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Cylinders for iterated rational maps

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by

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This thesis is dedicated to my grandfather, Dr. Michael Schneider.

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Chapter 1

Overview

1.1 Introduction

A *rational map* $f(z)$ is a holomorphic map of the Riemann sphere $\widehat{\mathbb{C}}$ to itself, and so defines a holomorphic dynamical system on the Riemann sphere. Given a point $z \in \widehat{\mathbb{C}}$, the *forward orbit orbit* of z under f is the set $\{z, f(z), f^{\circ 2}(z), f^{\circ 3}(z), \ldots\}$. The Riemann sphere decomposes into two dynamically distinguished sets. The *Fatou set* $F(f)$ of the map f is the set of points z for which the orbit of any nearby point w is close to that of z. The *Julia set* $J(f)$ of the map f is the complement of the Fatou set. Thus for points on the Julia set, there is *sensitive dependence on initial conditions.* Given a point $z \in J(f)$, there are points w arbitrarily close to z such that the orbits of z and w eventually look completely different.

The theory of iterated rational maps began with the investigations of Schröder into the local dynamical behavior of points near a fixed point of a holomorphic function. Fatou and Julia began the global theory by applying Montel's theory of normal families of holomorphic functions. For the most part, the subject was dormant until relatively recently. One reason for its resurgence is that computers now allow easy visualization of the complexity of dynamical behavior. For example, Figures 5.2, 5.4, 5.6, 5.7, 5.10, and 5.14 show examples of Julia sets. In each case, the Julia set is the boundary between the black and white regions.

Another reason is the richness of the theory. Many problems which are intractable in the general setting of smooth dynamics become much simpler when restricted to holomorphic maps in one complex dimension, since a wide variety of analytic tools apply. In addition to Montel's theory of normal families, one has at one's disposal tools from geometric function theory. Hyperbolic geometry also gives a convenient method of proving many theorems. Sullivan applied the theory of quasiconformal mappings to the study of conformal dynamics and completed the classification of the dynamics on the Fatou set begun by Fatou and Julia. As a result, the dynamics on the Fatou set of an arbitrary rational map is now completely understood.

Holomorphic families, or *parameter spaces*, of rational maps also have a rich structure coming from the the variation of dynamics within the family. A rational map f of degree d may be written in the form

$$f(z) = \frac{a_d z^d + a_{d-1} z^{d-1} + ... + a_0}{b_d z^d + b_{d-1} z^{d-1} + ... + b_0}$$

where the numerator and denominator are relative prime polynomials, and one of a_d, b_d is nonzero. This representation is unique up to multiplication of numerator and denominator by nonzero complex scalars. Sending f to the homogeneous coordinate $[a_d : b_d : ... : a_0 : b_0]$ identifies the set of rational maps of degree d with a Zariski open connected subspace Rat_d in the the complex projective space $\mathbb{C}¶^{2d+1}$ (see [Mil2], Appendix B, who follows [Seg].) We call the complex manifold Rat_d the *parameter space* of rational maps of degree d, equipped with the *algebraic* topology. The group of Möbius transformations acts on Rat_d by conjugation. The quotient $\mathcal{M}_\lceil$ we call the *moduli* space of rational maps of degree d; it is a Hausdorff complex orbifold. Thus a point in the moduli space of degree d maps determines a rational map, up to conjugacy. Where the discussion is independent of the representative of a conjugacy class, we will use the notation f for a point in moduli space. If we restrict our consideration to polynomial mappings and affine conjugacy, then we may similarly define the parameter and moduli spaces of polynomials of a given degree. For example, every quadratic polynomial is affine conjugate to a unique map of the form $f_c(z) = z^2 + c, c \in \mathbb{C}$.

Just as in dynamical space, parameter space decomposes into stable and unstable regions. Some maps f are *structurally stable*. This means that the dynamics of every map g in a sufficiently small neighborhood of f is topologically conjugate on $J(g)$ to the map f on $J(f)$. For other maps f, an arbitrarily small perturbation will produce a map g whose dynamics on its Julia set is very different.

Important examples of the first kind of parameter values are the *hyperbolic* rational maps. As dynamical systems, they are expanding on their Julia sets, and so are much easier to analyze. For example, the Julia set of a hyperbolic rational map has area zero and is locally connected if connected. Hyperbolic maps form an open subset of the set of all rational maps. The set of hyperbolic maps is conjectured to be dense in the space Rat_d. The property of a map being hyperbolic is an invariant of the conjugacy class of f. The image of a connected component of the space of hyperbolic rational maps under the projection to moduli space is called a *hyperbolic component.* Two points in the same hyperbolic component define, up to conjugacy, maps which are topologically conjugate on their Julia sets. Thus the hyperbolic component containing a conjugacy class of map f is a kind of deformation space of f. By varying the map f within its hyperbolic component, we retain qualitative features of the dynamics on the Julia set.

