Complex bounds for renormalization of one-dimensional dynamical systems

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Abstract of the Dissertation

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In the first part of this dissertation we prove complex *a priori* bounds for infinitely renormalizable real quadratic maps with essentially bounded combinatorics. This is the last missing ingredient in the problem of complex bounds for all infinitely renormalizable real quadratics. One of the corollaries is that the Julia set of any real quadratic map $z \mapsto z^2 + c$, $c \in [-2, 1/4]$, is locally connected.

In the second part we apply our techniques to critical circle maps to extend E. De Faria's complex *a priori* bounds to all critical circle maps with an irrational rotation number. The contracting property for renormalizations of critical circle maps

follows.

As another application of our methods we present a new proof of theorem of C. Petersen on local connectivity of some Siegel Julia sets.

Моему деду, Аграновичу Залману Сомоиловичу

Contents

Preface

This dissertation is concerned with the existence of complex *a priori* bounds for renormalizations of one-dimensional holomorphic dynamical systems. It is an adaptation of two papers, "Dynamics of quadratic polynomials: Complex bounds for real maps", written jointly with M. Lyubich, and "Complex bounds for renormalization of critical circle maps", whose earlier version has appeared as a Stony Brook preprint [Ya].

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I thank my family for their love and support.

Part I. Complex bounds for real quadratic-like maps

1. Complex bounds: Where they come from, and what they are good for: some history and statements of the results.

In the late 1970's Feigenbaum and independently Coullet and Tresser made a remarkable discovery of the "Universal Scaling Law" in transition to chaos in a one-parameter family of unimodal maps. Drawing an analogy with renormalization group methods in statistical physics, they defined a renormalization operator acting on the space of renormalizable unimodal maps, and conjectured that this operator has a hyperbolic invariant set with one-dimensional unstable foliation, and infinite-dimensional stable foliation.

Although the initial efforts to prove the conjecture were confined to the framework of real one dimensional dynamics (see [L5] for a brief review of the history of the subject), the connection with complex analytic dynamics was early realized. In particular, Epstein [E] observed the existence of an invariant analytic function space for the renormalization transformation, the *Epstein class*, and constructed periodic points of renormalization in this space. Moreover, as was later observed by Sullivan [S2, dMvS], the renormalizations of any sufficiently smooth infinitely renormalizable map are asymptotically in the Epstein class, and thus complex analysis naturally comes into play. The renormalization operator was complexified by Douady and Hubbard [DH], to explain the occurence of little copies of Mandelbrot set in various one complex parameter families of analytic maps. The complex renormalization acts on *quadratic-like maps*, which are analytic degree two branched coverings of simply connected domains $f : U \to V$ with the nesting property: $\mathrm{cl}\, U \subset V$.

In his address to ICM-86 in Berkeley Sullivan [S1] suggested a program of construction of the invariant set of quadratic-like renormalization and its stable set by means of Teichmüller theory. The program was carried out a few years later (see [S2, dMvS]). A different approach to the problem, employing the concept of geometric limits was then given by McMullen [McM2].

The stable set of an infinitely renormalizable quadratic-like map f in Sullivan's theory is its *hybrid class* $\mathcal{H}_f$. Two quadratic-like maps are hybrid equivalent (see [DH]) if they are quasiconformally conjugate on the neighborhoods of their filled Julia sets, with the conjugacy being conformal on the filled Julia sets. It has been established that all infinitely renormalizable real quadratic-like maps with the same combinatorics are contained in the same hybrid class (see [S2, dMvS, McM1, Sw2, L4]). The space $\mathcal{H}_f$ is equipped with the natural Teichmüller metric, and contains a unique quadratic polynomial. The distance from a quadratic-like map $f : U \to V$ to the polynomial map $p \in \mathcal{H}_f$ is controlled in terms of the modulus of the fundamental annulus $V \setminus U$. The key analytic point of Sullivan's argument are complex *a priori* bounds. By definition, an infinitely renormalizable map f has complex bounds if all its renormalizations $R^n f$ extend to quadratic-like maps with definite moduli of the fundamental annuli. Sullivan (see [S2, dMvS]) has shown that for an infinitely renormalizable unimodal map of Epstein class with bounded periods its sufficiently high renormalizations are quadratic-like with *a priori* bounds, which places them in a compact part of the corresponding hybrid class. He then used an elegant *non-coiling argument* to conclude that renormalization contracts the Teichmüller metric in a hybrid class. McMullen in his argument

[McM2] also used Sullivan's *a priori* bounds as a compactness condition, in the topology of geometric convergence.

The other application of complex *a priori* bounds is geometric information about the structure of Julia sets and the Mandelbrot set at an infinitely renormalizable point. One of the questions they shed light on is local connectivity of the Julia sets. This is of particular interest, since a quadratic polynomial with a locally connected Julia set admits a simple combinatorial model [Do3]. There are known examples of infinitely renormalizable (but not real) quadratic polynomials with non locally connected Julia sets (see [Sø]). Hu and Jiang [HJ] have used Sullivan's *a priori* bounds to demonstrate that the Julia set of Feigenbaum polynomial is locally connected. Their approach was later generalized by McMullen [McM1] to apply, in particular, to all real infinitely renormalizable quadratics with complex bounds.

Finally, complex bounds are related to rigidity questions. Lyubich's Rigidity theorem [L4] asserts that any two combinatorially equivalent infinitely renormalizable quadratics with complex *a priori* bounds satisfying secondary limbs condition (this condition is automatically satisfied for real maps) are the same up to an affine change of coordinates.

We hope that by now the reader is convinced that the existence of complex *a priori* bounds is one of the central issues in the dynamics of quadratic polynomials. After Sullivan [S2, dMvS] had established complex bounds for maps of bounded type, Lyubich demonstrated [L3, L4] that the map $R^n f$ has a big modulus provided the "essential period" $p_e(R^{n-1}f)$ (see §4 for the precise definition) is big. Thus the gap between [S2] and [L4] consists of quadratics of "essentially bounded but unbounded type". Loosely speaking for such maps the high renormalization periods are due to saddle-node behavior of the return maps. The first part of this thesis is based on the paper of M. Lyubich and the author [LY], where this specific phenomenon was analyzed.

Let us formulate the main result of [LY]. Given a quadratic-like map f, denote by $\mathrm{mod}(f)$ the supremum of the moduli of various fundamental annuli of f. We say that a real quadratic-like map f is *close to the cusp* if it has an attracting fixed point with the multiplier greater than $1/2$ (one can replace $1/2$ with $1-\epsilon$ for a fixed but otherwise arbitrary $\epsilon > 0$). Note that a renormalizable map has no attracting fixed points and therefore is not close to the cusp.

Theorem 1.1. *Let $f : z \mapsto z^2 + c$, $c \in \mathbb{R}$, be any n times renormalizable real quadratic polynomial, $0 \le n \le \infty$. Let*

$$\max_{1 \le k \le n-1} p_e(R^k f) \le \bar{p}_e.$$

Then

$$\mathrm{mod}(R^n f) \ge \mu(\bar{p}_e) > 0,$$

unless the last renormalization is of doubling type and $R^n f$ is close to the cusp.

This fills the above mentioned gap:

Complex Bounds Theorem. *There exists a universal constant $\mu > 0$ with the following property. Let f be any n times renormalizable real quadratic, $0 \le n \le \infty$. Then*

$$\mathrm{mod}(R^n f) \ge \mu,$$

unless the last renormalization is of doubling type and $R^n f$ is close to the cusp. In particular, infinitely renormalizable real quadratics have universal complex a priori bounds.

This result opens the possiblity to extend Sullivan's theory to maps with arbitrary combinatorics. Moreover, since the paper [LY] appeared, Lyubich [L6] has shown that the renormalization operator acting on real quadratic-like maps has a hyperbolic invariant set with a codimension one stable foliation, thus completing the proof of the Renormalization conjecture for all combinatorics.

Also, by above mentioned work of Hu and Jiang [HJ, J] and McMullen [McM2], the complex *a priori* bounds we obtained imply local connectivity of the Julia set $J(f)$ for any real infinitely renormalizable f. On the other hand, the Yoccoz Theorem gives local connectivity of $J(f)$ for at most finitely renormalizable quadratic maps (see [Hu, M1]). Thus we have

Local Connectivity Theorem. *The Julia set of any real quadratic map $z \mapsto z^2 + c$, $c \in [-2, 1/4]$, is locally connected.*

The methods of the proof of Theorem 1.1 are closer to [S2] rather than to [L4]. However, the base of Sullivan's argument, the so-called Sector Lemma (see [S2, dMvS]) does not hold for essentially bounded, but unbounded combinatorics; the pullback of the plane with two slits is not necessarily contained in a definite sector. What turns out to be true instead is that the *little Julia sets* $J(R^n f)$ are contained in a definite sector.

We will derive Theorem 1.1 from the following quadratic estimate for the renormalizations (appropriately normalized):

$$(1.1) \qquad |R^n f(z)| \geq c|z|^2,$$

with some $c > 0$ depending on the bound on the essential period. The main technical point of this work is to prove (1.1). In particular, this estimate implies that the diameters of the little Julia sets $J(R^n f)$ shrink to zero (see the discussion in §5), which already yields local connectivity of $J(f)$ at the critical point.

A quadratic-like map with a big modulus is close to a quadratic polynomial which is one of the reasons why it is important to analyze when the renormalizations have big moduli. It was proven in [L4] that $\text{mod}(Rf)$ is big if and only if f has a big essential period, which together with Theorem 1.1 implies:

Big Space Criterion. *There is a universal constant $\gamma > 0$ and two functions $\mu(p) > \nu(p) > \gamma > 0$ tending to ∞ as $p \to \infty$ with the following property. For an n times renormalizable quadratic polynomial f,*

$$\nu(p_e(R^{n-1}f)) \leq \text{mod}(R^n f) \leq \mu(p_e(R^{n-1}f)),$$

unless the n-th renormalization is of doubling type and $R^n f$ is close to the cusp.

The structure of the first part of the thesis is as follows: §2, contains some background and technical preliminaries. In §4 and §7 we describe the essentially bounded combinatorics and the related saddle-node phenomenon. In §5 we state the main technical lemmas, and derive from them our results. In §6 we give a quite simple proof of complex bounds in the case of bounded combinatorics, which will model the following argument. The proofs of the main lemmas are given in the final section, §8.

Remarks: 1. When Theorem 1.1 was proven the authors received a manuscript by Levin and van Strien [LS] with an independent proof of the Complex Bounds Theorem. The method of [LS] is quite different; instead of a detailed combinatorial analysis it is based on specific numerical estimates for the real geometry. It does not address the phenomenon of big space.

Another proof of the Complex Bounds Theorem along the same lines was recently announced by D. Sands [Sa]. The improved numerical estimates Sands obtained allowed him to greatly streamline the argument.

Also, the gap between [S2] and [L4] was independently filled by Graczyk & Swiatek [GS2]. The method of the latter work is specifically adopted to essentially bounded but unbounded combinatorics. Note also that a related analysis of the big space phenomenon for real quadratics was independently carried out in [GS1].

2. All the above results will actually be proven for maps of Epstein class $\mathcal{E}_\lambda$ (see §5). In this case the quadratic-like extension with a definite modulus (independent of λ) appears after skipping first $N = N(\lambda)$ renormalization levels.

2. Preliminaries

2.1. General notations and terminology. Let $\mathbb{D}_r = \{z : |z| < r\}$.

We use $|J|$ for the length of an interval J, dist and diam for the Euclidean distance and diameter in $\mathbb{C}$. Notation $[a, b]$ stands for the closed interval with endpoints a and b without specifying their order.

We call two real numbers a and b *K-commensurable* or simply *commensurable* if $K^{-1} \leq |a|/|b| \leq K$ for some $K > 1$. Two sets X and Y in $\mathbb{C}$ are K-commensurable, if their diameters are. A configuration of points $x_1, \dots x_n$ is called *K-bounded* if any two intervals $[x_i, x_j]$, and $[x_k, x_l]$ are K-commensurable.

Given a univalent function ϕ in a domain $U \subset \mathbb{C}$, the *distortion* of ϕ is defined as $\sup_{z,\zeta \in U} \log |\phi'(z)/\phi'(\zeta)|$.

We say that an annulus A has a *definite modulus* if $\operatorname{mod} A \geq \delta > 0$, where δ may also depend only on the specified combinatorial bounds.

For a pair of intervals $I \subset J$ we say that I is contained *well inside* of J if for any of the components $L \subset J \setminus I$, $|L| \geq K|I|$ where the constant $K > 0$ may depend only on the specified quantifiers.

A smooth interval map $f : I \to I$ is called *unimodal* if it has a single critical point, and this point is an extremum. A C^3 unimodal map is called *quasi-quadratic* if it has negative Schwarzian derivative, and its critical point is non-degenerate.

Given a unimodal map f and a point $x \in I$, x' will denote the dynamically symmetric point, that is, such that $fx' = fx$. Notation $\omega(z) \equiv \omega_f(z)$ means as usual the limit set of the forward orbit $\{f^n z\}_{n=0}^{\infty}$.

Set $Q_c(z) = z^2 + c$.

2.2. Hyperbolic disks. Given an interval $J \subset \mathbb{R}$, let $\mathbb{C}_J \equiv \mathbb{C} \setminus (\mathbb{R} \setminus J)$ denote the plane slit along two rays. Let $\overline{\mathbb{C}_J}$ denote the completion of this domain in the path metric in $\mathbb{C}_J$ (which means that we add to $\mathbb{C}_J$ the banks of the slits).

By symmetry, J is a hyperbolic geodesic in $\mathbb{C}_J$. Consider the *hyperbolic neighborhood* of J of radius r that is the set of all points in $\mathbb{C}_J$ whose hyperbolic distance to J is less than r. One verifies directly that a hyperbolic neighborhood is the union of two Euclidean discs symmetric to each other with respect to the real axis, with a common chord J. We will denote such a neighborhood $D_\theta(J)$, where $\theta = \theta(r)$ is the outer angle the boundaries of the two discs form with the real axis (an elementary computation yields $r = \log\tan(\pi/2 - \theta/4)$). Note, that in particular the Euclidean disc $D(J) \equiv D_{\pi/2}(J)$ can be interpreted as a hyperbolic neighborhood of J.

These hyperbolic neighborhoods were introduced into the subject by Epstein [E] and Sullivan [S2]. They are a key tool for getting complex bounds due to the following version of the Schwarz Lemma:

Schwarz Lemma. *Let us consider two intervals $J' \subset J \subset \mathbb{R}$. Let $\phi : \mathbb{C}_J \to \mathbb{C}_{J'}$ be an analytic map such that $\phi(J) \subset J'$. Then for any $\theta \in (0, \pi)$, $\phi(D_\theta(J)) \subset D_\theta(J')$.*

Let $J = [a, b]$. For a point $z \in \bar{\mathbb{C}}_J$, the *angle between z and J*, $\widehat{(z, J)}$ is the least of the angles between the intervals $[a, z]$, $[b, z]$ and the corresponding rays $(a, -\infty]$, $[b, +\infty)$ of the real line, measured in the range $0 \leq \theta \leq \pi$.

The following consequence of the Schwarz Lemma will provide us a key to control the inverse branches expansion.

Lemma 2.1. *Under the circumstances of the Schwarz Lemma, assume that ϕ admits a univalent extension $(\mathbb{C}_T, T) \to (\mathbb{C}_{T'}, T')$, where both components of $T \setminus J$ have length $2\rho|J|$. Let us consider a point $z \in \mathbb{C}_J$ such that $\widehat{(z, J)} \geq \epsilon$.*

Then

$$\frac{\operatorname{dist}(\phi z, J')}{|J'|} \le C \frac{\operatorname{dist}(z, J)}{|J|}$$

for some constant $C = C(\rho, \epsilon)$.

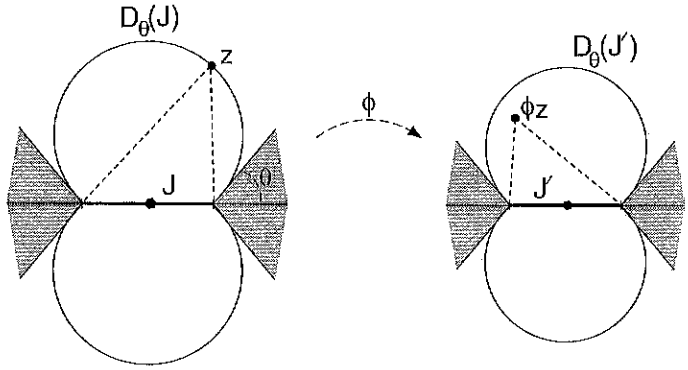


FIGURE 1

Proof. Let us normalize the situation in such a way: $J = J' = [0, 1]$. Since the space of univalent maps normalized at two points is compact (by the Koebe Theorem), the statement is true if $\operatorname{dist}(z, J) \le \rho$. So assume that $\operatorname{dist}(z, J) \ge \rho$.

Observe that the smallest (closed) geodesic neighborhood $\operatorname{cl} D_\theta(J)$ enclosing z satisfies: $\operatorname{diam} D_\theta(J) \le C(\epsilon) \operatorname{dist}(z, J)$ (cf Figure 1). Indeed, if $\theta \ge \epsilon/2$ then $\operatorname{diam} D_\theta(J) \le C(\epsilon)$, which is fine since $\operatorname{dist}(z, J) \ge \rho$.

Otherwise the intervals $[0, z]$ and $[1, z]$ cut out segments of angle size at least ϵ on the circle $\partial D_\theta(J)$. Hence the lengths of these intervals are commensurable with $\operatorname{diam} D_\theta(J)$ (with a constant depending on ϵ). On the other hand, these lengths are at most $(1 + \rho^{-1}) \operatorname{dist}(z, J)$, provided that $\operatorname{dist}(z, J) \ge \rho |J|$.

Together with the Schwarz Lemma this yields:

$$\operatorname{dist}(\phi z, J') \le \operatorname{diam}(D_{\theta'}(J')) \le \operatorname{diam}(D_\theta(J)) \le C(\rho, \epsilon) \operatorname{dist}(z, J),$$

and the claim follows. $\square$

2.3. Elementary properties of roots. We summarize here for future reference some elementary facts about the square root and the cube root maps. First, let $\varphi(z) = \sqrt{z}$ be the branch of the square root mapping the slit plane $\mathbb{C} \setminus \mathbb{R}_-$ into itself.

Lemma 2.2. *Let $K > 1$, $\delta > 0$, $K^{-1} \le a \le K$, $T = [-a, 1]$, $T' = [0, 1]$. Then:*

- $\varphi(D_\theta(T) \setminus \mathbb{R}_-) \subset D_{\theta'}(T')$, *with θ' depending on θ and K only.*
- *If $z' \in \varphi D(T) \setminus D([-\delta, 1+\delta])$, then*

$$\widehat{(z', T')} > \epsilon(\delta) > 0 \quad \text{and} \quad C(K,\delta)^{-1} < \operatorname{dist}(z', T') < C(K, \delta).$$

Lemma 2.3. *Let $\zeta \in \mathbb{C}$, $J = [a, b] \subset [0, +\infty)$. $\zeta' = \varphi(\zeta)$, $J' = [a', b'] = \varphi J$. Then:*

- *If $\operatorname{dist}(\zeta, J) > \delta|J|$ then*

$$\frac{\operatorname{dist}(J', \zeta')}{|J'|} < C(\delta)\frac{\operatorname{dist}(J, \zeta)}{|J|}.$$

- *Let θ denote the angle between $[\zeta, a]$ and the ray of the real line which does not contain J; η' denote the angle between $[\zeta', b']$ and the corresponding ray of the real line. If $\theta \le \pi/2$ then $\eta' \ge \pi/4$.*

(According to our convention, in the last statement we do not assume that $a < b$.)

Now let $\phi(z)$ be the branch of the cube root mapping the slit plane $\mathbb{C} \setminus \mathbb{R}_-$ into $\{z : |\arg(z)| < \pi/3\}$. The next Lemma is parallel to Lemma 2.2, but since its conclusion is somewhat more involved we choose to illustrate it in Figure 2.

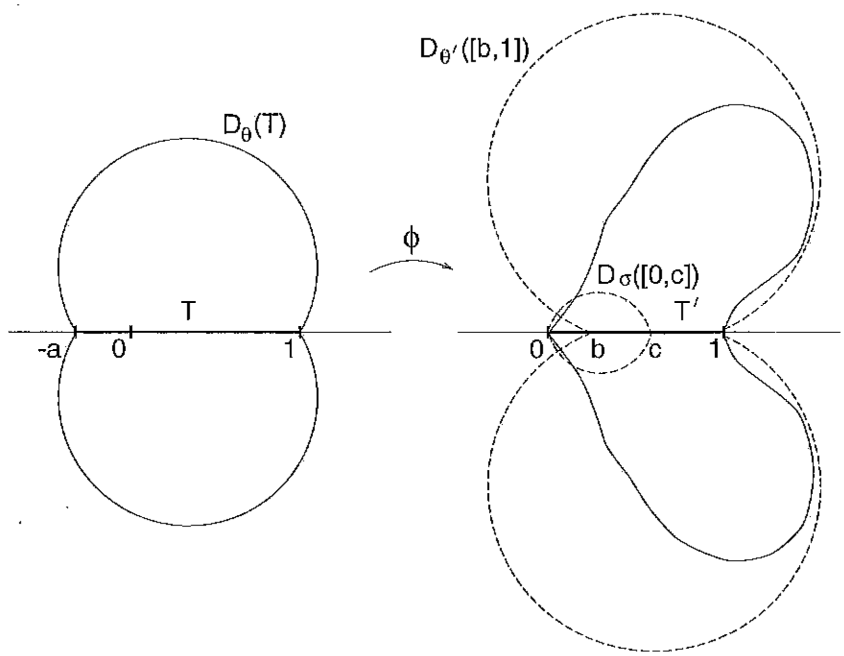


FIGURE 2

Lemma 2.4. *let $K > 1$, $\delta > 0$, $K^{-1} \leq a \leq K$, $T = [-a, 1]$, $T' = [0, 1]$. Then:*

- *$\phi D_\theta(T) \subset D_{\theta'}(T')$, with θ' depending on θ and K only.*
- *Moreover, there exist $b, c \in [0, 1]$, such that $0, b, c, 1$ form a $C(K)$-bounded configuration, and $\phi D_\theta(T) \subset D_{\theta'}([b, 1]) \cup D_\sigma([0, c])$ for $\sigma < \pi/2$.*
- *If $z' \in \phi D(T) \setminus D([-\delta, 1 + \delta])$, then*

$$\widehat{(z', T')} > \epsilon(K, \delta) > 0 \text{ and } C(K, \delta)^{-1} < \operatorname{dist}(z', T') < C(K, \delta).$$

3. Renormalization of unimodal maps in the Epstein class

3.1. Branched coverings. Let $0 \in U' \subset U \subset \mathbb{C}$ be two topological disks different from the whole plane, and $f : U' \to U$ be an analytic double branched covering map with critical point at 0. Let $\mathcal{B}$ denote the space of such double branched coverings.

For $f \in \mathcal{B}$, the *filled Julia* set $K(f)$ is naturally defined as the set of non-escaping points of f, $K(f) = \cap_{n \geq 0} f^{-n} U$, and the Julia set is defined as its boundary, $J(f) = \partial K(f)$. These sets are not necessarily compact and may change as the map is restricted to a smaller topological disk V' (such that this restriction is still a map of class $\mathcal{B}$). The Julia set (and the filled Julia set) are connected if and only if the critical point itself is non-escaping, $0 \in K(f)$.

If additionally $\mathrm{cl}\, U' \subset U$ then the map f is called *quadratic-like*. If the Julia set $J(f)$ of a quadratic-like map is connected then it does not change as the map is restricted to a smaller domain V' (such that this restriction is still quadratic-like), see [McM1], Theorem 5.11. Moreover, the Julia set of a quadratic-like map is compact, and this is actually the criterion for admitting a quadratic-like restriction:

Lemma 3.1 (compare [McM2], **Proposition 4.10).** *Let $U' \subset U$ be two topological disks, and $f : U' \to U$ be a double branched covering with non-escaping critical point and compact Julia set. Then there are topological discs $U \supset V \supset V' \supset K(f)$ such that the restriction $g : V' \to V$ is quadratic-like. Moreover, if* $\mathrm{mod}(U \setminus K(f)) \geq \epsilon > 0$ *then* $\mathrm{mod}(V \setminus V') \geq \delta(\epsilon) > 0$.

Proof. Let us consider the topological annulus $A = U \setminus K(g)$. Let $\phi : A \to R = \{z : 1 < |z| < r\}$ be its uniformization by a round annulus. It conjugates g to a map $G : R' \to R$ where R' is a subannulus of R with the same inner boundary, unit circle S^1. As G is proper near the unit circle, it is continuously extended to it, and then can be reflected to the symmetric annulus. We obtain a double covering map $\hat{G} : \hat{R}' \to \hat{R}$ of the symmetric annuli preserving the circle. Moreover $\hat{R}$ is a round annulus of modulus at least 2ϵ.

Let l denote the hyperbolic metric on $\hat{R}$, $\hat{V}$ denote the hyperbolic 1-neighborhood of S^1, and $\hat{V}' = \hat{G}^{-1}\hat{V} \subset \hat{V}$. As $\hat{G} : S^1 \to S^1$ is a double covering, we have:

$$2l(S^1) = \int_{S^1} \|Df(z)\| dl \leq \max_{S^1} \|Df(z)\| l(S^1),$$

so that $\max_{S^1} \|Df(z)\| \geq 2$. As $\operatorname{mod} \hat{R} \geq 2\epsilon$, $l(S^1) \leq L(\epsilon)$. Hence $\|Df(z)\| \geq \rho(\epsilon) > 1$ for all $z \in \hat{V}$. It follows that $\hat{V}'$ is contained in $(1/\rho(\epsilon))$-neighborhood of S^1. But then each component of $V \setminus V'$ is an annulus of modulus at least $\delta(\epsilon) > 0$.

