mathrm{cl}(U)$ such that $h(\mathrm{cl}(U)) = \mathrm{cl}(U)$. Let us examine the possibilities from Proposition 5.2.22.

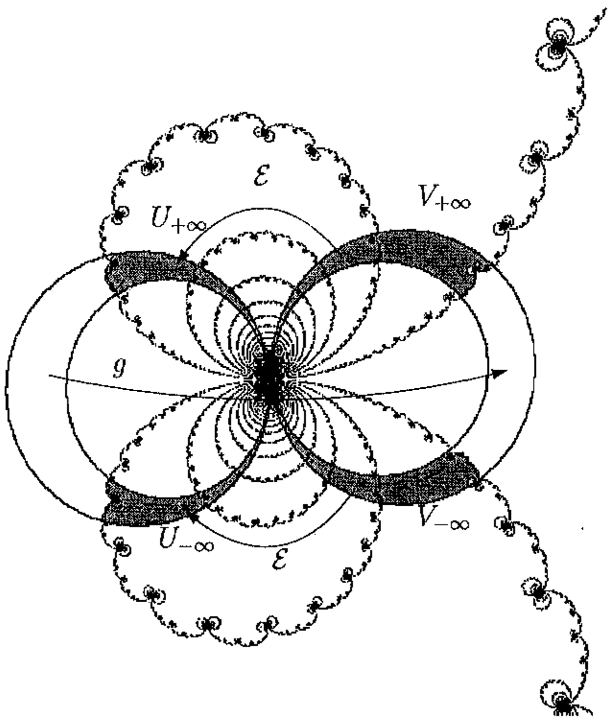

Figure 5.1: A blow-up of the Julia set of $f_0 = z^2 + 1/4$ with pre-images by f_0, g and $\mathcal{E}$ highlighting the sets $U_{\pm\infty}$ and $V_{\pm\infty}$.

Since $\mathrm{cl}(U)$ is disjoint from $P(\mathcal{T})$, Lemma 5.2.4 implies any periodic point in $\mathrm{cl}(U)$ must be repelling. Thus there cannot be an superattracting or attracting orbits. Suppose all iterates tend to ∂U. Now the set of points on ∂U fixed by h are isolated, since otherwise h would be the identity on an open set and that would contradict Proposition 5.2.3. Applying Proposition 5.2.23 again contradicts Lemma 5.2.4.

The other possibilities in Proposition 5.2.22 are ruled out because h expands the hyperbolic metric on $U^0 \setminus P^0$ and any map conjugate to a rotation will have high iterates arbitrarily close to the identity.

Now suppose there is a component U' from $\mathrm{orb}(U)$ such that $\mathrm{cl}(U')$ is not iterable. To simplify the exposition we will assume $\mathcal{T}$ is a real-symmetric tower. Since U is periodic we may assume $U = U'$. Since $U \subset K(f_0)$ there must be an $n \in S_{\mathcal{P}}$ such that $\mathrm{cl}(U) \cap \partial(B_n) \neq \emptyset$. Since $U \cap J(\mathcal{T}) = \emptyset$ it follows that $U \subset B_n$ and if $n' \in S_{\mathcal{P}}$ is the next parabolic level after n then $\mathrm{cl}(U) \cap B_{n'} = \emptyset$.

Let $K = \mathrm{cl}(U)$, $f = f_n|_{U_0^n}$ and $\xi = \beta(f)$. Since B_n and ∂B_n are invariant by f, it follows $K_k = f^k(K) \subset \mathrm{cl}(B_n) \setminus B_{n'}$ and $\partial K_k \cap \partial B_n \neq \emptyset$ for all $k \geq 0$. Let

$$\mathcal{B} = \mathrm{CC}(B_n \setminus (\cup_{k \geq 0}^{\infty} f^{-k}(\mathbb{R})))$$

be the collection of components of the partition pictured in Fig. 5.2.