What happens as we deform a hyperbolic map through hyperbolic maps? To make sense of the question, we must work in moduli space rather than parameter space. For we can always deform a map f by conjugating with a sequence A_n of automorphisms of the sphere tending to infinity in $\mathrm{Aut}(\widehat{\mathbb{C}})$.

To understand what happens as we tend to the boundary of a hyperbolic component, we first make some necessary definitions. There are several kinds of points with special dynamical behavior under the iteration of a rational map. A point z is called a *periodic point of* f if $f^{\circ n}(z) = z$ for some $n > 0$. A *cycle* is the forward orbit of a periodic point. A periodic point is called *attracting, indifferent, or repelling* according to whether the product of the derivatives of f along the cycle is less than, equal to, or greater than one in modulus. An indifferent cycle whose multiplier is a root of unity is called a *parabolic cycle.* A rational map of degree d also has $2d-2$ critical points, counting with multiplicity. The *postcritical set* $P(f)$ is defined as

$$P(f) = \overline{\bigcup_{n>0, f'(c)=0} f^{\circ n}(c)}.$$

If $|P(f)| < \infty$ we say that the map f is *postcritically finite.* The postcritical set plays a fundamental role in the theory of iterated rational maps. For example, if f is postcritically finite, then the Julia set of f is connected. A rational map is hyperbolic if and only if $|P(f) \cap J(f)| = \emptyset$. Multipliers of periodic cycles and the property of being hyperbolic or postcritically finite are all invariants of the conformal conjugacy class.

Recall that a hyperbolic component containing a conjugacy clas of map f is thought of as a kind of deformation space of f. Consider a sequence $\{f_n\}_{n=0}^{\infty}$ of conjugacy classes of maps in $\mathcal{M}_{\lceil}$. There are essentially three possibilities for the limiting behavior of the sequence $\{f_n\}_{n=0}^{\infty}$. One possibility is that during the deformation, the modulus of the multiplier of an attracting cycle of period p approaches a root of unity. The limit f_∞ then has a parabolic cycle. Suppose that the period of this parabolic cycle is equal to p. Then there exist arbitrarily small perturbations of f_∞ to hyperbolic maps g with an attracting periodic cycle of period $k \cdot p$ for some $k > 1$. The parabolic cycle of f_∞ *bifurcates* into an attracting cycle of higher period and a repelling cycle of the same period for the map g. The closure of the hyperbolic component containing g intersects the hyperbolic component containing f. We say that the hyperbolic components containing f and g *touch.* Other kinds of deformations reverse this process. This first possibility for the limit is distinguished by the fact that for the limiting map f_∞, the postcritical set $P(f_\infty)$ and the Julia set $J(f_\infty)$ intersect only in a parabolic cycle. The second possibility is that in the limit, the postcritical set and the Julia set intersect in a complicated manner. A third possibility is that the limit does not exist as a rational map of degree d.

We will now describe a heuristic way of thinking about what happens in the limit. This heuristic has been confirmed experimentally in many settings, and has been well-established for degree two maps (see Section 1.4). The goal of this work is to make precise the intuitive ideas sketched below.

The limiting map f_∞ in the first possibility is thought of as produced from f by "pinching along a lamination". Suppose that each Fatou component of

f is an open disc, i.e. that $J(f)$ is connected. Into each component of the Fatou set of f, consider gluing a collection L of chords such that the interiors of no two chords intersect, and such that the collection is invariant under the dynamics of f. The collection L is called a *lamination.* Intuitively, as we tend to the boundary within the hyperbolic component $H(f)$, the chords forming some lamination L gradually collapse to points. The Julia set of f_∞ is obtained from the Julia set of f by identifying any two points which are endpoints of the same chord.

Another way to think about the limiting map f_∞ is as a combination of f with another conformal dynamical system. Consider a finite family $\mathcal{P}$ of polynomials acting on disjoint copies of the complex plane. This means that each copy is mapped via a polynomial. By gluing in the dynamics of $\mathcal{P}$ into an invariant set of Fatou components of f, we obtain a new *topological* map $f * \mathcal{P}$ of the sphere to itself called the *tuning* of f by $\mathcal{P}$. Intuitively, the gradually collapsing chords L uniquely determine a finite family $\mathcal{P}$ such that the tuning $f * \mathcal{P}$ leaves the set L of chords invariant. Conjecturally, collapsing every chord to a point gives a semiconjugacy from $f * \mathcal{P}$ to f_∞.

