Conformal measures in polynomial dynamics

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Abstract of the Dissertation

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We study some geometric and analytic properties of polynomial dynamical system using conformal measures.

Our main results are:

1) We show that conformal measures are ergodic for polynomials having definite complex bounds. That implies ergodicity of conformal measures for all real quadratic polynomials.

2) We show that for certain classes of infinitely renormalizable polynomials (namely Sullivan and Lyubich polynomial), the infimum of all exponents for which there exists a conformal measure is equal to the Hausdorff dimension of the Julia set of the polynomial in consideration.

3) We show that for a certain class $\mathcal{C}$ of generalized polynomial-like maps (which includes Yoccoz, Lyubich, Sullivan and Fibonacci generalized polynomial-like map), the Teichmüller pseudo-distance is a distance. In other words, we show that if two generalized polynomial-like maps in $\mathcal{C}$ are quasi-conformally conjugated by conjugacies having arbitrarily small dilatation, then those maps are holomorphically conjugated.

Aos meus Pais e à Renata

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Chapter 1

Introduction

Our goal in this work is to study geometric and analytic properties of polynomial (or polynomial-like) dynamical systems using conformal measures.

Let $f : \mathbf{C} \to \mathbf{C}$ be a polynomial. Sullivan showed in [Sul80] that it is possible to construct a conformal measure for f with support on $J(f)$, the Julia set of f, for at least one positive exponent δ. By a conformal measure (or δ-conformal measure, to be more precise) we understand a Borel probability measure μ satisfying the following condition:

$$\mu(f(A)) = \int_A |Df(z)|^\delta d\mu(z),$$

whenever f restricted to the set A is one to one.

We say that μ is ergodic if $\mu(X) = 0$ or $\mu(X) = 1$ whenever we have $X = f^{-1}(X)$. Notice that usually when one talks about ergodicity of a measure it is assumed that the measure is invariant. In our case, due to the definition of conformal measure we are not dealing with an invariant measure but rather a quasi-invariant measure.

A quadratic polynomial is a Yoccoz polynomial if it is at most finitely many times renormalizable, without indifferent periodic points. A Lyubich polynomial is an infinitely many times renormalizable quadratic polynomial with sufficiently high combinatorics as described in [Lyu93] (see a precise definition of such polynomials in Subsection 2.2.2). A real unimodal polynomial is a polynomial of the form $f(z) = z^l + c$ where l is even and c is real.

Our goal in Chapter 3 is to show the following:

Theorem 1 *Let f be either a Yoccoz, a Lyubich or a real unimodal infinitely many times renormalizable polynomial. Let μ be any conformal measure for f. Then μ is ergodic.*

If we remember that any real quadratic polynomial can be holomorphically conjugate to a polynomial of the form $f(z) = z^2 + c$, c real, then we get the following Corollary from the previous Theorem:

Corollary 2 *If f is any real quadratic polynomial and μ is any conformal measure for f, then μ is ergodic.*

Ergodicity of conformal measures is known if f is expanding on $J(f)$ (see [Bow75], [Sul80] and [Wal78]). If $J(f)$ is disconnected (and f has just one critical point) then f is an expanding polynomial when restricted to $J(f)$. So we will assume that $J(f)$ is connected.

The situation studied in Chapter 3 is the complex counterpart for the ergodicity result in [BL91] where the Lebesgue measure is showed to be ergodic under S-unimodal maps.

Conformal measures are one of the tools for the study of the Hausdorff dimension of Julia sets. From the work of Bowen, Sullivan, and Walters (see [Bow75], [Sul80] and [Wal78]) we know that if f is expanding on $J(f)$, then the Hausdorff measure (which is finite and non-zero) is the only conformal measure on $J(f)$. In other words, there exists only one exponent δ for which a δ-conformal measure for f exists. This δ is the Hausdorff dimension of $J(f)$. Denker and Urbański showed in [DU91] (with a technical problem solved in [Prz93]) that the hyperbolic dimension of the Julia set of any rational function f is equal to $\inf\{\delta > 0 : \exists \ \ \delta - conformal\ measure\ for\ f\}$ (see section 4.1). Let us call this last quantity δ_{inf}.

Urbański showed in [Urb94] that if f is a rational function with no recurrent critical point then the Hausdorff dimension of the Julia set of f is equal to δ_{inf}. In this case critical points are allowed to be inside $J(f)$. In [Prz93] Przytycki showed this same result if f is a non-renormalizable quadratic polynomial. The goal of Chapter 4 is to extend Przytycki's result to some infinitely renormalizable polynomials: the Lyubich polynomials (see Section 2.2.2 for the definition of such polynomials) and Sullivan polynomials (real infinitely renormalizable quadratic polynomials with bounded combinatorics, see section 2.2.3). We use Przytycki's techniques in order to do that. In other words we show the following:

Theorem 3 *For any Lyubich or Sullivan polynomial f we have the following* *equality:*

$$\inf\{\delta : \exists\ \delta - conformal\ measure\ for\ f\} = \mathrm{HD}(J(f)),$$

where HD$(J(f))$ *stands for the Hausdorff dimension of the Julia set of f.*

We will also give a more geometric proof of the Denker-Urbański Theorem mentioned above concerning hyperbolic dimension and conformal measures for the specific case of Lyubich and Sullivan polynomials (see section 4.1).

In order to prove the renormalization conjecture for infinitely renormalizable real polynomials of bounded combinatorics, Sullivan in [Sul92] used a space of analytic maps where the renormalization operator is defined: the space of polynomial-like maps of degree two and bounded combinatorics modulo holomorphic conjugacies (see [DH85] and [dMvS93]). In this space it is possible to define a distance, d_T called the Teichmüller distance. This distance measures how far two polynomial-like maps are from being holomorphically conjugated (see Definition 5.1.4).

It is obvious from the definition that the Teichmüller distance is a pseudo-distance. It is not obvious that this pseudo-distance is actually a distance. To prove this is a distance it is necessary to show that if two polynomial-like maps f and g are such that $d_T(f,g) = 0$ then they are holomorphically conjugated (this can be viewed as a rigidity problem). Sullivan showed in [Sul92] that for real polynomials with connected Julia set this is true. He makes use of external classes of polynomial-like maps (as defined in [DH85]) to reduce the original rigidity problem to a rigidity problem of expanding maps of the circle, previously studied in [SS85]. The last result concerning expanding maps of the circle depends on the theory of Thermodynamical formalism.

In Chapter 5 we will show that the Teichmüller metric for a class $\mathcal{C}$ of generalized polynomial-like maps (see Definition 5.1.1) is actually a metric, as in the case Sullivan studied. The class $\mathcal{C}$ contains several important example of generalized polynomial-like maps, namely: Yoccoz, Lyubich, Sullivan and Fibonnaci. We will show the following:

Theorem 4 *Let f and g be two (generalized) polynomial-like maps belonging to the class $\mathcal{C}$. Suppose that $\mathrm{d_T}(f,g) = 0$. Then f and g are conformally conjugate on a neighborhood of their Julia sets.*

In our proof we can not use external arguments (like external classes). Instead we use hyperbolic sets inside the Julia sets of our generalized polynomial-like maps. Those hyperbolic sets will allow us to use our main analytic tool, namely Sullivan's rigidity Theorem for non-linear analytic repellers stated in Section 2.3.2.

Let us denote by m the probability measure of maximal entropy for the system $f : J(f) \to J(f)$. In [Lyu83] Lyubich showed how to construct a maximal entropy measure m for $f : J(f) \to J(f)$ for any rational function f. Zdunik classified in [Zdu90] exactly when $\mathrm{HD}(m) = \mathrm{HD}(J(f))$ and when $\mathrm{HD}(m) < \mathrm{HD}(J(f))$. The following is a particular case of Zdunik's result if we consider f as a polynomial. It is however an extension of Zdunik's result if f is a generalized polynomial-like map. The proof follows from the methods in Chapter 5.

Corollary 5 *If f belongs to the class $\mathcal{C}$ and m is the measure of maximal entropy for f, then* $\mathrm{HD}(m) < \mathrm{HD}(J(f))$.

Chapter 2

Background material

2.1 Construction of Conformal Measures

We will describe how to obtain conformal measures with support on $J(f)$ as proposed in [Sul80]. This construction was first described in [Pat76] for Fuchsian groups (see also [Nic91]). We will fix our attention to quadratic polynomials.

Let $f : \mathbf{C} \to \mathbf{C}$ be a quadratic polynomial and let $z \notin J(f)$. Define the following function:

$$\phi_\delta(z) = \sum_{n>0} \sum_{w \in f^{-n}(z)} \frac{1}{|D(f^n)(w)|^\delta}$$

There exists $\delta_0 = \inf\{\delta : \phi_\delta(z) < \infty\}$. One can show that $0 < \delta_0 \leq 2$. The number δ_0 is called the critical exponent of the map f. It's easy to see that if $\delta < \delta_0$ then $\phi_\delta(z) = \infty$ and if $\delta > \delta_0$ then $\phi_\delta(z) < \infty$. We say that f is of divergence type if $\phi_{\delta_0}(z) = \infty$ and of convergence type if $\phi_{\delta_0}(z) < \infty$.

Now let us consider the following family of probability measures:

$$\nu_\delta = \frac{\sum_{n>0}\sum_{w\in f^{-n}(z)} \frac{1}{|D(f^n)(w)|^\delta} m_w}{\phi_\delta(z)}$$

where m_w is the probability measure concentrated on w and $\delta > \delta_0$.

If f is of divergence type, then any weak limit of ν_δ as $\delta \to \delta_0$ is a conformal measure of exponent δ_0 concentrated on the Julia set of f.

If f is of convergence type, then one can find $h : \mathbf{R}_+ \to \mathbf{R}_+$ a continuous, monotone function with a proper asymptotic behavior near infinity (see [Nic91] for details on that) such that the new function

$$\tilde{\phi}_\delta(z) = \sum_{n>0}\sum_{w\in f^{-n}(z)} \frac{1}{(|D(f^n)(w)|^\delta} h(|D(f^n)(w)|^\delta)$$

has the same critical exponent, but with the divergence property at δ_0. Similarly to what was done before, we can create a family of probability measures $\tilde{\nu}_\delta$ such that any weak limit of such family as $\delta \to \delta_0$ is a conformal measure of exponent δ_0 concentrated on the Julia set of f.

2.2 Renormalization and combinatorics

2.2.1 Yoccoz polynomials

We will briefly describe how to construct the Yoccoz puzzle pieces for a quadratic polynomial. See [Hub] and [Mil91] for a complete exposition of such construction.

In this section we will just consider quadratic polynomials f with repelling periodic points.

We say that $g : U \to U'$ is a *quadratic – like map* if it is a double branched covering and U and U' are open topological discs with U properly contained in U'. In addition to that we require the filled in Julia set of g to be connected. By *filled in Julia set* of g we understand the set $\{z \in U : g^n(z)$ is defined for all natural numbers $n\}$. There are two fixed points of g inside its filled in Julia set. One of them, the dividing fixed point, disconnects the filled in Julia set of g in more than one connected component. The other does not. Usually the dividing fixed point is denoted by α.

Remember that f is *renormalizable* if there exist open topological discs $U \subset U'$ with $0 \in U$ with $R(f) : U \to U'$ being a quadratic-like map, $R(f) = f^k|_U$, with k the smallest natural number bigger than 1 satisfying this statement (we call k the *period of renormalization*). Here $R(f)$ stands for the renormalization of f. We can ask whether $R(f)$ is renormalizable or not and then define renormalizations of f of higher orders. So, each renormalization of f defines a quadratic polynomial-like map.

Let f be a degree two non-renormalizable polynomial and let G be the Green function of the filled Julia set of f. There are q external rays landing at the dividing fixed point of f, where $q \geq 2$. The q *Yoccoz puzzle pieces of depth zero* are the components of the topological disc defined by $G(z) < G_0$, where G_0 is any fixed positive constant, cut along the q external rays landing at the dividing fixed points. We denote $Y^0(x)$ the puzzle piece of depth zero containing x. We define the *puzzle pieces of depth n* as being the connected components of the pre-images of any puzzle piece of depth zero under f^n. Again, if x is an element of a given puzzle piece of depth n we denote such puzzle piece by $Y^n(x)$.

Suppose now that f is at most finitely renormalizable without indifferent periodic points. Let α be the dividing fixed point of the last renormalization of f. Let G be the Green function of the filled in Julia set of f. In that case we define the puzzle pieces of depth zero as being the components of the topological disc $G(z) < G_0$, G_0 a positive constant, cut along the rays landing at all points of the f-periodic orbit of α. As before we define the puzzle pieces of depth n as being the connected components of the pre-images under f^n of the puzzle pieces of depth zero. The puzzle piece at depth n containing x is denoted by $Y^n(x)$.

We will consider the Yoccoz puzzle pieces as open topological discs. Under this consideration the Yoccoz partition will be well defined over the Julia set of the polynomial f minus the set of pre-images of the dividing fixed point of the last renormalization of f (which is f itself in the non-renormalizable case).

A quadratic polynomial is a *Yoccoz polynomial* if it is at most finitely renormalizable without indifferent periodic points. We will need the following result:

Theorem 2.2.1 (Yoccoz) *If f is a Yoccoz polynomial then $\bigcap_{n\geq 0} Y^n(x) = \{x\}$ for any x where the Yoccoz partition is defined.*

2.2.2 Lyubich polynomials

Let us pass to the second class of polynomials that we will be considering, namely the Lyubich polynomials. See [Lyu93] for a detailed exposition on this matter. We will need some technical definitions.

Let us start with a quadratic polynomial f without indifferent periodic points. Given a Yoccoz puzzle piece Y_i^n of f and a point x such that $f^j(x)$ belongs to Y_i^n. We define the *pull back of* Y_i^n *along the orbit of* x as being the only connected component of $f^{-j}(Y_i^n)$ containing x. If moreover x belongs to Y_i^n and j is minimal and non-zero, then we say that j is *the first return time of* x *to* Y_i^n. A puzzle piece is said to be critical if it contains the critical point. Notice that if we pull back a critical puzzle piece $Y^n(0)$ along the first return of the critical point to $Y^n(0)$ we get a new critical puzzle piece.

Under certain conditions (referred in [Lyu93] as not Douady-Hubbard immediately renormalizable), it is possible to find a first critical puzzle piece such that the closure of its pull back along the first return of the critical point to itself is properly contained in itself. Let us denote such puzzle piece $V^{0,0}$. We say that this is our puzzle piece of level zero. The pull back of $V^{0,0}$ along the first return of the critical point to $V^{0,0}$ will be denoted $V^{0,1}$. We keep repeating this procedure: define $V^{0,t+1}$, the puzzle piece of level $t+1$, as being the pull back of $V^{0,t}$, the puzzle piece of level t, along the first return of the critical point to $V^{0,t}$. This procedure stops if the critical point does not return to a certain critical puzzle piece. If we assume that the critical point is combinatorially recurrent, then we can repeat this procedure forever, so let us assume that (as the opposite case is a well understood case). The collection $V^{0,t}$ for t being a natural number is the *principal nest of the first renormalization level.*

Now we have a sequence of first return maps $f^{l(t)} : V^{0,t+1} \to V^{0,t}$. By definition $V^{0,0}$ properly contains $V^{0,1}$. This implies that each $V^{0,t}$ properly contains $V^{0,t+1}$. It is also easy to see that each $f^{l(t)} : V^{0,t+1} \to V^{0,t}$ is a quadratic-like map.