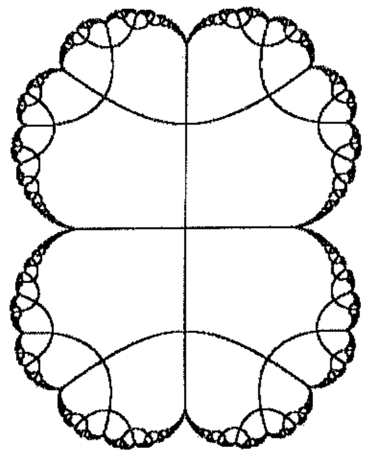

Figure 5.2: The tiling of B_n.

First we claim that $U \cap \mathbb{R} = \emptyset$. Let $n' \in S_{\mathcal{Q}}$ be the largest quadratic-like level before n. Let $B_{n''}$ be the central basin of the first level $n'' \in S_{\mathcal{P}}$ after n'. Then the $f_{n'}$ pre-images of $B_{n''}$ cover a dense subset of $\mathbb{R} \cap K(\mathcal{F}_{n'})$. It follows that the pre-images by $\mathcal{F}(\mathcal{T})$ cover a dense subset of $\mathbb{R} \cap B_n$ and accumulate at ξ. Since $\partial B_{n''} \subset J(\mathcal{T})$ the claim is established. Since U is periodic under $\mathcal{T}$, we can assume $U \subset A$ where $A \in \mathcal{B}$ satisfies $\xi \in \partial A$. Without loss of generality assume $A \subset \mathbb{H}$.

Let $\gamma = \partial A$. Let

$$\gamma_1 = \bigcup_{\tilde{g}_n} \tilde{g}_n^{-1}(\gamma).$$

Since g_n is a real translation, $\gamma_1 \subset \mathbb{H}$. Let

$$\gamma_2 = \bigcup_{k \geq 0} (f^{-1})^k(\gamma_1)$$

where the branch of f^{-1} is chosen so that $f^{-1}(\mathbb{H} \cap B_n) \subset \mathbb{H} \cap B_n$ (see Fig. 5.3). It follows from Lemma 3.1.4 that U is contained in the domain A_1 bounded by γ_2.

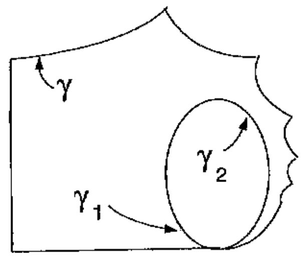

Figure 5.3: The curves γ, γ_1 and γ_2.

Continue this process. That is, the pre-image of γ_2 by $\widetilde{g}_n$ is contained in A_1 and pulling back by f^{-1} we see that U is contained in a domain $A_2 \subset A_1$. By Lemma 5.2.24,

$$\bigcap_{m \geq 1}^{\infty} A_m = \emptyset$$

and so a non-iterable periodic component U cannot exist. $\square$

Wandering Components

A component $U \in \mathcal{O}$ that is neither periodic nor pre-periodic is called *wandering.*

Proposition 5.2.26 *No $U \in \mathcal{O}$ is wandering.*

Proof: Suppose $U \in \mathcal{O}$ is wandering. Let $K \subset U$ be compact and connected. Then K is iterable and all iterates of K are iterable. Fix an $z \in \text{int}(K)$. Since each map h from the orbit of K is defined on a neighborhood of K and since $Q_h = h^{-1}(P(\mathcal{T})) \subset J(\mathcal{T})$, it follows from Proposition 5.2.3 that

$$\sup_h \|Dh(z)\| < \infty. \tag{5.3}$$

Suppose there is an $\epsilon > 0$ such that there are an infinite number of iterates h_n satisfying

$$d(h_n(z), P(\mathcal{T})) > \epsilon$$

where the distance is just the Euclidean distance. Order the h_n to match the ordering on the orbit. That is, if $n < m$ then $h_m(z) \in \text{orb}(h_n(z))$. Since each $h_n(z)$ lies in a compact subset of the hyperbolic surface $V^0 \setminus P(\mathcal{T})$,

$$d(h_n(z), Q_{f_0}) \leq C(\epsilon),$$

for some function $C(\epsilon)$, and so from Proposition 5.2.3,

$$\|Df_0(h_n(z))\| \geq C > 1. \tag{5.4}$$

But then

$$\|Dh_{n+1}(z)\| \geq \|D(f_0 \circ h_n)(z)\| = \|Df_0(h_n(z))\| \cdot \|Dh_n(z)\| \geq C\|Dh_n(z)\| \tag{5.5}$$

which as $n \to \infty$ contradicts equation 5.3.