The ideas in the previous two paragraphs can also be used to describe the second possibility. The third possibility can also be explained using the above ideas. What can go wrong is that the lamination L might separate the sphere. For example, if some Fatou component Ω does not have Jordan curve boundary, then one can insert a chord l into Ω which separates the sphere. If this happens, the domain of the map f_∞ is no longer a sphere. If we restrict the map f_∞ to some piece of the quotient which is a sphere, then we lose a definite portion of the dynamics in the limit, and so the degree of f_∞ must decrease. A natural question, then, is to determine when a Fatou component does not have Jordan curve boundary, and when the closures of two Fatou components intersect in such a manner that one can glue in a lamination whose chords separate the sphere. In the former case, we say that the boundary of a Fatou component is *pinched.* If the closures of two Fatou components intersect, we say that they *touch.*

Another intuitive explanation of why the limit may not exist comes from Thurston's characterization of postcritically finite rational maps as branched coverings of the sphere. The tuning $f * \mathcal{P}$ may have a topological obstruction to being *combinatorialy equivalent* to a rational map, as defined by Thurston. A topological map is combinatorially equivalent to a rational map if and only if it does not have a topological obstruction called a *Thurston obstruction.* A Thurston obstruction is a finite set of curves which have a special invariance property. Basically, if a topological map has a Thurston obstruction, then it cannot be deformed to a conformal map while at the same time retaining all of the topology.

Let $\mathcal{P}$ be the finite family of polynomials determined by some deformation of f which collapses a lamination L. Suppose $f * \mathcal{P}$ has a Thurston obstruction. Sometimes, but not always, a Thurston obstruction for $f * \mathcal{P}$ is formed by a finite set of periodic chords in L which separates the sphere. In this case, the obstruction is a very special one called a *Levy cycle.* Since the chords separate the sphere, we do not expect the limiting map to exist. In this case, we say that there is an *obvious* obstruction to the tuning $f * \mathcal{P}$ being combinatorially equivalent to a rational map. Thus, there is an obvious obstruction for some tuning $f * \mathcal{P}$ only if some periodic Fatou component of f does not have Jordan curve boundary, or if there is a collection of periodic Fatou components which touch. However, it is not always the case that an obstruction to a tuning $f * \mathcal{P}$ being combinatorially equivalent to a rational map is formed in this way. That is, it is possible for the tuning $f * \mathcal{P}$ to have a Thurston obstruction even though the quotient space obtained by collapsing every chord to a point is a sphere. For this reason, we say that there is a *non-obvious* obstruction to the tuning $f * \mathcal{P}$.

Since the goal of this work is to make these intuitive ideas precise, we investigate the relationship between

- combining a hyperbolic rational map f with a finite family of polynomials $\mathcal{P}$, using the combinatorics developed by Thurston;
- the topology of the Julia set of f;
- compactness properties of the hyperbolic component containing f.

Our approach is guided by analogous known results in the theory of hyperbolic structures on three-manifolds. There, the key link relating combinations of hyperbolic three-manifolds, deformation spaces of hyperbolic three-manifolds, and the topology of fractal objects naturally associated to hyperbolic three-manifolds is the notion of a *cylinder*. See Section 1.8 for further details.

In this work, we define a *cylinder* for an iterated rational map which is both hyperbolic and postcritically finite. This is like considering hyperbolic three-manifolds which are convex cocompact. For such maps, our main results are:

- a theorem which asserts that a rational map without cylinders can always be combined with a finite family $\mathcal{P}$ of topological polynomials satisfying a special property which we call *starlike* (Theorem 7.1).

  Conjecturally, a degree d starlike polynomial is a postcritically finite hyperbolic polynomials lying in a hyperbolic component which touches the hyperbolic component containing the map $z \mapsto z^d$.

- a combinatorial characterization of when and how the closures of two periodic Fatou components intersect, and when and how a Fatou component is not a Jordan domain. This will determine much (but not all) of the topology of the Julia set. As applications (Section 5.4), we give combinatorial characterizations of rational maps whose Julia sets are Sierpinski carpets and of maps all of whose Fatou components are Jordan domains.
- a characterization of cylindrical rational maps in terms of the dynamics and topology of their Julia sets (Theorem 6.1) .
- (with Tan Lei) a combinatorial construction which yields new examples of rational maps whose Fatou components have interesting topology (Section 5.6.2).