We obtain now the desired domains by going back to U: $V = \phi^{-1}\hat{V} \cup K(f)$, $V' = \phi^{-1}\hat{V}' \cup K(f)$. $\square$

Let us supply the space $\mathcal{B}$ of double branched coverings with the *Carathéodory topology* (see [McM1]). Convergence of a sequence $f_n : U_n' \to U_n$ in this topology means Carathéodory convergence of $(U_n, 0)$ and $(U_n', 0)$, and compact-open convergence of f_n.

3.2. Epstein class. Let us consider a quasi-quadratic interval map $f : I = [\beta, \beta'] \to I$ with $f(\beta') = f(\beta) = \beta$, where β is a non-attracting fixed point: $f'(\beta) \geq 1$. By definition, f belongs to Epstein class (see [E, S2]) if it admits an analytic extension to a double branched covering $f : U' \to U$ such that $U = \mathbb{C}_T$ and U' is an $\mathbb{R}$-symmetric topological disk meeting the real line along an interval T' containing I. (For reasons which will become clear in §3.3 we do not assume that $T' \subset T$.) Any map f in Epstein class admits a representation

$$f(z) = (\phi(z))^2 + c \equiv Q_c \circ \phi, \tag{3.1}$$

where $\phi : U' \to \Delta(\phi)$ is a univalent map onto the complex plane with four slits, which double covers $\mathbb{C}_T$ under the quadratic map $z \mapsto z^2 + c$. As the range $\Delta(\phi)$ is determined by T and c, we will also denote it as $\Delta_{T,c}$.

For purely notational convenience we will also assume the maps f of the Epstein class to be even: $f(z) = f(-z)$. Then the map ϕ is odd, and the intervals I, T' and the domain U' are symmetric about 0. Moreover, the interval T and hence the domain $U = \mathbb{C}_T$ can also be assumed symmetric: just shrink T to make it symmetric and adjust T' accordingly.

Remark. Of course, all the maps of Epstein class associated to a quadratic map (restricted iterates of a quadratic map) are automatically symmetric. To carry the argument through in the non-symmetric case, one should just observe that the dynamical involution $z \mapsto z'$, $f(z) = f(z')$, has bounded distortion on compact subsets of U.

Let $\mathcal{E}$ stand for the Epstein class modulo affine conjugacy (that is, rescaling of I). We will always normalize $f \in \mathcal{E}$ so that 0 is its critical point. Given a $\rho > 1$, let $\mathcal{E}^\rho \subset \mathcal{E}$ denote the space of maps of Epstein class modulo affine conjugacy such that $|T| > \rho|I|$.

Lemma 3.2. *For any $\rho > 0$, the space $\mathcal{E}^\rho$ is Carathéodory compact.*

Proof. Let us normalize a map f in $\mathcal{E}^\rho$ so that $I = [-1, 1]$, $\beta = 1$. Then $T \supset [-\rho, \rho]$. Moreover, since the modulus of the topological annulus $\mathbb{C}_T \setminus I$ is at most twice the modulus of $\mathbb{C}_{T'} \setminus I$, $T' \supset [-\rho', \rho']$ with $\rho' > 1$ depending only on ρ. Since the critical value c divides T into two intervals of length at least $\rho - 1$, the range $\Delta(\phi) \equiv \Delta_{T,c}$ covers the disk $\mathbb{D}_r$ with $r = \sqrt{\rho - 1}$.

Let us now have a sequence $f_n = Q_{c_n} \circ \phi_n$ of normalized maps of Epstein class (3.1). Clearly we can select a subsequence such that the slit domains $\mathbb{C}_{T_n}$ and $\Delta(\phi_n)$ Carathéodory converge respectively to some $\mathbb{C}_T$ and $\Delta_{T,c}$, where $T \supset [-\rho, \rho]$ and $\Delta_{T,c} \supset \mathbb{D}_r$.

Moreover, since $Q_c(\phi_n I) \subset [-1,1]$, we have: $|\phi_n I| \leq 2\sqrt{2}$, so that we can make $\phi_n I$ converge to some interval $J = [-a, a]$. This interval is contained in $\Delta_{T,c}$, since the intervals $\phi_n I$ are well inside $\Delta(\phi_n)$.

Since $\phi_n(\beta) \leq \sqrt{2}$ and $f_n'(\beta) \geq 1$, $\phi_n'(\beta)$ stays away from 0. So, the points $\phi_n\beta \to a$ stay definite distance from the boundary of $\Delta_{T,c}$ and $(\phi_n^{-1})'(\phi_n\beta)$ are bounded from above. By the Koebe Theorem, the family of univalent maps ϕ_n^{-1} is normal on $\Delta_{T,c}$.

Let us select a subsequence ϕ_n^{-1} uniformly converging on compact subsets of $\Delta_{T,c}$. Since $\phi_n I$ are intervals of bounded length staying away from the boundary of $\Delta_{T,c}$, the limit of the ϕ_n^{-1} is non-constant, and hence is a univalent function ϕ^{-1}. It follows that the domains U_n' of the maps ϕ_n Carathéodory converge to $U' = \phi^{-1}\Delta_{T,c}$.

Let us now observe that by the Koebe Theorem, the sequence of direct functions ϕ_n is normal on any domain $\Omega \supset I$ compactly contained in U'. Indeed, this is a family of univalent functions bounded on I, with the derivatives $\phi_n'(\beta)$ bounded away from 0 (since $f_n'(\beta) \geq 1$). It follows that $\phi_n \to \phi$ uniformly on compact sets of U'.

Since $c_n \to c$, we conclude that $f_n \to Q_c \circ \phi$. □

The above proof also yields:

Lemma 3.3. *Given a $\rho > 1$, there is a domain $O_\rho \supset [-1, 1]$ with the following* *property. For any $f \in \mathcal{E}^\rho$ normalized so that $I = [-1, 1]$, the univalent map ϕ in (3.1) is well-defined and has bounded distortion on O_ρ. Moreover, in scale ϵ the distortion of ϕ is bounded by $C(\rho)\epsilon$.*

We will refer to the above property by saying that f is a *quadratic map up to bounded distortion.* The last statement (which certainly follows from the Koebe Distortion Theorem) shows, in particular, that in some scale ϵ depending only on ρ the distortion of ϕ is bounded by 2.

We will mostly be concerned with a subset of Epstein maps specified by a stronger condition. Given a $\lambda \in (0, 1)$, let $\mathcal{E}_\lambda \subset \mathcal{E}$ be space of maps of Epstein class modulo affine conjugacy such that $T' \subset T$ and each component J of $T \setminus T'$ is λ^{-1}-commensurable with T'. Note that there exists $\lambda \in (0, 1)$, such that all real quadratics Q_c, $c \in [-2, 1/4]$, belong to the Epstein class $\mathcal{E}_\lambda$ (with T selected as a fixed large 0-symmetric interval).

Lemma 3.4. *Given a $\lambda \in (0, 1)$, let $f \in \mathcal{E}_\lambda$ and $[-1, 1] = I \subset T' \subset T$ be as above,*

- *the space $\mathcal{E}_\lambda$ is Carathéodory compact;*
- *both T and T' are $K(\lambda)-$commensurable with I, and I is contained well inside T';*
- *denote by J_i^n, $i = 1, 2$ the components of $f^{-n}(T \setminus T')$. If f is not close to the cusp then $|J_i^n|$ is $K(\lambda)$-commensurable with $\text{dist}(J_j^n, \partial I)$.*

Proof. As $\mathcal{E}_\lambda$ is a closed subset of some $\mathcal{E}^{\rho(\lambda)}$, the first property follows from Lemma 3.2.

Furthermore, there exists $\mu = \mu(\lambda) > 0$ such that the annulus $A = D(T) \setminus D(T')$ has modulus at least μ. Since $\text{mod}(f^{-n}A) = 2^{-n}\mu(\lambda)$, there exist

$K_n = K_n(\lambda)$ such that $|J_i^n| \geq K_n|T \setminus T'|$. Using the fact that both components of $T \setminus T'$ are λ^{-1}-commensurable with $|T'|$ we have $|J_i^n| > L_n|T'|$. As J_i^1 are contained in $T' \setminus I$, $|I|/|T'|$ is bounded from above.

Set $T'' = [\gamma', \gamma] = (f|_{\mathbb{R}})^{-1}T'$ where γ lies on the same side of 0 as the fixed point $\beta = 1$. Commensurability of the J_i^0 with T' and Koebe Distortion Theorem imply that ϕ in (3.1) has a $C(\lambda)$-bounded distortion on this interval. Hence $|T'| \geq |fT''| \asymp |T''|^2$. It follows that $|T''|/|T'| \to 0$ as $|T'| \to \infty$. Since J_i^2 is commensurable with T'', the length of T' must be bounded. Thus I is commensurable with T' (and T).

To prove the last statement, let us consider the interval $S = [\beta/2, \gamma]$. Bounded distortion of $\phi|S$ and elementary distortion properties of the quadratic map imply that f has bounded distortion on S.

By compactness of $\mathcal{E}_\lambda$, for $f \in \mathcal{E}_\lambda$ which is not close to the cusp the multiplier of β is bounded away from 1. Moreover, we can take a point $a \in (\beta, \gamma)$ which divides (β, γ) into $K(\lambda)$-commensurable parts and such that $f'(x) \geq q = q(\lambda) > 1$ for $x \in (\beta, a)$. Let $J^n \equiv J_1^n$ stand for the intervals lying on the side of the fixed point $\beta \in \partial I$. Then only bounded number of intervals J^n may be outside (β, a). For the rest of them, $|J^n| \geq (q-1)\operatorname{dist}(J^n, \beta)$ which finishes the proof of the lemma. $\square$

In what follows all maps are assumed to belong to the Epstein class $\mathcal{E}$.

3.3. Renormalization. Let us briefly recall the notion of renormalization in unimodal dynamics; for a detailed account the reader is referred to [dMvS].

A quasi-quadratic unimodal map f is called *renormalizable* if there exists an $n \geq 2$ and a closed interval P around the critical point 0, such that $f^n(P) \subset P$,

$f^n|P$ is unimodal, and the intervals $P, f(P), f^2(P), \ldots, f^{n-1}(P)$ have disjoint interiors. The smallest such n is called the *period* of f, and is denoted n_1. The corresponding periodic interval P^1 is not canonically defined: the minimal choice is $P^1 = [f^{n_1}(0), f^{2n_1}(0)]$; and the maximal choice is $P^1 = B^1 = [\beta_1, \beta_1']$, where β_1 is the appropriate fixed point of $f_1 \equiv f^{n_1}$. By definition, the *renormalization* Rf is equal to $f^{n_1}|P^1$ up to the choice of P^1 and rescaling. To be definite, we will assume that it is normalized so that B^1 is rescaled to $[-1, 1]$:

$$Rf(z) = q^{-1} f^{n_1}(qz), \text{ where } q = \beta_1.$$

If this procedure can be repeated $0 \le k \le \infty$ times, the map f is called *k times renormalizable.* In this case there is a nested sequence of periodic renormalization intervals around 0, $P^1 \supset P^2 \supset \ldots \supset P^k$. We let n_l denote the period of P^l, and let P_i^l be the component of $f^{-(n_l - i)} P^l$ containing $f^i 0$. We say that the intervals P_i^l, $i = 0, 1, \ldots, n_l - 1$, form the *cycle of level l*. We use notation f_l for the iterate f^{n_l}. Again, the maximal choice for P^l is $P^l = B^l = [\beta_l, \beta_l']$, where β_l is fixed under f_l.

Let $p_l = n_l / n_{l-1}$ be the relative periods, $\bar{p}_k(f) = \max_{1 \le l \le k} p_l$. We say that an infinitely renormalizable map f has *bounded combinatorics* if the sequence of relative periods is bounded.

If $n_1 = 2$ then f is called *immediately renormalizable*, and the corresponding renormalization is called *doubling*. In this case the maximal periodic intervals $P^1 = [\beta_1, \beta_1']$ and P_1^1 touch at their common fixed point β_1 (which coincides with the fixed point α of f with negative multiplier). In all other cases the periodic intervals P_i^1 are disjoint.

Besides β_l, the quasi-quadratic map f_l has one more fixed point on B^l which will be denoted by α_l. At the cusp (i.e., when $f_l'(\beta_l) = 1$) these two points coincide. Note that if $l < k$ (so that $R^l f$ is renormalizable), then $f_l'(\alpha_l) < -1$.

Let $S_1^l \supset P_1^l$ be the maximal interval such that the restriction of $f^{n_l - 1}$ to it is monotone. Set $T^l = f^{n_l-1}(S_1^l)$ and $S^l = f^{-1}(S_1^l)$. Then $f_l : S^l \to T^l$ is a unimodal map.

Let us now state some basic geometric properties of infinitely renormalizable maps usually referred to as *real bounds* (see [G, BL1, BL2, S2, dMvS] for the proofs). Below we assume that f is a k times renormalizable quasi-quadratic map of Epstein class $\mathcal{E}_\lambda$, $0 \le k \le \infty$.

Lemma 3.5. *For a quasi-quadratic map $f \in \mathcal{E}_\lambda$ as above:*

- *The interval P^k is well inside T^k and S^k. Moreover, after skipping initial $N(\lambda)$ levels, the space in between these intervals becomes absolute (i.e., independent of λ).*
- *The renormalizations $R^k f$ belong to some class $\mathcal{E}^r$ with $r = r_k(\lambda) \le r(\lambda) < 1$ which becomes absolute after skipping the initial $N(\lambda)$ levels.*
- *If α_k has negative multiplier then $S^k \subset T^k$.*
- *If $f'(\alpha_k) \le -\epsilon < 0$ then S^k is well inside T^k (with the space depending on ϵ).*

Proof. The first statement is proven in the above quoted works (see e.g., [dMvS, Lemma VI.2.1]). The second statement is the consequence of the first one.

Let us consider the component J of $S^k \setminus (\beta, \beta')$ containing β. If $S^k \supset T^k$ then f_k monotonically maps J into itself. Hence it has an attracting fixed point $\gamma \in J$ with positive multiplier. Since the critical point is attracted by the cycle of γ, γ belongs to some interval P_j^k. It follows that $R^k f$ also has an attracting fixed point with positive multiplier contradicting the assumption. This proves the third statement.

The last statement follows by compactness of $\mathcal{E}^\rho$. $\square$

Lemma 3.6. *The map $f^{n_k-i} : P_i^k \to P^k$, $0 < i < n_k$, of a non-central interval onto the central one is a diffeomorphism whose distortion is bounded by an absolute constant.*

Let G_s^k be the *gaps* of level k, that is the components of $P_j^{k-1} \setminus \cup P_i^k$. *Geometry* of f is said to be *δ-bounded* (up to level n) if there is a choice of periodic intervals P_i^k, such that for any intervals $P_i^k, G_s^k \subset P_j^{k-1}$, we have: $|P_i^k|/|P_j^{k-1}| \geq \delta$ and $|G_s^k|/|P_j^{k-1}| \geq \delta$, $k = 1, \ldots, n$. In other words, all the intervals and the gaps of level k contained in some interval of level $k-1$ are commensurable with the latter.

Let $\bar{p}$ be an upper bound on the essential periods of the first k renormalizations of f: $\bar{p}_i(f) \leq \bar{p}$, $i = 0, 1, \ldots, k$.

Theorem 3.7. *Any map f as above has a δ-bounded geometry, where δ depends only on $\bar{p}$. In particular, infinitely renormalizable maps with bounded combinatorics have bounded geometry.*

Corollary 3.8. *Let P_i^k be a non-central interval which belongs to the central interval P^{k-1}. Then the map $f^{n_k-i} : P_i^k \to P^k$ has a derivative bounded away from 0 and ∞ by constants depending on $\bar{p}$ only.*

Proof. Indeed, by Theorem 3.7, the intervals P_i^k and P^k are commensurable, while by Lemma 3.6, the map between them has a bounded distortion. $\square$

Corollary 3.9. *Let $f'(\alpha_k) \leq -\epsilon < 0$. Then S^k is $K(\bar{p})$-commensurable with T^k, and the renormalization $R^k f$ belongs to some class $\mathcal{E}_\mu$, with μ depending only on λ, $\bar{p}$ and ϵ.*

Proof. Given the last statement of Lemma 3.5, we only need to show that $|S^k|/|T^k|$ is bounded from below. But $S^k \subset P^{k-1}$, since the map $f_k = f_{k-1}^2$ is 0-symmetric and at least 3-modal on P^{k-1}. As P^{k-1} is f_k-invariant, $T^k \subset P^{k-1}$ as well. As by Theorem 3.7 P^k and P^{k-1} are $K(\bar{p})$-commensurable, we are done. □

3.4. Bounds and unbranching. Let us state a result which gives an estimate of the modulus of a quadratic-like map after one renormalization:

Theorem 3.10 ([L4], **Corollary 5.6**). *Let f be a renormalizable quadratic-like map with* $\operatorname{mod} f \geq \rho > 0$. *Then Rf is also quadratic-like, and*

$$\operatorname{mod} Rf \geq \delta(\rho) > 0$$

unless the renormalization is a doubling and Rf is close to the cusp.

A fundamental annulus A of a renormalization $R^k f$ is called *unbranched* if $A \cap \omega_f(0) = \emptyset$.

Lemma 3.11. [L4, Lemma 9.3] *Let f be an infinitely renormalizable $\mathbb{R}$-symmetric quadratic-like map with a priori bounds. Then every other renormalization $R^n f$ has an unbranched fundamental annulus with a definite modulus (depending on a priori bounds only).*

4. Essentially bounded combinatorics and geometry

Let f be a renormalizable quasi-quadratic map.

Recall that $\beta \equiv \beta_0$ and α stand for the fixed points of f with positive and negative multipliers correspondingly. Let $B \equiv B(f) = [\beta, \beta']$, $A \equiv A(f) = [\alpha, \alpha'] \subset B$.

If f is immediately renormalizable then A is a periodic interval with period 2. Otherwise let us consider the principal nest $A \equiv I^0 \equiv I^0(f) \supset I^1 \equiv I^1(f) \supset \dots$ of intervals of f (see [L3]). It is defined in the following way. Let $t(m)$ be the first return time of the orbit of 0 back to I^{m-1}. Then I^m is defined as the component of $f^{-t(m)}I^{m-1}$ containing 0. Moreover $\bigcap_m I^m = B(Rf)$.

For $m > 1$, let

$$g_m : \bigcup_i I_i^m \to I^{m-1}$$

be the *generalized renormalization* of f on the interval I^{m-1}, that is, the first return map restricted onto the intervals intersecting the postcritical set (here $I^m \equiv I_0^m$). Note that $g_m \equiv f^{t(m)} : I^m \to I^{m-1}$ is unimodal with $g_m(\partial I^m) \subset \partial I^{m-1}$, while $g_m : I_i^m \to I^{m-1}$ is a diffeomorphism for all $i \neq 0$.

Let us consider the following set of levels:

$$X = \{m : t(m) > t(m-1)\} \cup \{0\} = \{0 = m(0) < m(1) < m(2) < \dots < m(\chi)\}.$$

A level $m = m(k)$ belongs to X iff the return to level $m-1$ is *non-central*, that is $g_m 0 \in I^{m-1} \setminus I^m$. For such a moment the map $g_{m+1}|I^{m+1}$ is essentially different from $g_m|I^m$ (that is not just the restriction of the latter to a smaller domain). Let us use the notation $h_k \equiv g_{m(k)+1}$, $k = 1, \dots \chi$. The number $\chi = \chi(f)$ is called the *height* of f. (In the immediately renormalizable case set $\chi = -1$).

The nest of intervals

$$(4.1) \qquad I^{m(k)+1} \supset I^{m(k)+2} \supset \ldots \supset I^{m(k+1)}$$

is called a *central cascade.* The *length* l_k of the cascade is defined as $m(k+1)-m(k)$. Note that a cascade of length 1 corresponds to a non-central return to level $m(k)$.

A cascade (4.1) is called *saddle-node* if $h_k I^{m(k)+1} \not\ni 0$ (see Figure 3). Otherwise it is called *Ulam-Neumann.* For a long saddle-node cascade the map h_k is combinatorially close to $z \mapsto z^2 + 1/4$. For a long Ulam-Neumann cascade it is close to $z \mapsto z^2 - 2$.

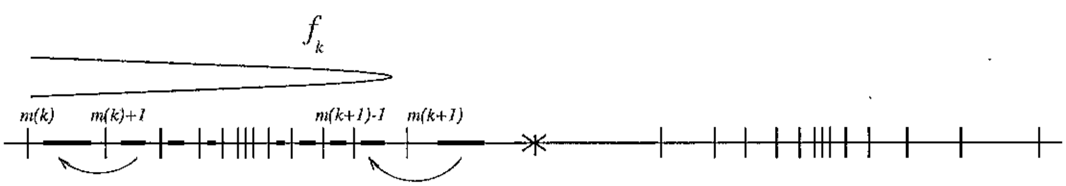


FIGURE 3. A long saddle-node cascade

Given a cascade (4.1), let

$$(4.2) \qquad K_j^{m(k)+i} \subset I^{m(k)+i-1} \setminus I^{m(k)+i}, \; i = 1, \ldots, m(k+1) - m(k) - 1,$$

denote the pullbacks of $I_r^{m(k)+1}$ under $h_k^{i-1} = g_{m(k)+1}^{i-1}$ (i.e., the connected components of the $I_r^{m(k)+1}$ under the corresponding inverse map). Clearly, $K_j^{m(k)+i+1}$ are mapped by h_k onto $K_r^{m(k)+i}$, $i = 1, \ldots, m(k+1) - m(k) - 1$, while $K_j^{m(k)+1} \equiv I_j^{m(k)+1}$ are mapped onto the whole $I^{m(k)}$. This family of intervals is called the *Markov family* associated with the central cascade.

For $x \in \omega(0) \cap (I^{m(k)} \setminus I^{m(k)+1})$ set

$$d(x) = \min\{j - m(k), m(k+1) - j\},$$

if $h_k x \in I^j \setminus I^{j+1}$ for $m(k) \leq j \leq m(k+1) - 1$ and $d(x) = 0$ otherwise (i.e., when $h_k x \in I^{m(k+1)}$). This parameter shows how deep the orbit of x lands inside the cascade. Let us now define d_k as the maximum of $d(x)$ over all $x \in \omega(0) \cap (I^{m(k)} \setminus I^{m(k)+1})$.

Given a saddle-node cascade (4.1), let us call all levels $m(k) + d_k < l < m(k+1) - d_k$ *neglectable.*

Let us now define *the essential period* $p_e = p_e(f)$. Let p be the period of the periodic interval $J = B(Rf)$. Let us remove from the orbit $\{f^k J\}_{k=0}^{p-1}$ all the intervals whose first return to some $I^{m(k)}$ belongs to a neglectable level. The essential period is the number of the intervals which are left.

We say that an infinitely renormalizable map f has *essentially bounded combinatorics* if $\sup_n p_e(R^n f) < \infty$.

Remark. Bounded essential period is equivalent to a bound on the following combinatorial factors: the height, the return times of the I_k^m to I^{m-1} under iterates of g_{m-1}, the lengths of the Ulam-Neumann cascades, and the depths d_k of landing at the saddle-node cascades.

Theorem 4.1. [L4, Theorem V] *Let $f \in \mathcal{E}_\lambda$ be a renormalizable quasi-quadratic map of Epstein class. There is $\bar{p}_\lambda > 0$ and a function $\nu_\lambda(p) \to \infty$ as $p \to \infty$ with the following property. If $p_e(f) \geq \bar{p}_e \geq \bar{p}_\lambda$ then Rf has an unbranched fundamental annulus A such that* $\text{mod}(A) \geq \nu_\lambda(\bar{p}_e)$.

Let $\sigma(f) = |B(Rf)|/|B(f)|$. Let us say that f has *essentially bounded* *geometry* if $\inf_n \sigma(R^n f) > 0$.

By the *gaps* G_j^m of level m we mean the components of $I^{m-1} \setminus \cup I_j^m$. We say that a level m is *deep inside the cascade* if $m(k)+\bar{p}_e \le m \le m(k+1)-\bar{p}_e$. The following lemma says that the maps with essentially bounded combinatorics have essentially bounded geometry (the inverse is true by Theorem 4.1).

Lemma 4.2. [L4, Lemma 8.8] *Let $f \in \mathcal{E}_\lambda$ be a renormalizable quasi-quadratic map with $p_e(f) \le \bar{p}_e$. Then all the intervals I^m in the principal nest of f are $C(\bar{p}_e, \lambda)$-commensurable. Moreover, the non-central intervals I_i^m, $i \neq 0$, and the gaps G_j^m of level m are $C(\bar{p}_e, \lambda)$-commensurable with $I^{m-1} \setminus I^m$. This is also true for the central interval I^m, provided m is not deep inside the cascade.*

Note that the last statement of the lemma is definitely false when m is deep inside a cascade: then I^m occupies almost the whole of I^{m-1}. So we observe commensurable intervals in the beginning and in the end of the cascade, but not in the middle. This is the saddle-node phenomenon which is in the focus of this work.

Corollary 4.3. *Let $f \in \mathcal{E}_\lambda$ be a renormalizable quasi-quadratic map with $p_e(f) \le \bar{p}_e$ such that Rf is not close to the cusp. Then $Rf \in \mathcal{E}^\rho$ with $\rho = \rho(\lambda, \bar{p}_e)$. If Rf has no attracting fixed points then $Rf \in \mathcal{E}_\mu$ with $\mu = \mu(\lambda, \bar{p}_e)$.*

Proof. In view of Lemmas 3.5 and 4.2 it is enough to notice that $I^{m(\chi-1)} \supset T^1 \supset S^1 \supset I^{m(\chi)}$ where $\chi = \chi(f)$ is the height of f (compare Corollary 3.9). □

The following important distortion result replaces Lemma 3.6 in the case of unbounded combinatorics:

Theorem 4.4 (see [GJ, Ma]). *For any quasi-quadratic map f, the return map $g_m : I_j^{m+1} \to I^m$ is a composition of the quadratic map $z \mapsto z^2$ and a map h with bounded distortion. Moreover, h^{-1} has a definite Koebe extension around I^m.*

The following two statements extend Corollary 3.8 and 3.9 to the case of essentially bounded combinatorics.