So we can assume

$$\limsup_h d(h(z), P(\mathcal{T})) = 0.$$

Let $\mathcal{T}_n = \mathcal{T}|_{S_n}$ be the restriction of $\mathcal{T}$ to levels $m \geq n$. Let $\mathcal{K}_n$ be the collection of little filled Julia sets $\mathcal{K}_n = \mathrm{orb}(\mathcal{T}, K(\mathcal{T}_n))$.

From Lemma 5.2.24 we see $\mathrm{orb}(z)$ must accumulate on some $z' \notin \xi_0$ where ξ_0 is the parabolic orbit of f_0. But then z' is contained in a little filled Julia set in $\mathcal{K}_1$. By iterating forward we can assume $z' \in K(\mathcal{T}_1)$. It follows that there is a $y_1 \in \mathrm{orb}(z)$ such that $y_1 \in K(\mathcal{T}_1)$. Now again there is an accumulation point $\mathrm{orb}(y_1)$ disjoint from ξ_1, the parabolic orbit of f_1, and, repeating the whole argument inductively, there is a sequence of iterates $y_n \in K(\mathcal{T}_n)$.

Each y_n has a moment $x_n \in \mathrm{orb}(z)$ when $\mathrm{orb}(z)$ enters the collection of little filled Julia sets $\mathcal{K}_n$. Once $\mathrm{orb}(z)$ enters $\mathcal{K}_n$ it never leaves. It can happen that different y_n have the same moment x_n. However, since

$$\bigcap_{n \geq 0} K(\mathcal{T}_n) = \{0\}$$

there must be an infinite number of distinct entry moments x_{n_i}.

Let $z_n \in \mathrm{orb}(z)$ satisfy $f_0(z_n) = x_n$. Thus the relation between the points z, x_n, y_n and z_n is given by: $z_n \in \mathrm{orb}(z)$, $x_n = f_0(z_n)$ is the time $\mathrm{orb}(z)$ enters $\mathcal{K}_n$ and $y_n \in \mathrm{orb}(x_n)$ is the first time x_n enters $K(\mathcal{T}_n)$. Claim

$$d(z_n, Q_{f_0}) \leq C'.$$

Let K_n' be the component of $f_0^{-1}(\mathcal{K}_n) \setminus \mathcal{K}_n$ containing z_n. The set K_n' is called a *companion* filled Julia set of level n. Since $Q_{f_0} \cap K_n' \neq \emptyset$, it is enough to show

$$\mathrm{diam}_\rho(K_n') \leq C'.$$

Consider the sets U_n' and V_n' containing K_n' which are pull-backs of U^n and V^n by the map sending z_n to y_n. By the unbranched property this pull-back is univalent. Since $\mathrm{mod}\,(V^n, U^n) \geq m$, we have $\mathrm{mod}\,(V_n', U_n') \geq m$ and so, from [McM1, Theorem 2.4], the diameter D_n of U_n' in the hyperbolic metric on V_n' is bounded. But $V_n' \subset (V^0 \setminus P^0)$. Thus

$$\mathrm{diam}_\rho(K_n') \leq \mathrm{diam}_\rho(U_n') \leq D_n \leq C$$

and the claim is established.

But then equations 5.4 and 5.5 hold along the sequence $z_{n_i} = h_{n_i}(z)$, and we again get a contradiction to 5.3. □

5.2.9 No invariant line fields

A *line field* is a measurable Beltrami differential $\mu = u(z)d\bar{z}/dz$ with $|u(z)| = 1$ on a set of positive measure and $|u(z)| = 0$ otherwise. A line field is *invariant* under $\mathcal{T}$ iff for every $h \in \mathcal{T}$, Dh maps the line at x to the line at $h(x)$ for almost every $x \in Dom(h)$. Using Proposition 5.2.5 and Proposition 5.2.21 we can rephrase Proposition 5.2.6 in terms of invariant line fields.

