Local Connectivity and Lebesgue Measure of Polynomial Julia Sets

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Abstract of the Dissertation

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For every complex polynomial P of degree $d \geq 2$, there is a closed, perfect set in the plane, called the Julia set, on which the dynamics of P are chaotic. In 1984, Douady and Hubbard presented a combinatorial description of the dynamics for a polynomial Julia set provided the Julia set is locally connected. In 1990, Yoccoz showed that a large class of quadratic polynomials, the finitely renormalizable ones, have locally connected Julia sets. In the first part of this dissertation, we extend some of Yoccoz' methods to certain classes of higher degree polynomials to study the local connectivity of Julia sets.

It was a long-standing question whether there were any polynomials with positive Lebesgue measure Julia sets. In 1994, Nowicki and VanStrien answered the question in the positive by showing that certain polynomials of

very high degree have positive measure Julia sets. However the question remains open for quadratic polynomials. Lyubich and Shishikura independently showed that finitely renormalizable quadratic polynomials have zero measure Julia sets. Still, little is known about the measure of quadratic Julia sets in the infinitely renormalizable case. In the second part of this dissertation, we shall construct a class of infinitely renormalizable quadratic polynomials whose Julia sets have zero Lebesgue measure.

For my wife, Ailyna.

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1. Introduction

Dynamical systems is the study of how systems, physical or theoretical, evolve through time. As our dynamical system we shall consider the Riemann sphere $\hat{\mathbb{C}}$ with a polynomial $P : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ representing the change in the system through one unit of time.

The Riemann sphere $\hat{\mathbb{C}}$ can be divided into two totally invariant sets with respect to P: a stable set, on which the dynamics of P is predictable; and an unstable set, on which the dynamics of P is chaotic. In the language of complex analysis, the stable set for P is the set of points $z \in \hat{\mathbb{C}}$ for which the family of iterates of P is normal in some open neighborhood of z. The stable set of P is called the Fatou set.

The chaotic set, or the Julia set, of P is the complement in $\hat{\mathbb{C}}$ of the Fatou set. It has several characterizations, including the absence of normality, as well as being the closure of the set of repelling periodic orbits, or the topological boundary of the unbounded Fatou component.

In this dissertation, we shall study some topological and measure theoretical questions of the Julia set of a polynomial. In particular, we shall study whether the Julia set of a polynomial is locally connected and whether it has zero Lebesgue measure.

1.1 A Bit of History

In 1965, Hans Brolin published a paper ([Bro]) in which he introduced the Green's function associated to a complex polynomial. The Green's function can be dynamically defined to produce an invariant foliation of the unbounded component of the Fatou set.

Branner and Hubbard ([BH]) in 1988 used the Green's function to partition the Julia set of certain cubic polynomials into "puzzle pieces". The puzzle pieces were defined to be the bounded components of the complement of an equipotential. The invariance of equipotentials gave the puzzle pieces certain Markov properties so that they could be used to define a symbolic representation of the dynamics of the Julia set.

A year later, J.C. Yoccoz extended the definition of puzzle pieces by combining equipotentials with a second invariant foliation, the external rays. Using these puzzle pieces, Yoccoz showed that any quadratic polynomial with no irrationally indifferent periodic orbits, which has a connected Julia set, and which is not infinitely renormalizable will necessarily have a locally connected Julia set. Because each puzzle piece intersects the Julia set in a connected set, his method of proof was to show that for each point in the Julia set, there is a nested sequence of puzzle pieces which shrink to the point.

In 1991, Lyubich ([Lyu3]) and Shishikura worked independently to show that all quadratic polynomials satisfying the three conditions of the previous paragraph had Julia sets with zero measure. Lyubich used puzzle pieces and Yoccoz' results with local connectivity to show that there exists a recursively defined covering of almost all of the Julia set which is shrinking in measure.

In his 1994 paper [Lyu2], Lyubich studied the rates at which puzzle pieces shrink for quadratic polynomials. In this paper, he shows that not only are the puzzle pieces shrinking to points, but they are doing so at an increasing rate. He then uses this strengthened version to show that there exists a class of infinitely renormalizable quadratic polynomials with locally connected Julia sets.

Through the tool of puzzle pieces, it becomes natural to study the questions of local connectivity and Lebesgue measure of a polynomial Julia set simultaneously. Because each puzzle piece intersects the Julia set in a connected set, they are a powerful tool for studying local connectivity. But by studying how the puzzle pieces shrink to points, we can also develop estimates on the geometry and, therefore, the Lebesgue measure, of the Julia set.

At this point we should note that there are examples of polynomials which have Julia sets which are connected but not locally connected. If the polynomial has a Crémer periodic orbit, then its Julia set can not be locally connected ([Mil1]), and there are examples of certain infinitely renormalizable quadratic polynomials with non-locally connected Julia sets ([Mil2]).

A problem dating back to the time of Fatou in the early part of this century was whether a rational map with a nowhere dense Julia set could have positive measure. Based on a computer experiment of Lyubich, Sutherland, and Tangerman from 1992, Lyubich and Sutherland conjectured in 1994 that certain polynomials, called Fibonacci polynomials, of degree ≥ 32 should have Julia sets which were nowhere dense but with positive measure. Also in 1994, Nowicki and van Strien ([NvS]) proved that Fibonacci polynomials of sufficiently high degree did have positive measure Julia sets, where sufficiently high was very large (> 1000).

1.2 Overview

We shall extend some of the results of Yoccoz, Lyubich, and Shishikura concerning local connectivity and Lebesgue measure for quadratic Julia sets. Throughout this dissertation, we will only be concerned with polynomials which have only repelling periodic orbits. Under this restriction, the Fatou set of the corresponding polynomials has no bounded components, and thus the Julia set is a "dendrite". We will extend these results in two directions. First, we will show that these results can be extended to certain classes of higher degree polynomials. The second direction will be to show that there exists a class of infinitely renormalizable quadratic polynomials with zero measure Julia sets.

To extend the results to higher degree non-renormalizable polynomials, we will use a classification of critical point orbits. The classification specifies that all critical point orbits are either non-recurrent, reluctantly recurrent, or persistently recurrent, depending on how frequently their orbit returns to a neighborhood of the original critical point. In chapter 3, we will deal with the case when all the critical points have orbits which are either non-recurrent or reluctantly recurrent. In this case, the Julia set of the corresponding polynomial is locally connected.

After that, we will only be concerned with polynomials which have only one critical point. In chapter 4, we will look at the family of non-renormalizable polynomials of the form $z \mapsto z^3 + c$. As was stated in the previous paragraph, we already know about the local connectivity of the Julia set if the critical orbit is either non-recurrent or reluctantly recurrent. In the persistently recurrent case, we can associate to the polynomial a different mapping called a "generalized polynomial-like mapping" which reflects the combinatorics of the original polynomial. This new mapping will be infinitely renormalizable. We specify the conditions on the combinatorics of the renormalizations of the generalized polynomial-like mapping that will be sufficient to imply the local connectivity and zero measure of the Julia sets for the original polynomial.

Finally, we will return to a class of infinitely renormalizable quadratic polynomials in chapter 5. This class, introduced by Lyubich in his paper [Lyu2], has locally connected Julia sets. This family is characterized by controlling where the renormalizations can occur within the Mandelbrot set and by establishing a lower bound on the combinatorial depth between renormalization levels (these terms will be made precise in chapter 5). We will show that polynomials in Lyubich's class have Julia sets with zero Lebesgue measure.

2. Background

2.1 Holomorphic Dynamics

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a holomorphic mapping from the Riemann sphere to itself. If $z_0 \in \widehat{\mathbb{C}}$, let $z_1 = f(z_0)$ be the image of z_0 under the mapping f, and let $z_n = f(z_{n-1})$. Then the sequence $\{z_0, z_1, z_2, \ldots\}$ will be called the *forward orbit* (or just *orbit*) of z under the mapping f. We will also use the notation $f^n(z)$ to denote the n-fold iterate of f:

$$f^n(z) = \underbrace{(f \circ \cdots \circ f)}_{n\ times}(z) \quad . \tag{1}$$

If for $z \in \widehat{\mathbb{C}}$ there exists a positive integer n such that $f^n(z) = z$, then z is called a *periodic point.* If m is the minimal positive integer such that $f^m(z) = z$, then $\underline{z} = \{z, z_1, \ldots, z_{m-1}\}$ is called a *periodic orbit*, and m is called the *period* for $\underline{z}$. If z is periodic of period 1, then z is called a *fixed point.* The *multiplier* Λ for a periodic orbit $\underline{z} = \{z, z_1, \ldots, z_{m-1}\}$ is defined as

$$\Lambda(z) = (f^m)'(z) = \frac{d}{dz}(f^m)(z) = \prod_{i=1}^{m} f'(f^{i-1}(z)) \quad . \tag{2}$$

Thus we can talk about the multiplier of the periodic orbit $\Lambda(\underline{z})$.

The multiplier provides us with a tool used in the classification of periodic orbits. A periodic orbit $\underline{z}$ is called *attracting* if $|\Lambda(\underline{z})| < 1$. The orbit is called *neutral* if $|\Lambda(\underline{z})| = 1$. And $\underline{z}$ is called *repelling* if $|\Lambda(\underline{z})| > 1$. One special case to note: if $|\Lambda(\underline{z})| = 0$, then the orbit $\underline{z}$ is called *super-attracting.*

Consider the function

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots \tag{3}$$

which is defined and holomorphic in some neighborhood of the origin, with a fixed point of multiplier a_1 at $z = 0$. The following two theorems are classical results whose proofs can be found in Milnor's notes [Mil1].

THEOREM (KŒNIG). *If the multiplier a_1 satisfies $|a_1| \neq 0, 1$, then there exists a local holomorphic change of coordinate $w = \phi(z)$, with $\phi(0) = 0$, so that $\phi \circ f \circ \phi^{-1}$ is the linear map $w \mapsto a_1 w$ for all w in some neighborhood of the origin. Furthermore, ϕ is unique up to multiplication by a non-zero constant.*

THEOREM (BÖTTCHER). *Suppose that*

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots \tag{4}$$

where $n \geq 2$, and $a_n \neq 0$. Then there exists a local holomorphic change of coordinate $w = \phi(z)$, which conjugates f to the n-th power map $w \mapsto w^n$ throughout some neighborhood of $\phi(0) = 0$. Furthermore, ϕ is unique up to multiplication by an (n-1)-st root of unity.

The set of all points whose orbit accumulates at the orbit $\underline{z}$ will be called the *basin of attraction* for the orbit $\underline{z}$.

A family of holomorphic functions $f_n : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is called *normal* if every sequence $\{f_n\}$ of functions in this family has a subsequence $\{f_{n_i}\}$ which converges uniformly on compact subsets of S. We can use the idea of normality to define a partition of S. The point $z \in S$ is in the *Fatou set* of f if the family of iterates of f, $\{f^n\}$ is a normal family on some neighborhood of z. The complement of the Fatou set is called the *Julia set*.

THEOREM (MONTEL). *Let S be any Riemann surface. If a collection F of holomorphic maps from S to $\hat{\mathbb{C}}$ takes values in some hyperbolic open subset $U \subset \hat{\mathbb{C}}$, then this collection F is normal.*

2.2 Some Results from Analysis

The results of this section are presented more formally in [Ah1], [Ah3], or [CG].

We will call a positive measurable function $\rho : \mathbb{C} \to R^+$ *admissible* if

$$A(\rho) = \iint_{\mathbb{C}} \rho^2 \, dxdy < \infty \quad . \tag{5}$$

Let Γ be any family of (possibly disconnected) rectifiable curves $\{\gamma_i\}$, with $\gamma_i : [0,1] \mapsto \mathbb{C}$. Let $L(\gamma, \rho) = \int_0^1 \rho(\gamma(t))dt$ for any $\gamma \in \Gamma$. We define the *extremal length* of Γ to be

$$\sigma(\Gamma) = \sup_{\rho \; admissible} \frac{\inf_{\gamma \in \Gamma}(L(\gamma,\rho))^2}{A(\rho)} \quad . \tag{6}$$

PROPOSITION 2.2.1. *$\sigma(\Gamma)$ is a conformal invariant.*

If $U, V \subset \mathbb{C}$ are open, and $\overline{U} \subset V$, let $A = V \setminus \overline{U}$ be the topological annulus, and let $\Gamma(A)$ be the family of rectifiable curves whose images is contained in A and has one end point in each boundary component of A. The *modulus* of A is defined as

$$mod(A) = \sigma(\Gamma(A)) \quad . \tag{7}$$

Suppose $U, V, W \subset \mathbb{C}$ are open, and $\overline{U} \subset V$, and $\overline{V} \subset W$. Let $A_1 = V \setminus \overline{U}$, $A_2 = W \setminus \overline{V}$, and $A = W \setminus \overline{U}$.

Proposition (Grötzch Inequality).

$$mod(A) \geq mod(A_1) + mod(A_2) \quad . \tag{8}$$

Proposition 2.2.2. *Suppose $f : V \to V'$ is a holomorphic degree d covering map which maps the topological annulus $A = V \setminus \overline{U}$ onto $A' = V' \setminus \overline{U'}$. Then*

$$mod(A) \geq \frac{1}{d} mod(A') \quad . \tag{9}$$

Let us turn our attention to the distortion of conformal maps. Distortion can be thought of as a measure of the non-linearity of a function. We define the distortion $dist(f, V)$ of a conformal mapping $f : V \to \mathbb{C}$ as

$$dist(f, V) \equiv \sup_{x,y \in V} \; log \, \frac{|f'(x)|}{|f'(y)|} \quad . \tag{10}$$

The proofs of the results in the next paragraphs can be found in [CG]. Let S bet the family of conformal maps on the unit disk such that $f(0) = 0$ and $f'(0) = 1$.

Theorem (Koebe Distortion). *If $f \in S$, then*

$$\frac{1 - |z|}{(1 + |z|)^3} \leq |f'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3} \tag{11}$$

and

$$\frac{|z|}{(1 + |z|)^2} \leq |f(z)| \leq \frac{|z|}{(1 - |z|)^2} \quad . \tag{12}$$

The Koebe distortion theorem gives an upper and lower bound for the derivative of a conformal map, and therefore provides a bound on the distortion of the conformal map. Using the properties of the modulus of an annulus, we have the following corollary, which is an immediate consequence of the Koebe distortion theorem.

COROLLARY 2.2.3. *Suppose $f : U \to U'$ is a conformal map between two topological disks, and V open such that $\overline{V} \subset U$. Then there exists $K \equiv K(mod(U \setminus \overline{V}))$ such that*

$$dist(f, V) \leq K \quad . \tag{13}$$

This corollary provides us with bounds on the non-linearity of a conformal mapping, provided the mapping can be extended conformally to an open set which is "definitely" bigger than our domain, where "definitely" is defined in terms of the modulus of an annulus.

2.3 Quasiconformal Maps

Quasiconformal maps have become an increasingly powerful tool in the study of complex dynamics. Let us mention a few properties of quasiconformal maps, and refer the reader to [Ah1] or [LV] for the details.

DEFINITION 2.3.1. *Let $U, V \subset \mathbb{C}$ open, and $f : U \to V$ a homeomorphism. The mapping f is K-quasiconformal if for every annulus $A \subset U$,*

$$\frac{1}{K} mod(A) \leq mod(f(A)) \leq K\, mod(A) \quad . \tag{14}$$

Such a mapping has locally integrable distributional derivatives $\partial_{\bar{z}} f$ and $\partial_z f$ which satisfy

$$\left|\frac{\partial_{\bar{z}} f}{\partial_z f}\right| \leq \frac{K-1}{K+1} < 1 \quad almost\ \ everywhere \quad . \tag{15}$$

Define the *Beltrami differential* to be the measurable function

$$\mu(z) = \frac{\partial_{\bar{z}}(f(z))}{\partial_z(f(z))} \tag{16}$$

which satisfies $\|\mu\|_\infty < 1$.

A mapping f is 1-quasiconformal if and only if f is conformal in the usual sense. For a conformal mapping, $\mu(z) \equiv 0$.

The following is an important property of quasiconformal maps that we will need on several occasions in this paper.

PROPOSITION 2.3.2. *Quasiconformal maps are absolutely continuous with respect to Lebesgue measure.*

2.4 Polynomial Dynamics

Let us choose our holomorphic function to be a polynomial $P : \mathbb{C} \to \mathbb{C}$ of degree ≥ 2. If we consider the polynomial P as acting on the Riemann sphere $\hat{\mathbb{C}}$, then P has a super-attracting fixed point at ∞. Therefore there exists a positive real number R such that if $|z| > R$, then z is in the basin of attraction of the point at ∞. Define the *filled Julia set* $K(P)$ to be the set of points in $\mathbb{C}$ whose orbit remains bounded.

LEMMA 2.4.1. *For any polynomial P, $K(P)$ is a compact set consisting of the Julia set $J(P)$ and all bounded Fatou components. These bounded components are all simply connected, and the Julia set $J(P)$ is equal to the topological boundary ∂K.*

THEOREM 2.4.2. *For any polynomial $P : \mathbb{C} \to \mathbb{C}$, there are two mutually exclusive possibilities: the filled Julia set $K(P)$ contains all finite critical points of P, and $K(P)$ and $J(P)$ are connected; or $\mathbb{C}\backslash K(P)$ contains at least one finite* *critical point, and both $J(P)$ and $K(P)$ have uncountably many connected components.*

The proofs of these two results can be found in [Mil1].

Let us now mention the relevance of the local connectivity of the Julia set with understanding the dynamics of the polynomial. The proof of this theorem can be found in either [CG] or [Mil1].

THEOREM (CARATHÉODORY). *Let D be a simply connected domain in $\hat{\mathbb{C}}$ whose boundary has at least two points. The ∂D is locally connected if and only if the Riemann mapping $\psi : \Delta \to D$ extends continuously to the closed disk $\overline{\Delta}$.*

It follows immediately from this theorem that if $K(P)$ is connected, the Böttcher coordinate ϕ for $\mathbb{C} \setminus K(P)$ extends continuously to $J(P)$ if and only if $J(P)$ is locally connected.

In a paper by Douady and Hubbard ([DH2]), an important tool for studying polynomial dynamics was introduced using techniques from quasiconformal mappings.

DEFINITION 2.4.3. *A polynomial-like mapping of degree d is a triple g, U, V which satisfies*

1) U, V are open in $\mathbb{C}$ and isomorphic to disks, and $\overline{U} \subset V$,

2) $g : U \to V$ is a proper holomorphic branched covering map of degree d.

The *filled Julia set* $K(g)$ is the set of points in U whose orbit under g never leaves U.

Two polynomial-like mappings $g_1 : U_1 \to V_1$ and $g_2 : U_2 \to V_2$ are called *hybrid equivalent* if there exists open neighborhoods W_1, W_2 about $K(g_1)$ and $K(g_2)$ respectively, and a quasiconformal map $h : W_1 \to W_2$ such that

$$h \circ g_1 \circ h^{-1}|_{W_2} = g_2 \tag{17}$$

and

$$\partial_{\bar{z}} h = 0 \ \ on \ \ K(g) \quad . \tag{18}$$

The equivalence class of all polynomial-like mappings hybrid equivalent to a polynomial-like mapping g is called the *internal class* of g, and is usually denoted by $c(g)$.

The following theorem will be used repeatedly throughout this paper, and is generally referred to as the *straightening theorem* ([DH2]).

THEOREM (STRAIGHTENING). *Every polynomial-like mapping $g : U \to V$ of degree d is hybrid equivalent to a polynomial P of degree d. Furthermore, if $K(g)$ is connected, then P is unique up to affine conjugation.*

2.5 Non-renormalizable Quadratic Polynomials

All quadratic polynomials are conformally conjugate to a polynomial of the form $P_c : z \mapsto z^2 + c$ for some $c \in \mathbb{C}$. By the previous section, $J(P_c)$ is connected if and only if the orbit of 0 is bounded; that is, $0 \in K(P_c)$

DEFINITION 2.5.1. *The Mandelbrot set $\mathcal{M}$ is the set of all parameters $c \in \mathbb{C}$ such that $J(P_c)$ is connected, or equivalently, the set of all parameters such that the orbit of 0 remains bounded.*

Assume $c \in \mathcal{M}$. Using the Böttcher coordinate for the point at ∞, we can find a conformal change of coordinates $\phi : \mathbb{C} \setminus K(P_c) \to \mathbb{C} \setminus \overline{\Delta}$ such that

$$(\phi \circ P_c \circ \phi^{-1})(z) = z^2 \tag{19}$$

for every $z \in \mathbb{C} \setminus \overline{\Delta}$, and $\lim_{z \to \infty} \phi(z) = z$. Using this mapping, we can define the *Green's function* $G(z)$ for P_c as

$$G(z) = log|\phi(z)| = \lim_{n \to \infty} log \frac{|P^n(z)|}{2^n} \quad . \tag{20}$$

Clearly if $z \in K(P_c)$, then $G(z) = 0$. For any $r > 0$, define the *equipotential* of value r as the set

$$\Gamma_r = \{z \in \mathbb{C} : G(z) = r\} \quad . \tag{21}$$

Note that if $z \in \Gamma_r$, then $P_c(z) \in \Gamma_{2r}$.

Also, define the *external ray* R_γ with angle $\gamma \in [0, 1)$, or just the *γ-ray*, for the polynomial P_c to be the set

$$R_\gamma = \{z \in \mathbb{C} \setminus K(P_c) : \phi(z) = re^{2\pi i \gamma} \ \ with \ \ r > 1\} \quad . \tag{22}$$

The ray R_γ is said to *land* at the point $z \in J(P_c)$ if

$$\lim_{r \to 1^+} re^{2\pi i\gamma} = \{z\} \quad . \tag{23}$$

It follows directly from their definitions that the set of external rays and the set of equipotentials form two invariant foliations of $\mathbb{C} \setminus K(P_c)$.

Using equipotentials and external rays, Yoccoz created a partition of the Julia set which has certain "Markov" properties. This partition is now referred to as the Yoccoz puzzle piece partition.

Suppose both fixed points of P_c are repelling. One of these fixed points, called the β-fixed point, is the landing point of the 0-ray. The other fixed point, called the α-fixed point, therefore can not be the landing point of a fixed ray.