We say that $f^{l(t)} : V^{0,t+1} \to V^{0,t}$ is a *central return* or that $t+1$ is a *central return level* if $f^{l(t)}(0)$ belongs to $V^{0,t+1}$. A *cascade of central returns* is a set of subsequent central return levels. More precisely, a cascade of central returns is a collection of central return levels $t = t_0 + 1, ..., t_0 + N$ followed by a non-central return at level $t_0 + N + 1$. In this case we say that the above cascade of central returns has length N. We could also have an infinite cascade of central returns. Notice that with the above terminology a non-central return level is a cascade of central return of length zero.

It is possible to show that the principal nest of the first renormalization level ends with an infinite cascade of central returns if and only if f is renormalizable. In that case, denote the first level of this infinite cascade of central returns by $t(0)+1$. Then we define the first renormalization $R(f)$ of f as being the quadratic-like map $f^{l(t(0))} : V^{0,t(0)+1} \to V^{0,t(0)}$. The filled-in Julia set of $R(f)$ is connected (it is also possible to show that $\bigcap V^{0,n} = J(R(f))$). Again we can find the dividing fixed point of the Julia set of $R(f)$, some external rays landing at it and define new puzzle pieces over the Julia set of $R(f)$. The rays landing at the new dividing fixed point are not canonically defined (remember that $R(f)$ is a polynomial-like map). We are not taking the external rays of the original polynomial. Instead we need to make a proper selection of those rays to be able to state the Theorem at the end of this Subsection (see [Lyu93]). As before we can construct the principal nest for $R(f)$, provided that $R(f)$ is not Douady-Hubbard immediately renormalizable. The elements of this new principal nest are denoted by $V^{1,0}, V^{1,1}, ..., V^{1,t}, ...$ and the nest is called the *principal nest of the second renormalization level.* If this new principal nest also ends in an infinite cascade of central returns, we repeat the procedure just described and construct a third principal nest. We repeat this process as many times as we can.

Now we define the *principal nest* of the polynomial f as being the set of critical puzzle pieces

$$V^{0,0} \supset V^{0,1} \supset ... \supset V^{0,t(0)} \supset V^{0,t(0)+1} \supset V^{1,0} \supset V^{1,1} \supset ..., V^{1,t(1)} \supset$$

$$\supset V^{1,t(1)+1} \supset ... \supset V^{m,0} \supset V^{m,1} \supset ... \supset V^{m,t(m)} \supset V^{m,t(m)+1} \supset ...$$

In order to go ahead with the definition of the class of polynomials we are interested in, we need the notion of a truncated secondary limb. A *limb* in the Mandelbrot set M is the connected component of $M \setminus \{c_0\}$ not containing 0, where c_0 is a bifurcation point on the main cardioid. If we remove from the limb a neighborhood of its root c_0, we get a *truncated limb.* A similar object corresponding to the second bifurcation from the main cardioid is a *truncated secondary limb.*

A Lyubich polynomial is defined as being an infinitely many times renormalizable polynomial satisfying the two following properties:

(i) First select in the Mandelbrot set a finite number of truncated secondary limbs. We require all the quadratic-like renormalizations to be in these limbs.

(ii) We also require that in between two quadratic-like renormalization levels we have a sufficiently high number of non-central returns (called height) which depends on the a priori selection of the limbs.

Such class of polynomials was originally introduced and studied in [Lyu93].

Before we go to the main theorem that, we will be using concerning Lyubich polynomials we will need one last notation. If $V^{i,k}$ is an element of the principal nest then we denote by $n(k)$ the number of central cascades in between $V^{i,0}$ and $V^{i,k}$. Remember that a non-central return is viewed as a central cascade of length zero.

The following result in [Lyu93] will allow us to create the "Koebe space" for Yoccoz and Lyubich polynomials:

Theorem 2.2.2 (Lyubich) *The principal modulus* $\mathrm{mod}(V^{i,k} \setminus V^{i,k+1})$ *grows linearly with n(k) for any Lyubich polynomial, if* $V^{i,k}$ *is on top of a cascade of central returns. Moreover,* $\mathrm{mod}(V^{i,0} \setminus V^{i,1}) \geq c \geq 0$.

2.2.3 Sullivan polynomials

Let $f(z) = z^2 + c$ be a quadratic polynomial with c being a real number. We say that f is a Sullivan polynomial if it is infinitely many times renormalizable and has bounded combinatorial type. See [dMvS93] for a more detailed definition.

We will describe one specific example of a Sullivan polynomial, namely the Feigenbaum map. For this map we will describe the construction of the

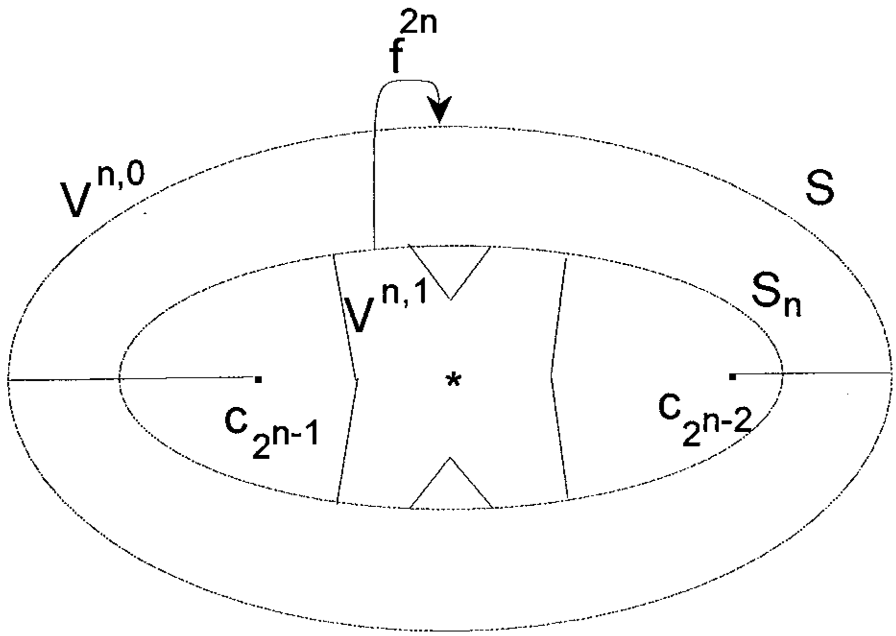

Figure 2.1: Renormalization of the Feigenbaum polynomial

principal nest as we did with Lyubich polynomials (see [HJ]). A similar construction can be carried out for any other Sullivan polynomial following the Feigenbaum example.

The Feigenbaum map $F(z) = z^2 + c_F$ is the only infinitely renormalizable real polynomial with all periods of renormalization equal to 2. Let $c_i = F^i(0)$ and $K_{n,0} = [c_{2^n}, c_{2^{n+1}}]$. Denote $K_{n,i} = F^i(K_{n,0})$, $i = 1, ..., 2^n - 1$. Let us consider a homeomorphism $h_n = F^{2^n-1} : K_{n,1} \to K_{n,0}$ with inverse g_n. Let $P_{n,0} = \mathbf{C} \setminus (\mathbf{R} \setminus [c_{2^n-1}, c_{2^n-2}])$. Then g_n can be extended to a schlicht function G_n on $P_{n,0}$. Let $P_{n,-1} = G_n(P_{n,0})$ and P_{n-2} be the closure of the union of the pre-image of $P_{n,-1}$ under the two inverse branches of F. Let S be a large equipotential of the Julia set of F and $S_n = F^{-2^n}(S)$. Define $V^{n,1}$ as being the region in P_{n-2} bounded by S_n and $V^{n,0}$ as being the region in $P_{n,0}$ bounded by S. One can see that $F^{2^n} : V^{n,1} \to V^{n,0}$ is a quadratic-like map. This is the quadratic-like renormalization of the Feigenbaum map (see Figure 2.1). We have the following properties (which follows from Sullivan's complex a priori bounds, see [dMvS93]):

(i) $\mathrm{mod}(V^{n,0} \setminus V^{n,1}) \geq c > 0$

(ii) $\mathrm{diam}(V^{n,0}) \to 0$ as $n \to \infty$

2.2.4 Infinitely renormalizable real unimodal polynomials

Let $f(z) = z^l + c$ be an infinitely renormalizable polynomial with c a real number and l an even integer (the class of Sullivan polynomials is a subclass of this class of polynomials). We say that an infinitely renormalizable polynomial admits *complex bounds* if each renormalization of f admits a polynomial-like restriction with modulus bounded away from zero. In [LvS95] Levin and van Strein showed that this class of polynomials admits complex bounds (see next paragraph). The works [LY95] together with [Lyu93] also give a proof of complex bounds in the case $l = 2$. This sort of result was first shown by Sullivan in case f is a Sullivan polynomial (see [dMvS93] and previous Subsection).

Theorem 2.2.3 *Let $f(z) = z^l + c$ be an infinitely renormalizable real polynomial of even degree l. If a_n is the period of the n^{th} renormalization of f, then there exist topological discs $V^{n,0}$ and $V^{n,1}$ such that:*

(i) $0 \in V^{n,1}$;

(ii) $\mathrm{cl}(V^{n,1}) \subset V^{n,0}$;

(iii) $\mathrm{mod}(V^{n,0} \setminus V^{n,1}) \geq c > 0$;

(iv) $f^{a_n} : V^{n,1} \to V^{n,0}$ *is a polynomial-like map of degree* l;

(v) $\mathrm{diam}(V^{n,0}) \to 0$ *as* $n \to \infty$.

2.2.5 Unbranched renormalization

We take the following from [McM94].

Definition 2.2.4 Let f *be a polynomial which is renormalizable and admits the polynomial-like map* $f^n : U_n \to V_n$ *as renormalization. We say that this renormalization is unbranched if* $V_n \bigcap \mathrm{Post}(f) = \mathrm{Post}_n$. *Here* $\mathrm{Post}(f)$ *stands for the post-critical set of* f *and* Post_n *for the post-critical set of the renormalization of* f, $f^n : U_n \to V_n$.

Let us fix some notations: $V_n(i)$ stands for $f^i(V_n)$ and if X is any subset of the complex plane, $X' = -X$. We will need the following basic fact concerning unbranched renormalization:

Proposition 2.2.5 *If* $f^n : U_n \to V_n$ *is an unbranched renormalization then* $V_n'(i)$ *is disjoint from the post-critical set of* f *for* $i \neq n$.

Proof. The small post-critical sets are disjoint. We also have the inclusion: $f(\mathrm{Post}_n(i-1)) \subset \mathrm{Post}_n(i)$. So when $f^n : U_n \to V_n$ is unbranched we have $V_n(i) \bigcap \mathrm{Post}(f) = \mathrm{Post}_n(i)$. Since $V_n(i)$ and $V_n'(i)$ have the same image under

f, any point in $V_n'(i) \bigcap \mathrm{Post}(f)$ must lie in $\mathrm{Post}_n(i)$. But $V_n(i)$ and $V_n'(i)$ are disjoint whenever we have $i \neq n$. □

Lemma 2.2.6 *If f is an infinitely renormalizable real unimodal polynomial or a Lyubich polynomial, then infinitely many renormalizations of f are unbranched and admit complex bounds.*

Proof. For Sullivan polynomials we can find a nest of domains of renormalization in the complex plane (admitting complex bounds in all renormalization levels) such that their intersections with the real line is a nest $K_{n,0}$ of symmetric f-periodic intervals around the critical point (see Subsection 2.2.3 and [HJ]). The result follows because the intersection of the orbit of those complex domains with the real line is the same as the orbits of the intervals $K_{n,0}$. It is possible to take the intervals $K_{n,0}$ such that $K_{n,0}, f(K_{n,0}), ..., f^{a_n-1}(K_{n,0})$ are disjoint intervals, where a_n is the n^{th} period of renormalization of f. The Lemma now follows for Sullivan polynomials because the critical orbit is contained inside the real line.

For all other infinitely renormalizable real unimodal polynomials we can make a similar construction. Some extra care should be taken in this general case where we need to exclude the renormalization levels close to parabolic bifurcations. See [LvS95] and [LY95] for that.

For Lyubich polynomial we used puzzle pieces to build the renormalized polynomial-like maps. The Lemma follows from the Markov property of puzzle pieces and from Theorem 2.2.2. □

2.3 Some facts from thermodynamical formalism

2.3.1 Classical results

We refer the reader to [Bow75] for a detailed introduction to the classical theory of thermodynamical formalism. See also [PU] for a more modern exposition of the subject. The goal of this Section is to introduce notations and classical facts.

Definition 2.3.1 Let f be any conformal map. In what follows by a hyperbolic or expanding set for f we understand, as usual, a closed set X such that $f(X) \subset X$ and $|D(f^n)(x)| \geq c\kappa^n$, for any x in X and for $n \geq 0$, where $c > 0$ and $\kappa > 1$.

Consider a hyperbolic set X as defined above. If $\phi : X \to \mathrm{R}$ is a Hölder continuous function, we say that the probability measure μ_ϕ is a *Gibbs measure* associated to ϕ if:

$$\sup_{\nu}\{\mathrm{h}_\nu(f) + \int_X \phi d\nu\} = \mathrm{h}_{\mu_\phi}(f) + \int_X \phi d\mu_\phi$$

where $\mathrm{h}_\nu(f)$ is the entropy of f with respect to the measure ν and the supremum is taken over all ergodic probability measures ν of the system $f : X \to X$. In this context we call ϕ a *potential function.* The *pressure v* of the potential ϕ is denoted $\mathrm{P}(\phi)$ and defined as $\mathrm{P}(\phi) = \sup_\nu\{\mathrm{h}_\nu(f) + \int_X \phi d\nu\}$, the supremum is taken over all ergodic probability measures.

The following Theorem assures us the existence of Gibbs measures.

Theorem 2.3.2 (Ruelle-Sinai) *Given $f : X \to X$ hyperbolic and a Hölder continuous potential $\phi : X \to \mathrm{R}$, there exists one Gibbs measure μ_ϕ associated to this potential. Moreover this Gibbs measure is unique.*

One needs to know when two potentials generate the same Gibbs measure. We have the following definition and Theorem to take care of that:

Definition 2.3.3 We say that two real valued functions $\phi, \psi : X \to \mathrm{R}$ are cohomologous (with respect to the system $f : X \to X$) if there exists a continuous function $s : X \to \mathrm{R}$ such that $\phi(x) = \psi(x) + s(f(x)) - s(x)$.