Before doing so, we need the following

Lemma 5.2.27 (**[L1]**) *Let $\mathcal{T} \in Tow_0$. The group G of homeomorphisms of $J(\mathcal{T})$ that commute with all maps $h \in \mathcal{T}$ is totally disconnected.*

Proof: Let $\phi \in G$ be a map in the connected component of the identity. Suppose z_0 is a repelling periodic point with $h(z_0) = z_0$ for some $h \in \mathcal{F}(\mathcal{T})$. Since the solutions to $h(z) = z$ are isolated ϕ must fix z_0. The lemma follows from density of repelling cycles: Proposition 5.2.5. □

We now prove the following version of forward tower rigidity:

Proposition 5.2.28 *Let $\mathcal{T}$ be an unbranched tower in $Tow_0(m, B, C, K)$ and let μ be a line field invariant under $\mathcal{T}$. Then $\mu|_{K(\mathcal{T})} = 0$ a.e.*

Proof: By Proposition 5.2.1 it suffices to consider a forward tower $\mathcal{T}$ having a base map of the form $z^2 + c_0$. From Proposition 5.2.21 we know $K(\mathcal{T}) = J(\mathcal{T})$. Suppose by contradiction that $\mathcal{T}$ did admit an invariant line field

$$\mu = u(z)d\bar{z}/dz$$

supported on $J(\mathcal{T})$. For any $w \in \mathbb{D}$ consider the invariant Beltrami differential

$$\mu_w = w \cdot u(z)d\bar{z}/dz$$

on $\widehat{\mathbb{C}}$. Let ϕ_w be a solution to the corresponding Beltrami equation normalized so that the map

$$f_{w,0} = \phi_w \circ f_0 \circ \phi_w^{-1}$$

is again a rational map of the form $z^2 + c_w$ for some $c_w \in \mathbb{C}$. Let $\mathcal{T}_w$ be the tower

$$\mathcal{T}_w = \{\phi_w \circ h \circ \phi_w^{-1} : h \in \mathcal{T}\}.$$

From Proposition 2.2.2 and the uniqueness of root points, $c_w = c_0$ for all $w \in \mathbb{D}$. Proposition 5.2.6 implies $f_{w,n} = f_n$ for all $n \in S$ and $g_{w,n} = g_n$ for all $n \in S_{\mathcal{C}}$ and $w \in \mathbb{D}$. So ϕ_w is a holomorphic family of quasi-conformal maps with $\phi_0 = id$ and ϕ_w mapping $J(\mathcal{T})$ homeomorphically to itself commuting with the dynamics of $\mathcal{T}$. From Lemma 5.2.27 $\phi_w|_{J(\mathcal{T})} = id$. But then the complex dilatation of ϕ_w is zero at all points of Lebesgue density of $J(\mathcal{T})$ and so μ is not supported on $J(\mathcal{T})$, a contradiction. □

5.3 Bi-infinite Towers

In this section we move from studying properties of forward towers to studying bi-infinite towers. The plan of attack again follows [McM2] and the goal is to prove combinatorial rigidity of bi-infinite towers with bounds. The idea behind the proof, namely to blow up a non-trivial deformation to obtain a contradiction, was pioneered by Mostow and Sullivan in the theory of Kleinian groups.