LEMMA 2.5.2. *If $J(P_c)$ is connected, then every repelling or parabolic periodic point is the landing point of at least one external ray, which is necessarily periodic.*

The proof can be found in [Mil1].

It follows from this theorem that the α-fixed point is the landing point of $q \geq 2$ rays which are permuted cyclically by P_c. The *Yoccoz configuration $\mathcal{Y}^0$ of depth 0* is defined as the 1-equipotential Γ_1 along with the segments of the q external rays which join Γ_1 with the α-fixed point. See Figure 2.1. The *Yoccoz puzzle pieces of depth 0* are then defined as the q bounded components of the complement of $\mathcal{Y}^0$ in $\mathbb{C}$, which we shall label $Y_0^0, \ldots Y_{q-1}^0$. The notation

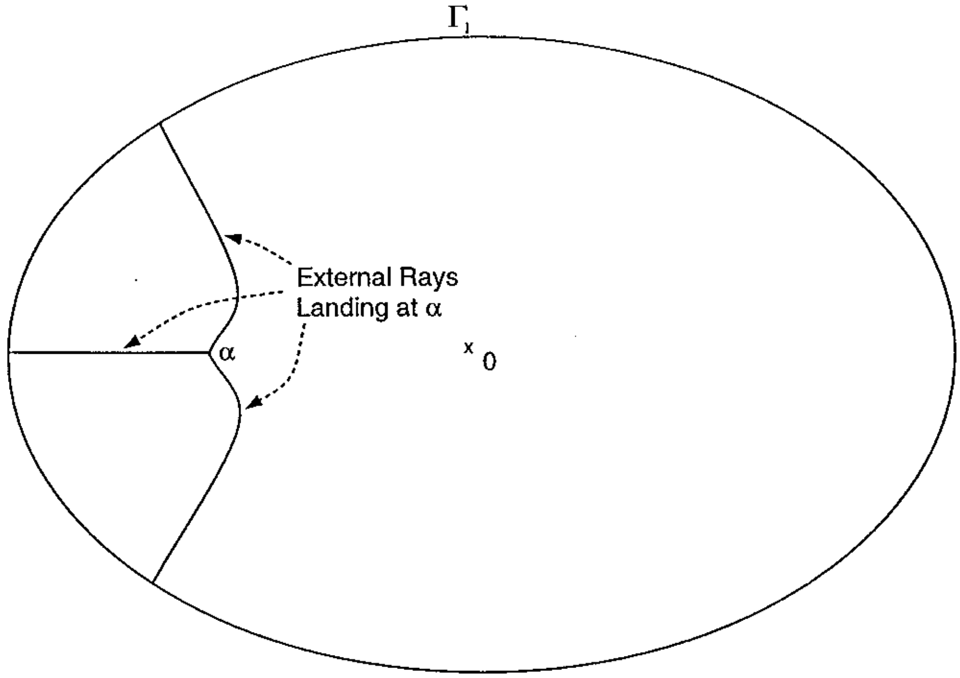

Figure 2.1. The Depth 0 Yoccoz Configuration: Γ_1 along with the external rays which land at α.

Y_0^0 will be used to denote the unique depth 0 puzzle piece which contains the critical point.

Puzzle pieces of depths greater than zero are defined by "pulling-back" the depth 0 puzzle pieces. That is, if $z \in J(P_c)$, and $P_c(z) \in Y_i^0$, then $Y^1(z)$ is defined as the unique open connected component of $P_c^{-1}(Y_i^1)$ such that $z \in Y^1(z)$. By the invariance of equipotentials and external rays, it follows that if $Y_i^1 \cap Y_j^0 \neq \emptyset$, then $Y_i^1 \subset Y_j^0$. Thus the puzzle pieces satisfy a Markov property. Furthermore, if z is not a pre-image of the α-fixed point, and $P_c^n(z) \in Y_i^0$, then let $Y^n(z)$ be the unique connected component of $P_c^{-n}(Y_i^0)$ containing z.

Two immediate properties of Yoccoz puzzle pieces: for any z which is not a pre-image of α, $Y^n(z) \subset Y^{n-1}(z)$, and $Y^n(z) \cap J(P_c)$ is a connected set for all

$n \in \mathbb{N}$. It follows immediately from these two properties that if $Y^n(z) \to \{z\}$ as $n \to \infty$, then $J(P_c)$ is locally connected at z.

DEFINITION 2.5.3. *If there exists an open set U about 0 and an integer $n \geq 2$ such that $P^n|_U : U \to V$ is a polynomial-like mapping, and $P^{kn}(0) \in U$ for every $k \in \mathbb{N}$, then P_c is called renormalizable of order n.*

Applying the straightening theorem, we can find a polynomial $P_{c'}$ such that $P^n|_U$ is hybrid equivalent to $P_{c'}$ for a renormalizable polynomial. If there are only finitely many values of n for which P_c is renormalizable, then P_c is called *finitely renormalizable*, and if there are infinitely many such values of n, then P_c is called *infinitely renormalizable.*

The following theorem is due to Yoccoz.

THEOREM 2.5.4. *Let P_c be a quadratic polynomial with a connected Julia set and no indifferent periodic orbits. If P_c is at most finitely renormalizable, then $J(P_c)$ is locally connected.*

The proof of this theorem can be found in [Mil2] or [H].

M. Lyubich [Lyu3] and Shishikura independently studied the question of the Lebesgue measure of these finitely renormalizable quadratic polynomials. In particular, they proved the following theorem.

THEOREM 2.5.5. *Let P_c be a polynomial with a connected Julia set and no indifferent periodic orbits. If P_c is not infinitely renormalizable, then*

$$\lambda(J(P_c)) = 0 \tag{24}$$

where λ denotes Lebesgue measure.

For any $z \in \mathbb{C}$, define the *ω-limit set* of z as the set

$$\omega(z) = \{\zeta \in \mathbb{C} : \exists\ \{n_i\}_{i \in \mathbb{N}} \subset \mathbb{N} \textit{ with } n_i \to \infty \textit{ such that } P^{n_i}(z) \to \zeta\} \quad . \tag{25}$$

The ω-limit set plays an important role in the study of the Lebesgue measure Julia sets. This is demonstrated in the following two propositions by M. Lyubich.

PROPOSITION 2.5.6. *Let C be the set of critical points for the polynomial P. Then $\omega(z) \subset \omega(C)$ for almost every $z \in J(P)$.*

The proof of this last proposition can be found in his paper [Lyu5]. The next proposition is a bit stronger, and a bit more specialized. Its proof can be found in [Lyu3].

LEMMA 2.5.7. *Let $P_c : z \mapsto z^2 + c$ be a quadratic polynomial which is at most finitely renormalizable. If $J(P_c) > 0$, then for almost every $z \in J(P_c)$, $\omega(z) = \omega(c) \ni c$. So $\lambda(J(P_c)) = 0$ if the orbit of c is non-recurrent.*

We will need both of these results and the techniques used to prove them in chapter 3.

3. Non-recurrent and Reluctantly Recurrent Polynomials

3.1 Preliminaries

Let $P : \mathbb{C} \to \mathbb{C}$ be a complex polynomial of degree d, with only repelling periodic orbits. Throughout this chapter, let $C = \{c_1, \ldots, c_k\}$ denote the set of critical points of P. We will call a critical point $c \in C$ *simple* if $P''(c) \neq 0$, where P'' denotes the second derivative of P.

Recall that for any $z \in \mathbb{C}$, the *ω-limit set* of z is defined as

$$\omega(z) = \left\{ \zeta \in \mathbb{C} : \exists \, \{n_i\}_{i \in \mathbb{N}} \subset \mathbb{N} \ \textit{such that} \ P^{n_i}(z) \to \zeta \right\} \quad . \qquad (26)$$

For $z \in \mathbb{C}$, $\omega(z)$ is the set of accumulation points of the forward orbit of z. For this reason, if $\zeta \in \omega(z)$, we will say that the orbit of z *accumulates* at ζ, or that it is *recurrent* about ζ. Clearly if $z \notin K(P)$, then $\omega(z) = \{\infty\}$, or more generally, if z is in the basin of attraction of an attracting or parabolic periodic orbit $\{\zeta_1, \ldots, \zeta_m\}$, then $\omega(z) = \{\zeta_1, \ldots, \zeta_m\}$.

When there is more than one critical point, or the critical point is degenerate, many of the methods that Yoccoz used to show the local connectivity for quadratic polynomials break down. For example, the "Two-Kids" Lemma (see [Mil2]) was valuable in the quadratic case, but is not true in general when there is more than one critical point.

However, in certain cases, the tools that do still work are enough to get local connectivity. These cases are classified by the behavior of the orbits of the critical points. In section 2, we will define a partial ordering on the critical points, which will allow us to define what it means for a critical orbit to be maximal. In section 3, we will define what it means for a polynomial to be renormalizable at a critical point. Also, we will need the orbit of the critical points to not return to themselves "too often" (the reluctantly recurrent condition).

THEOREM 3.1.1. *Let P be a polynomial of degree $d \geq 3$ such that $J(P)$ is connected, every periodic orbit is repelling, P is not renormalizable at any critical point, and every fixed ray of P lands at a distinct fixed point. If either*

1) P has no maximal critical orbits, or

2) Every maximal critical orbit of P is reluctantly recurrent.

Then the Julia set of P is locally connected.

There is one point that should be noted about this theorem. To prove the theorem as stated, the partial ordering on the critical points is not necessary. If every maximal critical orbit is reluctantly recurrent, then every critical orbit is either non-recurrent or reluctantly recurrent. However, the above theorem could be strengthened mildly in the following manner. If P is as stated in the theorem, but P has a simple critical point c whose orbit is maximal, persistently recurrent, and $\omega(c) \bigcap C = \{c\}$, then the result still holds. However to prove this version of the theorem would require a general review of Yoccoz' work with quadratic polynomials. For simplicity's sake, we will only prove the stated version of the theorem.

An immediate consequence of the above theorem is the following corollary.

COROLLARY 3.1.2. *Let P be a polynomial of degree $d \geq 3$ such that $J(P)$ is connected, every periodic orbit is repelling, P is not renormalizable at any critical point, and every fixed ray lands at a distinct fixed point. If every* *critical orbit is either non-recurrent or reluctantly recurrent, then $J(P)$ is locally connected.*

3.2 A Partial Ordering on Critical Points

Let us define a partial ordering on C using $\omega(z)$ as follows: for $c_i, c_j \in C$, if $c_j \in \omega(c_i)$, then $c_i \prec c_j$. We then use this partial ordering to define a relation $\sim$ as well: $c_i \sim c_j$ if $c_i \prec c_j \prec c_i$.

PROPOSITION 3.2.1.

1) $c_i \prec c_j \quad \Rightarrow \quad \omega(c_j) \subset \omega(c_i)$.

2) $\prec$ *is associative.*

3) $\sim$ *is symmetric.*

Proof: The proofs of 2) and 3) follow as an immediate consequence of 1). To prove 1), let $z \in \omega(c_j)$, and let U be any open neighborhood of z. Let $P^l(c_j)$ be the first moment in the orbit of c_j which enters U. Then U pulls back along the orbit of c_j to an open neighborhood V of c_j. Because $c_j \in \omega(c_i)$, it follows that there exists a positive integer k such that $P^k(c_i) \in V$. Therefore $P^{k+l}(c_i) \in U$, and it follows that $z \in \omega(c_i)$, and thus $\omega(c_j) \subset \omega(c_i)$. $\square$

Let us point out a few additional facts about $\prec$ and $\sim$. The first fact which is obvious is that c_i and c_j can be unrelated in terms of $\prec$, thus making $\prec$ only a partial ordering. The second not-so-obvious fact is that for $c_i, c_j, c_k \in C$, it is possible that $c_i \prec c_j$ and $c_i \prec c_k$, yet c_j and c_k are unrelated. An immediate consequence of this fact is the possibility that $c_i \sim c_i$, and $c_i \prec c_j$, but $c_i \not\sim c_j$.

We should mention that $\sim$ is not an equivalence relation because of the possibility that $c_i \not\sim c_i$. An example of this occurrence was mentioned previously when c_i is in the basin of attraction of an attracting or parabolic periodic orbit. However we still shall refer to the *relative class* of c_i as the set

$$[c_i] = \{c_j \in C : c_j \sim c_i\} \tag{27}$$

and accept that the relative class of c_i may be empty. If $[c]$ is not empty, then the orbit of c is called *recurrent*, and if $[c]$ is empty, then the orbit of c is called *non-recurrent.*

Finally, let us call $c_i \in C$ *maximal* if the relative class of c_i is not empty, and for every $c_j \in C$ with $c_i \prec c_j$, then $c_i \sim c_j$. In other words, if c_i is maximal and the orbit of c_i accumulates at c_j, then the orbit of c_j must accumulate at c_i. The orbits of the maximal critical points determine much of the geometry of the Julia set of P.

3.3 Yoccoz Puzzle Pieces for Maps of Degree $d \geq 3$

Because $\mathbb{C}$ is algebraically complete, $P : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 3$ has exactly d fixed points counted with multiplicity. However there are only $d - 1$ fixed rays (the map $z \mapsto z^d$ has only $d - 1$ fixed points on the unit circle). To make puzzle pieces, we'll need the following proposition.

PROPOSITION 3.3.1. *The polynomial $P : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 2$ has d distinct fixed points if and only if every fixed point w satisfies $P'(w) \neq 1$.*

Proof: w is a fixed point of P if and only if

$$P(z) - z = (z - w)^k Q(z) \tag{28}$$

for some $1 \leq k \leq d$ and some polynomial $Q(z)$ with $Q(w) \neq 0$, and $deg(Q(z))+k = d$, where $deg(Q(z))$ denotes the degree of the polynomial $Q(z)$. Now observe that $P'(w) = 1$ if and only if $k \geq 2$. So $P'(w) = 1$ if and only if $deg(Q(z)) \leq d-2$, and $deg(Q(z)) \leq d-2$ if and only if there are at most $d-1$ distinct fixed points. $\square$

So by assuming that there are no fixed points w with $P'(w) = 1$, we know that there are d distinct fixed points. However we have already observed that there are only $d-1$ fixed rays for a polynomial of degree d. Thus there exists a fixed point which is not the landing point of a fixed ray. Let us call this fixed point the α-fixed point for the polynomial P, just as in the quadratic case. One of our assumptions on the class of polynomials that we are dealing with is that every fixed ray lands at a distinct fixed point. Therefore the α-fixed point is uniquely defined.

Recall from Lemma 2.5.2 that at any repelling fixed point, there are a finite but positive number of rays landing at the fixed point which are permuted cyclically by the action of the polynomial P. It follows from this lemma and the preceding paragraph that there are $q \geq 2$ rays landing at α which are permuted cyclically by P. We now have everything we need to form Yoccoz puzzle pieces. The procedure for creating Yoccoz puzzle pieces in this situation is analogous to the procedure for quadratic polynomials. Let Γ_1 be the equipotential of value 1, and $R_1(\alpha)$ the union of the rays landing at α and truncated at Γ_1. We call this compact set the *Yoccoz configuration* of depth 0, and denote it by $\mathcal{Y}^0$. The bounded components of the complement are called the *depth 0 Yoccoz* *puzzle pieces*, and are denoted by Y_i^0, with the special subscript 0 reserved for the depth 0 Yoccoz puzzle piece which contains the critical point.

For each $z \in J(P)$ which is not the α-fixed point of P, there exists a unique depth 0 puzzle piece $Y^0(z) \ni z$. We define the depth n puzzle piece $Y^n(z)$ to be the unique connected open set containing z such that $P^n(Y^n(z)) = Y_i^0$ for some depth 0 puzzle piece Y_i^0. It is clear from the definition of puzzle pieces that if $m < n$, then $Y^n(z) \subset Y^m(z)$.

LEMMA 3.3.2. *Suppose P is a polynomial with $J(P)$ connected and no fixed points with multiplier 1. Then $\overline{Y_i^n} \bigcap J(P)$ is connected for every puzzle piece Y_i^n.*

Proof: Suppose $\overline{Y_i^0} \bigcap J(P) = V_1 \bigcup V_2$ with $V_1 \cap V_2 = \emptyset$. Assume $\alpha \in V_1$. The rays $R_1(\alpha)$ landing at α intersect $J(P)$ only at α, and Γ_1 does not intersect $J(P)$ at all. So $\overline{Y_i^0} \cap \mathcal{Y}^0 = \{\alpha\}$. Therefore

$$J(P) = \left\{ \left\{ \bigcup_{j \neq i} \overline{Y_j^0} \cap J(P) \right\} \cup V_1 \right\} \bigcup V_2 \tag{29}$$

and

$$\left\{ \left\{ \bigcup_{j \neq i} \overline{Y_j^0} \cap J(P) \right\} \cup V_1 \right\} \bigcap V_2 = \emptyset \quad . \tag{30}$$

By the connectedness of $J(P)$, either $\{\{\bigcup_{j \neq i} \overline{Y_j^0} \cap J(P)\} \bigcup V_1\}$ or V_2 is empty. But $\alpha \in \{\{\bigcup_{j \neq i} \overline{Y_j^0} \cap J(P)\} \bigcup V_1\}$, and therefore V_2 must be empty, which implies that $\overline{Y_i^0} \cap J(P)$ is connected for every i.

Suppose $V_i^n \cap J(P)$ is a connected set, and let V_j^{n+1} map by P onto V_i^n. Suppose $V_j^{n+1} \cap J(P) = B_1, \ldots, B_k$ are the various connected components. Because $B_1, \ldots, B_k$ are compactly contained in V_j^{n+1}, there exist open sets $U_1, \ldots, U_k$ with $U_l \subset V_j^{n+1}$, and $B_l \subset U_l$ for each l.

Suppose U_1 can be extended to an open set $\tilde{U}_1$ such that $\tilde{U}_1 \cap B_l = \emptyset$ for $l = 2, \ldots, k$ and $P(\tilde{U}_1) = V_i^n$. Then it follows the $\tilde{U}_1 = V_j^{n+1}$ and $B_2, \ldots, B_k$ are empty. Then we are done.

So assume U_1 can not be extended to such an open set. Then as U_1 is being extended, it must intersect some B_m, $m = 2, \ldots, k$. Let $x \in B_m$ be one of the points it intersects. Then we can find $x' \in B_1$ such that $P(x) = P(x')$ and x and x' can be connected by a path γ which is contained in the extension of U_1. That is, $\gamma' = P(\gamma) \subset V_i^n$. Furthermore, γ' is a loop, with endpoint $P(x)$.

Let $\tilde{\gamma}'$ denote the path γ' along with all bounded components of its complement. Then $\tilde{\gamma}'$ pulls back to a compact connected set $\tilde{\gamma}$ which maps onto $\tilde{\gamma}'$ as a degree d covering map for some degree $d \geq 2$. But then $\tilde{\gamma}$ must contain a critical point, and if we choose γ carefully to begin with, this critical point will not be able to be in the Julia set of P. See Figure 3.1. But then by Theorem 2.4.2, the Julia set of P will not be connected, contradicting one of our initial assumptions. Therefore $V_j^{n+1} \cap J(P)$ must have been connected to begin with, and we are done. $\square$

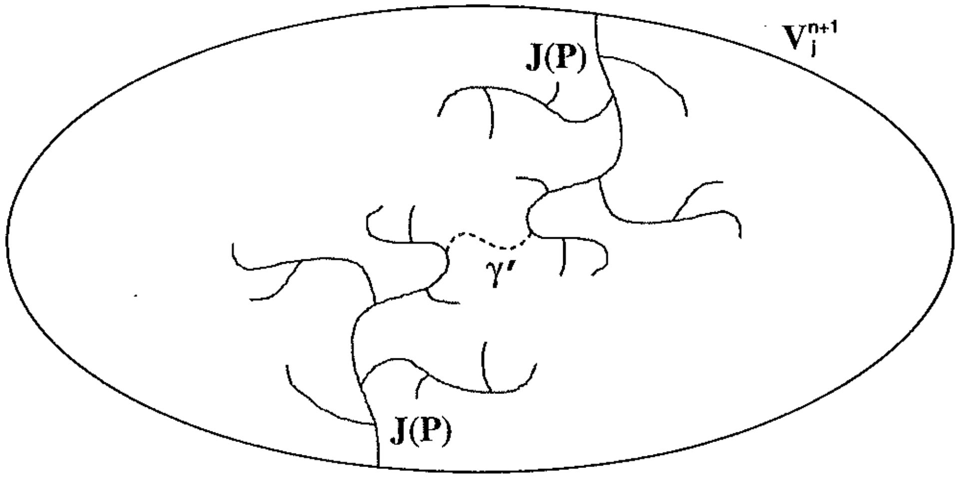

Figure 3.1. The path γ' connecting the two connected components of the Julia set inside the puzzle piece V_j^{n+1}.

3.4 Renormalization

Let $c \in C$ for the polynomial P.

DEFINITION 3.4.1. *The polynomial P is called renormalizable at the critical point c if there exists an open neighborhood U of c and an integer $n \geq 2$ such that $P^n|_U : U \to P^n(U)$ is a polynomial-like mapping, and $P^{kn}(c) \in U$ for every $k \in \mathbb{N}$.*

If there is only one critical point in U, then by applying the Straightening Theorem, we see that P is renormalizable at c to a polynomial with a connected Julia set. However, if there are other critical points in U, and the orbit of any of these other critical points escapes U, then P is renormalizable to a polynomial with a disconnected Julia set.

