Scaling Laws for Quadratic Maps

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Abstract of the Dissertation
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We explore two cases of scaling for one-dimensional quadratic dynamics. The first case involves the parameter structure of the standard complex quadratic family $z^2 + c$ around the real Fibonacci point. The second case involves the geometry of the critical orbit for quadratic maps under particular combinatorial assumptions.

In Chapter 2, we show that the scaling and asymptotic geometry of the principal nest Yoccoz parapuzzle pieces around the real Fibonacci parameter point is similar to that of the principal nest Yoccoz puzzle pieces for the Fibonacci Julia set. We also show that the Mandelbrot set is "hairy", i.e., the Mandelbrot set densely fills

the plane after dilating by factors determined by the parapuzzle scaling.

In Chapter 3, we consider classes of topologically conjugate unimodal maps with quadratic critical point and C^2 regularity. Under certain combinatorial conditions we prove that there is a finite upper bound on the number of parameters influencing the geometry of the closure of the critical orbit within this class. In fact, in a given class we assign each map a parameter vector which in turn determines the critical orbit geometry of the map. We then show that the parameter vector may be effectively varied. In other words, to every map f in a given class, its parameter vector has a neighborhood in which any vector is a parameter vector for some map in the class. Exploring various examples we show that for any positive integer N, there are combinatorics such that the critical orbit geometry is influenced by exactly N parameters.

For my Mom, Dad, Eric and Beth

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Chapter 1

Introduction to the Main Theorems

In this introduction we outline the text of the thesis as well as state the main theorems to be proven. A basic review of the necessary background material may be found in the first sections of Chapters 2 and 3. Chapter 2 focuses on complex dynamics, in particular, the Mandelbrot set. Chapter 3 is independent of Chapter 2 and explores the geometry of the closure of the critical orbit for certain unimodal maps of the interval.

In Chapter 2, we focus on the small scale similarities between the dynamical space and parameter space for the Fibonacci point in the family of maps $z \mapsto z^2 + c$. There is a general philosophy in complex dynamics that the structure we see in the parameter space around the parameter value c should be the "same" as that around the critical value 'c' in dynamical space [DH85a]. In the case where the critical point is pre-periodic, Tan Lei [Tan90] proved such asymptotic similarities by showing that the Mandelbrot set and Julia set exhibit the same limiting geometry. For parameters in which the critical point is recurrent (i.e., it eventually returns back to any neighborhood of itself), the Mandelbrot and Julia sets are much more complicated. Milnor, in [Mil89], made a number of conjectures (as well as pictures!) for the case of infinitely renormalizable points of bounded type. Dilating by factors determined by the renormalization, the resulting computer pictures demonstrate a kind of self-similarity, with each successive picture looking like a "hairier" copy of the previous. McMullen [McM94b] has proven that, for these points, the Julia set densely fills the plane upon repeated rescaling, i.e., hairiness; and Lyubich has recently proven hairiness of the Mandelbrot set for Feigenbaum like points. We focus on a primary example of dynamics in which we have a recurrent critical point and the dynamics is non-renormalizable: the Fibonacci map.

The dynamics of the real quadratic Fibonacci map, where the critical point returns closest to itself at the Fibonacci iterates, has been extensively studied (especially see [LM93]). Maps with Fibonacci type returns were first discovered in the cubic case by Branner and Hubbard [BH92] and have since been consistently explored because they are a fundamental combinatorial type of the class of non-renormalizable maps. The Fibonacci map was used by Lyubich and Milnor in developing the *generalized renormalization* procedure which has proven very fruitful. The Fibonacci map was also highlighted in the work of Yoccoz as it was in some sense the worst case in the proof of local connectivity of non-renormalizable Julia sets with recurrent critical point [Hub93], [Mil92].

The local connectedness proof of Yoccoz involves producing a sequence of partitions of the Julia set, now called Yoccoz puzzle pieces. These Yoccoz puzzle pieces are then shown to exhibit the *divergence property* and in particular nest down to the critical point, proving local connectivity there. Yoccoz then transfers this divergence property to the parapuzzle pieces around the parameter point to demonstrate that the Mandelbrot set is locally connected at this parameter value. Lyubich further explores the Yoccoz puzzle pieces of Fibonacci maps and demonstrates that the *principal nest* of Yoccoz puzzle pieces has rescaled asymptotic geometry equal to the filled-in Julia set of $z \mapsto z^2 - 1$ and that the moduli of successive annuli grow at a linear rate [Lyu93c].

We prove that the same geometric and rescaling results hold for the principal nest of parapuzzle pieces for the Fibonacci parameter point in the Mandelbrot set. Let the notation $mod(A, B)$ (where $B \subset A$) indicate the modulus of the annulus $A \setminus B$. (See Appendix for the definition of the modulus.)

Theorem A: (Parapuzzle scaling and geometry)

The principal nest of Yoccoz parapuzzle pieces, P^n, for the Fibonacci point c_{fib} has the following properties.

1. They scale down to the point c_{fib} in the following asymptotic manner:

$$\lim_{n\to\infty} mod(P^{n-1}, P^n) \;/\; n = \frac{2}{3}\ln 2.$$

2. The rescaled P^n have asymptotic geometry equal to the filled-in Julia set of $z \mapsto z^2 - 1$.

Remark. Concerning part 1 of Theorem A, we point out that in the paper [TV90], Tangerman and Veerman showed that in the case of circle mappings

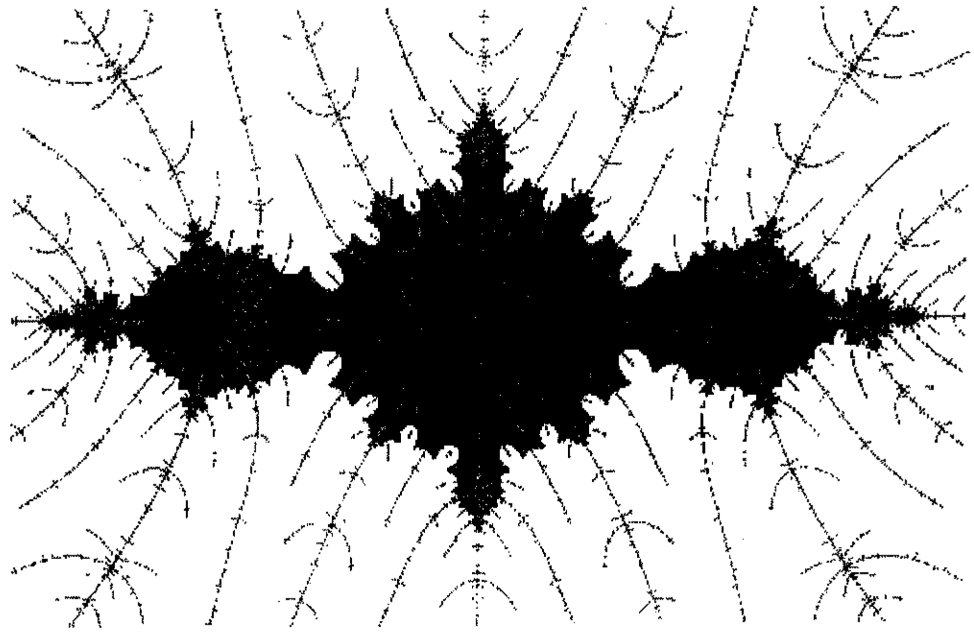

Figure 1.1: P^7 for the Fibonacci point, the seventh level parapuzzle piece with a part of the Mandelbrot set.

with a non-flat singularity, the parameter scaling and dynamical scaling agree for a large class of systems. They have real methods comparing the dynamical derivatives and parameter derivatives along the critical value orbit. Here, we use a complex technique for the unimodal scaling case since a direct derivative comparison appears to have extra difficulties. This is due to the changes in orientation, i.e., the folding which occurs for such maps, complicating the parameter derivative calculations.

Figures 1.1 and 1.2 illustrate item 2 of the theorem. The reader is also encouraged to compare Theorem A with Theorem 2.2.1 of Lyubich on page 21.

When dilating by the scaling factors given by the Fibonacci renormaliza-

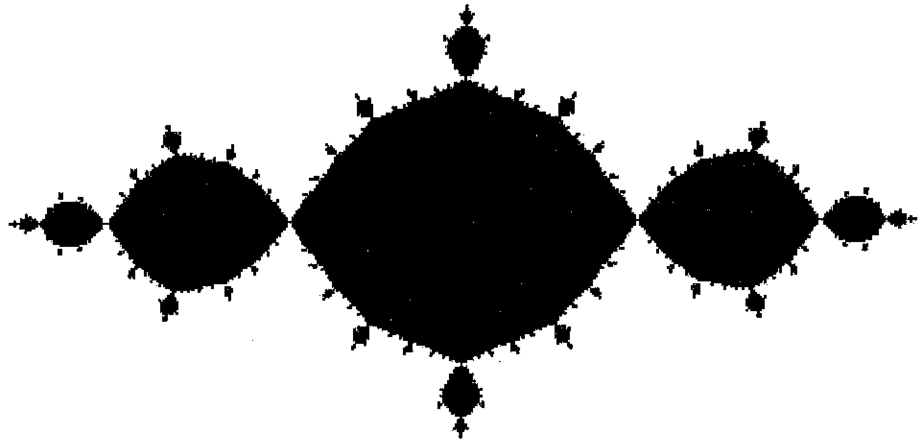

Figure 1.2: The filled-in Julia set of $z \mapsto z^2 - 1$.

tion procedure, the computer pictures around the Fibonacci parameter also exhibit a hairy self-similarity. (Compare Figure 1.1 with Figure 2.5 on page 40.) Using the main construction of the proof of Theorem A, we demonstrate this hairiness. The appropriate scaling maps are denoted by R_n, and the Mandelbrot set by $\mathbf{M}$.

Theorem B: (Hairiness for the Fibonacci parameter)

Given any disc $D(z, \epsilon)$ with center point z and radius $\epsilon > 0$ in the complex plane, there exists an N such that for all $n > N$ we have that

$$D(z, \epsilon) \cap R_n(\mathbf{M}) \neq \emptyset.$$

The sections of Chapter 2 are organized as follows. In Section 1, we review some basic material of quadratic dynamics and the role of equipotentials and external rays. In Section 2, we review the generalized renormalization procedure, where we define the principle nest in the dynamical plane as well as in the parameter plane. In Section 3, we prove dynamical scaling and geometry results for the principal nest for parameter points which are Fibonacci renormalizable n-times. These results and proofs are analogous to those given by Lyubich ([Lyu93b], [Lyu93c]) for the Fibonacci point. In Section 4, we construct a map of parameter space which allows us to compare it to the dynamical space and prove Theorem A, part 2. In Section 5, we complete the proof of Theorem A. Finally, in Section 6 we prove Theorem B.

In Chapter 3, the focus is changed to unimodal maps of the interval with non-degenerate critical point, i.e., the second derivative at the critical point is non-zero. The main theorem of Chapter 3 deals with the geometry of the critical orbit. Let us denote the closure of the critical orbit by $\overline{\mathcal{O}}$. Sullivan has shown that for Feigenbaum maps, where the regularity condition for each map is C^1 plus Zygmund, the geometry of $\overline{\mathcal{O}}$ is rigid [Sul93]. Lyubich and Milnor have shown that for Fibonacci maps with regularity assumption C^2, the geometry of $\overline{\mathcal{O}}$ is not rigid but depends on one parameter [LM93]. This parameter may be effectively varied. In the paper [Lyu93a], Lyubich analyzes a much larger class of maps and demonstrates some scaling properties of $\overline{\mathcal{O}}(f)$ similar to that of the Fibonacci map. In this larger class of maps, we wish to determine which maps have a bounded number of parameters that can influence the geometry of $\overline{\mathcal{O}}$.

The main result of Chapter 3 is concerned with the relationship between the combinatorics of a non-renormalizable map and the number of parameters influencing the geometry of the closure of the critical orbit. The regularity assumption in this chapter is that the maps are C^2. The combinatorics considered are those which satisfy two properties.

First, we define a type of recurrence for the critical point called *very-persistently recurrent*. This definition is to ensure a "fast" return time for the critical point at all generalized renormalization levels. We also assume the combinatorics is of *stationary type*. Stationary type combinatorics for non-renormalizable maps means that the combinatorics of the return map at each generalized renormalization level is independent of the level. The Fibonacci map is an example of a unimodal map which has a very-persistently recurrent critical point and is of stationary type.

We consider topologically conjugate non-renormalizable unimodal maps f which are C^2, have non-degenerate critical point, and the dynamics of which are of very-persistent type and of stationary type. A collection of such maps is denoted by the combinatorial class $\mathcal{F}$.

Remark. In [Lyu93a], where all possible types of combinatorics for non-renormalizable maps are considered, the condition on the maps is not C^2 but negative Schwarzian derivative (a stronger condition). For the Fibonacci case considered in [LM93], the assumption needed was just C^2. The maps we consider (very-persistently recurrent critical point with stationary type generalized renormalization scheme) are combinatorially very similar to the Fibonacci case and the maps with the C^2 assumption still exhibit a shrinking of the scaling factors with nearly identical proof. One first shows that if the scaling factors do not shrink there is some limiting geometry where all intervals and gaps of the generalized renormalization levels are comparable. Then, just as in [Lyu93a], there is convergence to some Epstein class (or negative Schwarzian derivative class) where the shrinking of the scaling factors holds for some maps and hence all maps due to quasi-conformal conjugacies.

Before stating the main theorem we need one extra piece of notation. For a fixed non-renormalizable map f with interior fixed point, we denote by I_0^1 the interval whose boundary points are the interior fixed point and its conjugate. Then we can renormalize in the generalized sense and create a nested sequence of central intervals I_0^n, the principal nest, each containing the critical point. We show the following.

Theorem C (Critical Orbit Geometry)

For a given combinatorial class $\mathcal{F}$, with very-persistent and stationary type combinatorics, we can find (independent of the maps in the class) an $N \times N$ real-valued matrix A and constants K_n, so that we have the following results.

1. Given any map $f \in \mathcal{F}$, there exists an $N \times 1$ parameter vector $\vec{v}(f)$ such that the scaling factors satisfy

$$\frac{|I_0^n(f)|}{|I_0^{n-1}(f)|} \sim e^{-[A^n(\vec{v}(f))]_1 + K_n}. \qquad (1.1)$$

The symbol $\sim$ indicates that the logarithm of their ratio converges to zero at exponential rate and the operation $[\vec{w}]_1$ on a vector $\vec{w}$ means to take the first entry.

2. For any two maps $f, g \in \mathcal{F}$,

$$\pi_e \vec{v}(f) = \pi_e \vec{v}(g)$$

if and only if these two maps are smoothly conjugate on $\overline{\mathcal{O}}$. The operator π_e projects a vector onto the non-contracting space of the matrix A.

3. In the class $\mathcal{F}$, the parameters may be effectively varied. For each f and $\vec{v}(f)$, there is a neighborhood around $\vec{v}$ in the non-contracting directions of the matrix A around $\vec{v}(f)$ such that, for any vector $\vec{w}$ in this neighborhood, there is a map $g \in \mathcal{F}$ for which $\vec{w} = \vec{v}(g)$.

Remark. The term $-[A^n(\vec{v}(f))]_1 + K_n$ tends to negative infinity at least at linear rate since Lyubich in [Lyu93a] has proven that the scaling factors shrink at least at exponential rate.

The sections of Chapter 3 are organized as follows. Section 1 reviews the combinatorial model of the the generalized renormalization scheme. Section 2 gives the combinatorial definitions necessary to our Theorem C. Section 3 gives results concerning the geometry of the generalized renormalization intervals. Section 4 develops a recurrent equation for the scaling factors. Section 5 proves part 1 of Theorem C. Section 6 proves part 2. In Section 7, we work through some examples. In Section 8, we prove part 3 of Theorem C. From the examples of Section 7 of this chapter, we conclude that given any positive integer N there exists a class with precisely N effective parameters.

Chapter 2

Parameter Scaling for the Fibonacci Point

2.1 Introductory Material

We outline some of the basics of complex dynamics of quadratic maps from the Riemann sphere to itself so that we may build the puzzle and parapuzzle pieces. We will consider the normalized form, $f_c(z) = z^2 + c$ with parameter value $c \in \mathbb{C}$. The *basin of attraction* for infinity, $A(\infty)$, are all the points z which converge to infinity under iteration. The dynamics near infinity and the corresponding basin of attraction has been understood since Böttcher (see [Mil90]). Notationally we have that $\mathbb{D}_r$ is the disc centered at 0 with radius r.

Theorem 2.1.1 *The map $f_c : A(\infty) \to A(\infty)$ is complex conjugate to the map $w \mapsto w^2$ near infinity. There exists a unique complex map Φ_c defined on $\widehat{\mathbb{C}} \setminus \bar{\mathbb{D}}_{r(c)}$, where $r(c)$ represents the smallest radius with the property that*

$$f_c \circ \Phi_c(w) = \Phi_c(w^2),$$

and normalized so that $\Phi_c(w) \sim w$ as $|w| \to \infty$.

By Brolin [Bro65] the conjugacy map Φ_c satisfies

$$\lim_{n\to\infty} \log^+(|f_c^n(z)|/2^n) = \log|\Phi_c^{-1}(z)|. \tag{2.1}$$

In fact, the left hand side of equation (2.1) is defined for all $z \in A(\infty)$ and is the Green's function for the Julia set.

Equipotential curves are images of the circles with radii $r > r(c)$, centered at 0 in $\widehat{\mathbb{C}} \setminus \bar{\mathbb{D}}_r$ under the map Φ_c. Actually, the moment that the above conjugacy breaks down is at the critical point 0 if it is in $A(\infty)$. In this case, if we try to extend the above conjugacy we see that the image of the circle with radius $r(c)$ passing through the critical point is no longer a disc but a "figure eight". Despite the conjugacy difficulty, we may define equipotential curves passing through any point in $A(\infty)$ to be the level set from Brolin's formula. *External rays* are images of half open line segments emanating radially from $\mathbb{D}_{r(c)}$, i.e., $\Phi_c(re^{i\theta})$ with $r > r(c)$ and θ constant. In fact, these are the gradient lines from Brolin's formula. So again, we may extend these rays uniquely up to the boundary of $A(\infty)$ or up to where the ray meets the critical point or some preimage, i.e., the "root" of a figure eight.

An external ray is referred to by its angle; for example the $\frac{1}{3}$-ray is the image of the ray with $\theta = \frac{1}{3}$. A central question to ask is whether a ray extends continuously to the boundary of $A(\infty)$. The following guarantees that some points (and their preimages) in the Julia set are such landing points.

Theorem 2.1.2 (Douady and Yoccoz, see [Mil90]) *Suppose z is a point in the Julia set which is periodic or preperiodic and the periodic multiplier is* *a root of unity or has modulus greater than* 1*, then it is the landing point of some finite collection of rays.*

One of the main objects of study in quadratic dynamics is the set of all parameters c such that the conjugacy Φ_c is defined for the whole immediate basin of infinity.

Definition. The *Mandelbrot set* **M** consists of all values c whose corresponding Julia set is connected.

The combinatorics of the Mandelbrot set have been extensively studied. In [DH85a], Douady and Hubbard present many important results, some of which follow below.

Theorem 2.1.3 (Douady and Hubbard, [DH85a])

1. *The Mandelbrot set is connected.*
2. *The unique Riemann map* $\Phi_M : \hat{\mathbb{C}} \setminus \bar{\mathbb{D}} \to \hat{\mathbb{C}} \setminus M$*, with* $\Phi_M(z) \sim z$ *as* $|z| \to \infty$*, satisfies the following relation with the Böttcher map:*

$$\Phi_M^{-1}(c) = \Phi_c^{-1}(c).$$

With the Riemann map Φ_M, we can define equipotential curves and external rays in the parameter plane analogous to the dynamical case above. From the second result of Theorem 2.1.3, it can be seen that the external rays and equipotentials passing through c (the critical value) in the dynamical space are combinatorially the same external rays and equipotentials passing through c

in the parameter space. Since the Yoccoz puzzle pieces have boundary which include rays that land, it is essential for the construction of the parapuzzle pieces that these same external rays land in parameter space. Before stating such a theorem, we recall some types of parameter points. *Misiurewicz points* are those parameter values c such that the critical point of f_c is pre-periodic. A *parabolic point* is a parameter point in which the map f_c has a periodic point with multiplier some root of unity. For these points, their corresponding external rays land.

Theorem 2.1.4 (Douady and Hubbard, [DH85a]) *If c is a Misiurewicz point then it is the landing point of some finite collection of external rays $R_M(\theta_i)$, where the θ_i represent the angle of the ray. In the dynamical plane, external rays of the same angle, $R_{f_c}(\theta_i)$, land at c (the critical value of f_c).*

Theorem 2.1.5 (Douady and Hubbard, [DH85a]) *If c is a parabolic point then it is the landing point of two external rays (except for $c = \frac{1}{4}$ which has one landing ray). In the dynamical plane for this c, these external rays land at the root point of the Fatou component containing the critical value.*

Using rays and equipotentials in dynamical space, Yoccoz developed a kind of Markov partition, now called Yoccoz puzzle pieces, for non-renormalizable (or at most finitely renormalizable) Julia sets with recurrent critical point and no neutral cycles [Hub93]. Combining Theorems 2.1.3 and 2.1.4, Yoccoz constructed the same (combinatorially) parapuzzle pieces for the parameter points of the non-renormalizable maps.

For our purposes we will now focus on the maps exhibiting initial behavior similar to the dynamics of the Fibonacci map. The Fibonacci parameter value lies in what is called the $\frac{1}{2}$-wake. The $\frac{1}{2}$-wake is the connected set of all parameter values with boundary consisting of the $\frac{1}{3}$- and $\frac{2}{3}$-rays (which meet at a common parabolic point) and does not contain the main cardioid. Dynamically, all such parameter points have a fixed point which is a landing point for the same angle rays, $\frac{1}{3}$ and $\frac{2}{3}$. In fact, for all parameter points in the $\frac{1}{2}$-wake, the two fixed points are stable; we may follow them holomorphically in the parameter c. We are now in a good position to review generalized renormalization in the $\frac{1}{2}$-wake. We point out that this procedure, developed in [Lyu93b], is not restricted to the $\frac{1}{2}$-wake and the construction given below is readily generalized from the following description.