Let $Tow^\infty = \{\mathcal{T} \in Tow : S(\mathcal{T}) = \mathbb{Z}\}$ denote the set of bi-infinite towers in Tow and let $Tow^\infty(m, B)$ denote the set of combinatorially real bi-infinite towers $\mathcal{T} = \{f_n, g_n\}$ with $\operatorname{mod} f_n \geq m$ and essential combinatorics bounded by B. Let $Tow^\infty(m, B, C, K) \subset Tow^\infty(m, B)$ the unbranched towers with

$$\operatorname{diam} V^n \leq C \operatorname{diam} K(f_n)$$

and each $U \in \mathrm{CC}(Dom(f_n))$ and $Range(f_n)$ K-quasidisks. For a given bi-infinite tower $\mathcal{T}$ define the post-critical set as

$$P(\mathcal{T}) = \cup_{n \leq 0} P(\mathcal{F}_n).$$

We say two bi-infinite towers $\mathcal{T}$ and $\mathcal{T}'$ are *equal* if $\mathcal{T}_n$ is equal to $\mathcal{T}_n'$ on $K(\mathcal{T}_n)$ for all $n \in \mathbb{Z}$ and we say $\mathcal{T}$ and $\mathcal{T}'$ with $S(\mathcal{T}) = S(\mathcal{T}')$ are *quasi-conformally equivalent* if there is a quasi-conformal map ϕ and an $\epsilon > 0$ such that

1. ϕ is a quasi-conformal conjugacy of f_n and f_n' on an ϵ-scaled neighborhood of $K(f_n)$ to a neighborhood of $K(f_n')$ for all $n \in S$,
2. ϕ induces a quasi-conformal conjugacy of the transit maps g_n and g_n' for $n \in S_{\mathcal{P}}$.

The following are straightforward generalizations of Lemma 5.2.17 and Proposition 5.2.16:

Lemma 5.3.1 *Let $m > 0$, $B \in \mathbb{N}$ and $\mathcal{T} \in Tow^\infty(m, B)$. Then* $\operatorname{diam} K(f_n) \to \infty$ *as $n \to -\infty$.*

Proposition 5.3.2 *For any $m > 0$, $C < \infty$, $K < \infty$ and $B \in \mathbb{N}$ the space of normalized towers $\mathcal{T} \in Tow(m, B, C, K)$ is compact.*

Given $\mathcal{T} \in Tow^\infty(m, B, C, K)$ define $S_\mathcal{N} \subset S_Q$ as follows. Let $S_{\mathcal{N},0} = \{0\}$. Then inductively let $S_{\mathcal{N},n+1} = S_{\mathcal{N},n} \cup \{m_{n+1}\}$ where

$$m_{n+1} = \max\{m \in S_Q |\ m < k = \min S_{\mathcal{N},n},\ U^m \supset V^k\}.$$

Define $S_\mathcal{N} = \cup_{n \to \infty} S_{\mathcal{N},n}$. That is, $S_\mathcal{N}$ is the minimal set of nested levels approaching $-\infty$ and starting at 0. From Lemma 5.3.1 we see $S_\mathcal{N}$ is unbounded below. Define the depth of $z \neq 0$ by

$$\operatorname{depth}(z) = \max\{m \in S_\mathcal{N} : z \in U^m\}.$$

For a point $z \in \mathbb{C}$ we say a (possibly finite) sequence $(z_0, z_1, z_2, \ldots)$ is a *sub-orbit* of z (in $\mathcal{T}$) if the following conditions are satisfied:

- $z_0 = z$
- if $z_i = 0$ then $z_{i+1} = 0$
- if $z_i \in Dom(\widetilde{g}_n)$ then $z_{i+1} = \widetilde{g}_n(z_i)$ for some local lift $\widetilde{g}_n \in \mathcal{T}$
- otherwise $z_{i+1} = f_{\mathrm{depth}(z_i)}(z_i)$

Let $\rho_{-\infty}$ be the hyperbolic metric on $\mathbb{C} \setminus P(\mathcal{T})$ and as in §5.2.2 let ρ_n be the hyperbolic metric on $V^n \setminus P^n$. From Lemma 5.3.1 and the unbranched property the metrics ρ_n, for $\mathcal{T} \in Tow^{\infty}(m, B, C, K)$, converge uniformly on compact sets to $\rho_{-\infty}$. Using the expansion from §5.2.2, we now prove

Theorem 1.2.1. *For any $\mathcal{T} \in Tow^{-\infty}(m, B, C, K)$*

$$\lim_{n \to -\infty} J(\mathcal{T}_n) = \widehat{\mathbb{C}}$$

in the Hausdorff topology.