Corollary 4.5. *Let f be a quasi-quadratic map with $p_e(f) \le \bar{p}_e$.*

- *For a non-central interval $I_j^{m(k)+1} \subset I^{m(k)} \setminus I^{m(k)+1}$ the derivative of the restriction $h_k|_{I_j^{m(k)+1}}$ is bounded away from 0 and ∞.*
- *For any $m(k) \le l < s < m(k+1)$, which are not deep inside the cascade, the derivative of the transition map*

$$h_k^{s-l} : I^s \setminus I^{s+1} \to I^l \setminus I^{l+1}$$

is bounded away from 0 and ∞.

The constants depend only on $\bar{p}_e$.

Proof. By Lemma 4.2, any non-central interval $I_j^{m(k)+1}$ is commensurable to its distance to 0. Hence the quadratic map has bounded distortion on $I_j^{m(k)+1}$. By Theorem 4.4, the return map $g_m : I_j^{m+1} \to I^m$ has bounded distortion as well. Since its domain and range are commensurable (by Lemma 4.2 again), we see that its derivative is bounded away from 0 and ∞.

Furthermore, the Koebe Principle easily implies that the transition map along the cascade has bounded distortion. Hence by essentially bounded geometry, it must have bounded derivative. □

Corollary 4.6. *Under the assumptions of the previous Corollary, let f be renormalizable. Let $P^i \subset I^{m(k+1)} \setminus I^{m(k+2)}$ be a non-central periodic interval. Consider its first return $f^s P^i \equiv P^{i+s}$ back to $I^{m(k+1)}$. Then P^{i+s} is $K(\bar{p}_e)$-commensurable with P^i.*

This is also true for the intermediate returns to $I^{m(k)}$, that is the intervals P_{i+j} satisfying $0 < j < s$ and $f^j P^i \subset I^m \setminus I^{m+1}$ with $m(k) \leq m \leq m(k+1)+1$, provided m is not deep inside the cascade.

Proof. The first statement follows from the previous lemma.

The second statement follows in a similar way from Theorem 4.4 and the second part of Corollary 4.5. □

5. Reductions to the main lemmas

In this section we will state the Main Lemmas and will derive all the results from them. The lemmas will be proven in the following sections. As everything will be done in the setting of the Epstein class, let us start with the corresponding version of Theorem 1.1.

Theorem 5.1. *For any $\lambda \in (0,1)$ there exists $N = N(\lambda)$ with the following property. Let $f \in \mathcal{E}_\lambda$ be an $n-$times renormalizable map, $N \leq n \leq \infty$. Let*

$$\max_{1 \leq k \leq n-1} p_e(R^k f) \leq \bar{p}_e.$$

Then $R^n f$ has a quadratic-like extension with

$$\mathrm{mod}(R^n f) \geq \mu(\bar{p}_e) > 0,$$

unless the last renormalization is of doubling type and $R^n f$ is close to the cusp.

Main Lemmas Let P^k, $f_k \equiv f^{n_k}$ etc. be as in §3.3. Set $S \equiv S^0$, $T \equiv T^0$, so that $f : S \to T$ is unimodal. Let us consider the decomposition:

$$f_k = \psi_k \circ f, \tag{5.1}$$

where ψ_k is a univalent map from a neighborhood U^k of P_1^k onto $\mathbb{C}_{T^k}$.

Lemma 5.2. *Let $f : [-1,1] \to [-1,1]$ be a k times renormalizable quasi-quadratic map of Epstein class $\mathcal{E}_\lambda$. Assume that $p_e(R^l f) \le \bar{p}_e$ for $l = 0, 1, \ldots, k-1$. Then there exist $C = C(\bar{p}_e) > 0$ and $t = t(\lambda, \bar{p}_e) \in \mathbb{N}$, such that $\forall z \in D(T^t) \cap \mathbb{C}_{T_k}$ the following estimate holds:*

$$\frac{\operatorname{dist}(\psi_k^{-1} z, P_1^k)}{|P_1^k|} \le C \frac{\operatorname{dist}(z, P^k)}{|P^k|}. \tag{5.2}$$

Thus the maps ψ_k^{-1} after appropriate rescaling (that is normalizing $|P^k| = |P_1^k| = 1$) have at most linear growth depending on λ and $\bar{p}$ only. This implies, in particular, that for sufficiently big l (depending on λ and $\bar{p}_e$ only), $\psi_k^{-1}(D(T^l))$ is contained in the range where f^{-1} is the square root map up to bounded distortion (see Lemma 3.3). This yields the quadratic estimate (1.1) stated in the Introduction:

$$\frac{\operatorname{dist}(f^{n_k}(z), P^k)}{|P^k|} > c\left(\frac{\operatorname{dist}(z, P^k)}{|P^k|}\right)^2, \quad z \in \Omega^k \equiv f^{-1}\psi_k^{-1}(D(T^l)), \tag{5.3}$$

where c and l depend only on $\bar{p}$ and λ.

Corollary 5.3. *Under the circumstances of Lemma 5.2, there exists $N = N(\lambda, \bar{p}_e)$ with the following property. For any $k > N$, $f_k : P^k \to P^k$ admits a quadratic-like extension whose little Julia set is $K(\bar{p}_e)$-commensurable with the interval P^k.*

Proof. The above estimate (5.3) implies that for a sufficiently large r we have $|f^{n_k}(z)| > 2|z|$, provided $\text{dist}(f^{n_k}(z), P^k) > r|P^k|$. By real bounds there exists s depending only on λ and $\bar{p}_e$, such that $\text{dist}(\zeta, P^k) > r|P^k|$ for $k > s + l$ and any $\zeta \in \partial D(T^{k-s})$.

Set $V^k = D(T^{k-s}) \cap \mathbb{C}_{T^k}$, $\Delta^k = (f_k|\Omega^k)^{-1}V^k$. Then by the above estimate, $\partial\Delta^k$ cannot touch $\partial D(T^{k-s})$. Neither can it touch $T^{k-s} \setminus T^k$ since $\partial\Delta^k \cap \mathbb{R} = S^k$. Hence Δ^k is compactly contained in V^k, so that the restriction $f^{n_k} : \Delta^k \to V^k$ is quadratic-like. Its little Julia set is contained in $D(T^{k-s})$ which is commensurable with P^k. □

Carrying the argument for Lemma 5.2 further, we will prove the following result:

Lemma 5.4. *Under the circumstances of the previous lemma, the little Julia set $J(f_k)$ (for the quadratic-like extension of $f_k : P^k \to P^k$) is contained in the hyperbolic disk $D_\epsilon(B^k)$, where $\epsilon > 0$ depends only on $\bar{p}_e$, unless the k-th renormalization is of doubling type and $R^k f$ is close to the cusp.*

Proof of the main results.

Proof of Theorem 5.1. Choose N as in Corollary 5.3. Let us assume first that $R^k f$ does not have an attracting fixed point. Then by Lemma 3.5, B^k is well inside of T^k. Hence the hyperbolic disk $D_\epsilon(B^k)$ is well inside the slit plane $\mathbb{C}_{T^k}$. By Lemma 5.4, the Julia set $J(R^n f)$ is also well inside $\mathbb{C}_{T^k}$, and the desired follows from Lemma 3.1.

If $R^k f$ has an attracting cycle, let us go one level up. As $R^{k-1} f$ does not have attracting points, it has a definite modulus. By Theorem 3.10 its first renormalization, $R^k f$, also has a definite modulus, unless it is of doubling type and close to the cusp. □

Proof of Theorem 1.1. For $n > N$ the claim follows from Theorem 5.1. As $\mathrm{mod}\, f = \infty$ for any quadratic polynomial f, for all preceding levels $n \leq N$ we have bounds by Theorem 3.10. □

The statement of the Complex Bounds Theorem needs an obvious adjustment for maps of Epstein class (where one should skip first $N(\lambda)$ levels), or for quadratic-like maps (where the bounds depend on $\mathrm{mod}(f)$). Note also that due to the Straightening Theorem (see [DH, McM1]), the latter case follows from the quadratic one.

Proof of the Complex Bounds Theorem. By Lemma 3.5, all the renormalizations $R^m f$, $N(\lambda) \leq m < k$, belong to a class $\mathcal{E}_\theta$ with an absolute θ. Without loss of generality we can assume that $N(\lambda) = 0$ (taking into account Theorem 5.1 in the quadratic-like case).

Take a $\mu > 0$, e.g. $\mu = 1$. By Theorem 4.1, there is a $\tilde{p} = \tilde{p}(\mu)$ such that $\mathrm{mod}(Rf) \geq \mu$ for all renormalizable maps f of Epstein class $\mathcal{E}_\theta$ with $p_e(f) \geq \tilde{p}$. So we have complex bounds for all renormalizations $R^{n+1} f$ such that $p_e(R^n f) \geq \tilde{p}$. For all intermediate levels we have bounds by Theorem 3.10 and Theorem 5.1 (except perhaps for the first N levels with an absolute $N = N(\theta)$).

The latter bounds depend on $\tilde{p}$. But with the choice $\mu = 1$, $\tilde{p}$ and hence the bounds are absolute. □

Now the Complex Bounds Theorem and Lemma 3.11 yield:

Lemma 5.5. *Let $f \in \mathcal{E}_\lambda$. Then for every other level $k \geq N(\lambda)$, the renormalization $R^k f$ has an unbranched fundamental annulus with a definite modulus.*

By a *puzzle piece* we mean a topological disk bounded by rational external rays and equipotentials (compare [Hu, L4, M1]).

Proof of the Local Connectivity Theorem. By [HJ, J, McM2], unbranched *a priori* bounds imply local connectivity of the Julia set. For the sake of completeness we will supply the argument below.

For now, f is an infinitely renormalizable map of class $\mathcal{E}_\lambda$. Since quadratic-like maps (considered up to rescaling) with a priori bounds form a compact family, the Julia set $K(g)$ depends upper semi-continuously on g, and the β-fixed point depends continuously on the map (see [McM1, §4] for all these properties), the little Julia sets $J(f_k)$ are commensurable with the intervals B^k. Hence the $J(f_k)$ shrink to the critical point. By the Douady and Hubbard renormalization construction (see [Do1, L4, M2]), each little Julia set is the intersection of a nest of puzzle pieces. As each of these pieces contains a connected part of the Julia set, $J(f)$ is locally connected at the critical point.

Let us now prove local connectivity at any other point $z \in J(f)$ (by a standard "spreading around" argument). Take a puzzle piece $V \ni 0$. The set of points which never visit V, $Y_V = \{\zeta : f^n\zeta \notin V,\ n = 0, 1, \dots\}$, is expanding. (Cover this set by finitely many non-critical puzzle pieces, thicken them a bit, and use the fact the branches of the inverse map are contracting with respect to the Poincaré metric in these pieces). It follows that if $z \in Y_V$ then there is a nest of puzzle pieces shrinking to z, and we are done.

Let now $z \notin Y_V$, for any critical puzzle piece V. Take an unbranched level k.

Then there is a puzzle piece $V^k \supset J(f_k)$ with a definite space in between it and the rest of the postcritical set. Take the first moment l_k such that $f^{l_k} z \in V^k$. Then there exists a single-valued inverse branch $f^{-l_k} : \mathbb{C}_{(1+\epsilon)B^k} \to \mathbb{C}$ whose image contains z (where ϵ depends on λ only).

Furthermore, there is an $r \in (0,1)$ depending on λ only such that the hyperbolic disk Ω^k in $\mathbb{C}_{(1+\epsilon)B^k}$ of radius r (centered at 0) contains V^k. Moreover, by the Koebe Theorem this disk has a bounded shape.

Using the Koebe theorem once more, we see that the f^{-l_k} have a bounded distortion on Ω^k. Hence the pullbacks $U^k = f^{-l_k}\Omega^k$ have a bounded distortion as well. As they cannot contain a disk of a definite radius (as any disk $B(z,\epsilon)$ must cover the whole Julia set under some iterate of f), we conclude that $\operatorname{diam} U^k \to 0$. All the more, the pullbacks of the V^k under f^{-l_k} shrink. This is the desired nest of puzzle pieces about z. □

Proof of the Big Space Criterion. It follows from Theorem 4.1 and Theorem 1.1.□

6. Bounded combinatorics case

We first prove the complex bounds in the case when the map f has bounded combinatorics. The result is well-known in this case [dMvS, S2], but we give a quite simple proof which will be then generalized onto the case of essentially bounded combinatorics.

The ϵ-jumping points. Given an interval $T \in \mathbb{R}$ let $f : U' \to \mathbb{C}_T$ be a map of Epstein class.

For a point $x \in \mathbb{R} \cap U'$ which is not critical for f^n, let $V_n(x) \equiv V_n(x, f)$ denote the maximal domain containing x which is univalently mapped by f^n onto $\mathbb{C}_T$. Its intersection with the real line is the monotonicity interval $H_n(x) \equiv H_n(x, f)$ of f^n containing x. Let $f_x^{-n} : \mathbb{C}_T \to V_n(x)$ denote the corresponding inverse branch of f^{-n} (continuous up to the boundary of the slits, with different values on the different banks). If J is an interval on which f^n is monotone, then the notations $V_n(J)$ and $H_n(J)$ and f_J^{-n} make an obvious sense.

Take an $x \in \mathbb{R}$ and a $z \in \mathbb{C}_T$. If we have a backward orbit of $x \equiv x_0, x_{-1}, \dots, x_{-l}$ of x which does not contain 0, the *corresponding* backward orbit $z \equiv z_0, z_{-1}, \dots, z_{-l}$ is obtained by applying the appropriate branches of the inverse functions: $z_{-n} = f_x^{-n} z$. The same terminology is applied when we have a monotone pullback $J \equiv J_0, \dots, J_{-l}$ of an interval J.

Let $H \supset J$ be two intervals. Let $S_{\theta,\epsilon}(H, J)$ denote the union of two 2ϵ-wedges with vertices at ∂J (symmetric with respect to the real line) cut off by the neighborhood $D_\theta(H)$ (cf. Figure 4). Let $C_\epsilon(J)$ denote the complement

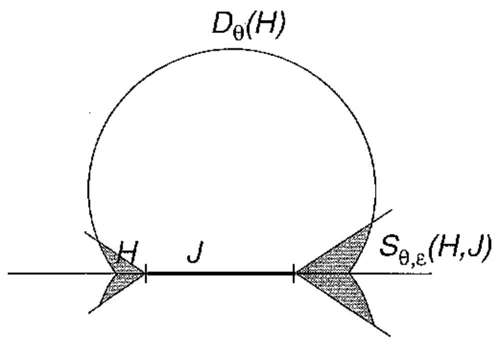


FIGURE 4

of the above two wedges (that is, the set of points looking at J at an angle at least ϵ).

Lemma 6.1. *Let f be a quadratic map. Let $J \equiv J_0, J_{-1}, \dots, J_{-l} \equiv J'$ be a monotone pullback of an interval J, $z \equiv z_0, z_{-1}, \dots, z_{-l} \equiv z'$ be the corresponding backward orbit of a point $z \in \mathbb{C}_T$. Then for all sufficiently small $\epsilon > 0$ (independent of f), either $z_{-k} \in C_\epsilon(J_{-k})$ at some moment $k \le l$, or $z' \in S_{\theta,\epsilon}(H_l(J'), J')$ with $\theta = \pi/2 - 0(\epsilon)$.*

If the first possibility of the lemma occurs we say that the backward orbit of z *ϵ-jumps.*

Proof. Assume that the backward orbit of z does not "ϵ-jump", that is, z_{-k} belongs to an $\mathbb{R}$-symmetric 2ϵ-wedge centered at $a_{-k} \in \partial J_{-k}$, $k = 0, 1 \dots, l$. By the second statement of Lemma 2.3, $f a_{-(k+1)} = a_{-k}$. Let $M_{-k} = f^{l-k} H_n(J')$, and b_{-k} be the boundary point of M_k on the same side of J_{-k} as a_{-k}. Let us take the moment k when $b_{-k} = 0$. At this moment the point z_{-k} belongs to a right triangle based upon $[a_{-k}, b_{-k}]$ with the ϵ-angle at a_{-k} and the right angle at b_{-k}. Hence $z_{-k} \in D_\theta(M_{-k})$ with $\theta = \pi/2 - 0(\epsilon)$. It follows by Schwarz Lemma that $z' \in D_\theta(H_l(J'))$, and we are done. □

In view of Lemma 3.3, the above lemma admits the following straightforward extension onto the Epstein class:

Lemma 6.2. *The conclusion of Lemma* 6.1 *still holds, provided f is a map of Epstein class $\mathcal{E}_\lambda$, and the backward orbit of z stays sufficiently close to the real line (depending on λ).*

Proof of Lemma 5.2 (for bounded combinatorics). For technical reasons we consider a new family of intervals $\tilde{S}^k$ and $\tilde{T}^k$, for which $P^k \subset \tilde{S}^k \subset S^k \subset$

$\tilde{T}^k \subset T^k$, each of the intervals is commensurable with the others and contained well inside the next one, and $f_k(\tilde{S}^k) = \tilde{T}^k$.

Let us fix a level k, and set $n \equiv n_k$,

$$J_0 \equiv P_0^k, J_{-1} \equiv P_{n-1}^k, \dots , J_{-(n-1)} \equiv P_1^k. \tag{6.1}$$

Take now any point $z_0 \in D(T^t) \cap \mathbb{C}_{T^k}$ with sufficiently big $t = t(\lambda)$. Let $z_{-1}, \dots , z_{-(n-1)}$ be its backward orbit corresponding to the above backward orbit of J_0. Our goal is to prove that

$$\frac{\operatorname{dist}(z_{-(n-1)}, J_{-(n-1)})}{|J_{-(n-1)}|} \leq C(\bar{p}) \frac{\operatorname{dist}(z_0, J_0)}{|J_0|}. \tag{6.2}$$

Take a big quantifier $\bar{K} > 0$. Let us say that s is a "good" moment of time if J_{-s} is $\bar{K}$-commensurable with J_0. For example, let $J_{-s} \subset P^l$ and $s < n_{l+1}$, that is s is a moment of backward return to P^l preceding the first return to P^{l+1}. Thus J_{-s} is contained in one of the non-central intervals $P_i^{l+1} \subset P^l$. By Corollary 3.8 we see that the moment s is good, provided $\bar{K}$ is selected sufficiently big.

We proceed inductively:

Lemma 6.3. *Let $J = J_{-s}$ and $J' = J_{-(s+n_l)}$ be two consecutive returns of the backward orbit (6.1) to a periodic interval P^l, $l < k$. Let z and z' be the corresponding points of the backward orbit of z_0. If $z \in D(\tilde{T}^l)$ then $\operatorname{dist}(z', J') \leq C(\bar{p})|\tilde{T}^l|$. Moreover, either $z' \in D(\tilde{T}^l)$, or $\widehat{(z', J')} > \epsilon(\bar{p}) > 0$.*

Proof. Let us consider the decomposition (5.1). By Lemma 3.5 the space between the intervals T and $\tilde{T}$ depends only on $\bar{p}$. Applying Koebe Distortion Theorem to the map ψ_l^{-1} we see that its distortion on $\tilde{T}^l$ is $C(\bar{p})$-bounded.

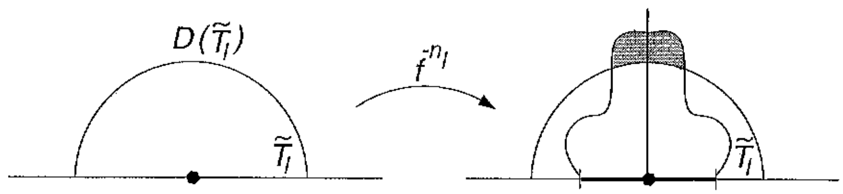


FIGURE 5

Set $\tilde{Z}^l = \psi_l^{-1}\tilde{T}^l$. By bounded geometry, the point $f_l 0$ divides $\tilde{T}^l$ into commensurable parts. Hence the critical value $f0 = \psi_l^{-1}(f_l 0)$ divides $\tilde{Z}^l$ into commensurable parts; let $A = A(\bar{p})$ stand for a bound of the ratio of these parts.

By the Schwarz lemma, domain $V = \psi_l^{-1}(D(\tilde{T}^l))$ is contained in $D(\tilde{Z}^l)$. By Lemma 3.3 and Lemma 2.2 its pullback, $f^{-1}V$ is contained in a domain $W = f^{-1}D(\tilde{Z}^l)$ intersecting the real line by $\tilde{S}^l$, with $\operatorname{diam} W \leq K(\bar{p})|\tilde{S}^l|$; moreover, $W \setminus D(\tilde{T}^l)$ is contained in a sector $Q_\epsilon(\tilde{S}^l)$ with ϵ depending only on A (see Figure 3), and thus the proof is completed.

□

Let us now give a more precise statement:

Lemma 6.4. *Let $J = J_{-s}$ and $J' = J_{-s'}$ be two returns of the backward orbit (6.1) to P^l, where $s' = s + tn_l$. Let z and z' be the corresponding points of the backward orbit of z_0. Assume $z \in D(\tilde{T}^l)$. Then either for some $0 \leq i \leq t$, a point $z_{-(s+in_l)}$ ϵ-jumps and $|z_{-(s+in_l)}| \leq C|T^l|$, or $z_{-s'} \in D_{\theta'}(H')$, where H' is the monotonicity interval of f^{tn_l} containing J', and $\theta' = \pi/2 - O(\epsilon)$.*

Proof. Assume that the above points do not ϵ-jump. Then by Lemma 6.3 they belong to the disk $D(\tilde{T}^l)$. As the map ψ_l^{-1} from (5.1) has bounded distortion, none of the points z_{-m} δ-jumps for $s \leq m \leq s'$, where $\delta = O(\epsilon)$ as $\epsilon \to 0$.

Now the claim follows from Lemma 6.1. □

The following lemma will allow us to make an inductive step:

Corollary 6.5. *Let $J = J_{-(n_l)}$, $J' = J_{-n_{l+1}}$, and z, z' be the corresponding points of the backward orbit of z_0. Assume $z \in D(\tilde{T}^{l-1})$. Then either there is a good moment $-m \in (-n_l, -n_{l+1})$ when the point z_{-m} ϵ-jumps and $|z_{-m}| \le C|T^l|$, or $z' \in D(\tilde{T}^l)$.*

Proof. Note that by bounded geometry (Corollary 3.8) all the moments

$$-n_l, -(n_l + n_{l-1}), -(n_l + 2n_{l-1}), \dots, -n_{l+1},$$

when the intervals of (6.1) return to P^{l-1} before the first return to P^{l+1}, are good (provided the quantifier $\bar{K}$ is selected sufficiently big). Hence by Lemma 6.4 either the first possibility of the claim occurs, or $z' \in D_{\theta'}(L')$, where L' is the monotonicity interval of $f^{n_{l+1}-n_l}$ containing J', and $\theta' = \pi/2 - O(\epsilon)$. As $n_{l+1} - n_l \ge n_l$, L' is contained in S^l, which is well inside $\tilde{T}^l$. Thus $D_{\theta'}(L') \subset D(\tilde{T}^l)$, provided ϵ is sufficiently small. □

We are ready to carry out the inductive proof of (6.2). Let j be the smallest level for which

$$z_0 \in D(\tilde{T}^j). \tag{6.3}$$

By Lemma 6.3, either $z_{-n_j} \in D(\tilde{T}^j)$, or z_{-n_j} ϵ-jumps. Moreover, in the latter case $|z_{-n_j}| \le C|z_0|$, and the map ψ_j^{-1} from (5.1) admits a univalent extension to $\mathbb{C}_{T^j}$. So Lemma 2.1 yields (6.2).

In the former case we will proceed inductively. Assume that either $z_{-n_l} \in D(\tilde{T}^{l-1})$, or z_{-t} ϵ-jumps at some good moment $-t \ge s$. If the latter happens, we are done. If the former happens, we pass to $l+1$ by Corollary 6.5. Lemma 5.2 is proved.

Proof of Lemma 5.4 (for bounded combinatorics). By Corollary 5.3, $\operatorname{diam} J(f_k) \le C|T^k|$, with a $C = C(\bar{p})$. Hence $J(f_k) \subset D(\tilde{T}^l)$, where $l \ge k - N(\bar{p})$. Let $\zeta' \in J(f_k)$, $\zeta = f_k\zeta'$, and $\zeta = \zeta_0, \zeta_{-1}, \dots, \zeta_{-n} = \zeta'$ be the corresponding backward orbit under iterates of f_l.

By Lemma 6.3, either ζ_{-j} ϵ-jumps at some moment, or $\zeta' \in D(\tilde{T}^k)$. If the former happens then $\zeta_{-j} \in D_\theta(J_{-jn_l})$, where $\theta = \theta(\epsilon) > 0$, and J_{-m} are the intervals from (6.1). But then by the Schwarz Lemma $\zeta' \in D_{\theta'}(P^k)$ with some θ' depending on λ and $\bar{p}$ only. Thus $J(f_k) \subset D_{\theta'}(P^k) \cup D(\tilde{T}^k)$, and we are done.

7. Saddle-node cascades

Let $f \in \mathcal{E}_\lambda$ be a map of Epstein class.

Let us note first for a long saddle-node cascade 4.1, the map $h_k : I^{m(k)+1} \to I^{m(k)}$ is a small perturbation of a map with a parabolic fixed point.