2.2 A Review of Puzzles and Parapuzzles

Initial Yoccoz Puzzle Pieces

We now review the Yoccoz puzzle piece construction essentially without proofs. (See [Hub93] or [Lyu93b] for more details.) For each parameter in the $\frac{1}{2}$-wake, we begin with the two fixed points commonly called the α and β fixed points. The β fixed point is the landing point of the $0 - ray$ (the only ray which maps to itself under one iterate). The α point is the landing point of the $\frac{1}{3}$- and $\frac{2}{3}$-rays for all parameters in the $\frac{1}{2}$-wake. By the Böttcher map, it is easy to see that the $\frac{1}{3}$- and $\frac{2}{3}$-rays are permuted by iterates of f_c. The initial Yoccoz puzzle pieces are constructed as follows. Fix an equipotential E. The top level Yoccoz puzzle pieces are the bounded connected sets in the plane with boundaries made up of parts of the equipotential E and external rays. (See Figure 2.1.) For the generalized renormalization procedure described below, the top level Yoccoz puzzle piece containing the critical point is labeled V_0^0.

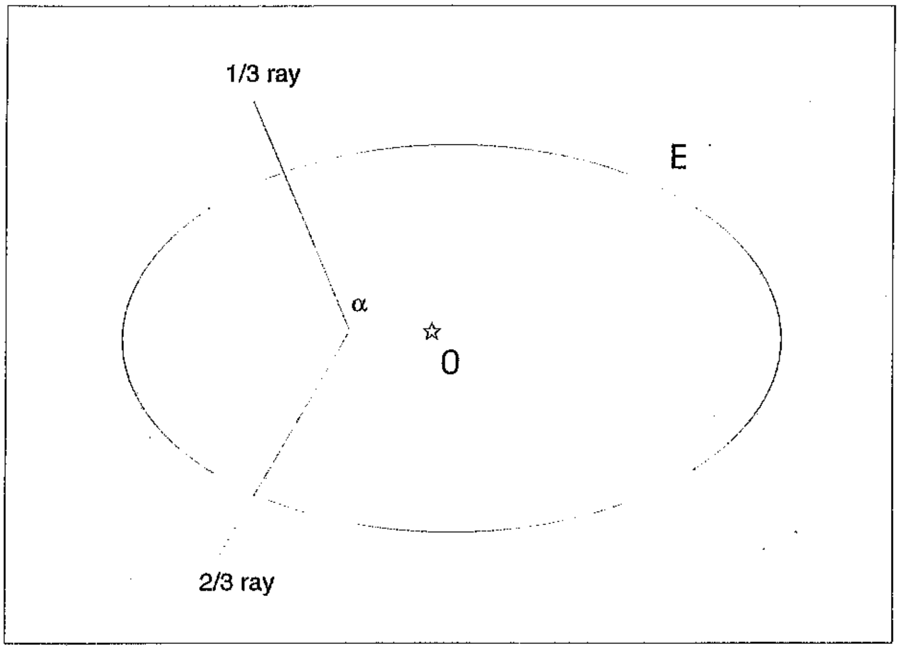

Figure 2.1: Beginning generalized renormalization.

The Principal Nest

The generalized renormalization procedure for quadratic maps with recurrent critical point proceeds as follows. For each parameter value c, iterate the critical point 0 by the map f_c until it first returns back to the set V_0^0. In fact, this will be two iterates. Take the largest connected set around 0, denoted V_0^1, such that $f^2(V_0^1) = V_0^0$. Note that we suppress the parameter c in this discussion, $V_0^n = V_0^n(c)$. This is the level 1 central puzzle piece and we label the return map f_c^2 restricted to the domain V_0^1 by g_1 $(= g_{1,c})$. It is easy to see that $V_0^1 \subset V_0^0$ and that g_1 is a two-to-one branched cover. The boundary of V_0^1 is made up of pieces of rays landing at points which are preimages of α, as well as pieces of some fixed equipotential. Now we proceed by induction. Iterate the critical point until it first returns to V_0^n, say in m iterates, and then take the largest connected set around 0, denoted V_0^{n+1}. This gives $f_c^m(V_0^{n+1}) = V_0^n$. Inductively we get a collection of nested connected sets $V_0^0 \supset V_0^1 \supset V_0^2 \supset V_0^3 ...$, and return maps $g_n(V_0^n) = V_0^{n-1}$. Each of the V_0^i has boundary equal to some collection of pieces of rays landing at preimages of α and pieces of some equipotential. Each g_i is a two-to-one branched cover. The collection of V_0^i is called the *principal nest* of Yoccoz puzzle pieces around the critical point.

To define the principal nest of Yoccoz parapuzzle pieces in the parameter space it is easiest to view the above procedure around the critical value. In this case, the principal nest is just the image of the principal nest for the critical point, namely $f_c(V_0^0) \supset f_c(V_0^1) \supset f_c(V_0^2) \supset f_c(V_0^3)....$ Notice that again the puzzle pieces are connected and we have that the boundary of each puzzle piece to be some parts of a fixed equipotential and parts of some external rays landing at preimages of α. If we consider these same combinatorially equipotentials and external rays in the parameter space we get a nested collection of Yoccoz parapuzzle pieces. By combinatorially the same we mean external rays with the same angle and equipotentials with the same values.

Definition. Given a parameter point c, the *parapuzzle piece of level* n, denoted by $P^n(c)$, is the set in parameter space whose boundary consists of the same (combinatorially) equipotentials and external rays as that of $f_c(V_0^n)$.

We mention the essential properties about the sets P^n used by Yoccoz. (The reader may wish to consult [Hub93] or [GM93].) The sets P^n are topological discs. For all points c in P^n, the Yoccoz puzzle pieces of the principal nest (up to level n) are combinatorially the same. This structural stability also applies to the off-critical pieces (up to level n) which are defined below. Hence, all parameter points in P^n may be renormalized in the same manner combinatorially up to level n. We also point out that the set P^n $(n > 0)$ intersects the Mandelbrot set only at Misiurewicz points.

Off-critical Puzzle Pieces

If a quadratic map is non-renormalizable, then at some level the principal nest is non-degenerate. In other words, there is some N such that for all $n \geq N$, $\operatorname{mod}(V_0^n, V_0^{n+1})$ is non-zero. For these same $n \geq N$, we may iterate the critical point by the map g_n some finite number of times until landing in $V_0^{n-1} \setminus V_0^n$ (otherwise the map would be renormalizable). Hence, to keep track of the critical orbit the generalized renormalization incorporates the following procedure. Let us fix a level n. For any point x in the closure of the critical orbit contained in $V_0^{n-1} \setminus V_0^n$, we iterate by f_c until it first returns back to the V_0^{n-1} puzzle piece. Denoting the number of iterates by l, we then take the largest connected neighborhood of x, say X, such that $f^l(X) = V_0^{n-1}$. We only save those sets X, denoted V_i^n $(i > 0)$, which intersect some point of the critical orbit. We point out that the collection of V_i^n are pairwise disjoint for

$n > 1$. The return map $f^l(V_i^n)$ restricted to the set V_i^n will still be denoted by g_n. The boundary of each V_i^n must be a union of external rays landing at points which are some preimage of α and pieces of some equipotential. Also, the return maps g_n restricted to V_i^n $(i \neq 0)$ are univalent. To review, for each level n we have a collection of disjoint puzzle pieces V_i^n and return maps,

$$\begin{aligned} \bigcup_i V_i^n &\subset V_0^n, \\ g_n(V_i^n) &= V_0^n. \end{aligned}$$

The Fibonacci Combinatorics

Let us denote the Fibonacci sequence by $u(n)$, where $u(n)$ represents the n-th Fibonacci number. The Fibonacci numbers are defined inductively: $u(0) = 1, u(1) = 1$ and $u(n) = u(n-1) + u(n-2)$. The dynamical condition for $f_{c_{fib}}$ (recall c_{fib} is real) is that for all Fibonacci numbers $u(n)$, we have $|f^{u(n)}(0)| < |f^{u(n-1)}(0)| < |f^i(0)|$, $u(n-1) < i < u(n)$. So the Fibonacci combinatorics require that the critical point return closest to itself at the Fibonacci iterates.

The generalized renormalization for the Fibonacci case is as follows. (See [LM93] and [Lyu93c] for a more detailed account. There is only one off-critical piece at every level, V_1^n. The return map of V_1^n to V_0^{n-1} is actually just the restriction of the map $g_{n-1} : V_0^{n-1} \to V_0^{n-2}$. We point out that the map g_{n-1} is the iterate $f^{u(n)}$ with restricted domain. In short we have

$$g_n \,(\simeq g_{n-2} \circ g_{n-1}) \;:\; V_0^n \to V_0^{n-1} \;(\textit{analytic double cover}),$$

$$g_{n-1} \;:\; V_1^n \to V_0^{n-1} \;(\textit{univalent}).$$

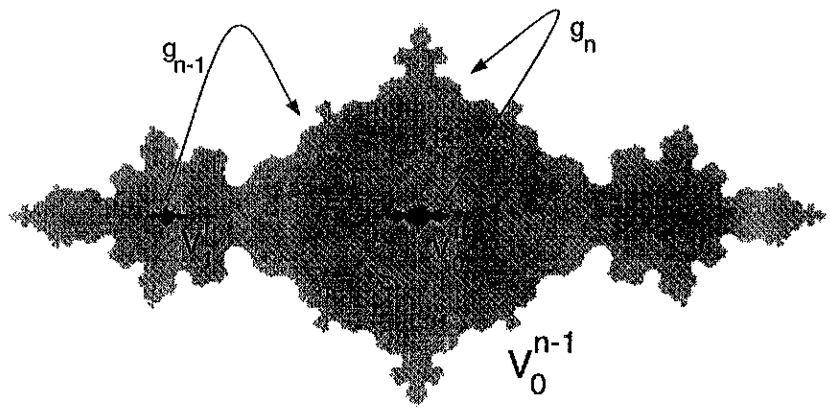

Figure 2.2: Generalized renormalization: Fibonacci type return with $n = 7$.

Finally, we define the puzzle piece $\widetilde{V}_1^{n+1}$ to be the set V_1^{n+1} which map to the central puzzle piece of the next level down under g_n. Namely, $\widetilde{V}_1^{n+1}$ is the set such that

$$\widetilde{V}_1^{n+1} \subset V_1^{n+1},$$

$$g_n(\widetilde{V}_1^{n+1}) = V_0^{n+1}.$$

Fibonacci Parapuzzle Pieces

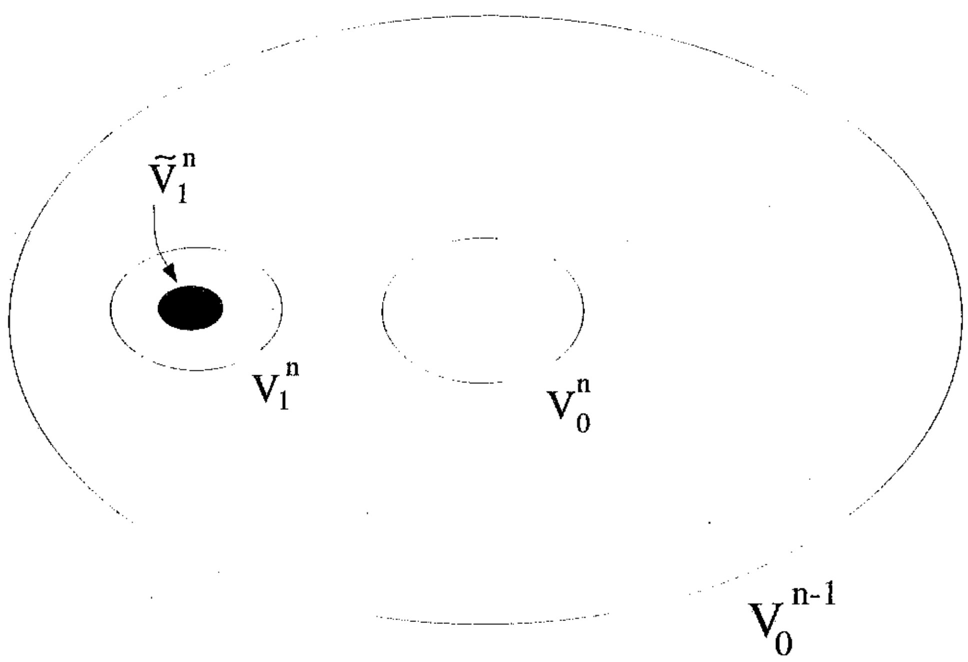

Figure 2.3: Fibonacci puzzle piece nesting.

A Fibonacci parapuzzle piece, P^n, is defined as the set with the same combinatorial boundary as that of $f(V_0^n)$. We also define an extra puzzle piece, Q^n. In particular, Q^n is a subset of P^{n-1} and hence may be renormalized in the Fibonacci way $n-1$ times. The boundary of the set Q^n is combinatorially the same as $f(V_1^{n-1})$. Finally observe that $P^n \subset Q^n \subset P^{n-1}$. Properties for P^n and Q^n are given below.

$$\begin{aligned} c \in P^n &\implies g_n(0) \in V_0^{n-1} \\ c \in Q^{n+1} &\implies g_n(0) \in V_1^n \\ c \in P^{n+1} &\implies g_n(0) \in \tilde{V}_1^n \end{aligned}$$

We warn the reader that the parameter value c has been suppressed as an index for the maps g and puzzle pieces V. Also, it is useful to use Figure 2.2 when tracing through the above properties of Q^n, P^n, and P^{n+1}, keeping in mind that $\widetilde{V}_1^n$, although too small for this picture, is contained in V_1^n (see Figure 2.3).

Lyubich's Motivating Result

The main motivating result of Lyubich is stated below. We will give a brief review of the proofs and also point out that the proof of part 2 of the theorem may be found in Lemma 4 of [Lyu93c], while the proof of part 1 is a direct consequence of part 2 and the scaling results of [Lyu93b] (see pages 11-12 of this paper). Finally, we point out that similar scaling results were obtained on the real line in Lemma 5.4 of [LM93].

Theorem 2.2.1 (Lyubich) *The principal nest of central Yoccoz puzzle pieces for the Fibonacci map has the following properties.*

1. The puzzle pieces scale down to the critical point in the following asymptotic manner:

$$\lim_{n\to\infty} \operatorname{mod}(V_0^{n-1}, V_0^n) \,/\, n = \frac{1}{3}\ln 2.$$

2. The rescaled puzzle pieces V_0^n have asymptotic geometry equal to the filled-in Julia set of $z \mapsto z^2 - 1$.

The scaling factor in the theorem is exactly half that for the parameter scaling. This is because here the scaling is done around the critical point as opposed to the critical value.

2.3 Beginning Geometry and Scaling

In studying the parameter space of complex dynamics, one first needs a strong command of the dynamics for all the parameter points involved. Hence, before proceeding in the parameter space we shall first study the geometry of the central puzzle pieces, $V_0^n(c)$, for all $c \in Q^n$.

Before stating a result similar to Theorem 2.2.1 for $c \in Q^n$, we indicate precisely how the rescaling of $V_0^n(c)$ is to be done. For the parameter point c_{fib}, we dilate (about the critical point) the set $V_0^n(c_{fib})$ by a positive real constant so that the boundary of the rescaled $V_0^n(c_{fib})$ intersects the point $(1+\sqrt{5})/2$, the non-dividing fixed point for the map $z^2 - 1$. If we dilate all central puzzle pieces $V_0^n(c_{fib})$ this way, we can then consider the rescaled return maps of g_n, denoted G_n, which map the dilated $V_0^n(c_{fib})$ to the dilated $V_0^{n-1}(c_{fib})$ as a two-to-one branched cover. The map G_n restricted to the real line either has a minimum or maximum at the critical point. To eliminate this orientation confusion, let us always rescale (so now possibly by a negative number) $V_0^n(c_{fib})$ so that the map G_n always has a local minimum at the critical point.

The point which maps to $(1+\sqrt{5})/2$ for $V_0^n(c_{fib})$ under this dilation we label β_n. Note that it must be some preimage of our original fixed point α and hence a landing point of one of the boundary rays. (Puzzle pieces may only intersect a Julia set at preimages of α.) This point β_n is parameter stable in that it may be continuously (actually holomorphically) followed for all $c \in Q^{n+1}$. Hence we may write $\beta_n(c)$. So for $c \in Q^{n+1}$, the rescaling procedure for $V_0^n(c)$ is to linearly scale (now possibly by a complex number) by taking $\beta_n(c)$ to $(1+\sqrt{5})/2$.

The geometric lemma below gives asymptotic structure results for the central puzzle pieces $V_0^n(c)$ for all $c \in Q^n$. In particular, the lemma indicates that as long as we can renormalize, the rescaled central puzzle pieces converge to the Julia set of z^2-1. The notation $J\{z^2-1\}$ is used to indicate the Julia set for the map $z \mapsto z^2-1$.

Before proceeding we give a brief review of the Thurston Transformation (see [DH93]) needed in the next lemma. Consider the Riemann sphere punctured at $\infty, -1, 0$, and $\frac{1+\sqrt{5}}{2}$. The map $\theta : z \mapsto z^2-1$ fixes ∞ and $\frac{1+\sqrt{5}}{2}$ while -1 and 0 form a two cycle. Consider any conformal structure ν on the Riemann sphere punctured at these points. We can pull this conformal structure back by the map θ. This induces a map T on the Teichmüller space of the four punctured sphere. A main result of this transformation T is as follows.

Theorem 2.3.1 (Thurston) *Given any conformal structure ν we have that $T^n(\nu)$ converges at exponential rate to the standard structure in the Teichmüller space.*

Lemma 2.3.2 (Geometry of central puzzle pieces) *Given $\epsilon > 0$, there exists an $N > 0$ such that for all $c \in Q^i$ where $i \geq N$ we have that the rescaled $\partial V_0^j(c)$ is ϵ-close in the Hausdorff metric around $J\{z^2-1\}$, where $i \geq j+1 \geq N$.*

Proof: First observe that the Julia set for $\Theta : z \mapsto z^2-1$ is hyperbolic. This means that given a small δ-neighborhood of the Julia set there is some uniform contraction under preimages. More precisely, there exists an integer m and value $K > 1$ such that for any point in the δ-neighborhood of $J\{z^2 - 1\}$ we have

$$\max_{y \in \Theta^{-m}(x)} \operatorname{dist}(y, J\{z^2 - 1\}) < \frac{1}{K} \operatorname{dist}(x, J\{z^2 - 1\}). \tag{2.2}$$

Returning to our Fibonacci renormalization, it is a consequence of the main theorem of Lyubich's paper [Lyu93b] that the moduli of the nested central puzzle pieces, i.e., $\operatorname{mod}(V_0^n, V_0^{n-1})$, grow at least at a linear rate independent of c. Hence, independent of our parameters c (although we must be able to renormalize in the Fibonacci sense), we have a definite growth in Koebe space for the map g_n. (This is somewhat misleading as the map g_n is really a quadratic map composed with some univalent map. Thus, when we say the map g_n has a large Koebe space, we really mean that the univalent return map has a large Koebe space.) The growing Koebe space implies that the rescaled maps $G_{n,c}$ have the following asymptotic behavior:

$$G_{n,c}(z) = (z^2 + k(n,c))(1 + O(p^n)). \tag{2.3}$$

The bounded error term $O(p^n)$ comes from the Koebe space and hence, by the above discussion, is independent of c.

We claim that $k(n,c) \to -1$ at an exponential rate in n, i.e., $|k(n,c)+1|$ exponentially decays. This result was shown to be true for $k(n, c_{fib})$ in Lemma 3 of [Lyu93c]. We use this result as well as its method of proof to show our claim. First, we review the method of proof used by Lyubich in the Fibonacci case. This was to apply Thurston's transformation on the tuple ∞, $G_n(0)$, 0, and $\frac{1+\sqrt{5}}{2}$. Pulling back this tuple by G_n results in a new tuple: ∞, the negative preimage of $G_n^{-1}(0)$, 0, and $\frac{1+\sqrt{5}}{2}$. Next, two facts are used concerning the negative preimage of $G_n^{-1}(0)$. The first is that it is bounded between 0 and $-\frac{1+\sqrt{5}}{2}$. (This is shown in [LM93].) The second is that the puzzle piece V_1^{n+1} is exponentially small compared to V_0^n (a consequence of the main result of [Lyu93b]); therefore, after rescaling, the points $G_n^{-1}(0)$ and $G_{n+1}(0)$ are exponentially close. Hence, the tuple map

$$(\infty, G_n(0), 0, \frac{1+\sqrt{5}}{2}) \mapsto (\infty, G_{n+1}(0), 0, \frac{1+\sqrt{5}}{2}) \tag{2.4}$$

is exponentially close to the Thurston transformation since the pull-back by G_n is exponentially close to a quadratic pull-back map (in the C^1 topology). The Thurston transformation is strictly contracting; hence, the tuple must converge to its fixed point $(\infty, -1, 0, \frac{1+\sqrt{5}}{2})$. Hence, we get $k(n, c_{fib}) \to -1$ at a uniformly exponential rate. This concludes the summation of the Fibonacci case.