Proof: Let $z \notin \cup_{s \leq 0} J(\mathcal{T}_s)$. Without loss of generality we may assume $z \in U^0$. Then $\mathrm{orb}(\mathcal{F}_s, z)$ escapes U^s for any $s \in S_{\mathcal{N}}$. Let $z_s = h_s(z)$ be the orbit point just before the first moment of escape on level s. That is, $f_s(z_s) \in V^s \setminus U^s$ and if $z' \in \mathrm{orb}(z)$ also satisfies $f_s(z') \in V^s \setminus U^s$ then $z' \in \mathrm{orb}(z_s)$. For a given $s \in S_{\mathcal{N}}$ let γ_s' be a hyperbolic geodesic in $V^s \setminus P^s$ connecting z_s with $J(\mathcal{T}_s)$. From Lemma 5.2.20, there is a C independent of s such that $\ell_s(\gamma_s') \leq C$. But h_s has an extension $h \in \mathcal{F}(\mathcal{T})$ that is a covering map onto $V^s \setminus P^s$. Let γ_s be the connected component of $h^{-1}(\gamma_s')$ containing z.

We now argue $\ell_s(\gamma_s)$ shrinks as $s \to -\infty$. The proposition would follow since ρ_s converges to $\rho_{-\infty}$ near z and since Julia sets are backward invariant. Fix an $s \in S_{\mathcal{N}}$ and let $N_s = |\{s, \ldots, 0\} \cap S_{\mathcal{N}}|$ be the minimal number of moments when the orbit of z escapes a nested level. It follows from Lemma 5.2.20 and Corollary 5.2.19 that there is a $C > 1$ such that

$$C \leq \|Df_t(z_t)\|_s$$

for any $t \in \{s, \ldots, 0\} \cap S_{\mathcal{N}}$. Hence

$$C^{N_s} \leq \|Dh_s(z)\|_s. \tag{5.6}$$

Hence the derivative at the endpoint z grows exponentially in N_s. From Proposition 5.2.3, there exists a $C > 1$ such that equation 5.6 holds along γ_s and hence the length of γ_s shrinks as $s \to -\infty$. □

A measurable line field μ on an open set U is called *univalent* if there is a univalent map $h : U \to \mathbb{C}$ such that $\mu = h^*(d\bar{z}/dz)$. The main statement in this section is the following extension of Proposition 5.2.28, which immediately implies Theorem 1.2.2.

Theorem 5.3.3 *Let $\mathcal{T} \in Tow_{\infty}(m, B, C, K)$. There does not exist a measurable line field μ in the plane such that $h_*(\mu) = \mu$ for all $h \in \mathcal{F}(\mathcal{T}_n)$, $n \in S_{\mathcal{N}}$.*

Proof: Suppose to the contrary that $\mu = u(z)d\bar{z}/dz$ is a measurable invariant line field which is non-zero on a set, B, of positive measure. Let $z \in B$ be a point of almost continuity of u and satisfying $|u(z)| = 1$. That is, for each $\epsilon > 0$, the chance of randomly choosing a point y a distance r from z that satisfies $|u(y) - u(z)| > \epsilon$ tends to 0 as r tends to 0:

$$\lim_{r \to 0} \frac{\text{area}(\{y \in B(z,r) \ : |u(y) - u(z)| > \epsilon\})}{\text{area}\, B(z,r)} = 0$$

where $B(z,r)$ is the Euclidean ball of radius r centered at z. By Proposition 5.2.28, we can assume $z \notin K(\mathcal{T}_n)$ for any n. Let z_n be an infinite sub-orbit from z and for each $s \in S_{\mathcal{N}}$ let $z_{n_s} = h_{n_s}(z)$ denote the moments in the sub-orbit when z_{n_s+1} first satisfies $z_{n_s+1} \in V^s \setminus U^s$.