Lemma 7.1. [L4] *Let h_k be a sequence of maps of Epstein class $\mathcal{E}_\lambda$ having saddle-node cascades of length $l_k \to \infty$. Then any limit point $f : I' \to I$ of this sequence (in the Caratheodory topology) has on the real line topological type of $z \mapsto z^2 + 1/4$, and thus has a parabolic fixed point.*

Proof. It takes l_k iterates for the critical point to escape $I^{m(k)+1}$ under iterates of h_k. Hence the critical point does not escape I' under iterates of f. By the kneading theory [MT] f has on the real line topological type of $z^2 + c$ with $-2 \le c \le 1/4$. Since small perturbations of f have escaping critical point, the choice for c boils down to only two boundary parameter values, $1/4$ and -2. Since the cascades of h_k are of saddle-node type, $fI' \not\ni 0$, which rules out $c = -2$.

□

Remark 7.1. Thus the plane dynamics of h_k with a long saddle node cascade resembles the dynamics of a map with a parabolic fixed point: the orbits follow horocycles (cf. Figure 6).

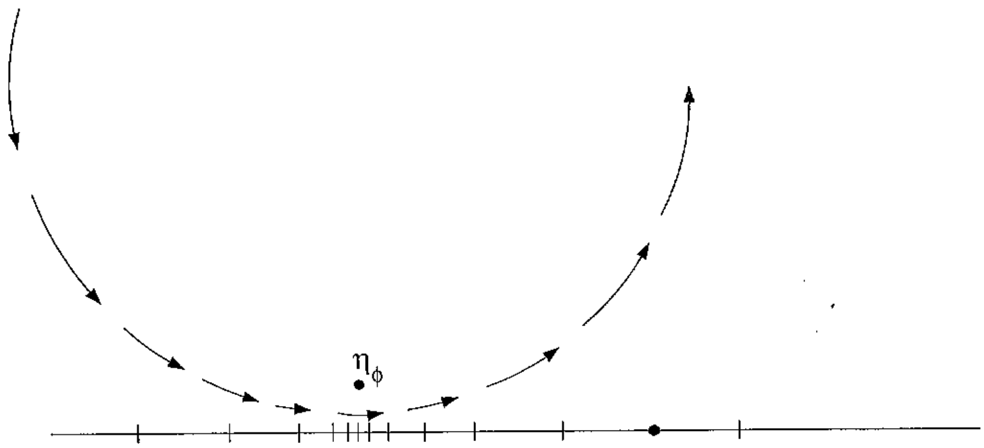


FIGURE 6. The backward trajectory of a point corresponding to a saddle-node cascade

Lemma 7.2. *Let us consider a saddle-node cascade* 4.1 *generated by a return map* h_k. *Let us also consider a backward orbit of an interval* $E \subset I^{m(k)} \setminus I^{m(k)+1}$ *under iterates of* h_k:

$$E \equiv E_0,\ E_{-1} \subset I^{m(k)+1} \setminus I^{m(k)+2}, \dots, E_{-j} \equiv E' \subset I^{m(k)+j} \setminus I^{m(k)+j+1},$$

where $m(k) + j + i \leq m(k+1)$. *Let* $z = z_0, z_{-1}, z_{-2}, \dots, z_{-j} = z'$ *be the corresponding backward orbit of a point* $z \in D(I^{m(k)})$. *If the length of the* *cascade is sufficiently big, then either* $z' \in D(I^{m(k)})$, *or* $\widehat{(z', J')} > \epsilon$ *and* $\operatorname{dist}(z', J') \le C(\bar{p})|I^{m(k)}|$.

Proof. To be definite, let us assume that the intervals E_{-i} lie on the left of 0 (see Figure 4). Without loss of generality, we can assume that $z \in \mathbb{H}$. Let $\phi = h_k^{-1}$ be the inverse branch of h_k for which $\phi E_{-i} = E_{-(i+1)}$. As ϕ is orientation preserving on $(-\infty, h_k 0]$, it maps the upper half-plane $\mathbb{H}$ into itself: $\phi(\mathbb{H}) \subset \{z = r^{ei\theta} | \ r > 0, \ \pi > \theta > \pi/2\}$.

By Lemma 7.1, if the cascade 4.1 is sufficiently long, the map ϕ has an attracting fixed point $\eta_\phi \in \mathbb{H} \cap D(I^{m(k)+2})$ (which is a perturbation of the parabolic point for some map of type $z^2 + 1/4$). By the Denjoy-Wolf Theorem, $\phi^n(\zeta) \underset{n\to\infty}{\longrightarrow} \eta_\phi$ for any $\zeta \in \mathbb{H}$, uniformly on compact subsets of $\mathbb{H}$. Thus for a given compact set $K \Subset \mathbb{H}$, there exists $N = N(K, \phi)$ such that $\phi^N(K) \subset D(I^{m(k)+1})$. By a normality argument, the choice of N is actually independent of a particular ϕ under consideration.

By Lemma 2.2 the set $K = \phi(D(I^{m(k)})) \backslash D(I^{m(k)}) \cap \mathbb{H}$ is compactly contained in $\mathbb{H}$, and $\operatorname{diam} K \le C|I^{m(k)}|$. For N as above we have $z' \in \cup_{i=0}^{N-1} \phi^i(K) \cup D(I^{m(k)})$ and the lemma is proved.

□

8. Proofs of the main lemmas

The case of essentially bounded combinatorics is more involved than the bounded case treated above (§6). Above we needed only quite rough combinatorial information in between two consecutive renormalization levels. Below we will need to pull the point more carefully through the principal nest waiting until it jumps. A difficulty arises if the jump occurs at a "bad" moment. Then the corresponding iterate of the periodic interval is deep inside of a cascade and hence is not commensurable with its original size. The analysis of saddle-node behavior given in §7 will allow us to handle this problem.

Proof of Lemma 5.2. In view of Lemma 3.5, we can assume without loss of generality that all the renormalizations $R^l f$, $l = 0, \dots, k-1$, belong to a class $\mathcal{E}_\lambda$ with an absolute λ. Let us start with a little lemma:

Lemma 8.1. *Let $f \in \mathcal{E}_\lambda$ be a map of Epstein class which is not close to the cusp. Then both components of $B \setminus A$ contain an f-preimage of 0 which divides them into commensurable parts.*

Proof. The interval $[\alpha, \beta']$ is mapped by f onto $[\beta, \alpha] \ni 0$. Denote by $\eta = f^{-1}(0) \cap [\alpha, \beta']$. Under our assumption this point is clearly different from α and β'. As the space of maps of Epstein class $\mathcal{E}_\lambda$ which are not close to the cusp is compact, η divides $[\alpha, \beta']$ into commensurable parts. The analogous statement is certainly true for the symmetric point $\eta' \in [\beta, \alpha']$. □

As in §6, let us fix a level τ, let $n = n_\tau$, and set

$$J_0 \equiv P^\tau, J_1 \equiv P^\tau_{n-1}, \dots, J_{-(n-1)} \equiv P^\tau_1. \tag{8.1}$$

Let $z \in D(T^t) \cap \mathbb{C}_{T^\tau}$ with sufficiently big $t = t(\lambda)$.

$$z \equiv z_0, z_{-1}, z_{-2}, \dots, z_{-(n-1)} \tag{8.2}$$

the backward orbit of z corresponding to the orbit (8.1). We should prove that

$$\frac{\operatorname{dist}(z_{-(n-1)}, J_{-(n-1)})}{|J_{-(n-1)}|} \le C(\bar{p}_e) \frac{\operatorname{dist}(z_0, J_0)}{|J_0|}. \tag{8.3}$$

We will proceed inductively along the principal nest. Namely, we will show below that the backward z-orbit (8.2) either ϵ-jumps at some good moment, or follows the backward J-orbit (8.1) with at most one level delay.

In what follows we work with a fixed renormalization level l and skip index l in the notations: $f \equiv f_l \equiv R^l(f_0)$, $S = S^l$, $A \equiv A^l, B \equiv B^l$. We will use notations of §4 for different combinatorial objects. Let $H_s(x)$ be the monotonicity intervals as defined in §6.

Lemma 8.2 (Return to A**).** *Let $E = E_0, E_{-1}, \dots, E_{-s} = E'$ be consecutive returns of the backward orbit (8.1) to B, between two consecutive returns to A. Let $\zeta = \zeta_0, \zeta_{-1}, \dots \zeta_{-s} = \zeta'$ be the corresponding points of the backward orbit (8.2). Assume $\zeta \in D(S)$. Then either $\zeta' \in D(B)$, or there is a moment when $-i \in [-s, 0]$ when the point ζ_{-i} ϵ-jumps: $(\widehat{\zeta_{-i}, E_{-i}}) > \epsilon(\bar{p}_e) > 0$ and moreover*

$$\frac{\operatorname{dist}(\zeta_{-i}, E_{-i})}{|E_{-i}|} \le C(\bar{p}_e)\frac{\operatorname{dist}(\zeta_0, E_0)}{|E_0|}. \tag{8.4}$$

Proof. By definition of the essential period, $s \le \bar{p}_e$. Note that the interval $f^{-1}(S)$ is contained well inside S. By Schwarz Lemma and Lemma 2.2, if a point $\zeta_{-i} \notin D(S)$, then it ϵ-jumps. Combining Lemma 2.3 and Lemma 3.3 we see that (8.4) holds up to the first moment $-i$ when ζ_{-i} ϵ-jumps.

By Lemma 8.1 each component of $B \setminus A$ contains an f-preimage of 0 which divides B into K-commensurable intervals, with $K = K(\bar{p}_e)$. Hence the monotonicity interval of f, $H = H_s(E_{-s})$, is well inside of B. As $f : B \to B$ has an extension of Epstein class $\mathcal{E}_{\mu(\lambda)}$ (Corollary 3.9), we can apply Lemma 6.2. It follows that if none of the points ζ_{-i} ϵ-jumps, then $\zeta_{-i} \in D_\theta(H)$, $0 \ge -i \ge -s$, with $\theta = \pi/2 - O(\epsilon)$. Thus $\zeta_{-s} \in D(B)$ for sufficiently small $\epsilon < \epsilon(\bar{p}_e)$, and the proof is completed. □

We say that a point/interval is deep inside of the cascade (4.1) if it belongs to $I^{m(k)+\bar{p}_e} \setminus I^{m(k+1)-\bar{p}_e}$. (In the case of essentially bounded combinatorics this cascade must be of saddle node type). Recall that a moment $-i$ is called *good* if the interval J_{-i} is commensurable with J_0. By Lemma 4.6, this happens, e.g., when for some k, the interval J_{-i} lies in $I^{m(k)} \setminus I^{m(k+1)}$ before the first entering to $I^{m(k+1)}$ but is not deep inside the corresponding cascade.

Lemma 8.3 (First return to $I^{m(1)}$). *Assume that f is not immediately renormalizable. Let $E \equiv E_0, E_{-1}, \dots, E_{-s} \equiv E'$ be the consecutive returns of the backward orbit (8.1) to A until the first return to $I^{m(1)}$. Let $\zeta \in \mathbb{C}_A \cap D(B)$, and let $\zeta \equiv \zeta_0, \zeta_{-1} \dots \zeta_{-s} \equiv \zeta'$ be the corresponding points in the backward orbit of ζ_0. Then either $\zeta' \in D(A)$, or $(\widehat{\zeta_{-i}, E_{-i}}) > \epsilon(\bar{p}_e) > 0$ and $\mathrm{dist}(\zeta_{-i}, E_{-i}) \leq C(\bar{p}_e)|B|$ at some good moment $0 \geq -i \geq -s$.*

Proof. Let $H = H_s(E_{-s})$.

As f is not immediately renormalizable, we have the interval $I^1 = [p, p']$, which is contained well inside of A by Lemma 4.2. If p is chosen on the same side of 0 as α, then $f^2[\alpha, p] \supset [\alpha, \alpha']$. Denote by η the f^2-preimage of 0 in $[\alpha, p]$. Since f is quadratic up to bounded distortion (Lemma 3.3), the map $f^2|_{[\alpha,p]}$ is quasi-symmetric (that is, maps commensurable adjacent intervals onto commensurable ones). It follows that η divides $[\alpha, p]$, and hence A, into $K = K(\bar{p}_e, \lambda)$-commensurable parts. Hence $H \subset [\eta, \eta']$ is well inside A.

By Lemma 6.2 and Lemma 8.2, either $\zeta' \in D_\theta(H)$ with $\theta = \pi/2 - O(\epsilon)$, or there is a moment $i \leq s$ such that

$$(\widehat{\zeta_{-i}, E_{-i}}) > \epsilon \quad \text{and} \quad \mathrm{dist}(\zeta_{-i}, E_{-i}) \leq C(\bar{p}_e)|B|. \tag{8.5}$$

In the former case we are done as $D_\theta(H) \subset D(A)$ for sufficiently small ϵ.

Let the latter case occur. Then we are done if the moment $-i$ is good. Otherwise E_{-i} is deep inside the cascade $A = I^0 \supset I^1 \supset \cdots \supset I^{m(1)}$. Consider the largest r such that $E_{-(i+q)} \subset I^{t+q-1} \setminus I^{t+q}$ for all $0 \le q \le r$. Note that by essentially bounded combinatorics (Corollary 4.6), the moment $-j = -(i+r)$ has to be good. By Lemma 7.2, either (8.5) occurs for ζ_{-j}, and we are done, or $\zeta_{-j} \in D(A)$.

In the latter case let $\tilde{K} \subset I^{m(1)-1} \setminus I^{m(1)}$ be the interval containing $E_{-(s-1)}$ which is homeomorphically mapped under h_1^{s-1-j} onto A (to see that such an interval exists, consider the Markov scheme described in §5). By the Schwarz lemma $\zeta_{-(s-1)} \in D(\tilde{K}) \subset D(A)$. Now the claim follows from Lemmas 2.2 and 3.3. □

Now we are in a position to proceed inductively along the principal nest: Note that the assumption of the following lemma is checked for $k = 1$ in Lemma 8.3.

Lemma 8.4 (Further returns to $I^{m(k)}$). *Let E and E' be two consecutive returns of the backward orbit (8.1) to the interval $I^{m(k)}$. Let ζ and ζ' be the corresponding points of the backward orbit of z_0. Assume that $\zeta \in D(I^{m(k-1)})$. Then, either $\zeta' \in D(I^{m(k)})$, or $\widehat{(\zeta', E')} > \epsilon(\bar{p}_e) > 0$, and* $\operatorname{dist}(\zeta', E') < C(\bar{p}_e)|I^{m(k-1)}|$.

Proof. Denote by $\tilde{E}$ the last interval in the backward orbit (8.1) between E and E', which visits $I^{m(k-1)}$ before returning to $I^{m(k)}$. Then $h_k E' = \tilde{E}$ and $h_k^{\circ j} \tilde{E} = E$ for an appropriate j.

The Markov scheme (4.2) provides us with an interval $\tilde{K} \subset I^{m(k-1)} \setminus I^{m(k)}$ containing $\tilde{E}$ which is homeomorphically mapped under $h_k^{\circ j}$ onto $I^{m(k-1)}$. By essentially bounded geometry $\tilde{K}$ is well inside $I^{m(k)-1} \setminus I^{m(k)}$. Repeating the argument of Lemma 4.6, we see that the iterate $h_k^{\circ j}$ has bounded distortion on $\tilde{K}$, and thus the critical value of h_k divides $\tilde{K}$ into commensurable parts.

Let $K' \supset E'$ be the pull-back of $\tilde{K}$ by $h_k|I^{m(k)}$. It follows that K' is contained well inside $I^{m(k)}$.

Let $\tilde{\zeta} = h_k\zeta'$ be the point of the orbit (8.2) corresponding to $\tilde{E}$. By the Schwarz lemma, $\tilde{\zeta} \in D(\tilde{K})$. By the previous remarks and Lemma 2.2, $\zeta' \in D(I^{m(k)})$, or $\widehat{(\zeta', E')} > \epsilon(\bar{p}_e)$ and $\text{dist}(\zeta', E') < C(\bar{p}_e)|I^{m(k-1)}|$. □

Lemma 8.4 is not enough for making inductive step since the jump can occur at a bad moment. The following lemma takes care of this possibility in the way similar to Lemma 8.3.

Lemma 8.5 (First return to $I^{m(k+1)}$, $k \geq 1$). *Let $E \equiv E_0, E_{-1}, \ldots, E_{-s} \equiv E'$ be the consecutive returns of the orbit (8.1) to $I^{m(k)}$ until the first return to $I^{m(k+1)}$. Let $\zeta \equiv \zeta_0, \zeta_{-1} \ldots, \zeta_{-s} \equiv \zeta'$ be the corresponding points in the backward orbit of ζ. Assume that $\zeta_{-1} \in \mathbb{C}_{I^{m(k+1)}} \cap D(I^{m(k)})$. Then either $\zeta' \in D(I^{m(k)})$, or $\widehat{(\zeta_{-i}, E_{-i})} > \epsilon(\bar{p}_e) > 0$ and $\text{dist}(\zeta_{-i}, E_{-i}) < C(\bar{p}_e)|I^{m(k)}|$ at some good moment $-1 \geq -i \geq -s$.*

Proof. Let $H \supset E'$ be the maximal interval on which $f_k^{\circ s}$ is monotone. Note, that both components of $I^{m(k)} \setminus I^{m(k)+1}$ contain pre-critical values of h_k, which divide $I^{m(k)}$ into $K(\bar{p}_e, \lambda)$-commensurable parts. Hence, H is well inside of $I^{m(k)}$.

By Lemma 6.2, either $\zeta' \in D_\theta(H)$ with $\theta = \pi/2 - O(\epsilon)$, or there is a moment $1 \le i \le s$ such that

$$(8.6) \qquad (\widehat{\zeta_{-i}, E_{-i}}) > \epsilon \quad \text{and} \quad \operatorname{dist}(\zeta_{-i}, E_{-i}) \le C(\bar{p}_e)|I^{m(k)}|.$$

In the former case we are done as $D_\theta(H) \subset D(I^{m(k)})$ if ϵ is sufficiently small.

Let the latter case occur. Then we are done if the moment $-i$ is good. Otherwise E_{-i} is deep inside the cascade $I^{m(k)} \supset I^{m(k)+1} \supset \cdots \supset I^{m(k+1)}$. Consider the largest r such that $E_{-(i+q)} \subset I^{m(k)+t+q-1} \setminus I^{m(k)+t+q}$ for all $q \le r$. Note that by essentially bounded combinatorics, the moment $-j = -(i+r)$ has to be good. By Lemma 7.2, either (8.6) occurs for ζ_{-j}, and we are done, or $\zeta_{-j} \in D(I^{m(k)})$.

In the latter case, the Markov scheme (4.2) provides us with an interval $\tilde{K} \subset I^{m(k+1)-1} \setminus I^{m(k+1)}$ containing $E_{-(s-1)}$ which is mapped homeomorphically onto $I^{m(k)}$ by h_k^{s-1-j}. By the Schwarz Lemma $\zeta_{-(s-1)} \in D(\tilde{K}) \subset D(I^{m(k)})$. The claim now follows from Lemmas 2.2 and 3.3. □

The following lemma will allow us to pass to the next renormalization level. Note that the statement is almost identical to that of Lemma 8.2. Let us now restore the label l for the renormalization level.

Lemma 8.6 (To the next renormalization level: period>2 case). *Suppose f_l is not immediately renormalizable. Let$E = E_{-1}, \ldots, E_{-r} = E', \ldots E_{-(r+s)} = E''$ be the returns of the backward orbit (8.1) to B^{l+1}, and let E', E'' be two consecutive returns to A^{l+1}. Let $\zeta = \zeta_{-1}, \ldots, \zeta', \ldots, \zeta_{-(r+s)} = \zeta''$ be the corresponding points of the backward orbit (8.2), and suppose $\zeta \in D(I^{m(\chi-1)})$, where $\chi = \chi(f_l)$ is the height of f_l. Then either $\zeta'' \in D(B^{l+1})$, or $(\widehat{\zeta_{-i}, E_{-i}}) >$*

$\epsilon(\bar{p}_e) > 0$ *and* $\mathrm{dist}(\zeta_{-i}, E_{-i}) \leq C(\bar{p})|B^{l+1}|$ *for some* $1 \leq i \leq r+s$. *Moreover, all these moments are good.*

Proof. First, $r + s \leq 2\bar{p}_e$ by definition of the essential period $\bar{p}_e$, and the last statement follows from Lemma 3.3.

By Lemma 8.4, either $(\widehat{\zeta_{-2}, E_{-2}}) > \epsilon$, $\mathrm{dist}(\zeta_{-2}, E_{-2}) \leq C(\bar{p}_e)|B^{l+1}|$, or $\zeta_{-2} \in D(I^{m(x)})$.

By the Schwarz lemma and Lemma 2.2, if $\zeta_{-i} \in D(I^{m(x)})$, then either $\zeta_{-(i+1)} \in D(I^{m(x)})$, or $\mathrm{dist}(\zeta_{-(i+1)}, E_{-(i+1)}) \leq C(\bar{p}_e)|B^{l+1}|$ and $(\widehat{\zeta_{-(i+1)}, E_{-(i+1)}}) > \epsilon(\bar{p}_e) > 0$. In the latter case we are done.

If the former case occurs for all $i < r+s$ then by Lemma 6.2, $\zeta'' \in D_\theta(H)$, where $H = H_{r+s-1}(E'', f_{l+1})$ and $\theta = \pi/2 - O(\epsilon)$. By Lemma 8.1, H is well inside B^{l+1}, and hence $D_\theta(H) \subset D(B^{l+1})$ for sufficiently small $\epsilon > 0$. □

Our last lemma takes care of the case when the map f_l is immediately renormalizable.

Lemma 8.7 (To the next renormalization level: period 2 case). *Assume that* f_l *is immediately renormalizable, so* $A^l = B^{l+1}$. *Let* $E \subset B^{l+1}$, $E \equiv E_0, E_{-1}, \ldots, E_{-s} \equiv E'$ *be the consecutive returns of the backward orbit (8.1) to* B^l, *until the first return to* A^{l+1}. *Let* $\zeta \equiv \zeta_0, \ldots, \zeta_{-s} \equiv \zeta'$ *be the corresponding points of the backward orbit (8.2). Assume also that* $\zeta \in \mathbb{C}_{A^l} \cap D(B^l)$. *Then either* $\zeta' \in D(B^{l+1})$, *or*

$$(\widehat{\zeta_{-i}, E_{-i}}) > \epsilon \quad \textit{and} \quad \mathrm{dist}(\zeta_{-i}, E_{-i}) < C(\bar{p}_e)|B'|$$

for some $0 \geq -i \geq -s$. *Moreover, all these moments are good.*

Proof. By essentially bounded combinatorics, $s \le 2\bar{p}_e$ which yields the last statement.

Further, by Lemma 8.1, the monotonicity interval $H_s(E_{-s}, f_l)$ is contained well inside of B^{l+1}, and the claim follows from Lemma 6.2. □

Let us now summarize the above information. When $f_{\tau-1}$ is immediately renormalizable, set $V_\tau = B^{\tau-1}$. Otherwise let $V_\tau = I^{m(\chi-1)}(f_{\tau-1})$ where $\chi = \chi(f_{\tau-1})$ is the height of $f_{\tau-1}$.

Lemma 8.8. *Let $f_\tau = R^\tau f$. Let us consider the backward orbit (8.1) of an interval J and the corresponding orbit (8.2) of a point z. Then there exist $\epsilon = \epsilon(\bar{p}_e) > 0$ such that either one of the points z_{-s} ϵ-jumps at some good moment, or $z_{-(n-1)} \in D(V_\tau)$.*

Proof of Lemma 4.1. If the former possibility of Lemma 8:8 occurs than Lemma 2.1 yields (8.3) (note that the assumptions of Lemma 2.1 are satisfied due to Theorem 4.4). In the latter possibility happens then

$$\frac{\operatorname{dist}(z_{-(n-1)}, J_{-(n-1)})}{|J_{-(n-1)}|} \le C(\bar{p}_e)$$

by essentially bounded geometry, and we are done again.

The lemma is proved. □

Proof of Lemma 5.4 Let us first show that $J(f_k) \subset D_\theta(S^k)$ with a $\theta = \theta(\bar{p}_e)$ (recall that $S^k \ni 0$ is the maximal interval on which f_k is unimodal).

By Corollary 5.3, $\operatorname{diam} J(f_\tau) \le C(\bar{p}_e)|B^\tau|$. Take $\zeta'' \in J(f_\tau)$. Let $\zeta' = f_\tau(\zeta'')$, $\zeta = f_\tau(\zeta')$, and $\zeta = \zeta_0, \zeta_{-1}, \ldots, \zeta_{-n} = \zeta', \ldots, \zeta_{-2n} = \zeta''$ be the corresponding backward orbit.

Let the first possibility of Lemma 8.8 occur and ζ_{-s} ϵ-jumps at a good moment for $s \leq n-1$. Then $\zeta_{-s} \in D_\delta(J_{-s})$ with $\delta = \delta(\bar{p}_e) > 0$, since $\operatorname{dist}(\zeta_{-s}, J_{-s})$ is commensurable. But then by the Schwarz lemma and Lemma 2.2, $\zeta'' \in Q_\theta(S_\tau)$ with a $\theta = \theta(\bar{p}_e) > 0$,.

Let the second possibility of Lemma 8.8 occur.

Let us first consider the case when $f_{\tau-1}$ is not immediately renormalizable. Then $\zeta' \in D(I^{\tau-1,m(\chi-1)})$. By Lemma 8.4, $\zeta'' \in D(I^{\tau-1,m(\chi)}) \subset D(S^\tau)$. Thus $J(f_\tau) \subset Q_\epsilon(S_\tau)$, and we are done.