Just as in the quadratic case, we can talk about P being *finitely renormalizable* or *infinitely renormalizable.*

LEMMA 3.4.2. *If c is a recurrent but non-renormalizable critical point of P, then for every $N \in \mathbb{N}$, there exists an integer $n > N$ such that*

$$mod(Y^n(c) \setminus \overline{Y^{n+1}(c)}) > 0 \quad . \tag{31}$$

Such an annulus is called *non-degenerate.*

Proof: Let $p \geq 2$ be the number of external rays landing α. We claim that in order for P to not be renormalizable at c, there must exist a positive integer $t > 0$ such that $P^{tp}(c) \in \bigcup Y_i^1$ for one of the depth one Yoccoz puzzle pieces which does not touch α. Let us prove the claim by contradiction: that it, assume no such $t > 0$ exists. Let $\tilde{Y}$ be a slight "thickening" of $Y^1(c)$; that is, let $\tilde{Y}$ be the union of $\overline{Y^1(c)}$ and a small open neighborhood of $\overline{Y^1(c)}$. If this neighborhood is chosen small enough, then $P^p(\tilde{Y})$ contains the closure of $\tilde{Y}$. Further, $P^{kp}(c) \in Y^1(c) \subset \tilde{Y} \forall k \in \mathbb{N}$. And $P_c^p(\tilde{Y})$ will be a finite degree branched covering map onto its image. Therefore P_c is renormalizable, which is a contradiction, and thus $P_c^{tp}(0) \in \bigcup_i Y_i^1$ for some $t \in \mathbb{N}$ and some depth 1 puzzle piece not touching α.

But the depth one puzzle pieces which do not touch α are compactly contained in some Y_j^0, and therefore the annulus $Y_j^0 \setminus Y_i^1$ has positive modulus. By Proposition 2.2.2 and the fact that there exists a puzzle piece $Y^n(c)$ which maps as a finite degree branched covering map onto Y_j^0, the annulus $Y_j^0 \setminus Y_i^1$ pulls back to an annulus about c which has positive modulus.

Finally, because the critical point c is recurrent about itself, we can get annuli with positive modulus of arbitrary depth about the critical point c. $\square$

3.5 Reluctant vs. Persistent Recurrence

Let $c \in C$ for the polynomial P, and suppose P is non-renormalizable at c. Recall that the orbit of c is *non-recurrent* if $[c]$ is empty. Equivalently, the orbit of c is non-recurrent if there exists an open neighborhood U of c such that the forward orbit of c never enters U.

Now suppose c is maximal.

DEFINITION 3.5.1. *If there exists positive integers $k, N \in \mathbb{N}$, and an increasing sequence of positive integers $\{n_i\}_{i \in \mathbb{N}} \subset \mathbb{N}$ such that $P^{n_i - k}(Y^{n_i}(c))$ has degree $\leq N$, then the orbit of c is called reluctantly recurrent.*

LEMMA 3.5.2. *If the orbit of c is reluctantly recurrent, then there exists $\epsilon > 0$ such that for every $n \in \mathbb{N}$, there exists a pullback $\underline{z} = \{z_0, z_{-1}, \ldots, z_{-n}\}$ in $\omega(c)$ satisfying that the pullback of $B(z_0, \epsilon)$ along $\underline{z}$ is univalent.*

This property is sometimes referred to as the "long univalent pull-backs" property of $\omega(c)$.

Proof: Because the critical point is reluctantly recurrent, there exist puzzle pieces of some level k which have arbitrarily long pull-back of degree less than N. Note that the bound N is uniform, but the pull-backs are arbitrarily long. Pull-back the puzzle pieces along the critical orbit until it hits its first critical point. If these pull-backs can be arbitrarily long, then we're done. If not, then we can begin again from this Yoccoz depth. We only have to repeat this procedure at most N times to find a Yoccoz depth which has pieces with arbitrarily long univalent pull-backs along $\omega(c)$.

If $\alpha \in \omega(c)$, then we can certainly find a ball of radius ϵ about α which shrinks uniformly to α. However if $\alpha \notin \omega(c)$, then the forward orbit of c stays a definite distance away from the boundaries of the puzzle pieces on any given depth. So find an ϵ such that for some point $z \in \omega(c)$, the ball of radius ϵ is contained in the Yoccoz puzzle piece of the depth described in the preceding paragraph. Clearly this ϵ satisfies the conditions of the lemma. $\square$

DEFINITION 3.5.3. *Suppose the orbit of the critical point c is recurrent, but not renormalizable nor reluctantly recurrent. Then the orbit of c is called persistently recurrent.*

LEMMA 3.5.4. *Let $\{z_0 \mapsto z_1 \mapsto \cdots\}$ be an orbit of of P. If z_k is the first point in the orbit of z_0 which enters $Y^n(z_k)$, then*

$$Y^{n+i}(z_{k-i}) \bigcap Y^{n+j}(z_{k-j}) = \emptyset \tag{32}$$

for $0 \leq i < j \leq k$.

Proof: Suppose $Y^{n+i}(z_{k-i}) \cap Y^{n+j}(z_{k-j}) \neq \emptyset$ for some $0 \leq i < j \leq k$. Then

$$Y^{n+j}(z_{k-j}) \subset Y^{n+i}(z_{k-i}) \quad . \tag{33}$$

Therefore $P^i(Y^{n+j}(z_{k-j})) \subset Y^n(z_k)$, and therefore $P^{n-(j-i)}(z) \in Y^n(z_k)$ for $n-(j-i) < n$. This is a contradiction with the assumption that z_k is the first moment in the orbit of z which enters $Y^n(z_k)$. $\square$

LEMMA 3.5.5. *Suppose $c_1 \prec c_2$, with c_1 recurrent and c_2 maximal. Then if c_2 is reluctant, so is c_1.*

Proof: Let k, N and $\{n_i\}$ be as in the definition of reluctantly recurrent. Then $P^{n_i-k}(Y^{n_i}(c))$ has degree less than N. Let $P^m(c_l)$ be the first moment when the orbit of C_l enters $Y^{n_i}(c)$. Pull $Y^{n_i}(c)$ back along the orbit of C_l to $Y^{n_i+m}(c_l)$.

Claim: this pullback of $Y^{n_i}(c)$ hits every other critical point of P at most once. Suppose not. Suppose there exists a critical point c_q such that the pull-back of $Y^{n_i}(c)$ hits c_q twice before hitting c_l. Let $0 < r < s$ be such that $Y^{n_i+r}(c_q)$ and $Y^{n_i+s}(c_q)$ are the two corresponding puzzle pieces. Then $Y^{n_i+s}(c_q) \subset Y^{n_i+r}(c_q)$. Therefore $P^r(Y^{n_i+s}(c_q)) \subset Y^{n_i}(c)$. But then $P^{m-(s-r)}(c_l) \in Y^{n_i}(c)$ for $m-(s-r) < m$, contradicting our assumption that $P^m(c_l)$ was the first moment that the orbit of c_l entered $Y^{n_i}(c)$, and the claim is proved.

To conclude the proof of the lemma, note that there are only finitely many critical points for a polynomial. Therefore there exists $N_2 \in \mathbb{N}$ and an increasing sequence of integers $\{n_t\}_{t\in\mathbb{N}}$ such that $P^{n_t-k}(Y^{n_t}(c_l)$ has degree less than N_2, and therefore the orbit of c_l is reluctantly recurrent. So if one critical point $c \in [c]$ is reluctant, they all are. $\square$

COROLLARY 3.5.6. *The orbits of all critical points in a relative class are simultaneously reluctant or persistent.*

THEOREM 3.5.7. *If the orbit of c is reluctantly recurrent, and $c \in \omega(z)$, then $Y^n(z) \to \{z\}$ as $n \to \infty$.*

Let $[c] = \{c, c_1, \ldots, c_p\}$, and k, N, and $\{n_i\}$ be as in the definition of reluctantly recurrent. Because none of the $c_j \in [c]$ are non-renormalizable, by Lemma 3.4.2, there exists a minimal $l_j > 0$ such that $mod(Y^{k+l_j-1}(c_j) \setminus \overline{Y^{k+l_j}(c_j)}) > 0$. Let $L = max\{l_j\}$, and

$$M = \min_j mod(Y^{k+l_j-1}(c_j) \setminus \overline{Y^{k+l_j}(c_j)}) \quad . \tag{34}$$

By the definition of reluctant recurrence, $P^{n_i-k}(Y^{n_i}(c))$ is a map of degree at most N. Push $P^{n_i-k}(c)$ forward until the first time its orbit enters $Y^{k+l_j}(c_j)$ for some $c_j \in [c]$. Claim: $A^{k+l_j}(c_j)$ pulls back along the orbit of c to a critical annulus $\widetilde{A}_i(c)$ satisfying

$$mod(\widetilde{A}_i(c)) \geq \frac{M}{N\, d^L} \tag{35}$$

where d is the degree of P.

To prove the claim, note that if A_1 is an annulus such that $P(A_1) = A^{k+l_j}(c_j)$, then

$$mod(A_1) \geq \frac{1}{d} mod(A^{k+l_j}(c_j)) \geq \frac{M}{d} \tag{36}$$

because of Proposition 2.2.2. Similarly, if $P^L(A_L) = A_1$, then

$$mod(A_L) \geq \frac{M}{d^L} \quad . \tag{37}$$

Now we have assumed that $Y^{k+l_j}(c_j)$ is the first time the orbit of $P^{n_i-k}(c)$ has entered one of these puzzle pieces. Therefore if A_L denotes the pull-back of $A^{k+l_j}(c_j)$ along this orbit, then A_L pulls back to an annulus $\widetilde{A}_i$ of c by a map of degree at most N. Therefore

$$mod(\widetilde{A}_i(c)) \geq \frac{1}{N} mod(A_L) \geq \frac{M}{N\, d^L} \tag{38}$$

and the claim is proved.

Because there are infinitely many such annuli about c, each with modulus greater than or equal to $\frac{M}{N\, d^L}$, we can apply the Grötzch inequality to get that $Y^n(c) \to \{c\}$ as $n \to \infty$. □

Proof: (of Theorem 3.1.1)

If the orbit of z accumulates at a critical point, then by Theorem 3.5.7, $Y^n(z) \to \{z\}$ as $n \to \infty$. But if the orbit of z does not accumulate at some critical point, then we can apply Lemma 2.5.7 to get that $Y^n(z) \to \{z\}$ as $n \to \infty$. Therefore the puzzle pieces shrink to every point in the Julia set. By Lemma 3.3.2, each puzzle piece intersects the Julia set in a connected set. Thus the Julia set is locally connected. $\square$

4. Local Connectivity and Lebesgue Measure of Julia Sets in the Family $z \mapsto z^3 + c$.

4.1 Introduction

Let $P(z)$ be a cubic polynomial of the form $P : z \mapsto z^3 + c$. Thus there is only one critical point as in the quadratic case. However this critical point is degenerate. In this chapter, we want to examine how far the methods of Yoccoz and Lyubich for non-renormalizable quadratic polynomials can be pushed in studying the local connectivity and Lebesgue measure of the Julia set of such cubic polynomials.

In the first two sections of this chapter, we shall not introduce anything new, but rather review what has already been done for quadratic polynomials in this new setting. Let us assume that all periodic cycles of P are repelling.

Recall that a polynomial with one critical point c_0 is called *renormalizable* if there exists a neighborhood U of c_0 and an integer $n \geq 2$ such that $P^n|_U : U \to P^n(U)$ is a polynomial-like mapping with the forward orbit of c_0 under P^n never escaping U. Let $K(P^n|_U)$ denote the filled-in Julia set of the corresponding polynomial-like mapping.

The class of cubic polynomials P we want to study are those with the following properties:

1) $J(P)$ is connected,

2) P has only one critical point,

3) P has only repelling periodic orbits,

4) P is non-renormalizable.

A polynomial P with only one critical point c is called *recurrent* if $c \in \omega(c)$. P is called *reluctantly recurrent* if there exists an open neighborhood U of c such that for every $n \in \mathbb{N}$, there exists a point c_k in the forward orbit of the critical point satisfying

1) $c_k \in U$, and

2) $P^i(c_k) \notin U$ for every $1 \leq i \leq n$.

If the orbit of the critical point of P is recurrent but not reluctantly recurrent, than P is called *persistently recurrent.*

From the previous chapter, if the orbit of the critical point for such a polynomial is either non-recurrent or reluctantly recurrent, then the Julia set of P is locally connected (provided it is connected). We want to deal here with the case when the orbit of the critical point is persistently recurrent. Recall from the previous chapter that persistent recurrence of the orbit of the critical point 0 is actually a condition on $\omega(0)$. that is, if $\omega(0)$ satisfies the "no long univalent pull-backs" condition, then the orbit of 0 is persistent.

A *generalized cubic-like mapping* is a collection of disjoint open disks V_i contained in an open disk V^0, along with a collection of analytic functions $g_i : V_i \to V^0$ such that g_i is univalent for $i \neq 0$ and $g_0 : V_0 \to V^0$ is a degree three holomorphic covering map with a single branch point, and the orbit of this branch point never escapes the collection of V_i. We will denote such a map as $g : \cup V_i \to V^0$. Furthermore, we'll assume in this chapter that the orbit of the critical point enters every V_i^n.

In this chapter, we shall prove the following theorem.

THEOREM 4.1.1. *Let $g : \cup V_i^1 \to V^0$ be a generalized polynomial-like mapping with one cubic critical point. If there exists some renormalization level n of g such that $g_n : \cup V_i^n \to V_0^{n-1}$ has admissible families on every renormalization level following n, then $K(g)$ is a totally disconnected measure zero Cantor set.*

Admissible families are families of puzzle pieces which have certain special properties. We shall define admissible families more precisely a little later in this chapter.

The next theorem is actually a corollary to the previous theorem. However it is this result that we really want, so we shall state it as its own theorem. To every non-renormalizable polynomial $P_c : z \mapsto z^3 + c$ which has a persistently recurrent critical orbit, we can associate in a canonical way a generalized cubic-like mapping which is infinitely renormalizable.

THEOREM 4.1.2. *Let $P_c : z \mapsto z^3 + c$ be a non-renormalizable polynomial with a connected Julia set and a persistently recurrent critical orbit. If the generalized polynomial-like mapping associated to P_c satisfies the conditions of Theorem 4.1.1, then $J(P_c)$ is locally connected and has zero Lebesgue measure.*

4.2 Generalized Polynomial-like Mappings

To every non-renormalizable polynomial P with a recurrent critical point, one can associate a mapping which reflects the local combinatorics of the original polynomial about this recurrent critical point. Further, this new mapping is infinitely renormalizable in a sense to be made precise shortly. This section is dedicated to the construction of this new mapping, which we call *the generalized polynomial-like mapping associated to the polynomial* P_c, and the development of some of its properties. The results in this section are essentially those of M. Lyubich from [Lyu3].

Let $V^0 \subset \mathbb{C}$ be an open topological disk, and let $\cup V_i$ be a countable collection of disjoint open topological disks each of which is compactly contained in V^0. That is

$$\cup \overline{V_i} \subset V^0 \; and \; V_i \cap V_j = \emptyset \; for \; i \neq j \quad . \tag{39}$$

To each V_i, let $g_i : V_i \to V^0$ be a branched finite degree holomorphic covering map. Then $g : \cup V_i \to V^0$ defined as

$$g|_{V_i} = g_i \tag{40}$$

is called a *generalized polynomial-like mapping* if it has only finitely many branch points. If g has only one branch point $z_0 \in V_0$, and $g : V_0 \to V^0$ is a degree three covering map, then we shall call g a *generalized cubic-like mapping.* We shall assume that the orbit of this lone critical point never escapes $\cup V_i$.

Let $P_c : z \mapsto z^3 + c$ be a non-renormalizable polynomial with only repelling periodic orbits, a connected Julia set, and a recurrent critical orbit. Let $Y_0^1, \dots, Y_{q-1}^1$ denote the depth 1 puzzle pieces whose closures touch the

α-fixed point of P_c, and let $Z_1^1, \ldots, Z_{2q-2}^1$ denote the depth 1 puzzle pieces which do not touch α. See diagram 1.

Claim: the orbit of the critical point must eventually enter $\cup Z_i^1$. Suppose not. Then $P_c^{qk}(0) \in Y_0^1$ for every $k \in \mathbb{N}$. Let $\tilde{Y}$ be an open set with $\overline{Y_0^1} \subset \tilde{Y}$ and $P_c(0), \ldots, P_c^{q-1}(0) \notin \tilde{Y}$. Then $P_c^q : \tilde{Y} \to P_c^q(\tilde{Y})$ is a polynomial-like mapping in the sense of Douady and Hubbard, and therefore P_c is renormalizable, contradicting our assumption that P_c is non-renormalizable. So the claim is proven.

Suppose the critical orbit first enters Z_i^1. The annulus $Y_0^0 \setminus Z_i^1$ is non-degenerate; that is, $mod(Y_0^0 \setminus Z_i^1) > 0$. Pulling this annulus back along the critical orbit, we obtain a non-degenerate critical annulus $A_0^k = Y_0^{k-1} \setminus Y_0^k$. Let $V^0 = Y_0^k$.

Consider all returns of the orbit of the critical point to V_0, and pull V_0 back along the orbits to obtain puzzle pieces V_i^1. Claim: each V_i^1 is compactly contained in V^0. Otherwise $\partial V_i^1 \cap \partial V^0 \neq \emptyset$ for some i. Let $l > k$ be the Yoccoz depth of V_i^1, and $j = l - k$. Then

$$\partial V^0 \cap \partial(P_c^j V^0) \supset P_c^j(\partial V_i^1 \cap \partial V^0) \neq \emptyset \tag{41}$$

which contradicts the fact that $\overline{Y_0^k} \subset Y_0^{k-1}$. Clearly the critical orbit never leaves $\cap V_i^1$ by their very definition. This completes the construction of the generalized polynomial-like mapping $g_1 : \cup V_i^1 \to V^0$ associated to P_c.

To renormalize g_1, note that the critical orbit of P is assumed to be recurrent. Therefore the critical orbit returns to V_0^1 infinitely many times. Let c_k be any point in the forward orbit of the critical point c such that $c_k \in V_0^1$, and let c_{k+l} be the first time the orbit of c_k returns to V_0^1. Then pull-back

V_0^1 along the orbit of c_k to obtain the level 2 puzzle piece $V^2(c_k)$. The unique level two puzzle piece containing the critical point is denoted V_0^2. Thus we have defined a new generalized polynomial-like mapping $g_2 : \cup V_i^2 \to V_0^1$ which we call the *renormalization* of g_1.

We can rèpeat this construction as often as well like to get generalized polynomial-like mappings of all levels.

4.3 Combinatorics of the Return Map and the Asymmetric Modulus

Let $g_n : \cup V_i^n \to V_0^{n-1}$ be some renormalization level for a generalized cubic-like mapping. V_0^n shall denote the unique puzzle piece of level n which contains a branch point. Let Γ^n denote the free semi-group generated by the formal symbols V_i^n representing the off-critical puzzle pieces of level n. That is,

$$\Gamma^n = \{\gamma = \{\gamma_1, \ldots, \gamma_k\} : \gamma_i = V_j^n \textit{ for some } j \neq 0\} \quad . \tag{42}$$

Let $\mathbb{I}$ denote the identity element of this semi-group, and let $|\gamma|$ denote the length of the word γ. Define $|\mathbb{I}| = 0$.

We can associate to every word $\gamma \in \Gamma^n$ an open set in V_0^{n-1} as follows. If $|\gamma| = 0$, then $\gamma = 1$ and we define its associated open set $U(\mathbb{I}) = V_0^n$. If $|\gamma| = 1$, then $\gamma = \{V_i^n\}$ for some off-critical puzzle piece V_i^n. Associate to γ the open set $U(\gamma) = V_i^n$ where this V_i^n is not the formal symbol but the open subset in V_0^{n-1}. Now suppose to every γ with $|\gamma| \leq k-1$ we have associated some open set $U(\gamma)$ in V_0^{n-1}. Let $\gamma = \{\gamma_1, \ldots, \gamma_k\}$. Then $\gamma_1 = V_j^n$ for some $j \neq 0$. So associate to γ the unique open set $U(\gamma) \subset V_j^n$ such that $g_n(U(\gamma)) = U(\{\gamma_2, \ldots, \gamma_k\})$. It follows from this definition that for every $\gamma \in \Gamma^n$, $U(\gamma)$ maps conformally onto V_0^{n-1} under $g_n^{|\gamma|}$.

By the definition of the open set $U(\gamma)$, if $\gamma = \{\gamma_1, \ldots, \gamma_k\}$, then $g(\gamma) = \{\gamma_2, \ldots, \gamma_k\}$. Therefore g acts as a shift map on the words $\gamma \in \Gamma^n$ considered in conjunction with its action the open set $U(\gamma)$.

Let $\{\gamma^j\}_{j \in J} \subset \Gamma^n$ be any collection of words in Γ^n such that for $i, j \in J$,

$$U(\gamma^i) \cap U(\gamma^j) \neq \emptyset \Leftrightarrow i = j \quad . \tag{43}$$

For each $j \in J$, define the annulus $R_J(U(\gamma^j))$ as follows: $R_J(U(\gamma^j))$ is the topological annulus of maximal modulus such that

i) $R_J(U(\gamma^j)) \subset V_0^{n-1} \setminus \overline{V_i^n}$,

ii) V_j^n is contained in the unbounded component of the complement of $R_J(U(\gamma^j))$ for $j \neq i$, and

iii) V_i^n is contained in the bounded component of the the complement of $R_J(U(\gamma^j))$.

See Figure 4.1. We know such an annulus exists by Montel's Theorem. Note that the definition of $R_J(U(\gamma^j))$ depends on the family J, and thus we include J as a subscript.

Let $\{\gamma^j\}_{j \in J} \subset \Gamma^{n-1}$ be a family of words in Γ^{n-1}, with $\gamma^j = \{\gamma_0^j, \ldots, \gamma_{k(j)}^j\}$.

DEFINITION 4.3.1. *We will say that the family J is separated if $U(\gamma_0^j) \neq U(\gamma_0^i)$ for $i \neq j$.*

Now let $\{\gamma^j\}_{j \in J} \subset \Gamma^n$ be any separated family of puzzle pieces in Γ^n, with $\gamma^j = \{\gamma_0^j, \ldots, \gamma_{k(j)}^j\}$. If $\gamma^j = \mathbb{I}$, then $U_0(\gamma^j) = U(\gamma^j) = V_0^n$. If $\gamma^j \neq \mathbb{I}$, then $g_{n-1}^{k(j)+1}$ maps γ^j conformally onto V_0^{n-1}. Therefore there exists an open set in $U(\gamma^j)$ which maps conofrmally onto V_0^n under $g_{n-1}^{k(j)+1}$. Let $U_0(\gamma^j)$ denote this set.