To prove a similar result for our parameter values c, let us choose some large level n for which the c_{fib} tuple is close to its fixed point tuple and such that the Koebe space for g_n is large, i.e., $G_{n,c}$ is very close to a quadratic map. Then we can find a small neighborhood around c_{fib} in parameter space for which we still have a large Koebe space for g_n (notationally g_n is now c dependent) and its respective tuple is also close to the fixed point tuple. Then as long as the values c in this neighborhood are Fibonacci renormalizable, we claim the $|k(n,c)+1|$ exponentially decays. We know that the Koebe space growth is at least linear and independent of the value c; hence by the strict contraction of the Thurston transformation we get our claim of convergence for $k(n,c)$. Thus we may replace Equation (2.3) with

$$G_{n,c}(z) = (z^2 - 1)(1 + O(p^n)). \tag{2.5}$$

Returning to Equation (2.2), we can state a similar contraction for the maps $G_{n,c}$. In particular, in some small δ-neighborhood of $J\{z^2-1\}$, we can find a value k $(K > k > 1)$ and large positive integer N_1 so that for the same value m as in Equation (2.2) and for all $n > N_1$, we have

$$\max_{y \in G^{-1}_{n+m-1,c} \circ G^{-1}_{n+m-2,c} \circ \ldots \circ G^{-1}_{n,c}(x)} \operatorname{dist}(y, J\{z^2-1\}) < \frac{1}{k} \operatorname{dist}(x, J\{z^2-1\}), \tag{2.6}$$

as long as c is renormalizable in the Fibonacci sense, i.e., $n+m$ times.

From Equation (2.6), we conclude the lemma. We take the rescaled V_0^n and note that it contains the critical point and critical value. Hence we see that the topological annulus with boundaries $\partial \mathbb{D}_r$, r large, and ∂V_0^n under pull backs of $G_{n,c}$ must converge to the required set. This concludes the lemma. ⊙

The geometry of the puzzle pieces provides us with sufficient dynamical scaling results for the central puzzle pieces as well as for the off-critical puzzle pieces for $c \in Q^n$.

Lemma 2.3.3 *Given $\epsilon > 0$ there exists an N so that for all $c \in Q^n$, $n > N$, we have the following asymptotics for the moduli growth of the principal nest*

$$\left|\frac{\mathrm{mod}(V_0^n(c), V_0^{n+1}(c))}{n} - \frac{1}{3}\ln 2\right| < \epsilon. \qquad (2.7)$$

Proof: Notationally we will suppress the dependence of the parameter c. By Lyubich ([Lyu93b], page 12), the moduli growth from $\mathrm{mod}(V_0^{n-1}, V_0^n)$ to $\mathrm{mod}(V_0^n, V_0^{n+1})$ approaches $\frac{1}{2}(-\mathrm{cap}_\infty(J\{z^2-1\}) - \mathrm{cap}_0(J\{z^2-1\}))$. (See Appendix for the definition of capacity.) The proof of the growth relies only on the geometry of the puzzle pieces. The map g_n takes the annulus $V_0^n \setminus V_0^{n+1}$ as a two-to-one cover onto the annulus $V_0^{n-1} \setminus \widetilde{V}_1^n$. Hence we have the equality

$$\mathrm{mod}(V_0^n, V_0^{n+1}) = \frac{1}{2}\mathrm{mod}(V_0^{n-1}, \widetilde{V}_1^n). \qquad (2.8)$$

Using the Grötzsch inequality on the right hand modulus term, we have

$$\mathrm{mod}(V_0^{n-1}, \widetilde{V}_1^n) = \mathrm{mod}(V_0^{n-1}, V_1^n) + \mathrm{mod}(V_1^n, \widetilde{V}_1^n) + a(V_0^{n-1}, \widetilde{V}_1^n), \qquad (2.9)$$

where the function $a(V_0^{n-1}, \widetilde{V}_1^n)$ represents the Grötzsch error. By applying the map g_{n-1} to V_1^n we see that $\mathrm{mod}(V_1^n, \widetilde{V}_1^n)$ is equal to $\mathrm{mod}(V_0^{n-1}, V_0^n)$. The term $\mathrm{mod}(V_0^{n-1}, V_1^n)$ converges to $\mathrm{mod}(V_0^{n-2}, V_0^{n-1})$. This is easily seen by applying the map g_{n-1} which is a two-to-one branched cover with the critical point image being pinched away from V_0^{n-1} as $n \to \infty$. Finally, the Grötzsch error a depends only on the geometry of V_0^n because of the linear increase in modulus between both V_0^{n-1} and V_0^{n+1} and hence is approaching $-\mathrm{cap}_\infty(V_0^n) - \mathrm{cap}_0(V_0^n)$ (see Lemma A.0.4 in the Appendix). But this is approaching $-\mathrm{cap}_\infty(J\{z^2-1\}) - \mathrm{cap}_0(J\{z^2-1\})$ by Lemma 2.3.2 and the fact that the capacity function preserves convergence in the Hausdorff metric (see Lemma A.0.3). Finally, $-\mathrm{cap}_\infty(J\{z^2-1\}) - \mathrm{cap}_0(J\{z^2-1\})$ is shown to be equal to $\ln 2$ in the Appendix. Using the notation $m_n = \mathrm{mod}(V_0^{n-1}(c), V_0^n(c))$, we may rewrite Equation (2.9) as

$$m_{n+1} = \frac{1}{2}m_n + \frac{1}{2}m_{n-1} + \frac{1}{2}a + o(1),$$

where $a = \ln 2$. The asymptotics of this equation give the desired result. $\odot$

2.4 The Parameter Map

Dynamical puzzle piece rescaling

Now that we have a handle on the geometry of the central puzzle pieces for values c in our parapuzzle, let us consider rescaling the V_0^n in a slightly different manner. For each $c \in Q^n$ dilate V_0^n so that the point $g_n^{-1}(0)$ maps to -1. Notice this is just an exponentially small perturbation of our previous rescaling since there we had $G_n^{-1}(0)$ approaching -1 uniformly in n for all $c \in Q^n$. Hence Lemmas 2.3.2 and 2.3.3 still hold for this new rescaling. Let us denote this new rescaling map by $r_{n,c}$. Therefore, fixing $c \in Q^n$, the map $r_{n,c}$ is the complex linear map $x \mapsto (1/g_n^{-1}(0)) \cdot x$.

Lemma 2.4.1 *The rescaling map $r_{n,c}$ is analytic in c. In other words, $g_n^{-1}(0)$ is analytic in $c \in Q^n$.*

Proof: The roots of any polynomial vary analytically without branching provided no two collide. We claim the root in question does not collide with any other. But for all $c \in Q^n$ we have that the piece $V_1^{n+1}(c)$ can be followed univalently in c. Hence, we have our claim. $\odot$

We remind the reader that the map $g_{n,c}$ is just a polynomial in c. Let us define the analytic parameter map which allows us to compare the dynamical space and the parameter space.

The Parameter Map: The map $M_n(c)$ is defined as the map $c \mapsto r_{n,c} \cdot g_{n+1,c}(0)$ with domain $c \in Q^n$.

Since the map $r_{n,c}$ is just a dilation for fixed c, we see that if $M_n(c) = 0$ then this parameter value must be superstable. This superstable parameter value, denoted c_n, is the unique point which is Fibonacci renormalizable n times, and for the renormalized return map, the critical point returns precisely back to itself, i.e., $g_n(0) = 0$. Equivalently, this is the superstable parameter whose critical point has closest returns at the Fibonacci iterates until the $n+1$ Fibonacci iterate when it returns to itself, $f_{c_n}^{u(n+1)}(0) = 0$.

Lemma 2.4.2 (Univalence of the parameter map.) *For sufficiently large n, there exists a topological disc S^n such that $P^n \subset S^n \subset Q^n$, the map $M_n(c)$ is univalent in S^n, and* $\mathrm{mod}(S^n, P^n)$ *grows linearly in n.*

The proof of Lemma 2.4.2 is technical so we give an outline for the reader's convenience. We first show that the winding number is exactly 1 around the image -1 for the domain P^n. This will be a consequence of analysis of a finite number of Misiurewicz points along the boundary of P^n. Using Lemma 2.3.2 we will locate the positions (up to some small error) these selected Misiurewicz points must map to under $M_n(c)$. Then we prove that the image of the segments in ∂P^n between these Misiurewicz points is small, where "between" is defined by the combinatorial order of their rays and equipotentials. Hence, the $c \in \partial P^n$ have to follow the combinatorial order of the points of $J\{z^2-1\}$ without much error. Since we wind around -1 only once when traveling around $J\{z^2-1\}$ the only way we could have more than one preimage of -1 for the map $M_n(c)$ would be for one of these segments of ∂P^n to stretch a "large" distance and go around the point -1 a second time. But this cannot happen if the segments follow the order of $J\{z^2-1\}$ without much error. Finally, we show that this degree one property extends to some increasingly large image around -1 in Lemma 2.4.3.

Proof: We will again use the map $\Theta(z) = z^2 - 1$. Let $b_0 = \frac{1+\sqrt{5}}{2}$ be the non-dividing fixed point for the Julia set of Θ. The landing ray for this point is the 0-ray. Taking a collection of pre-images of b_0 under the map Θ we may order them by the angle of the ray that lands at each point. (Note that there is only one angle for each point.) The notation for this combinatorial order of preimages will be $b_0, \ldots b_i, b_{i+1}, \ldots, b_0$.

Since the point b_0 is in the Julia set of Θ, the set of all preimages of b_0 is dense in the Julia set. Given that this Julia set is locally connected we have the following density property of the preimages of b_0: given any $\epsilon > 0$, we can find an l so that the collection of preimages $\Theta^{-l}(b_0)$ is such that the Julia set between any two successive points (in combinatorial order) is compactly contained in an ϵ-ball. In other words, for this set $\Theta^{-l}(b_0)$, given any b_i and

b_{i+1}, the combinatorial section of the Julia set of Θ between these two points is compactly contained in an ϵ-ball.

For each $c \in Q^n$ we define an analogous set of points $b_{i,n}(c)$ along the boundary of the rescaled puzzle pieces $V_0^n(c)$. First let us return to our old way of rescaling $V_0^n(c)$, taking the point $\beta_n(c)$ to $1 + \sqrt{5}/2$ (see page 22). For our value l above we take a set of points to be preimages of $(1+\sqrt{5})/2$ under the map $G_{n-1} \circ ... \circ G_{n-l}$ for each c. These points are on the boundary of the rescaled $V_0^n(c)$ and in particular are endpoints of some of the landing rays which make up some of the boundary of the rescaled V_0^n. In particular, we may label and order this set of preimages $b_{i,n}(c)$ by the angles of their landing rays. Hence we may also refer to a piece of the rescaled boundary of $V_0^n(c)$ as a piece of the boundary that is combinatorially between two successive $b_{i,n}(c)$.

We claim that for n large enough we have that for all $c \in Q^n$ these combinatorial pieces of $r_{n,c}(V_0^n)$, say from $b_{i,n}$ to $b_{i+1,n}$, is in the exact same ϵ-ball as their b_i to b_{i+1} piece counterpart. For this claim we first want $b_{i,n}(c) \to b_i$ as $n \to \infty$. But this is true (for this rescaling) by the proof of Lemma 2.3.2 since the rescaled maps G_n converge to Θ exponentially.

Now that we have a nice control of where the Misiurewicz points of ∂P^n are landing, we focus on the boundary segments of P^n between them. Note that by Theorem 2.1.3 of Douady and Hubbard, we have a good combinatorial description of ∂P^n in terms of rays and equipotentials. Combinatorially the image of these boundary segments under the map M_n will be in the appropriate boundary segments of the dynamical puzzle pieces. Therefore, we focus on controlling the combinatorial segments between the $b_{i,n}$ along the central puzzle pieces in dynamical space. With precise information on where these combinatorial segments are in dynamical space we make conclusions on the image of ∂P^n.

Now we prove that the combinatorial piece between $b_{i,n}(c)$ and $b_{i+1,n}(c)$ converges to the combinatorial piece from b_i to b_{i+1} in the Hausdorff metric. Let us take a small neighborhood around c_{fib} such that the rescaled V_0^n are in some small neighborhood around $J\{z^2-1\}$. For all c in this neighborhood, take the combinatorial piece $b_{i,n}(c)$ to $b_{i+1,n}(c)$ such that the distance (in the Hausdorff metric) is greatest from b_i to b_{i+1}. Suppose this distance is δ, then after m preimages (the value m being the same as in Lemma 2.3.2, see Equations (2.2) and (2.6)), the distances between these preimages is less than δ/λ^m, where $\lambda > 1$ and is independent of the parameter. Finally, notice that for the b_i segments, any preimages of a combinatorial segment must be contained in another b_i segment (the Markov property). Hence, we actually get convergence at an exponential rate.

To review, the points $c \in \partial P^n$ under the map $M_n(c)$ must traverse around the point -1 with each appropriate Misiurewicz point landing very near b_i since for all c, $b_{i,n}(c) \to b_i$. But each combinatorial piece is also very near the combinatorial piece for the Julia set of Θ and the Julia set has winding number 1 around the point -1 which completes the winding number argument for this rescaling. Now if we rescale by $r_{n,c}$ instead of the old way (they are exponentially close) the same result holds. This completes the proof of the univalence of the map at least in some small image containing -1. The lemma below will complete the proof of this lemma. $\odot$

Lemma 2.4.3 *For all sufficiently large n, there exists $R(n) \to \infty$ as $n \to \infty$ such that the map $c \mapsto M_n(c)$ is univalent onto the disc $D(-1, R(n))$.*

Proof: The image of any point $c \in \partial Q^n$ under the map $M_n(c)$ is contained in the set $r_{n,c}(V_0^{n-1}(c))$. But the boundary of $r_{n,c}(V_0^{n-1}(c))$ under the rescaling of $r_{n,c}$ is very far from $r_{n,c}(V_0^n(c))$ by the modulus growth proven in Lemma 2.3.3 (see Appendix, Proposition A.0.5 and reference). Let R equal the minimum distance from the image of ∂Q^n to the origin. Note that $Q^n \setminus P^n$ cannot contain the point -1 in its image under $M_n(c)$ since the closest these points can map to -1 is when they map into a small neighborhood of $J\{z^2 - 1\}$. Since we showed in the proof above that the winding number around -1 for $M_n(\partial P^n)$ is one, we must have the same result for $M_n(\partial Q^n)$ since -1 can have no new preimages in this domain $Q^n \setminus P^n$. Hence, the winding number is one for all points in the disc of radius R. Thus, the map M_n must be univalent in some domain with image (at least) the disc centered at 0 and radius R. Taking the preimage of this disc will define the desired set in parameter space, S^n. The result follows and hence does Lemma 2.4.2. $\odot$

Lemma 2.4.2 also allows us to give the geometric result of the Main Theorem. As n increases we have an increasingly large Koebe space around the image of ∂P^n. Since the image of ∂P^n under the map $M_n(c)$ must asymptotically approach that of $J\{z^2 - 1\}$, the parapuzzle pieces must also asymptotically approach this same geometry by application of the Koebe Theorem. Hence, by Lemma 2.4.2, we get the geometric result of the Main Theorem.

Theorem 2.4.4 (Theorem A, part 2) *The rescaled parapuzzle pieces P^n converge at an exponential rate in the Hausdorff metric to the filled-in Julia set of $z^2 - 1$.*

2.5 Parapuzzle Scaling Bounds

To understand the scaling in parameter space, we focus on the image of the parapuzzle pieces P^n and P^{n+1} under the parameter map M_n. Since M_n is nearly a linear map for the domain P^n, we are in a good position to prove the scaling results of the Main Theorem A.

Theorem 2.5.1 (Theorem A, part 1.) *The principal nest of Yoccoz parapuzzle pieces P^n for the Fibonacci point c_{fib} scale down in the following asymptotic manner:*

$$\lim_{n\to\infty} \operatorname{mod}(P^n, P^{n+1}) \;/\; n = \frac{2}{3}\ln 2.$$

Proof: We begin by defining two bounding discs for the $J\{z^2-1\}$. Take as a center the point 0 and fix a radius T so that the disc $D(0,T)$ compactly contains $J\{z^2-1\}$. Also take a radius t so that the disc $D(0,t)$ is strictly contained in the immediate basin of 0 for $J\{z^2-1\}$. (See Figure 2.4.) This gives

$$J\{z^2-1\} \subset D(0,T) \setminus D(0,t). \tag{2.10}$$

Let us calculate the scaling properties of the image of ∂P^{n+1} under the same map M_n. Again we will have that the image of ∂P^{n+1} "looks" like

$J\{z^2 - 1\}$ although at a much smaller scale. We remind the reader that M_n maps the point c_{n+1} to -1. Now we claim that the point c_{n+1} acts as the "center" of ∂P^{n+1} in the following sense:

$$M_n(\partial P^{n+1}) \subset D(-1, \frac{M_n'}{M_{n+1}'}T) \setminus D(-1, \frac{M_n'}{M_{n+1}'}t), \tag{2.11}$$

where M_n' represents the derivative of M_n at the point c_n. To prove the claim we note that $M_{n+1}(\partial P^{n+1}) \subset D(0,T) \setminus D(0,t)$. Pulling this image back by the univalent map $M_n \circ M_{n+1}^{-1}$ and noting that this map has increasing Koebe space for our domain proves this claim.

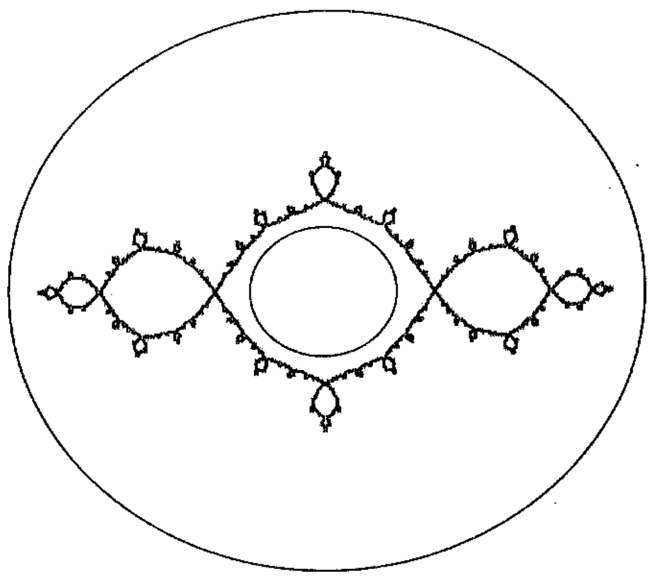

Figure 2.4: The centering property for the Julia set of $z^2 - 1$.

Now let us observe what is happening dynamically for all $c \in P^{n+1}$. We have that $r_{n,c}(\tilde{V}_1^n)(c)$ is also centered around -1 by the construction of $r_{n,c}$. Hence we have a result similar to that in expression (2.10), although perhaps with different radii. Most importantly, however, the different radii must preserve the same centering ratio seen in expression (2.11), i.e., $\frac{T}{t}$.

To compare the centerings of the dynamical and parameter sets above, we focus on the Fibonacci point c_{fib}. We have that the point $M_n(c_{fib})$ is contained in the topological annulus of expression (2.11). But this image must also be contained in the centering annulus of $r_{n,c_{fib}}(\partial \widetilde{V}_1^n)$ in the dynamical space. Geometrically the point $M_n(c_{fib})$ is to the sets $r_{n,c_{fib}}(\partial \widetilde{V}_1^n)$ and $M_n(\partial P^{n+1})$ as the -1 point is to the Julia set of $z \rightarrow z^2 - 1$ up to exponentially small error. Hence we have the following equivalent centerings

$$r_{n,c_{fib}}(\partial \widetilde{V}_1^n)(c_{fib}) \subset D(-1, \frac{M_n'}{M_{n+1}'}T) \setminus D(-1, \frac{M_n'}{M_{n+1}'}t), \tag{2.12}$$

$$M_n(\partial P^{n+1}) \subset D(-1, \frac{M_n'}{M_{n+1}'}T) \setminus D(-1, \frac{M_n'}{M_{n+1}'}t). \tag{2.13}$$

Let us rewrite the scaling estimate of Equation (2.8) from Lemma 2.3.3,

$$\lim_{n\to\infty} \operatorname{mod}\left(V_0^{n-1}(c_{fib}), \widetilde{V}_1^n(c_{fib})\right) \ / \ n = \frac{2}{3}\ln 2.$$

Since the modulus function is preserved under rescalings, we apply $r_{n,c_{fib}}$ to get

$$\lim_{n\to\infty} \operatorname{mod}\left(r_{n,c_{fib}}(V_0^{n-1}(c_{fib})), r_{n,c_{fib}}(\widetilde{V}_1^n(c_{fib}))\right) \ / \ n = \frac{2}{3}\ln 2. \tag{2.14}$$

Expressions (2.12) and (2.13) and Equation (2.14) combined with Lemma 2.4.2 give

$$\lim_{n\to\infty} \text{mod}\left(M_n(P^n), M_n(P^{n+1})\right) \ / \ n = \frac{2}{3}\ln 2, \tag{2.15}$$

which completes the proof of Theorem A, part 2. ⊙

2.6 Hairiness at the Fibonacci Parameter

Let us define the Mandelbrot dilation for the Fibonacci point given by the renormalization. We wish to dilate the Mandelbrot set, M, about the Fibonacci parameter point by taking the approximating superstable parameter points c_n to some fixed value for each n. Of course, we have been doing a similar kind of dilation in the previous section so we will take advantage of this work and rescale in the following more well-defined manner.

Mandelbrot rescaling: Let R_n be the linear map acting on the parameter plane which takes c_{fib} to -1 and c_n to 0. Notice that this is nearly the same map as our parameter map M_n. The maps M_n have an increasing Koebe space, take c_n to 0, and asymptotically takes c_{fib} to -1.

The proof of hairiness will be a consequence of the geometry of the external rays which make up pieces of the boundary of the principal nest puzzle pieces, $V_0^n(c)$. Before proving this theorem, we first give a combinatorial description of how these rays lie in the dynamical space for the Fibonacci parameter.

We remind the reader that $\beta_{n,0}$ is on the boundary of V_0^n and is the landing point of two external rays. We label the union of these two rays of

$\beta_{n,0}$ as $\gamma(\beta_{n,0})$. The curve $\gamma(\beta_{n,0})$ divides the complex plane into two regions. We label the region which does not contain the piece puzzle V_0^n as $\Gamma(\beta_{n,0})$.