In the case when $f_{\tau-1}$ is immediately renormalizable $\zeta' \in D(B^{\tau-1})$. Consider the interval of monotonicity of $f_{\tau-1}$, $H = H_2(\zeta'') \subset S_\tau$. By Lemma 6.2, $\zeta'' \in D_\theta(H)$ with $\theta = \pi/2 - O(\epsilon)$, and the claim follows.

Let us now show how to replace S^τ by B^τ. By essentially bounded geometry, the space $S^\tau \setminus B^\tau$ is commensurable with $|B^\tau|$ (see Corollary 4.3 and the second statement of Lemma 3.4). By the last statement of Lemma 3.4, for any $\delta > 0$, there is an $N = N(\bar{p}_e, \delta)$ such that the N-fold pull-back of S^τ by f_τ is contained in $(1+\delta)B^\tau$. By the Schwarz lemma and Lemma 2.2, $J(f_\tau) \subset D_\rho((1+\delta)B^\tau)$, with a $\rho = \rho(\delta, \bar{p}_e)$.

By the compactess Lemma 3.2, for some $\delta > 0$ (independent of τ) the map f_τ is linearizable in the $\delta|B^\tau|$-neighborhood of the fixed point β_τ. In the corresponding local chart the Julia set $J(f_\tau)$ is invariant with respect to $f'_\tau(\beta_\tau)$-dilation. Hence further pull-backs will keep it within a definite sector. $\square$

Part II. Complex bounds for critical circle maps

9. Some more history, and statements of the results

The renormalization theory for critical circle mappings has developed alongside the theory for unimodal maps. The appropriate renormalization transformation has been introduced by Feigenbaum, Kadanoff and Shenker [FKS] and also by Ostlund, Rand, Sethna and Siggia [ORSS] to explain the universality of the scaling ratios for critical circle maps with golden mean rotation number.

By definition, a critical circle mapping is a smooth orientation-preserving self-homeomorphisms of the circle $T \cong \mathbb{R}/\mathbb{Z}$ with one critical point, which is usually placed at 0. For the length of our discussion we will assume that the critical point is cubic, although all our results will be valid for any other odd degree. As a circle homeomorphism, every critical circle map f has a well-defined rotation number, which we will further denote by $\rho(f)$. Examples of critical circle maps with any given rotation number are found in the analytic family of Arnold (or standard) maps:

$$A_\theta : x \mapsto x + \theta - \frac{1}{2\pi}\sin(2\pi x).$$

The renormalization operator for critical circle maps is defined in the language of *commuting pairs* of homeomorphisms (see the next section for the precise definition), which has first appeared in [ORSS]. Commuting pairs correspond to conjugacy classes of critical circle maps, and thus posess well defined rotation numbers, on which renormalization acts as the classical number-theoretic Gauss map. As in the unimodal case, the renormalization transformation was expected to posess a hyperbolic structure (compare with [La1, La2, Ra1, Ra2]).

In its most general formulation, the hyperbolicity conjecture is due to Lanford (see [La2]), and states that the renormalization operator is globally hyperbolic with one dimensional unstable and infinite-dimensional stable foliations.

Epstein and Eckmann [EE] have observed that similarly to the interval case, the circle renormalization has an invariant space of commuting pairs of complex-analytic maps (the appropriate *Epstein class*), and presented periodic points of renormalization in this space. Again, it follows from distortion estimates, that the Epstein class contains all limits of renormalization of smooth circle maps (see, for instance, [dFdM1]). The techniques of complex analytic dynamics were brought to the subject of critical circle maps by de Faria, who transferred Sullivan's renormalization theory to this setting [dF1, dF2]. De Faria has defined a complexified renormalization operator acting on the space of *holomorphic commuting pairs* (the precise definition is somewhat involved and will be postponed until §10.2). These complex-analytic dynamical systems are analogues of quadratic-like maps in the unimodal setting which are appropriate for the application of Sullivan's methods.

The analytic cornerstone of de Faria's argument is the existence of complex *a priori* bounds for renormalization of maps in the Epstein class, which are again the bounds on the moduli of the holomorphic extensions of renormalizations of commuting pairs. Modifying Sullivan's argument, de Faria has shown the existence of complex a priori bounds for critical circle mappings of Epstein class with *bounded combinatorics.* This last condition is equivalent to rotation number $\rho(f)$ being Diophantine of order 2, that is having an infinite continued fraction expansion with bounded elements. He concluded that the renormalization operator acting on the Epstein circle maps with bounded combinatorics indeed posesses a global attractor, thus proving a part of the hyperbolicity conjecture.

It turns out that the geometric properties of critical circle maps with an arbitrary irrational rotation number are analogous to those of unimodal maps with *essentially bounded combinatorics*, which were the subject of the inverstigation in the first part of this dissertation. We were able to carry over the method developed with M. Lyubich in [LY] to the setting of critical circle maps to prove the following:

Theorem 9.1. *Let f be a critical circle map of Epstein class. Then f has complex a priori bounds.*

As in the interval case, the theorem follows from a power estimate for the appropriately rescaled renormalizations of the map f:

$$|R^k f(z)| \geq c|z|^3, \tag{9.1}$$

with a universal $c > 0$.

From Theorem 1.1 we derive the following conclusion:

Theorem 9.2. *Let f_1 and f_2 be two critical circle mappings in the Epstein class with the same irrational rotation number. Then there exists $N > 0$ such that the $n-$th renormalizations of f_1 and f_2 have holomorphic extensions which are $K-$quasiconformally conjugate with a universal bound K, for all $n > N$.*

Note that in the case of real quadratic-like maps, the existence of a bound on the moduli of fundamental annuli of two maps with the same combinatorics immediately implies the existence of a quasiconformal conjugacy with bounded

"K" (see for example [McM1]). In the circle case, however, we need an additional argument which is presented in §13. Sullivan's theory of Riemann surface laminations adapted to circle mappings by de Faria ([dF2, Chapters VIII, IX]) can now be applied to obtain the following renormalization contraction result, which complements the results of de Faria for bounded combinatorics case:

Theorem 9.3. *Let f_1 and f_2 be two critical circle maps of Epstein class with the same irrational rotation number. Then*

$$\mathrm{dist}_{C^r}(R^n f_1, R^n f_2) \to 0,$$

for all $0 \leq r < \infty$, where dist_{C^r} denotes the distance in C^r metric.

Let us also note that in a very recent work E. de Faria and W. de Melo [dFdM1, dFdM2] have enhanced the results in [dF1, dF2]. They have incorporated McMullen's approach to renormalization theory [McM2] into de Faria's theory of holomorphic pairs, proving exponential convergence of renormalization on the space of Epstein critical circle mappings of bounded type. Their work culminates in showing that any two analytic critical circle mappings with the same irrational rotation number of bounded type are $C^{1+\alpha}$-conjugate, thus proving a rigidity conjecture for such maps. The complex *a priori* bounds obtained in this dissertation open a possiblity of generalizing the results of de Faria and de Melo to the case of an arbitrary irrational rotation number.

In the quadratic-like case the existence of complex *a priori* bounds has immediate implications for local connectivity of the Julia set, as seen from the proof of Local Connectivity Theorem in the first part. Somewhat unexpectedly, we find a similar application for our methods for critical circle maps. Namely, in §14 we present a new proof of a remarkable result of C. Petersen on local connectivity of Julia sets of Siegel quadratics with rotation numbers of bounded type [P]. The structure of our proof bears a strong similarity to the argument of Hu, Jiang and McMullen presented in Part I.

10. Renormalization of critical circle maps

10.1. Real commuting pairs. We present here a brief summary on renormalization of critical circle mappings. A more detailed exposition can be found for example in [dF2].

Let f be a critical circle mapping with rotation number $\rho(f)$. If $\rho(f)$ is irrational, it can be represented as an infinite continued fraction

$$\rho(f) = [r_0, r_1, r_2, \ldots] = \cfrac{1}{r_0 + \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cdots}}}.$$

We say that ρ is of *bounded type* if $\sup r_i < \infty$, which is also equivalent to $\rho(f)$ being Diophantine of order 2.

For a critical circle map f denote by q_m the moments of closest returns of the critical point 0. The numbers q_m appear as the denominators in the irreducible form of the m-th truncated continued fraction expansion of $\rho(f)$, $\frac{p_m}{q_m} = [r_0, \ldots, r_{m-1}]$. Set $I_m \equiv [0, f^{q_m}(0)]$. As a consequence of Świątek - Herman real *a priori* bounds ([Sw1, He]), the intervals I_m and I_{m+1} are $K-$commensurable, with a universal constant K provided m is large enough.

The dynamical first return map of the union of intervals $I_m \cup I_{m+1}$ is given by f^{q_m} on I_{m+1} and by $f^{q_{m+1}}$ on I_m. The consideration of the pair of maps

$$(f^{q_{m+1}}|I_m, f^{q_m}|I_{m+1})$$

leads to the following general definition:

Definition 10.1. A *commuting pair* $\zeta = (\eta, \xi)$ consists of two smooth orientation preserving interval homeomorphisms $\eta : I_\eta \to \eta(I_\eta)$, $\xi : I_\xi \to \xi(I_\xi)$, where

- $I_\eta = [0, \xi(0)]$, $I_\xi = [\eta(0), 0]$;
- Both η and ξ have homeomorphic extensions to interval neighborhoods of their respective domains which commute, i.e. $\eta \circ \xi = \xi \circ \eta$ where both sides are defined;
- $\xi \circ \eta(0) \in I_\eta$;
- $\eta'(x) \neq 0 \neq \xi'(y)$, for all $x \in I_\eta \setminus \{0\}$, and all $y \in I_\xi \setminus \{0\}$.

The notion of renormalization acting on the space of commuting pairs first appeared in a paper by Ostlund, Rand, Sethna, and Siggia [ORSS], and was developed by Lanford and Rand (cf. [La1, La2, Ra1, Ra2]).

A *critical commuting pair* is a commuting pair (η, ξ), which maps can be decomposed as $\eta = h_\eta \circ Q \circ H_\eta$, and $\xi = h_\xi \circ Q \circ H_\xi$, where $h_\eta, h_\xi, H_\eta, H_\xi$ are real analytic diffeomorphisms and $Q(x) = x^3$. Given a commuting pair $\zeta = (\eta, \xi)$ we will denote by $\tilde{\zeta}$ the pair $(\tilde{\eta}|\widetilde{I_\eta}, \tilde{\xi}|\widetilde{I_\xi})$ where tilde means rescaling by a linear factor $\lambda = \frac{1}{|I_\eta|}$.

For a critical circle mapping f one obtains a critical commuting pair from the pair of maps $(f^{q_{m+1}}|I_m, f^{q_m}|I_{m+1})$ as follows. Let $\bar{f}$ be the lift of f to the real line satisfying $\bar{f}'(0) = 0$, and $0 < \bar{f}(0) < 1$. For each $m > 0$ let $\bar{I}_m \subset \mathbb{R}$ denote the closed interval adjacent to zero which projects down to the interval I_m. Let $\tau : \mathbb{R} \to \mathbb{R}$ denote the translation $x \mapsto x + 1$. Let $\eta : \bar{I}_m \to \mathbb{R}$, $\xi : \bar{I}_{m+1} \to \mathbb{R}$ be given by $\eta \equiv \tau^{-p_{m+1}} \circ \bar{f}^{q_{m+1}}$, $\xi \equiv \tau^{-p_m} \circ \bar{f}^{q_m}$. Then the pair of maps $(\eta|\bar{I}_m, \xi|\bar{I}_{m+1})$ forms a critical commuting pair corresponding to $(f^{q_{m+1}}|I_m, f^{q_m}|I_{m+1})$. Henceforth, we shall abuse notation and simply denote this commuting pair by

$$(f^{q_{m+1}}|I_m, f^{q_m}|I_{m+1}). \tag{10.1}$$

We see that return maps for critical circle homeomorphisms give rise to a sequence of critical commuting pairs (10.1). Conversely, regarding I_η as a circle (identifying $\xi(0)$ and 0), we can recover a smooth conjugacy class of critical circle mappings $f^\phi = \phi \circ f_\zeta \circ \phi^{-1}$, where $\phi : I_\eta \to I_\eta$ is a smooth orientation preserving homeomorphism with a cubic critical point at 0, and f_ζ is a circle homeomorphism defined by

$$f_\zeta(x) = \begin{cases} \xi \circ \eta(x), & if\ 0 \le x \le \eta^{-1}(0) \\ \eta(x), & if\ \eta^{-1}(0) \le x \le \xi(0) \end{cases} \tag{10.2}$$

We can define the *rotation number of a commuting pair* ζ to be equal to $\rho(\zeta) = \rho(f_\zeta)$. If $\rho(\zeta) = [r, r_1, r_2, \ldots]$, one verifies that the mappings $\eta|[0, \eta^r(\xi(0))]$ and $\eta^r \circ \xi|I_\xi$ again form a commuting pair.

Definition 10.2. The *renormalization* of a real commuting pair $\zeta = (\eta, \xi)$ is the commuting pair

$$\mathcal{R}\zeta = (\widetilde{\eta^r \circ \xi}|\widetilde{I}_\xi,\ \widetilde{\eta}|[0, \widetilde{\eta^r(\xi(0))}]).$$

It is easy to see that renormalization acts as a Gauss map on rotation numbers, that is if $\rho(\zeta) = [r, r_1, r_2, \ldots]$ then $\rho(\mathcal{R}\zeta) = [r_1, r_2, \ldots]$. The renormalization of the real commuting pair (10.1), associated to some critical circle map f, is the rescaled pair $(\widetilde{f^{q_{m+2}}}|\widetilde{I_{m+1}}, \widetilde{f^{q_{m+1}}}|\widetilde{I_{m+2}})$. Thus for a given critical circle map f the renormalization operator recovers the (rescaled) sequence of the first return maps: $\{(\widetilde{f^{q_{i+1}}}|\widetilde{I}_i, \widetilde{f^{q_i}}|\widetilde{I_{i+1}})\}_{i=1}^{\infty}$.

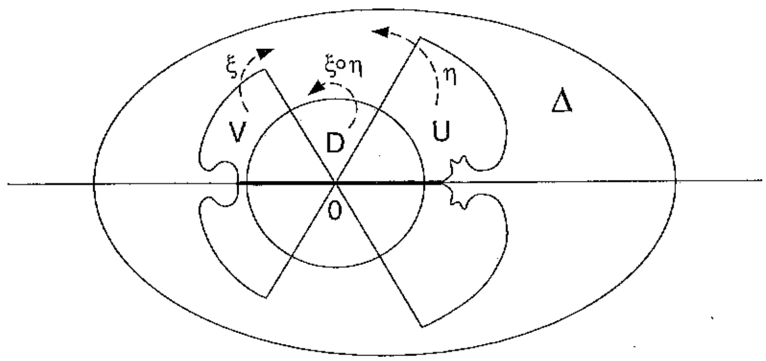


FIGURE 7

10.2. Holomorphic commuting pairs. Following [dF1, dF2] we say that a real commuting pair (η, ξ) extends to a *holomorphic commuting pair* $\mathcal{H}$ (cf. Figure 7) if there exist four $\mathbb{R}$-symmetric domains Δ, D, U, V, such that

- $\bar{D}$, $\bar{U}$, $\bar{V} \subset \Delta$, $\bar{U} \cap \bar{V} = \{0\}$; $U \setminus D$, $V \setminus D$, $D \setminus U$, and $D \setminus V$ are nonempty connected sets, $U \supset I_\eta$, $V \supset I_\xi$;
- mappings $\eta : U \to \Delta \cap \mathbb{C}_{\eta(J_U)}$ and $\xi : V \to \Delta \cap \mathbb{C}_{\xi(J_V)}$ are onto and univalent, where $J_U = U \cap \mathbb{R}$, $J_V = V \cap \mathbb{R}$;
- η and ξ have holomorphic extensions to D which commute, $\eta \circ \xi(z) = \xi \circ \eta(z)$ $\forall z \in D$; $\eta \circ \xi : D \to \Delta \cap \mathbb{C}_{\eta\circ\xi(J_D)}$, where $J_D = D \cap \mathbb{R}$, is a

three-fold branched covering with the only critical point at 0 .

Note that $J_D = (\eta^{-1}(0), \xi^{-1}(0))$.

Set $\Omega = D \cup U \cup V$. The *shadow* of the holomorphic commuting pair is the piecewise holomorphic mapping $F : \Omega \to \Delta$, given by

$$F(z) = \begin{cases} \eta(z), \ z \in U \\ \\ \xi(z), \ z \in V \\ \\ \xi \circ \eta(z), \ z \in D \setminus (U \cup V) \end{cases}$$

Proposition 10.1 (Prop. II.4. [dF2]**).** *Given a holomorphic commuting pair $\mathcal{H}$ as above, consider its shadow F. Let $I = \Omega \cap \mathbb{R}$, and $X = I \cup F^{-1}(I)$. Then:*

- *The restriction of F to $\Omega \setminus X$ is a regular three fold covering onto $\Delta \setminus \mathbb{R}$.*
- *F and $\mathcal{H}$ share the same orbits as sets.*

Definition 10.3 (Complex Bounds). We say that a critical circle map f has *complex* a priori *bounds* if there exists M and $\mu > 0$, such that for all $m > M$ the real commuting pair (10.1) extends to a holomorphic commuting pair $(\Delta_m, D_m, U_m, V_m)$ with $\mathrm{mod}(\Delta_m \setminus (D_m \cup U_m \cup V_m)) > \mu > 0$.

10.3. Epstein class. An interval map $g|I$ belongs to the *Epstein class*, if the restriction of g to I can be decomposed as $g \equiv h \circ Q$, where $Q(z) = z^3$, and $h : Q(I) \to J = g(I)$ is an orientation preserving diffeomorphism, which inverse h^{-1} extends to a univalent mapping $\mathbb{C}_{\tilde{J}} \to \mathbb{C}$, where $\tilde{J} \supset J$. Thus a map g of Epstein class can be extended to a 3-fold analytic branched covering of a domain $U \subset \mathbb{C}$ onto $\mathbb{C}_{\tilde{J}}$. Let us denote by $\mathcal{E}$ the collection of all Epstein mappings g together with their domains U, equipped with Carathéodory topology (see [McM1]). The convergence of a sequence $g_n : U_n \to \mathbb{C}_{J_n}$ in this topology is means Carathéodory convergence of the pointed domains $(U_n, 0)$, and compact-open convergence of g_n. We further denote by $\mathcal{E}_s$ the subspace of maps in $\mathcal{E}$, for which both $|I|$ and $\text{dist}(I, J)$ are s^{-1}-commensurable with $|J|$, and the length of each component of $\tilde{J} \setminus J$ is at least $s|J|$. We will often refer to the space $\mathcal{E}$ as *the* Epstein class, and to each $\mathcal{E}_s$ as *an* Epstein class.

Remark 10.1. The reader may have noticed that the above definition of the Epstein class is more restrictive than that used in Part I (a univalent map precomposed with a polynomial instead of a univalent map postcomposed with a polynomial). This is explained by some purely technical differences in the proofs, which would have unduly complicated our arguments, were we to use a more general definition.

We say that a commuting pair (η, ξ) belongs to the (an) Epstein class if both of its maps do. It immediately follows from the definitions that:

Lemma 10.2. *The space of commuting pairs in the Epstein class $\mathcal{E}$ is invariant under renormalization.*

The following compactness statement will be of key importance for us:

Lemma 10.3. *For each $s > 0$ the quotient space $\mathcal{E}_s$ modulo affine conjugacy is sequentially compact.*

Proof. Consider a sequence of maps $g_n = h_n \circ Q : U_n \to \mathbb{C}_{J_n}$ in $\mathcal{E}_s$ normalized so that $I_n = [0, 1]$. Let us ensure by passing to a subsequence that all J_n's contain some fixed subinterval T. By Koebe theorem the derivatives of the inverse maps $h_n^{-1} : J_n \to [0, 1]$ are bounded away from 0 and ∞. The maps

h_n^{-1} have univalent extensions to $\mathbb{C}_T$ which form a normal family by Koebe theorem. Any limit in this family is non-constant, and hence univalent. The convergence of the direct maps with the domains readily follows. □

The following statement is a weakened version of the convergence to an Epstein class of the renormalizations of a smooth map, discussed in [dFdM1]:

Lemma 10.4. *Let $f \in C^r$, $(r \geq 3)$ be a critical circle map with an irrational rotation number. Then the collection of real commuting pairs $(\widetilde{f^{q_{m+1}}}|_{\widetilde{I_m}}, \widetilde{f^{q_m}}|_{\widetilde{I_{m+1}}})$ is precompact in C^r topology, and all its partial limits are contained in $\mathcal{E}_s$, for some universal constant $s > 0$, independent on the original map f.*

In particular for a critical circle map $f \in \mathcal{E}$ there exists $\sigma > 0$ such that all its renormalizations are contained in $\mathcal{E}_\sigma$. Moreover, the constant σ can be chosen independent on f, after skipping first few renormalizations.

We will further only work with maps in the Epstein class. It will be shown that a renormalization of such a map has an extension to a holomorphic commuting pair with modulus bound depending only on the Epstein constants of the few previous renormalizations, and thus, in the view of the above lemma, eventually universal. Therefore we obtain universal complex *a priori* bounds for critical circle maps. Note that even for maps of bounded type this improves the complex *a priori* bounds obtained by E. de Faria, which depend also on the rotation number of the map.

11. The main lemma

Let f be an analytic map which restricts to a self-homeomorphism of the circle T. We reserve the notation $f^{-i}(z)$ for the i−th preimage of $z \in T$ under

$f|_T$.

Let f be a critical circle map. If $f \in \mathcal{E}$, it admits a restriction

$$f^{q_{n+1}} : U_n \to \mathbb{C}_{f^{q_{n+1}}(I_n)}$$

which is a three-fold branched covering. Let us decompose

$$f^{q_{n+1}} = \psi_n \circ Q, \tag{11.1}$$

where $Q(z) = z^3$, and ψ_n univalently maps $Q(U_n)$ onto $\mathbb{C}_{f^{q_{n+1}}(I_n)}$. Denote by D_m the Euclidean disc with diameter $[f^{q_{m+1}}(0), f^{q_m - q_{m+1}}(0)]$. Let $\sigma(m)$ be the smallest positive number such that $\mathcal{R}^i(f) \in \mathcal{E}_{\sigma(m)}$ for all $i \geq m$.

Lemma 11.1 (Main Lemma). *Fix integers $n > M > 0$, and let $z \in \mathbb{C}_{f^{q_{n+1}}(I_n)}$ $\cap D_M$. Then the following estimate holds:*

$$\frac{\operatorname{dist}(\psi_n^{-1} z, Q(I_n))}{|Q(I_n)|} \leq C \frac{\operatorname{dist}(z, I_n)}{|I_n|}, \tag{11.2}$$

where ψ_n is the map from (11.1), and the constant $C = C(\sigma(M))$.

Thus the maps ψ_n^{-1} after appropriate rescaling (that is normalizing $|I_n| = |Q(I_n)| = 1$) have at most linear growth. Note that if $\widehat{(z, I_n)} > \epsilon > 0$, then the inequality (11.2) follows directly from Lemma 2.1 Our strategy of proving Lemma 11.1 will be to monitor the inverse orbit of a point z together with the interval I_n until the conditions of Lemma 2.1 are satisfied.

Lemma 11.1 immediately yields the cubic estimate stated in the introduction:

(11.3)
$$\frac{\operatorname{dist}(f^{q_{n+1}}(z), f^{q_{n+1}}(I_n))}{|f^{q_{n+1}}(I_n)|} \geq c \left(\frac{\operatorname{dist}(z, I_n)}{|I_n|} \right)^3, \; f^{q_{n+1}}(z) \in D_M \cap \mathbb{C}_{f^{q_{n+1}}(I_n)},$$

with $c = c(\sigma(M))$, and thus universal for M sufficiently large. Now fixing M and choosing l sufficiently large (depending only on $\sigma(M)$) we have $\Omega_n = f^{-q_{n+1}}(D_l) \cap U_n$ is compactly contained in U_n and $\mathrm{mod}(D_l \setminus \Omega_n) > \mu = \mu(\sigma(M)) > 0$, and Theorem 1.1 follows.

For future reference let us note:

Remark 11.1. It follows from our argument that the domains Δ_m, D_m, U_m, and V_m in the definition of complex bounds (10.3) can be chosen K-commensurable with I_m, with a universal constant K, for all m sufficiently large.

12. Proof of the main lemma

Let us begin by introducing some notation. For a map $g : U \to \mathbb{C}_T$ in the Epstein class and $x \in \mathbb{R} \cap U$ which is not a critical point of g^n, let $V_n(x, g)$ denote the maximal domain containing x which is univalently mapped by g^n to $\mathbb{C}_T$. Let $g_x^{-n} : \mathbb{C}_T \to V_n(x, f)$ denote the corresponding inverse branch of g^{-n} (continuous up to the boundary of the slits, with different values on different banks). Take an $x \in \mathbb{R}$, and $z \in \mathbb{C}_T$. If we have a backward orbit of $x \equiv x_0, x_{-1}, \ldots, x_{-l}$ of x which does not contain 0, the *corresponding* backward orbit $z \equiv z_0, z_{-1}, \ldots, z_{-l}$ is obtained by applying the appropriate inverse branches: $z_{-i} = g_{x_{-i}}^{-1}(z_{-(i-1)})$. The same terminology is applied if we have a monotone pull-back of an interval $J \equiv J_0, J_{-1}, \ldots, J_{-l}$.