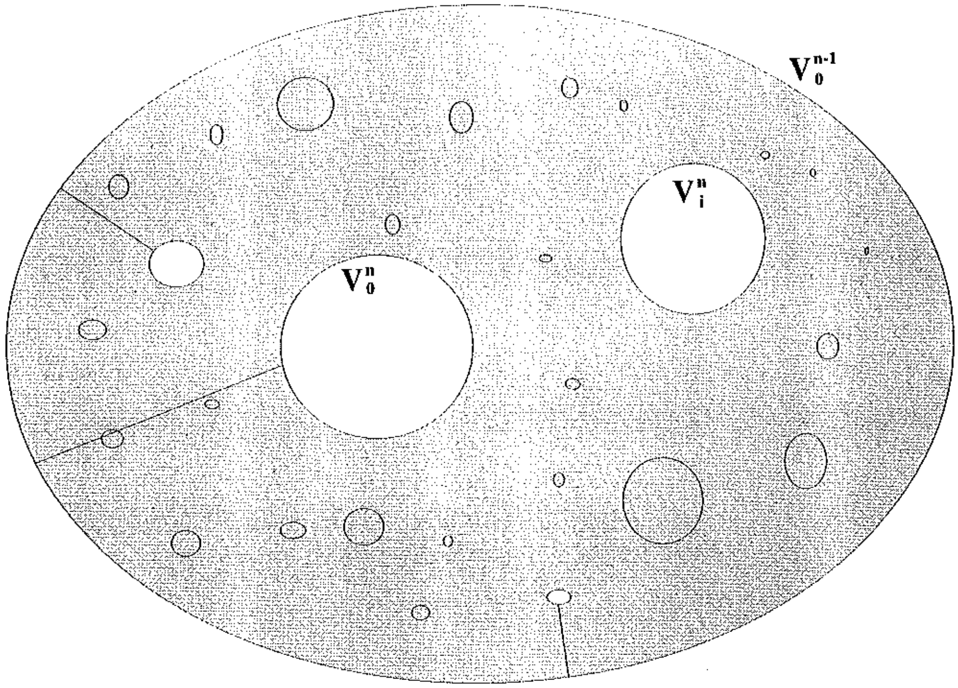

Figure 4.1. R_i^n

Let $\{V_i^n\}_{i\in I}$ be a family of puzzle pieces of level n, one of which must be the critical puzzle piece V_0^n.

DEFINITION 4.3.2. *The asymmetric modulus of the family I is defined as*

$$\sigma_n(I) = mod(R_I(V_0^n)) + \frac{1}{3}\sum_{i\neq 0} mod(R_I(V_i^n)) \quad . \tag{44}$$

We will see later that the asymmetric modulus defines a way of measuring space between two renormalization levels.

Again, let $\{V_i^n\}_{i\in I}$ be a family of puzzle pieces and $\{\gamma^j\}_{j\in J} \subset \Gamma^n$ be any separated family of puzzle pieces in Γ^n as above.

DEFINITION 4.3.3. *The family J is said to be subjugate to the family I if for every $i \in I$, either*

1) there exists $j \in J$ and an integer $1 \le m \le k(j)$ such that $U(\gamma^j_m) = V^n_i$,

or

2) $V^n_i = \gamma^j_0$ for some $j \in J$, and $\cup_{j\in J}(U(\gamma^j_0)) \subset \cup_{i\in I} V^n_i$.

LEMMA 4.3.4. *If the separated family $\{\gamma^j\}_{j\in J}$ has at least 3 elements and is subjugate to the family $\{V^n_i\}_{i\in I}$, then*

$$\frac{1}{3}\sum_j mod(R_J(U_0(\gamma^j))) \ge \sigma_n(I) \quad . \tag{45}$$

Proof: By the definition of the asymmetric modulus,

$$\sigma_n(I) = mod(R_I(V^n_0)) + \frac{1}{3}\sum_{i\ne 0} mod(R(V^n_i)) \quad . \tag{46}$$

So first suppose that for every $i \in I$ with $i \ne 0$, there exists $j \in J$ and an integer $1 \le m \le k(j)$ such that $U(\gamma^j_m) = V^n_i$. Then

$$mod(U(\gamma^j_{m-1} \setminus \overline{\gamma^j_m})) \ge mod(V^{n-1} \setminus \overline{V^n_0}) \ge mod(R_I(V^n_i)) \quad . \tag{47}$$

Also, if $\gamma_j \ne 1$, then $mod(U(\gamma^j) \setminus \overline{U_0(\gamma^j)}) \ge mod(V^{n-1}_0 \setminus \overline{V^n_0})$.

Now suppose there exists $i \in I$ such that the only $j \in J$ and $0 \le m \le k(j)$ with $V^n_i = U(\gamma^j_m)$ is $U(\gamma^j_0)$ for some $j \in J$. Then by the definition of subjugate, $\cup_{j\in J}(U(\gamma^j_0)) \subset \cup_{i\in I} V^n_i$. Therefore

$$mod(R_J(U(\gamma^j_0)) \ge mod(R_I(V^n_i)) \quad . \tag{48}$$

This completes the proof of the lemma. $\square$

LEMMA 4.3.5. *Suppose the separated family J has only two elements, which we shall call γ^1 and γ^2. If J is subjugate to the family I, then*

$$\frac{2}{3}mod(R_J(U_0(\gamma^1))) + \frac{1}{3}mod(R_J(U_0(\gamma^2))) \ge \sigma_n(I) \quad . \tag{49}$$

Proof: Let $\gamma^1 = \{\gamma_0^1, \ldots, \gamma_k^1\}$ and $\gamma^2 = \{\gamma_0^2, \ldots, \gamma_l^2\}$ For V_i^n with $i \neq 0$, there exists $0 \leq m \leq k$ such that $V_i^n = U(\gamma_m)$. If $m = 0$, then

$$mod(R_J(U(\gamma_m))) \geq mod(R_I(V_i^n)) \quad , \tag{50}$$

and if $m \neq 0$, then

$$mod(U(\gamma_{m-1}^j \setminus \overline{\gamma_m^j})) \geq mod(V^{n-1} \setminus \overline{V_0^n}) \geq mod(R_I(V_i^n)) \quad . \tag{51}$$

And again, for $\gamma \neq \mathbb{1}$, $mod(U(\gamma) \setminus \overline{U_0(\gamma)}) \geq mod(V_0^{n-1} \setminus \overline{V_0^n})$. The result of the proof now follows. $\square$

4.4 Admissible Families

Let us now define what it means for a family of puzzle pieces of level n to be admissible. The definition will be given inductively on the level n. We start with a generalized cubic-like mapping $g_1 : \cup V_i^1 \to V^0$ which has at least three level one puzzle pieces.

Let $g_2 : \cup V_j^2 \to V_0^1$ be the renormalization of g_1. A family I^2 of puzzle pieces of level two is called *admissible* if

1) I^2 has exactly three puzzle pieces, the critical puzzle piece and two off-critical puzzle pieces, and

2) there exists an integer $k \geq 0$ such that $g_1^l(I^2)$ is contained in the same level one puzzle piece for all $0 \leq l \leq k$, and $g_1^{k+1}(I^2)$ is a separated family.

Now suppose we have admissible families on levels two through $n-1$ for $n \geq 3$. A family I^n of puzzle pieces of level n is called *admissible* if

1) I^n has exactly three puzzle pieces, the critical puzzle piece and two off-critical puzzle pieces,

2) there exists an integer $k \geq 0$ such that $g_1^l(I^n)$ is contained in the same level one puzzle piece for all $0 \leq l \leq k$, and g_1^{k+1} maps (I^n) to a separated family family, and

3) if $\{\gamma^i\}_{i \in I}$ is the family of words such that $U_0(\gamma^i) = g_{n-1}^{k+1}(V_i^n)$, then $\{\gamma^i\}_{i \in I}$ is subjugate to an admissible family of level $n-1$.

This final condition is precisely why we need to define admissibility inductively on the level. Unfortunately, if there is a level on which there are no admissible families, we need to begin our definition of admissibility from this level onwards.

We can now define the asymmetric modulus of level n. Suppose levels two through $n \geq 2$ all contain admissible families. Then the *asymmetric modulus of level n* is defined as

$$\sigma_n = \min_{I \text{ } admissible} \sigma(I) \quad . \tag{52}$$

LEMMA 4.4.1. *Let $g_1 : \cup V_i^1 \to V^0$ be a generalized cubic-like mapping with admissible families on levels 2 through $n-1$ for $n \geq 3$. If there exists an admissible family on level n, then*

$$\sigma_n \geq \sigma_{n-1} \quad . \tag{53}$$

Proof: Let I^n be any admissible family on level n. By the definition of admissibility, there exists an integer $k \geq 1$ such that g_{n-1}^{k+1} separates I^n. Let $\{\gamma^j\}_{j \in J} \subset \Gamma^{n-1}$ denote the family of words corresponding to the image of I^n

under g_{n-1}^{k+1}. The separated family J can have either two or three puzzle pieces (the two off-critical puzzle pieces could have the same image).

If there are three words in J, then it follows from the definition of the asymmetric modulus that

$$\begin{aligned} \sigma_n(I^n) &= mod(R_{I^n}(V_0^n)) + \frac{1}{3}\sum mod(R_{I^n}(V_i^n)) \\ &\geq \frac{1}{3}\sum mod(R_J(U_0(\gamma^j))) \quad . \end{aligned} \tag{54}$$

By the definition of admissibility, the family J is subjugate to an admissible family I^{n-1} of level $n-1$. Therefore we can apply Lemma 4.3.4 to get

$$\frac{1}{3}\sum mod(R_J(U_0(\gamma^j))) \geq \sigma_{n-1}(I^{n-1}) \geq \sigma_{n-1} \quad . \tag{55}$$

Now suppose there are only two words in J, say γ^1 and γ^2. Then it follows that one of these two words corresponds with the image of both off-critical puzzle pieces in I^n. Say this word is γ^1. Then

$$\begin{aligned} \sigma_n(I^n) &= mod(R_{I^n}(V_0^n)) + \frac{1}{3}\sum mod(R_{I^n}(V_i^n)) \\ &\geq \frac{2}{3}mod(R_J(U_0(\gamma^1))) + \frac{1}{3}mod(R_J(U_0(\gamma^2))) \quad . \end{aligned} \tag{56}$$

Again, by the definition of admissibility, the family J is subjugate to an admissible family I^{n-1} of level $n-1$. Therefore we can apply Lemma 4.3.5 to get

$$\frac{2}{3}mod(R_J(U_0(\gamma^1))) + \frac{1}{3}mod(R_J(U_0(\gamma^2))) \geq \sigma_{n-1}(I^{n-1}) \geq \sigma_{n-1} \quad . \tag{57}$$

Because this is true for every admissible family of level n, the proof is complete.

□

4.5 Proofs of the Main Theorems

Before proving the two main theorems of this chapter, we need to mention two useful lemmas. The first Lemma is from Milnor's exposition [Mil2], and its proof can be found there.

LEMMA 4.5.1. *Suppose that some orbit $z_0 \mapsto z_1 \mapsto \cdots$ in the Julia set never reaches the puzzle pieces $Y^N(0)$ of the critical point. Then the intersection $\bigcap Y^n(z_0)$ of the puzzle pieces containing z_0 reduces to the single point z_0.*

The second lemma is from Lyubich's paper [Lyu3].

LEMMA 4.5.2. *Let D be a topological disk and $K \subset D$ a compact subset consisting of finitely many components. If Γ is the family of non-connected rectifiable curves separating K for ∂D, and $\sigma(\Gamma)$ is the extremal length of this family, then*

$$\frac{\lambda(D)}{\lambda(K)} \geq 1 + \frac{4\pi}{\sigma(\Gamma)} \quad . \tag{58}$$

Proof: (of Theorem 4.1.1)

Note that the boundary of every puzzle piece does not intersect $K(g)$. Therefore, if two points $z_1, z_2 \in K(g)$ are contained in different puzzle pieces of some depth, then they cannot be in the same connected component of $K(g)$. In particular, if we can show that the nested sequence of puzzle pieces $V^n(z) \to \{z\}$ for every point $z \in K(g)$, then $K(g)$ must be totally disconnected.

Step 1: Show the critical puzzle pieces V_0^n shrink to the critical point as $n \to \infty$. By assumption, there exists a renormalization level n such that $g_n : \cup V_i^n \to V_0^{n-1}$ has admissible families on every renormalization level following it. By the Lemma 4.4.1, it follows that every renormalization level $m \geq n$ satisfies $\sigma_m \geq \sigma_n > 0$. Therefore on any renormalization level $m \geq n$, there exists a family I of three puzzle pieces V_0^m, V_1^m, V_2^m such that

$$mod(R_I(V_0^m)) + \frac{1}{3}mod(R_I(V_1^m)) + \frac{1}{3}mod(R_I(V_2^m)) \geq \sigma_n > 0 \quad . \tag{59}$$

Therefore $mod(R_I(V_i^m)) \geq b > 0$ for some $i = 0, 1, 2$ and a definite $b > 0$. If $i = 0$, then we're done. So suppose $i \neq 0$. Let $g_m^k(0) \in V_i^m$ be the first moment when the critical orbit enter V_i^m. Then $R_I(V_i^m)$ pulls back to a critical annulus A which maps as a three-to-one covering map on $R_I(V_i^m)$. Therefore

$$mod(A) \geq \frac{1}{3}mod(R_I(V_i^m)) \geq \frac{b}{3} \quad . \tag{60}$$

Thus we again have a critical annulus with a definite modulus. Because we can find such an annulus on infinitely many levels, it follows from the Grötzch inequality that the critical puzzle pieces are shrinking to the critical point.

Step 2: Let us show that for every $z \in K(g)$ that

$$\sum mod(V^n(z) \setminus \overline{V^{n+1}(z)}) = \infty \quad . \tag{61}$$

If $0 \notin \omega(z)$, then there is some level n such that the orbit of z enters V_0^{n-1}, but never enters V_0^n. There are only finitely many off-critical puzzle pieces of level n, and the closures of these sets are pairwise disjoint. Therefore if $\{V_i^n\}_{i \in I}$ is the family of off-critical puzzle pieces, than there exists $B > 0$ such that

$$mod(R_I(V_i^n)) \geq B \tag{62}$$

for every $i \in I$. But the orbit of z passes through some V_i^n infinitely many times. Each time, conformally pulling back $R_I(V_i^n)$ along the orbit of z gets us an annulus about z with definite modulus. Therefore

$$\sum mod(V^n(z) \setminus \overline{V^{n+1}(z)}) = \infty \quad . \tag{63}$$

Now suppose $0 \in \omega(z)$. By Step 1, there are infinitely many levels l such that $mod(V^l(0) \setminus \overline{V^{l+1}(0)})$ is bounded away from zero.

Let $g^k(z)$ be the first moment when the orbit of z enters $V^l(0)$. then $g^k(z) \in V_i^{l+1}$ for some i. If $i = 0$, then $A^{l+1}(0) = V^l(0) \setminus \overline{V^{l+1}(0)}$ pulls back conformally to an annulus $A^{k+l+1}(z)$ about z. If $i \neq 0$, then $g^k(z) \in U_0(\gamma)$ for some $\gamma \in \Gamma^{l+1}$. It follows that $U(\gamma) \setminus \overline{U_0(\gamma)}$ pulls-back conformally to an annulus about z. But $mod(U(\gamma) \setminus \overline{U_0(\gamma)}) = mod(V^l(0) \setminus \overline{V^{l+1}(0)})$.

Because we can repeat these steps for infinitely many levels l, it follows that

$$\sum mod(V^l(z) \setminus \overline{V^{l+1}(z)}) = \infty \quad . \tag{64}$$

Applying the Grötzch inequality, we have the $V^n(z) \to \{z\}$ for every $z \in K(g)$, and thus $K(g)$ is totally disconnected.

Step 3: Let us show using steps 1 and 2 that $\lambda(K(g)) = 0$. This argument is identical to that of Lyubich in [Lyu3].

Let us organize the set of pieces V_k^n in a tree joining V_k^n with V_i^{n+1} in the case when $V_i^{n+1} \subset V_k^n$. Let us assign to each edge $[U, W]$ of the tree a number

$$\nu[U, W] \equiv \nu(U) = \min(mod(W \setminus \overline{U}), \frac{1}{2}), \tag{65}$$

and to each branch ξ a number $\nu(\xi)$ which is the sum of $\nu[U, W]$ over all edges of ξ. Denote by Ω_n the family of all branches of length n, $n \leq \infty$ (saying "branch" we mean a path in the tree beginning at the root vertex V^0). It follows from steps 1 and 2 that

$$\nu(\xi) = \infty \tag{66}$$

for any $\xi \in \Omega_\infty$. Let us show that

$$M_n \equiv \min_{\xi \in \Omega_n} \nu(\xi) \to \infty. \tag{67}$$

Given a C, consider a subtree of vertices W such that $\nu[V^0, W] \leq C$ where $[V^0, W]$ is the branch ending at W. Now use Kœnig's lemma (this is not the same as Kœnig's theorem from chapter 2): if a tree with finitely many branches at any vertex has arbitrarily long branches then it has an infinite branch. Along this branch the divergence condition fails.

By Lemma 4.5.2, for any vertex U of level n

$$\frac{\lambda(V^{n+1} \cap U)}{\lambda(U)} \leq \exp(-b\nu(U)) \tag{68}$$

with an appropriate constant b. Now one can easily derive from here (induction in n) that

$$\lambda(V^n) \leq \exp(-bM_n)\lambda(V^0), \tag{69}$$

and by the growth of M_n, this value goes to 0. $\square$

4.6 Examples

Let us give some examples which demonstrate the combinatorics described in this chapter. These examples should elucidate the type of generalized renormalizations required for this theory to work.

Example 1: A Good Case

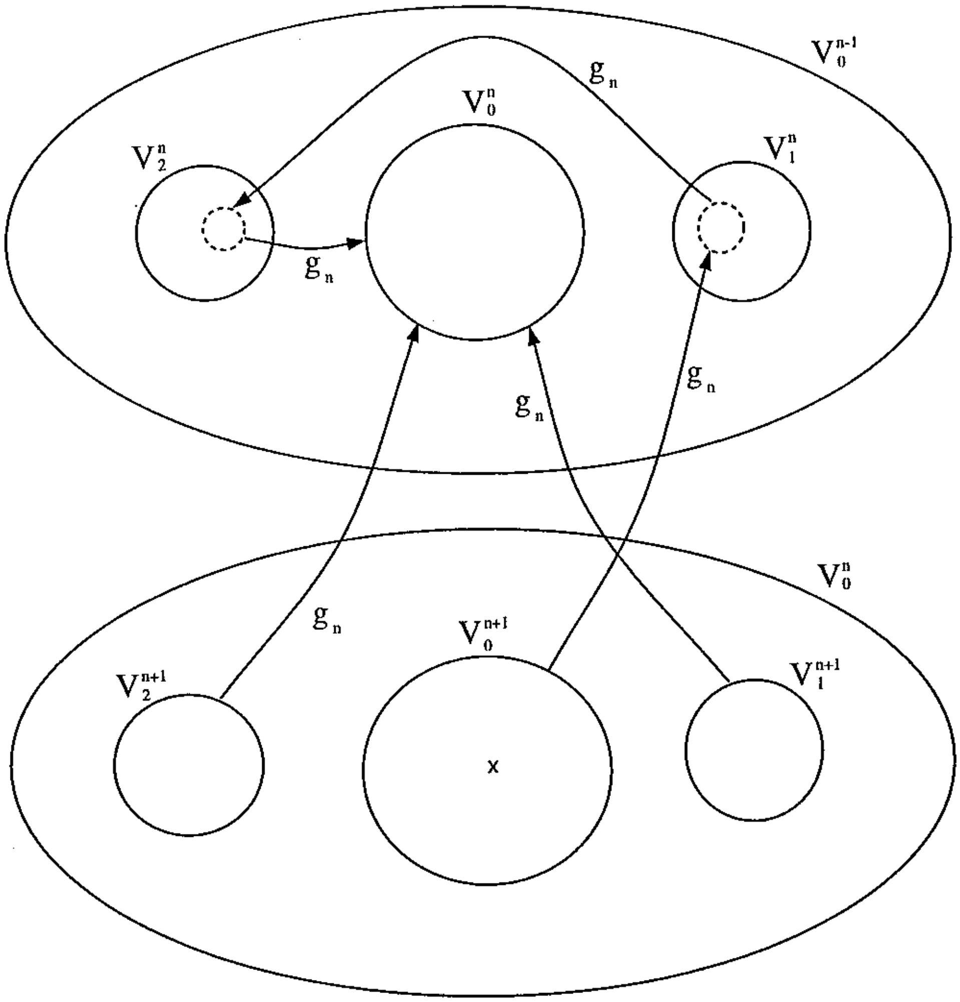

Figure 4.2. A Good Case.

Assume these combinatorics are repeated on all renormalization levels. Then there are three puzzle pieces on each renormalization level. Because the two off-critical puzzle pieces map directly onto the critical puzzle piece of the previous level while the critical puzzle pieces maps through each off-critical puzzle piece before returning, we can choose $k = 1$ and the three puzzle pieces form an admissible family. Therefore the Julia set of a polynomial with these combinatorics is locally connected and has zero measure.