We also define similar objects $\Gamma(x)$ and $\gamma(x)$ for the other Julia set points x on the boundary of V_0^n. To start, we have the symmetric point $\beta_{n,1}$ of $\beta_{n,0}$, and note $g_n(\beta_{n,1}) = g_n(\beta_{n,0})$. We can exhaust all other Julia set points on the boundary of V_0^n, denoting them as $\beta_{n,i}$ where $g_{n-i+1} \circ g_{n-i-1} \circ \ldots \circ g_n(\beta_{n,i}) = \beta_{n-i,0}$ for $2 \leq i \leq n$. Of course this representation is not unique in the variable i but we will not need to distinguish between these various $\beta_{n,i}$ points. For each of the $\beta_{n,i}$ points we can define $\gamma(\beta_{n,i})$ as the union of the two external rays which land there. Similarly we define the $\Gamma(\beta_{n,i})$ region as we did for $\beta_{n,0}$. In particular, $\Gamma(\beta_{n,i})$ has boundary $\gamma(\beta_{n,i})$ and does not contain V_0^n.

The combinatorial properties for the γ and Γ sets are easy to determine for the Fibonacci parameter. First we have that $|\beta_{n,0}| < |\beta_{n-1,0}|$ where the absolute values are necessary since the β's change orientation (see page 22). If the $\beta_{n,0}$ and $\beta_{n-1,0}$ have the same sign then $\Gamma(\beta_{n,0}) \supset \Gamma_{n-1}(\beta_{n-1,0})$, otherwise we replace $\beta_{n,0}$ with its symmetric point to achieve this inclusion. By application of pull-backs of g_n it is easy to see that

$$\bigcup_i \Gamma(\beta_{n,i}) \supset \bigcup_i \Gamma(\beta_{n-1,i}). \tag{2.16}$$

Since this is just a combinatorial property depending on the first n Fibonacci renormalizations, this property holds as we vary our parameter c in Q^n. As a direct consequence of expression (2.16), we conclude that

$$J_c \cap (V_0^{n-1}(c) \setminus V_0^n(c)) \subset \bigcup_i \Gamma_n(\beta_{n,i}(c)). \tag{2.17}$$

By the dynamical scaling results we know that if we rescale the left side of expression (2.17) by $r_{n,c}$ then ∂V_0^{n-1} tends to infinity while $V_0^n(c)$ stays bounded (see Appendix, Proposition A.0.5). Hence for connected Julia sets the appropriate connected pieces must "squeeze through" the Γ regions in $V_0^{n-1} \setminus V_0^n$. We will be able to conclude the hairiness theorem by application of our map M_n and by the geometry of the Γ regions, i.e., the controlled "hairiness" of J_c. We show that the rescaled Γ regions, i.e., $r_{c,n}(\Gamma_n(\beta_{n,0}(c)))$ are converging to the 0-ray of $J\{z^2-1\}$. (Compare Figures 2.5 and 2.6 with 2.7.) Also we show that $r_{c,n}(\Gamma_{n+1}(\beta_{n+1,0}(c)))$ converges to the inner 0-ray of the Fatou component containing 0 for $J\{z^2-1\}$.

Lemma 2.6.1 *For $c \in P^n$ the linear rescaling maps $r_{n,c}$ and $r_{n,c_{fib}}$ have asymptotically the same argument, $|\arg(r_{n,c}) - \arg(r_{n,c_{fib}})| \to 0$ modulo π.*

Proof: The return maps g_n are asymptotically z^2-1 post-composed and pre-composed by a linear dilation. When our return maps have a large Koebe space we see that the rescaling argument difference (as in the Lemma) converges to a constant modulo π. For the Fibonacci parameter case we are always rescaling by a real value so the difference is 0 modulo π. Since we are scaling down to the Fibonacci parameter we get the desired result. $\odot$

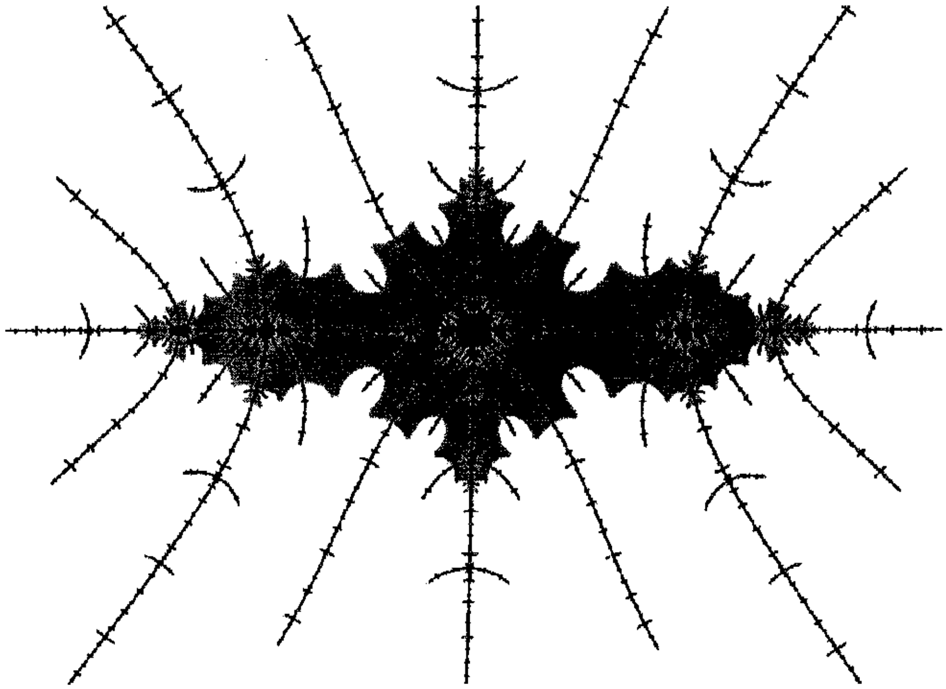

Figure 2.5: Parapuzzle piece P^6 with Mandelbrot set.

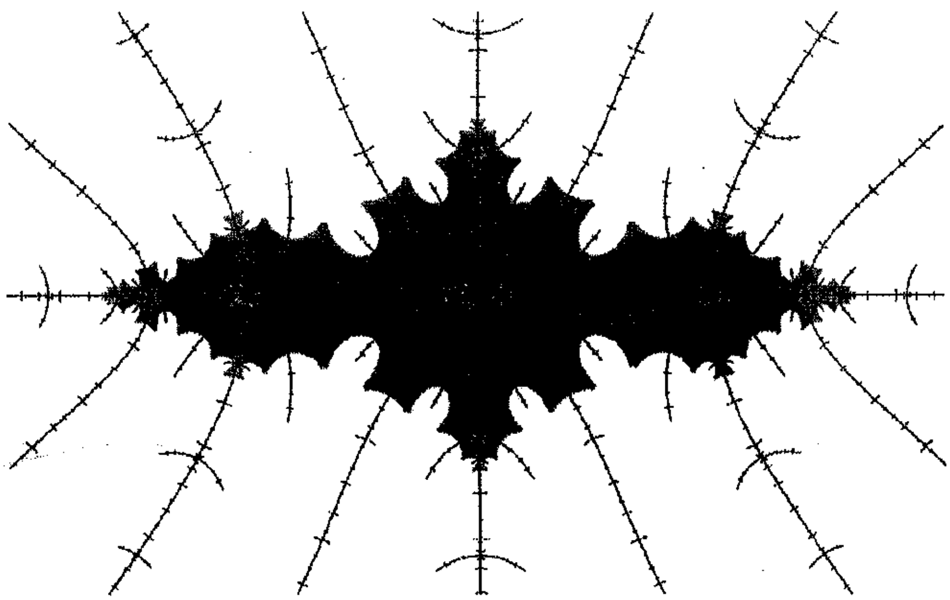

Figure 2.6: Dynamical central puzzle piece V_0^6 for the Fibonacci Julia set.

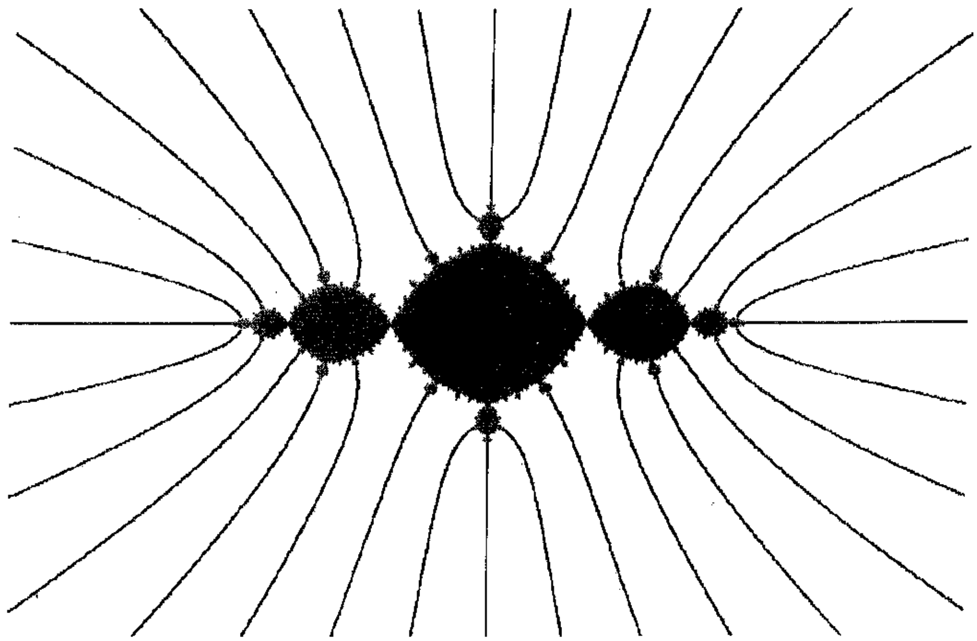

Figure 2.7: The 0-ray and some of its preimages for the Julia set of $z^2 - 1$.

Lemma 2.6.2 *For discs $D(0,\rho)$ in the plane, there exists an $N(\rho) > 0$ so that for all $n > N$ the curves $r_{n,c}(\Gamma(\beta_{n,0}(c)))$ converge to the 0-ray of the Julia set of $z \mapsto z^2 - 1$ in the Hausdorff metric in $D(0,\rho)$. Also, the curves $r_{n,c}(\Gamma(\beta_{n+1,0}(c)))$ converge to the inner 0-ray of the Fatou component containing 0 for the Julia set of $z \mapsto z^2 - 1$.*

Proof:

By Lemma 2.6.1 the rescaling maps $r_{n,c}$ converge to a real dilation. Hence there is a decreasing amount of "rotation" in the return map $g_{n,c}$. In particular, the return maps g_n are close to $z^2 - 1$ post-composed and pre-composed with a real rescaling in the C^1 topology. Let us focus on the curves $r_{n,c}(\Gamma(\beta_{n,0}(c)))$. Since we know that the pull-backs are essentially $z^2 - 1$, the curves should converge as stated in the theorem. However, there are two difficulties. First, our G_n pull-backs are not defined in all of $\mathbb{C}$ and second, z^2-1 is contracting under preimages. Hence, we check that after pulling back our curves $r_{n,c}(\Gamma(\beta_{n,0}(c)))$ by G_n that their extensions (i.e., the rescaled pull-back of the whole curve by the appropriate f iterate) have some a priori bounds.

Let us take the set $\gamma_n \cap V_0^{n-2}$ and pull-back by $g_n \circ g_{n-1}$. Taking the appropriate branches we get $(\gamma(\beta_{n+1,0}) \cup \gamma(\beta_{n+1,1}) \cup \gamma(\beta_{n+1,2})) \cap V_0^n$. In particular, the endpoints of $\gamma(\beta_{n+1,0})$ lie on the boundary of V_0^n. Hence their extension is determined by property (2.16) (the geometry of the rays of the previous level). In particular, we have that $\gamma(\beta_{n+1,0}) \cap (V_0^{n-1} \setminus V_0^n)$ is combinatorially between $\Gamma(\beta_{n,0})$ and $\Gamma(\beta_{n,2})$. The piece of $\gamma(\beta_{n+1,0})$ contained in V_0^n is controlled by the nearly z^2-1 pull-backs (the maps $g_{n-1} \circ g_{n-2}$ after rescaling) of again lesser level rays as constructed above. So let us assume the sets $\gamma(\beta_{j,0})$, $\gamma(\beta_{j,1})$ and $\gamma(\beta_{j,2})$, $j = \{n, n-1\}$ nicely lie in the appropriate half-planes, where nice means that $r_{c,n}(\gamma(\beta_{j,0}))$ is in the right-half plane, $r_{c,n}(\gamma(\beta_{j,1}))$ in the left-half plane, and $r_{c,n}(\gamma(\beta_{j,2}))$ in the upper-half plane. Then by the above argument we have that the collection $r_{n+1,c}(\gamma(\beta_{j,0}))$, $r_{n+1,c}(\gamma(\beta_{j,1}))$ and $r_{n+1,c}(\gamma(\beta_{j,2}))$ with $j = n+1$ is also nice in that they lie in the appropriate half-planes. This completes the induction step.

The initial step comes from the fact that the geometry is nice in the Fibonacci case. More precisely we have that $r_{c,n}(\gamma(\beta_{n,0}))$ is contained in the right-half plane just by symmetry. If we pull-back as above we see that $r_{c,n}(\gamma(\beta_{n,0}))$ must be contained in the right-half plane. Hence we may perturb this set-up in a small parameter neighborhood to start the induction process.

Because the return maps G_n uniformly (in parameter c) approach z^2-1, we may use the a priori bounds and the coordinates from the Böttcher map of z^2-1 to conclude that the rescaled rays $r_{c,n}(\Gamma(\beta_{n,0})$ must uniformly approach the 0-ray of z^2-1 in compact sets. Finally, viewing this same pull-back argument inside of $r_{n,c}(V_0^n)$ for the curves $r_{n,c}(\Gamma(\beta_{n+1,0}(c)))$ yield convergence to the inner 0-ray and completes the lemma.

$\odot$

We are now in a good position to prove hairiness in an arbitrary disc $D(z,\epsilon) \subset \mathbb{C}$. We point out that if z is in $J\{z^2-1\}$, the theorem holds by Lemma 2.4.2. In this lemma we showed that the Misiurewicz points on the boundary of P^n under our rescaling map, M_n, converge to the preimages of the β fixed point of z^2-1. Note that the preimages of the β fixed point are dense in $J\{z^2-1\}$. Given that the map $M_n(c)$ is an exponentially small perturbation of $R_n(c)$ we must have hairiness for neighborhoods of such z and this claim is proven. In fact, the above argument shows that it suffices to show that images $M_n(\mathbf{M})$ satisfy the Theorem B.

Proof of hairiness:

Proof: We first focus on the structure of J_c for parameters c in Q^n. By Lemma 2.6.2, we have that $r_{c,n}(\Gamma(\beta_{n,0}(c)))$ converges to the 0-ray of $J\{z^2-1\}$ in bounded regions. Hence, for $c \in \mathbf{M} \cap P^n$ we must have that its Julia set in this region, i.e., $r_{c,n}(\Gamma(\beta_{n,0}(c))) \cap J_c$, also converges to the 0-ray (compare property (2.17)). Now the image of $M_n(\mathbf{M} \cap Q^n)$ must map into the set

$\cup_c r_{c,n}(\Gamma(\beta_{n,0}(c)) \cap J_c)$. Also, this domain contains the Misiurewicz point, say c', which lands at the rescaled β point $r_{c',n}(\beta_{n-1,0})$. But $r_{c',n}(V_0^{n-1})$ is growing at an exponential rate while $r_{c,n}(\beta_{n,0})$, the "other" end of this image, converges to the β fixed point of $z^2 - 1$ for all $c \in Q^n$. Note we must have a Misiurewicz point landing near this β point as well. Because the Mandelbrot set is connected we get that a piece of the image $M_n(\mathbf{M} \cap Q^n)$ converges to the 0-ray of $J\{z^2-1\}$. Similarly we have convergence of $\cup_c r_{c,n}(\Gamma(\beta_{n+1,0}(c)) \cap J_c)$ to the inner 0-ray of $J\{z^2-1\}$. Hence pieces of $M_n(\mathbf{M} \cap Q^n)$ also have convergence to this inner 0-ray.

So given an arbitrary disc $D(z, \epsilon)$, we iterate it forward by $z^2 - 1$ until it intersects the 0-ray or inner 0-ray of the Julia set of $z \mapsto z^2 - 1$. By the above we have that this image will eventually intersect all Julia sets of $P^n \cap \mathbf{M}$. Pulling back by our almost $z \mapsto z^2 - 1$ maps shows that all Julia sets $P^n \cap \mathbf{M}$ must eventually intersect $D(z, \epsilon)$. Applying our parameter map and arguing as above yields hairiness. $\odot$

Chapter 3

Geometry of the Critical Orbit

A *unimodal* map is a map which takes an interval into itself, mapping boundary into boundary, and has exactly one interior global maximum or minimum. The Fibonacci map of the previous chapter, when restricted to the interval with appropriate boundary (the beta fixed point and its conjugate), is an example of a unimodal map. We assume our unimodal maps have C^2 regularity, are symmetric, and have their critical point at 0. We also assume that the critical point is non-degenerate, i.e., $f''(0) \neq 0$.

One particular class of maps studied in [Lyu93a] (page 9) are the *non-renormalizable persistently recurrent* maps. A *renormalizable* map is one in which there is some interior interval containing the critical point which maps unimodally into itself with boundary mapping to boundary for some iterate of f. To define *persistently recurrent* we remark that it is the opposite of *reluctantly recurrent.* A unimodal map is called *reluctantly recurrent* if there exists a point x in $\overline{\mathcal{O}}(f)$ and neighborhood U such that there are arbitrarily long monotone pull-backs of U under the backwards orbit of points in $U \cap \overline{\mathcal{O}}(f)$.

To understand the combinatorics for non-renormalizable maps we will review the *generalized renormalization* procedure developed in [LM93] and [Lyu93a].

3.1 Generalized Renormalization

A *unimodal* map is a map which takes an interval into itself, mapping boundary into boundary, and has exactly one interior global maximum or minimum. The Fibonacci map of the previous chapter, when restricted to the interval with appropriate boundary (the beta fixed point and its conjugate), is an example of a unimodal map. We assume our unimodal maps have C^2 regularity, are symmetric, and have their critical point at 0. We also assume that the critical point is non-degenerate, i.e., $f''(0) \neq 0$.

One main class of maps studied in [Lyu93a] (page 9) are the *non-renormalizable persistently recurrent* maps. A *renormalizable* map is one in which there is some interior interval containing the critical point which maps unimodally into itself with boundary mapping to boundary for some iterate of f. To define *persistently recurrent*, we remark that it is the opposite of *reluctantly recurrent.* A unimodal map is called *reluctantly recurrent* if there exists a point x in $\overline{\mathcal{O}}(f)$ and neighborhood U such that there are arbitrarily long monotone pull-backs of U under the backwards orbit of points in $U \cap \overline{\mathcal{O}}(f)$. To understand the combinatorics for non-renormalizable maps, we review the *generalized renormalization* procedure developed in [LM93] and [Lyu93a].

The generalized renormalization procedure was first developed for the Fibonacci case in [LM93] and was enhanced for the recurrent non-renormalizable case in section 2 of [Lyu93a]. This procedure gives a nice combinatorial description of the dynamics of $\overline{\mathcal{O}}$. We briefly describe generalized renormalization and review the main results of [Lyu93a] as it pertains to this chapter.

The generalized renormalization procedure is a kind of Markov partition for the dynamics of f. To begin take the interval whose boundary consists of the interior fixed point and its symmetric point. This is an example of a *nice* interval ([Mar94]), an interval in which the forward orbit of the boundary never lands in the interior of the interval. We denote this interval by I_0^1. We now iterate the critical point until it first lands back inside I_0^1. Under this iterate take the domain around 0 which maps unimodally into I_0^1 with boundary mapping into boundary. This interval is denoted by I_0^2 and is contained in I_0^1. Notice that I_0^2 must be *nice* as well. Now consider the set $I_0^1 \setminus I_0^2$. Take any point x in this set which lands in I_0^2 under some iterate. Take the interval containing the point x which maps onto I_0^2 under this first return for x, with boundary mapping to boundary. After applying this procedure for all $x \in I_0^1 \setminus I_0^2$ which return to I_0^2 we save only those intervals which contain points of $\overline{\mathcal{O}}(f)$. We index these intervals by I_i^2 ($i \neq 0$). All of these intervals map diffeomorphically to I_0^1 under some iterate of f. The inductive procedure is to start with I_0^n as the nice interval and construct intervals I_i^{n+1} (where the interval I_0^{n+1} always denotes the interval containing the critical point). This completes the generalized renormalization procedure.

The intervals of the form I_i^n are said to be of *level n*. The return maps for the level n intervals will be denoted by $f^{(n,i)}$. Hence, we have for $i \neq 0$ the diffeomorphism $f^{(n,i)}$ mapping I_i^n onto I_0^{n-1}. For $i = 0$, the return map

$f^{(n,0)}$ of I_0^n to I_0^{n-1} is unimodal with the endpoints of I_0^n mapping to one of the endpoints of I_0^{n-1}. The collection of return maps for level n are denoted by g_n. Hence we have,

$$g_n : \bigcup I_i^n \to I_0^{n-1}. \tag{3.1}$$

We state without proof properties of the intervals I_i^n and their return maps. (Again see [Lyu93a] or [Lyu93b] for details.) It is important to notice that there are extensions for these return maps $f^{(n,i)}$ to a much larger domain that preserve the diffeomorphic or unimodal properties (see **3** and **4**).