Theorem 2.3.4 (Livshitz) *Given $f : X \to X$ hyperbolic and two Hölder continuous functions $\phi, \psi : X \to \mathrm{R}$, the following are equivalent:*

(i) $\mu_\phi = \mu_\psi$;

(ii) ϕ and ψ are cohomologous;

(iii) For any periodic point x of $f : X \to X$ we have:

$$\sum_{i=0}^{n-1} \phi(f^i(x)) - \sum_{i=0}^{n-1} \psi(f^i(x)) = n(\mathrm{P}(\phi) - \mathrm{P}(\psi))$$

where n is the period of x.

Of special interest is the one parameter family of potential functions given by $\phi_t(x) = -t\log(|Df(x)|)$. Notice that by the definition of hyperbolic set, the functions ϕ_t are Hölder continuous. One can study the function $\mathrm{P}(t) = \mathrm{P}(\phi_t)$. Here are some properties of this function:

(i) $\mathrm{P}(t)$ is a convex function;

(ii) $\mathrm{P}(t)$ is a decreasing function;

(iii) $\mathrm{P}(t)$ has only one zero exactly at $t = \mathrm{HD}(X)$;

(iv) $\mathrm{P}(0) = \mathrm{h}(f) =$ topological entropy of $f : X \to X$.

One can show that if $f : X \to X$ is hyperbolic, as we are assuming, then the Hausdorff measure of X is finite and non-zero. That is because one can show that the Gibbs measure associated to the potential given by $\phi_{\mathrm{HD}(X)} = -\mathrm{HD}(X) \cdot \log(|Df|)$ is equivalent to the Hausdorff measure of X. Notice that $\mathrm{P}(\phi_{\mathrm{HD}(X)}) = 0$. The Gibbs measure $\mu_{\phi_0 - \mathrm{P}(\phi_0)}$ associated to the potential $\phi_0 - \mathrm{P}(\phi_0) \equiv -\mathrm{P}(\phi_0) =$ constant is the measure of maximal entropy for the system $f : X \to X$. Notice that $\mathrm{P}(\phi_0 - \mathrm{P}(\phi_0)) = 0$. Instead of denoting this measure by $\mu_{\phi_0 - \mathrm{P}(\phi_0)}$ we will simply write μ_{const}.

Let us denote $m = \mu_{const}$ and $\nu = \mu_{\phi_{\mathrm{HD}(X)}}$. The following is a consequence of the previous paragraph and Theorem 2.3.4.

Corollary 2.3.5 *Let $f : X \to X$ be hyperbolic. The measures m and ν are equal if and only if there exists λ such that for any periodic point x of $f : X \to X$ we have $|Df^n(x)| = \lambda^n$, where n is the period of x.*

2.3.2 Sullivan's rigidity Theorem

We refer the reader to [Sul86] and [PU] in order to get a complete proof of the results in this Section.

Definition 2.3.6 An invariant affine structure for the system $f : X \to X$ is an atlas $\{(\sigma_i, U_i)\}_{i \in I}$ such that $\sigma_i : U_i \to \mathrm{C}$ is conformal injection for each i

where $X \subset \bigcup_i U_i$ and all the maps $\sigma_i \sigma_s^{-1}$ and $\sigma_i f \sigma_s^{-1}$ are affine (whenever they are defined).

Definition 2.3.7 We say that $f : X \to X$ is topologically transitive if there exists a dense orbit inside X.

Suppose that $f : X \to X$ is hyperbolic. Then transitivity is equivalent to the following: for every non-empty set $V \subset X$ open in X, there exists $n \geq 0$ such that $\bigcup_n f^n(V) = X$. That is due to the existence of Markov partition for $f : X \to X$.

Lemma 2.3.8 (Sullivan) *Let $f : X \to X$ be a transitive and hyperbolic system. The potential $\log(|Df|)$ is cohomologous to a locally constant function if and only if $f : X \to X$ admits an invariant affine structure.*

We call $f : X \to X$ a *non-linear system* if it does not admit an invariant affine structure. Let $g : Y \to Y$ be another system and let $h : X \to Y$ be a conjugacy between f and g. Then we say that h *preserves multipliers* if for every f-periodic point of period n we have $|Df^n(x)| = |Dg^n(h(x))|$.

Theorem 2.3.9 (Sullivan) *Let $f : X \to X$ and $g : Y \to Y$ be two non-linear transitive and hyperbolic systems. Suppose that f and g are conjugated by a homeomorphism $h : X \to Y$ preserving multipliers. Then h can be extended to an analytic isomorphism from a neighborhood of X onto a neighborhood of Y.*

Chapter 3

Ergodicity of conformal measures

3.1 Density Estimates

From now on μ will denote a conformal measure with exponent δ concentrated on the Julia set of f.

The analytic tool that we will use is the well known Koebe distortion Theorem:

Theorem 3.1.1 (Koebe) *Let $A \subset B$ be two topological discs contained in the complex plane. Suppose that f is univalent when restricted to B. Also suppose that $B \setminus A$ is a topological annulus with positive modulus m. Then*

$$\frac{1}{K} \leq \frac{|Df(z_1)|}{|Df(z_2)|} \leq K$$

for all z_1 and z_2 in A, where the constant K depends only on the number m.

Let f be a Yoccoz polynomial. Notice that if a periodic point of f in $J(f)$ is expanding then the set of all its pre-images has zero μ-measure. As we used just expanding periodic points to construct puzzle pieces, given any closed subset X of $J(f)$, we can create a cover K_i of X (up to a set of zero measure) built up by puzzles pieces and with $\lim \mu(K_i) = \mu(X)$. This follows from Yoccoz Theorem (see Theorem 2.2.1) and the regularity of conformal measures.

Definition 3.1.2 The density of a set X inside a set Y is defined as the following: $\mathrm{dens}(X|Y) = \frac{\mu(X \bigcap Y)}{\mu(Y)}$.

Lemma 3.1.3 *Let f be a Yoccoz polynomial and $X \subset J(f)$ be any measurable subset. If $\mu(X) > 0$, there is $x \in X$ such that* $\limsup(\mathrm{dens}(X|Y^n(x))) = 1$.

Proof. Assume $\mu(X) > 0$. If X is not closed, take $K \subset X$ compact with $\mu(X \setminus K)$ small. Notice that $\mathrm{dens}(X|Y^n(x)) \geq \mathrm{dens}(K|Y^n(x))$ for any $Y^n(x)$. For all $\varepsilon > 0$, there exists $i(\varepsilon)$, such that $1 - \varepsilon \leq \frac{\mu(K \bigcap K_i)}{\mu(K_i)} \leq 1$ if $i > i(\varepsilon)$ (remember that K_i are the covers of X made out of puzzle pieces). So we have for i big $\mathrm{dens}(K|K_i) = \frac{\mu(K \bigcap K_i)}{\mu(K_i)} \geq 1 - \varepsilon$. As K_i is the union of puzzle pieces we can certainly find a puzzle piece in K_i, say $Y^{n(i)}(x_i)$ such that $\mathrm{dens}(K|Y^{n(i)}(x_i)) \geq 1 - \varepsilon$. Now replacing X by $X \bigcap Y^{n(i)}(x_i)$ and repeating this argument we will end up with the desired result. □

Definition 3.1.4 The point $x \in X$ obtained in the previous Lemma is called a weak density point of X.

Proposition 3.1.5 *Let $A \subset B$ be two μ-measurable subsets of the complex plane. Suppose that f restricted to an open neighborhood of B is one to one. Also suppose that there exists a positive constant K such that*

$$\frac{1}{K} \leq \frac{|Df(z_1)|}{|Df(z_2)|} \leq K$$

for all z_1 *and* z_2 *in* B*, then*

$$\frac{1}{K^\delta}\text{dens}(A|B) \leq \text{dens}(f(A)|f(B)) \leq K^\delta \text{dens}(A|B).$$

Proof. Follows from the definition of conformal measure and the definition of dens$(A|B)$. □

If U is a subset of the complex plane, we will denote by U^c the complement of U inside the complex plane.

Lemma 3.1.6 *Let* f *be any Yoccoz and* μ *a conformal measure for* f*. Let* U *be any neighborhood of the critical point. Then the set*

$$\{x \in \mathbf{C} : f^n(x) \in U^c,\ for\ all\ positive\ n\}$$

has zero μ*-measure.*

Proof. It is enough to show this Lemma for $Y^i(0)$ because by Yoccoz's Theorem any neighborhood of the critical point contains some $Y^i(0)$, for i sufficiently big. Suppose that the set $A = \{x \in \mathbf{C} : f^n(x) \in Y^i(0)^c$, for all n positive $\}$ has positive measure, for some i fixed. Then this set has a point of weak density x, according to Lemma 3.1.3. So we can find some sequence $n(j) \to \infty$ such that dens$(A|Y^{n(j)}(x)) \to 1$.

Notice that $f^{n(j)-i}(Y^{n(j)}(x))$ is a puzzle piece of depth i and none of the puzzle pieces $Y^{n(j)}(x), f(Y^{n(j)}(x)) \ldots, f^{n(j)-i}(Y^{n(j)}(x))$ contains the critical point. That is because of the Markov property of puzzle pieces and the fact that $Y^{n(j)}(x)$ contains elements of the set A. So for all $Y^{n(j)}(x)$, $f^{n(j)-i}(Y^{n(j)}(x))$ is a puzzle piece of depth i distinct from $Y^i(0)$. As there exist just finitely many puzzle pieces of depth i then there is a fixed puzzle piece $Y^i(y)$ (distinct from the one containing the critical point) such that $f^{n(j)-i}(Y^{n(j)}(x)) = Y^i(y)$ for infinitely many $n(j)$. Passing to a subsequence and keeping the same notation we will assume that the above property is true for all $n(j)$.

We will construct a neighborhood of $Y^i(y)$ where the inverse branch $f^{-(n(j)-i)}$ along the orbit $x, f(x), \ldots, f^{n(j)-i}(x)$ is defined as an isomorphism (remember that x is a weak density point of A).

Let $i_1 > i$ such that $\text{mod}(Y^i(0) \setminus Y^{i_1}(0))$ is positive. This is possible by Yoccoz's Theorem.

The boundary of $Y^i(y)$ is composed by pairs of external rays landing at points in the Julia set and equipotentials. The intersection of this boundary with the Julia set is finite. Let z be a point of such finite intersection. Consider all puzzle pieces of depth i_1 having z as vertex. The closure of the union of those puzzle pieces is a neighborhood of z in the plane. Let us call such neighborhood V_z. Notice that each equipotential and the pieces of external rays landing at z outside V_z are at some definite distance from the Julia set. Take a small tubular neighborhood (not intersecting the Julia set) of each one of the equipotentials and pieces of external rays belonging to the boundary of $Y^i(y)$. Now we define the neighborhood N of $Y^i(y)$ as being the union of each V_z with all tubular neighborhoods described above and $Y^i(y)$ itself (see Figure 3.1). Notice that we can make N into a topological disc if i_1 is big and the tubular neighborhoods small. Also notice that since the distance between the boundaries of $Y^i(y)$ and N is strictly positive, we get that $\mathrm{mod}(N \setminus Y^i(y))$ is strictly positive.

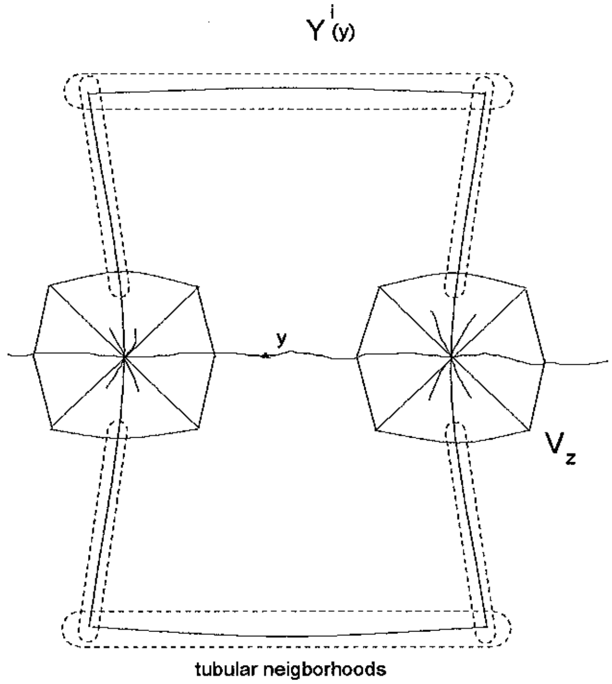

Figure 3.1: Construction of the neighborhood N of $Y^i(y)$

Now let us prove that we can pull N back isomorphically along the orbit $x, \ldots, f^{n(j)-i}(x)$ for any $n(j)$.

The pull back of $Y^i(y)$ cannot hit the critical point because $Y^{n(j)}(x)$ contains points in the set A (by the Markov property of puzzle pieces and the f-invariance of the set A). None of the pull backs of the tubular neighborhoods can hit the critical point because those neighborhoods are outside the Julia set. The pull backs of the sets V_z do not touch the critical point because each puzzle piece in each V_z is at depth i_1. So if it would enter $Y^i(0)$ when we pull it back it would touch the boundary of $Y^i(0)$ because the pull back of $Y^i(y)$ never enters $Y^i(0)$. By the choice of i_1 (i.e., $\mathrm{mod}(Y^i(0) \setminus Y^{i_1}(0))$ is positive) the backward orbit of V_z can not hit the critical point.

So we can pull N back isomorphically along the orbit $x, \ldots, f^{n(j)-i}(x)$ for any $n(j)$. By the construction of N we have: $\mathrm{mod}(N \setminus Y^i(y)) > 0$. So we conclude that $f^{n(j)-i} : Y^{n(j)}(x) \to Y^i(y)$ has bounded distortion with the Koebe constant not depending on $n(j)$.

Using the above bounded distortion property, Proposition 3.1.5 and the fact that x is a density point for A, we conclude that $\mathrm{dens}(A|Y^i(y))$ is arbitrarily close to one. On the other hand there exists some pre-image of $Y^i(0)$ inside $Y^i(y)$, so $\mathrm{dens}(A|Y^i(y))$ is bounded away from 1. Contradiction! □

Let us prove a similar result for the classes of infinitely renormalizable polynomials that we are dealing with (Lyubich and real unimodal infinitely renormalizable):

Lemma 3.1.7 *Let f be any Lyubich or any real unimodal infinitely renormalizable quadratic polynomial and μ a conformal measure for f. Let U be any neighborhood of the critical point. Then the set*

$$\{x \in \mathbf{C} : f^n(x) \in U^c, \; for\ all\ positive\ n\}$$

has zero μ-measure.