Example 2: "Fibonacci" Combinatorics

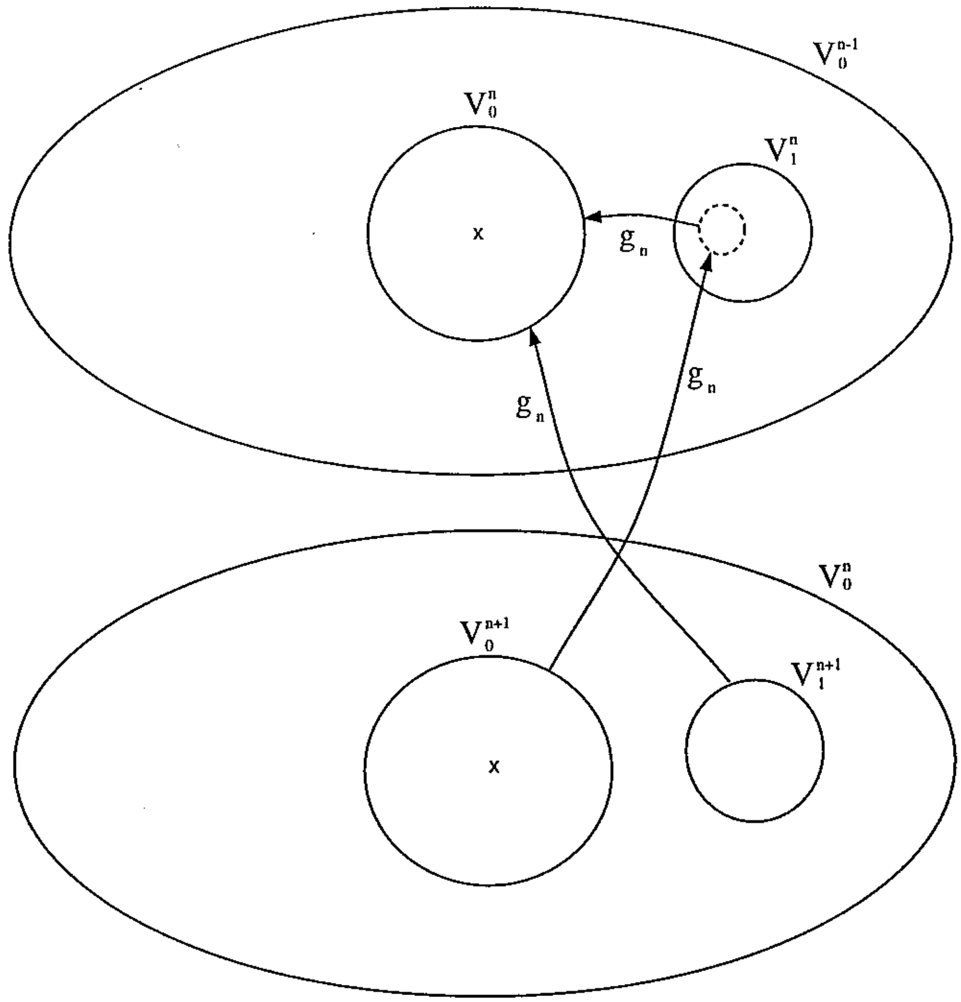

Figure 4.3. "Fibonacci" Combinatorics.

This example of renormalization is sometimes referred to as "Fibonacci" combinatorics, because the combinatorics are the same as those exhibited by the generalized polynomial-like mapping associated to a real Fibonacci polynomial of even degree.

Assume the renormalization pictured in the diagram is repeated on all renormalization levels. Because there are only two puzzle pieces on each level, we do not have admissible families on all levels, and therefore can not apply Theorem 4.1.1. The local connectivity of the Julia for a cubic polynomial with these combinatorics remains an open question.

Example 3: Reluctant Recurrence

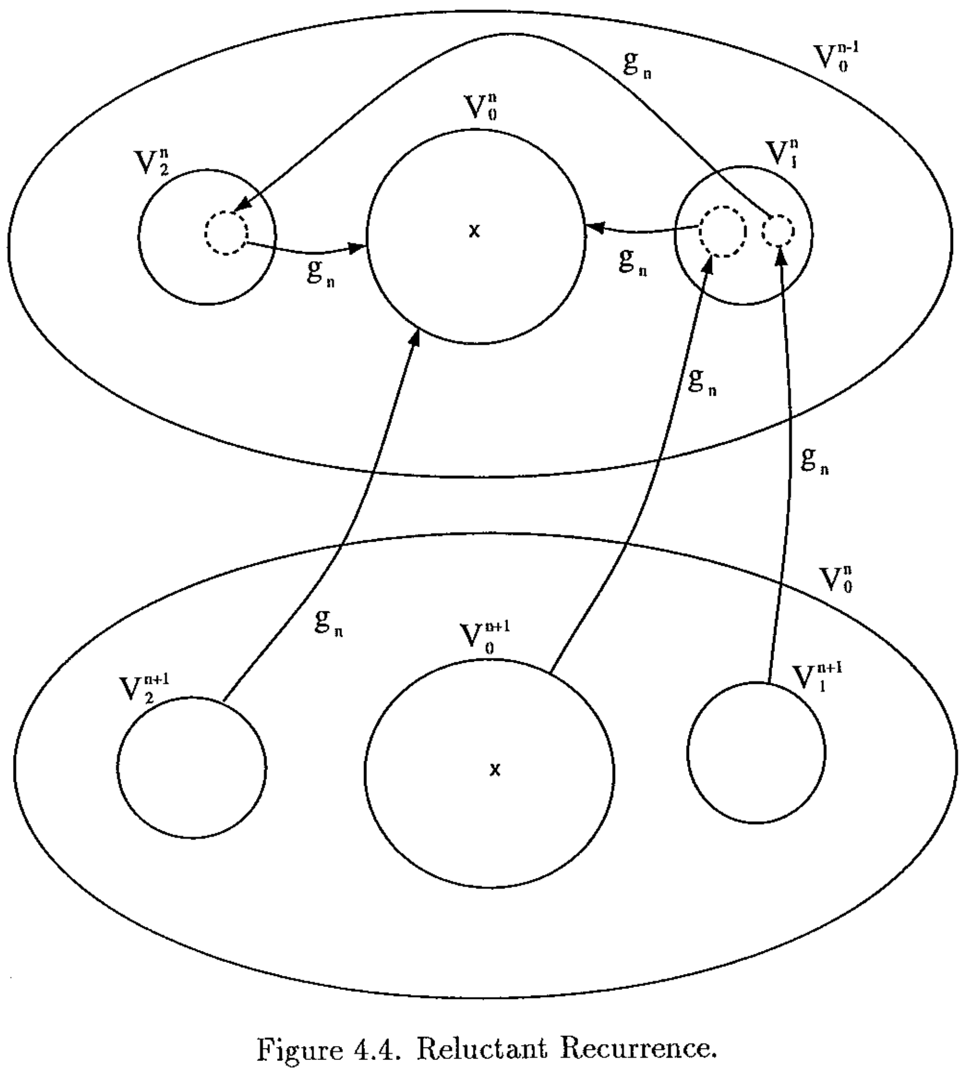

Figure 4.4. Reluctant Recurrence.

Because V_0^{n+1} and V_1^{n+1} map into V_1^n while V_2^{n+1} maps directly onto V_0^n, it follows that there can be no $k \geq 1$ which will separate the three puzzle pieces. Therefore our method from this chapter does not apply.

However, if these combinatorics were repeated on every renormalization elvel, then the critical orbit would be reluctantly recurrent. Reluctant recurrence follows from the observation that V_1^{n+1} maps into V_1^n, and therefore there exists arbitrarily long univalent pull-backs of V_1^n in $\omega(0)$. It is well known that a polynomial with only one critical point which is reluctantly recurrent has a Julia set which is locally connected and has zero Lebesgue measure. See chapter 3 for more information.

Example 4: An Annoying Example

This example remains the foremost obstacle to proving the following conjecture:

CONJECTURE 4.6.1. *If g is a generalized polynomial-like mapping with one cubic critical point and at least three puzzle pieces on every renormalization level, then $K(g)$ is totally disconnected and has zero Lebesgue measure.*

If the combinatorics showed in this diagram are repeated on every renormalization level (in cycles of two), then our method breaks down. Because of the combinatorics of the mapping $g_{n+1} : \bigcup V_i^{n+2} \to V_0^{n+1}$, there is no admissible family on that level. Because of the combinatorics of $g_n : \bigcup V_i^{n+1} \to V_0^n$, the critical orbit is not reluctantly recurrent. Therefore we can not yet conclude anything about the local connectivity or Lebesgue measure of the Julia set of a polynomial with such combinatorics.

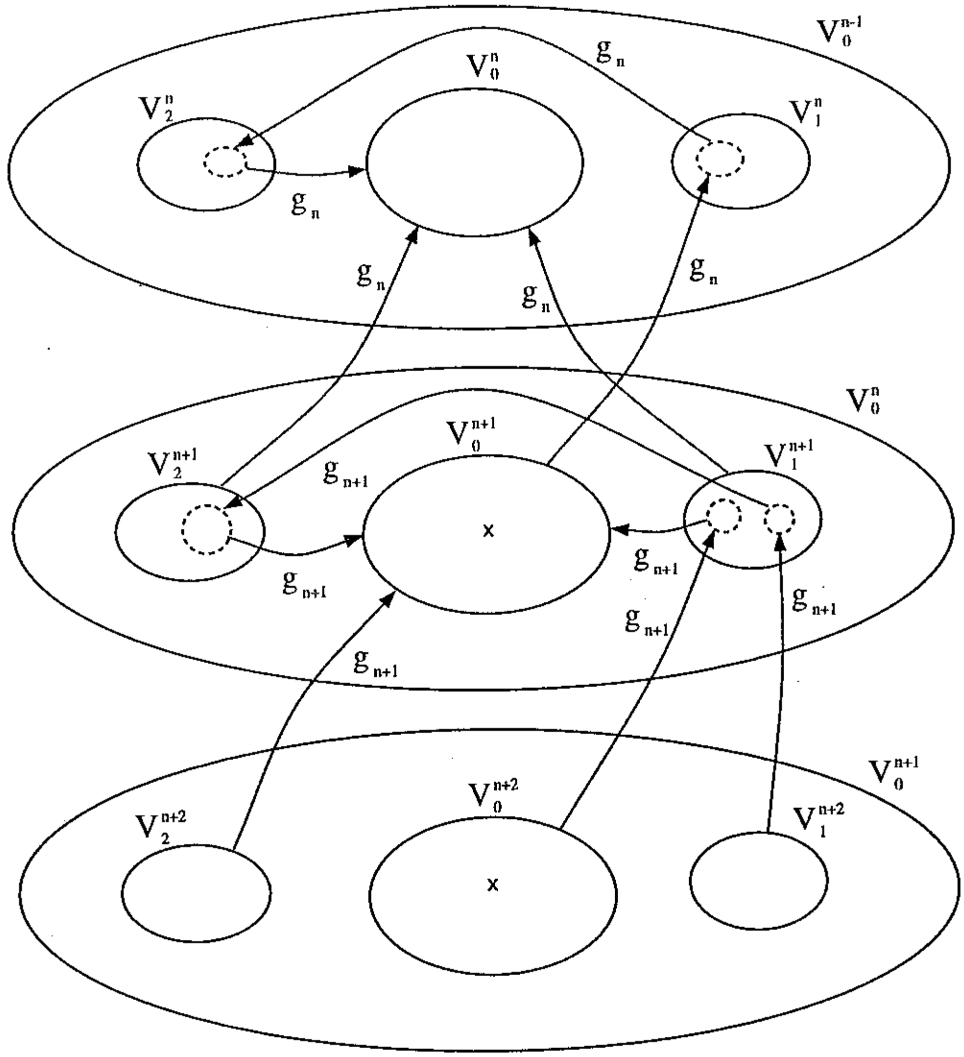

Figure 4.5. An Annoying Example.

5. A Class of Infinitely Renormalizable Quadratic Polynomials with Zero Measure Julia Sets

5.1 Introduction

Let $P_c : z \mapsto z^2 + c$ be a quadratic polynomial. If the critical point 0 is not in the Julia set $J(P_c)$, then it is known that the Julia set $J(P_c)$ has zero Lebesgue measure. In 1991, Lyubich ([Lyu3]) and Shishikura independently showed that if P_c is at most finitely renormalizable, then the Julia set $J(P_c)$ has zero Lebesgue measure as well. However, little is known about the Lebesgue measure of $J(P_c)$ when P_c is infinitely renormalizable.

In [Lyu2], Lyubich introduced a class $\mathcal{L}$ of infinitely renormalizable quadratic polynomials which have particularly nice "geometry." In particular, Lyubich showed that the Julia sets for polynomials in this class are locally connected.

In this chapter, we shall study the question of the Lebesgue measure λ of the Julia sets for quadratic polynomials P_c in this class $\mathcal{L}$.

THEOREM 5.1.1. *For $P_c \in \mathcal{L}$, $\lambda(J(P_c)) = 0$.*

It should be noted that our class $\mathcal{L}$ is not identical to Lyubich's class: qualitatively it is the same class, however there are some subtle quantitative differences.

To give an accurate description of how the class $\mathcal{L}$ is defined, we must first discuss some aspects of the dynamics of quadratic polynomials in a bit more detail. In particular, let us give some additional background on the Mandelbrot set and on renormalization theory.

First, let us discuss some aspects of the Mandelbrot set $\mathcal{M}$ (see [DH2] for more details). A connected component H of the interior of the Mandelbrot set for which each polynomial $P_c \in H$ has an attracting or super-attracting periodic orbit is called a *hyperbolic component.* The attracting periodic orbit in every polynomial in H has the same period, which we shall denote by n_H. The hyperbolic component which contains 0 is called the *main cardioid* and is denoted by $H_\heartsuit$. Each hyperbolic component H has a unique point $c_H \in \partial H$ for which P_{c_H} has a periodic orbit of period n_H which has multiplier 1. This point is called the *root point* of the hyperbolic component H.

To each hyperbolic component H other than the main cardioid, the root point separates the Mandelbrot set. That is, $\mathcal{M} \setminus \{c_H\}$ consists of two connected components. The connected component not containing 0 is called the *limb* L_H associated to H or to c_H. If $c_H \in \partial H_\heartsuit$, then L_H is called a *primary limb.* If $c_H \in \partial H_\heartsuit$, and $c_{H'} \in \partial H$, then $L_{H'}$ is called a *secondary limb.* A secondary limb can be thought of as a limb which is "once removed" from the main cardioid. Finally, a *truncated limb* L_H^{tr} is a limb with a small open neighborhood of the root point c_H removed from it. Examples of truncated secondary limbs can be found in Figure 5.1.

Let us now give more information on the renormalization theory of quadratic polynomials which will be pertinent to this chapter. There is a canonical method of forming a generalized quadratic-like mapping associated to our polynomial P_c which carries much of the combinatorial information we shall need. This construction will be the focus of section 2 of this chapter. This generalized quadratic-like mapping may be quadratic-like in the sense of Douady and Hubbard, however we shall assume this is not the case for polynomials in

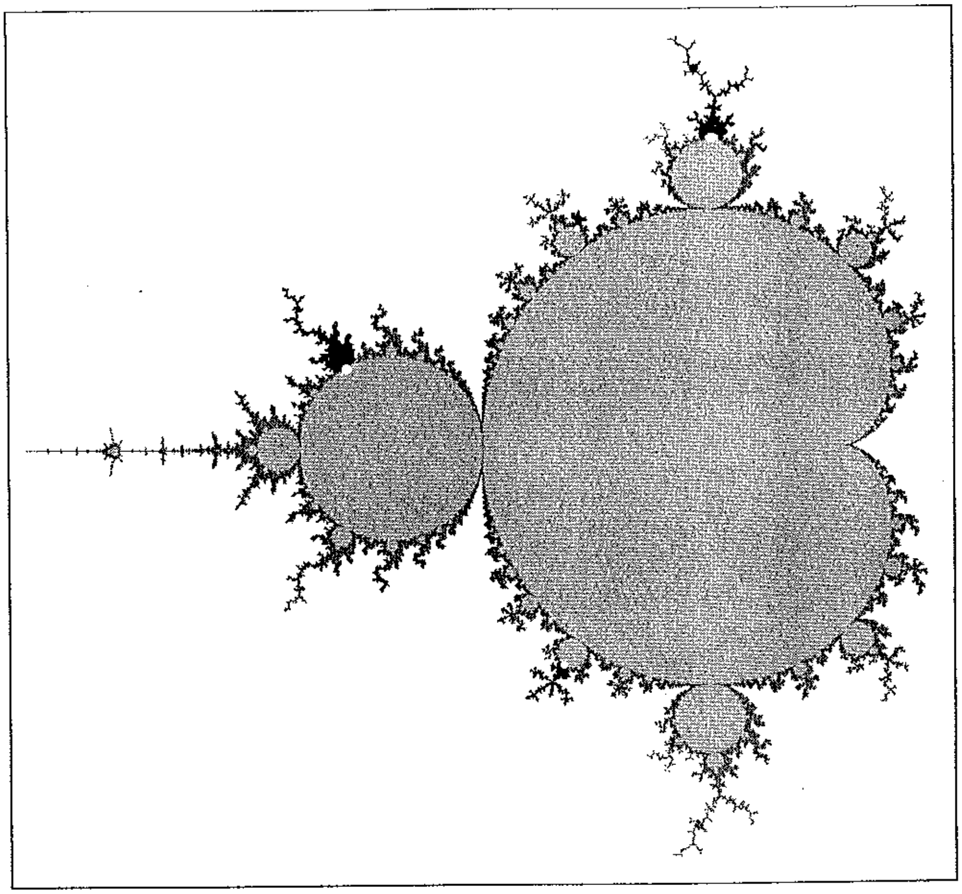

Figure 5.1. The Mandelbrot set, with four examples of truncated secondary limbs, the regions shaded in black.

our class $\mathcal{L}$. In either case, the generalized quadratic-like mapping will have at most finitely many puzzle pieces, only one of which can contain the critical point. See Figure 5.2.

If the generalized quadratic-like mapping is not quadratic-like in the sense of Douady and Hubbard, then we can perform a generalized renormalization to obtain a new generalized quadratic-like mapping. This method of renormalization was first introduced by Lyubich in [Lyu3]. Again, the new mapping obtained may be quadratic-like in the sense of Douady and Hubbard.

Under the new mapping, the critical point may land in any of the finitely

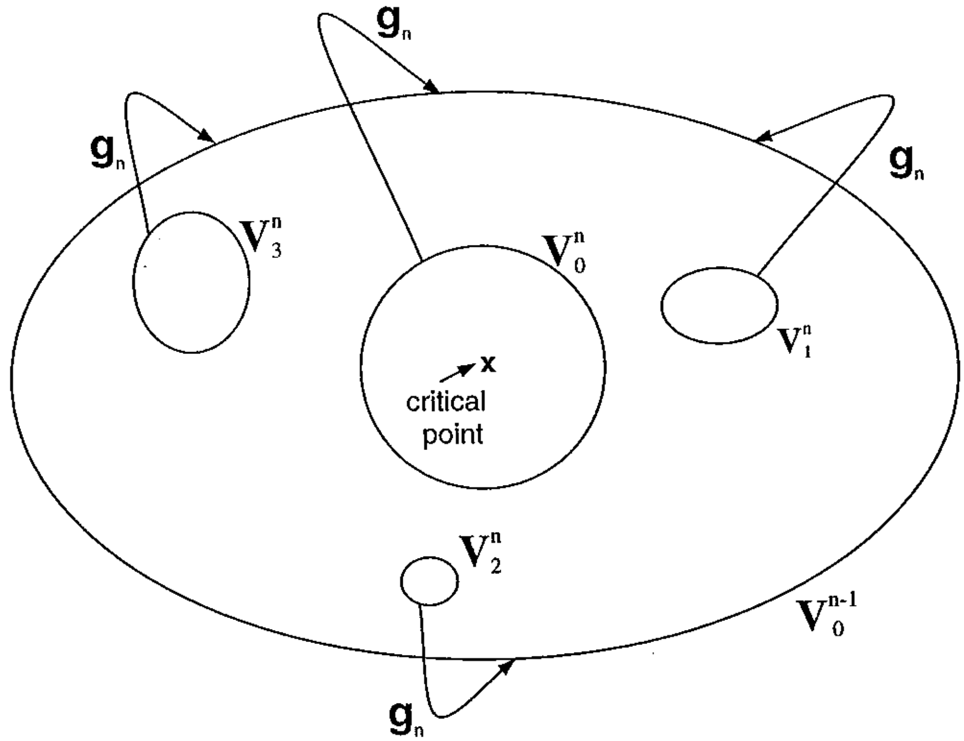

Figure 5.2. An Example of a Generalized Polynomial-like Mapping.

many puzzle pieces. If it lands in the puzzle piece containing the critical point, then this renormalization level will be called a *central return level.* As we continue to renormalize, we categorize all levels as either central or non-central return levels. Finally, because our polynomial is infinitely renormalizable, there will eventually be an infinitely long cascade of central return levels. The number of non-central return levels before arriving at the infinite string of central returns is used as a measure of the "combinatorial depth" between renormalization levels.

We can now define our class of quadratic polynomials.

DEFINITION 5.1.2. *The infinitely renormalizable quadratic polynomial* P_c :

$z \mapsto z^2 + c$ *is in the class* $\mathcal{L}$ *if*

i) the internal class of every renormalization of P_c *contains a polynomial* $P_{c'}$ *which is located in one of a finite number of truncated secondary limbs, and*

ii) there is sufficient combinatorial depth between every renormalization level.

The exact definition of "sufficient combinatorial depth" is dependent only on the finite number of truncated secondary limbs. This also specifies the "subtle quantitative difference" between Lyubich's class and the class we will be dealing with: for certain limbs, we shall require more combinatorial depth then Lyubich required. We will give a more precise definition of "sufficient depth" later in the chapter.

Let us now describe the structure of this chapter.

In section 2, we shall create the initial Markov construction for a polynomial in a truncated secondary limb. This construction can be generalized to all quadratic-like mappings whose internal class is in a truncated secondary limb. This Markov construction will create puzzle pieces which cover Lebesgue almost every point in the Julia set. Furthermore, the density of these puzzle pieces (in a certain sense) will be bounded by a constant depending only on the truncated secondary limb.

In section 3, we shall refine the Markov partition created in section 2 by means of a first-returns mapping. This refinement will still cover almost all of the Julia set. It is in this section that we shall further elaborate on the importance of central returns.

In section 4, we shall show that the density of the puzzle pieces of one refinement inside of the pieces of the previous refinement will be shrinking. Thus if there are sufficiently many refinements, the measure of all the puzzle pieces will be getting quite small.

Finally, in section 5, we shall deal with the case when our refinement arrives at the infinite string of central returns which was discussed earlier. Such an infinite cascade corresponds with a renormalization level in the sense of Douady and Hubbard. At this point, we shall use a combination of the initial Markov construction and the methods of section 4 to pass through to the next renormalization level without incurring "too much" damage to our density bounds.