Combinatorics for level n:

1. For a fixed n, (n¿1), the intervals I_i^n are pairwise disjoint and contained in I_0^{n-1}.

2. The union of the intervals I_i^n contain the set $\overline{\mathcal{O}}(f) \bigcap I_0^{n-1}$.

3. (Koebe space for $f(I_i^n)$) The diffeomorphic return map $f^{(n,0)-1}$ for the interval $f(I_i^n)$ extends to a larger domain. There exists an interval J_i^n containing $f(I_i^n)$ such that $f^{(n,i)-1} : J^n \to I_0^{n-2}$ is a diffeomorphism mapping boundary into boundary.

4. (Nesting property) For any interval I_i^n and positive integer $j < (n, i)$, if $f^j(I_i^n)$ is a strict subset of I_0^k for some k, then there exists an l such that $f^j(I_i^n) \subset I_l^{k+1}$.

Before stating a result of Lyubich regarding the scaling factors, we need to remark on a special kind of return. This special case is when $g_n(0) \subset I_0^n$, i.e., the critical point immediately returns to the n-th level. This is called a *central return* of level n. We denote the number of levels, less than or equal to n, with non-central returns by $\kappa(n)$. If the return for level n is non-central, then $\kappa(n) = \kappa(n-1)+1$; otherwise, $\kappa(n) = \kappa(n-1)$. We now state one of the main results of [Lyu93a] concerning the scaling factors $|I_0^n|/|I_0^{n-1}|$, which are denoted by μ_n.

Theorem 3.1.1 (Lyubich)

The scaling factors, μ_n, shrink to 0 at least exponentially in $\kappa(n)$.

The fact that the scaling factors tend to zero allows for a control of the non-linearity of the maps $f^{(n,i)}$ for the domains I_i^n. This is because there is a large Koebe space around the image. (See rules **2** and **3** above.) In particular, given a non-renormalizable map f (as above), there exists a $p < 1$ such that the map $f^{(n,0)}$ may be rewritten as the composition of a quadratic map and an approximately linear map. In other words we have that $f^{(n,i)} = h \circ \Phi$, where $\Phi(x) = x^2 + c$ and $h(x) = Cx(1 \pm O(p^{\kappa(n)})$. The constant error term $O(p^{\kappa(n)})$ depends only on the non-linearity of the original map f.

3.2 Combinatorial Assumptions

In this section, we define the sufficient combinatorial conditions for Theorem C. Notice that by rule **1** of Section 3.1, every interval of level $n+1$ is contained in the central piece of level n. Therefore, we may apply the map $f^{(n,0)}$ to any interval of level $n+1$. Let us take the central interval I_0^{n+1} and inductively apply iterates of the type $f^{(j,0)}$, $j < n$, as "*efficiently*" as possible, i.e., by applying the following inductive procedure. To start, we apply $f^{(n-1,0)}$ to the interval I_0^{n+1}. Next suppose we have $L = f^{(j,0)} \circ \ldots \circ f^{(n,0)}(I_0^{n+1})$, and $L \subset I_0^k$, but $L \not\subset I_0^{k+1}$ for $k < n$ (note that this may not be a monotone decreasing sequence of $(i, 0)$). Then we apply the map $f^{(k,0)}$ to L. (It can happen that our interval L lies in I_0^1 but not I_0^2. In this case, we use the return map $f^{(2,i)}$ where $L \subset I_i^2$.) We continue this induction until the interval I_0^{n+1} returns to I_0^n. This procedure results in an *efficient decomposition* of $f^{(n,0)}$ with respect to I_0^n. The next lemma shows that this is in fact a decomposition.

Lemma 3.2.1 *The map $f^{(n,0)}$ is equal to $f^{(j,0)} \circ \ldots \circ f^{(n,0)}$ where this is the efficient decomposition of $f^{(n,0)}$.*

Proof: The map $f^{(n,0)}$ represents exactly the number of iterates for I_0^{n+1} to first return to I_0^n. So we must show that this is the same for the efficient decomposition of $f^{(n,0)}$. There must be at least as many iterates of f in the efficient decomposition of $f^{(n,0)}$ as in $f^{(n,0)}$ itself since we do not stop the efficient decomposition until I_0^{n+1} lands in I_0^n. But it is easy to see that we cannot apply more iterates since when L (notation from above) is a subset of I_0^k but not I_0^{k+1} we apply the map which first allows L be in I_0^{k-1}, i.e., $f^{(k,0)}$. The result follows. $\odot$

We are now in a good position to define a particular type of return time for the critical point. Choose a level n. The *efficient return time* of level n is the number of iterates of the form $f^{(j,0)}$ used, counted with multiplicity, for our efficient decomposition of $f^{(n,0)}$.

Definition. The map f has *very-persistent* combinatorics if there is an upper bound on the efficient return time of the central intervals. This upper bound is the *very-persistent type.*

Example. The Fibonacci map has very-persistent combinatorics. For any level the efficient return time is always 2 as $f^{(n-1,0)} \circ f^{(n,0)}(I_0^{n+1}) \subset I_0^n$. Hence the very-persistent type would also be 2.

The other combinatorial assumption imposed on the maps we are considering is that of stationary type. First let us label the intervals I_i^n of the n-th generalized renormalization level so that the index i numbers the I_i^n from left to right with I_0^n being the central piece. For example, I_{-2}^n would be the second interval to the left of I_0^n. We point out that in the next section, under our combinatorial assumptions there are only a finite number of intervals I_i^n for each level.

Definition. A map f has *stationary type* combinatorics if the interval I_i^n has the same itinerary through $\cup_i I_i^{n-1}$ under iterates of g_{n-1} until the first return to I_0^{n-1} independently of n.

Example. The following combinatorics demonstrate both the very-persistent property as well as the stationary type property. For all generalized renormalization levels, i.e., for all n, suppose we have

$$
\begin{aligned}
I_{-2}^n &\rightarrow I_1^{n-1}, I_1^{n-1}, I_0^{n-1} \\
I_{-1}^n &\rightarrow I_1^{n-1}, I_0^{n-1}
\end{aligned}
$$

$$\begin{aligned} I_0^n &\to I_{-1}^{n-1}, I_{-2}^{n-1}, I_1^{n-1}, I_0^{n-1} \\ I_1^n &\to I_0^{n-1} \end{aligned}$$

where $\to$ indicates the itinerary of the interval via the map g_{n-1}. For example, the first line indicates the following: $g_{n-1}(I_{-2}^n) \subset I_1^{n-1}$; $g_{n-1} \circ g_{n-1}(I_{-2}^n) \subset I_1^{n-1}$; and $g_{n-1} \circ g_{n-1} \circ g_{n-1}(I_{-2}^n) \subset I_0^{n-1}$.

3.3 Generalized Renormalization Geometry

Let us fix a combinatorial class $\mathcal{F}$ as defined in the Introduction before Theorem C.

Lemma 3.3.1 *For $f \in \mathcal{F}$, there are no central returns.*

Proof: If there is a central return at some level then, because of the stationary type combinatorics, there must be a central return at every level. This implies that the map is renormalizable in the classical sense and we have a contradiction. $\odot$

Due to Lyubich's Theorem 3.1.1, Lemma 3.3.1 above implies that the scaling factors μ_n tend to 0 at exponential rate in n. Also, using Lyubich's Theorem 3.1.1 and rule **3** on page 48, it is easy to see that the lengths of the I_i^n for all i are becoming exponentially small as compared to I_0^{n-1}. In short, with our combinatorial assumptions, we have the following immediate corollary of Lyubich's theorem.

Corollary 3.3.2 *Given a map $f \in \mathcal{F}$, there exists a value ρ, $0 < \rho < 1$ such that for all i,*

$$\frac{|I_i^n|}{|I_0^{n-1}|} < \rho^n.$$

Now we wish to show that there is a finite number of intervals on each level independent of the level. In fact, this is really a statement about very-persistence and hence the following two lemma do not assume stationary type combinatorics.

Lemma 3.3.3 *Every interval I_i^n contains a point of the critical point orbit of some efficient decomposition.*

Proof: We define the set $\overline{\mathcal{O}}_{(l,0)}(0)$ to be the set of points in the orbit of 0 along the efficient decomposition of $f^{(l,0)}(0)$. Note that the number of points in $\overline{\mathcal{O}}_{(l,0)}(0)$ is the same as the value of the efficient return time of level l. We will show that if $f^q(0) \cap I_i^n \neq \emptyset$ for some $q \leq (l,0)$ then we must have $\overline{\mathcal{O}}_{(b,0)}(0) \cap I_i^n \neq \emptyset$ for some $b \leq l$. This will prove the claim.

Suppose we have an I_i^n where the claim is not true. Because I_i^n contains some part of the orbit of 0 for the map f, there exists an l such that $f^q(0) \cap I_i^n = \emptyset$ for all $q \leq (l-1,0)$ and $f^q(0) \cap I_i^n \neq \emptyset$ for some $q \leq (l,0)$.

Let us take an interval J which is an iterate of I_0^{l+1} with respect to the efficient decomposition of $f^{(l,0)}$. We denote the next iterate of the efficient decomposition for the interval J as $f^{(j,0)}$. So J is chosen so that for some q with $0 < q < (j,0)$, we have

$$f^q(J) \subset I_i^n,$$

$$f^{(j,0)}(J) \not\subset I_i^n.$$

Now take the interval I_t^s such that $f^{(j,0)}(J) \subset I_t^s$, where s is chosen as large as possible. Hence we must have that $s \geq j-1$, as an interval may move to a lesser level by just one step per efficient iterate.

Our contradiction argument has two cases. First we consider the case of $n \geq j$. We have a contradiction of the existence of q since $f^{(j,0)}$ is the first return map for the interval J to the central interval $I_0^{j-1}(\supset J)$. The second case is $n < j$. Then we have $J \subset I_0^j$ with $j < l$. Hence we contradict our first assumption since we have some $q < (j,0) < (l,0)$ such that $f^q(I_0^{j-1}) \cap I_i^n \neq \emptyset$. This concludes the claim that $\overline{\mathcal{O}}_{(l,0)}(0) \cap I_i^n \neq \emptyset$ for some l, and hence the fact that each I_i^n is "hit" by some efficient decomposition. ⊙

Lemma 3.3.4 *Given a unimodal map f which is of very-persistence type, the number of intervals I_i^n on a given level n is uniformly bounded depending on the type.*

Proof: By the previous lemma we know that at a fixed level n that each interval I_i^n contains some point of $\overline{\mathcal{O}}_{(l,0)}(0)$. Also notice that in an efficient decomposition of $f^{(l,0)}$, an interval can move back to at most to one lesser level for each iterate of $f^{(j,0)}$, $k \leq l$. Starting with any central interval I_0^n, and taking the very-persistence type to be M, the condition implies that this interval can at best reach intervals of level $n - M$ and no lesser level under efficient decomposition. Putting these two facts together, each I_i^n, $i \neq 0$, for a fixed level n must be visited by some central interval I_0^{n+k}, $k \leq M$ by an efficient decomposition. Since this is a collection of finite central intervals they may visit no more than $M \cdot (M - 1)$ intervals for any level n under efficient iterates. This completes the proof. $\odot$

Now that we have established that there are a uniformly bounded number of intervals at each level (in fact a constant number for stationary type), we will establish some asymptotic geometry of the intervals I_i^{k-1} inside I_0^k independent of the map f in a given class $\mathcal{F}$. In particular, the main lemma of this section deals with the limiting geometry of the intervals I_i^n as they lie inside I_0^{n-1}. There is a special case of this limiting geometry due to *islands.* So before stating the lemma we review the notion of islands and also define *ghost islands.*

The definition of an island is introduced in [Lyu93b]. An *island* is an interval J contained in some I_0^k and containing at least two intervals of the form I_i^{k+1}. A ghost island is a special preimage of an island.

Definition. A *ghost island* is an interval J contained in some level k, i.e., $J \subset I_0^k$, and containing at least two distinct intervals, I_i^{k+1} and I_j^{k+1}. This interval J is mapped monotonically or unimodally into an island with boundary mapping to boundary under the composition of decreasing level returns, i.e., $f^{(n-j,0)} \circ \ldots \circ f^{(n-2,0)} \circ f^{(n-1,0)}(J)$. The value j is the *rank* of the ghost island.

Definition. A *critical ghost island* is a ghost island which contains the critical point.

We would like to define the value j above as the rank for an interval I_i^n which lies in a ghost island. But because ghost islands (as well as islands) may be nested we must avoid any ambiguity. Hence, given an interval I_i^n which lies in a critical ghost island we define the *rank* of I_i^n to be ghost island rank for the smallest ghost island containing I_i^n. In other words, the critical rank j represents the minimum number of (decreasing) level returns (as in the ghost island definition) needed until I_i^n and any other I_k^n of level n map into different intervals.

The first step for proving a limiting rescaled geometry of the intervals I_i^{n+1} in I_0^n (except for those in critical ghost islands) are establishing a priori bounds. The following lemma says that the geometry of some gaps, or more precisely the distances between an I_i^n, $i \neq 0$, and the point 0, are uniformly comparable to the interval I_0^{n-1}, provided I_i^n does not lie in a critical ghost island.

Lemma 3.3.5 (A priori bounds) *Given $f \in \mathcal{F}$ there exist constants k_1 and k_2 such that*

$$0 < k_2 < \frac{|\text{dist}(I_i^n, 0)|}{|I_0^{n-1}|} < k_1 < 1, \tag{3.2}$$

for all levels n and intervals I_i^n, $i \neq 0$, which do not lie in critical ghost islands.

Remark. We prove the limiting geometry Lemmas 3.3.5 and 3.3.7 without the requirement that the class of maps $\mathcal{F}$ be of stationary type.

Since we are concerned primarily with the small scale structure around the critical point, we adjust our maps $f^{(n,0)}$ by rescaling (by a positive constant) so that each central interval I_0^n is equal to the interval $[-1,1]$. The rescaled maps are of the form $F_{n,0}(x) = (cx^2 + b)(1 + O(p^n))$ for some b and c, with the restrictions $-1 < b < 1$ and $0 < c < 2$ or $-2 < c < 0$, depending on the orientation. In other words, for $\mathrm{x} \in I_i^n$ we have

$$f^{(n,0)}(\mathrm{x}) = \frac{|I_0^{n-1}|}{2} \cdot F_{n,0}\left(\mathrm{x} \cdot \frac{2}{|I_0^n|}\right), \quad x \in I_0^n$$

.

We denote the images of I_i^n under the rescaling of I_0^{n-1} by $\ddot{I}_i^n$. The proof of Lemma 3.3.5 is be broken up into two parts. First we prove the upper bound.

Proof: **(Existence of k_1)**

We now consider any interval I_i^n with $i \neq 0$. We begin by applying the efficient decomposition of central maps for this interval until an iterate fails to lessen the image one level. In other words $f^{(n-l,0)} \circ f^{(n-(l-1),0)} \ldots \circ f^{(n-1,0)}(I_i^n) \subset I_0^{n-l}$. Now using our rescaled maps $F_{j,0}$ it is easy to that $F_{n-l,0} \circ F_{n-(l-1),0} \circ \ldots \circ F_{n-1,0}(\ddot{I}_i^n) = O(\rho^n)$ with $\rho < 1$ (compare Corollary 3.3.2).

It is also easy to see that preimages of the point 0 uniformly stay away from the boundary points -1 and 1 for a uniformly bounded number of composed maps $F_{j,0}$. This is because the derivatives of the endpoints -1 and 1 under preimages of $F_{j,0}$ are bounded below by $\frac{1}{4}(1 \pm O(p^n))$. The intervals $\ddot{I}_i^n$ are nearly points as compared to $[-1,1]$ when n is large. The preimages of $\ddot{I}_0^{n-(l-1)}$ under the inverse of the map $F_{n-l,0} \circ F_{n-(l-1),0} \circ \ldots \circ F_{n-1,0}$ must also be uniformly bounded away from -1 and 1 since very-persistence gives a uniform bound for the number of $F_{n-j,0}$ in this composition. Finally, we note that the existence of k_1 is certainly scale invariant and hence we have the desired result. $\odot$

The existence of k_1 implies that our maps $F_{n,0}(x) = (cx^2+b)(1+O(p^n))$ are further restricted by $-1 < -k_1 < b < k_1 < 1$. Now we complete the proof of Lemma 3.3.5 with the following stronger *a priori bounds* lemma which provides information on the gaps between intervals.

Lemma 3.3.6 (Existence of k_2)

For maps $f \in \mathcal{F}$ there exists a value $K > 0$ such that for any level n, the distance between any two intervals of level n has the property that

$$\mathrm{dist}(I_i^n, I_j^n) > K \cdot |I_0^{n-1}|,$$

provided they do not belong to a common island or ghost island.

Proof: Take any two intervals $\ddot{I}_i^n$ and $\ddot{I}_j^n$. Then it is easy to see that, if they are some bounded distance apart, then under pull-back of $F_{n,0}$ they are still bounded apart with the distance depending only on their initial distance. Now take preimages until one is pre-central, where pre-central means the critical point first maps to that preimage under $F_{k,0}$ for some k. Then the other interval is not pre-central at that same number of preimages. This uniformly finite pull-back can have a contraction on the gap between them by at most some value C. More precisely $C \leq (4 \cdot (1 \pm O(p^n)))^{-N}$ since the maximum derivative of any point $x \in [-1, 1]$ for $F_{n,0}$ is less then or equal to 4 times the

Koebe error. But once we pull-back by this last $F_{n,0}$, we have an expansion in distance by the square root map. Hence if the distance was originally x, then after some uniformly finite pull-back the distance is now $\sqrt{x/C}$. This map $x \to \sqrt{x/C}$ is definitely expanding near zero and we have the desired result provided the original intervals $\ddot{I}_i^n$ and $\ddot{I}_j^n$ are not created arbitrarily close together (independent of n), i.e., they are not in a common ghost island. $\odot$

Using the a priori bounds given by the above lemmas we wish to show that the rescaled picture for any two combinatorially similar maps is nearly the same when n is sufficiently large. More precisely, after rescaling the central interval of any level n, the placement of the intervals I_j^{n+1} is asymptotically independent of the choice of f within combinatorial types.

Lemma 3.3.7 *Given two maps $f, \tilde{f} \in \mathcal{F}$, there exists a $q < 1$ such that the following equation holds for any two conjugating intervals I_i^n and $\tilde{I}_j^n$ which do not lie in a critical ghost island:*

$$\frac{\operatorname{dist}(I_i^n, 0)}{|I_0^{n-1}|} \cdot \frac{|\tilde{I}_0^{n-1}|}{\operatorname{dist}(\tilde{I}_i^n, 0)} = 1 + O(q^n).$$

Proof: The proof of this lemma uses Thurston's contraction principle for rational maps of the sphere [DH93]. For a more detailed account of the Thurston transformation, see Appendix B.

Using our rescaled central intervals and maps $F_{n,0}$ we again treat the associated rescaled intervals $\ddot{I}_i^n$ of level n inside of the rescaled I_0^{n-1} as virtual points. Also the ghost islands, being just a uniformly bounded number of pull-backs of these "points" under the map $F_{n,0}$, are also asymptotically points.

To use the Thurston contraction map we construct a punctured sphere for each level n. In particular our sphere will be punctured at 0, 1, ∞ and one arbitrarily chosen point from each $\ddot{I}_j^n$ unless there is a ghost island containing it. For the case of ghost islands we just choose one point from the ghost island of least rank and no other points in this ghost island. Hence for maps in $\mathcal{F}$ we have a uniformly bounded number of punctures at each level and the punctures are uniformly bounded apart by Lemma 3.3.5.

We now pull back along the composition $F_{n+N,0} \circ ... \circ F_{n+1,0}$. If each $F_{j,0}$ is exactly quadratic, then there is a definite contraction in all directions of the Teichmüller space of the punctures unless all poles of the quadratic differential pull back to poles. But our finite string of pull-backs can produce only a uniformly bounded number of punctures at each level by Lemma 3.3.4. Hence, there must be a contraction since we run out of punctures for the poles to pull-back to. So in fact, for type N we would have a definite contraction at least after N preimages in every direction of Teichmüller space. Of course, our chosen punctures do not necessarily pull-back to the punctures on the previous level but they converge at exponential rate to them.

Now since the $F_{j,0}$ are asymptotically quadratic and the intervals are asymptotically points, it is enough to point out that since our punctured spaces for f and $\tilde{f}$ remain in a compact set in Teichmüller space (by Lemma 3.3.5 and the above construction of punctures), we must have a definite contraction for all sufficiently large levels in the rescaled geometry between the two maps

f and $\tilde{f}$. This concludes the proof of the lemma. ⊙

3.4 A Recurrent Equation for Scaling Factors

For our fixed class $\mathcal{F}$ we will suppose throughout this section that the very-persistent type is $M+1$. To start the proof of part 1 of Theorem C we establish a recurrent equation for the $\mu_i = |I_0^n|/|I_0^{n-1}|$. We begin by defining the following geometry constants motivated by Lemma 3.3.7.

Definition. Given a class $\mathcal{F}$, we choose an arbitrary map f in this class and define the constants C_i below for all i in which I_i^n does not lie in a critical ghost island.

$$C_i = \lim_{n\to\infty} \frac{\mathrm{dist}(I_i^n, 0)}{(1/2)\cdot |I_0^{n-1}|}$$

It is clear from the definition that for the arbitrarily chosen f, the constants C_i represent the rescaled distance from the interval I_i^n to the critical point 0. We know by Lemma 3.3.7 that if we had chosen a different f within the fixed combinatorial class $\mathcal{F}$, then we would obtain the same constants and for each f we obtain convergence at an exponential decaying rate. Finally, we point out that Lemma 3.3.7 also gives convergence at an exponential rate for the above limit.