Proof. Let us denote the set in the statement of this Lemma by A. We have $A = J(f) \setminus \bigcup_k f^{-k}(U)$. So A is a nowhere dense forward invariant set. Notice that $A \bigcap \overline{\mathcal{O}}$ is empty (because of the definition of A and minimality of $\overline{\mathcal{O}}$ in the case where f either a Lyubich or a real infinitely renormalizable polynomial). In view of the Lebesgue density Theorem (see Theorem 2.9.11 in [Fed69]), the set of density points of A has full measure inside A. Here by density points we mean $x \in A$ such that $lim_{r\to\infty}\text{dens}(A|B(x,r)) = 1$, where $B(x,r)$ is the Euclidean ball with center at x and radius r. Suppose that $\mu(A)$ is positive. Then we conclude that there exists a density point x in A. There also exists y inside A and a sequence of natural numbers $k_j \to \infty$ such that $f^{k_j}(x) \to y$. We can pull back a ball of definite size centered in y along $x, f(x), ..., f^{k_j}(x)$ (to be more precise, the size of this ball is $\text{dist}(y, \overline{\mathcal{O}})$). That implies that we can fix a positive number η and pull back the ball $B(f^{k_j}(x), \eta)$, for k_j big along $x, f(x), ..., f^{k_j}(x)$. Since A is nowhere dense and μ is positive on non-empty subsets of the Julia set, for large n_j we have:

$$\mu(B(f^{k_j}(x), \frac{\eta}{2}) \setminus A) \geq \mu(B(y, \frac{\eta}{4}) \setminus A) > 0.$$

As a consequence of Koebe's Theorem, the definition of conformal measure and the invariance of A we have:

$$K\mu(B(x, K^{-1}\frac{\eta}{2}|Df^{k_j}(x)|^{-1}) \setminus A) \leq |Df^{k_j}(x)|^{-\delta}\mu(B(f^{k_j}(x), \frac{\eta}{2}) \setminus A) \leq$$

$$\leq K\mu(B(x, K\frac{\eta}{2}|Df^{k_j}(x)|^{-1}) \setminus A).$$

Let us denote $r = K\frac{\eta}{2}|Df^{k_j}(x)|^{-1}$. From the above and from the definition of conformal measure we get:

$$\frac{\mu(B(x,r) \setminus A)}{\mu(B(x,r))} \geq \frac{|Df^{k_j}(x)|^{-\delta}K^{-1}\mu(B(f^{k_j}(x), \frac{\eta}{2}) \setminus A)}{\mu(B(x,r))} \geq$$

$$\geq \frac{K|Df^{k_j}(x)|^{\delta}}{\mu(f^{k_j}(B(x,r)))}|Df^{k_j}(x)|^{-\delta}K^{-1}\mu(B(f^{k_j}(x),\frac{\eta}{2})\setminus A)\geq$$

$$\geq \frac{\mu(B(f^{k_j}(x),\frac{\eta}{2})\setminus A)}{\mu(f^{k_j}(B(x,r)))}\geq \frac{\mu(B(f^{k_j}(x),\frac{\eta}{2})\setminus A)}{1}\geq \mu(B(y,\frac{\eta}{4})\setminus A)\geq c>0.$$

That implies that

$$\text{limsup}_{r\to 0}\frac{\mu(B(x,r)\setminus A)}{\mu(B(x,r))}>0,$$

which contradicts the choice of x as a density point of A. □

Note that in Lemma 3.1.7 we used the fact that f restricted to $\overline{\mathcal{O}}$ is minimal which is not necessarily true for a Yoccoz polynomial. On the other hand, in Lemma 3.1.6 we used the fact that we have a partition for the entire Julia set by puzzle pieces whose pre-images shrink to points. We do not have that for Lyubich and real unimodal infinitely renormalizable polynomial.

From the previous Lemmas we conclude that the set

$$\mathcal{W}=\{z\in J(f):0\in w(z)\}$$

has full measure, i. e., $\mu(\mathcal{W})=1$. Here $w(z)$ denotes the w-limit set of z. Let $X\subset\mathcal{W}$ be any measurable set. If f is a Yoccoz polynomial we can create a cover of X by puzzle pieces as follows: fix $V^{0,n}$ (remember that in the case of a Yoccoz polynomial the first index of the principal nest is always 0. That is because we don't have renormalization levels). For every $x\in X$ there exists a first time m such that $f^m(x)\in V^{0,n}$. So we can pull $V^{0,n}$ along the orbit of x back to a puzzle piece containing x. Changing $x\in X$ we will obtain the desired cover. Let us call this cover O_n. We can make a similar construction for any Lyubich and any real unimodal infinitely renormalizable polynomials using the sets $V^{n,1}$ (constructed in sub-section 2.2.2 and section 2.2.4). We have the following properties:

(i) O_n is an open cover;

(ii) $O_n \subset O_{n-1}$;

(iii) $\bigcap O_n = X$;

(iv) $\mu(O_n) \to \mu(X)$ as $n \to \infty$.

The first and the second properties are trivial. The third one is a consequence of Yoccoz Theorem, if f is a Yoccoz polynomial. The same fact is also true for Lyubich and real unimodal infinitely renormalizable polynomials due to the complex bounds (Lyubich Theorem and Theorem 2.2.3, respectively) together with Lemmas 3.2.1 and 3.2.2. The last one follows by regularity of the measure μ.

To simplify the notation, elements of O_n will be denoted by the letter U (indexed in some convenient fashion). In other words, if we say that U is an element of O_n we mean that U is some univalent pull back of $V^{0,n}$, in the case f is a Yoccoz polynomial. If f is either a Lyubich or an infinitely many times renormalizable unimodal real polynomial and U is an element of O_n then U is some univalent pull back of $V^{n,1}$.

Lemma 3.1.8 *For all i, there exists U^i in O_i such that* $\mathrm{dens}(X|U^i) \to 1$, *as* $i \to \infty$.

Proof. Similar to Lemma 3.1.3. □

3.2 Proof of Theorem 1

Before we pass to the next Lemmas, remember that by definition $V^{i,k+1}$ is the pull back of $V^{i,k}$ along the first return of the critical point to $V^{i,k}$ (see sub-section 2.2.2). We will denote the critical point of f by 0.

Lemma 3.2.1 *Let f be a Yoccoz polynomial, U be in O_n and let m be the smallest time that $0 \in f^m(U) = V^{0,n}$. Then we can univalently pull $V^{0,n-1}$ back along the orbit $x, \ldots, f^m(x)$, $x \in U$.*

Proof. If not, $f^{-t}(V^{0,n-1})$ would contain the critical point, for some t less than m (here f^{-t} means the branch of f^{-t} along the orbit of x). That would mean that t is greater or equal to the first return time of 0 to $V^{0,n-1}$. That would imply $f^{-t}(V^{0,n-1}) \subset V^{0,n}$ by the Markov property of puzzle pieces. In other words, x would hit $V^{0,n}$ on a time strictly less than m, contradicting the definition of m. □

Lemma 3.2.2 *Let f be any Lyubich or any infinitely renormalizable unimodal real polynomial. Let U be in O_n and let m be the smallest time that $0 \in f^m(U) = V^{n,1}$. Then we can univalently pull $V^{n,0}$ back along the orbit $x, \ldots, f^m(x)$, $x \in U$.*

Proof. Suppose that $f^{j_n} : V^{n,1} \to V^{n,0}$ is the first return map of the critical point to $V^{n,0}$. Remember that the above renormalization of f is unbranched (see Lemma 2.2.6).

Suppose we cannot pull back $V^{n,0}$ univalently along the orbit of x. In that case the pull back of $V^{n,0}$ would hit the critical point before reaching U. Then there would be a time t such that $f^{-t}(V^{n,0})$ would contain the critical point, for a time $t < m$ (here the pull back is made along the orbit of $x, ..., f^m(x)$). Let us follow the orbit of $f^{-t}(V^{n,0})$. Remember that as we are assuming that the critical point belongs to $f^{-t}(V^{n,0})$ we would have:

$$f^j(f^{-t}(V^{n,0})) \bigcap \mathrm{Post(f)} \neq \emptyset \; for \; any \; j \tag{3.1}$$

Obviously, $f^t(f^{-t}(V^{n,0})) = V^{n,0}$, so $f^{t-1}(f^{-t}(V^{n,0}))$ is either $f^{j_n-1}(V^{n,1})$ or $-f^{j_n-1}(V^{n,1})$. From Proposition 2.2.5 and from (3.1) we conclude that $f^{t-1}(f^{-t}(V^{n,0})) = f^{j_n-1}(V^{n,1})$. Finite induction with the same argument would shows the following equality: $f^{t-(j_n-1)}(f^{-t}(V^{n,0})) = f(V^{n,1})$. If we make one more pull back we will see that the only possibility is to have $j_n = t$ (remember that $t \geq j_n$ because j_n is the first return of the critical point to $V^{n,0}$) and $V^{n,1} = f^{-t}(V^{n,0})$. That would mean that $f^{m-t}(U) \subset V^{n,1}$ which is a contradiction with the definition of m as the first time the orbit of z reaches $V^{n,1}$. □

Let us prove Theorem 1.

Let $Y \subset \mathcal{W} = \{z \in J(f) : 0 \in w(z)\} \subset J(f)$ be an f-invariant set (remember that $\mathcal{W}$ has full measure). Suppose that $\mu(Y) > 0$.

If f is a Yoccoz polynomial we use Lyubich's Theorem to conclude that $\mathrm{mod}(V^{0,n-1} \setminus V^{0,n}) > c > 0$, for all n. From Lemma 3.1.8 we can find U^n in O_n such that $\mathrm{dens}(Y|U^n) \to 1$. Let us assume that $f^{t(n)} : U^n \to V^{0,n}$ is an isomorphism (given by the definition of O_n). That means by Lemma 3.2.1 and by Koebe's Theorem that $f^{t(n)}$ has bounded distortion, i.e.:

$$\frac{1}{K} \leq \frac{|D(f^{t(n)})(z_1)|}{|D(f^{t(n)})(z_2)|} \leq K$$

for all z_1 and z_2 in U^n, where K depends just on c, the constant that appears in the statement of Lyubich's Theorem.

Now let us apply Proposition 3.1.5 to the sets $Y^c \bigcap U^n$ and U^n with respect to the map $f^{t(n)}$. Due to the fact that the set Y is f-invariant and that $f^{t(n)}(U^n) = V^{0,n}$ we get $\frac{1}{K^\delta}\mathrm{dens}(Y^c|U^n)) \leq \mathrm{dens}(Y^c|V^{0,n}) \leq K^\delta \mathrm{dens}(Y^c|U^n))$.

We know that $\mathrm{dens}(Y|U^n) \to 1$. Passing to the complement of Y we get $\mathrm{dens}(Y^c|U^n) \to 0$. From this and the above inequalities we conclude that $\mathrm{dens}(Y^c|V^{0,n}) \to 0$ or $\mathrm{dens}(Y|V^{0,n}) \to 1$.

Notice that if $\mu(Y^c) > 0$ then we can repeat the argument changing Y by Y^c. Doing this we get $\mathrm{dens}(Y^c|V^{0,n}) \to 1$ and that contradicts the previous limit because $\mathrm{dens}(Y|V^{0,n}) + \mathrm{dens}(Y^c|V^{0,n}) = 1$.

So we conclude that $\mu(Y^c)$ =0, or equivalently, $\mu(Y)$ =1. This finishes the proof of the Theorem if f is a Yoccoz polynomial.

If f is either a Lyubich polynomial or an infinitely renormalizable real unimodal polynomial the proof of Theorem 1 is basically the same. The only differences are that we need to use both Lyubich's Theorem and Theorem 2.2.3

in order to get bounds for $\mathrm{mod}(V^{n,0}\backslash V^{n,1})$, and we need to refer to Lemma 3.2.2 instead of Lemma 3.2.1. This finishes the proof of Theorem 1.

Chapter 4

Conformal measure and Hausdorff dimension

4.1 Conformal measures and hyperbolic dimension

Definition 4.1.1 ([DU91] [Shi91]) We define the hyperbolic dimension of f as:

$$\text{hypdim}(f) = \sup\{\text{HD}(X) \ : \ X \subset J(f) \ \textit{is hyperbolic for} \ f\}$$

Our goal in this section is to give an alternative (more geometric) demonstration of a Theorem due to Denker and Urbański (see [DU91]). The original Theorem is true for any rational map and depends on a result from [Prz93]. Here we will proof it in the particular case of Lyubich and Sullivan polynomials:

Theorem 4.1.2 *If f is either a Lyubich or a Sullivan polynomial, then*

$$\inf\{\delta : \exists \, a \ \delta - conformal \ measure \ for \ f\} = \text{hypdim(f)}.$$

In order to prove this result, we will describe a way of constructing conformal measures. Let U be a neighborhood of the critical point. Then we define

$$A_U = \{z \in J(f) : f^j(z) \in U^c, \forall j \geq 0\}.$$

Here U^c means the complement of U inside C. Notice that the set A_U is forward f-invariant.

Lemma 4.1.3 *The map f restricted to A_U is hyperbolic if f is a Lyubich or a Sullivan polynomial.*

The above Lemma is a special case of Lemma 5.2.1. We will not prove it here.

Let U_n be a sequence of neighborhoods of the critical point such that $\text{diam}(U_n)$ goes to zero. Define the sets $A_n = A_{U_n}$.

Lemma 4.1.3 allows us to use the classical theory of thermodynamical formalism to construct probability conformal measures for the hyperbolic systems $f : A_n \to A_n$. We will call those measures ν_n (see [Bow75] and [PU] as references to thermodynamical formalism). Notice that ν_n is a conformal measure for $f : A_n \to A_n$ but not for $f : J(f) \to J(f)$ because A_n is not backward invariant.

If the Hausdorff dimension of A_n is δ_n then again by thermodynamical formalism ν_n is a δ_n-conformal measure on A_n. Notice that as $A_n \subset A_{n+1}$ then δ_n is an increasing sequence. Let $\delta_* = \sup \delta_n$. We claim that if μ is any weak limit of the sequence ν_n, then it is a δ_*-conformal measure on $J(f)$ for

f. To see that we need to show that the Jacobian of μ is $|Df|^{\delta_*}$. Let $z \in J(f)$ and let $B_r(z)$ be a ball not containing the critical point in its closure. Take n big enough so that $B_r(z) \bigcap U_n$ is empty. Notice that in this case we have $f(B_r(z) \bigcap A_n) = f(B_r(z)) \bigcap A_n$. If we consider ν_n as a measure on $J(f)$, then $\nu_n(f(B_r(z))) = \nu_n(f(B_r(z) \bigcap A_n)) = \int_{B_r(z) \bigcap A_n} |Df|^{\delta_n} d\nu_n = \int_{B_r(z)} |Df|^{\delta_n} d\nu_n$ so if r is sufficiently small and n is sufficiently big (and then δ_n is close to δ_*) we get $\nu_n(B_r(z))(|Df(z)|^{\delta_*} - \varepsilon) \leq \nu_n(f(B_r(z)) \leq \nu_n(B_r(z))(|Df(z)|^{\delta_*} + \varepsilon)$. We get that $\mu(B_r(z))(|Df(z)|^{\delta_*} - \varepsilon) \leq \mu(f(B_r(z)) \leq \mu(B_r(z))(|Df(z)|^{\delta_*} + \varepsilon)$, if we pass to the limit, provided that $\mu(\partial B_r(z)) = 0$. So we conclude that μ is δ_*-conformal on any set not containing the critical point. We now just need to show that μ is δ_*-conformal at the critical point. That is the same as to show that $\mu(f(0)) = 0$. This follows from the following Lemma:

Lemma 4.1.4 *If f is either a Lyubich or a Sullivan polynomial, then we have:* $\limsup |Df^n(f(0))| \to \infty$.