The initial Markov construction along with the lemmas from sections 4 and 5 will be sufficient to complete the proof of the main Theorem (Theorem 5.1.3). The proof will rely on showing first that the critical point can not be a Lebesgue density point of the Julia set, and then showing that in fact there can not be any Lebesgue density points of the Julia set. Therefore the Julia set must have measure zero.

5.2 The Initial Markov Partition

Construction

The following is a construction of a Markov partition for the polynomial P_c using the Yoccoz puzzle pieces. We will use this Markov partition to construct a generalized quadratic-like mapping associated to the polynomial P_c. The construction is general enough that it can be used on any quadratic polynomial with only repelling periodic points and which is not "immediately renormalizable" (we shall define what this means shortly). But under our assumptions for the class $\mathcal{L}$, we will get much more information from it.

Let us look first at the puzzle pieces of depth 1. Let's relabel these puzzle pieces as follows: let $Y_0^1, \ldots, Y_{p-1}^1$ denote the puzzle pieces of depth 1 which have $\alpha \in \partial Y_i^1$. Then let $Z_1^1, \ldots, Z_{p-1}^1$ be the puzzle pieces of depth 1 for which $-\alpha \in \partial Y_i^1$, and $\alpha \notin \partial Y_i^1$. The only puzzle piece which has both $\pm\alpha$ in its boundary is the critical puzzle piece Y_0^1. See Figure 5.3.

From the rotation of the external rays landing at α, we know that $P_c^p(0) \in Y_0^1 \cup \bigcup_i Z_i^1$.

Definition 5.2.1. *If the α-fixed point of P_c has p external rays landing at it, and P_c is renormalizable or order p, then P_c is said to be immediately renormalizable.*

As an exceptional case, we shall say that a polynomial which is immediately renormalizable has zero combinatorial depth before its first renormalization. We shall say that in order for a polynomial to have sufficient combinatorial depth between renormalization levels to be in our class $\mathcal{L}$, its combinatorial

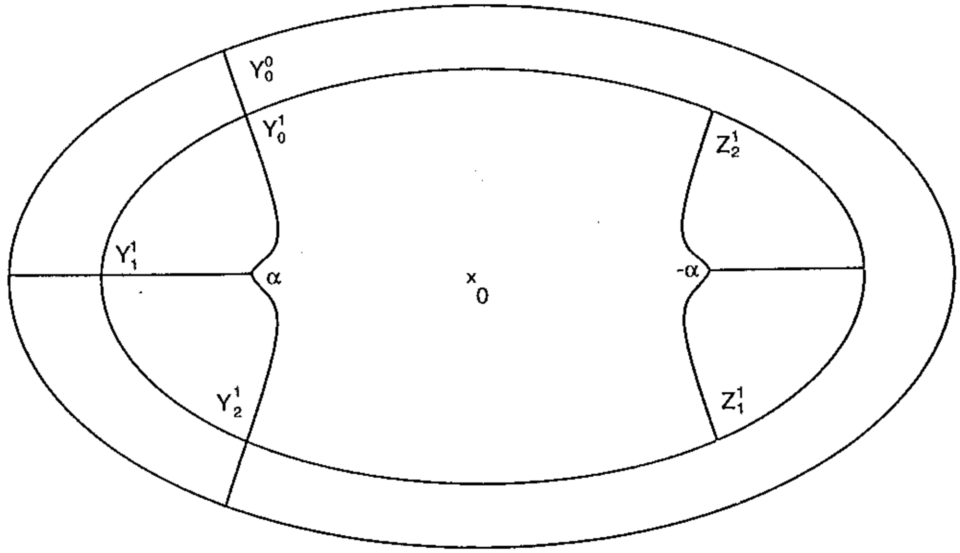

Figure 5.3. The depth 1 Yoccoz puzzle pieces.

depth must be at least positive. Therefore we impose the condition that no polynomials in $\mathcal{L}$ can be immediately renormalizable.

We claim that in order for P_c to not be immediately renomalizable, there must exist a positive integer $t > 0$ such that $P_c^{tp}(0) \in \bigcup Z_i^1$, assuming that the α-fixed point has p external rays landing at it. Let us prove the claim by contradiction: assume no such $t > 0$ exists. Let $\tilde{Y}$ be a slight "thickening" of Y_0^1; that is, let $\tilde{Y}$ be the union of $\overline{Y_0^1}$ and a small open neighborhood of $\overline{Y_0^1}$. If this neighborhood is chosen small enough, then $P_c^p(\tilde{Y})$ contains the closure of $\tilde{Y}$. Further, $P_c^{kp}(0) \in Y_0^1 \subset \tilde{Y} \forall k \in \mathbb{N}$. And $P_c^p(\tilde{Y})$ will be a finite degree branched covering map onto its image. Therefore P_c is immediately renormalizable, which is a contradiction, and thus $P_c^{tp}(0) \in \bigcup_i Z_i^1$ for some $t \in \mathbb{N}$.

Let $z \in \mathbb{C}$, and suppose $P_c^k(z) \in U$ where U is any puzzle piece. For each i, $0 < i \leq k$, there exists a unique component W_i of $P_c^{-i}(U)$ such that $P_c^{k-i}(z) \in W_i$. Thus we have a string of k puzzle pieces $W_1, \ldots, W_k$ satisfying

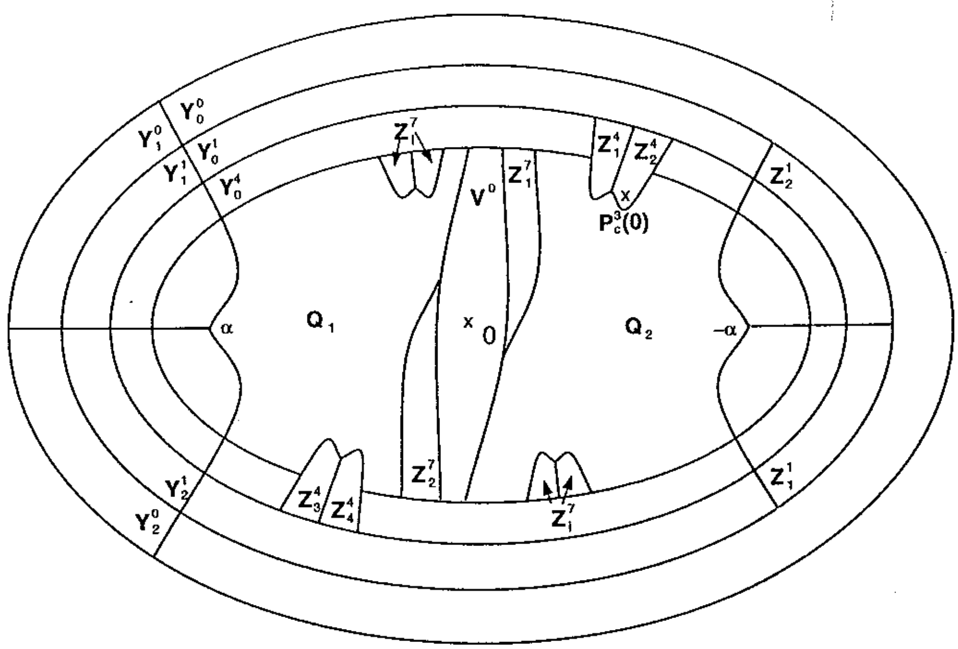

Figure 5.4. The initial Markov construction.

$P_c^{k-i}(z) \in W_i$ for each i, $0 < i \leq k$. The process of pulling back U to W_1 to W_2 etc., is called *pulling back U along the orbit of z*. The puzzle piece $W_k \ni z$ defined from the above process is called the *pull-back of U along the orbit of z*.

From the combinatorial dynamics of the Yoccoz puzzle pieces, it is clear that $Y_0^1 \cup \bigcup Z_i^1 \subset P_c^p(Y_0^1)$. Now suppose $P_c^p(0) \in Y_0^1$. Pulling back Y_0^1, we get Y_0^{1+p}, and pulling back $P_c^p(Y_0^1)$ along the orbit of 0 gets us $Z_1^{1+p}, \ldots, Z_{2p-2}^{1+p}$ which are the pre-images of $Z_1^1, \ldots, Z_{p-1}^1$ under P_c^p. Because each $\overline{Z_i^1} \subset P_c^p(Y_0^1)$, it follows that $\overline{Z_k^{1+p}} \subset Y_0^1$ for each $1 \leq k \leq 2p-2$. If $P_c^p(0) \in Y_0^{1+p}$, then we can repeat this procedure to get $0 \in Y_0^{1+2p}$ and Z_t^{1+2p} for $1 \leq t \leq 4p-4$. See Figure 5.4.

From the previous discussion, we know that the forward orbit of the critical point must eventually enter Z_i^1 for some i. This statement is equivalent to saying $P_c^p(0) \in Z_j^{tp+1}$ for some j and some $t \geq 1$. Furthermore, Z_j^{tp+1} maps conformally under some iterate of P_c onto a set which contains Y_0^1. Let $k \geq (t+1)p+1$ be the first point in the forward orbit of the critical point after the orbit enters Z_i^1 for which $P_c^k(0) \in Y_0^1$. Then pull Y_0^1 back along the critical orbit to obtain V^0. The set V^0 is the first piece of our Markov partition.

For every point $z \in J(P_c)$ whose forward orbit enters V^0, pull V^0 back along the orbit of z to obtain $V^1(z)$. The sets $\bigcup V_i^1$ of all puzzle pieces obtained in this manner shall be called the *initial Markov partition* for the polynomial P_c.

The Density of the Markov Partition

Our goal in this subsection is to show that there is a constant dependent only on the truncated secondary limb which bounds the density of those V_i^1 which are contained in V^0.

First note that if there are p external rays landing at the α-fixed point of P_c, then there exists a repelling periodic point $\zeta \in Y_0^1$ of period p. A property of secondary limbs is that the external rays landing at ζ vary continuously with the parameter throughout the secondary limb.

Recall that the depth 1 Yoccoz puzzle pieces are the bounded components of the complement of the external rays landing at $\pm\alpha$ and the $\frac{1}{2}$-equipotential. Define the *central domain* D_0 to be the set containing the critical point which is bounded by external rays landing at $\pm\zeta$ and the $\frac{1}{4}$-equipotential. See Figure 5.5.

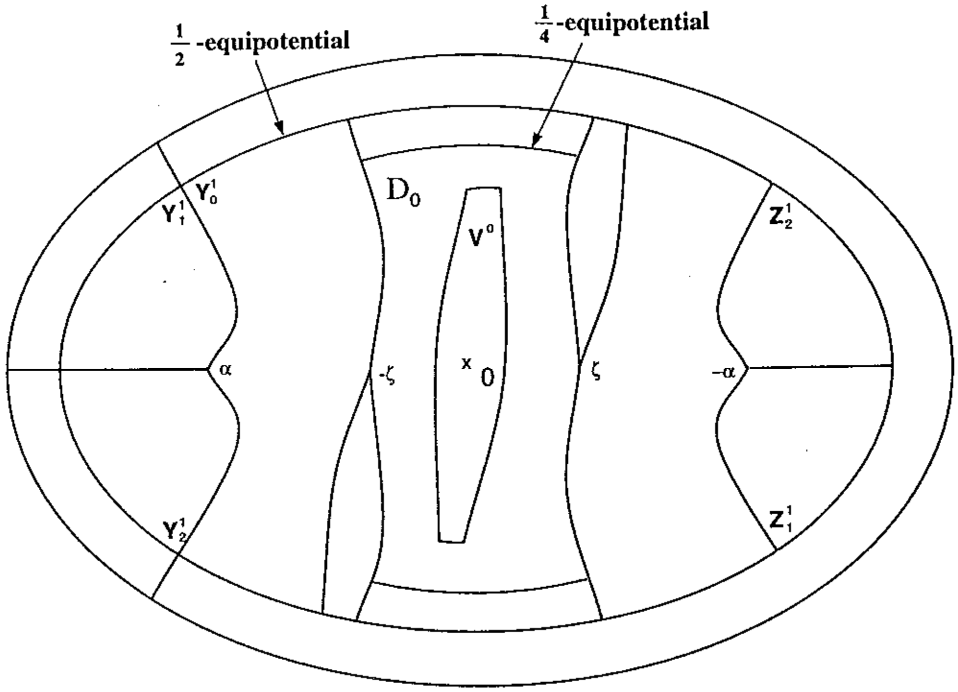

Figure 5.5. The Central Domain Near α.

PROPOSITION 5.2.2. *There exists a constant $\xi_0 < 1$ depending only on the truncated secondary limb L_b^{tr} such that*

$$\frac{\lambda(\{\bigcup V_i^1 \cup V^0\} \cap D_0)}{\lambda(D_0)} < \xi_0 \quad . \tag{70}$$

Proof: The truncated secondary limb L_b^{tr} is a compact set, and the equipotentials along with the external rays landing at α and ζ move continuously with the parameter throughout the limb. Therefore there is a definite space in D_0 which is not in any V_i^1. $\square$

There are two pre-images of D_0 in Y_0^1 for the p^{th} iterate of the polynomial: one near the α-fixed point and one near $-\alpha$. Let $\pm D^*$ denote these two pre-images. It follows that for any point $z \in J(P_c)$ near $\pm\alpha$ (but not α itself), its forward orbit will eventually pass through $\pm D^*$. It also follows that for any such point z that there will be a conformal pull-back of D_0 to an open neighborhood of z.

For each point $z \in Y_0^1 \cap R(P_c)$, push z forward until the first time its orbit enters D_0. Then pull D_0 back along the orbit of z to get D_i for some index i.

PROPOSITION 5.2.3. *There exists a constant $m > 0$ dependent only on L_b^{tr} such that for $D_i \subset Y_0^1$,*

$$mod(Y_0^1 \setminus D_0) \geq m \quad . \tag{71}$$

Proof: For any $\epsilon > 0$, there exists a $\delta > 0$ such that if $U \subset Y_0^1$ is the set of all points in Y_0^1 which are at least δ away from the boundary, then if $D_i \not\subset U$, then $d(D_i, \pm\alpha) < \epsilon$. If $D_i \subset U$, then the statement is proven. If $D_i \not\subset U$, then it will map conformally onto D_0 without ever leaving Y_0^1. Because the external rays landing at α and ζ are varying continuously throughout the truncated secondary limb, it is clear that we can find an m which satisfies the statement of the proposition. □

PROPOSITION 5.2.4. *There exists a constant $d < \infty$ such that every D_i maps conformally onto D_0 with distortion bounded by d.*

Proof: Again, the rays landing at α and ζ, along with the equipotentials, are varying continuously with c throughout the truncated secondary limb. Then we can safely "thicken" $\pm D^*$ independent of the parameter within the truncated secondary limb. Because any D_i returning to D_0 without escaping to some Z_i^1 must pass through $\pm D^*$, each will map with distortion bounded by a constant depending only on the truncated secondary limb. But if D_i escapes to some Z_i^1 before returning to D_0, then it will get Koebe space from Z_i^1.

Therefore it too will map with distortion bounded by a constant depending only on the truncated secondary limb. $\square$

Let us now prove the density result for the initial Markov partition that we shall need in this paper.

LEMMA 5.2.5. *Let $\bigcup V_i^1$ denote the initial Markov partition for a polynomial $P_c \in L_b^{tr}$. There exists a constant $\xi < 1$ dependent only on the truncated secondary limb L_b^{tr} such that*

$$\frac{\lambda(\bigcup V_i^1 \cap V^0)}{\lambda(V^0)} \leq \xi \quad . \tag{72}$$

Proof: Look at the map corresponding to the iterate of P_c which maps V^0 as a degree two branched covering onto Y_0^1. The critical value of this mapping must be located in some D_i. By Proposition 5.2.3, $mod(Y_0^1 \setminus D_i) \geq m$ for an m depending only on L_b^{tr}. Therefore D_i pulls back to a set D' satisfying

$$mod(V^0 \setminus D') \geq \frac{m}{2} \quad , \tag{73}$$

and therefore there exists a constant $\eta_1 > 0$ dependent only on L_b^{tr} such that

$$\frac{\lambda(V^0 \setminus D')}{\lambda(V^0)} \geq \eta_1 \quad . \tag{74}$$

For each $V_j^1 \not\subset D'$, there exists a D_i such that the image of V_j^1 under the degree two branched covering map $V^0 \to Y_0^1$ is in D_i. But by Proposition 5.2.4, D_i maps onto D_0 with bounded distortion. Furthermore, by Proposition 5.2.2, $\bigcup V_i^1 \cup V^0$ has bounded density in D_0. Therefore there exists a constant $\eta_2 > 0$ dependent only on L_b^{tr} such that

$$\frac{\lambda(V^0 \setminus \{D' \cup \bigcup V_i^1\})}{\lambda(V^0 \setminus D')} \geq \eta_2 \quad . \tag{75}$$

Combining the above two equations completes the proof. $\square$

The Markov Partition for Polynomial-like Mappings

The initial Markov construction, done previously for a polynomial, can be generalized to quadratic-like mappings in a straightforward way. Let $g : U \to V$ be a quadratic-like mapping in the sense of Douady and Hubbard. Then by the Straightening theorem, there exists a polynomial $P_{c'}$ which is hybrid equivalent to g.

Suppose this polynomial $P_{c'}$ is located in the truncated secondary limb L_b^{tr} and is not immediately renormalizable. Then perform the construction of the initial Markov partition for $P_{c'}$. By Lemma 5.2.5, this partition has a density bounded by a constant dependent only on the truncated secondary limb L_b^{tr}.

Because g and $P_{c'}$ are hybrid equivalent, there exist open neighborhoods U_1, U_2 of $K(g), K(P_{c'})$ respectively, and a quasiconformal map $h : U_1 \to U_2$ satisfying

$$(h \circ g) \mid_{U_1} = (P_{c'} \circ h) \mid_{U_1} \quad . \tag{76}$$

Therefore we can pull-back the puzzle piece Y_0^1 and the Markov partition for $P_{c'}$ by the quasiconformal map h to get a puzzle piece $\widetilde{Y}_0^1$ and a Markov partition for $K(g)$. This will be called the *initial Markov construction* for the polynomial-like mapping g.

The bound on the density obtained for $P_{c'}$ is useless unless we also have a bound on the maximal dilatation for the quasiconformal map h. Suppose $g, U, V, h, P_{c'}, U_1$, and U_2 are as in the previous paragraphs.

THEOREM (MCMULLEN). *There exists a constant L depending only on mod$(U \setminus V)$ such that the maximal dilatation of h on U_1 is bounded from above by L.*

The bound obtained from McMullen's theorem on the dilatation of the quasiconformal conjugacy finishes the proof of the following lemma.

LEMMA 5.2.6. *Let $g : U \to V$ be a quadratic-like mapping with g hybrid equivalent to a quadratic polynomial $P_{c'}$ located in the truncated secondary limb L_b^{tr}, and let $\widetilde{Y}_0^1 \subset U$ be the image of the puzzle piece Y_0^1 for $P_{c'}$ under the quasiconformal conjugacy. Then there exists a constant K dependent only on L_b^{tr} and mod$(V \setminus U)$ such that if $\bigcup \widetilde{V}_i^1$ is the initial Markov partition for g, then*

$$\frac{\lambda(\bigcup \widetilde{V}_i^1 \cap \widetilde{V}^0)}{\lambda(\widetilde{V}^0)} \leq K < 1 \quad . \tag{77}$$

5.3 The First-returns Map

The Markov Partition and the Recurrent Set

The Markov partition defined in the previous section covers all points in $J(P_c) \cap Y_0^1$ whose forward orbit enters V^0. For each off-critical depth 1 puzzle piece Y_i^1 or Z_i^1, there is an iterate of P_c which will map it conformally onto a set which covers Y_0^1. Therefore this Markov covering can be extended to all points in the Julia set whose orbit enters V^0.

Let us define the *recurrent set* $R(P_c)$ of the polynomial P_c to be the set of points whose orbit accumulates at the critical point. That is,

$$R(P_c) = \left\{ z \in J(P_c) : 0 \in \omega(z) \right\} \quad . \tag{78}$$

The Markov covering we have defined covers all of $R(P_c)$. Fortunately, that is all we need to cover.

PROPOSITION 5.3.1. *For any polynomial P_c,*

$$\lambda(J(P_c) \setminus R(P_c)) = 0 \quad . \tag{79}$$

The proof of this proposition follows immediately from Proposition 2.5.6. It follows from this proposition that our Markov covering covers Lebesgue almost every point of the Julia set. Therefore if $\lambda(R(P_c)) = 0$, then $\lambda(J(P_c)) = 0$.

The First-returns Mapping

For each point $z \in R(P_c)$, its orbit must eventually enter V_0^1. Therefore we can pull back V_0^1 along the orbit of z to get a new puzzle piece $V^2(z)$. Each $V^2(z)$ is a conformal pull back of V_0^1 except for the unique level 2 puzzle piece containing the critical point. We denote this critical puzzle piece by V_0^2, which is a degree two pull back of V_0^1. Note that $\bigcup V_i^2$ still covers all of $R(P_c)$. Because each V_i^2 is contained in some $\bigcup X_i \cup \bigcup Z_j$, we have defined a refinement of our Markov covering. Further, we call the mapping g_2 which maps each V_i^2 onto V_0^1 the *first-returns mapping of level 2.*

We can repeat the construction we just performed for level 2 as often as we like to get increasing levels. That is, for any $z \in R(P_c)$, we can pull-back V_0^n along the orbit of z to obtain the puzzle piece $V^{n+1}(z)$. The mapping g_n which carries each V_i^{n+1} onto V_0^n is called *the level $n+1$ first-returns mapping.* And again, the new covering covers all of $R(P_c)$, so that we have refined the level n Markov covering.

It should be noted that the first-returns mappings we have defined in the previous paragraphs are the "global" first-returns mappings. That is, the Markov covering covers all of $R(P_c)$ and the mapping is defined for all of $R(P_c)$. However in the next section, we shall only be concerned with the "central" first-returns mapping. That is, we shall restrict the domain of the first-returns mapping g_n only to those puzzle pieces V_i^n contained in V_0^{n-1}. The central first-returns mapping carries most of the density information we shall need, and can be "spread" around to all of $R(P_c)$. This "spreading" shall be done explicitly in section 5. Also note that we shall use the same notation for both the central and global first-returns map. We shall attempt to make it clear which is being referred to from the context in which it is used.