Case 1: No ghost islands

To set up a recurrence relation for the μ_n, we analyze the long term derivative of $f^{(n,0)-1}(f(0))$, i.e., the derivative of the orbit of the critical value until its return to the central interval I_0^{n-1}. Then using the chain rule to differentiate the efficient decomposition of $f^{(n,0)-1}(f(0))$ we compare these two long-term derivatives. When we decompose $f^{(n,0)}(0)$ into our efficient iterates we count the multiplicity of each $f^{(j,0)}$, $j < n$, and denote this by a_j. (This doesn't depend on n because of the stationary type assumption on our class.) Just counting the total number of iterates of f gives

$$(n,0) = \sum_{i=n-M}^{n-1} a_i \cdot (n-i,0) \tag{3.3}$$

The sum starts at $n - M$ because our return is very-persistent of type $M + 1$. Also note that the a_i are non-negative.

To calculate the long term derivative of $f^{(n,0)-1}(f(0))$, we take any point $f(x)$ where $x \in I_0^n$. Because our return is non-central one of the two following estimates must hold by application of the Koebe Theorem. Equation (3.4) is for *high returns*, i.e., the image $f^{(n,0)}(I_0^n)$ covers more than half of I_0^{n-1}, and Equation (3.5) is for *low returns*. The denominator for each represents the length of the domain $f(I_0^n)$, while the numerator represents the length of the image $f^{(n,0)-1}(f(I_0^n))$.

$$\frac{d}{dx} f^{(n,0)-1}(f(x)) = \frac{(C_i + \frac{1}{2})|I_0^n|}{|\frac{1}{2} \cdot I_0^{n+1}|^2}(1 + O(p^n)), \tag{3.4}$$

$$\frac{d}{dx} f^{(n,0)-1}(f(x)) = \frac{(-C_i + \frac{1}{2})|I_0^n|}{|\frac{1}{2} \cdot I_0^{n+1}|^2}(1 + O(p^n)). \tag{3.5}$$

The error term incorporates both the Koebe error for the map $f^{(n,0)-1}(f(0))$ as well as the error in the constants C_i. Hence, the above equations hold independently of the map f, although the value of $p < 1$ may differ.

We wish to compare the estimates (3.4) and (3.5) with estimates obtained when differentiating the efficient decomposition of $f^{(n,0)-1}$, i.e., $f^{(n-1,0)-1} \circ f^{(n-2,0)} \circ f^{(n-l,0)} \circ \ldots \circ f^{(n-j),0)}$. So Suppose I_i^{n-j} is not in a critical ghost island; then for $x \in I_i^{n-j}$, $i \neq 0$, we have

$$\frac{d}{dx} f^{(n-j,0)}(x) = C_i \cdot |I_0^{n-j+1}| \cdot \frac{d}{dx} f^{(n-j,0)-1}(f(x)) \cdot (1 + O(p^n)). \tag{3.6}$$

The right side of Equation (3.6) may now be easily simplied by using Equation (3.4) or Equation (3.5), depending on whether the return is high or low. In any case, we have the following simplification, where the value K_i is some non-zero constant independent of the map f:

$$\frac{d}{dx} f^{(n-j,0)}(x) = K_i \cdot \frac{|I_0^{n-j}|}{|I_0^{n-j+1}|} \cdot (1 + O(p^n)) \quad x \in I_i^{n-j}, i \neq 0. \tag{3.7}$$

Using Equations (3.3), (3.4), (3.5) and (3.7), we can compare the derivative estimate of $f^{(n,0)-1}(f(0))$ with the derivative estimates of its efficient decomposition. Combining all of the constant factors, we get

$$\frac{|I_0^n|}{|I_0^{n+1}|^2} = K \cdot \left(\frac{|I_0^{n-1}|}{|I_0^n|^2}\right)^{a_1} \cdot \left(\frac{|I_0^{n-2}|}{|I_0^{n-1}|}\right)^{a_2} \cdot \left(\frac{|I_0^{n-3}|}{|I_0^{n-2}|}\right)^{a_3} \cdots \cdot \left(\frac{|I_0^{n-(M+1)}|}{|I_0^{n-M}|}\right)^{a_{M+1}} (1 + O(p^n)). \tag{3.8}$$

The constant K is positive and independent of f. The error term $1 + O(p^n)$ incorporates the Koebe error of the map f and the exponential small error obtained from the constants C_i.

Let us make the substitution $\lambda_i = -\ln \mu_i$ and create a recurrent equation for the λ_i. We simplify Equation (3.8) by first multipling both sides by $|I_0^n|$. Then taking the natural logarithm of both sides, dividing by 2, and using our λ_i substitution yields

$$\lambda_n = \left(\sum_{i=1}^{M-1} a_i \cdot \lambda_{n-i} \right) - K + O(p^n), \tag{3.9}$$

where our new constant K and non-negative constants a_i are independent of our choice of $f \in \mathcal{F}$.

Case 2: Ghost Islands

The ghost island case is more technical but we can still establish a recurrent equation like (3.9) although the a_i will not match the efficient decomposition iterate as before. Let us take the collection of I_i^n contained in a critical ghost island. Then we apply iterates of the efficient decomposition of $f^{(n,0)}$ until these I_i^n cover I_0^{n-1}. Then the large Koebe space allows the following estimate for $x \in I_i^n$:

$$\begin{aligned} &\operatorname{dist}(I_i^n, 0) \\ &\quad = \frac{1}{2} \cdot |I_0^{n-1}| \cdot \left| \frac{d}{dx} f^{(n-j,0)} \circ \ldots \circ f^{(n-1,0)}(f(x)) \right|^{-1} (1 + O(p^n)), \end{aligned} \tag{3.10}$$

where $f^{(n-p,0)} \circ \ldots \circ f^{(n-1,0)}$ represents the efficient decomposition of $f^{(n,i)}$. To estimate this derivative we can make use of Equations (3.4) and (3.5), as long as this efficient decomposition does not land in any ghost islands. Hence, we take care to inductively compute this derivative estimate of (3.10) by the natural hierarchy of the intervals I_i^n which lie in ghost islands. In other words we compute those in order of least rank. After doing this Equation (3.10) becomes

$$\mathrm{dist}(I_i^n, 0) \;=\; K_i \cdot |I_0^{n-l}| \cdot \prod_{j=1}^{M} (\mu_{n-j})^{b_{j,i}} \cdot (1 + O(p^n)), \tag{3.11}$$

for non-positive $b_{j,i}$. This may be further simplified to

$$\mathrm{dist}(I_i^n, 0)/|I_0^n| \;=\; K_i \cdot \prod_{k=1}^{l} \mu_{n-j} \cdot \prod_{j=1}^{M} (\mu_{n-j})^{b_{j,i}}. \tag{3.12}$$

Now using Equations (3.4) and (3.5), we can estimate the derivatives along the efficient decomposition just as before and obtain

$$\begin{aligned} &\frac{d}{dx} f^{(k-1,0)}(x) \\ &\qquad = \frac{(\pm C_1 + \frac{1}{2})|I_0^k|}{|\frac{1}{2} \cdot I_0^{k+1}|^2} \cdot 2 \cdot K_i \cdot |I_0^k| \cdot \prod_{k=1}^{l} (\mu_{n-j})^{-1} \cdot \prod_{j=1}^{M} (\mu_{n-j})^{b_{j,i}} \cdot (1 + O(p^n)) \end{aligned}$$

(Compare with Equation (3.7).) By simplifying the constants we have for $x \in I_i^k$

$$\frac{d}{dx} f^{(k-1,0)}(x) = K_i \cdot \prod_{j=0}^{M} (\mu_{n-j})^{d_{i,j}} \cdot (1 + O(p^n)). \tag{3.13}$$

where K_i is some constant independent of the map f and $d_{i,j}$ are some non-positive integers also independent of f. In particular, this may now be easily incorporated into Equation (3.8) and the recurrent Equation (3.9) for the case of critical ghost islands. Also, we may now use this estimate for simplifying Equation (3.10) for intervals of higher rank completing the induction.

Proceeding as in the non-ghost island case, we can determine our recurrent equation for the $\lambda_n = -\ln \mu_n$. This yields

$$\lambda_n = \left(\sum_{i=1}^{M} B_i \cdot \lambda_{n-i} \right) \ + \ K \ + \ O(p^n), \tag{3.14}$$

with constant K and non-negative constants B_i depending only on the class $\mathcal{F}$.

3.5 Scaling Factor Parameters

This section is motivated by the recurrent equation for the scaling factors, i.e., Equation (3.9) or (3.14). These equations essentially give a linear model for the asymptotics of the scaling factors. Let us define an $M \times M$ matrix A equal to

$$\begin{pmatrix} B_M & B_{M-1} & . & . & 0 & B_1 \\ 1 & 0 & . & . & 0 & 0 \\ 0 & 1 & . & . & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & . & . & 1 & 0 \end{pmatrix}$$

where the entries B_i come from Equation (3.14). Let us define the vector $\vec{v}_n = (\lambda_n, ..., \lambda_{n-M+1})$. Then we may rewrite Equation (3.14) as

$$\vec{v}_{n+1} = A\vec{v}_n + K + O(\vec{p}^{\,n}), \tag{3.15}$$

where $O(\vec{p}^n)$ represents the vector $(O(p^n), 0, ..., 0)$.

Showing that we have finitely many parameters controlling the scalings of the λ_n is equivalent to solving the following linear algebra problem. (Since K does not depend on f we have excluded it for convenience.)

Lemma 3.5.1 *Given a vector $\vec{v}$ and maps $A_n(\vec{v}) = A\vec{v} + \vec{Q}_n$, where the constant vectors $\vec{Q}_n$ we have $\|\vec{Q}_n\| = O(p^n)$, $p < 1$, there exists a constant vector $\vec{v}^*$ such that*

$$\|A_n \circ A_{n-1} \circ ... \circ A_1(\vec{v}) - A^n(\vec{v}^*)\| \to 0, \tag{3.16}$$

where $\to 0$ indicates convergence to zero at an exponential rate. As $\vec{v}$ varies, the minimum number of free parameter of $\vec{v}^*$ is equal to the number of eigenvalues, counted with multiplicity, of our matrix A, which have modulus greater than or equal to one.

Proof: Without loss of generality we assume that our matrix A is in Jordan canonical form. Given a vector $\vec{v}$ we claim that the following limit exists and satisfies Equation (3.16):

$$\vec{v}^* = \pi_e(\vec{v}) + \sum_{i=1}^{\infty} A^{-i}\pi_e(\vec{Q}_i), \tag{3.17}$$

where the map π_e is projection onto the non-contracting Jordan block spaces, i.e., those corresponding to the eigenvalues with modulus greater than or equal to one. Hence, the inverse map, A^{-n}, makes sense there.

We show that $\vec{v}^*$ exists by showing that the sequence

$$\vec{v}^{(n)} = \pi_e(\vec{v}) + \sum_{i=1}^{n} A^{-i}\pi_e(\vec{Q}_i)$$

forms a Cauchy sequence. But this is easy to see as

$$\left\|\vec{v}^{(n)} - \vec{v}^{(n+1)}\right\| = \left\|A^{-n}\pi_e(\vec{Q}_n)\right\| = O(p^n)$$

with the last equality coming from the fact the inverse iterates of A (in the non-contracting direction) can expand vectors at only a polynomial rate.

Using the triangle inequality,

$$\begin{aligned} &\|A_n \circ A_{n-1} \circ \ldots \circ A_1(\vec{v}) - A^n(\vec{v}^{\,*})\| \qquad (3.18) \\ &\quad\leq \left\|A_n \circ A_{n-1} \circ \ldots \circ A_1(\vec{v}) - A^n(\vec{v}^{(n)})\right\| + \left\|A^n(\vec{v}^{(n)}) - A^n(\vec{v}^{\,*})\right\|. \end{aligned}$$

The second term of the right-hand side of (3.18) is seen to be converging at exponential rate as the following simplifications show:

$$\begin{aligned} &\left\|A^n(\vec{v}^{(n)}) - A^n(\vec{v}^{\,*})\right\| \\ &\quad = \left\|A^n\left(\sum_{i=n}^{\infty} A^{-i}\pi_e(\vec{Q}_i)\right)\right\| \leq \sum_{i=n}^{\infty}\left\|\vec{Q}_i\right\| \leq O(p^n). \end{aligned}$$

We analyze the first term of the right-hand side of (3.18) and obtain

$$\begin{aligned} &\left\|A_n \circ A_{n-1} \circ \ldots \circ A_1(\vec{v}) - A^n(\vec{v}^{(n)})\right\| \\ &\quad = \left\|A^n\left(\vec{v} + \sum_{i=1}^{n} A^{-i}(\vec{Q}_i)\right) - A^n\left(\pi_e(\vec{v}) + \sum_{i=1}^{n} A^{-i}\pi_e(\vec{Q}_i)\right)\right\| \\ &\quad = \left\|A^n\left(\pi_c\left(\vec{v} + \sum_{i=1}^{n} A^{-i}(\vec{Q}_i)\right)\right)\right\|. \end{aligned}$$

The notation π_c represents projection of the vector onto the contracting Jordan block spaces, i.e., those corresponding to the eigenvalues with modulus less than one. We let r be the largest eigenvalue of A with modulus less than one. Distributing the π_c and A^n gives

$$\begin{aligned}
&\left\| A^n \left(\pi_c \left(\vec{v} + \sum_{i=1}^{n} A^{-i}(\vec{Q}_i) \right) \right) \right\| \\
&\quad \le \; O(r^n) + \sum_{i=1}^{n} \left\| A^{(n-i)} \pi_c(\vec{Q}_i) \right\| \\
&\quad = \; O(r^n) + \sum_{i=1}^{n} O(r^{n-i}) \cdot O(p^i) \\
&\quad = \; O(\varrho^n),
\end{aligned}$$

where $\varrho = \max(r, p)$. Hence the first term of the right-hand side of (3.18) also converges to zero at exponential rate as n goes to infinity. This completes the proof of the lemma. ⊙

Applying Lemma 3.5.1 to our recurrence Equation (3.14) and incorporating the constant K gives us the following asymptotic description of the scaling factors:

$$\frac{|I_0^n(f)|}{|I_0^{n-1}(f)|} \;\sim\; e^{-[A^n \vec{v}(f)]_1 + K_n}. \tag{3.19}$$

For a given f, the parameter vector $\vec{v}(f)$ is defined as the $\vec{v}^*$ from the above lemma. The term K_n, $n \ge N+1$, comes from the following equation

$$K_n = [\sum_{i=0}^{n-N+1} A^i \vec{K}]_1 \tag{3.20}$$

where $\vec{K} = (K, 0, 0, ..., 0)$ with K from Equation (3.14) or (3.9) This completes the proof of Theorem C, part 1.

3.6 Smooth Conjugacies of $\overline{\mathcal{O}}$

Using Theorem C part 1 we will prove part 2. We also point out that the lemmas and proofs contained in this section essentially follow [LM93], in particular, Lemmas 3.7, 5.5, and 5.7 of that paper. The main idea is to push the central scaling results to all points of $\overline{\mathcal{O}}$. We review the definition of a smooth conjugacy between two Cantor sets $\overline{\mathcal{O}}(f)$ and $\overline{\mathcal{O}}(g)$.

Definition. A conjugacy ϕ between $\overline{\mathcal{O}}(f)$ and $\overline{\mathcal{O}}(g)$ is called *smooth* if for any point $x \in \overline{\mathcal{O}}(f)$ the following limit exists:

$$\lim \frac{|\phi(x) - \phi(y)|}{|x - y|} \neq 0$$

as $y \to x$ along $\overline{\mathcal{O}}(f)$, and depends continuously on x.

First we construct a covering of $\overline{\mathcal{O}}(f)$. We let the set M^n be the union of the intervals I_i^n and their forward orbit until just before their return to I_0^{n-1}.

$$M^n = \bigcup_i \bigcup_{k=0}^{(n,i)-1} f^k(I_i^n)$$

Theorem 3.6.1 *For the map f, the sets M^n form a nested sequence of closed sets $M^1 \supset M^2 \supset M^3 \supset \cdots$ whose intersection is equal to the set $\overline{\mathcal{O}}(f)$.*

Proof: The intervals of level n, I_i^n, are contained in I_0^{n-1}. By construction of the I_i^n we know that $I_0^{n-1} \cap \overline{\mathcal{O}}(f) \subset \cup_i I_i^n$. Since we are iterating each I_i^n until just before covering I_0^{n-1}, we must have $M^{n-1} \supset M^n \supset \overline{\mathcal{O}}(f)$. The scaling results of Lyubich prove that the limiting intersection of the M^n is $\overline{\mathcal{O}}(f)$. ⊙

We wish to develop some notation to indicate different intervals of the collection M^n. Consider the infinite bi-cylinder $\Sigma = [s_2, s_3, s_4...]$ where each s_l corresponds to some pair of integers $\{k, i\}$. The integer k must satisfy $0 \leq k < (n, i)$. (Recall that (n, i) represents the number of interates of f needed for the first return of I_i^n to I_0^{n-1}.) In what follows we use the notation $M^n_{s_2,...,s_n}$ to represent the interval $f^{k_n}(I^n_{i_n})$ where $s_n = \{k_n, i_n\}$ and the added restriction that for $s_j = \{k_j, i_j\}$, $j < n$, we have the following containment:

$$f^{k_j}(I^j_{i_j}) \subset f^{k_{j+1}}(I^{j+1}_{i_{j+1}}).$$

Given a point $x \in \overline{\mathcal{O}}$ we can assign the string $s_2, s_3, s_4...$ such that for each $s_j = \{k, i\}$ we have $x \in f^k(I_i^j)$. It may happen, however, that this representative string is not unique. Suppose we have two representations $s_2, s_3, s_4, ...$ and $t_2, t_3, t_4, ...$ For the first term in which they disagree, say s_n and t_n suppose we have $M^n_{...,s_n} \supset M^n_{...,t_n}$; then we only use the $..., s_n, ...$ representation in our infinite bi-cylinder. If the two sets are always equal, i.e., $M^n_{...,s_n} = M^n_{...,t_n}$ for all n we mod out our infinite bi-cylinder by this relation. With this restricted bi-cylinder $\tilde{\Sigma}$, every point x in $\overline{\mathcal{O}}(f)$ has a unique representation.

We wish to estimate the ratio of any two intervals $M^n_{s_2,...,s_n} \supset M^{n-1}_{s_2,...,s_{n-1}}$. Theorem C (part 1) gives an asymptotic formula if all $s_j = (0, 0)$. So we make estimates for $|I_i^n|/|I_0^{n-1}|$. Let us take a point $x \in f(I_i^n)$. Then we have the following estimate because of the large Koebe space around the interval $f(I_i^n)$ for the map $f^{(n,i)}(I_i^n) = I_0^n$.

$$\frac{|I_i^n|}{|I_0^{n-1}|} \sim \left|\frac{d}{dx} f^{(n,i)}(x)\right|^{-1}, \quad x \in f(I_i^n). \tag{3.21}$$

To estimate the right side of Equation (3.21), we may use the efficient decomposition methods for the map $f^{(n,i)}$. In particular, we know that this decomposition has at most $M+2$ iterates of the form $f^{(k,0)}$ with $k < n$ (by very-persistence type $M+1$). The estimates for these maps are contained in Equation (3.6). Using the same method as in Section 4 to calculate this long-term derivative estimate we obtain Equation (3.22) in the following lemma. Before the statement we need some extra notation. In this lemma, a map of the form L_{s_n} acts on a vector to return a real value and is linear in each of the components.

Lemma 3.6.2 *Given a map $f \in \mathcal{F}$ we have linear maps L_{s_n} and constants K_{s_n} with $s_n = \{i, 0\}$ depending only on the class $\mathcal{F}$, so that the following estimates hold:*

$$\frac{|I_i^n|}{|I_0^{n-1}|} \sim e^{-L_{s_n}(\lambda_n,\ldots,\lambda_{n-N})-K_{s_n}} \tag{3.22}$$

$$\sim e^{-L_{s_n}(A^n\vec{v}(f))-K_{s_n}}. \tag{3.23}$$

Proof: Equation (3.22) follows directly from the preceding paragraph. To prove Equation (3.23) we first recall that in the previous section we established that each λ_n equal to $[A^n\vec{v}(f)]_1$ up to some exponentially small error. Substituting these linear combinations into Equation (3.22) completes the proof as long as the coefficients of L_i are bounded for all n and i. But we claim this is true because the map is of very-persistent type. Our efficient decomposition has at most $M+1$ iterate terms of the form $f^{(k,0)}$ (counting multiplicity). Hence the coefficients for the linear maps L_i are bounded by $M+2$ for all n and i. This completes the claim and hence the proof since Estimate (3.23) has at most some exponentially small error in n. ⊙

To this point we have estimated the ratios of the size of the intervals $|M^n_{s_2,\dots,s_n}|/|M^{n-1}_{s_2,\dots,s_{n-1}}|$ with $s_j = (0,0)$ for $j < n$ and $s_n = (i,0)$ using just the combinatorics of the map and the parameter vector $\vec{v}(f)$. The next lemma shows that we may also make similar claims for the ratios of any two nested intervals of M^n and M^{n-1}.