Proof. Let f be a Sullivan polynomial. One can find a sequence of symmetric intervals containing the critical point I_n such that $\frac{|I_n|}{|I_{n+1}|}$ is asymptotic to a constant as n goes to infinity. Such intervals have the property that their diameters shrink to zero and that $f^{a(n)} : I_{n+1} \to I_n$ is a unimodal map (i.e., the map has only one critical point) for an appropriate integer $a(n)$ (see [dMvS93]). One can also prove that the map $f^{a(n)-1} : f(I_{n+1}) \to I_n$ is a diffeomorphism with bounded distortion (not depending on n). Putting all the previous information together we conclude that $|D(f^{a(n)-1}(f(0)))| \to \infty$.

(this fact follows because we have hyperbolicity, and then bounded distortion for f^n inside A_n). In that case, if we take any cover of A_n by ball $B_{r_i}(z_i)$ with r_i sufficiently small, we get $\sum r_i^\delta \leq C \sum \nu(B_{r_i}(z_i))$. As $\nu(A_n)$ is zero (see Lemma 3.1.7) then we can make the last sum as close to zero as we want. That implies that the Hausdorff dimension δ_n of A_n is at most δ. As the previous argument is true for any A_n, we get that $\delta_* = \sup \delta_n \leq \delta$. Taking the infimum over all δ's we get $\delta_* \leq \delta_{inf}$. So we conclude that $\delta_* = \delta_{inf}$. This finishes the proof of the Theorem.

4.2 Modified principal nest

The goal of this section is to construct a new principal nest that we will call *the modified principal nest*, starting from the principal nest constructed in [Lyu93] and described in section 2.2.2. The elements of the modified principal nest will be related to each other via maps which are compositions of a quadratic map and an isomorphism with bounded distortion (depending just on the map f). To simplify notation we will denote the elements of the principal nest of the first renormalization level (see section 2.2.2) by $V^0, V^1, ..., V^n, ...$ We can divide as usual this principal nest in disjoint unions of cascades of central returns (remember that a non-central return level is a cascade of central returns of length zero or a trivial cascade). Let $n(k)$ be the number of cascades of central returns before the level k.

Beginning of the construction of the modified principal nest: The elements of the modified principal nest will be denoted by W^i. Let $n + 1$

be the first level of the first non-trivial cascade of central returns. We define $W^i = V^i$ for all $i = 1, \dots, n+2$. Suppose that this first non-trivial cascade has its last element at level $n+k$, i. e., the first non-central return appears on $f^{l(n+k)} : V^{n+k} \to V^{n+k-1}$. We will construct the next element W^{n+3} of our modified principal nest as being a puzzle piece satisfying:

(i) $V^{n+k+1} \subset W^{n+3} \subset V^{n+k}$ and $0 \in W^{n+3}$;

(ii) the puzzle piece W^{n+3} is mapped as a branched covering of degree two onto $V^{n+2} = W^{n+2}$;

(iii) The above map is the composition of a pure quadratic map and an isomorphism with bounded distortion;

(iv) $\operatorname{mod}(V^{n+k} \setminus W^{n+3}) \geq \frac{1}{2}\mu_n = \mu_{n+1}$.

We can obtain W^{n+3} by the procedure described in what follows (see Figure 4.1). As we are in a cascade of central returns there exists a number p such that $f^p = f^{l(i)} : V^i \to V^{i-1}$ for all $i = n+1, \dots, n+k$. It is also true that $f^{ip}(0) \in V^{n+k-i} \setminus V^{n+k-i+1}$ for $i = 1, \dots, k$ (where 0 is the critical point). In particular $f^{(k-1)p}(0) \in V^{n+1} \setminus V^{n+2}$.

We know that the orbit of $f^{(k-1)p}(0)$ should enter V^{n+2}. That is because the orbit of V^{n+k+1} should enter V^{n+2} in order to return to V^{n+k}. Let us call S^{k-1} the pull back of V^{n+2} along the orbit of $f^{(k-1)p}(0)$. To be more precise, let $t > 0$ be the first time that $f^t(f^{(k-1)p}(0)) \in V^{n+2}$. Then $S^{k-1} = f^{-t}(V^{n+2})$. Here we are considering the branch of f^{-t} that takes $f^t(f^{(k-1)p}(0))$ to $f^{(k-1)p}(0)$. Now we define $S^i = f^{-(k-i-1)p}(S^{k-1})$, where

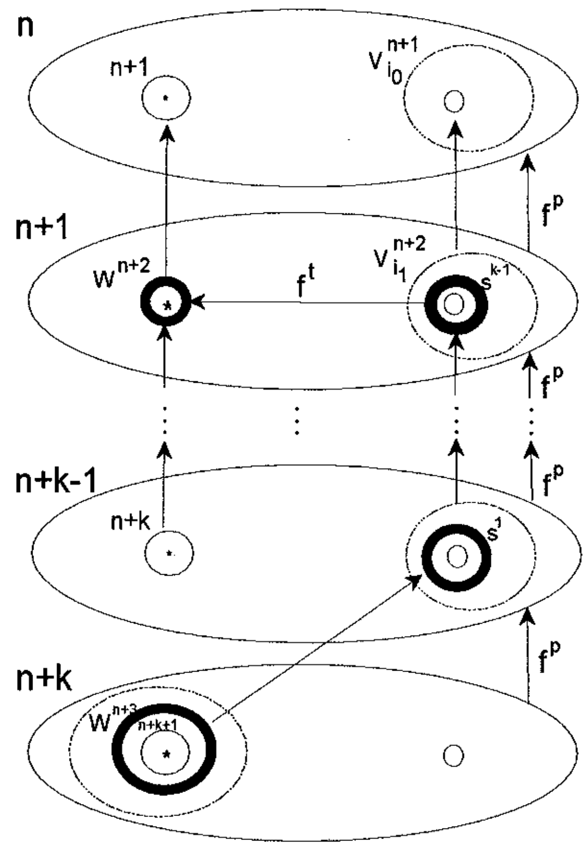

Figure 4.1: Construction of W^{n+3}

we consider $f^{-(k-i-1)p}$ as being the branch taking $f^{(k-1)p}(0)$ to $f^{ip}(0)$, for $i = 1, ..., k-1$.

Notice that $f^p(0) \in S^1$. So we finally define $W^{n+3} = f^{-p}(S^1)$, where we understand f^{-p} as the branch taking $f^p(0)$ to 0.

Having this definition of W^{n+3}, the first property is obvious by construction and from the Markov property of puzzle pieces. The next property follows from the fact that each one of the maps $f^p : S^i \to S^{i+1}, i = 1, ..., k-2$ is an isomorphism. That is true because we are inside a cascade of central returns.

Let us prove the third property. There exists a puzzle piece $V_{i_1}^{n+2}$ of level $n+2$ not containing the critical point such that $S^{k-1} \subset V_{i_1}^{n+2}$. This is true because otherwise we would find a non-central return at some level between $n+1$ and $n+k-1$. There also exists a puzzle piece $V_{i_0}^{n+1}$ of level $n+1$ such that $f^p(V_{i_1}^{n+2}) \subset V_{i_0}^{n+1}$. Again, $V_{i_0}^{n+1}$ is not critical (because we are inside of a cascade of central returns).

We have $\mathrm{mod}(V_{i_0}^{n+1} \setminus f^p(V_{i_1}^{n+2})) \geq \mu_n$ (it follows from the Bernoulli property of the principal nest, see [Lyu95]). The map $f^{(k-1)p} : V^{n+k-1} \to V^n$ has all its critical points inside V^{n+k} (because we are inside a cascade of central returns). As $V_{i_0}^{n+1} \neq V^{n+1}$, we conclude that we can isomorphically pull $V_{i_0}^{n+1}$ back along the orbit $S^1, S^2, ..., S^{k-1}, f^p(S^{k-1})$. That means that $\mathrm{mod}(f^{-(k-1)p}(V_{i_0}^{n+1}) \setminus S^1) \geq \mu_n$. If we make one extra pull back we will get $\mathrm{mod}(f^{-kp}(V_{i_0}^{n+1}) \setminus W^{n+3}) \geq \frac{1}{2}\mu_n$.

Remember that f is a quadratic polynomial. Putting this fact together with the information from the last paragraph we conclude that we can decompose $f^{pk} : W^{n+3} \to S^{k+1}$ into a pure quadratic map followed by an analytic isomorphism of bounded distortion (by Koebe's Theorem). The distortion depends only on the principal modulus μ_n, which is definite.

The inverse of the map $f^t : S^{k-1} \to V^{n+2}$ can be extended to V^{n+1} (see Lemma 3.2.2). That means, by Koebe's Theorem, that the distortion of f^t is bounded (depending just on μ_{n+1}) when restricted to S^{k-1}.

Putting the information of the last paragraphs together we can show the third property. Property number four also follows from the previous argument.

We will now define the element W^{n+4} following W^{n+3} in our modified principal nest. This new element W^{n+4} will be defined as a certain pull-back of W^{n+3}.

Immediately after the first non-trivial cascade of central returns we will find either another non-trivial cascade of central returns or a trivial cascade of central return. We need to consider both cases.

Continuation of the modified principal nest through a non-trivial cascade: Suppose first that we have another non-trivial cascade of central returns. In that case, we would find central returns on all levels from $n+k+1$ to $n+k+m-1$ (so this new cascade of central returns has length m). We can find W^{n+4} such that:

(i) $V^{n+k+m+1} \subset W^{n+4} \subset V^{n+k+m}$ and $0 \in W^{n+4}$;

(ii) W^{n+4} is mapped as a branched covering of degree four onto W^{n+3};

(iii) The above map is the composition of a pure quadratic map followed by an isomorphism with bounded distortion, an other pure quadratic map and an isomorphism with bounded distortion;

(iv) $\mathrm{mod}(V^{n+k+m} \setminus W^{n+4}) \geq \frac{1}{2}\mu_{n+k}$.

The above properties and their justifications are similar to the ones for W^{n+3} stated before (see Figure 4.2). This finishes the construction of W^{n+4} in the case we have a non-trivial cascade of central returns following the first non-trivial cascade of central returns.

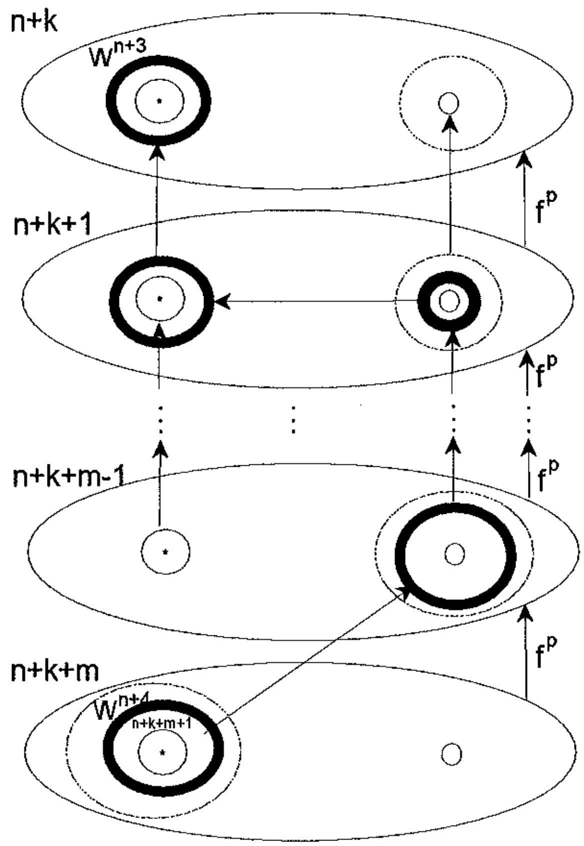

Figure 4.2: First construction of W^{n+4}

Continuation of the modified principal nest through a trivial cascade: In this case we suppose that we have a trivial cascade following the first non-trivial cascade of central returns. In other words the first return map $f^{l(n+k+1)} : V^{n+k+1} \to V^{n+k}$ is a non-central return. In that case we can find W^{n+4} such that:

(i) $V^{n+k+2} \subset W^{n+4} \subset V^{n+k+1}$ and $0 \in W^{n+4}$;

(ii) W^{n+4} is mapped as a branched covering of degree two onto W^{n+3};

(iii) The above map is the composition of a pure quadratic map and an isomorphism with bounded distortion;

(iv) $\mathrm{mod}(V^{n+k+1} \setminus W^{n+4}) \geq \frac{1}{2}\mathrm{mod}(V^{n+k} \setminus W^{n+3}) \geq \frac{1}{4}\mu_n$.

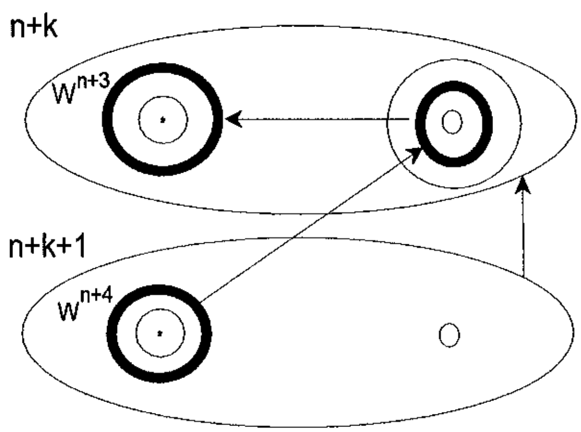

Figure 4.3: Second construction of W^{n+4}

Again with the same type of argument we used to show properties of W^{n+3} we show the above properties (see Figure 4.3).

The definition of the modified principal nest: It follows now by induction. We assume that we have the definition of one of its elements, say W^i. Such element is inside the last level of a given cascade of central returns. We can construct the next one, W^{i+1} following the constructions just as we described in the previous two cases.

The modified principal nest as constructed has the following properties. Given any cascade of central returns with last element V^{n+1}:

(i) There exist W in the modified principal nest such that $V^{n+1} \subset W \subset V^n$ with $\text{mod}(V^n \setminus W)$ growing linearly with the number of cascades of central returns;

(ii) If W' is the element following W in the modified principal nest, then $\text{mod}(W \setminus W')$ is growing linearly (with the number of cascades of central returns);

(iii) W' is mapped as a branched covering either of degree two or degree four onto W;

(iv) The map in the previous item is the composition of a pure quadratic map and an isomorphism with bounded distortion in the degree two case. In the degree four case, this map is a composition of two maps as in the degree two case.