- A reformulation of results due to Makienko into the language of cylinders, which then imply that hyperbolic component which contain maps with special kinds of cylinders have noncompact closure in moduli space (Chapters 6 and 8).

  There are no examples of hyperbolic components in the moduli space of degree d rational maps which are known to have compact closure.

- A theorem which says that for a postcritically finite rational map f with exactly two critical points which not conjugate to a polynomial, every Fatou component of f has Jordan curve boundary (Chapter 9).

1.2 Contents of this chapter

In Section 1.3 of this chapter we discuss what is meant by combinations and decompositions of rational maps.

In Section 1.4 we discuss some related known results for degree two rational maps. We conclude the section by explaining what goes wrong when trying to naively generalize the results for degree two to higher degrees.

In Section 1.5 we discuss how restricting to tunings by starlike polynomials allows us to generalize some of the techniques used in degree two.

In Section 1.6 we discuss how we combinatorially detect intersections of closures of Fatou components, and define cylinders for an iterated rational map.

In Section 1.7 we state a conjecture relating cylinders, tuning, compactness properties of deformation spaces, and the topology and dynamics of Julia sets. We call this conjecture the *Limiting Map conjecture*. This conjecture will generalize the known results for degree two maps. Our results will prove this conjecture in part.

In Section 1.8 we discuss the connection with the theory of hyperbolic three-manifolds. We summarize known results from this theory into the Limiting Manifold Theorem.

In Section 1.9 we conclude with a brief description of the contents of each chapter.

1.3 Combinations and decompositions of rational maps

1.3.1 Rational maps as combinatorial objects

For postcritically finite rational maps $f(z)$, there is a well-defined notion of combinatorics due to Thurston. A postcritically finite rational map may be regarded as a branched covering of the sphere to itself, up to a kind of isotopy called *combinatorial equivalence*. Thurston [DH2] has combinatorially characterized those postcritically finite branched coverings which are combinatorially equivalent to rational maps, and has shown that in all except a handful of completely understood special cases, a postcritically finite rational map is determined by its combinatorial class, up to conjugacy by automorphisms of the sphere. A postcritically finite branched covering f is combinatorially equivalent to a rational map if and only if there does not exist a set of disjoint simple closed curves Γ which are invariant under f in a special manner. The collection Γ is called a *Thurston obstruction* to the existence of a rational map combinatorially equivalent to f.

1.3.2 Combining rational maps

Douady and Hubbard [DH1] have defined a way of combining a postcritically finite hyperbolic rational map with a postcritically finite family $\mathcal{P}$ of polynomials to yield a new combinatorial class of branched covering of the sphere. This process is called *tuning f by $\mathcal{P}$*. Let B be forward invariant subset of $f^{-1}P(f)$ containing every critical point which eventually lands in B. Remove a neighborhood of B, and glue in a finite family $\mathcal{P}$ of discs mapping by the action of a finite family of polynomials on copies of the extended complex plane. The result $f * \mathcal{P}$ is the tuning of f by $\mathcal{P}$. An open question is to determine combinatorial conditions on f, $\mathcal{P}$, and the gluing data for the tuning $f * \mathcal{P}$ to be combinatorially equivalent to a rational map.

1.3.3 Decomposing rational maps

In [McM3], McMullen defines an inverse process to tuning which we call *collapsing.* Though defined for arbitrary branched coverings, we will specialize to the postcritically finite case. Let f be a postcritically finite branched covering of the sphere to itself. To form a quotient g of f, find a set of discs $\mathcal{D}$ which is forward-invariant up to isotopy fixing $P(f)$. Collapse each component of $\mathcal{D}$ to a point. The result is a new branched covering g, well-defined up to combinatorial equivalence, called a *quotient* of f. We prove (Theorem 3.13) that if R and f are combinatorial classes of branched covers, then f is a quotient of R if and only if R is the tuning of f by some finite family $\mathcal{P}$ of topological polynomials. McMullen has proved that the quotient of a rational map is always again a rational map (Theorem 4.33). Using his result and the previous theorem, we prove that a postcritically finite hyperbolic rational map R admits a quotient if and only if it is the tuning $f * \mathcal{P}$ of a rational map by a finite family of conformal polynomials (Theorem 4.34).

1.4 Known results for quadratic rational maps

Mating is a special kind of tuning. To define mating, let f and g be two monic postcritically finite hyperbolic quadratic polynomials of the form $z^2 + c$. The Julia sets $J(f), J(g)$ are connected. There is a canonical extension of f and g to maps of the complex plane compactified by the circle at infinity. The maps f and g may be glued together as follows: identify $\overline{\Delta}$ with the domain of the extension of f, and identify $\widehat{\mathbb{C}} - \Delta$ with the domain of the extension of g via reflection $z \mapsto 1/\overline{z}$ through the unit circle. The result is a postcritically finite branched covering of the sphere which is called the *mating* of f and g. The dynamics of f near the fixed critical point at infinity is replaced with the dynamics of g acting on the extended complex plane.