Let $f \in \mathcal{E}$ be a critical circle map. Fix an integer $M > 0$ and $n \geq M$. Recall from the previous section, that D_m denotes the Euclidean disc $D([f^{q_{m+1}}(0), f^{q_m - q_{m+1}}(0)])$. Consider the inverse orbit:

$$(12.1) \qquad J_0 \equiv f^{q_{n+1}}(I_n), J_{-1} \equiv f^{q_{n+1}-1}(I_n), \ldots, J_{-(q_{n+1}-1)} \equiv f(I_n)$$

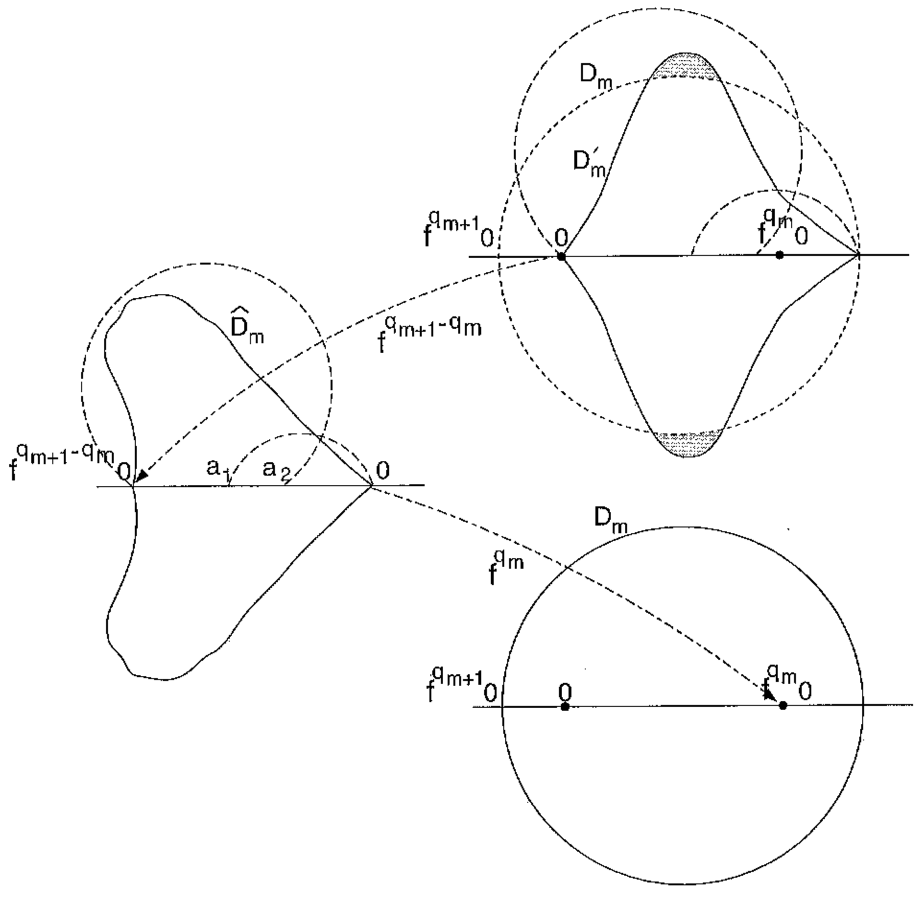
For a point $z \in D_M \cap \mathbb{C}_{f^{q_{n+1}}(I_n)}$ look at the corresponding inverse orbit

$$z_0 \equiv z, z_{-1}, \ldots, z_{-(q_{n+1}-1)} \tag{12.2}$$

We say that an element of the orbit z_{-i} *ϵ-jumps* if $\widehat{(z_{-i}, J_{-i})} > \epsilon$. Let us say that $-i$ is a *good moment* if the interval J_{-i} is commensurable with J_0 with a quantifier depending only on $\sigma(M)$. It follows from compactness of an Epstein class that the first few returns of the orbit (12.1) to each I_m with $m \geq M$ happen at good moments. We would like to argue that the points of the orbit (12.2) either ϵ-jump at a good moment, or follow closely the corresponding intervals of the orbit (12.1). The first step towards this assertion is the following Lemma:

Lemma 12.1. *Let $J \equiv J_{-k}, J_{-k-q_{m+1}} \equiv J'$ be two consecutive returns of the backward orbit* (12.1) *to I_m, for $m \geq M$, and let ζ and ζ' be the corresponding points of the orbit* (12.2)*. Suppose $\zeta \in D_m$, then either $\zeta' \in D_m$, or $\widehat{(\zeta', J')} > \epsilon$, and* $\operatorname{dist}(\zeta', J') < C|I_m|$*, where the quantifiers ϵ and C depend only on the Epstein constant $\sigma(M)$.*

Proof. Let D'_m denote the pull-back of D_m corresponding to the piece of backward orbit $J_{-k}, \ldots, J_{-k-q_{m+1}}$, and let $\hat{D}_m$ denote the pull-back of D_m along the piece of the orbit $J_{-k} \to \cdots \to J_{-k-q_m}$ (cf. Figure 8). By compactness of an Epstein class, the points 0 and $f^{q_m}(0)$ divide the interval $D_m \cap \mathbb{R}$ into $B(\sigma(M))$-commensurable pieces. By Schwarz Lemma and by Lemma 2.4, there exist points $a_1, a_2 \in [f^{q_{m+1}-q_m}(0), 0]$, such that $f^{q_{m+1}-q_m}(0), a_1, a_2$, and 0 is a $K = K(\sigma(M))$-bounded configuration, and angles θ and $\sigma < \pi/2$, also depending only on $\sigma(M)$, for which $\hat{D}_m \subset D_\theta([f^{q_{m+1}-q_m}(0), a_1]) \cup D_\sigma([a_2, 0])$.



D_m
D'_m
$f^{q_{m+1}}0$
0
$f^{q_m}0$
$\widehat{D}_m$
$f^{q_{m+1}-q_m}$
$f^{q_{m+1}-q_m}0$
a_1
a_2
0
D_m
f^{q_m}
$f^{q_{m+1}}0$
0
$f^{q_m}0$

FIGURE 8

Applying Schwarz Lemma, we obtain that $D'_m \subset D_m \cup D_\theta([0, f^{-q_{m+1}+q_m}(a_1)])$ and the claim immediately follows. $\square$

A saddle-node phenomenon. The above Lemma puts us in the position to apply Lemma 2.1 if the inverse orbit (12.2) ϵ-jumps during one of the first few (or the last few) returns of (12.1) to some I_m. There is a danger however that when q_{m+1}/q_m is large, some of the returns of the inverse orbit of J_0 to I_m are small as compared to the original size. If the orbit (12.2) ϵ-jumps at such a bad moment $-i$, although in view of the previous Lemma $\text{dist}(z_{-i}, J_{-i}) < C|I_m|$, the ratio $\text{dist}(z_{-i}, J_{-i})/|J_{-i}|$ may grow unbounded.

The situation when q_{m+1}/q_m is large is, however, well understood (see for example [He]). By Lemma 10.4, the family of rescaled restrictions $\{\widetilde{g_j^{q_{m_j+1}}}|\widetilde{I_{m_j}}\}$ of Epstein circle maps, for which $g^{q_{m+1}}$ belongs to a fixed $\mathcal{E}_s$, is pre-compact in Carathéodory topology. Any partial limit of sequence of maps in this family with $q_{m+1}/q_m \to \infty$ has a fixed point in the interval. Since this fixed point is destroyed by a small perturbation, it is necessarily parabolic, and by Schwarz lemma it is unique.

Let us now handle the possibility of a jump at a bad moment:

Lemma 12.2. *Let us consider the map $f^{q_{m+1}}|I_m$. Let $P_0, P_{-1}, \dots, P_{-k}$ be the consecutive returns of the backward orbit (12.1) to I_m, and denote by $\zeta_0, \dots, \zeta_{-k}$ the corresponding moments of the backward orbit of a point $\zeta_0 = z \in D_m$. Then either $z' \equiv \zeta_{-k} \in D_m$, or $(\widehat{z', P_{-k}}) > \epsilon$ and $\text{dist}(z', P_{-k}) \leq C|I_m|$, where $C = C(\sigma(M))$.*

Proof. To be definite, let us assume that the intervals P_{-i} lie on the left of

0. Without loss of generality we can assume that $z \in \mathbb{H}$. Let $\phi = f^{-q_{m+1}}$ be the branch of the inverse for which $\phi P_{-i} = P_{-(i+1)}$. As ϕ is orientation preserving on $(-\infty, f^{q_{m+1}+q_m}(0)]$, it maps the upper half-plane $\mathbb{H}$ into itself : $\phi(\mathbb{H}) \subset \{z = re^{i\theta} | r > 0, \pi > \theta > 2\pi/3\}$. By Denjoy-Wolf theorem, ϕ has an attracting fixed point $\eta_\phi \in \mathbb{H}$, whose basin is the whole of $\mathbb{H}$. As we have seen, for q_{m+1}/q_m large enough the map ϕ is a small perturbation of a parabolic map. A compactness argument implies that there exists $B = B(\sigma(M))$ such that the fixed point $\eta_\phi \in D(I_m) \subset D_m$ provided $q_{m+1}/q_m > B$.

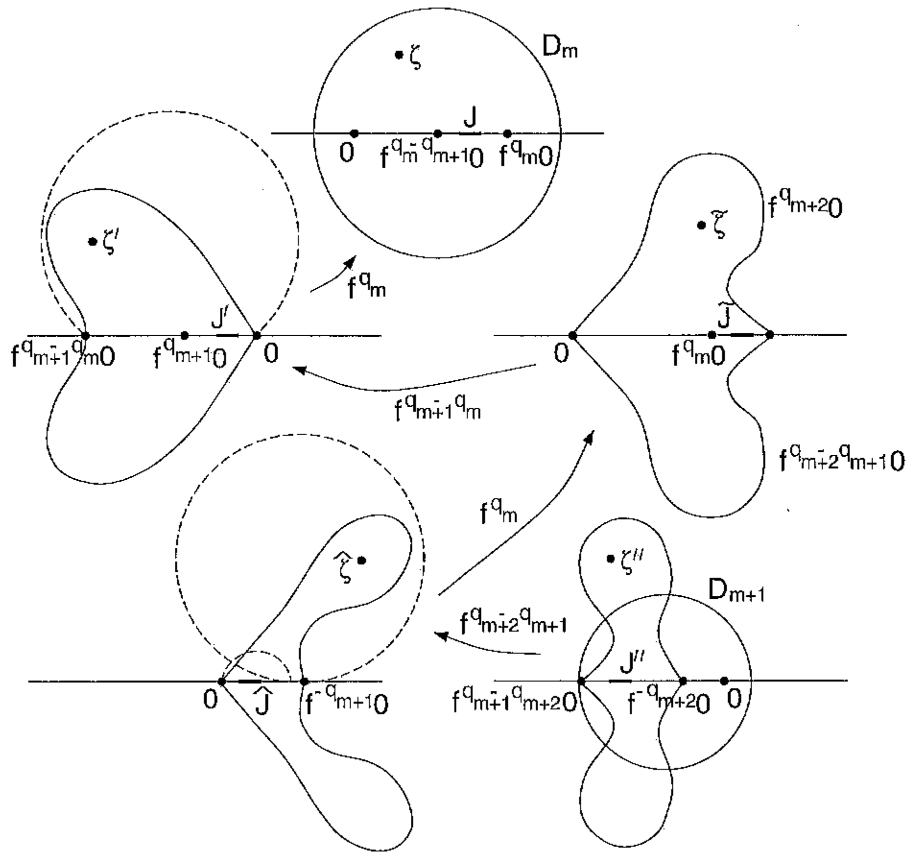
If $q_{m+1}/q_m < N$ the Lemma readily follows from Lemmas 12.1 and 2.1. Otherwise, by Denjoy-Wolf theorem the iterates of ϕ uniformly converge to η_ϕ on any compact subset $K \Subset \mathbb{H}$. By compactness, there exists $N = N(K, \sigma(M))$ such that $\phi^N(K) \subset D_m$.

Suppose $\zeta_{-i} \notin D_m$. By Lemma 12.1 the set $(D_m \setminus \phi(D_m)) \cap \mathbb{H}$ is contained in a compact region $K = K(\sigma(M)) \Subset \mathbb{H}$. For N as above we have $z' \in \cup_{i=0}^{N-1} \phi^i(K) \cup D_m$, and the lemma is proved. □

The inductive step. The next lemma provides a step of induction for our argument:

Lemma 12.3. *Let J be the last return of the backward orbit (12.1) to the interval I_m before the first return to I_{m+1}, and let J' and J'' be the first two returns of (12.1) to I_{m+1}. Let ζ, ζ', ζ'' be the corresponding moments in the backward orbit (12.2), $\zeta = f^{q_m}(\zeta')$, $\zeta' = f^{q_{m+2}}(\zeta'')$.*

Suppose $\zeta \in D_m$. Then either $(\widehat{\zeta'', I_{m+1}}) > \epsilon(\sigma(M))$ and $\operatorname{dist}(\zeta'', J'') < C(\sigma(M))|I_{m+1}|$, or $\zeta'' \in D_{m+1}$.



ζ
D_m
J
0
$f^{q_m - q_{m+1}}0$
$f^{q_m}0$
ζ'
J'
$f^{q_{m+1}-q_m}0$
$f^{q_{m+1}}0$
0
f^{q_m}
$\tilde{\zeta}$
$f^{q_{m+2}}0$
$\tilde{J}$
0
$f^{q_m}0$
$f^{q_{m+1}-q_m}$
$f^{q_{m+2}-q_{m+1}}0$
f^{q_m}
$\hat{\zeta}$
ζ''
D_{m+1}
$f^{q_{m+2}-q_{m+1}}$
J''
0
$\hat{J}$
$f^{-q_{m+1}}0$
$f^{q_{m+1}-q_{m+2}}0$
$f^{-q_{m+2}}0$
0

FIGURE 9

Proof. Note that $J \subset [f^{q_m+q_{m+1}}(0), f^{q_m}(0)]$. By Lemma 2.4, we have $\zeta' \in D_\theta([f^{q_{m+1}-q_m}(0), 0])$ for some θ depending only on the Epstein constants. Denote by $\tilde{J}$, and $\hat{J}$ the intervals of (12.1), such that $f^{q_{m+1}-q_m}(\tilde{J}) = J'$, $f^{q_m}(\hat{J}) = \tilde{J}$, and let $\tilde{\zeta}$, $\hat{\zeta}$ be the corresponding points in the orbit (12.2) (cf. Figure 9) . The interval $\tilde{J} \subset [f^{q_m}(0), f^{q_m - q_{m+1}}(0)]$, $\tilde{\zeta} \in D_\theta([f^{q_m}(0), f^{q_m-q_{m+1}}(0)])$. By Schwarz Lemma and Lemma 2.4 there are points $b_1, b_2 \in [0, f^{-q_{m+1}}(0)]$, such that $0, b_1, b_2$, and $f^{-q_{m+1}}(0)$ form a $K(\sigma(M))$-bounded configuration, and $\hat{\zeta} \in D_\delta([0, b_1]) \cup D_\gamma([b_2, f^{-q_{m+1}}(0)])$ for $\gamma = \gamma(\sigma(M))$, and $\delta = \delta(\sigma(M)) < \pi/2$. The claim now follows by Schwarz Lemma. □

Inductive argument. We start with a point $z \in D_M$. Consider the largest m such that D_m contains z. We will carry out induction in m. Let $P_0, \ldots, P_{-k}$ be the consecutive returns of the backward orbit (12.1) to the interval I_m until the first return to I_{m+1}, and denote by $z = \zeta_0, \ldots, \zeta_{-k} = \zeta'$ the corresponding points of the orbit (12.2). By Lemma 12.1 and Lemma 12.2, ζ_{-i} either ϵ-jumps at a good moment when P_{-i} is commensurable with J_0, and $\text{dist}(\zeta_{-i}, P_{-i}) \leq C|I_m|$, or $\zeta' \in D_m$.

In the former case we are done by Lemma 2.1. In the latter case consider the point ζ'' which corresponds to the second return of the orbit (12.1) to I_{m+1}. By Lemma 12.3, either $(\widehat{\zeta'', I_{m+1}}) > \epsilon$, and $\text{dist}(\zeta'', I_{m+1}) \leq C|I_{m+1}|$, or $\zeta'' \in D_{m+1}$.

In the first case we are done again by Lemma 2.1. In the second case, the argument is completed by induction in m.

Let us make several remarks:

Remark 12.1. It follows directly from the argument, that every point $z \in$

$f^{-q_{n+1}}(D_M) \cap U_n$ is either conatined in the disc $D([0, f^{q_n - q_{n-1}}(0)])$ or satisfies $\widehat{(z, I_n)} > \epsilon = \epsilon(\sigma(M))$.

Remark 12.2. In Lemma 11.1 and the estimate (11.3) the domain D_M can be replaced with $D_\alpha([f^{q_{M+1}}(0), f^{q_M - q_{M+1}}(0)])$ for some $\alpha > \frac{\pi}{2}$, following an obvious change in the argument.

13. Construction of a quasiconformal conjugacy

We proceed to prove Theorem 9.2. Let f_1, f_2 be two critical circle mappings in the Epstein class, having the same irrational rotation number ρ. We will use the subscript i to denote the objects corresponding to the mapping f_i for $i = 1, 2$. By choosing M sufficiently large, we can ensure that all renormalizations $\mathcal{R}^n f_i$ for $n \geq M$ are contained in an Epstein class $\mathcal{E}_s$ with a universal $s > 0$. Futhermore, in view of Remark 11.1, there exists a universal constant $N > 0$, such that the real commuting pair $(f_i^{q_{n+1}}, f_i^{q_n})$ extends to a holomorphic commuting pair $\mathcal{H}_i^n$ with complex *a priori* bounds (10.3), ranging over $\Delta_i^n \equiv D([f_i^{q_{n-N+1}}(0), f_i^{q_{n-N}}(0)])$, provided $n - N \geq M$. We will denote the shadow of this holomorphic pair, as defined in §10.2, by $F_i^n : \Omega_i^n \to \Delta_i^n$. The precise formulation of the statement of Theorem 9.2 is the following:

For F_i^n as above, there exists a $K-$quasiconformal $\mathbb{R}$-symmetric homeomorphism $H^n : \Delta_1^n \to \Delta_2^n$, conjugating the piecewise-holomorphic maps F_1^n and F_2^n, with a universal dilatation bound K.

By a theorem of Herman [He] there exists a K_1-quasisymmetric map

$$h : [f_1^{q_{n-N+1}}(0), f_1^{q_{n-N}}(0)] \to [f_2^{q_{n-N+1}}(0), f_2^{q_{n-N}}(0)]$$

conjugating the real commuting pairs $(f_1^{q_{n+1}}, f_1^{q_n})$ and $(f_2^{q_{n+1}}, f_2^{q_n})$, with a universal bound on K_1. We will show that this map can be extended to a quasiconformal conjugacy with the required properties.

As we work with a fixed value of n, let us omit the superscript n where possible, to simplify the notation. The mapping h can be extended to an $\mathbb{R}$-symmetric quasiconformal homeomorphism $\psi : \Delta_1 \to \Delta_2$ whose dilatation bound depends only on K_1 and therefore is universal. As follows from Proposition 10.1, the map ψ can be lifted to an $\mathbb{R}$-symmetric mapping $\tilde{\psi} : \Omega_1 \to \Omega_2$, such that $F_2 \circ \tilde{\psi} = \psi \circ F_1$, which is still an extension of h. The Theorem 9.2 will follow via the standard pull-back argument (as presented in [dF2, pp. 15-17]) from the following statement:

Proposition 13.1 (Quasiconformal Interpolation). *There exists a K-quasiconformal map $H_1(z) : \Delta_1 \to \Delta_2$, with a universal K, satisfying the following:*

$$H_1(z) = \begin{cases} \psi(z), \textit{ if } z \in \partial\Delta_1 \\ \tilde{\psi}(z), \textit{ if } z \in \Omega_1 \end{cases}$$

Let us comment on the nature of difficulties we are faced with in proving the Proposition. Recall first that in the case of quadratic-like maps, complex *a priori* bounds imply the existence of a fundamental annulus with modulus bounded below, both of whose boundary curves are quasicircles (see for example [McM1]). The existence of a quasiconformal interpolation between two such annuli is standard.

Consider, however, a maximal extension $\mathcal{H} : \Omega \to \Delta$ of a real commuting pair $(f^{q_{m+1}}|I_m, f^{q_m}|I_{m+1})$. It is easy to see that the intersection $\Omega \cap \mathbb{R}$ is the interval $T = [f^{q_{m-1}-q_m}(0), f^{q_m - q_{m+1}}(0)]$, and its image $T' = [f^{q_{m-1}}(0), f^{q_m}(0)] \subset T'$. Thus on the real line we see exactly the opposite to what happens in the complex plane: a region which is compactly contained in its preimage. This precludes us from finding a fundamental annulus for a holomorphic commuting pair.

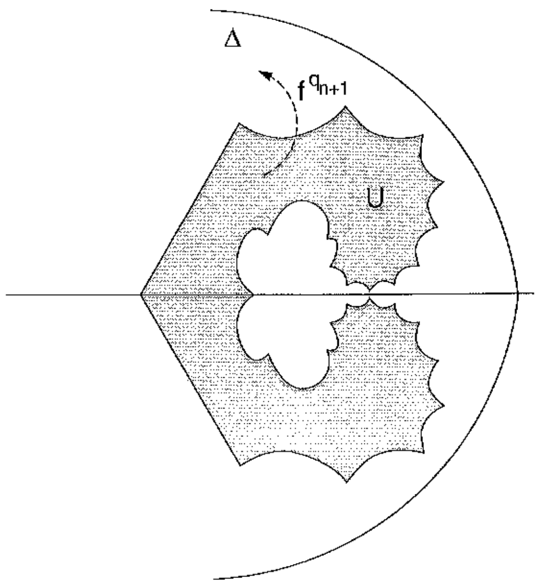


FIGURE 10. The domains Δ and U when q_{n+1}/q_n is large.

Instead we should construct the quasiconformal interpolation in the fundamental regions $(\Delta_i \setminus \Omega_i) \cap \pm\mathbb{H}$. This does not create a problem in the case of ρ of bounded type, as in this case these regions are $K(B)$-quasidiscs, where B is the bound on q_{n+1}/q_n (see [dF1, dF2]). As seen before, if we select the rotation number ρ, for which the ratio q_{n+1}/q_n is large, the restriction $f_i^{q_n}|I_{n-1}$ is a small perturbation of a parabolic map. Following the logic of Lemma 12.2, we see that for the corresponding holomorphic pair, the domain U_i is pinched in a neighborhood of the ghost of the parabolic point (see Figure 10). Thus the bound on the quasicircle $(\Delta_i \setminus \Omega_i) \cap \pm\mathbb{H}$ spoils, and we require a new argument to show the existence of the quasiconformal interpolation.

We will establish Proposition 13.1 by analysing in some detail the geometry of domains Ω_i and showing that the pinching occurs in the same way for all such domains.

Some auxiliary lemmas. Let $f \in \mathcal{E}$ be a critical circle mapping. By the main result of §12, the pair of maps $(f^{q_{n+1}}|I_n, f^{q_n}|I_{n+1})$ extends to a holomorphic commuting pair $\mathcal{H}$ with domains Δ, D, U, and V, commensurable with I_n, and having complex bounds for all n sufficiently large.

Consider the inverse orbit:

$$(13.1) \qquad J_0 \equiv f^{q_{n+1}}(I_n), J_{-1} \equiv f^{q_{n+1}-1}(I_n), \ldots , J_{-(q_{n+1}-1)} \equiv f(I_n),$$

and the corresponding inverse orbit for the domain $\Delta_0 \equiv \Delta \cap \mathbb{H}$:

$$(13.2) \qquad \Delta_0, \Delta_{-1}, \ldots , \Delta_{q_{n+1}-(-1)} \equiv f(U) \cap \mathbb{H}.$$

We make several observations.

Lemma 13.2. *Let $J' \equiv J_{-k}, J'' \equiv J_{-k-q_m}$ be two consecutive returns of the backward orbit (13.1) to the interval I_{m-1} before the first return to I_m, and denote by Δ', Δ'' the corresponding moments in the orbit (13.2). Then*

$$f^{q_m}(\partial\Delta'' \cap \mathbb{H}) = (\partial\Delta' \cap \mathbb{H}) \cup [f^{q_{m-1}-q_m}(0), f^{q_{m-1}}(0)].$$

The following two Lemmas are illustrated by Figure 11.

Lemma 13.3. *Let $J' = J_{-q_{m+1}}$ and $J'' = J_{-2q_{m+1}}$ be the first two returns of the backward orbit (13.1) to the interval I_m, and let Δ', Δ'' be the corresponding* *domains in the orbit* (13.2)*. Then for any* $z \in \Delta''$,

$$\widehat{(z, P)} > \epsilon,$$

for some fixed angle $\epsilon > 0$*, where* $P = [f^{-q_{m+1}}(0), f^{q_m - q_{m+1}}(0)] = \Delta'' \cap \mathbb{R}$.