For a given $s \in S_{\mathcal{N}}$ let $\mathcal{T}'_s$ denote $\mathcal{T}$ shifted so that level s is moved to level 0 and let w_s and u_s denote z_{n_s} and u shifted by s. That is, if $|B(f_s)| = \alpha_s$, then $w_s = \alpha_s^{-1} z_{n_s}$ and $u_s(z) = u(\alpha_s z)$. Then since $Tow^\infty(m, B, C, K)$ is compact the sequence $\mathcal{T}'_s$ has a subsequence which as $s \to -\infty$ converges to some $\mathcal{T}' \in Tow^\infty(m, B, C, K)$. By choosing a further subsequence we may assume w_s converges to a

$$w \in \text{cl}((f'_0)^{-1}(Range(f'_0) \setminus Dom(f'_0)))$$

and, from [McM2], μ_s converges weak*, and hence pointwise almost everywhere, to a measurable line field μ' invariant by $\mathcal{T}'$ in the sense that $h_*(\mu) = \mu$ for all $h \in \mathcal{F}(\mathcal{T}'_n)$, $n \in S_{\mathcal{N}}(\mathcal{T}')$.

Let D be a small disk around w in $A = Range(f'_0) \setminus P(\mathcal{T}'_0)$. The hyperbolic diameter of D in A is close to that of $D_s = \alpha_s^{-1}(D)$ in the metric on $V^s \setminus P^s$ for s near $-\infty$. Since D_s is disjoint from P^s, there is, by the argument given in Proposition 1.2.1, a univalent pullback D'_s of D_s by the map h_{n_s}. By equation 5.6 and the variation of expansion in Proposition 5.2.3, we see D'_s is a sequence of open sets containing z such that in the Euclidean metric $\text{diam}(D'_s) \to 0$ and $B(z, C\,\text{diam}(D'_s)) \subset D'_s$ as $s \to -\infty$ for some constant C. Therefore from [McM1, Theorem 5.16] we can choose μ' to be univalent on D.

By Proposition 1.2.1, there is an $s \in S_{\mathcal{N}}(\mathcal{T}')$ such that $J(\mathcal{T}'_s) \cap D \neq \emptyset$. By invariance, if $Dh(z) \neq 0$ and μ' is locally univalent around z then μ' agrees almost everywhere with a locally univalent line field around $h(z)$ for any composition $h \in \mathcal{F}(\mathcal{T}'_s)$. From Proposition 5.2.5, the orbit of D by $\mathcal{T}'_s$ covers all of V'_s. So μ' agrees almost everywhere with a line field that is locally univalent on the set $Range(f'_s) \setminus P(\mathcal{T}'_s)$. Since f'_s is injective on $P(\mathcal{T}'_s)$ every point in $P(\mathcal{T}'_s)$ except $f'_s(0)$ has an f'_s pre-image around which μ' agrees (a.e.) with a locally univalent line field. Hence μ' agrees (a.e.) with a locally univalent line field around $(f'_s)^2(0)$ and 0, which is a contradiction, since then we obtain contradictory behavior of μ' around $f'_s(0)$. □

As a corollary we obtain

Theorem 1.2.2. *If $\mathcal{T}, \mathcal{T}' \in Tow^\infty(m, B, C, K)$ are normalized combinatorially equivalent towers then $\mathcal{T}$ and $\mathcal{T}'$ are equal.*

Proof: Let $S_{\mathcal{N}}$ be the set of nested levels of $\mathcal{T}$ as constructed above. For each $n \in S_{\mathcal{N}}$, let ϕ_n be a hybrid equivalence between $\mathcal{T}_n$ and $\mathcal{T}_n'$ coming from straightening (see Corollary 5.2.10). The dilatation of ϕ_n is bounded above by a constant depending only on m and B and ϕ_n fixes 0 and ∞ and maps $\beta(f_0)$ to $\beta(f_0')$. Thus we can pass to a convergent subsequence $\phi_{n_k} \to \phi$ as $n \to -\infty$. Since ϕ_n restricts to a quasi-conformal equivalence of f_s and f_s' for $s > n$, $s \in S_{\mathcal{N}}$, on a definite neighborhood of $K(f_s)$, it follows that ϕ is a quasi-conformal equivalence. Let μ be the line field defined by ϕ and μ_n the line field defined by ϕ_n. Since $h_*(\mu_n) = \mu_n$ for all $h \in \mathcal{F}(\mathcal{T}_n)$ it follows that $h_*(\mu) = \mu$ for all $h \in \mathcal{F}(\mathcal{T}_n)$, $n \in S_{\mathcal{N}}$.