Cascades of Central Returns and the Principal Nest

For each first-returns level n, there exists a minimum strictly positive integer k such that $g_{n-1}^k(0) \in V_0^{n-1}$. If $l = 1$, then we call the level n a *central return level.* For such a level, $g_n|_{V_0^n} \equiv g_{n-1}|_{V_0^n}$. If several levels in succession are all central return levels, then we shall call these levels a *cascade of central returns.* Further, if there exists a level m such that every level $n \geq m$ is a central return level, then we shall say that the level m is the beginning of an *infinite cascade of central returns.*

LEMMA 5.3.2. *If $g_m : \cup V_i^m \to V_0^{m-1}$ is the beginning of an infinite cascade of central returns, then $g_m|_{V_0^m} : V_0^m \to V_0^{m-1}$ is a quadratic-like mapping in the sense of Douady and Hubbard, and corresponds to a renormalization of our original polynomial P_c.*

Proof: The map $g_m|_{V_0^m}$ maps V_0^m onto V_0^{m-1} as a two-to-one holomorphic covering map, and therefore is quadratic-like. The infinite cascade condition implies that $g_m^k(0) \in V_0^m$ for every $k \in \mathbb{N}$. Thus the critical point never escapes V_0^m. $\square$

From the first-returns mapping, we have a nested sequence of puzzle pieces containing the critical point. These critical puzzle pieces are called the *principal nest*:

$$V^0 \supset V_0^1 \supset V_0^2 \supset \cdots \quad . \tag{80}$$

The *principal modulus* μ_n is the modulus of the associated annulus:

$$\mu_n = mod(V_0^{n+1} \setminus V_0^n) \quad . \tag{81}$$

The principal nest can be divided into cascades of central returns. A non-central return level which is not immediately following a central return level is considered to be a cascade of depth 1. We consider the level immediately following a central return level to be part of the cascade. A non-renormalizable polynomial will have infinitely man cascades in its principal nest, while a renormalizable polynomial will have finitely many cascades followed by an infinite cascade of central returns.

It will be important in the following section to be able to count the non-cascade levels. If level 1 is a non-central return level, then $n(1) = 1$. Otherwise, if k is the first non-central return level, the $n(1) = k+1$. Recall that we define the first non-central return level following a sequence of successive central return levels to be part of the cascade. In general, if $n(i) = m$, and level $m+1$ is a non-central return level, then $n(i+1) = m+1$; if level $m+1$ is a central return level, and $m+k$ is the next non-central return level, then $n(i+1) = m+k+1$. The strictly increasing sequence of integers $\{n(i)\}$ is said to count *the non-cascade levels.*

Let us now distinguish the "short" and "long" principal nests. As mentioned earlier, to our first-returns mapping we can associate a nested sequence of critical puzzle pieces,

$$V^0 \supset V_0^1 \supset V_0^2 \supset \cdots \tag{82}$$

called the principal nest. Because our polynomial is renormalizable, this sequence will eventually end in an infinite cascade of central returns. Suppose level m_1 is the first level of the infinite cascade of central returns. We shall call the finite nest of puzzle pieces terminating at $V_0^{m_1}$ the *short principal nest.*

Because all renormalizations of P_c are located in a finite number of truncated secondary limbs, we can construct a new Markov partition for $K(g_{m_1}|_{V_0^{m_1}})$ as described in the previous section. This new partition enables us to define a new short principle nest

$$V^{2,0} \supset V_0^{2,1} \supset V_0^{2,2} \supset \cdots \supset V_0^{2,m_2} \tag{83}$$

where m_2 is the first level of a new infinite cascade of central returns. Repeating this procedure indefinitely, we have constructed the *long principal nest,*

$$V^0 \supset V_0^1 \supset \cdots \supset V_0^{m_1} \supset V^{2,0} \supset V_0^{2,1} \cdots V_0^{2,m_2} \supset V^{3,0} \supset \cdots \quad . \tag{84}$$

5.4 First-Returns and Density

Let $G^n = V_0^{n-1} \backslash \cup V_i^n$. The set G^n is called the "gap space" of level n. Let $\delta_n = \frac{\lambda(G^n)}{\lambda(V_0^{n-1})}$. In this section, we will show that δ_n increases with first returns level, provided the level is "nice". Let $\rho_n = \frac{\lambda(V_0^n)}{\lambda(V_0^{n-1})}$ and $\mu(n) = mod(V_0^{n-1} \setminus \overline{V_0^n})$.

For V_i^n with $i \neq 0$, let $dist(V_i^n)$ be the Koebe distortion of the map $g_n|_{V_i^n} : V_i^n \to V_0^{n-1}$, and let $dist(n) = \sup_i dist(V_i^n)$.

This section is going to be devoted to proving the following lemma.

LEMMA 5.4.1. *Let $n(k)$ count the non-central return levels. Then there exists a constant A depending only on the first modulus $\mu_1 = mod(V_0^0 \setminus \overline{V_0^1})$ such that*

$$\frac{\delta_{n(k+1)}}{(1 - \rho_{n(k+1)}) - \delta_{n(k+1)}} \geq A \frac{\delta_{n(k)}}{\rho_{n(k)}} \quad . \tag{85}$$

Preliminaries

Let us state two results of M. Lyubich from [Lyu2] which will be important in the proof of Lemma 5.4.1.

THEOREM 5.4.2. (LYUBICH) *Let $n(k)$ count the non-central levels in the principal nest $\{V^n\}$. Then*

$$mod(V_0^{n(k)+1} \setminus \overline{V_0^{n(k)}}) \geq B\, k \tag{86}$$

where the constant B depends only on the first modulus $\mu_1 = mod(V_0^0 \setminus \overline{V_0^1})$.

Let us call a cascade of central returns *long* if it is of length greater than N_* for a fixed positive integer N_*. Also, for notational purposes, assume

$$g_n\, |_{V_0^n} = \phi \circ h_n$$

where ϕ is a purely quadratic map, and h_n is a conformal map.

THEOREM 5.4.3. (LYUBICH) *Given a generalized quadratic-like map g_1, we have the following bounds for the geometric parameters within its principal nest.*

1) *The asymmetric moduli σ_n grow monotonically and hence stay away from 0 on all levels (until the first Douady-Hubbard renormalization level) : $\sigma_n \geq \bar{\sigma} > 0$.*

2) *The principal moduli μ_n stay away from 0 (that is, $\mu_n \geq \bar{\mu} > 0$) everywhere except for the case when $n-1$ is in the tail of a long cascade (the bound $\bar{\mu}$ depends the choice of N_*).*

3) *The non-critical puzzle pieces V_i^n are well inside V_0^{n-1} (that is, $mod(V_0^{n-1} \setminus V_i^n) \geq \bar{\mu} > 0$) except for the case when V_i^n is pre-critical and $n-2$ is the last level of a long cascade.*

4) *The distortion of h_n is uniformly bounded on all levels by a constant $\bar{K}$.*

All bounds depend only on the first principal modulus μ_1 and (as $\bar{\mu}$ is concerned) on the choice of N_*.

Distortion Bounds

Let us start with a fixed first-returns level n and $g_n : \bigcup V_i^n \to V_0^{n-1}$. For any fixed off-critical puzzle piece V_i^n, define T_i^n to be the annulus of maximal modulus about V_i^n in V_0^{n-1} with V_0^n in the unbounded component of the complement of T_i^n, and similarly define S_i^n to be the annulus of maximal modulus about V_i^n in V_0^{n-1} with V_0^n in the unbounded component of the complement of S_i^n.

PROPOSITION 5.4.4. *Let $g_n : \bigcup V_i^n \to V_0^{n-1}$ be a first-returns map which does not correspond to a level which is in or immediately following a cascade of central returns. If $mod(V_0^{n-1} \setminus V_0^n) \leq B$, there exists a constant $K \equiv K(B)$ such that for every off-critical V_i^n, $dist(V_i^n) \leq K$.*

Proof: Suppose there exists M dependent only on B such that

$$mod(T_i^n) \geq M$$

for all off-critical V_i^n. Then by the Koebe distortion theorem we are done.

So suppose we can not find such an M. That is, for every $M > 0$, there exists V_i^n with $mod(T_i^n) < M$. Let $M_1 < \frac{\bar{\sigma}}{3}$, where $\bar{\sigma}$ is constant from Theorem 5.4.3.

Because $mod(V_0^{n-1} \setminus V_0^n) \leq B$, for every $\epsilon > 0$, there exists a constant δ dependent only on ϵ and B such that if $d_{Hyp}(V_0^n, V_i^n) < \delta$, then $mod(S_i^n) < \epsilon$. Let $\epsilon < \frac{\bar{\sigma}}{3}$.

By Theorem 5.4.3, there exists a constant $\bar{\mu}$ such that $mod(V_0^{n-1} \setminus V_i^n) \geq \bar{\mu}$ for every V_i^n. Because $mod(V_0^{n-1} \setminus V_i^n) \geq \bar{\mu}$, it follows that there exists $M_2 > 0$ such that if $mod(T_i^n) < M_2$, then $d_{Hyp}(V_0^n, V_i^n) < \delta$. Let $M_0 = min\{M_1, M_2\}$. Then for V_i^n with $mod(T_i^n) < M_0$,

$$\sigma_n \leq mod(S_i^n) + \frac{1}{2} mod(T_i^n) \leq \frac{\bar{\sigma}}{3} + \frac{\bar{\sigma}}{6} < \bar{\sigma} \quad .$$

This contradiction of Theorem 5.4.3 completes the proof of the proposition. $\square$

Let $B_n(0, r)$ denote the hyperbolic ball of radius r about the origin in V_0^{n-1}.

PROPOSITION 5.4.5. *Suppose n is not a level in or immediately following a long cascade of central returns. Then there exist constants $r > 0, K > 0$ dependent only on $\bar{\mu}, \bar{\sigma}$, such that if $V_i^n \not\subset B_n(0,r)$, then $dist(V_i^n) \leq K$.*

Proof: For every V_i^n, $mod(V_0^{n-1} \setminus V_i^n) > \bar{\mu}$. For any $s > 0$, we can find $r > 0$ such that if $V_i^n \not\subset B^n(0,r)$, then $d_{Hyp}(0, V_i^n) \geq s$. The fact that for every s we can find such an r follows from the solution of Grötzch's extremal problem (See [Ah1], page 35).

Suppose $V_0^n \not\subset B_n(0, s/2)$. Then again by the solution of Grötzch's extremal problem, there exists a constant $B > 0$ such that $mod(V_0^{n-1} \setminus V_0^n) \leq B$. Applying Proposition 5.4.4, we have a constant $K \equiv K(B)$ with $dist(V_i^n) \leq K$ for every off-critical V_i^n.

If $V_0^n \subset B_n(0, s/2)$, then $d_{Hyp}(V_i^n, V_0^n) \geq s/2$. Therefore there exists a constant C depending only on s and $\bar{\mu}$ such that $mod(T_i^n) \geq C$. Applying the Koebe distortion theorem finishes the proof. $\square$

Density Bounds

Let Γ_n be the semi-group generated by the set of off-critical puzzle pieces of level n. This is the same semi-group as we used in chapter 4. To each word $\gamma \in \Gamma_n$ of finite length is associated an open set in V_0^{n-1} as follows: let $\gamma = V_1 V_2 \ldots V_k$. Then z is in the open set associated with γ if for every $i \leq k$, $g_n^i(z) \in V_i$. As opposed to how we denoted this set in chapter 4, we shall let γ denote both the word and the open set.

For each word γ, the open set γ maps conformally onto V_0^{n-1}. Therefore there exists a subset $\gamma_e \subset \gamma$ which corresponds to the pre-image of G^n (the subscript "e" stands for "escaping"). There is also a pre-image of V_0^n in γ, which we shall denote by γ_r, (the "r" represents "returning").

Now define the two sets

$$E^n = G^n \cup \bigcup_{\gamma \in \Gamma_n} \gamma_e, \tag{87}$$

and

$$R^n = V_0^n \cup \bigcup_{\gamma \in \Gamma_n} \gamma_r.$$

We claim that E^n and R^n partition V_0^{n-1} up to a set of measure 0. To see this, let z be any point in $\cup V_i^{n-1}$ which is not in either E^n or R^n. It follows that z must correspond to an infinite-length word: that is, z remains in $\bigcup_{i \neq 0} V_i^n$ throughout its entire forward orbit. Therefore $0 \notin \omega(z)$ and so $z \notin R(g_1)$. From Lemma 2.5.7, we know that the set of all such $z \in K(g)$ has measure zero. Therefore the claim is proved, and E^n and R^n partition V_0^{n-1} up to a set of measure zero.

PROPOSITION 5.4.6. *Let $\gamma \in \Gamma^n$. If γ maps onto V_0^{n-1} with distortion bounded by a constant k, then*

$$\frac{\lambda(\gamma_e)}{\lambda(\gamma_r)} \geq k^2 \, \frac{\lambda(G^n)}{\lambda(V_0^n)} \quad . \tag{88}$$

Proof: The result follows from the Koebe distortion theorem. $\square$

For $U \subset V$, let $dens(U \mid V)$ be defined as

$$dens(U \mid V) = \frac{\lambda(U \cap V)}{\lambda(V)} \quad . \tag{89}$$

Proposition 5.4.7. *Suppose the for every V_j^n off-critical that $dist(V_j^n) \leq k$ for some k. Let g_n be as above, and let V_i^n be any off-critical puzzle. Then*

$$\frac{dens(E^n \mid V_i^n)}{dens(R^n \mid V_i^n)} \geq k^2 \frac{\lambda(G_0^n)}{\lambda(V_0^n)} \quad . \tag{90}$$

Proof: For any $\gamma^1, \gamma^2 \in \Gamma_n$, $\gamma_e^1 \bigcap \gamma_e^2 = \emptyset$, and $\gamma_r^1 \bigcap \gamma_r^2 = \emptyset$. Let Γ_n' denote the subset of Γ_n for which the open set $\gamma \subset V_i^n$. We now have

$$\begin{aligned} \frac{dens(E^n \mid V_i^n)}{dens(R^n \mid V_i^n)} &= \frac{\sum_{\gamma\in\Gamma'} \lambda(\gamma_e)}{\sum_{\gamma'\in\Gamma'} \lambda(\gamma_r')} \\ &= \sum_{\gamma\in\Gamma'} \left(\frac{\lambda(\gamma_e)}{\lambda(\gamma_r)} * \frac{\lambda(\gamma_r)}{\sum_{\gamma'\in\Gamma'} \lambda(\gamma_r')} \right) \\ &\geq k^2 \frac{\lambda(G^n)}{\lambda(V_0^n)} \quad . \end{aligned} \tag{91}$$

□

Shrinkage when not in a cascade of central returns

Suppose $g_n(0) \in V_l^n$. Define the subset G_H^{n+1} of G^{n+1} as the set

$$G_H^{n+1} = \left\{ z \in G^{n+1} : g_n(z) \in \gamma \text{ } for \text{ } some \text{ } \gamma \in \Gamma^n \text{ , } \gamma \neq \{V_l^n\} \right\} \quad . \tag{92}$$

Choose any $B > \bar{\mu} > 0$. Let us break the argument into three cases.

Case 1: $mod(V_0^{n-1} \setminus V_0^n) \leq B$.

Then by Proposition 5.4.4, every off-critical V_i^n maps onto V_0^{n-1} with distortion bounded by a constant depending only on the initial modulus. Therefore applying Proposition 5.4.7,

$$\frac{dens(E^n \mid V_i^n)}{dens(R^n \mid V_i^n)} \geq K \frac{\lambda(G_0^n)}{\lambda(V_0^n)} \quad . \tag{93}$$

with K dependent only on the initial principle modulus. Pulling this back to V_0^n, we have

$$
\begin{aligned}
\frac{\lambda(G^{n+1})}{\lambda(V_0^n)} &> \frac{\lambda(G_H^{n+1})}{\lambda(V_0^n)} \\
&= \frac{\lambda(G_H^{n+1})}{\lambda(\bigcup_{i\neq 0} V_i^n)} * \left\{\frac{\lambda(\bigcup_{(all\ i)} V_i^{n+1}) - \lambda(V_0^{n+1})}{\lambda(V_0^n)}\right\} \\
&\geq K\,\frac{\lambda(G^n)}{\lambda(V_0^n)} * \left\{\frac{\lambda(V_0^n) - \lambda(G^{n+1}) - \lambda(V_0^{n+1})}{\lambda(V_0^n)}\right\} \\
&= K\,\frac{\lambda(G^n)}{\lambda(V_0^n)} * \left\{1 - \frac{\lambda(G^{n+1})}{\lambda(V_0^n)} - \frac{\lambda(V_0^{n+1})}{\lambda(V_0^n)}\right\} \quad .
\end{aligned}
$$

Switching to δ_n and ρ_n notation, the previous inequality is equivalent to

$$\delta_{n+1} > K\,\frac{\lambda(G^n)}{\lambda(V_0^n)} * \left\{1 - \delta_{n+1} - \rho_{n+1}\right\} \tag{94}$$

which is the same inequality as in the statement of the theorem.

Now suppose $mod(V_0^{n-1} \setminus V_0^n) \geq B$. Let r, k be as in Proposition 5.4.5. There are two possibilities: either

$$\frac{\lambda(G^n \cap B_n(0,r))}{\lambda(G^n)} > \frac{1}{2} \quad , \tag{95}$$

or

$$\frac{\lambda(G^n \cap B_n(0,r))}{\lambda(G^n)} \leq \frac{1}{2} \quad . \tag{96}$$

Case 2: $mod(V_0^{n-1} \setminus V_0^n) > B$, and

$$\frac{\lambda(G^n \cap B_n(0,r))}{\lambda(G^n)} > \frac{1}{2} \quad . \tag{97}$$

Then for each $\gamma \in \Gamma^n$, there is a pre-image of $B_n(0,r)$ which maps onto $B_n(0,r)$ with distortion bounded by a constant dependent only on the initial principle modulus. Define the set $\gamma_{e'}$ as the set which maps onto $G^n \cap B_n(0,r)$. The hyperbolic ball $B_n(0,r)$ has definite space in V_0^{n-1}: that is

$$mod(V_0^{n-1} \setminus B_n(0,r)) \geq D$$

for a D dependent only on r. Therefore applying the Koebe distortion theorem to the mapping which sends γ onto V_0^{n-1}, we have

$$\frac{\lambda(\gamma_{e'})}{\lambda(\gamma_r)} \geq K\frac{\lambda(G^n \cap B_n(0,r))}{\lambda(V_0^n)} \geq \frac{K}{2}\frac{\lambda(G^n)}{\lambda(V_0^n)} \quad .$$

We define the subset $G_{H'}^{n+1}$ of G^{n+1} to be the set

$$G_{H'}^{n+1} = \left\{z \in G^{n+1} : g_n(z) \in V_i^n \ , \ and \ z \in \gamma_{e'} \ for \ some \ \gamma \in \Gamma^n\right\} \quad .$$

Applying the same argument to $G_{H'}^{n+1}$ as in the previous case, we have

$$\begin{aligned}\frac{\lambda(G^{n+1})}{\lambda(V_0^n)} &> \frac{\lambda(G_{H'}^{n+1})}{\lambda(V_0^n)} \\ &> K\,\frac{\lambda(G^n)}{\lambda(V_0^n)} * \left\{1 - \frac{\lambda(G^{n+1})}{\lambda(V_0^n)} - \frac{\lambda(V_0^{n+1})}{\lambda(V_0^n)}\right\} \quad .\end{aligned}$$

Case 3: mod$(V_0^{n-1} \setminus V_0^n) > B$, *and*

$$\frac{\lambda(G^n \cap B_n(0,r))}{\lambda(G^n)} \leq \frac{1}{2} \quad . \tag{98}$$

Let $\widetilde{\Gamma}^n$ denote the semi-group generated by all off-critical puzzle pieces $V_i^n \not\subset B_n(0,r)$. By Proposition 5.4.4, each $\gamma \in \widetilde{\Gamma}^n$ maps onto V_0^{n-1} with distortion bounded by a constant K depending on $\bar{\mu}$, $\bar{\sigma}$, and L_b^{tr}. For each γ, define $\gamma_{e''}$ to be the pre-image of $\{G^n \setminus B(0,r)\}$. Then it follows that

$$\begin{aligned}\frac{\lambda(\gamma_{e''})}{\lambda(\gamma_r)} &\geq K\frac{\lambda(G^n \setminus B(0,r))}{\lambda(V_0^n)} \\ &\geq \frac{K}{2}\frac{\lambda(G^n)}{\lambda(V_0^n)} \quad .\end{aligned} \tag{99}$$

Now let $\widetilde{E}^n$ be the set

$$\widetilde{E}^n = \{G^n \setminus B(0,r)\} \cup \bigcup_{\gamma \in \widetilde{\Gamma}^n} \gamma_{e''} \quad . \tag{100}$$

Applying Proposition 5.4.7, we have

$$\begin{aligned}\frac{\lambda(G^n)}{\lambda(R^n)} &\geq \frac{\lambda(\tilde{G}^n)}{\lambda(R^n)} \\ &\geq K\frac{\lambda(G^n \setminus B(0,r))}{\lambda(V_0^n)} \\ &\geq \frac{K}{2}\frac{\lambda(G^n)}{\lambda(V_0^n)} \quad .\end{aligned}$$

Applying the argument given for the previous case again results in the inequality

$$\frac{\lambda(G^{n+1})}{\lambda(V_0^n)} > K\,\frac{\lambda(G^n)}{\lambda(V_0^n)} * \left\{1 - \frac{\lambda(G^{n+1})}{\lambda(V_0^n)} - \frac{\lambda(V_0^{n+1})}{\lambda(V_0^n)}\right\} \quad .$$

for a K independent of the level.

This completes the proof of Lemma 5.4.1 when we are not in a cascade of central returns.