Proof. It follows from Remark 12.1 that the domain Δ' is contained in $W = D_\theta([0, f^{q_m - q_{m-1}}(0)])$ for some fixed θ. Note that the smallest iterate of the critical point 0 contained in W is $f^{q_n(0)}$. The claim now follows from Schwarz Lemma and Lemma 2.4. □

Lemma 13.4. *Let* $J = J_{-q_{m+1}+q_{m-1}}$ *be the last return of the backward orbit* (13.1) *to the interval* I_{m-1} *before the first return to* I_m*, and denote again by* J' *and* J'' *the first two returns of the orbit* (13.1) *to* I_m*. Let* Δ*,* Δ' *and* Δ'' *be the corresponding domains in the orbit* (13.2)*. Let* $\zeta'' \in \partial\Delta''$*, and let* $f^{q_{m+1}}(\zeta'') = \zeta' \in \partial\Delta'$*,* $f^{q_{m-1}}(\zeta') = \zeta \in \partial\Delta$*.*

Suppose, that $\zeta \in \mathbb{H}$*. Then,*

$$\operatorname{dist}(\zeta'', I_m) > K|I_m|.$$

Proof. Note, that the disk $D([f^{q_{m+1}-q_m}(0), f^{q_m}(0)])$ is pulled back univalently along the piece of the inverse orbit (13.1)

$$J_0, J_{-1}, \dots, J_{-q_{m+1}+q_{m-1}} = J.$$

By Świątek- Herman real *a priori* bounds, we can choose an interval

$$[f^{q_{m+1}}(0), f^{q_m+q_{m+1}}(0)] \subset T_0 \subset [f^{q_{m+1}-q_m}(0), f^{q_m}(0)]$$

such that each of the intervals is contained well inside the next one. Let T, and $D \subset \Delta$ denote the pullbacks of the interval T_0 and the disk $D(T_0)$ correspondingly along the orbit $J_0, J_{-1}, \dots, J_{-q_{m+1}+q_{m-1}} = J$. By Koebe distortion theo-

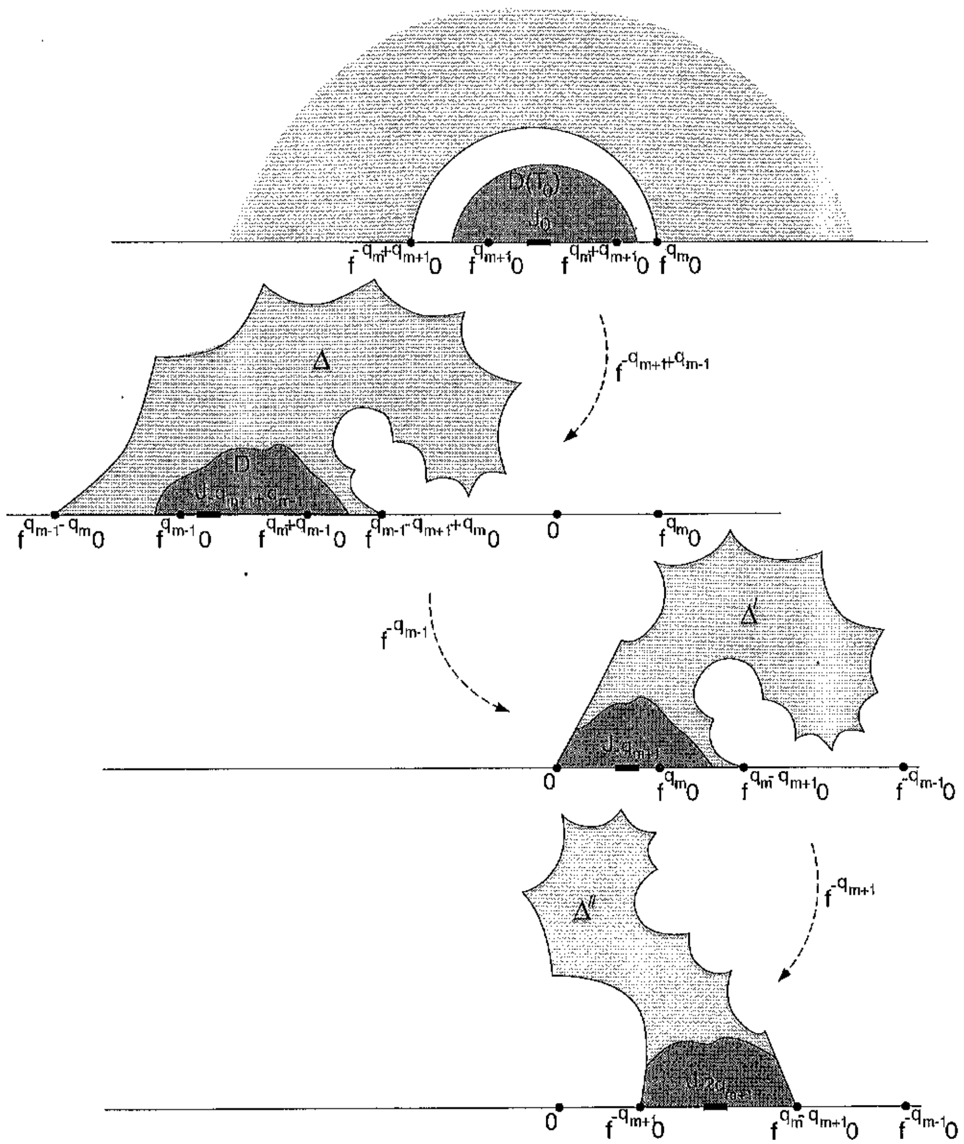


FIGURE 11

rem, $D \supset D_\theta(T)$ for some fixed value of θ. Thus, if $(\widehat{\zeta, [f^{q_{m-1}}(0), f^{q_m+q_{m-1}}(0)]}) > \epsilon$, then $\text{dist}(\zeta, [f^{q_{m-1}}(0), f^{q_m+q_{m-1}}(0)]) > K|I_{m-1}|$, and the claim follows by Koebe distortion theorem.

Otherwise, the point ζ is contained in an ϵ-wedge, with the vertex at an end point of the interval $[f^{q_{m-1}}(0), f^{q_m+q_{m-1}}(0)]$. Assume first that ζ is contained in the wedge attached to $f^{q_{m-1}}(0)$. Then ζ' is contained in a triangle with angle $\frac{\pi}{3}$ at 0, at a distance $K|I_m|$ from the interval I_m. The claim follows from Koebe distortion theorem and the elementary properties of the cube root map.

In the other case, ζ' is contained in an ϵ'-wedge at the pont $f^{q_m}(0)$, at a distance $C|I_m|$ from the point $f^{q_m}(0)$. The Lemma follows for the same reason. □

Perturabations of parabolic maps. As we know, when q_{m+1}/q_m is large, the restriction $f^{q_m}|I_{m-1}$ is a small perturbation of a parabolic map. We summarize here several facts about such mappings. A detailed account may be found, for example, in [Sh].

Consider the consecutive returns of the backward orbit of 0 to the interval I_{m-1} before the first return to I_m,

$$f^{-q_m}(0), f^{-2q_m}(0), \ldots, f^{q_{m-1}-q_{m+1}+q_m}(0), f^{q_{m-1}-q_{m+1}}(0).$$

For a bounded value of s, the interval $[f^{q_{m-1}-q_{m+1}+sq_m}(0), f^{-sq_m}(0)] \subset I_{m-1}$ is K=K(s)-commensurable with I_{m-1}.

Lemma 13.5 (Perturbed Fatou Coordinate). *There exists a holomorphic $\mathbb{R}$-symmetric mapping Ψ from a neighborhood $W \supset [f^{q_{m-1}-q_{m+1}+sq_m}(0), f^{-sq_m}(0)]$*

to a vertical strip $\{0 < \mathrm{Re}(z) < (q_{m+1} - q_{m-1})/q_m - 2s\}$, *conjugating* f^{-q_m} *to the unit translation* $z \mapsto z + 1$.

By a compactness argument, the constant s in the above statement can be chosen independent on f.

A direct transcription of the proof of Lemma 12.2 yields the following statement.

Lemma 13.6. *Consider the sequence of returns of the backward orbit* (13.1) *to the interval* I_{m-1} *until the first return to* I_m, $J_{-q_m}, J_{-2q_m}, \ldots, J_{-l_m q_m}$, *and let* $\zeta_{-1}, \ldots, \zeta_{-l_m}$ *be the corresponding inverse orbit of a point* $\zeta_0 \in \mathbb{H}$ *under the mapping* f^{q_m}.

Let $\mathrm{dist}(\zeta_{-2}, J_{-2q_m})$ *be* K*-commensurable with* I_{m-1} *and* $(\widehat{\zeta_{-2}, J_{-2q_m}}) > \epsilon$. *Then* $\zeta_{-s} \in W$ *for some* s *depending only on* K *and* ϵ.

Consider now a curve $t \subset \mathbb{C}$ such that for every $z \in t$, $\mathrm{dist}(z, J_{-2q_m})$ is K-commensurable with I_{m-1} and $(\widehat{z, J_{-2q_m}}) > \epsilon$. Let $T = J_{-q_m} \cup J_{-2q_m} \cup J_{-3q_m}$. There exists an annulus $A \subset \mathbb{C}_T$ with $\mathrm{mod}\, A > \mu > 0$, containing the curve t together with the interval J_{-2q_m}. The domain $\mathbb{C}_T$ can be pulled back univalently along the inverse orbit $J_{-2q_m}, J_{-2q_m-1}, \ldots, J_{-(l_m-3)q_m}$. Applying the Koebe distortion theorem, and noting that the intervals J_{-2q_m} and $J_{-(l_m-3)q_m}$ are commensurable, we obtain the following corollary.

Corollary 13.7. *Let a curve* $t \subset \mathbb{C}$ *be as above. Denote by* t' *the pullback of the curve* t *along the backward orbit* $J_{-2q_m}, J_{-2q_m-1}, \ldots, J_{-l_m q_m}$. *Then* $\mathrm{dist}(t', I_{m-1}) > K|I_{m-1}|$, $(\widehat{\zeta, I_{m-1}}) > \delta(\epsilon) > 0$, *for all* $\zeta \in t'$, *and the curves* t *and* t' *have commensurable lengths.*

The shape of the domain of a holomorphic pair. Let $L \subset \mathbb{C}$ be a rectifiable simple closed curve. For any two points z_1, z_2 contained in L denote by $l(z_1, z_2)$ the length of the shorther subarc of L connecting z_1 and z_2. We say that the curve L is of *bounded* (by K) *turning*, if there exists $K > 1$, such that $l(z_1, z_2) < K \operatorname{dist}(z_1, z_2)$, for all pairs of points z_1, z_2. We recall the following fact ([LV, Theorem 8.6]):

There exists an increasing function $B(K) > 1$, such that a Jordan curve of K-bounded turning is a $B(K)$-quasicircle.

We will further simply say "bounded turning" implying the existense of some universal constant K. Consider the consecutive returns of the orbit (13.1) to I_{m-1} before the first return to I_m,

$$J_{-q_m}, J_{-2q_m}, \ldots, J_{-l_m q_m}. \tag{13.3}$$

Consider the curve segment $\gamma^m \subset f^{-q_m}([f^{q_{m-1}-q_m}(0), f^{q_{m-1}}(0)]) \cap \mathbb{H}$, $\operatorname{cl}\gamma^m \ni f^{q_{m-1}-q_m}(0)$, and let

$$\gamma^m \equiv \gamma^m_{-1}, \gamma^m_{-2}, \ldots, \gamma^m_{-l_m} \tag{13.4}$$

be the corresponding inverse orbit of the curve segment γ^m under f^{q_m}. Let the curve Γ^m be the union of the segments γ^m_i, for $i = -1, \ldots, -l_m$.

Lemma 13.8. *The curve Γ^m is of bounded turning.*

Proof. Let Ψ_m as in Lemma 13.5 be the perturbed Fatou coordinate for the mapping f^{-q_m}. The curve segments $\Psi_m(\gamma^m_i)$ are obtained by a horizontal translation of the segment γ^m, and therefore the whole curve $\Psi_m(\Gamma^m)$ is of bounded turning. An explicit estimate can be given for the mapping Ψ_m (see [Sh, pp. 19,30]), which shows that it is a small perturbation of a Möbius map. Thus the curve Γ^m itself is of bounded turning. $\square$

Lemma 13.9. *There exists a topological disk $\hat{D} \subset D([f^{sq_n - q_{n+1}}(0), f^{-q_{n-1}-sq_n}(0)])$ commensurable with I_{n-1}, such that*

- *The domain $U \setminus \hat{D}$ is a K-quasidisk for some fixed K.*
- *The intersection $\partial U \cap \hat{D} \cap \mathbb{H}$ is contained in the curve $f^{-(q_{n-1}-1)}(\Gamma^n)$, where Γ^n is as above.*

Proof. The proof of the Lemma consists of several steps. First we partition the boundary of the domain U into a finite number of curves with bounded turning, attached to each other at angles $\frac{\pi}{3}$. Each of these curves is "created" between the moments of the first return of the inverse orbit (13.1) respectively to I_{m-1} and to I_m, for some m. We proceed to show that after we cut out a piece of the boundary of U as in the statement of the Lemma, the union of the remaining curves is of bounded turning.

So let $\zeta \equiv \zeta_{-p} \in \partial U \cap \mathbb{H}$, and consider the forward orbit:

$$\zeta_{-(p-1)} \equiv f(\zeta), \zeta_{-(p-2)}, \dots, \zeta_0 \equiv f^p(\zeta).$$

Take the smallest $k = k(\zeta)$ such that $\zeta_{-k} \in \mathbb{H}$. Let the curve $t_m^+ \subset \partial U \cap \mathbb{H}$ consist of all ζ, such that the interval J_{-k} is contained in the backward orbit (13.1) between the first return to the interval I_{m-1} and the first return to I_m. Let t_m^- denote the corresponding curve in the lower half-plane. Thus we obtain a finite partition of the boundary of the domain U into a collection of curves $t_m^\pm$. If we let the curve Γ^m be as above, then by Lemma 13.2, $f^{p-1-(q_{m+1}-q_{m-1})}(t_m^+) =$

Γ^m.

As we know from Lemma 13.8, each curve Γ^m is of bounded turning. Suppose that $m < n$. Recall that $J_{-q_{m+1}+q_{m-1}}$ is the last return of the inverse orbit (13.1) to the interval I_{m-1} before the first return to I_m, and that $J_{-q_{m+1}}$, $J_{-2q_{m+1}}$ are the first two returns of the same orbit to I_m. Let $\tilde{\Gamma}^m$ be the pullback of Γ^m along the backward orbit $J_{-q_{m+1}+q_{m-1}}, \ldots, J_{-2q_{m+1}}$. By Lemmas 13.3 and 13.4, $\mathrm{dist}(\tilde{\Gamma}^m, I_m) > K|I_m|$, and $\widehat{(\zeta, I_m)} > \epsilon > 0$ for any $\zeta \in \tilde{\Gamma}^m$. Applying now Koebe distortion theorem to the further pullbacks of $\tilde{\Gamma}^m$, we conclude, that the curve t^+_m is of bounded turning as well.

The curve $\tilde{\Gamma}^m$ satisfies the assumptions of Corollary 13.7. Thus if we denote by $\hat{\Gamma}^m$ the further pullback of $\tilde{\Gamma}^m$ until the first return of the orbit (13.1) to the interval I_{m+1}, then the curves $\hat{\Gamma}^m$ and $\tilde{\Gamma}^m$ have commensurable sizes. Proceeding to apply Corollary 13.7 to $\hat{\Gamma}^m$, etc. we establish that the sizes of the curves $t^{\pm}_m$ are commensurable with I_n, and $\mathrm{dist}(t^{\pm}_m, I_n) > K|I_n|$, provided that $m > n$.

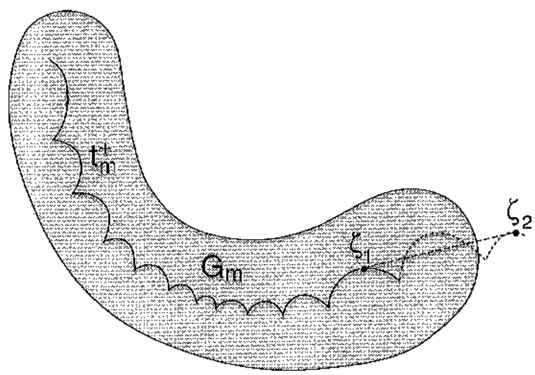


FIGURE 12

Choose a domain $\tilde{G}_m \supset \tilde{\Gamma}^m$, such that $\mathrm{mod}(\tilde{G} \setminus \tilde{\Gamma}^m) > \mu > 0$, and $\tilde{G}_m \subset \mathbb{H}$. Setting $G_m \supset t^+_m$ to be the pullback of $\tilde{G}_m$, we have $\mathrm{mod}(G_m \setminus t^+_m) = \mathrm{mod}(\tilde{G} \setminus$

$\tilde{\Gamma}^m) > \mu$. The domain G_m can be chosen in such a way as to intersect only the first segments of the adjacent curves t_i^+ (see Figure 12). Let ζ_1 and ζ_2 be any two points in the boundary of U contained in different curves $t_m^\pm$, with $m > n$. It follows from elementary considerations, that $\text{dist}(\zeta_1, \zeta_2)$ is commensurable with the shortest segment of the boundary of U between the points ζ_1, ζ_2. Therefore, the whole curve $\cup_{m>n} t_m^+$ is of bounded turning.

Since $\text{dist}(t_m^\pm, I_n) > K|I_n|$ for $m > n$, we can choose a bounded s, such that $D \equiv D([f^{sq_n - q_{n+1}}(0), f^{-q_{n-1}-sq_n}(0)]) \cap \partial U \subset t_n^+ \cup t_n^-$. By Lemma 13.6, all, but finitely many curves γ_i^n are contained in the disk D. Adjust the disk D slightly to obtain a smaller disk $\hat{D}$ with the same property, and such that $\partial(U \setminus \hat{D})$ is connected and is of bounded turning. Then the domain $U \setminus \hat{D}$ is a K-quasidisk, and the proof is complete.

□

The existence of a quasiconformal interpolation. We now present the proof of Proposition 13.1. Assume that q_{n+1}/q_n is large. For the map $f_i^{-q_n}$ consider its perturbed Fatou coordinate $\Psi_i|W_i$ as in Lemma 13.5. Consider the orbit (13.4) corresponding to $m = n$, and $f \equiv f_i$. By Lemma 13.6, we can choose r bounded and independent on the map, so that $\gamma_{-r}^n \subset W_i$. Let $c_i \subset \mathbb{H}$ be a curve connecting an endpoint of γ_{-r}^n with $f_i^{-q_{n+1}+q_{n-1}+sq_n}(0)$, such that c_i does not cross γ_{-r}^n, and the translate of c_i under $f_i^{-q_n}$ is disjoint from c_i. The segments $[f_i^{-q_{n+1}+q_{n-1}+sq_n}(0), f_i^{-q_{n+1}+q_{n-1}+(s+1)q_n}(0)]$, γ_{-r}^n, c_i and $f_i^{q_n}(c_i)$ bound a fundamental rectangle $R_i \subset W_i$ for the map Ψ_i (see Figure 13).

Since all four sides of the rectangle R_i are commensurable, there exists a K-quasiconformal mapping $\kappa : R_1 \to R_2$ with a bounded dilatation K, satisfying

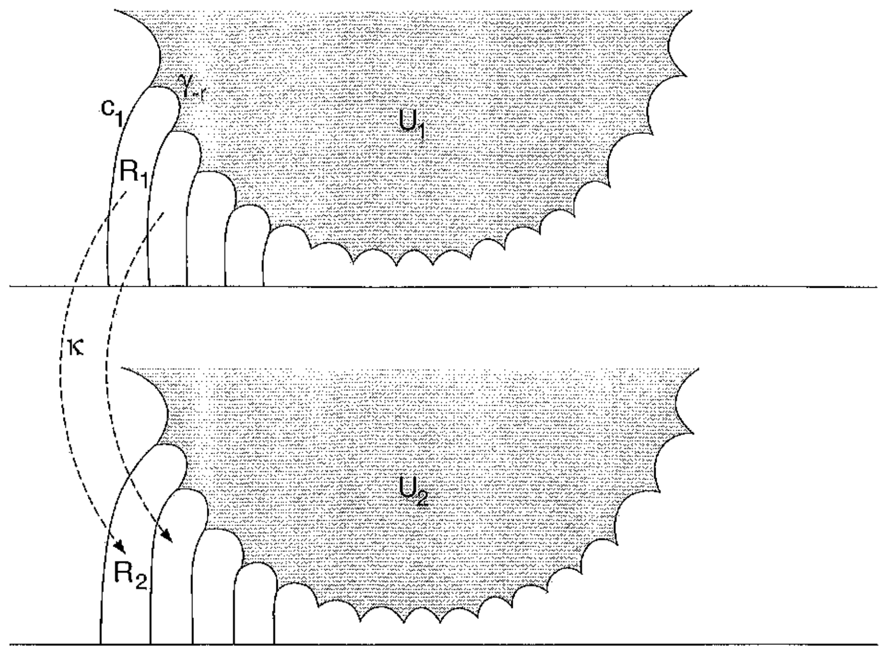


FIGURE 13

$\kappa \circ f_1^{q_n}|c_1 \equiv f_2^{q_n} \circ \kappa|c_2$. We extend κ to all domains $R_1^k \subset W_1$, $f_1^{kq_n}(R_1^k) = R_1$ by $\kappa|R_1^k \equiv f_2^{-kq_n} \circ \kappa \circ f_1^{kq_n}|R_1^k$, provided that $R_2^k \equiv \kappa(R_1^k) \subset W_2$. Thus we obtain the desired interpolation near the pinched region of the domain U_1. The pinched region of V_1 is treated in a similar fashion.

By Lemma 13.9, the interpolating map can be extended to the rest of $\Delta_1 \backslash \Omega_1$ with bounded dilatation, which completes the proof.

14. Local connectivity of Siegel Julia sets

In this section we will use the methods developed in §12 to give a new proof for the following theorem of C. Petersen:

Theorem 14.1 (Petersen, [P]**).** *Let P_θ denote the quadratic polynomial $z \mapsto e^{2\pi i\theta}z + z^2$. If θ is an irrational number of bounded type, then the Julia set $J(P_\theta)$ is locally connected and has Lebesgue measure zero.*

As a consequence of a classical result of Siegel, for an irrational θ of bounded type, the polynomial P_θ has a Siegel disc Δ around the origin. The action of P_θ on the domain Δ is conformally conjugated to that of the rigid rotation $z \mapsto e^{2\pi i\theta}z$ on the open unit disc $\mathbb{D}$. The methods of quasiconformal surgery allow in this case to relate the dynamics of P_θ to that of a rational critical circle map. Namely, consider the following one-parameter family of Blaschke products:

$$f^\tau(z) = e^{i2\pi\tau}z^2\frac{z-3}{1-3z}. \tag{14.1}$$

A map in this family is a degree three branched covering of the Riemann sphere with superattracting fixed points at 0 and ∞. The only other singularity of f_τ is a cubic critical point at 1. Since the unit circle T is invariant, the restriction $f^\tau|T$ is a homeomorphism with one cubic critical point, that is, a critical circle map. By the standard monotonicity considerations, for each irrational θ in $(0,1)$ there is a unique parameter $\tau = \tau(\theta) \in (0,1)$ for which the circle restriction f^τ has rotation number θ; and we set $f_\theta \equiv f^{\tau(\theta)}$. The connection between a Blaschke product f_θ and the quadratic polynomial P_θ is given by the following theorem:

Theorem 14.2 (Douady, Ghys, Herman, Shishikura, [Do2]**).** *Let θ be an irrational number of bounded type, and let Δ denote the Sigel disc of the quadratic polynomial P_θ. Then there exists a quasi-conformal homeomorphism $\phi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, conformal on the immediate basin of infinity of f_θ, such that $\phi(\mathbb{D}) = \Delta$, and $\phi \circ f_\theta = P_\theta \circ \phi$ on $\mathbb{C} \setminus \mathbb{D}$.*

Denote by W the f_θ^{-1}-preimage of the unit disc $\mathbb{D}$ not contained in $\mathbb{D}$. Set

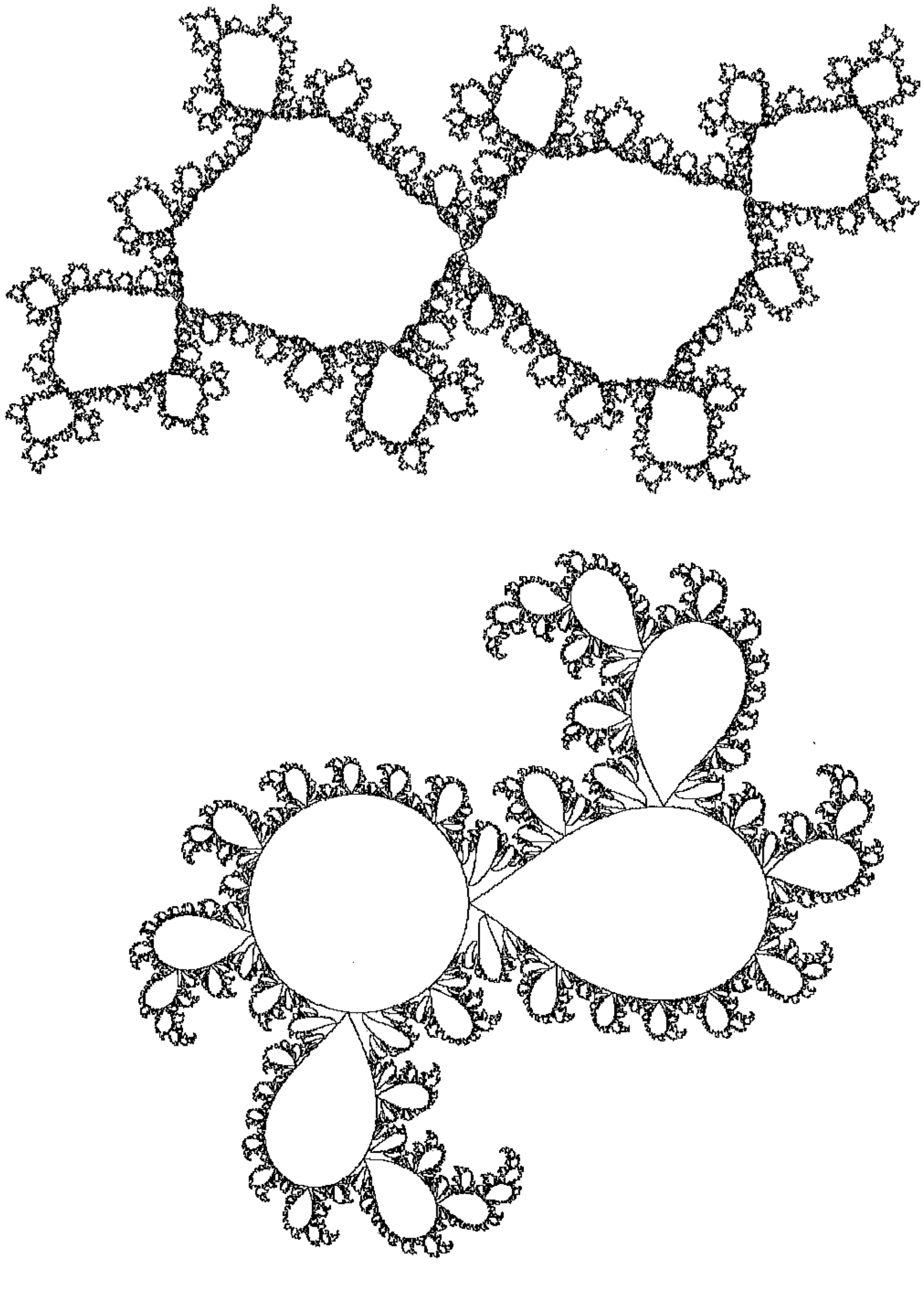


FIGURE 14. A quadratic Siegel Julia set (above) and the corresponding set J_θ

$J_\theta = \phi^{-1}(J(P_\theta))$. This set is obtained by removing from the Blaschke Julia set $J(f_\theta)$ all points whose orbits eventually land in the interior of $\mathbb{D}$:

$$J_\theta = J(f_\theta) \setminus (\cup_{n\geq 0} f_\theta^{-n}(W) \cup \mathbb{D}).$$

Petersen [P] proved the following result, which together with Theorem 14.2 implies Theorem 14.1:

Theorem 14.3 (Petersen, [P]). *For any irrational rotation number θ the set J_θ is locally connected and has Lebesgue measure zero.* [1]

Our proof of Theorem 14.3 bears a strong resemblance to the proofs of local connectivity of quadratic Julia sets with complex bounds developed in [HJ, McM2]. Following Petersen, we construct a plethora of *puzzle-pieces* for the map f_θ, which partition J_θ into connected subsets. By a version of the cubic estimate (11.3) for the map f_θ, the diameters of the puzzle-pieces around the critical point 1 shrink to zero. Thus we obtain a basis of connected neighborhoods around the critical point. Finally, we use bounded distortion considerations to transfer this basis to other points of the set J_θ.