From Theorem 5.3.3, $\mu = 0$ and so ϕ is conformal. Since $\lim_{n\to-\infty} U^n = \widehat{\mathbb{C}}$, ϕ is linear and since $\mathcal{T}$ and $\mathcal{T}'$ are normalized ϕ is the identity. □

5.4 Proof of Theorem I and Theorem II

Let $p > 1$. Let f be an ∞-renormalizable real quadratic polynomial with $\bar{p}_e(f) \leq p$. The first step in the proof of Theorem II is to construct a tower $\mathcal{T} \in Tow_0(m, B, C, K)$ from f.

It follows from Lemma 2.2.4 that there are m, B, C and K depending only on p, and a forward tower $\mathcal{T} = \{f_n\} \in Tow_0(m, B, C, K)$ with the following property. For $n \in S_{\mathcal{Q}}$ let $[f_n']$ be $[f_n]$ normalized and let $k(n) = |S_{\mathcal{Q}} \cap \{1, \ldots, n\}|$. Then

$$[f_n'] = \mathcal{R}^{k(n)} f.$$

Hence renormalization acts on towers by shifting. Let $\mathcal{T}_n$ denote the tower $\mathcal{T}$ shifted by n so that f_n is normalized and has index 0. By compactness there exists a limiting tower $\mathcal{T}'$ and by Theorem 1.2.2 the germ $[f_0]$ is uniquely specified by the combinatorics of $\mathcal{T}'$: a bi-infinite sequence of $\sigma \in \Omega_e^{cpt}(p)$. Hence if f has essentially period tripling combinatorics the germs $\mathcal{R}^k f$ converge to a unique germ F, which proves Theorem I.

To prove Theorem II suppose $\bar{\sigma}$ is a bi-infinite sequence of shuffles and ends in $\Omega_e^{cpt}(p)$. Let $\sigma_n = \pi_n(\bar{\sigma})$ denote the n-th element of $\bar{\sigma}$. For each σ_n let $\sigma_{m,n}$ be a sequence in $\Omega_e(p)$ converging to σ_n. Define the sequence $\bar{\tau} \in \Pi_0^\infty \Omega_e(p)$ by

$$\bar{\tau} = (\sigma_{0,0}, \sigma_{1,-1}, \sigma_{1,0}, \sigma_{1,1}, \sigma_{2,-2}, \sigma_{2,-1}, \sigma_{2,0}, \sigma_{2,1}, \sigma_{2,2}, \ldots, \sigma_{n,-n}, \ldots, \sigma_{n,n}, \ldots)$$

and let $\bar{\tau}_n = \theta^{j(n)}(\bar{\tau})$ where θ is the left-shift operator and $j(n) = 1 + 3 + 5 + \cdots + (2n-1) + n$. Then by construction $\bar{\tau}_n \to \bar{\sigma}$. Let f be a real quadratic polynomial with shuffle sequence $\bar{\tau}$ and let $\mathcal{T}$ be a tower in $Tow(m, B, C, K)$ constructed from f. By compactness of towers let $\mathcal{T}' = \{f_n', g_n'\}$ be a limiting tower of $\mathcal{T}_{j(n)}$. Define the function $h : \Pi_{-\infty}^\infty \Omega_e^{cpt}(p) \to \mathcal{G}$ by

$$h(\bar{\sigma}) = [f_0'].$$

From Theorem 1.2.2 h is well-defined and is continuous. The other properties of h are clear.

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