We now present some evidence supporting the heuristic picture sketched in the introduction. We begin by summarizing some well-known facts. Given a quadratic polynomial $f_c = z^2 + c$, if $J(f_c)$ is connected, there is a canonical Riemann map ϕ to the basin of infinity Ω sending $0 \in \Delta$ to ∞ and conjugating $z \mapsto z^2$ on Δ to f on Ω. An *external ray of angle* t, denoted R_t, is the image $\phi(\{z | z = re^{2\pi it}, r \in [0,1)\})$ of a radial segment of argument $2\pi it$. If this arc has a well-defined limit as $r \to 1$, we say that the ray *lands* at the limit point.

The *lamination* L of f is the equivalence relation on the circle defined by setting $s \sim t$ if the s and t-rays land at a common point. The boundaries of the convex hulls of the equivalence classes then form a set of chords in the open unit disc.

For polynomials, there is a finite set of rays landing at each periodic point. The landing point of the ray of angle 0 is called the β fixed point; the other (if it exists) is called the α*-fixed point*. Suppose $q > 1$ rays land at the α fixed point of f_c. The angles of the these rays form a finite set in S^1 which is permuted under f in a manner which agrees with an orientation-preserving homeomorphism of the circle. Thus this set has a well-defined *rotation number* $p/q \in (0,1)$. We then say that the map f has *combinatorial rotation number* p/q *at its* α*-fixed point*. The set of all c such that $J(f_c)$ is connected and f_c has combinatorial rotation number p/q is called the p/q*-limb* of the Mandelbrot set. The $1 - p/q$-limb is image of the p/q limb under the reflection $c \mapsto \overline{c}$. If f hyperbolic, and is in the p/q limb of the Mandelbrot set, then there is a unique attracting fixed point, and a unique attracting cycle of period larger than one.

Let f by a hyperbolic polynomial quadratic polynomial with a nonzero combinatorial rotation number at its α fixed point. Rees [Ree1] has proved that the hyperbolic component $H(f)$ in the degree two moduli space $\mathcal{M}_2$ of rational maps is naturally biholomorphically equivalent to $\Delta \times \Delta$. The map is given by sending a map f to the pair (λ, μ), where λ is the multiplier of the unique attracting fixed point and μ is the multplier of the unique attracting cycle of period larger than one.

The following theorem is a summary of known results.

Theorem 1.1 *Let f be a postcritically finite hyperbolic quadratic polynomial of the form $z \mapsto z^2 + c, c \neq 0$, and let $H(f)$ denote the hyperbolic component in $\mathcal{M}_2$ containing f. Let p/q be a rational number in $(0,1)$. Then the following are equivalent:*

1. *The map $f(z)$ has combinatorial rotation number p/q near its α-fixed point.*

2. *If g is a postcritically finite hyperbolic polynomial which does not have combinatorial rotation number $1 - p/q$, then the the mating of f and g is*

combinatorially equivalent to a rational map.

3. *For* $|\lambda| < 1$*, let* f_λ *denote the unique map in* $H(f)$ *with a fixed point at infinity of multiplier* λ *and a periodic finite critical point. Then for* $p'/q' \neq 1 - p/q$*, the radial limit*

$$\lim_{\lambda \to e^{-2\pi i p'/q'}} f_\lambda,$$

exists in M_2*.*

4. *Let* $L_{\overline{g}}$ *be the lamination for* $\overline{g}$ *as a subset of the unit disc. Let* $\phi : (\overline{\Delta}, 0) \to (\overline{\Omega}_f, \infty)$ *be the extension to the boundary of the canonical conjugacy from* $z \mapsto z^2$ *on* Δ *to* f *on its basin* Ω_f *of infinity. Then the quotient space* $\widehat{\mathbb{C}}/\sim$ *obtained by collapsing every chord in* $\phi(L_{\overline{g}})$ *to a point is not homeomorphic to a sphere.*

These conditions concern (1) the combinatorial dynamics of f, (2) combinations of f with other maps, (3) deformations of f, and (4) the topology of $J(f)$ and the dynamics of f on $J(f)$. The implication (1) implies (2) shows that there are only obvious obstructions to matings of postcritically finite quadatic polynomials being combinatorially equivalent to rational maps.