Lemma 3.6.3 *There exist linear maps L_{s_n} and constants K_{s_n} depending only on the tuple $s_n = (i,k)$, such that for any map $f \in \mathcal{F}$ and respective parameters $\vec{v}$ we have the following estimates:*

$$\frac{|M^n_{s_2,\dots,s_n}|}{|M^n_{s_2,\dots,s_{n-1}}|} \sim e^{-L_{s_n}(\lambda_n,\dots,\lambda_{n-N})-K_{s_n}} \tag{3.24}$$

$$\sim e^{-L_{s_n}(A^n\vec{v})-K_{s_n}}. \tag{3.25}$$

Proof: We consider the corresponding intervals $f^k(I_i^{n-1}) = M^{n-1}_{s_2,\dots,s_{n-1}}$ and $f^k(I_i^n) = M^n_{s_2,\dots,s_n}$. Iterating the first interval by $f^{(n-1,i)-k}$ results in a first return map, i.e., the first time it covers the critical point, with image being precisely I_0^{n-2}. Because this map has a large Koebe space we have the following estimate:

$$\frac{|M^n_{s_2,\ldots,s_n}|}{|M^n_{s_2,\ldots,s_{n-1}}|} \sim \frac{|f^{(n-1,i)-k}(M^n_{s_2,\ldots,s_n})|}{|I_0^{n-2}|}. \tag{3.26}$$

In particular, we have that $f^{(n-1,i)-k}(M^n_{s_2,\ldots,s_n})$ is contained in some interval of level $n-1$. If we iterate further, our interval will eventually cover I_0^{n-1}. Let us take the iterate p which does this. Because this map, f^p, has a large Koebe space around $f^{(n-1,i)-k}(M^n_{s_2,\ldots,s_n})$, we may estimate the size of $f^{(n-1,i)-k}(M^n_{s_2,\ldots,s_n})$ by

$$\left|f^{(n-1,i)-k}(M^n_{s_2,\ldots,s_n})\right| = |I_0^{n-1}| \cdot \left|\frac{d}{dx}f^p(x)\right|^{-1} (1 \pm O(p^n)), \tag{3.27}$$

where $s_n = \{i,k\}$ and $x \in f^{(n-1,i)-k}(M^n_{s_2,\ldots,s_n})$. The long-term derivative of the right-hand side may be estimated with exponentially small error using its efficient decomposition. The method of proof then follows exactly as that for Lemma 3.6.2. This completes the proof. ⊙

Theorem C *(part 2) Given two maps $f, g \in \mathcal{F}$ with the same parameter vector $\vec{v}(f) = \vec{v}(g)$ we have that the conjugacy map $\phi : \overline{\mathcal{O}}(f) \to \overline{\mathcal{O}}(g)$ is smooth.*

Proof: It follows from Theorem C (part 1), Lemma 3.6.2, and Lemma 3.6.3 that for any sequence $s = s_2, s_3, \ldots$ that the following limit exists, converges at an exponential rate, and depends continuously on s:

$$\lim_{n\to\infty} \frac{|M^n_{s_2,\ldots,s_n}(f)|}{|M^n_{s_2,\ldots,s_n}(g)|}. \tag{3.28}$$

⊙

3.7 Examples of Maps with Many Parameters

We now explore some examples of families $\mathcal{F}$ satisfying the assumptions of Theorem C. In particular, given an arbitrary positive integer M we will construct a family $\mathcal{F}$ in which the combinatorics of these maps has a matrix A in which there are M eigenvalues of modulus greater than 1.

To assist in the understanding of the combinatorics of the maps which will be discussed, we consider an adjustment of the labeling of the intervals. Previously, we denoted the non-central intervals as I_i^n where n indicates the level and i indicates its placement in relation to the central interval I_0^n. For example I_{-2}^n is the non-central interval second to the left (with respect to all other non-central interval of level n) of the central interval. Our adjusted indices will still indicate the central interval of level n as I_0^n. The adjustment to be made is in the labeling of the intervals I_i^n, $i \neq 0$. The subscript i will now refer to the order that the off-critical intervals of level n are visited by the critical point with respect to iterates of f. For example I_2^n will now indicate that this off-critical interval is visited by the orbit of the critical point after the interval I_1^n but before all other non-central intervals of level n. Through-out this section this adjusted labeling is used.

As mentioned on page 46, we can follow the orbit of I_i^n with respect to the map g_n by keeping track of which intervals of level $n-1$ it passes through before landing in I_0^{n-1}. To keep track of multiplicity let $I_0^n \rightarrow \eta \cdot I_1^{n-1}, I_0^{n-1}$

indicate that as the interval I_0^n travels through the intervals of level $n-1$ under the map g_n it lands in I_1^{n-1} η times in a row and then finally falls in I_0^{n-1}. Given this notation consider the following renormalization scheme below. Notice that there are two free constants η and M. The number $M+1$ is the number of intervals for each level and M will end up being equal to the exact number of parameter values influencing the geometry of the critical orbit, i.e., the number of eigenvalues (counted with multiplicity) of the matrix A with modulus greater than or equal to 1. The value η will be chosen once the recurrent equation for the scaling factors μ_i has been established and in particular will depend on M.

Renormalization Scheme for M parameters influencing $\overline{\mathcal{O}}$

$$\begin{aligned}
I_0^n &\rightarrow \eta \cdot I_1^{n-1}, I_2^{n-1}, \ldots, I_{M-1}^{n-1}, I_M^{n-1}, I_0^{n-1} \\
I_1^n &\rightarrow \eta \cdot I_2^{n-1}, I_3^{n-1}, \ldots, I_{M-1}^{n-1}, I_M^{n-1}, I_0^{n-1} \\
I_2^n &\rightarrow \eta \cdot I_3^{n-1}, I_4^{n-1}, \ldots, I_{M-1}^{n-1}, I_M^{n-1}, I_0^{n-1} \\
&\vdots \\
I_{M-1}^n &\rightarrow \eta \cdot I_M^{n-1}, I_0^{n-1} \\
I_M^n &\rightarrow I_0^{n-1}
\end{aligned} \tag{3.29}$$

Remark. The Fibonacci case has $M=1$ and $\eta=1$

Notice that this renormalization scheme has no islands since each interval of level n lands in a different interval of level $n-1$ with respect to one iterate of g_n. Hence there are also no ghost islands. We claim that this example is of very-persistent type. This will be made more clear after following the example below.

Combinatorial Realization Criterion

Before calculating the recurrent equation for the scaling factors, we briefly review the following conditions, established in Section 2 of [Lyu93a], for a generalized renormalization scheme to be in fact realizable: the monotonicity character of $g_n(I_i^n)$ for $i \neq 0$ (i.e., whether the map is monotone increasing or decreasing), and the linear order of the intervals I_i^n in I_0^{n-1}. The monotonicity character of the map $g_n(I_i^n)$ is determined by the intervals of level $n-1$ (and in particular the itinerary of I_i^n through these level $n-1$ intervals). Since all monotone characteristics are determined inductively, once the top level map g_1 is determined, this condition is unimportant for our problem. The second condition is important since the itinerary of all of the intervals I_i^n through the intervals of level $n-1$ is initially determined by the same map g_{n-1} (which is unimodal on I_0^{n-1}). So the unimodal condition of $g_{n-1}(I_0^{n-1})$ does put restrictions on the placement of the intervals I_i^n in I_0^{n-1}. Hence there is a realizability issue to resolve for the Renormalization Scheme (3.29).

If the placement of the intervals I_i^n mimics that of a realizable quadratic super-attracting orbit (up to reversing orientation), then we have realizability ([Lyu93a], Section 2). For example in Chapter 2, we have that the placement of the puzzle pieces V_1^n, V_0^n mimicked that of the super-attracting orbit for the map $z \to z^2 - 1$. A better example is the period 4 case where there are two possible realizable orders (again up to reversing orientation) for the super-attracting orbit of a quadratic map f: $f(0) < 0 < f^3(0) < f^2(0)$ or

$f(0) < f^3(0) < 0 < f^2(0)$. So for the scheme (3.29) above, with $n = 4$, we may have the placement of the intervals as $I_1^n < I_0^n < I_3^n < I_2^n$ or $I_1^n < I_3^n < I_0^n < I_2^n$ (where $I_j^n < I_i^n$ if $x_j < x_i$ for $x_j \in I_j^n, x_i \in I_i^n$). In general, the placement of the intervals in the renormalization scheme 3.29 must mimic some realizable super-attracting orbit of a unimodal map.

Scaling Factor Equation

First we claim that the recurrent equation for the scaling factors is

$$\lambda_{n+1} = \sum_{i=1}^{M} B_i \cdot \lambda_{n-i} \ + K \ + O(p^n). \tag{3.30}$$

To better understand the coefficients B_i for this example recurrent equation, we develop the following procedure. Since there are no ghost islands the B_i simply represent the number of times the critical point visits the set $I_0^{n-i} \setminus I_0^{n-(i+1)}$ for the efficient decomposition of $f^{(n,0)}$. Take the orbit of I_0^n through the $(n-1)$-st level, and replace each I_k^{n-1}, $k \neq 0$, with its itinerary through I_0^{n-2}. Repeat this procedure inductively replacing the remaining non-central levels, I_k^m, $k \neq 0$ and $m \leq n-2$, with its itinerary through I_0^{m-1} until all non-central intervals have been replaced with central ones. Then the resulting itinerary shows how many times each λ_{n-i} appears in the recurrent equation. A simple example is done below, and its resulting recurrent equation is written.

Example. Here is an example stationary renormalization scheme:

$$I_0^n \ \rightarrow \ \eta \cdot I_1^{n-1}, I_2^{n-1}, I_0^{n-1}$$

$$
\begin{aligned}
I_1^n &\rightarrow \eta \cdot I_2^{n-1}, I_0^{n-1} \\
I_2^n &\rightarrow I_0^{n-1}
\end{aligned}
$$

Working through the above suggested method of replacement to calculate the recurrent equation we have the following itinerary of the critical point with respect to central intervals only.

$$
\begin{aligned}
& I_0^n \rightarrow \eta \cdot I_1^{n-1}, I_2^{n-1}, I_0^{n-1} \qquad (3.31) \\
\Longrightarrow\ & I_0^n \rightarrow \eta \cdot (\eta \cdot I_2^{n-2},\ I_0^{n-2}),\ I_0^{n-2},\ I_0^{n-1} \\
\Longrightarrow\ & I_0^n \rightarrow \eta \cdot (\eta \cdot (I_0^{n-3}),\ I_0^{n-2}),\ I_0^{n-2},\ I_0^{n-1}.
\end{aligned}
$$

By disregarding the order of the central intervals appearing in the model (3.31), but respecting the coefficients η we see that the recursive equation is

$$2 \cdot \lambda_{n+1} = \lambda_n + \ (\eta+1)\lambda_{n-1} + \ (\eta^2)\lambda_{n-2} + \ K + O(p^n).$$

The number of times I_0^{n-i} appears in model (3.31) is equal to the coefficient for λ_{n-i-1}. The constant K is as in Equation (3.8) and, in particular, is independent of the map in this class.

One key element demonstrated by this example is that the coefficients have increasingly larger powers of η for the lesser level λ_{n-i}. This is true in general for our above renormalization scheme (3.29).

Lemma 3.7.1 *For a fixed positive integer M the coefficients B_i, $1 \leq i \leq M$, of the renormalization scheme (3.29) are polynomials of degree $i-1$ in η.*

Proof: The replacement scheme outlined above for the renormalization scheme (3.29) yields the following, independently of the level $n \geq M+1$:

$$I_0^n \to \eta \cdot (\eta \cdot ...(\eta \cdot (I_0^{n-M}), I_0^{n-(M-1)})... \; I_0^{n-2}), \; I_0^{n-1}. \tag{3.32}$$

Hence the orbit (with respect to the efficient decomposition) of I_0^n must fall in each set $I_0^k \setminus I_0^{k+1}$ the same number of times as I_0^k appears in Equation (3.32). The claim follows. ⊙

Lemma 3.7.1 will allow us to easily analyze the solution of the recurrent Equation (3.30). First let us drop the constant term and error term from the recurrent equation. This yields the equation

$$2 \cdot \lambda_{n+1} = \sum_{i=1}^{M} B_i \cdot \lambda_{n-i}. \tag{3.33}$$

Now an explicit solution of the above recurrent Equation (3.33) may be determined.

Theorem 3.7.2 *The recurrent Equation (3.33) has M independent solutions provided η is chosen sufficiently large.*

Proof: We need to find solutions of the recurrent equation

$$2 \cdot \lambda_{n+1} = \sum_{i=1}^{M} B_i \cdot \lambda_{n-i}.$$

To solve we need only replace λ_n by t^n and determine the roots of the polynomial equation $2t^M - B_1 t^{M-1} - B_2 t^{M-2} - ... - B_M = 0$. By Lemma 3.7.1 if η is sufficiently large our polynomial has roots which are just small perturbations of the roots of the equation $2t^M - B_M = 0$. It is now clear that we have M unique solutions with modulus greater than one for our polynomial if η is chosen sufficiently large. Let us denote the roots of our polynomial equation by α_1 through α_M. Then our recurrent Equation (3.33) has solutions

$$\lambda_n = k_1 \alpha_1^n + k_2 \alpha_2^n + \ldots + k_M \alpha_M^n$$

where the k_i's are free constants. Hence the claim is proven. $\odot$

It is easy to see that the eigenvalues of the matrix A are precisely the roots of the polynomial equation above. Since all roots have positive modulus, each will contribute in the scaling of the central intervals.

3.8 Robustness of the Parameters

In this section, we show that the parameters influencing the geometry of the critical orbit may be effectively varied. This will complete the proof of Theorem C. Let us take a combinatorial class $\mathcal{F}$ and any map f in this class with its parameter vector $\vec{v}(f)$. We demonstrate that there is a neighborhood around $\vec{v}(f)$ (in the non-contracting direction of matrix $A = A(\mathcal{F})$) such that for any vector $\vec{w}$ in this neighborhood there is a map $g \in \mathcal{F}$ which realizes this parameter vector. For simplicity, throughout the rest of this chapter, we will omit mentioning that the neighborhood of $\vec{v}(f)$ is actually just a neighborhood in the non-contracting direction of the matrix A.

We begin by outlining the construction. In order to show that there exists a neighborhood of $\vec{v}(f)$ in which all $\vec{w}$ in this neighborhood are realized for each n, we perform a continuous deformation of f for each of the generalized renormalization levels $n+1$ through $n+N$ for sufficiently large n, creating an N-dimensional parameter space of maps in $\mathcal{F}$. This N-dimensional continuous deformation space is constructed by rescaling the factors λ_{n+1} through λ_{n+N} without changing any λ_i, $i < n$. Given that the deformation space of maps is continuous, we show that the error term in Equation (3.9) varies continuously. So for each n we have a continuous change in Equation (3.9) and this will result in all vectors $\vec{w}$ in a neighborhood of $\vec{v}(f)$ being realized by our N-dimensional family.

Creating an N-parameter family of maps in $\mathcal{F}$ through f

At some renormalization level $n+1$ for the map f, we wish to create our N-parameter family of maps by linearly adjusting the sizes of $\lambda_{n+1}, ..., \lambda_{n+N}$ and their respective return maps $g_{n+i,0}$. Of course, if we simply rescale a λ_{n+i}, the resulting map may not be in $\mathcal{F}$ so we must take care to remain in this family. To start we show how to change the scaling factor λ_{n+1} continuously while remaining in $\mathcal{F}$.

Let us take the n-th renormalization level of f and denote the interval I_0^n by $[-a_0, a_0]$. Recall the map g_n which maps I_0^n unimodally to I_0^{n-1}. As before, there is an extension of g_n to an interval which we denote by $[-a_2, a_2]$ which maps unimodally into I_0^{n-2}. Finally, we take an intermediate interval $[-a_1, a_1] = [-\sqrt{a_0 a_2}, \sqrt{a_0 a_2}]$. Now we construct a 2-parameter family of maps $F_{T,c}$ and show there exists a 1-parameter family of maps $F_{T,c(T)}$ in $\mathcal{F}$ passing through f. For $x \in [\frac{-a_1}{\sqrt{T}}, \frac{a_1}{\sqrt{T}}]$, in place of the return map g_n we put

$$G_{T,c}(x) = T(g_n(x) - g_n(0)) + g_n(0) + c.$$

We point out that for our family $F_{T,c}$ that we do not change any other branches of g_n. For points $x \in [-a_2, -a_1]$ and $[a_1, a_2]$ we just smooth the map F, C^2 in the variables c and T, so that it agrees with the boundary map of g_n. Note that $g_n(x) = F_{1,0}(x)$ for $x \in [a_1, b_1]$. Finally, for the above to make sense in T and c, we require that $\sqrt{T} \in (\frac{a_0}{a_1}, \infty)$ and c satisfy $g_n(0) + c \in I_1^n$. (Recall that I_1^n has the property that it contains $g_n(0)$.) The parameter T varies the central interval I_0^n as follows: $|I_0^n(T,c)| \to 0$ as $T \to \infty$ and $I_0^n(T,c) \to [-a_1, a_1]$ as $\sqrt{T} \to a_0/a_1$.

Taking all maps $F_{T,c}$ which are renormalizable with respect to the combinatorics of $\mathcal{F}$, denoted by $R_{\mathcal{F}}$, we may rescale λ_{n+2} by a factor of T_2 just as above. Proceeding inductively, we apply $R^i_{\mathcal{F}}$, $i \leq N$, and rescale λ_{n+i} by a factor of T_i. This forms an $N+1$-parameter family of maps $F_{T_1,\ldots,T_N,c}$, with the parameter c further restricted by the combinatorial condition that we be able to renormalize N times. Now we show that there is an N-parameter subset which is in $\mathcal{F}$ and passes through our original map f.

Lemma 3.8.1 *For the variables $T_1, \ldots, T_N$ and c there is a N-parameter family $F_{T_1,\ldots,T_N,c(T_1,\ldots,T_N)}$ so that the resulting dynamics of the deformed f is in $\mathcal{F}$ and passes through f. Also, the map $c(T_1, \ldots, T_N)$ is smooth.*

Proof: We first work through the proof with $N = 1$. For each fixed T_1 if we vary the parameter c in its domain we get a "full family" in that all possible combinatorics exists between the extreme points of c [Lyu93a]. In particular, for each T_1 there exists at least one c such that after the resulting perturbation of g_n, $F_{T_1,c(T_1)}$ is in $\mathcal{F}$ as in the lemma. To prove that $c(T_1)$ is smooth we take long term parameter derivatives in c and T with respect to the orbit of the critical point. We then apply the implicit function theorem to a particular limiting sequence of parameter values, namely, those values, denoted by $c_m(T_1)$, for which the map has a super-attracting orbit following the combinatorics of $\mathcal{F}$ up to level m. In other words, we have that $R^m_{\mathcal{F}}(F_{T_1,c(T_1)})(0) = 0$, and we have created a sequence of submanifolds, each submanifold consisting of maps with the same super-attracting combinatorics, and the limit of these submanifolds is again a submanifold with our desired combinatorics.

So we fix T_1 and take all such c so that $F_{T_1,c}$ is renormalizable with respect to the combinatorics of $\mathcal{F}$ l times. We then show in following Lemma 3.8.2 that

$$\frac{d}{dc}\left(R^l_{\mathcal{F}}(F_{T,c}(0))\right) \text{ and } \frac{d}{dx}\left(R^l_{\mathcal{F}}(F_{T,c}(x))\,\big|_{x=F_{T,c}(0)}\right)$$

are comparable uniformly in l. In other words, the parameter derivative grows like the dynamical derivative. Then we show that

$$\frac{d}{dT}(R^l_{\mathcal{F}}(F_{T,c}(0)))$$

is at most comparable (if not smaller) than the above derivatives. Using this fact together with the implicit function theorem, we obtain a unique smooth function $c_m(T_1)$ for all T_1 in our domain. We also find that taking the limit in m of the solutions $c_m(T_1)$ gives a smooth 1-parameter family of maps F

with parameters T_1 and $c(T_1)$ all of which are in $\mathcal{F}$. This limit exists because the derivatives with respect to c are comparable (or much larger) than the derivatives with respect to T_1 and hence our solutions $c_m(T_1)$ have a uniformly bounded derivative in m. We will prove these derivative results below for all T_i provided we start at a sufficiently high renormalization level. Applying the same implicit function argument for the parameters $T_1, ..., T_N$ will complete the proof of the lemma. $\odot$

The following Lemma is motivated by the parameter and dynamical comparison theorems for circle mappings in [TV90].

Lemma 3.8.2 *Given any $K < 1$ there exists an n_0 so that for all $n > n_0$ if we perturb the map g_n as above we have the following estimates.*

$$K \cdot \left| \frac{d}{dx} \left(R^l_{\mathcal{F}}(F_{T_1,...,T_N,c}(x)) \right) \Big|_{x=F_{T_1,...,T_N,c}(0)} \right| < \frac{d}{dc} \left(R^l_{\mathcal{F}}(F_{T_1,...T_N,c}(0)) \right) \quad (3.34)$$

$$K \cdot \left| \frac{d}{dT_i} \left(R^l_{\mathcal{F}}(F_{T_1,...,T_N,c}(0)) \right) \right| < \frac{d}{dc} \left(R^l_{\mathcal{F}}(F_{T_1,...,T_N,c}(0)) \right) \quad (3.35)$$

for all $l \geq 0$.

Proof: We use the following notation:

- $Df(z) = d/dx(f(x))|_z$;
- $\triangle f_c(f^i(0)) = d/dc(f_c(x))$, evaluated at $x = f^i_c(0)$.

By the chain rule we may differentiate with respect to c using the following formula:

$$\triangle f_c^k(0) = \sum_{i=1}^{k} Df_c^{k-i}(f^i(0)) \cdot \triangle f_c(f_c^{i-1}(0)).$$

Using this formula, we calculate the long-term derivative in c for the $F_{T_1,\ldots,T_N,c}$ iterates of 0, to obtain

$$\begin{aligned}&\sum_{i=1}^{k} DF^{k-i}(F^i(0)) \cdot \triangle F(F^{i-1}(0)) \\ &\quad = DF^{k-1}(F^i(0)) \cdot \left(1 + \sum_{i=2}^{k} \frac{DF^{k-i}(F^i(0))}{DF^{k-1}(F^i(0))} \cdot \triangle F(F^{i-1}(0))\right) \quad (3.36)\end{aligned}$$

where we write F instead of $F_{T_1,\ldots,T_N,c}$ to avoid cumbersome equations. To obtain Equation (3.34), it suffices to show that the sum in Equation (3.36) is less than one in absolute value for a sufficiently large renormalization level. First note that $\triangle F_c(F_{T,c}^{i-1}(0))$ is either 0 or 1. Hence, it suffices to show that the sum of the amount of increase with increment of k, i.e.,

$$\sum_{k=2}^{\infty} \left(\sum_{i=2}^{k} \frac{DF^{k-i}(F^i(F(0)))}{DF^{k-1}(F^i(F(0)))} - \sum_{i=2}^{k+1} \frac{DF^{k-i}(F^i(F(0)))}{DF^{k-1}(F^i(F(0)))} \right) \quad (3.37)$$

is finite. Simplifying Equation (3.37), we get

$$\sum_{k=2}^{\infty} \frac{1}{DF^{k-1}(F(0))}.$$

But this converges by the exponential scaling results (compare [LM93], Lemma 5.9). We see that Equation (3.36) gives the first part of the lemma.