Let us choose an irrational $0 < \theta < 1$ and in what follows work with the fixed map $f \equiv f_\theta$.

For two points a and b on the unit circle T, $[a, b] \subset T$ will denote the shorter arc with these endpoints; $|[a, b]|$ will stand for its length. For $[a, b] \subset T$ let $D_\alpha([a, b])$ be the preimage of $D_\alpha([0, 1])$ under a Möbius map sending T to $\mathbb{R}$ and $[a, b]$ to $[0, 1]$. As ∞ is a superattracting fixed point of f, its immediate

[1] In [P] The original argument of Petersen showed measure zero for θ of bounded type only. The argument for the general case was suggested by M. Lyubich.

basin is parametrized by $\mathbb{C} \setminus \bar{\mathbb{D}}$ via Böttcher coordinate. The images of polar coordinate curves in this parametrization will be referred to as the external rays and equipotentials of f; the former have the same landing properties as in the polynomial case. For a point $\zeta \in T$ with $f^i(\zeta) = 1$ denote by $W(\zeta)$ the preimage of the domain W attached to ζ.

Construction of puzzle-pieces. Petersen's construction of puzzle-pieces begins as follows. Let $\gamma_0' \subset \partial W$, $\gamma_0 \subset \partial W$, and $f(\gamma_0') = [f(1), 1]$, $f(\gamma_0) = T \setminus [f(1), 1]$. Let $\gamma_1' \cap \gamma_0' \neq \emptyset$, $\gamma_1 \cap \gamma_0 \neq \emptyset$, and $f(\gamma_1') = \gamma_0'$, $f(\gamma_1) = f(\gamma_0)$. Inductively let $\gamma_i' \cap \gamma_{i-1}' \neq \emptyset$, $\gamma_i \cap \gamma_{i-1} \neq \emptyset$, and $f(\gamma_i') = \gamma_{i-1}'$, $f(\gamma_i) = f(\gamma_{i-1})$. Considerations of hyperbolic geometry (compare with Douady-Hubbard-Sullivan Landing theorem) imply that the curves γ_i converge to a fixed point of f which we will denote by β; this point is necessarily repelling.

Let $\Gamma' = \cup_i \gamma_i' \cup \beta$, $\Gamma = \cup_i \gamma_i \cup \beta$, and let $\hat{\Gamma}$ be the component of $f^{-1}(\Gamma')$ attached to $f^{-1}(1)$. Being the only fixed point of f in the complement of the unit disc, β is the landing point of the external ray R of the external argument 0. Let R' be the preimage of R landing at the end point of $\hat{\Gamma}$. Fix any equipotential curve E, and let the puzzle-piece $P_0 \supset W$ be the closed domain cut out by the curves $R \cup \Gamma \cup [1, f^{-1}(1)] \cup \hat{\Gamma} \cup R'$ and E (cf. Figure 15).

As before, denote the moments of the closest returns of the critical point 1 by q_i, these numbers appear as the denominators in the truncated continued fraction expansions of the rotation number θ. Let the *n-th central puzzle-piece* P_n be the univalent pull-back of P_{n-1} along the inverse orbit $[1, f^{q_n}(1)] \subset P_{n-1}, [f^{-1}(1), f^{q_n-1}(1)], \ldots, [f^{-q_n}(1), 1]$. Note that $P_n \cap T = [f^{-q_n}(1), 1]$, $f^{q_n}(P_n \cap \partial W) = [f^{q_n}(1), f^{-q_{n-1}}(1)]$, and $f^{q_n+q_{n-1}+q_{n-2}}(P_n \cap$

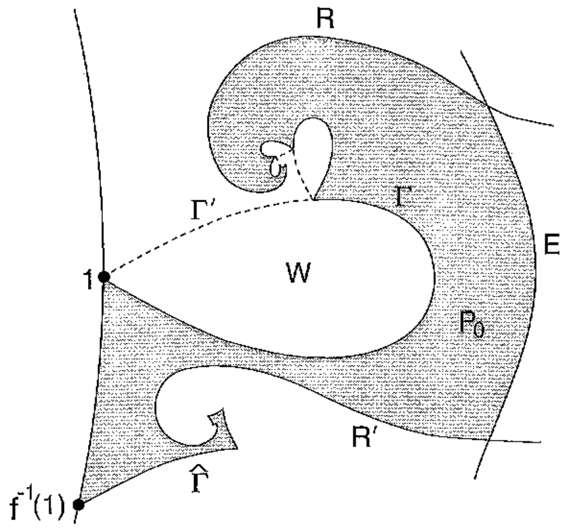


FIGURE 15

$\partial W(f^{-q_n}(1)) = [1, f^{q_{n-1}+q_{n-2}}(1)]$. As follows from disjointness properties of external rays and equipotentials, the central puzzle-pieces form two nested sequences inside the first two puzzle-pieces P_0 and P_1. By construction we have:

Proposition 14.4. *The intersection $P_n \cap J_\theta$ is connected.*

Let us also note:

Lemma 14.5. *The puzzle-piece P_n contains a Euclidean disc B with $B \cap J_\theta = \emptyset$ and* $\operatorname{diam} B > K \operatorname{diam} P_n$*, for some $K > 0$, independent of n.*

Proof. Note that by construction, $W(f^{-q_{n+2}}(1)) \subset P_n$. The claim now easily follows from Świątek - Herman real *a priori* bounds and Koebe distortion theorem. □

A cubic estimate for the map f. Our goal now is to adapt the methods of §12 to obtain a cubic estimate for the inverse branches of f, similar to (11.3).

Let us first observe that

$$f(z) - f(1) = e^{2\pi i \tau(\theta)} \frac{(z-1)^3}{1-3z}.$$

Therefore, for all points $z \in J_\theta$ we have

$$c|z-1|^3 < |f(z) - f(1)| < C|z-1|^3 \tag{14.2}$$

for a suitable choice of positive constants c and C.

The map f is not a self-map of the Riemann sphere with points 0 and ∞ removed, and therefore can not be lifted to the universal covering. Nevertheless, consider the multivalued meromorphic function

$$F(z) = \frac{1}{2\pi i} \operatorname{Log} f(e^{2\pi i z}).$$

It preserves the real axis and has singularities at the integer points, whose images are the integer translations of $\tau = \tau(\theta)$. By Monodromy Theorem, in the domain $\mathbb{C}_{(\tau+i,\tau+i+1)}$ with the critical values removed, we have well-defined univalent branches of the inverse map F^{-1}, mapping the open interval $(\tau + i, \tau + i + 1)$ homeomorphically onto an interval between two consecutive integers $(m, m+1)$. These maps range over the simply-connected regions $\mathbb{C}_{(m,m+1)} \setminus \cup \mathcal{W}_j$, where $\mathcal{W}_j$ denotes the component of $1/2\pi i \operatorname{Log}(W)$ attached to j.

Denote by $\Upsilon(z) : T \setminus \{f^{-1}(1)\}$ the branch of $1/2\pi i \operatorname{Log}(z)$ mapping 1 to 0. Consider the lifts of the inverse branches $f^{-q_1}|[f^{-1}(1), 1]$ and $f^{-1}|[1, f^{-q_1}(1)]$ given by $\Upsilon \circ f^{-q_1} \circ \Upsilon^{-1}$ and $\Upsilon \circ f^{-1} \circ \Upsilon^{-1}$ respectively. It is convenient for us to abuse notation and label these branches ϕ^{-q_1} and ϕ^{-1}. Thus we obtain a commuting pair of inverse maps

$$\Phi = (\phi^{-q_1}|\Upsilon([f^{-1}(1), 1]), \phi^{-1}|\Upsilon([1, f^{-q_1}(1)])).$$

As we have seen above, these maps have univalent extensions to the double slit plane, however, they are not inverses of Epstein maps, and we will re-state Lemmas 12.1-12.3 for them.

Set $I_m = \Upsilon([0, f^{q_m}(0)])$, $T_m = \Upsilon(f^{q_{n+1}}([0, f^{q_n}(0)]))$. Let D_m denote the hyperbolic neighborhood $D_\alpha(\Upsilon([f^{q_{m+1}}(0), f^{q_m - q_{m+1}}(0)]))$, where $0 < \alpha < \pi/2$ will be specified later. Set $J_0 = T_n$ and consider the orbit of this interval under Φ. For a point $z \in \mathbb{C}_{T_n}$ the *corresponding* inverse orbit is obtained by applying the appropriate univalent branches in the slit plane. We begin with the following version of Lemma 12.1.

Lemma 14.6. *Let J and J' be two consecutive returns of the orbit of J_0 to I_m, for $m > 1$, and let ζ and ζ' be the corresponding points of the inverse orbit of z as above. Suppose $\zeta \in D_m$, then either $\zeta' \in D_m$, or $\widehat{(\zeta', J')} > \epsilon$, and* $\operatorname{dist}(\zeta', J') < C|I_m|$; *where the quantifiers ϵ and C are independent on m.*

We remark that the constants ϵ and C will in general depend on the choice of the Blaschke product f.

Proof. Let D'_m denote the pull-back of D_m corresponding to the inverse orbit $J, \ldots, J'$, and let $\hat{D}_m$ denote the pull-back of D_m along the piece of the orbit $J, \ldots, \phi^{-q_m}(J)$. By Świątek - Herman real *a priori* bounds, the points 0 and $\Upsilon(f^{q_m}(1))$ divide the interval $D_m \cap \mathbb{R}$ into K-commensurable pieces, where K becomes universal for large m, and therefore can be chosen simultaneously for all m.

As the absolute value of the derivative of the logarythmic map is bounded away from 0 and ∞ on the set J_θ, the estimate (14.2) is still valid for the lifted map near the critical point. Together with Schwarz Lemma and Lemma 2.4

this implies that $\hat{D}_m \subset D_\beta([\Upsilon(f^{q_{m+1}-q_m}(1)),0])$ for some $\beta > 0$ independent of m. Moreover, since the boundary of $\hat{D}_m$ contains a segment of ∂W_0, which forms the angle $\pi/3$ with $\mathbb{R}$ at 0, we have $\hat{D}_m \subset D_\gamma([\Upsilon(f^{q_{m+1}-q_m}(1)),a_1]) \cup D_\sigma([a_2,0])$, where the points $\Upsilon(f^{q_{m+1}-q_m}(1)), a_1, a_2, 0$ form a B-bounded configuration; with B, $\gamma > 0$ and $\sigma < \pi/2 < \alpha$ independent of m. Applying Schwarz Lemma we have $D'_m \subset D_m \cup D_\gamma([0,\Upsilon(f^{-q_{m+1}+q_m}(a_1))])$ and the claim follows.

□

Lemma 12.3 is also re-formulated in the obvious way:

Lemma 14.7. *Let J be the last return of the orbit of J_0 to the interval I_m before the first return to I_{m+1}, and let J' and J'' be the first two returns to I_{m+1}. Let ζ, ζ', and ζ'' be the corresponding points in the inverse orbit of z, $\zeta' = \phi^{-q_m}(\zeta)$, $\zeta'' = \phi^{-q_{m+2}}(\zeta')$.*

Suppose $\zeta \in D_m$. Then either $(\widehat{\zeta'', I_{m+1}}) > \epsilon = \epsilon(f)$ and $\mathrm{dist}(\zeta'', J'') < C(f)|I_{m+1}|$, or $\zeta'' \in D_{m+1}$.

The proof is again obtained by a direct transcription of the proof of Lemma 12.3. Finally, let us address the saddle-node phenomenon:

Lemma 14.8. *Let $P_0, P_{-1}, \dots, P_{-k}$ be the consecutive returns of the orbit of J_0 to I_m under the iterates of $\phi^{-q_{m+1}}$, and denote by $\zeta_0, \zeta_{-1}, \dots, \zeta_{-k}$ the corresponding moments in the backward orbit of z. Suppose, that $\zeta_{-i} \in D_m$ for some $-k \le -i \le 0$. Then either $\zeta_{-k} \in D_m$, or $(\widehat{\zeta_{-k}, P_{-k}}) > \epsilon = \epsilon(f)$ and $\mathrm{dist}(\zeta_{-k}, P_{-k}) \le C(f)|I_m|$.*

Proof. Let $\widetilde{\phi^{-q_{m+1}}}$ denote the rescaling of the appropriate inverse branch by a linear factor $1/|I_m|$. Since the maps $\phi^{-q_{m+1}}$ do not assume values in $\mathcal{W}_0$, the sequence $\{\widetilde{\phi^{-q_{m+1}}}\}$ forms a normal family by Montel's Theorem. The partial limits of subsequences with $q_{m+1}/q_m \to \infty$ are necessarily parabolic maps. By Denjoy-Wolf Theorem each of the maps $\phi^{-q_{m+1}}$ has an attracting fixed point both in the upper and the lower half-plane, whose basin is the whole half-plane; by compactness, this point is contained in $D(I_m) \subset D_m$ provided q_{m+1}/q_m is large enough.

The proof is now completed as in Lemma 12.2. □

We are now in position to repeat the inductive argument of §12, to obtain the following estimate for the inverse branches:

$$\frac{\operatorname{dist}(\phi^{-q_{n+1}+1}(z), \phi^{-q_{n+1}+1}(T_n))}{|\phi^{-q_{m+1}+1}(T_n)|} \leq C\frac{\operatorname{dist}(z, T_n)}{|T_n|}, \tag{14.3}$$

where $z \in \mathbb{C}_{T_n} \cap D_1$. The constant C depends on the map f as well as on the choice of the angle α in the definition of D_m.

Let us now choose α in such a way that the components of $1/2\pi i\,\mathrm{Log}(P_2)$ and $1/2\pi i\,\mathrm{Log}(P_3)$ attached to 0 are contained in D_1. For $n > 3$ let $G_n \supset P_{n+1}$ be the univalent pull-back of $P_2 \cup P_3$ along the orbit $[1, f^{q_{n+1}}(1)], \dots, [f^{-q_{n+1}}(1), 1]$. Using the estimate (14.3), and the fact that the absolute value of the derivative of $1/2\pi i\,\mathrm{Log}(z)$ is bounded away from 0 and ∞ on J_θ, we arrive at the following estimate:

$$\frac{\operatorname{dist}(f(z), f([1, f^{q_n}(1)]))}{|f([1, f^{q_n}(1)])|} \leq c\frac{\operatorname{dist}(f^{q_{n+1}}(z), f^{q_{n+1}}([1, f^{q_n}(1)]))}{|f^{q_{n+1}}([1, f^{q_n}(1)])|}, \tag{14.4}$$

for any $z \in G_n \cap J_\theta$. Together with the cubic estimate (14.2) for the map f

this yields the desired cubic estimate for $f^{q_{n+1}}$:

$$\frac{\operatorname{dist}(f^{q_{n+1}}(z), f^{q_{n+1}}([1, f^{q_n}(1)]))}{|f^{q_{n+1}}([1, f^{q_n}(1)])|} \geq B\left(\frac{\operatorname{dist}(z, [1, f^{q_n}(1)])}{|[1, f^{q_n}(1)]|}\right)^3, \tag{14.5}$$

for some $B > 0$.

By Świątek-Herman real *a priori* bounds the arcs $[f^{-q_{n+1}}(1), 1]$, $[f^{q_n}(1), 1]$, $[1, f^{q_{n+1}}(1)]$ and $f^{q_{n+1}}([1, f^{q_n}(1)])$ are all $K-$commensurable, with a universal constant K for sufficiently large n. By the above cubic estimate (14.5), for $n > 3$, we have

$$\operatorname{diam} P_{n+1} \leq b\sqrt[3]{\frac{\operatorname{diam} P_n}{|[1, f^{-q_n}(1)]|}} \cdot |[f^{-q_{n+1}}(1), 1]|.$$

Hence, if $\dfrac{\operatorname{diam} P_n}{|[1, f^{-q_n}(1)]|} > K_1$ for a large K_1, then $\dfrac{\operatorname{diam} P_{n+1}}{|[f^{-q_{n+1}}(1), 1]|} < \dfrac{1}{2} \cdot \dfrac{\operatorname{diam} P_n}{|[1, f^{-q_n}(1)]|}$. It follows that for all sufficiently large n, the piece P_n is K_2-commensurable with $[f^{-q_n}(1), 1]$ with a universal constant K_2. Together with Proposition 14.4 this implies

Proposition 14.9. *The set J_θ is locally connected at the critical point* 1.

"Spreading around" argument. Choose any $z \in J_\theta$. Assume first that there exists n such that $f^i(z) \notin P_k$ for any $i \geq 0$ and $k \geq n$. As f has an irrational rotation number on the circle this implies that the forward orbit $z_0 \equiv z, z_1 \equiv f(z), z_2 \equiv f(z_1), \ldots$ does not accumulate on the circle, i.e. there exists $\epsilon > 0$, such that $z_i \in V_\epsilon \equiv \{\zeta,\ |\zeta| > 1 + \epsilon\}$.

As the set J_θ is locally connected at the critical point 1, there exist two external rays r_1 and r_2 landing at this point on different sides of W. For a point $\zeta \in J_\theta$ whose forward orbit lands at the critical point 1 let $r_1(\zeta)$, $r_2(\zeta)$ be the preimages of the rays r_1 and r_2, landing on ζ. Denote by $Z(\zeta)$ the component of $\mathbb{C} \setminus (r_1(\zeta) \cup r_2(\zeta) \cup \{\zeta\})$ not containing 1. The *limb* L_ζ of the set J_θ is the set $Z_\zeta \cap J_\theta$. Since external rays do not intersect the Julia set, L_ζ is a connected component of $J_\theta \setminus \{\zeta\}$. Let a be an accumulation point of the sequence $\{z_k\}$. Choose a limb $L \equiv L_\zeta$ containing a, with $L \cap T = \emptyset$. Denote by k_n the moments when $z_{k_n} \in L$. Let $L_n \ni z$ be the pull-back of L along the backward orbit $z_{k_n}, z_{k_n-1}, \dots, z_1, z_0 \equiv z$. We refer to the following general principle to assert that $\mathrm{diam}(L_n) \to 0$.

Lemma 14.10 ([L2], **Prop. 1.10**). *Let f be a rational map. Let $\{f_i^{-m}\}$ be a family of univalent branches of the inverse functions in a domain U. If $U \cap J(f) \neq \emptyset$, then for any V such that* $\mathrm{cl}\, V \subset U$,

$$\mathrm{diam}(f_i^{-m} V) \to 0.$$

This yields the desired nest of connected neighborhoods around z, and we are done.

In the complementary case, let z_k be the first point in the orbit $z_0, z_1, z_2, \dots$ contained in the puzzle-piece P_n. Denote by

$$\Pi_0 \equiv P_n, \Pi_{-1}, \dots, \Pi_{-k} \tag{14.6}$$

the preimages of P_n corresponding to the inverse orbit $z_k, z_{k-1}, \dots, z_0$.

Lemma 14.11. *The inverse orbit (*14.6*) hits the critical point* 1 *at most once.*

Proof. To be definite, assume that P_n is above the critical point 1. Note that if $\Pi_{-i} \cap T = \emptyset$ for some $i \leq q^{n+1}$, then the inverse orbit (14.6) never hits the critical point. Otherwise, denote by A and $B \equiv P_{n+1}$ the "above" and "below" $f^{q_{n+1}}$-preimages of P_n, i.e. $f^{q_{n+1}}(A) = P_n$, $A \cap T \neq \emptyset$, and A is above 1, and similarly for B. Notice that $A \cap T = [f^{-q_{n+1}-q_n}(1), 1] \subset [f^{-q_n}(1), 1]$. Let

$L_1 = P_n \cap \partial W$, and $L_2 = A \cap W$, then

$$\begin{aligned} f^{q_{n-1}+q_n+q_{n+1}}(L_1) &= f^{q_{n+1}}([f^{q_{n-1}+q_n}(1), 1]) = [f^{q_{n-1}+q_n+q_{n+1}}(1), f^{q_{n+1}}(1)] \\ &\supset [f^{q_{n-1}+q_n}(1), f^{q_{n-1}+q_n+q_{n+1}}(1)] = f^{q_{n-1}+q_n}([1, f^{q_{n+1}}(1)]) \\ &= f^{q_{n-1}+q_n+q_{n+1}}(L_2). \end{aligned}$$

Thus $L_1 \supset L_2$, and as two different preimages of W cannot cross, it follows that $A \subset P_n$. Hence $\Pi_{-q_{n+1}} \neq A$.

Finally, denote by $B' \cap T \neq \emptyset$ the f^{q_n} preimage of B, $f^{q_n}(B') = B$. $B' \cap T = [f^{-q_n}(1), f^{-q_n-q_{n+1}}(1)] \subset [f^{-q_n}(1), 1]$. Let $L_2 = P_n \cap W(f^{-q_n}(1))$, and $L_3 = B' \cap W(f^{-q_n}(1))$, then

$$\begin{aligned} f^{q_n+q_{n-1}+q_{n-2}}(L_2) &= [1, f^{q_{n-1}+q_{n-2}}(1)] \\ &\supset [f^{q_{n-1}+q_{n-2}-q_n-q_{n+1}}(1), f^{q_{n-1}+q_{n-2}}(1)] \\ &= f^{q_n+q_{n-1}+q_{n-2}}(L_3). \end{aligned}$$

Therefore, $L_2 \supset L_3$, and $B' \subset P_n$.

Thus, $\Pi_{-q_{n+1}-q_n} \cap T = \emptyset$ and the claim follows. □

As follows from the inductive argument (compare with Remark 12.1), for n large enough, the puzzle-piece P_{n+1} is contained in $D_\beta([1, f^{q_{n-1}-q_n}(1)])$ for some $\beta > 0$ independent of n. By Koebe theorem combined with real *a priori* bounds, the non-critical pull-backs of the puzzle-piece P_n along the circle have bounded distortion (compare Lemmas 13.3 and 13.4). Moreover, at the first moment when $\Pi_{-i} \cap T = \emptyset$, there is an annulus of a definite modulus around Π_{-i} which does not intersect the circle T and thus the whole postcritical set of f. By Lemma 14.5 the set $P_n(z) \equiv \Pi_{-k}$ contains a Euclidean disc B_n of commensurable size, with $B_n \cap J_\theta = \emptyset$. The diameters of B_n necessarily converge to zero, as a non-empty disc contained in an infinite subsequence of P_n's would eventually map over the whole $J(f)$. Thus $\cap P_n(z) = \{z\}$, and the set J_θ is locally connected at z.

Proof of measure zero statement. The following argument was suggested by M. Lyubich.

We will make use of the following ergodic principle of Lyubich ([L1]).

Let f be a rational map, whose Julia set $J(f)$ is not the whole Riemann sphere. Then the forward orbit of almost every $z \in J(f)$ converges to the postcritical set of f.

First consider the set of points $J_1 \subset J_\theta$, $J_1 \equiv \{\zeta \in J_\theta | \ \exists n,\ f^i(\zeta) \notin P_k,\ \forall i \geq 0, k \geq n\}$. Since $f|T$ has irrational rotation number, the points in J_1 do not accumulate on the critical orbit. Thus J_1 has zero Lebesgue measure

Consider now a point $z \in J_\theta \setminus J_1$, and let $P_n(z)$ be as above. We have shown that $P_n(z)$ contains a Euclidean disc B_n of commensurable diameter, not intersecting the set J_θ. As $\operatorname{diam} P_n(z) \to 0$, z can not be a Lebesgue density point of J_θ. This completes the proof of Theorem 14.3.

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COMPLEX BOUNDS FOR RENORMALIZATION
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