The last three properties of the modified principal nest are true by construction (the third one is a consequence of the first). The first property is obvious except perhaps in one case, namely when we have more than one trivial cascade of central returns together. Notice that in our construction of the modified principal nest through a trivial cascade (see Figure 4.3) we got the following estimate : $\text{mod}(V^{n+k+1} \setminus W^{n+4}) \geq \frac{1}{2}\text{mod}(V^{n+k} \setminus W^{n+3})$. One can ask whether we will keep dividing by two the bound for the modulus comparing the principal and the modified nests at a certain level, if we have several trivial cascades, one following the other. If that would be the case we would spoil our estimates concerning the modified principal nest.

Let us analyse what happens when we have two consecutive trivial cascades. On Figure 4.4, if V_i^{n+2} is a non-critical puzzle piece of level $n+2$, then $\mathrm{mod}(V^{n+1} \setminus V_i^{n+2}) \geq \frac{1}{2}\mu_{n+1}$, which is definite because μ_{n+1} is the principal modulus at the top of a (trivial) cascade of central returns. This is enough to show that $\mathrm{mod}(V^{n+2} \setminus W^{n+2}) \geq \frac{1}{4}\mu_{n+1}$, and then a definite number.

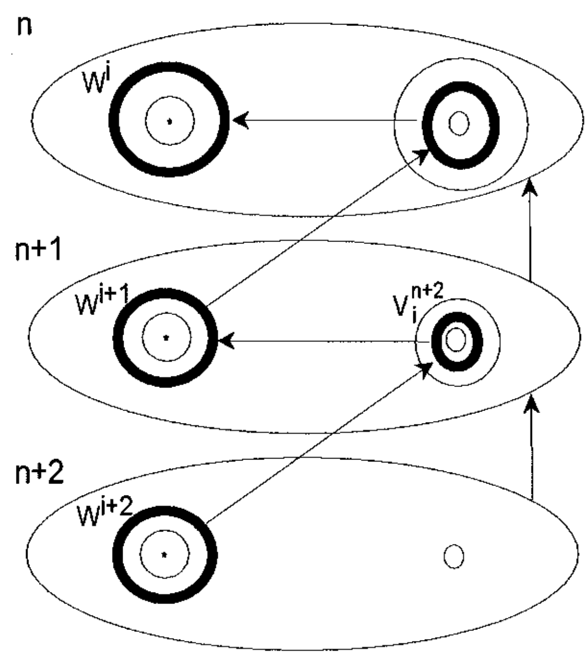

Figure 4.4: Two consecutive trivial cascades of central returns

From our considerations we see that for each cascade of central return we have one element of the modified principal nest. Therefore we can enumerate the elements of the modified principal nest counting the cascades of central returns. So W^n is the n^{th} element of the modified principal nest, i.e. , the element at the end of the n^{th} cascade of central return.

Suppose that the first principal nest of f ends in an infinite cascade of central returns. Then the modified principal nest also ends in an infinite cascade of central returns. The construction of the modified principal nest through the renormalization level is the same as the construction of the principal nest defined in [Lyu93] through the renormalization level. Now we complete the modified principal nest repeating the construction that we have just described forever.

4.3 Proof of Theorem 3

4.3.1 Lyubich polynomials

We will use the following notation: if a_n and b_n are two sequences of positive real numbers, then we write $a_n \asymp b_n$ if there exists K such that $K^{-1} \leq \frac{a_n}{b_n} \leq K$.

Let f be a Lyubich polynomial and μ a δ-conformal measure for f. Take W and W' being two consecutive elements of the modified principal nest and $f^a : W' \to W$ the first return map of the critical point to W'.

Lemma 4.3.1 *Let f be any Lyubich polynomial and assume that W and W' are two consecutive elements of the modified principal nest belonging to the same renormalization level. Then*

$$\frac{\mu(W)}{(\mathrm{diam}(W))^\delta} \leq K \frac{\mu(W')}{(\mathrm{diam}(W'))^\delta}$$

where K is a constant depending just on the selection of the secondary limbs.

Proof. Suppose first that $f^a : W' \to W$ is a degree two branched covering. Then the map $f^{a-1} : f(W') \to W$ has bounded distortion (by construction of the modified principal nest). That implies that $\frac{\mu(f(W'))}{(\text{diam}(f(W')))^\delta} \asymp \frac{\mu(W)}{(\text{diam}(W))^\delta}$. So we conclude that there is a constant K_1 such that:

$$\frac{(\text{diam}(f(W')))^\delta}{(\text{diam}(W))^\delta} \leq K_1 \frac{\mu(f(W'))}{\mu(W)}. \tag{4.1}$$

As $f : W' \to f(W')$ is pure quadratic we have the following inequality: $\text{diam}(W') \leq L|Df(x)|^{-1}\text{diam}(f(W'))$, where L is a constant that just depends on the degree of the critical point of f and x is any element in W'. As μ is conformal we find z_0 in W' such that $\mu(f(W')) = \frac{1}{2}|Df(z_0)|^\delta \mu(W')$.

Putting last two observations together we get:

$$\frac{(\text{diam}W')^\delta}{(\text{diam}(f(W')))^\delta} \leq L^\delta \frac{1}{2} \frac{\mu(W')}{\mu(f(W'))}. \tag{4.2}$$

Multiplying equations (4.1) and (4.2) we get the Lemma. Remember that we are assuming that $f : W \to W'$ has degree two.

Suppose now that $f^a : W' \to W$ is a degree four branched covering. Then it is a composition of two maps which are themselves the composition of a pure quadratic map followed by some map with bounded distortion. Then we repeat the previous argument twice to get the same result. □

For the next Lemma we will consider two consecutive elements of the modified principal nest W and W' such that $f^a : W' \to W$ (the first return map of the critical point to W') has connected Julia set. In other words, we will be considering the level of the modified principal nest corresponding to a change of renormalization level. Let W'' be the first element of the modified principal nest following W'.

Lemma 4.3.2 *Suppose that* $f^a : W' \to W$ *has a connected Julia set. Then*

$$\frac{\mu(W')}{(\operatorname{diam}(W'))^\delta} \leq K \frac{\mu(W'')}{(\operatorname{diam}(W''))^\delta}$$

where W'' *is the first element of the modified principal nest following* W' *and* K *depends just on the selection of the secondary limbs.*

Proof. Suppose that the α fixed point of $g = f^a : W' \to W$ is the landing point of p external rays. For each puzzle piece of level zero Y_i^0 for g we define $S_i = Y_i^0 \bigcap J(g)$. Let us define Y^1 as being the intersection of $J(g)$ with the critical puzzle piece of level zero for g. We also define $W_i = -S_i$.

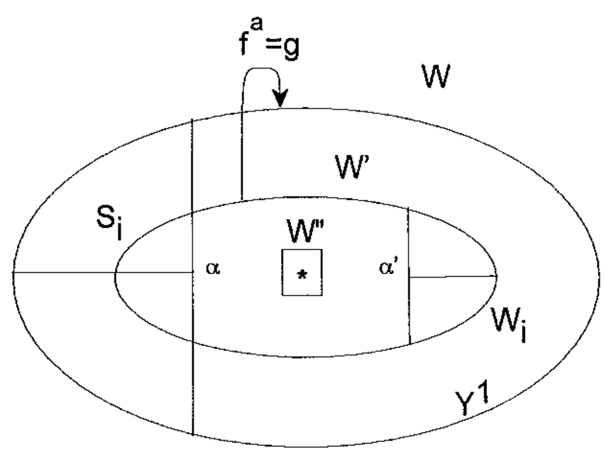

Figure 4.5: Renormalization level

The pieces S_i can be enumerated following the orbit of the critical point. It follows from bounded geometry of the puzzle pieces in Figure 4.5, as shown in [Lyu93], that $g : S_i \to S_{i+1}$ has bounded distortion for $i = 1, ..., p-1$ (depending only on the choice of the secondary limbs). The same happens to the map $g : S_{p-1} \to Y^1$. This implies that for $i = 1, ..., p-2$ we have:

$$\frac{\mu(S_i)}{\text{diam}(S_i)^\delta} \asymp \frac{\mu(S_{i+1})}{\text{diam}(S_{i+1})^\delta}$$

and

$$\frac{\mu(S_{p-1})}{\text{diam}(S_{p-1})^\delta} \asymp \frac{\mu(Y^1)}{\text{diam}(Y^1)^\delta}.$$

By the hypothesis of Lyubich polynomials we know that there is a time j (this being minimal) such that $g^j(0) \in W_i$, for some i. According to [Lyu93], W'' is the pull back of W_i along the critical orbit, back to the critical point. As this map is the composition of a pure quadratic polynomial with an isomorphism with bounded distortion (again by bounded geometry), we can repeat the proof of Lemma 4.3.1 to get:

$$\frac{\mu(W'')}{\text{diam}(W'')^\delta} \geq K_0 \frac{\mu(W_i)}{\text{diam}(W_i)^\delta} = \frac{\mu(S_i)}{\text{diam}(S_i)^\delta}.$$

Last equality follows from the fact that $S_i = -W_i$. Notice that K_0 depends just on the selection of secondary limbs. There exist an integer k such that $g^k : W_i \to Y^1$. This map has bounded distortion (again because of bounded geometry). Then we get:

$$\frac{\mu(W_i)}{\text{diam}(W_i)^\delta} \asymp \frac{\mu(Y^1)}{\text{diam}(Y^1)^\delta}.$$

Putting all the previous estimates together we get the following (remember that p is the number of external rays landing at α, which depends just on the selection of the secondary limbs):

$$p\frac{\mu(W'')}{\text{diam}(W'')^\delta} \geq K\frac{\mu(Y^1)}{\text{diam}(Y^1)^\delta} + K\sum_{i=1}^{p-1}\frac{\mu(S_i)}{\text{diam}(S_i)^\delta} \geq$$

$$\geq K\frac{\mu(Y^1) + \sum_i \mu(S_i)}{\text{diam}(W')^\delta} = K\frac{\mu(W')}{\text{diam}(W')^\delta}.$$

So we get the Lemma. □

Lemma 4.3.3 *The diameter of the puzzle-pieces of the modified principal nest decreases superexponentialy fast, i.e.,*

$$\text{diam}(W^n) < e^{-Ln}$$

for any L, if f is a Lyubich polynomial.

Proof. According to our construction of the modified principal nest, the principal modulus $A_n = W^{m,n} \setminus W^{m,n+1}$ grows linearly with n in between to renormalization levels. That means that $\text{mod}(W^{m,0} \setminus W^{m,n}) > C \cdot \sum_n L^n$ (remember that n counts the number of central cascades in a given renormalization level). The constant L is uniform according to [Lyu93]. The Lemma follows from that. □

Now if we put Lemmas 4.3.1, 4.3.2 and 4.3.3 together we conclude that for any element of the modified principal nest W^n we have (here we are enumerating the whole modified principal nest with just one index):

$$\frac{\mu(W^n)}{(\text{diam}(W^n))^\delta} \geq K(C)^n \frac{\mu(W^0)}{(\text{diam}(W^0))^\delta}.$$

In other words,

$$\frac{\mu(W^n)}{(\text{diam}(W^n))^{\delta}} \geq aK^n.$$

If we take $\delta' > \delta$ then

$$\frac{\mu(W^n)}{(\text{diam}(W^n))^{\delta'}} = \frac{\mu(W^n)}{(\text{diam}(W^n))^{\delta}} \frac{1}{(\text{diam}(W^n))^{\delta'-\delta}}.$$

As $\text{diam}(W^n)$ goes to zero superexponentialy fast, we have:

$$\frac{\mu(W^n)}{(\text{diam}(W^n))^{\delta'}} > C \geq 0.$$

This last inequality implies that $\text{HD}(J(f)) \leq \delta$. That is because we can now construct a family of covers $(U_n(x))$, n natural, of almost all $J(f)$ (with respect to the measure μ, see Chapter 3). This family of covers has two properties. One is that $\text{diam}(U_n(x))$ goes to zero as n goes to infinity. The second property is that $\frac{\mu(U_n(x))}{\text{diam}(U_n(x))^{\delta}} > C \geq 0$. That is due to bounded distortion properties (see in [Lyu93] the proof of local connectivity of $J(f)$, when f is a Lyubich polynomial). Remember that δ_{inf} is the infimum of the exponents of conformal measures. Putting this together with the result in Theorem 4.1.2 that says that $\delta_{inf} = \text{hypdim}J(f)$ we conclude that $\text{HD}(J(f)) = \delta_{inf}$. This finishes the proof of the Theorem 3 for Lyubich polynomials.

4.3.2 Sullivan polynomials

Let f be a Sullivan polynomial and μ a δ-conformal measure for f. As we described in section 2.2.3 $f^{l(n)} : V^{n,1} \to V^{n,0}$, is the renormalization map of f. Notice that $\bigcap_{n\geq 1} V^{n,0} = U$ contains an open set (see Figure 2.1). This open set intersects the Julia set of f in a non-empty set. As the conformal measure of any open set intersecting the Julia set is strictly positive, we conclude that $\mu(U) > 0$. In that case we have:

$$\frac{\mu(V^{n,0})}{(\mathrm{diam}(V^{n,0}))^{\delta'}} > C \geq 0.$$

Using exactly the same argument as in Lemma 4.3.1 we get:

$$\frac{\mu(V^{n,1})}{(\mathrm{diam}(V^{n,1}))^{\delta'}} > C \geq 0.$$

Now the conclusion of the Theorem follows in the same way as in the Lyubich case.

Chapter 5

Teichmüller metric

5.1 Statement of the result

Definition 5.1.1 Let U and U_i be open topological discs, $i = 0, 1, ..., n$. Suppose that $cl(U_i) \subset U$ and $U_i \bigcap U_j = \emptyset$ if i is different than j. A generalized polynomial-like map is a map $f : \cup U_i \to U$ such that the restriction $f|U_i$ is a branched covering of degree $d, d \geq 1$.

We will not use the above Definition in full generality. From now on, all generalized polynomial-like maps on this work will have just one critical point. We will fix our notation as follows: $f|U_0$ is a branched covering of degree d onto U (with only one critical point) and $f|U_i$ is an isomorphism onto U, if $i = 1, ..., n$.

The *filled in Julia set* of f, denoted by $K(f)$, is defined, as usual, as $K(f) = \bigcap f^{-n}(\bigcup U_i)$. The *Julia set* of f, denoted by $J(f)$, is defined as $J(f) = \partial(K(f))$. Douady and Hubbard introduced in [DH85] the notion of a polynomial-like map. Their definition coincides with the previous one when the domain of f has just one component (the critical one). They also showed that a polynomial-like map of degree d is hybrid conjugate to a polynomial of the same degree (in some neighborhoods of the Julia sets of the polynomial and the polynomial-like maps). The above definition was given in [Lyu91]. It was also showed that a generalized polynomial-like map is hybrid conjugate to a polynomial (generally of higher degree but with only one non-escaping critical point).