The second part of the lemma follows in the same manner by observing the inequality

$$\triangle F(F^{i-1}(0)) > \frac{d}{dT_1} F(F^{i-1}(0)) \tag{3.38}$$

and the fact that the following are comparable,

$$\frac{d}{dT_j} F(F^{i-1}(0)) \cdot (DR^j_{\mathcal{F}}(F(0)))^{-1}, \frac{d}{dT_1} F(F^{i-1}(0)). \tag{3.39}$$

The extra constant term $(DR^j_{\mathcal{F}}(F(0)))^{-1}$ is certainly bounded with respect to j, $1 \leq j \leq N$. This completes the proof of the lemma. $\odot$

Moving the parameters

Now that we have created our N-parameter family of maps in $\mathcal{F}$ passing through the given map f, we examine the scaling factors for our new maps $F(T_1, ..., T_N, c(T_1, ..., T_N))$. Starting with $F(1, ..., 1, 0)$, if we vary T_N then we get the scaling vector $\lambda_{n+N}(f) + \ln T_N, \lambda_{n+N-1}(f), ...\lambda_{n+1}(f)$. If we adjust T_{N-1}, this changes T_N by the amount given in the recurrent equation up to some small error (see Equation (3.14)). Inductively, we see that adjusting T_i changes T_{n+i} through T_{n+N} by some linear amount up to a small error. In this whole family of $F(T_1, ...T_N, c(T_1, ..., T_N))$ the error terms remain small because the Koebe error and the error in the C_i constants remain small. Hence, for now we ignore the error term and calculate how adjustment of the T_i changes the scaling factors at levels $n+1$ to $n+N$. We obtain the following formula with the constants $d_{i,j}$ defined below.

$$\begin{pmatrix} \lambda_{n+N}(F_{T_1,\dots,T_N}) \\ \lambda_{n+N-1}(F_{T_1,\dots,T_N}) \\ . \\ . \\ \lambda_{n+1}(F_{T_1,\dots,T_N}) \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & d_{12} & d_{13} & . & d_{1N} \\ 0 & 1 & d_{23} & . & d_{2N} \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & . & 1 \end{pmatrix} \begin{pmatrix} \ln T_N \\ \ln T_{N-1} \\ . \\ . \\ \ln T_1 \end{pmatrix} + \begin{pmatrix} \lambda_{n+N}(f) \\ \lambda_{n+N-1}(f) \\ . \\ . \\ \lambda_{n+1}(f) \end{pmatrix} \tag{3.40}$$

The $N \times N$ matrix is a constant matrix and independent of the level. Let us call this matrix D. The entries of matrix D may be calculated from the recurrent equation, (3.14).

$$d_{i,j} = 0 \text{ for } i > j, \tag{3.41}$$

$$d_{i,j} = 1 \text{ for } i = j, \tag{3.42}$$

$$\text{and } d_{i,j} = \sum_{l=1}^{j-i} b_l d_{i-l,j} \text{ for } i < j \tag{3.43}$$

Since D is invertible we see that by varying the T_i we obtain a neighborhood around $\lambda_{n+N}(f), \dots, \lambda_{n+1}(f)$. Pulling this neighborhood back (in the non-contracting direction of A) by our recurrent equation we obtain a neighborhood around $\vec{v}(f)$. We now estimate the size of this pulled-back neighborhood (still ignoring the error terms). Recall that $\sqrt{T_i}$ may vary over $(\frac{a_0}{a_1}, \infty)$.

Hence, the extremes of $\ln T_i$ (one negative, one positive) grow at least linearly in n by [Lyu93a]. Hence, for a Jordan block direction of A with eigenvalue r we see that the size of the pulled-back neighborhood around $\vec{v}(f)$ is of size $R(n)$ where $R(n)$ is bounded from below by $O(1/r^n)$. The error terms of Equation (3.14) at level n and beyond we perturb this neighborhood on the order of $O(\frac{p^n}{r^n})$, $(0 < p < 1, |r| > 1)$, uniformly over our created family. Note that this error term also depends continuously on our constructed family. By point-set topology, if we homotopically include this error term when pulling back we end up with a neighborhood which contains a ball of radius $R(n) - O(\frac{p^n}{r^n})$. Hence, for any n (sufficiently large) we have a neighborhood of the parameter vector $v(f)$ of size at least on the order of $O(1/r^n)$. Hence, any point in this neighborhood is realized by a map in our constructed family. This concludes Theorem C, part 3.

Appendix A

Geometry of Sets in the Plane

Topological discs in the plane.

We define capacity for sets in the plane and reference the perturbation result used in this paper. We point out that there are many equivalent definitions of capacity, many of which may be found in the book of Ahlfors (chapter 2, [Ahl73]). We give one such definition. Take a topological disc, U in the plane with boundary $\partial U = \Gamma$. Fix a point $z \in U$. Let $\mathcal{R}$ be the Riemann map of the unit disc onto U with $\mathcal{R}(0) = z$.

Definition. The *capacity* of U (or Γ) with respect to the point z is

$$cap_z(U) = \ln \mathcal{R}'(0).$$

We can calculate the capacities needed for this paper. For $cap_\infty(J\{z^2-1\})$ we proceed as follows. Using the Böttcher map and Brolin's formula, we see that the dynamics for the attracting basin is conjugate to the complement of the unit disc under the $z \mapsto z^2$ map. The conjugacy is in fact the Riemann mapping which has derivative precisely 1 at infinity (in the appropriate coordinate system). Hence, $cap_\infty(J\{z^2-1\}) = \ln 1 = 0$.

Similarly, we may calculate $cap_0(J\{z^2-1\})$. (Note we must only consider the connected component containing 0 for the capacity definition.) The dynamics around the critical point 0 is $z \mapsto 2z^2$ (two iterates of $z \mapsto z^2-1$). Again we can conjugate the immediate basin of attraction for the critical point to $z \mapsto z^2$ with domain the unit disc (again by the Böttcher map). Comparing the two maps, $z \mapsto 2z^2$ and $z \mapsto z^2$, we see that the conjugacy (Riemann map) must have derivative equal to 1/2. Hence $cap_0(J\{z^2-1\}) = \ln \frac{1}{2}$.

To state a perturbative result of capacity, consider all topological disc boundaries Γ in the plane with the Hausdorff metric d_H. The following result says that if we fix a point z bounded away from some Γ_∞, then exponential convergence to this curve in the Hausdorff metric yields exponential convergence in their capacities. The result is due to Schiffer and may be found in his paper [Sch38] or the book of Ahlfors [Ahl73] pages 98-99.

Theorem A.0.3 (Schiffer) *Given a sequence of disc boundaries Γ_i with convergence at an exponentially decreasing rate to some Γ_∞, $d_H(\Gamma_n, \Gamma_\infty) = O(p^n)$, and a point z bounded away from Γ_∞, then $cap_z(\Gamma_n) = cap_a(\Gamma_\infty) + O(p^n)$*

Topological annuli in the plane.

We take two topological open discs U_1 and U_2 in the plane such that U_2 is compactly contained in U_1. Then we may form the annulus $A = U_1 \setminus \bar{U}_2$. Every such annulus can be mapped (a *canonical map*) univalently to an annulus

$\{z|\ 0 < r_1 < |z| < r_2\}$. Although an annulus can be mapped to many different such annuli, there does exist a conformal invariant, namely the ratio of the radii r_2/r_1. There are many equivalent definitions for the modulus of an annulus, one of which is given here.

Definition. The *modulus* of an annulus A, $modA$, is the conformal invariant $\log \frac{r_2}{r_1}$ resulting from a canonical map.

Theorem (Koebe: Analytic version) *Take any two topological discs U_1, U_2 with $U_2 \subset\subset U_1$, and a univalent map g with domain U_1. Then independently of the map g, there exists ' a constant K such that*

$$\frac{|g'(x)|}{|g'(y)|} < K$$

for $x, y \in U_2$. Also $K = 1 + O(\exp(-mod(U_2 \setminus U_1)))$ as $mod(U_2 \setminus U_1) \to \infty$.

Theorem (Grötzsch Inequality) *Given three strictly nested topological ' discs, $U_3 \subset U_2 \subset U_1$ in $\mathbb{C}$,*

$$mod(U_1 \setminus U_2) + mod(U_2 \setminus U_3) \leq mod(U_1 \setminus U_3).$$

Now suppose we take a sequence of $U_3(i)$, containing 0 and converging to 0, and a sequence $U_1(i)$ with boundary converging to infinity. The set U_2 will remain fixed. Also, suppose $U_3(i) \subset U_2 \subset U_1(i) \subset \mathbb{C}$, then the equipotentials for the topological annuli $U_1(i) \setminus U_3(i)$ in compact regions of $\mathbb{C} \setminus 0$ converge to circles centered at 0. One consequence is the following proposition.

Proposition A.0.4 *Given* $U_1(i)$*,* U_2*, and* $U_3(i)$*, the deficit in the Grötzsch Inequality converges to* $cap_0(U_2) + cap_\infty(U_2)$.

Finally, we mention one extremal situation (see [LV73], Chapter 2 for actual estimates). Suppose we take a topological annulus $A \in \mathbb{C}$ with inner boundary Γ_1 and outer boundary Γ_2.

Proposition A.0.5 *If* A *is normalized so that the diameter of* Γ_1 *is equal to* 1 *then* $\text{dist}(\Gamma_1, \Gamma_2) \to \infty$ *as* $\text{mod}A \to \infty$.

Appendix B

Thurston's Teichmüller Transformation

We review some of the concepts treated in the paper of [DH93] concerning the contraction of Thurston's Teichmüller transformation. Some statements are slightly more general, although we restrict ourselves to the quadratic case. As seen in Chapter 2, the generality comes from considering compositions of different Thurston Transformations. However, the contraction theorem stated below is nearly identical to that of the Thurston Transformation. The reader may also wish to review the report titled "The Spider Algorithm" by Hubbard and Schleicher.

We outline the mathematics which follows. We first define a marked Teichmüller space of a finitely punctured sphere. We then define an ***admissible*** map (each a Thurston Transformation) from one such space to another, allowing for spaces of different dimension. Finally, we compose many admissible maps (and hence use potentially many different Teichüller spaces) and give a contraction result (Thurston's Contraction) depending on the number of compositions and the dimension of the Teichmüller spaces involved.

We denote the quadratic maps by $F_c(z) = (1-c)z^2 + c,\ c \in C \setminus \{1\}$. We wish to fix a set $P_{(i,j)}$ which will be our marked points for the Teichmüller space defined below. In particular, the fixed set $P_{(i,j)}$ is a set of ordered $(i+1+j)$ -tuples of real numbers such that

$$P_{(i,j)} = (p(-i), p(-(i-1)), ..., p(j-1), p(j) = 1),$$

$$p(0) = 0,\ p(j) = 1,$$

$$-1 < p(k) < p(k+1) \quad -i \leq k \leq j-1.$$

We define the space $\mathcal{T}_{(i,j)}$ as the space of all quasiconformal deformations of the sphere which fix the points $p(0) = 0, p(1) = 1$, and ∞. Observe that this space contains the space constructed in the proof of Lemma 3.3.7. The infinitesimal metric we put on the space $\mathcal{T}_{(i,j)}$ is the Finsler metric. In particular, we have that the complex tangent space is

$$T(\mathcal{T}_{(i,j)}) = \{\zeta\} := (\vec{\zeta}_{-i}, .., \vec{\zeta}_{-1},\ \vec{\zeta}_1, .., \vec{\zeta}_{j-1}) \mid \vec{\zeta}_k = \zeta_k \frac{\partial}{\partial p(k)} \in T_{p(k)}\bar{C}.$$

Finally, a point $P_{(i,j)} \in \mathcal{T}_{(i,j)}$ is represented by some $(i+1+j)$-tuple of complex numbers $(p(-i), p(-(i-1)), ..., p(0) = 0, ..., p(j-1), p(j))$.

We wish to form the map from one space $\mathcal{T}_{(i_1,j_1)}$, to another such space $\mathcal{T}_{(i_2,j_2)}$ which is induced by a quadratic map. To do this we first fix a space $\mathcal{T}_{(i_2,j_2)}$ with a marked tuple entry $p(k^*)$, $-i_2 \leq k^* \leq j_2$.

Definition. For a fixed space $\mathcal{T}_{(i_2,j_2)}$ with marked tuple entry $p(k^*)$ we say the pair $(\mathcal{T}_{(i_1,j_1)}, \chi)$ is *quadratic admissible* provided χ is a map from the integers in $[-i_1, k_1]$ to the integers in $[-i_2, j_2]$, which satisfies the following:

$$\chi(0) = k^*,$$

$$\chi(l) < \chi(l-1) \;\; l \leq 0,$$

$$\chi(l) < \chi(l+1) \;\; l \geq 0.$$

Now given a point $P_{(i_2,j_2)} \in \mathcal{T}_{(i_2,j_2)}$ with marked tuple entry $p(k^*)$, we may consider the quadratic map $F_{p(k^*)} = (1 - p(k^*))z^2 - p(k^*)$. We say the quadratic map $F_{p^*(k)}$ respects the quadratic admissible pair $(\mathcal{T}_{(i_1,j_1)}, \chi)$ if for any point $P_{(i_2,j_2)} \in \mathcal{T}_{(i_2,j_2)}$ it induces a unique point $P_{(i_1,j_1)} \in \mathcal{T}_{(i_1,j_1)}$ such that for each tuple entry $p_1(l) \in P_{(i_1,j_1)}$ we have $F_{p_2^*(k)}(p_1(l)) = (p_2(\chi(l))$.

Definition.Thurston Transformation Pulling-back points $P_{(i_2,j_2)}$ in the space $\mathcal{T}_{i_2,j_2}$ via the map $F_{p^*(k)}$ with respect to some quadratic admissible pair $(\mathcal{T}_{(i_1,j_1)}, \chi_1)$ results in the analytic map

$$\mathcal{T}_{(i_2,j_2)} \overset{\sigma_{\chi_1}}{\to} \mathcal{T}_{(i_1,j_1)}.$$

Given the notion of admissible pairs we now define an M-string of admissible pairs.

Definition. The following diagram is an *M-string* of admissible pairs

$$\mathcal{T}_{(i_M,j_M)} \overset{\sigma_{\chi_{M-1}}}{\to} \mathcal{T}_{(i_{M-1},j_{M-1})} \overset{\sigma_{\chi_{M-2}}}{\to} \dots \overset{\sigma_{\chi_2}}{\to} \mathcal{T}_{(i_2,j_2)} \overset{\sigma_{\chi_1}}{\to} \mathcal{T}_{(i_1,j_1)} \tag{B.1}$$

if $\mathcal{T}_{(i_{k+1},j_{k+1})} \overset{\sigma_{\chi_k}}{\to} \mathcal{T}_{i_k,j_k}$ is admissible for all integers $k \in [1, M-1]$.

Theorem B.0.6 (Contraction of the Thurston Transformation) *Suppose we have an M-string of admissible pairs (as in equation B.1) with the extra condition that* $i_k + 1 + j_k \leq M + 2$ *for all integers* $k \in [1, M]$. *Then the Thurston Transformation is strictly contracting from* $\mathcal{T}_{i_M, j_M}$ *to* $\mathcal{T}_{i_1, j_1}$, *i.e.,*

$$\| d(\sigma_{\chi_M} \circ \sigma_{\chi_{M-1}} \circ ... \circ \sigma_{\chi_1}(P_{(i_1,j_1)}) \| < 1.$$

Proof: (Outline) Just as in [DH93], in place of the map $\mathcal{T}_{(i_{k+1}, j_{k+1})} \stackrel{\sigma_{\chi_k}}{\rightarrow} \mathcal{T}_{(i_k, j_k)}$ one may instead look at the transpose map F_*. This is the map from the space of quadratic differentials $Q(\mathcal{T}_{(i_k,j_k)})$ to $Q(\mathcal{T}_{(i_{k+1},j_{k+1})})$ where the single poles of a quadratic differential exists only in the $(i + 1 + k)$ -tuples of points. Also each space $Q(\mathcal{T}_{(i_k,j_k)})$ is equipped with the L^1-norm.

The transpose map (outside of critical values) is defined as

$$F_*(q) = (q_1(z) + q_2(z))d(z)^2,$$

where q_1 and q_2 represent q at the two preimages of any point in C under the map $F_{p^*(k)}$. In particular, one has that $\| F_*(q) \| \leq \| q \|$. The only possibility for equality, i.e.,

$$\int |(q_1(z) + q_2(z))| = \int |q_1(z)| + \int |q_2(z))|$$

is if all preimages of poles of F_*q under the map $F_{p^*(k)}$ are also poles. But since all points except the critical value and infinity have two pre-images it is easy to see that provided one starts with $k \geq 3$ poles for q one will then create $k + 2$ poles for F_*q. Now given the facts that every q must have at least 4 poles and that each pole can occur only at the $(i_l + 1 + k_l)$ -tuples of points (which is bounded in number by M by assumption), we have our result. $\odot$

Bibliography

[Ahl73] L. Ahlfors, *Conformal Invariants, Topics in Geometric Function Theory*, McGraw-Hill, 1973.

[BH92] B. Branner and J.H. Hubbard, *The iteration of cubic polynomials, part II*, Acta Math. **169** (1992), 229–325.

[Bla84] P. Blanchard, *Complex analytic dynamics on the Riemann sphere*, Bull. Amer. Math. Soc. **11** (1984), 85–141.

[Bro65] H. Brolin, *Invariant Sets Under Iteration of Rational Functions*, Ark. Math. **6** (1965), 103–144.

[DH85a] A. Douady and J.H. Hubbard, *E'tude dynamique des polynomes complexes*, Publications Mathematiques d'Orsay, Universite' de Paris-Sud, 1984-1985.

[DH85b] A. Douady and J.H. Hubbard, *On the dynamics of polynomial-like maps.*, Ann. Sc. Ec. Norm. Sup. **18** (1985), 287–343.

[DH93] A. Douady and J.H. Hubbard, *A proof of Thurston's topological characterization of rational functions.*, Acta Math. **171** (1993), 263–297.

[Dou93] A. Douady, *Descriptions of compact sets in* $\hat{C}$., Topological Methods in Modern Mathematics, A Symposium in Honor of John Milnor's 60th Birthday, Publish or Perish, 1993.

[EL90] A. E. Eremenko and M. Yu. Lyubich, *The dynamics of analytic transformations.*, Leningrad Math. J. **1** (1990), no. 3, 563–634.

[GM93] L. Goldberg and J. Milnor, *Fixed point portraits of polynomial maps.*, Preprint 14, SUNY at Stony Brook, IMS, 1993.

[Hub93] J.H. Hubbard, *Local connectivity of Julia sets and bifurcation loci: three theorems of J. C. Yoccoz.*, Topological Methods in Modern Mathematics, A Symposium in Honor of John Milnor's 60th Birthday, Publish or Perish, 1993.

[LC93] T. W. Gamelin L. Carleson, *Complex dynamics*, Springer-Verlag, 1993.

[LM93] M. Lyubich and J. Milnor, *The fibonacci unimodal map.*, Journal of the AMS **6** (1993), 425–457.

[LV73] O. Lehto and K. I. Virtanen, *Quasiconformal mappings in the plane*, Springer-Verlag, 1973.

[Lyu93a] M. Lyubich, *Combinatorics, geometry and attractors of quasi-quadratic maps*, Ann. of Math. **140** (1993), 347–404.

[Lyu93b] M. Lyubich, *Geometry of Quadratic Polynomials: Moduli, Rigidity, and Local Connectivity*, Preprint 9, SUNY at Stony Brook, IMS, 1993.

[Lyu93c] M. Lyubich, *Teichmuller Space of Fibonacci maps*, Preprint 12, SUNY at Stony Brook, IMS, 1993.

[Mar94] M. Martens, *Distortion results and invariant cantor sets of unimodal maps.*, Ergod. Th. and Dynam. **14** (1994), 331–349.

[McM94a] C. McMullen, *Frontiers in Complex Dynamics*, Bull. Amer. Math. Soc. **2** (1994), 155–172.

[McM94b] C. McMullen, *Renormalization and 3-manifolds which fiber over the circle*, preprint, University of California at Berkley, 1994.

[Mil89] J. Milnor, *Self-similarity and hairiness in the Mandelbrot set*, Computers in Geometry and Topolgy (Lecture Notes PUre Appl. Math Tangora, ed.), no. 114, Beckker, 1989.

[Mil90] J. Milnor, *Dynamics in one complex variable: Introductory lectures*, Preprint 5, SUNY at Stony Brook, IMS, 1990.

[Mil92] J. Milnor, *Local connectivity of Julia sets: Expository lectures*, Preprint 11, SUNY at Stony Brook, IMS, 1992.

[Sch38] M. Schiffer, *A Method of Variation within the Family of Simple Functions*, Proc. London Math. Soc. **2** (1938), 450–452.

[Sul93] D. Sullivan, *Linking the universalities of Milnor-Thurston, Feigenbaum and Ahlfors-Bers*, Topological Methods in Modern Mathematics, A Symposium in Honor of John Milnor's 60th Birthday, Publish or Perish, 1993.

[Tan90] L. Tan, *Similarity between the Mandelbrot set and Julia sets*, Commun. Math. Phys. **134** (1990), 587–617.

[TV90] F. M. Tangerman and J. J. P. Veerman, *Scalings in Circle Maps (ii).*, Commun. Math. Phys. (1990), no. 141, 279–291.