That conditions (1) and (4) are equivalent follows immediately from the definition of combinatorial rotation number and lamination. That (2) implies (1) has been known for some time; see e.g. [Lev]. That (1) implies (2) is a special case of the main result in [Tan2]. That (3) implies (1) is proved in [Pet] using extremal length estimates for the multipliers of fixed points. This also follows from a theorem in [Mak] which does not give estimates. According to Rees (personal communication) the implication (1) implies (3) follows from techniques in [Ree1].

Rees [Ree2] and Ahmadi [Ahm] have extended the implication (1) implies (2) to analogous results for the general quadratic postcritically finite hyperbolic case. Their results also show that the only obstructions to the tuning of a hyperbolic quadratic polynomial being combinatorially equivalent to a rational map are obvious ones. That is, they show that the tuning of a postcritically finite hyperbolic rational map by a family of quadratic polynomials has a Thurston obstruction only when there are chords for the lamination which separate the sphere when glued into the Fatou components of f.

Tan's proof of (1) implies (2), and the proofs given by Rees and Ahmadi mentioned in the previous paragraph, rely on the fact that for degree two maps, a Thurston obstruction may always be reduced to a special one called a Levy cycle. They then use the expanding nature of hyperbolic rational maps to conclude that the existence of a Levy cycle implies the existence of an obvious obstruction. However, an example of Tan and Shishikura [ST1] shows that there are matings of cubic polynomials which are obstructed, but which have no Levy cycles or obvious obstructions.

1.5 Starlike polynomials

To circumvent the difficulties posed by the existence of non-obvious obstructions, we will consider only tunings by families of starlike polynomials. Using a generalization of a theorem of M. Shishikura and Tan Lei ([ST1]), which we call the *Shishikura-Tan theorem* (Theorem 4.29), we will show that if $\mathcal{P}$ is a finite family of starlike polynomials, any Thurston obstruction to a tuning $f * \mathcal{P}$ may be reduced to a Levy cycle. A postcritically finite hyperbolic polynomial p is starlike if there is a finite graph G whose vertices are the finite postcritical points of p, such that p maps edges of G to edges of G, up to isotopy fixing the postcritical set of p. The Shishikura-Tan theorem will imply that any Thurston obstruction which meets this graph must be a Levy cycle. We will use the existence of a Levy cycle to deduce the existence of a cylinder for f to prove that tunings of acylindrical maps by starlike polynomials are always unobstructed.

1.6 Cylinders and the characteristic subcomplex

Recall that there are obvious obstructions to the tuning $f * \mathcal{P}$ being combinatorially equivalent to a rational map only when there are Fatou components with non-Jordan curve boundary, or when there are Fatou components which touch. We now discuss a combinatorial characterization of when this occurs.

1.6.1 Jordan domain Fatou components

Two preliminary results in this direction are the following results, which we prove in Chapter 9. The techniques we use to prove these theorems, however, will not be used in our subsequent characterization of pinching and touching. In the following, note that the hypotheses of the theorems refer only to the dynamics of f restricted to the finite set $P(f)$.

Theorem 1.2 *Let f be a postcritically finite rational map (not necessarily hyperbolic) with exactly two critical points, not counting with multiplicity. Then exactly one of the following possibilities holds:*

- *f is conjugate to z^d and its set is a Jordan curve, or*
- *f is conjugate to a polynomial of the form $z^d + c, c \neq 0$, and the Fatou component containing infinity is the unique Fatou component which is not a Jordan domain, or*
- *f is not conjugate to a polynomial, and every Fatou component is a Jordan domain.*

In the next theorem, there are no hypothesis on the number of critical points.

Theorem 1.3 *Let f be a postcritically finite hyperbolic rational map for which every postcritical point is periodic. Then there is at least one cycle of Fatou components with Jordan curve boundary.*

Corollary 1.4 *Let f be a postcritically finite map for which every postcritical point lies in the same cycle. Then every Fatou component has Jordan curve boundary.*

The first theorem in this section, for special degree two cases, appears in [Ree2]. The general degree two case appears in [Ahm]. Their arguments assume without proof that a certain lamination associated to a postcritically finite degree two rational map has a kind of backward invariance property. In Section 5.6 we give examples of maps for which the corresponding laminations fail to be invariant. These examples provide motivation for using the dynamics of isotopy classes of arcs in studying touching and pinching of Fatou component boundaries, described below. Tan Lei (personal communication) has observed that these examples may be constructed by a process which we call *blowing up an arc*. We show in Section 5.6.2 that this construction provides another way of producing new rational maps from a given rational map. The proof that the blowing up construction is unobstructed depends on the Shishikura-Tan theorem.