Cascades of Central Returns

Suppose now that we are in the tail of a long cascade of central returns. That is, we assume the cascade is longer than N_*. Because the cascade is longer than N_*, the principle moduli $\mu_k = mod(V_0^{k-1} \setminus \overline{V_0^k})$ will become very small, and the inequalities of the previous subsection will no longer have definite bounds. Our method for dealing with these problems is to go "around" the cascade.

Let m be the first level of a cascade of central returns, and let $m+q-1$ be the first non-central return level following the cascade. We can assume that $q \geq N_*$ is arbitrarily large (or take N_* to be larger).

Let V_i^{m+q} be the off-critical puzzle piece containing $g_{m+q}(0)$. The critical point then passes all the way up past m before returning to V_0^{m+1}. Therefore there exists $W' \supset V_i^{m+q}$ such that W' maps conformally onto V_0^{m-1}. Further, W' has a definite Koebe space about it, and thus maps with bounded distortion onto V_0^{m-1}. This puzzle piece W' is in fact the pull-back of V_0^{m-1} along the orbit of the critical point.

From the construction of W', it is clear that $g_{m+q}(0) \in W'$. Therefore pull-back W' by g_{m+1} to get the set $W \subset V_0^{m+q}$. See Figure 5.6. Because $g_{m+q}|_{V_0^{m+q}} = \phi \circ h$, with ϕ quadratic and h conformal with bounded distortion, it follows that W maps with bounded distortion onto V_0^{m-1}. That is, the map which sends the 0-symmetric set W onto V_0^{m-1} is a purely quadratic map composed with a conformal map with bounded distortion. It follows from the Koebe distortion theorem that the Lebesgue measures of sets will change by only a bounded amount under this map.

From the argument in the preceding paragraph, we get the inequality

$$\frac{dens(G^{m+q+1}|W)}{dens(V_0^{m+q+1}|W)} \geq A\frac{\delta_m}{\rho_m} \tag{101}$$

with a definite A. The size of A is dependent on $dens(V_0^{m+q+1}|W)$, which in turn is dependent on μ_m and q.

We now want to pass to the next level, $m+q+2$. To do this, we need to define a new open set associated to each word $\gamma \in \Gamma_{m+q+1}$. The new set, denoted by γ^W, is just the pre-image of W under the conformal map which sends γ onto V_0^{m+q}. We also need to define the sets γ_e^W, which is the pre-image of $G^{m+q+1}|W$, and $\gamma_r^W = \gamma_r$.

Because W has a definite Koebe space about it in V_0^{m+1}, it follows that γ^W maps with bounded distortion onto W. Therefore the analogous statements of Proposition 5.4.6 and Proposition 5.4.7 hold, with γ replaced by γ^W. We

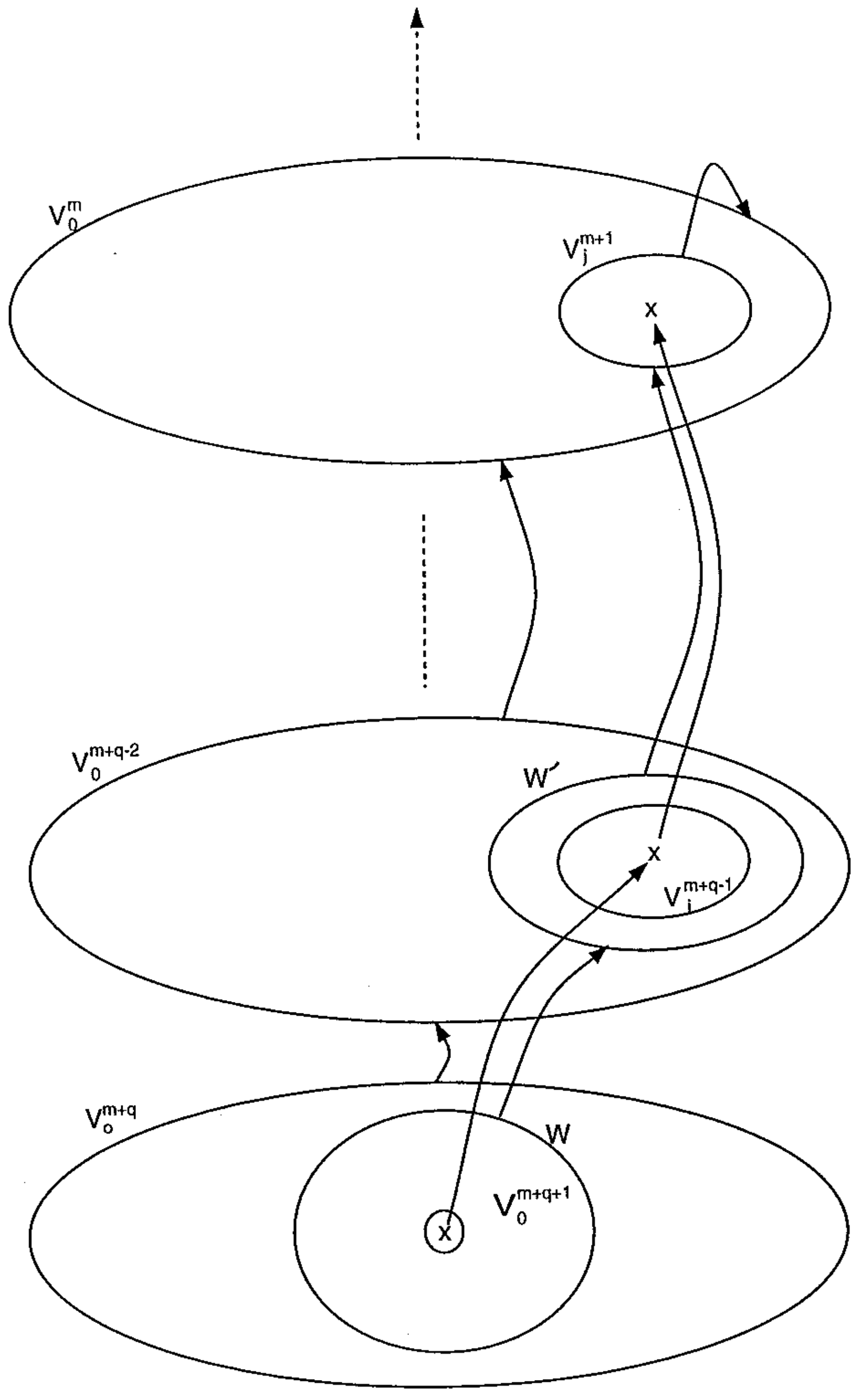

Figure 5.6. Long Cascades of Central Returns.

go further to define $E_W^{m+q+1} = \{G^{m+q+1}|W\} \bigcup \gamma_e^W$, and $R_W^{m+q+1} = R^{m+q+1}$, and $G_W^{m+q+2} = g_{m+q+1}^{-1}(E_W^{m+q+1})$. We now have

$$\begin{aligned} \frac{\delta_{m+q+2}}{1-\delta_{m+q+2}} &\geq \frac{dens(G_W^{m+q+2})}{1-dens(G_W^{m+q+2})} \\ &\geq a_1 * \frac{dens(E_W^{m+q+1}|V_0^{m+q})}{dens(R_W^{m+q+1}|V_0^{m+q})} \\ &\geq a_2 * \frac{dens(G^{m+q+1}|W)}{dens(V_0^{m+q+1}|W)} \\ &> a_2 * A * \frac{\delta_m}{\rho_m} \end{aligned} \tag{102}$$

with a definite a_1 and a_2, and A as above. It now follows that we have a definite bound on δ_{m+q+2} after an arbitrarily long cascade of central returns. Further, level $m + q + 2$ is not immediately following a cascade of central returns. Therefore we return to the construction in the non-central returns case.

This concludes the proof of Lemma 5.4.1.

5.5 Renormalization and Infinite Cascades of Central Returns

Because P_c is infinitely renormalizable, there exists some level n after which there will be an infinite length cascade of central returns. In other words,

$$\{g_{1,n}^k(0)\}_{k\in\mathbb{N}} \cap \{V_0^{1,n-1} \setminus V_0^{1,n}\} = \emptyset \quad . \tag{103}$$

By Lemma 5.3.2, an infinite cascade of central returns corresponds with a renormalizable map in the sense of Douady and Hubbard: that is, the restriction of $g_{1,n}$ to the central puzzle piece $V_0^{1,n}$ is a quadratic-like mapping (not generalized).

At this point, we need to define a first-returns map using a different method than the one we've been using up now. This is because if we continue renormalizing using the same method, we will not have definite bounds on the modulus of the critical annulus. Our estimates for Lemma 5.4.1 strongly relied on these definite bounds.

However, using the fact that $g_{1,n}|V_0^{1,n} : V_0^{1,n} \to V_0^{1,n-1}$ is a quadratic-like mapping, we can define a new first returns mapping which has all the properties we need.

LEMMA 5.5.1. *Let $g_{1,n} : \bigcup V_i^{1,n} \to V_0^{1,n-1}$ be a central first returns mapping, with $1, n$ being the first level of an infinite cascade of central returns. Further suppose*

a) the internal class $c(g_{1,n}|_{V_0^{1,n}})$ of the quadratic-like mapping $g_{1,n}|_{V_0^{1,n}} : V_0^{1,n} \to V_0^{1,n-1}$ ranges over a truncated secondary limb L_b^{tr},

b) the principle modulus $\mu_{1,n} = mod(V_0^{1,n-1} \setminus V_0^{1,n}) \geq L > 0$, and

c) $\frac{\lambda(\bigcup V_i^{1,n})}{\lambda(\bigcup V_j^{1,n-1})} \leq M < 1$ for a definite M.

Then there exists constants L', M' depending only on L, M, and L_b^{tr} such that if $V^{2,0}$ is the first puzzle piece of the initial Markov construction for $g_{1,n}|_{V_0^{1,n}} : V_0^{1,n} \to V_0^{1,n-1}$, and $g_{2,1} : \bigcup V_i^{2,1} \to V^{2,0}$ is its associated central first returns mapping, then

$$\frac{\lambda(\bigcup V_i^{2,1})}{\lambda(V^{2,0})} \leq M' < 1 \quad and \tag{104}$$

$$mod(V^{2,0} \setminus V_i^{2,1}) \geq L' > 0 \;\; for \;\; every \;\; V_i^{2,1} \quad . \tag{105}$$

Proof: For the quadratic-like mapping $g_{1,n}|_{V_0^{1,n}} : V_0^{1,n} \to V_0^{1,n-1}$, we can construct the initial Markov partition as in section 5.2. Let $\bigcup W_j^{2,1}$ denote the collection of puzzle pieces obtained from the initial Markov construction. By Lemma 5.2.6, there exists a constant ξ depending only on L_b^{tr} and L such that

$$\frac{\lambda(\{\bigcup W_i^{2,1}\} \cap V^{2,0})}{\lambda(V^{2,0})} \leq \xi < 1 \tag{106}$$

for all $W_i^{2,1}$ in the initial Markov partition associated to this quadratic-like mapping. However, the $W_i^{2,1}$ obtained from the initial Markov construction covers all of $R(P_c) \cap K(g_{1,n}|_{V_0^{1,n}})$, not all of $R(P_c)$. There will be points in $R(P_c) \cap V^{2,0}$ whose orbit escapes to some off-critical $V_i^{1,n}$ before returning to $V^{2,0}$.

Let $\bigcup X_i^{2,1}$ denote the set of all puzzle pieces which escape to some off-critical $V_i^{1,n}$ before returning to $V^{2,0}$. It follows that all puzzle pieces obtained from pulling back $V^{2,0}$ along orbits in $R(P_c)$ are either in the initial Markov partition $\bigcup W_j^{2,1}$ or are in $\bigcup X_i^{2,1}$. Let $G = V^{2,0} \setminus \bigcup W_j^{2,1}$. We need to show that there exists a constant ξ_1 depending only on L_b^{tr}, L, and M such that

$$\frac{\lambda(\{\bigcup X_i^{2,1}\} \cap V^{2,0})}{\lambda(V^{2,0} \setminus \{\bigcup W_j^{2,1}\})} \leq \xi_1 < 1 \quad . \tag{107}$$

The argument to show that the $\bigcup X_i^{2,1}$ are not too dense is very similar to the argument in the non-central cascade part of the proof from the previous section. We shall split the argument into the same three cases as in that proof.

Let us define a subset of G as follows:

$$G_H = \left\{ z \in G : \exists\ k\ g_{1,n}^k(z) \in \gamma_e \ for\ some\ \gamma \in \Gamma^{1,n} \right\} \quad . \tag{108}$$

Choose any $B > \bar{\mu} > 0$.

Case 1: $mod(V_0^{1,n-1} \setminus V_0^{1,n}) \leq B$.

Then by Proposition 5.4.4, every off-critical $V_i^{1,n}$ maps onto $V_0^{1,n-1}$ with distortion bounded by a constant depending only on the initial modulus. Therefore applying Proposition 5.4.7,

$$\frac{dens(E^n \mid V_i^{1,n})}{dens(R^n \mid V_i^{1,n})} \geq K\frac{\lambda(G^{1,n})}{\lambda(V_0^{1,n})} \quad . \tag{109}$$

with K dependent only on the initial principle modulus. Pulling this back to $V_0^{1,n}$, we have

$$\begin{aligned}
\frac{\lambda(G)}{\lambda(V_0^n)} &> \frac{\lambda(G_H)}{\lambda(V_0^{1,n})} \\
&= \frac{\lambda(G_H)}{\lambda(\bigcup_{i\neq 0} V_i^{1,n})} * \left\{\frac{\lambda(\bigcup_{(all\ i)} V_i^{1,n+1}) - \lambda(V_0^{1,n+1})}{\lambda(V_0^{1,n})}\right\} \\
&\geq K\ \frac{\lambda(G^{1,n})}{\lambda(V_0^{1,n})} * \left\{\frac{\lambda(V_0^{1,n}) - \lambda(G^{1,n}) - \lambda(V_0^{1,n+1})}{\lambda(V_0^{1,n})}\right\} \\
&= K\ \frac{\lambda(G^{1,n})}{\lambda(V_0^{1,n})} * \left\{1 - \frac{\lambda(G^{1,n})}{\lambda(V_0^{1,n})} - \frac{\lambda(V_0^{1,n+1})}{\lambda(V_0^{1,n})}\right\} \quad .
\end{aligned}$$

Therefore there is a lower bound on $\lambda(G)/\lambda(V_0^{2,1})$ depending only on L, M.

Now suppose $mod(V_0^{n-1} \setminus V_0^{1,n}) \geq B$. Let r, k be as in Proposition 5.4.5. There are two possibilities: either

$$\frac{\lambda(G^{1,n} \cap B_{1,n}(0,r))}{\lambda(G^{1,n})} > \frac{1}{2} \quad , \tag{110}$$

or

$$\frac{\lambda(G^{1,n} \cap B_{1,n}(0,r))}{\lambda(G^{1,n})} \leq \frac{1}{2} \quad . \tag{111}$$

Case 2: $mod(V_0^{n-1} \setminus V_0^{1,n}) > B$, and

$$\frac{\lambda(G^{1,n} \cap B_{1,n}(0,r))}{\lambda(G^{1,n})} > \frac{1}{2} \quad . \tag{112}$$

Then for each $\gamma \in \Gamma^n$, there is a pre-image of $B_{1,n}(0,r)$ which maps onto $B_{1,n}(0,r)$ with distortion bounded by a constant dependent only on the initial principle modulus. Define the set $\gamma_{e'}$ as the set which maps onto $G^{1,n} \cap B_{1,n}(0,r)$. The hyperbolic ball $B_{1,n}(0,r)$ has definite space in V_0^{n-1}: that is

$$mod(V_0^{n-1} \setminus B_{1,n}(0,r)) \geq D$$

for a D dependent only on r. Therefore applying the Koebe distortion theorem to the mapping which sends γ onto V_0^{n-1}, we have

$$\frac{\lambda(\gamma_{e'})}{\lambda(\gamma_r)} \geq K \frac{\lambda(G^{1,n} \cap B_{1,n}(0,r))}{\lambda(V_0^{1,n})} \geq \frac{K}{2} \frac{\lambda(G^{1,n})}{\lambda(V_0^{1,n})} \quad .$$

We define the subset $G_{H'}$ of G to be the set

$$G_{H'} = \left\{ z \in G : g_{1,n}^k(z) \in \gamma_{e'} \; for \; some \; \gamma \in \Gamma^n \right\} \quad .$$

Applying the same argument to $G_{H'}$ as in the previous case, we have

$$\begin{aligned} \frac{\lambda(G)}{\lambda(V_0^{1,n})} &> \frac{\lambda(G_{H'})}{\lambda(V_0^{1,n})} \\ &> K \frac{\lambda(G^{1,n})}{\lambda(V_0^{1,n})} * \left\{ 1 - \frac{\lambda(G^{1,n})}{\lambda(V_0^{1,n})} - \frac{\lambda(V_0^{1,n+1})}{\lambda(V_0^{1,n})} \right\} \quad . \end{aligned}$$

Case 3: $mod(V_0^{n-1} \setminus V_0^{1,n}) > B$, *and*

$$\frac{\lambda(G^{1,n} \cap B_{1,n}(0,r))}{\lambda(G^{1,n})} \leq \frac{1}{2} \quad . \tag{113}$$

Let $\tilde{\Gamma}^n$ denote the semi-group generated by all off-critical puzzle pieces $V_i^{1,n} \not\subset B_{1,n}(0,r)$. By Proposition 5.4.4, each $\gamma \in \tilde{\Gamma}^n$ maps onto V_0^{n-1} with distortion bounded by a constant K depending on $\bar{\mu}$, $\bar{\sigma}$, and L_b^{tr}. For each γ, define $\gamma_{e''}$ to be the pre-image of $\{G^{1,n} \setminus B(0,r)\}$. Then it follows that

$$\begin{aligned} \frac{\lambda(\gamma_{e''})}{\lambda(\gamma_r)} &\geq K \frac{\lambda(G^{1,n} \setminus B(0,r))}{\lambda(V_0^{1,n})} \\ &\geq \frac{K}{2} \frac{\lambda(G^{1,n})}{\lambda(V_0^{1,n})} \quad . \end{aligned} \tag{114}$$

Now let $\widetilde{E}^{1,n}$ be the set

$$\widetilde{E}^{1,n} = \{G^{1,n} \setminus B(0,r)\} \cup \bigcup_{\gamma \in \widetilde{\Gamma}^{1,n}} \gamma_{e''} \quad . \tag{115}$$

Applying Proposition 5.4.7, we have

$$\begin{aligned} \frac{\lambda(G^{1,n})}{\lambda(R^n)} &\geq \frac{\lambda(\widetilde{G}^n)}{\lambda(R^n)} \\ &\geq K \frac{\lambda(G^{1,n} \setminus B(0,r))}{\lambda(V_0^{1,n})} \\ &\geq \frac{K}{2} \frac{\lambda(G^{1,n})}{\lambda(V_0^{1,n})} \quad . \end{aligned}$$

Applying the argument given for the previous case again results in the inequality

$$\frac{\lambda(G)}{\lambda(V_0^{1,n})} > K \frac{\lambda(G^{1,n})}{\lambda(V_0^{1,n})} * \left\{ 1 - \frac{\lambda(G^{1,n})}{\lambda(V_0^{1,n})} - \frac{\lambda(V_0^{1,n+1})}{\lambda(V_0^{1,n})} \right\} \quad .$$

for a K independent of the level. □

With Lemma 5.5.1, we can give a precise definition of "sufficient combinatorial depth." A polynomial P_c will be said to have sufficient combinatorial depth between every renormalization level provided that there are enough non-central return levels for the generalized polynomial-like mapping $g_{n,1} : \bigcup V_i^{n,1} \to V^{n,0}$ so that

$$\frac{\lambda(\{\bigcup V_i^{n+1,1}\} \cap V^{n+1,0})}{\lambda(V^{n+1,0})} \leq \xi < 1 \tag{116}$$

for a constant ξ independent of the renormalization level. By Theorem 5.4.2, Lemma 5.4.1, and Lemma 5.5.1, the definition of sufficient combinatorial depth is dependent only on the truncated secondary limbs.

Let us now prove Theorem 5.1.3.

Proof: (of Theorem 5.1.3)

By a combination of Lemma 5.4.1 and Lemma 5.5.1, there exists a constant $\xi < 1$ such that

$$\frac{\lambda(\{\bigcup V^{m,n}\} \cap V_0^{m,n-1})}{\lambda(V_0^{m,n-1})} \leq \xi \tag{117}$$

for infinitely many values $m, n \in \mathbb{N}$. By Theorem 5.4.2, the long principle nest $V_0^{m,n}$ shrinks to the critical point 0. Therefore the critical point can not be a Lebesgue density point of the recurrent set $R(P_c)$.

Let $z \in R(P_c)$ be chosen arbitrarily. Let z_k be the first point in the forward orbit of z contained in $V_0^{m,n-1}$. Then $z_k \in \gamma_r$ for some $\gamma \in \Gamma^{m,n-1}$. Again by Lemma 5.4.1 and Lemma 5.5.1, there exists a constant $\xi < 1$ such that

$$\frac{\lambda(\{\bigcup V^{m,n}\} \cap \gamma)}{\lambda(\gamma)} \leq \xi \tag{118}$$

for infinitely many values of m, n. Here γ represents the puzzle piece, not the word. By Theorem 5.4.2 and Lemma 5.5.1, $mod(V_0^{m,n-1} \setminus \gamma) \geq \bar{\mu} > 0$ for a constant $\bar{\mu}$ independent of $m, n-1 \in \mathbb{N}$ (except when $m, n-1$ is in or immediately following a long cascade of central returns). Therefore γ pulls back along the orbit of z conformally and with distortion bounded by a constant which is independent of the level m, n. By Theorem 5.4.2, the nested sequence of puzzle pieces U_γ corresponding to the pull-backs shrinks to $\{z\}$. It now follows that z can not be a Lebesgue density point of the recurrent set $R(P_c)$.

Because z was chosen arbitrarily, it follows that there are no Lebesgue density points in $R(P_c)$, and therefore $\lambda(R(P_c)) = 0$. Combining this with Proposition 5.3.1 completes the proof of Theorem 5.1.3. $\square$

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