Definition 5.1.2 A generalized polynomial-like map(or a polynomial) f is said to be in the class $\mathcal{C}$ if for any neighborhood of the critical point, the set of points of $J(f)$ which avoid this neighborhood under the dynamics is hyperbolic. We also ask f to have its critical point inside of its Julia set and not to be conjugate to a Chebychev polynomial, in a neighborhood of its Julia set.

There are several important examples of generalized polynomial-like maps (or polynomials) belonging to the class $\mathcal{C}$. Some example are the following (see Lemma 5.2.1): Sullivan polynomials (see [dMvS93] and [Sul92]), Yoccoz polynomials (see [Mil91]), Lyubich polynomials (see [Lyu93]) and their respective analog classes of polynomial-like maps. Also Fibonacci generalized polynomial-like maps of even degree (see [LM93]) are elements of the class $\mathcal{C}$.

Definition 5.1.3 We say that two generalized polynomial-like maps are in the same conformal class if they are holomorphically conjugate in some neighborhoods of their Julia sets. The conformal class of f will be denoted by $[f]$.

Definition 5.1.4 Let $[f]$ and $[g]$ be two arbitrary conformal classes of generalized polynomial-like maps. Let h_0 be a homeomorphism conjugating f and g on their Julia sets (for other maps on the same conformal class, we just compose h_0 with the holomorphic conjugacies given by the definition of the conformal classes. We will also call this conjugacies by h_0). Suppose that there exist U and V neighborhoods of the Julia sets of f_1 in $[f]$ and g_1 in $[g]$ and $h : U \to V$ a conjugacy between f_1 and g_1. Assume that h is quasi-conformal with dilatation K_h and that it is an extension of h_0. Then we define the Teichmüller distance between $[f]$ and $[g]$ as $\mathrm{d_T}([f],[g]) = \inf_h \log K_h$, where the infimum is taken over all conjugacies h as described, between all the elements f_1 and g_1 in $[f]$ and $[g]$, respectively.

Notice that $\mathrm{d_T}([f],[g]) \geq 0$ and $\mathrm{d_T}([f],[g]) \leq \mathrm{d_T}([f],[t]) + \mathrm{d_T}([t],[g])$, where f, g and t are polynomial-like maps. In order to say that "$\mathrm{d_T}$" is a distance we need to show that if $\mathrm{d_T}([f],[g]) = 0$ then $[f] = [g]$. This is exactly what we will show, if $f \in \mathcal{C}$. We prove the following Theorem:

Theorem 4 *Let f and g be two (generalized) polynomial-like maps belonging to the class $\mathcal{C}$. Suppose that $\mathrm{d_T}(f,g) = 0$. Then f and g are conformally conjugate on a neighborhood of their Julia sets.*

5.2 Hyperbolic sets inside the Julia set

Let $f : \cup U_i \to U$ be any generalized polynomial-like map (or any polynomial) belonging to the class $\mathcal{C}$. Let U be any neighborhood of the critical point. As in Chapter 4 we define:

$$A_U = \{z \in J(f) : f^j(z) \in U^c, \forall j \geq 0\}$$

Here U^c means the complement of the set U inside the complex plane. Notice that the set A_U is forward f-invariant. According to the Definition of the class $\mathcal{C}$ (see Definition 5.1.2) we know that $f : A_U \to A_U$ is hyperbolic.

The next Lemma will show us that several important classes of polynomials and generalized polynomial-like maps are subclasses of the class $\mathcal{C}$.

Lemma 5.2.1 *The map f restricted to A_U is hyperbolic if f is either a Yoccoz polynomial-like map or a Lyubich polynomial-like map or a Sullivan polynomial-like map or a Fibonacci generalized polynomial-like map of even degree.*

Proof. This fact is true because we can construct puzzle pieces for the set A_U if f belongs to one of the classes mentioned in the statement of this Lemma. Let us describe how to do that. If f is a generalized polynomial-like map we will assume that the domain of f has just one component (we will treat the other case later). We can find neighborhoods U_n of the critical point such that their boundaries are made out of pieces of equipotentials and external rays landing at appropriate pre-images of periodic points of f and the diameter of U_n tends to zero as n grows (reference for this fact: [Hub] or [Mil91] if f is Yoccoz, [Lyu93] if f is Lyubich, [HJ] if f is Sullivan, [LvS95] if f is Fibonacci). Let us fix U_{n_0} such that $U_{n_0} \subset U$. Taking all the forward images of all the external rays and pre-images of the fixed point belonging to the boundary of U_{n_0} will give us a forward invariant set. So we have a Markov partition of our set A_U. Each connected component of the plane without this invariant set, intersecting A_U and bounded by a fixed equipotential of $J(f)$ is by definition a puzzle piece of level zero. We define the puzzle-pieces of level k as being the connected components of the k^{th} pre-image of the puzzle pieces of level zero that intersect A_U.

As we are at a positive distance from the critical point (by the definition of the set A_U) it is easy to see that the pull backs of the puzzle pieces shrink to points (using Schwarz Lemma, as in the case of off-critical points for a Yoccoz polynomial. It is shown for that case that the puzzle pieces containing one off-critical point shrink to this point. See [Mil91]).

Now one can show the hyperbolicity of $f : A_U \to A_U$. Take any point x in A_U. Let X be a puzzle piece containing x with arbitrarily small diameter. As X is a puzzle piece, then $X = f^{-n}(X_0)$, where X_0 is a puzzle piece of level zero and f^{-n} is some inverse branch of f^n. Notice that there are just finitely many puzzle pieces of level zero. From this fact we conclude that $\text{diam}(X_0) > \min\{\text{diam}(Y_0) : Y_0 \text{ is a puzzle piece of level zero }\} > C > 0$. So $f^n|X : X \to X_0$ maps a set of arbitrarily small diameter to a set of definite diameter. Shrinking X_0 a little, we can assume that f^n has bounded distortion. Those observations together with the fact that A_U is compact yield hyperbolicity.

The same type of argument can be carried out if f is a Fibonacci generalized polynomial-like map. If this is the case, then the domain of f has more than one component. In that case the puzzle pieces of level zero are the connected components of the domain of the map f. The puzzle pieces of higher levels are the pre-images of the puzzle pieces of level zero. It follows from [LM93] (in the degree two case) and [LvS95] (in the even degree greater then two case) that the puzzle pieces shrink to points. The rest of the proof is identical to the previous case. □

Let f be any polynomial-like map in the class $\mathcal{C}$. We will now construct a sequence of sets that we will call B_n. As the sets A_U, the sets B_n will also be f-invariant and hyperbolic. The dynamics when restricted to this new family of sets will be transitive. This is the main reason why we will need this new family of sets.

Let us pass to the construction of the sets B_n. In order to do that, for each non-postcritical periodic orbit of f inside $J(f)$, take one periodic point p_i belonging to it. Let the period of p_i be n_i. We denote the orbit of p_i by $\mathcal{O}(p_i)$.

We define the set B_1 as being simply $\mathcal{O}(p_1)$. We will now define the set B_2. Let U_i be a small neighborhood of p_i such that $f^{-n_i}(U_i) \subset U_i$ $i = 1, 2$. Here $f^{-n_i}(U_i)$ stands for the connected component of the pre-image of U_i under f^{-n_i} containing p_i. There exists a pre-image y_i of p_i (suppose that $f^{s_i}(y_i) = p_i$) inside U_j, $\{i, j\} = \{1, 2\}$. The orbit $y_i, f(y_i), ...f^{s_i}(y_i) = p_i$ will be called a bridge from $\mathcal{O}(p_i)$ to $\mathcal{O}(p_j)$, for $i \neq j$. There exists a small neighborhood $\widetilde{U_i} \subset U_i$ containing p_i such that $y_i \in f^{-s_i}(\widetilde{U_i}) \subset U_j$, $i \neq j$.

In what follows $i \in \{1, 2\}$. Consider the pull back of the set U_i along the periodic orbit $p_i, f(p_i), ..., f^{n_i}(p_i) = p_i$: $U_i = U_i^0, U_i^{-1}, ..., U_i^{-n_i+1}, U_i^{-n_i}$. Here $U_i^{-k} = f^{-k}(U_i)$ for $k = 0, 1, ..., n_i$. Consider also the pull back of the set $\widetilde{U_i}$ along the orbit $y_i, f(y_i), ..., f^{s_i}(y_i) = p_i$: $\widetilde{U_i} = \widetilde{U_i}^0, \widetilde{U_i}^{-1}, ...\widetilde{U_i}^{-s_i}$. Here $\widetilde{U_i}^{-k} = f^{-k}(\widetilde{U_i})$ for $k = 0, 1, ..., s_i$. We have the following collections of inverse branches of f: the first collection is $f^{-1} : U_i^{-l} \to U_i^{-l-1}$, for $l = 0, 1, ..., n_i - 1$ and the second is $f^{-1} : \widetilde{U_i}^{-l} \to \widetilde{U_i}^{-l-1}$ for $l = 0, 1, ..., s_i - 1$.

The union of the two collections of inverse branches of f described in the previous paragraph will be called our "selection" of branches of f^{-1} for B_2 (notice that we are specifying the domain and image of each one of the branches of f^{-1} in our "selection"). Consider now the set of all possible pre-images of p_i, $i = 1, 2$ under composition of branches of f^{-1} in our "selection" of branches. We define the set B_2 as being the closure of the set of all such pre-images.

We can define B_n in a similar fashion: instead of letting i in last paragraphs to be just in $\{1, 2\}$, we let i to be in $\{1, 2, ..., n\}$. For each p_i, U_i is as before a small neighborhood around p_i, $i = 1, 2, ..., n$. There exist $y_{i,j}$ pre-image of p_i (suppose that $f^{s_{i,j}}(y_{i,j}) = p_i$) contained inside U_j (those points define the bridges between any two distinct orbits). There exists a small neighborhood $\widetilde{U_{i,j}}$ of p_i contained in U_i such that $y_{i,j} \in f^{-s_{i,j}}(\widetilde{U_{i,j}}) \subset U_j$, $i \neq j$. As for B_2 now we can define the suitable "selection" of branches of f^{-1} for B_n in a similar way. Consider now the set of all possible pre-images of p_i, $i = 1, 2, ..., n$ under composition of branches of f^{-1} in our "selection" of branches. We define the set B_n as being the closure of the set of all pre-images just described. Notice that we can carry on our construction such that we have $B_{n-1} \subset B_n$.

The sets B_n have the following properties:

(i) Each set B_n is f-forward invariant;

(ii) Each B_n is compact;

(iii) $f|B_n : B_n \to B_n$ is hyperbolic;

(iv) If we fix i, the set of all pre-images of p_i is dense inside B_n, for $i = 1, 2, ..., n$;

(v) $f|B_n : B_n \to B_n$ is topologically transitive (see Definition 2.3.7);

(vi) $\bigcup_n B_n$ is dense inside $J(f)$.

The first and the second properties are true by construction. The third property follows because we are excluding from our construction periodic points which are images of the critical point by some iteration of f. That implies that the distance from the set B_n to the critical point of f is strictly positive (depending on n). Property (iii) follows now, if f belongs to the class $\mathcal{C}$ of generalized polynomial-like maps.

Let us show the forth property. It is clear that the pre-images of the set $\{p_1, p_2, ..., p_n\}$ under the system $f : B_n \to B_n$ is dense inside B_n. So, in order to show that the pre-images of some p_i are dense inside B_n, we just need to show that given any $1 \leq j \leq n$, there exist pre-images of p_i arbitrarily close to p_j. That is true because there exists a pre-image $y_{i,j}$ of p_i inside U_j, the neighborhood of p_j used in the construction of B_n. If we take all pre-images of $y_{i,j}$ along the periodic orbit of of p_j we will find pre-images of p_i arbitrarily close to p_j (remember that all periodic points are repelling).

Let us show (v). By (iv), inside any open set $V \neq \emptyset$, there exists a pre-image of p_i, for each $i = 1, 2, ..., n$. Then, for some m_i, $f^{m_i}(V)$ is a neighborhood of p_i, for each p_i. Let x be any point in B_n. There are j and k positive such that $f^{-k}(x) \in U_j$. That is because of the construction of B_n. So $f^{-k}(x) \in U_j$. Pulling $f^{-k}(x)$ back along the orbit of p_j sufficiently many times we will find a pre-image of x inside $f^{m_j}(V)$. That implies that for some positive s, $x \in f^s(V)$. So we conclude that $B_n \subset \bigcup_{k \geq 0} f^k(V_i)$.

The last property is obvious because $\bigcup_n B_n$ contains all the periodic points inside $J(f)$ with the exception of at most finitely many (in the case that the critical point is pre-periodic or periodic).

5.3 Non-existence of affine structure

In this Section we will show that if f is a map belonging to the class $\mathcal{C}$ then $f : B_n \to B_n$ does not admit an invariant affine structure (see Definition 2.3.6), for $n > n_0$, for some n_0 depending on f.

Lemma 5.3.1 *Suppose that $f : B_n \to B_n$ and $f : B_{n+1} \to B_{n+1}$ admit invariant affine structures. Then the invariant affine structure in B_{n+1} extends the invariant affine structure in B_n.*

Proof. We will start by taking $n = 1$. Let $\{(\phi_i, V_i)\}$ be a finite atlas of an invariant affine structure for $f : B_1 \to B_1$ and let $\{(\sigma_j, U_j)\}$ be a finite atlas of an invariant affine structure for $f : B_2 \to B_2$. We will show that the collection $\{(\sigma_j, U_j)\} \bigcup \{(\phi_i, V_i)\}$ is an atlas of an invariant affine structure for

$f : B_2 \to B_2$. Notice that the invariant affine structure for $f : B_1 \to B_1$ is unique, given by the linearization coordinates of p_1.

Let us suppose that $V_i \bigcap U_j \neq \emptyset$. We will check that the change of coordinates $\sigma_j(\phi_i)^{-1}$ is affine. We can assume that the closure of $B_2 \bigcap V_i \bigcap U_j$ is not empty, otherwise we can shrink U_j to $U_j \setminus V_i$. Let x be an element belonging to this intersection. We can assume for simplicity that V_i is a chart in B_1 containing p_1 and U_i is a chart in B_2 containing p_1 (remember that p_i is the enumeration of periodic points used to construct the sets B_n and that $B_1 \subset B_2$). As the affine structure for periodic orbits is unique, we conclude that (σ_i, U_i) and (ϕ_i, V_i) are the same (up to an affine map) in a neighborhood of p_1. So we can assume that $U_i \subset V_i$. We can pull x back by f^{n_1} l times along the (periodic) orbit of p_1 until the moment that we find $y \in B_2$ pre-image of x under f^{ln_1} belonging to U_i. Because (ϕ_i, V_i) is a linearization coordinate around the periodic point p_1, we conclude that $\phi_i f^{ln_1}(\phi_i)^{-1}$ is affine from a neighborhood of $\phi_i(y)$ to a neighborhood of $\phi_i(x)$. On the other hand, as $y \in B_2 \bigcap U_i$ and $x = f^{ln_1}(y) \in B_2 \bigcap U_j$, we conclude that $\sigma_j f^{ln_1}(\sigma_i)^{-1}$ is affine from a neighborhood of $\sigma_i(y)$ to a neighborhood of $\sigma_j(x)$. Keeping in mind that σ_i is the restriction of ϕ_i (up to an affine map), we get that the change of coordinate $\sigma_j(\phi_i)^{-1}$ is affine (see Figure 5.1). From that follows trivially that any composition of the form $\sigma_j f(\phi_i)^{-1}$ and $\phi_i f(\sigma_j)^{-1}$ is affine, whenever they are defined. So we proved the Lemma in the case $n = 1$.