1.6.2 Laminations for rational maps

It is essentially enough to decide when the closures of periodic Fatou components intersect, or when a periodic Fatou component has non-Jordan curve boundary, when trying to determine when there are Fatou components with non-Jordan curve boundary, or when two Fatou components touch.

We will reduce this question to the computation of the *lamination* associated to a postcritically finite hyperbolic rational map f. Let $Q(f) = f^{-1}(P(f))$. For $x \in Q(f)$, let Ω_x denote the Fatou component containing x. A classical theorem due to Böttcher implies that there exist Riemann mappings $\phi_x : (\Delta, 0) \to \Omega_x$ conjugating $z \mapsto z^{d_x}$ to f, where d_x is the local degree of f near x. The theorem also implies that there are finitely many such choices for the maps ϕ_x; fix one such choice. Since f is hyperbolic, $\partial\Omega_x$ is locally connected, so the ϕ_x extend to $\overline{\Delta}$, by a theorem of Carathéodory.

The *lamination* of f is the equivalence relation on $Q(f) \times S^1$ defined by $(x, s) \sim (y, t)$ if $\phi_x(e^{2\pi is}) = \phi_y(e^{2\pi it})$. The set $\phi_x([0, 1]e^{2\pi is})$ is called an *internal ray*. A *chord* is a pair of equivalent points. The *arc formed by a chord* is the union of the corresponding pair of internal rays.

Definition 1.5 (Preliminary definition) *A* **cylinder** *for a postcritically finite hyperbolic rational map f is a finite collection $\mathcal{R}$ of internal rays of f such that*

1. *$f(\mathcal{R}) = \mathcal{R}$ (periodicity),*
2. *the union of the rays in $\mathcal{R}$ separates the sphere (separation), and*
3. *no proper subset of $\mathcal{R}$ satisfies (1) and (2) above (minimality).*

We will now characterize combinatorially when this happens. One way to do this is to require the following: there exists a set W of arcs in the sphere with endpoints in $Q(f)$ and interiors disjoint from $Q(f)$, such that the preimage $f^{-1}(W)$ contains W, up to isotopy, and such that there is an embedded loop made up of arcs in W which separates points of $Q(f)$. We will take a different avenue and study arcs individually.

Define an *arc in* $(S^2, Q(f))$ to be an arc α with endpoints in $Q(f)$ and whose interior is disjoint from $Q(f)$. Two such arcs are said to be *isotopic* if one can be deformed to the other through such arcs. The set of isotopy classes of arcs in $(S^2, Q(f))$ we denote by $\mathcal{A}(Q(f))$. Given a chord, the arc α it forms is an arc in $(S^2, Q(f))$. Consider the forward orbit $\{f^{\circ n}(\alpha)\}_{n=0}^{\infty}$. This determines a sequence of arcs in the sphere with endpoints in $Q(f)$. Record the isotopy class of each term as an element of $\mathcal{A}(Q(f))$. We will show that this data suffices to determine the chord.

We will actually show more than this. We define the *pushforward relation* f_*^Q on $\mathcal{A}(Q(f))$. An isotopy class $[\beta]$ is in the image $f_*^Q([\alpha])$ if and only if the there is a representative of $[\alpha]$ whose image under f represents the element $[\beta]$. The set A of all eventually periodic elements under the relation f_*^Q is finite (Theorem 5.2). The restriction of f_*^Q to the product $A \times A$ determines a (not necessarily irreducible) subshift of finite type (Σ_f, σ_f). The underlying space Σ_f is the space of all possible forward orbits of elements in A under the relation. The map σ_f is the one-sided shift map which forgets the first term in a sequence. We call the pair (Σ_f, σ_f) the *combinatorial characteristic subcomplex (CCS)* of f. The CCS is isomorphic as a topological dynamical system to the action of f on the space of chords of its lamination (Theorem 5.8).

As an immediate application of this fact (Section 5.4), we can combinatorially characterize when and, roughly speaking, how two periodic Fatou components intersect: the intersection is finite, countable, or contains a Cantor set if and only if the corresponding portion of Σ_f is finite, countable, or contains a Cantor set. A periodic Fatou component is a Jordan domain if and only if a certain portion of Σ_f is empty. We also characterize when the intersection of two Fatou components is a Jordan curve and one component is also a Jordan domain.

Using this analysis, we characterize when a Julia set is a Sierpinski carpet: this occurs if and only if A is empty, if and only if a slightly simpler set is empty (one where we measure isotopy classes in $(S^2, P(f))$).