Now suppose that we have some invariant affine structure $\{(\phi_i, V_i)\}$ in B_n and $\{(\sigma_j, U_j)\}$ in B_{n+1}. We want to show that $\{(\phi_i, V_i)\} \bigcup \{(\sigma_j, U_j)\}$ is an invariant affine structure in B_{n+1}. Suppose that U_j intersects a chart of one

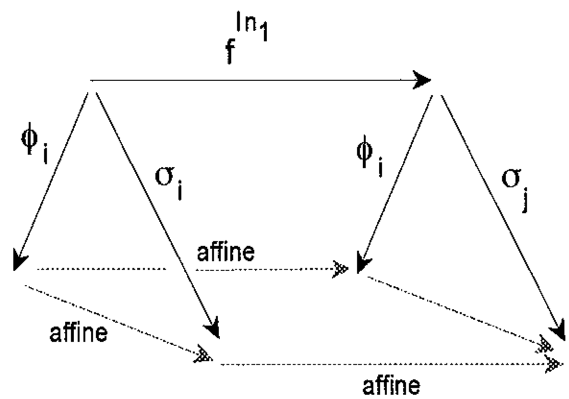

Figure 5.1: Commutative diagram of charts

of the periodic points $p_1, p_2, ..., p_n$ in B_n. Then the change of coordinates from U_j to one of those charts is affine (same as the proof for $n = 1$). Now let U_j and V_i be two arbitrary charts with non-empty intersection. We can assume that there exists x in the closure of $B_{n+1} \bigcap U_j \bigcap V_i$ (otherwise we can shrink U_j to $U_j \setminus V_i$). Let V_1 be the chart around p_1 in the affine structure for B_n and U_1 be the chart around p_1 in the affine structure for B_{n+1}. Then $V_1 \bigcap U_1$ is a neighborhood of p_1. Inside B_n we can pull back V_i until we find a pre-image (of V_i with respect to $f : B_n \to B_n$) strictly inside $U_1 \bigcap V_1$ (this is possible by hyperbolicity). So $f^{-l}(V_i) \subset V_1 \bigcap U_1$. Then it is clear that $\phi_i f^l(\phi_1)^{-1}$ is affine in $f^{-l}(V_i)$. On the other hand, $\sigma_j f^l(\sigma_1)^{-1}$ is affine in in a subset of $f^{-l}(V_i)$ containing the pre-image y of x via f^{-l} (remember that x is the element in B_{n+1} contained in $U_j \bigcap V_i$). As σ_1 and ϕ_1 are equal up to an affine transformation in a neighborhood of y (because both are linearizing coordinates around a periodic point), we conclude that the change of coordinates $\phi_i(\sigma_j)^{-1}$ is affine (just imagine an appropriate diagram similar to the one in Figure 5.1). It is trivial to check that the affine structure defined by $\{(\phi_i, V_i)\} \bigcup \{(\sigma_j, U_j)\}$ is invariant under f. □

We would like to point out that with exactly the same demonstration as above we show that if there exists an invariant affine structure for the system $f : B_n \to B_n$, then it is unique.

Lemma 5.3.2 *If f belongs to the class $\mathcal{C}$, then there is a positive number n_0 such that $f : B_n \to B_n$ does not admit an affine structure if $n > n_0$ (n_0 depends on f).*

Proof. Suppose that $f : B_n \to B_n$ admits an affine structure, for infinitely many n. Then all those structures coincide when defined in common subsets by Lemma 5.3.1. This implies that we can define the set $X = \bigcup_n B_n$ and an invariant affine structure for $f : X \to X$ (notice that X is f-invariant and dense inside $J(f)$). Let us denote the elements of the atlas defining such affine structure over X by (σ_i, U_i).

There exists n such that some element of $f^{-n}(0)$, say y_0, belongs to some U_β (here we need to have the critical point inside $J(f)$). There exists m such that some element of $f^{-m}(f^2(0))$, say y_1, which is not a pre-image of the critical point 0 belongs to U_α, for some α (notice that this is not true if f is a Chebychev polynomial-like map). We can take U_α and U_β small enough such that $f^m : U_\alpha \to f^m(U_\alpha) = U'_\alpha$ and $f^n : U_\beta \to f^n(U_\beta) = U'_\beta$ are isomorphisms (see Figure 5.2). We can also assume that $f^2(U'_\beta) = U'_\alpha$. We can find $x \in X \bigcap U'_\beta$ because X is dense inside $J(f)$. Then $f^2(x) \in U'_\alpha$. We can take charts from the atlas on X, say $(\sigma_\gamma, U_\gamma), U_\gamma \subset U'_\beta$ and $(\sigma_\nu, U_\nu), U_\nu \subset U_{\alpha'}$ containing x and $f^2(x)$ respectively. Let $\sigma'_\beta = \sigma_\beta f^{-n}$ and $\sigma'_\alpha = \sigma_\alpha f^{-m}$, where the inverse branches f^{-n} and f^{-m} are defined according to our previous discussion. Notice that σ'_β and σ'_α are isomorphisms onto their respective images. Let $A = \sigma_\nu f^2 \sigma_\gamma^{-1}$. The map A is affine (because A is the map f^2 viewed from the atlas over X).

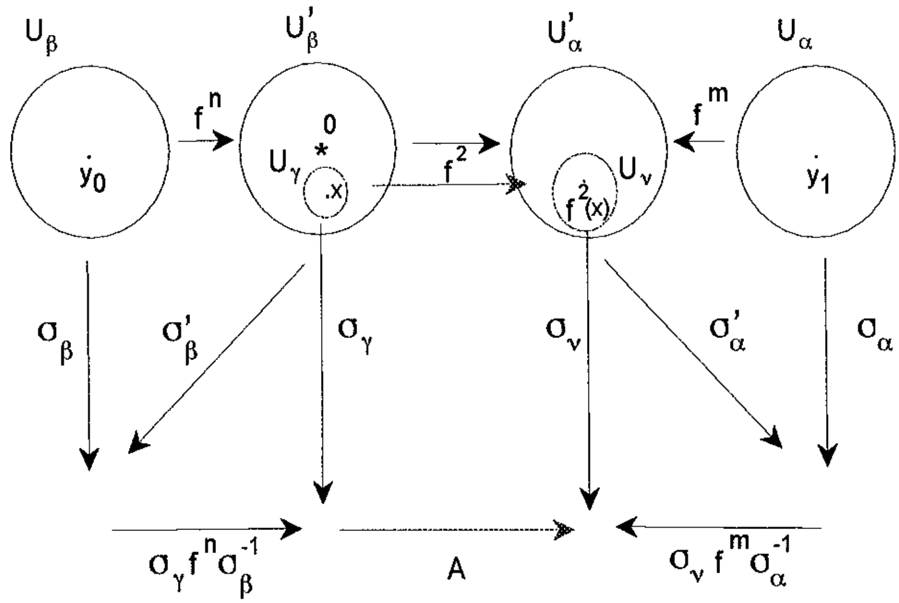

Figure 5.2: Commutative diagram

Notice that

$$\sigma'_\alpha f^2 (\sigma'_\beta)^{-1} = (\sigma_\nu f^m \sigma_\alpha^{-1})^{-1} A (\sigma_\gamma f^n \sigma_\beta^{-1})$$

when we restrict both sides of the equation to the set $\sigma'_\beta(U_\gamma)$.

The left-hand side of the above formula is a restriction of a degree two branched covering, namely $\sigma_\alpha f^{-m} f^2 f^n (\sigma_\beta)^{-1}$. If we restrict the right-hand side of our equation to $\sigma'_\beta(U_\gamma)$ we get an affine map. That is because the right-hand side of our equation is equal to $\sigma_\alpha f^{-m} f^2 f^n(\sigma_\beta)^{-1}$ (see Figure 5.2). This map is affine when restricted to $\sigma'_\beta(U_\gamma)$ because σ_α and σ_β belong to the atlas of the affine structure for $f : X \to X$ (remember that X is f-backward invariant). So the right-hand side of our equation has constant derivative. Contradiction! Therefore we conclude that it is impossible for $f : B_n \to B_n$ to admit an affine structure, for n arbitrarily big. The Lemma is proved. □

5.4 Proof of Theorem 4

We will present in this Section the proof of Theorem 4. Let $f : \cup U_i \to U$ and $g : \cup V_i \to V$ be two polynomial-like maps or two generalized polynomial-like maps belonging to the class $\mathcal{C}$ such that $\mathrm{d_T}(f, g) = 0$. This implies that there exists a homeomorphism $h : J(f) \to J(g)$ conjugating f and g that preserves multipliers. That means that if $x \in J(f)$ is an f-periodic point of period n, then $h(x) \in J(g)$ is a g-periodic point with the same period n and $|Df^n(x)| = |Dg^n(h(x))|$.

We define the hyperbolic sets $X_n = B_n \subset J(f)$ (as introduced in Section 5.2) and $Y_n = f(X_n) \subset J(g)$. By definition, the conjugacy h maps X_n onto Y_n. In other words we have the following family of homeomorphisms: $h_n = h|X_n : X_n \to Y_n$. It is clear that h_n conjugates $f|X_n$ and $g|Y_n$.

The systems $f : X_n \to X_n$ and $g : Y_n \to Y_n$ do not admit invariant affine structures if n is big (see Lemma 5.3.2). In other words, $f : X_n \to X_n$ and

$g : Y_n \to Y_n$ are non-linear systems, for n big. We also know that h_n preserves multipliers (because it is true for $h : J(f) \to J(g)$). So by Theorem 2.3.9 we know that there exist open neighborhoods O_n of X_n and O'_n of Y_n and holomorphic isomorphisms $H_n : O_n \to O'_n$ extending h_n. We can assume that $O_n \subset O_{n+1}$ and $O'_n \subset O'_{n+1}$. Notice that by analytic continuation we have that $H_n = H_{n+1}$ inside O_n.

We define two open sets, $O = \bigcup_n O_n$ and $O' = \bigcup_n O'_n$. We can define $H : O \to O'$ by the following: for any $z \in O$ there exists some n such that $z \in O_n$. Then we define $H(z) = H_n(z)$. The map H is well defined. The map H is holomorphic because locally it coincides with H_n, for some n. It is also injective. The map H conjugates $f|X$ and $g|Y$, where we define $X = \bigcup X_n$ and $Y = \bigcup Y_n$. The sets X and Y are dense subsets of $J(f)$ and $J(g)$, respectively. So the conjugacy H is defined in a open neighborhood of a dense subset of $J(f)$. Our goal is to extend H to a neighborhood of the whole Julia set.

Suppose that z is a point in $J(f)$ not belonging to O. If z is not the critical value, then there exists n and an element z_{-n} of $f^{-n}(z)$, such that $z_{-n} \in O$, and the iteration f^n restricted to a small ball around z_{-n} is injective. Consider the holomorphic map defined in a small neighborhood W of z by $\phi = g^n H f^{-n}$, where by f^{-n} we understand the branch of f^{-n} that takes z to z_{-n}. If W is sufficiently small, then ϕ is an isomorphism. It is clear that ϕ and h coincide where both are defined. By analytic continuation, that means that ϕ also coincides with H where both are defined. In this way we managed to extend H to an open neighborhood of $J(f) \setminus \{f(0)\}$. We will keep calling this extension H. If z is the critical value, then instead of looking for pre-images of z in order to repeat the previous reasoning, just look for the first image of z. Remember that now the second iterate of the critical point belongs to the domain of H. We can define an isomorphism in a small neighborhood of the critical value given by $\phi = g^{-1}Hf$. The same argument as before goes through to show that we have extended H to an open neighborhood of $J(f)$. This proves the Theorem.

5.5 Other consequences of the non-existence of affine structure

It follows from the results in Section 5.3 that if f is in the class $\mathcal{C}$ and n is a big natural number, then the system $f : B_n \to B_n$ does not admit an invariant affine structure. In this Section we will derive some other consequences from this non-existence of affine structure.

According to Lemma 2.3.8, the non-existence of invariant affine structure for $f : B_n \to B_n$ is equivalent to $\log(|Df)|$ not cohomologous to a locally constant function in B_n. In particular, the non-existence of an affine structure implies that $\log(|Df|)$ is not cohomologous to a constant function inside B_n. This last observation together with Theorem 2.3.4 implies the following:

Corollary 5.5.1 *If f belongs to the class $\mathcal{C}$, then there is no λ such that for any j and any f-periodic point p of period j, $|Df^j(p)| = \lambda^j$.*

Proof. By our previous comments, we conclude that if n is big, then there is no λ such that for any j and any f-periodic point p of period j, $|Df^j(p)| = \lambda^j$

inside B_n. That implies the Corollary. □

Definition 5.5.2 If μ is a Borel probability measure in J(f), *then we define the Hausdorff dimension of μ as* $\mathrm{HD}(\mu) = \inf \mathrm{HD}(Y)$ *where the infimum is taken over all sets* $Y \subset$ J(f) *with* $\mu(Y) = 1$.

Remember that the measure $m = \mu_{const}$ is the measure of maximal entropy for the hyperbolic system $f : X \to X$. In [Lyu83] Lyubich showed how to construct a maximal entropy measure m for $f : J(f) \to J(f)$ for any rational function f. Zdunik classified in [Zdu90] exactly when $\mathrm{HD}(m) = \mathrm{HD}(J(f))$ and when $\mathrm{HD}(m) < \mathrm{HD}(J(f))$. The following is a particular case of Zdunik's result if we consider f as a polynomial. It is however an extension of Zdunik's result if f is a generalized polynomial-like map:

Corollary 5.5.3 *If f belongs to the class $\mathcal{C}$ and m is the measure of maximal entropy for f, then* $\mathrm{HD}(m) < \mathrm{HD}(J(f))$.

Proof. It was shown in [PUZ89] that it is enough to check that $\log(|Df|)$ is not cohomologous to a locally constant function in $J(f)$. By that we mean the following: there is no real function h which is equal m-a.e. to a continuous function in a small neighborhood of any point in $J(f)$ without the post-critical set and $\log(|Df|) = c + h(f(x)) - h(x)$.

Suppose that $\log(|Df|)$ is cohomologous to a locally constant function, in the sense defined in the previous paragraph. Remember that the sets B_n are at a positive distance from the closure of the critical orbit. So we would conclude that $\log(|Df|)$ is cohomologous to a constant (in the sense of Definition 2.3.3). Lemma 2.3.8 and Lemma 5.3.2 imply that this is impossible. □

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