The CCS can be empty–this occurs if and only if the Julia set of f is a Sierpinski carpet. In [Mil4] an example is given which shows this can occur. Their example is a degree two map with seven postcritical points, and their proof uses polynomial-like mappings. We give an example in Section 5.6 of a degree three postcritically finite hyperbolic map with four postcritical points and show that its CCS is empty, thus proving that its Julia set is a Sierpinski carpet.

1.6.3 Cylinders

Given any set $S \subset \Sigma_f$, let $A(S)$ denote the union of all coordinates occurring in some sequence in S. Then $A(S)$ is a subset of A. A finite subset $E \subset \mathcal{A}(Q(f))$ is said to *separate* $Q(f)$ if, given any collection of representatives of E, at least two components of the complement of their union contains points of $Q(f)$.

Definition 1.6 (Cylinder) *A combinatorial cylinder is a finite set $C \subset \Sigma_f$ such that*

1. *$\sigma(C) = C$ (periodicity);*
2. *$A(C)$ separates points of $Q(f)$ (separation);*
3. *no proper subset of C satisfies (1) and (2) (minimality).*

We will show that this definition of cylinder agrees with the previous one (Theorem 6.5). A postcritically finite hyperbolic rational map for which $A = \emptyset$ we will call *strongly acylindrical.*

1.7 The Limiting Map conjecture

We propose the following conjecture which generalizes Theorem 1.1 .

Conjecture 1.7 (Limiting map conjecture) *Let $f(z)$ be a postcritically finite hyperbolic rational map of degree $d \geq 2$. Then the following are equivalent.*

1. *The map $f(z)$ is acylindrical.*
2. *For every postcritically finite family of starlike polynomials $\mathcal{P}$, the tuning $f * \mathcal{P}$ of f is combinatorially equivalent to a rational map.*
3. *The hyperbolic component $H(f)$ containing f has compact closure in the moduli space of rational maps of degree d.*
4. *There is no finite set $\mathcal{R}$ of internal rays of f for which $f(\mathcal{R}) = \mathcal{R}$ and which separates the sphere.*

Our main results establish the equivalence of (1) and (4) and the implication (2) implies (1). Our reformulation of Makienko's results becomes a partial proof of the implication (3) implies (1). Conjecturally, the tuning of an acylindrical map f by a finite starlike family $\mathcal{P}$ is again acylindrical. Thus one can continue tuning an acylindrical map by starlike polynomials. Thus at least conjecturally, for an acylindrical map, one can carry out any infinite sequence of bifurcations.

Alternatively, one might replace condition (2) with the following: the limit in $H(f)$ corresponding to pinching along the lamination corresponding to any finite family $\mathcal{P}$ exists. Conjecturally, this limiting map is topologically conjugate to the tuning $f * \mathcal{P}$.

If f is strongly acylindrical, then the Julia set of f is a Sierpinski carpet; compare this with the Limiting Manifold theorem below.

1.8 Motivation from hyperbolic three-manifolds

We propose viewing the connection between tuning, compactness of hyperbolic components, and the topology of the Julia set from the point of view of the extensive analogy between the theories of rational maps and Kleinian groups as conformal dynamical systems on the Riemann sphere.

For example, the limit set $\Lambda(\Gamma)$ of a Kleinian group Γ is the analog of the Julia set; the domain of discontinuity $\Omega(\Gamma)$ is the analog of the Fatou set of a rational map. While there is now a general notion of a holomorphic dynamical system (see [MS]), we will instead focus on aspects of this analogy which stem from the fact that Kleinian groups arise as the fundamental groups of hyperbolic three-manifolds.

Thurston has proved that all atoroidal Haken three-manifolds admit hyperbolic structures; see [Thu1], [Thu2], [Thu3], [Mor]. A *hyperbolic structure* on an oriented three-manifold M, with possibly nonempty boundary, is a pair (ψ, N) consisting of a Riemannian three-manifold N with constant sectional curvature -1, together with a homotopy equivalence $\psi : M \to N$ which preserves the peripheral structure in $\pi_1(M)$ if M has boundary.

There is a natural topology on the set of hyperbolic structures on M called the *algebraic topology*, defined as follows. The map ψ determines a discrete faithful representation of the fundamental group of M into the group $\mathrm{Isom}^+(\mathbb{H}^3)$ of orientation-preserving isometries of hyperbolic three-space, up to conjugation. The group $\mathrm{Isom}^+(\mathbb{H}^3)$ is naturally identified with the group $\mathrm{Aut}(\widehat{\mathbb{C}})$ of conformal automorphisms of the Riemann sphere, since the Riemann sphere is naturally the boundary at infinity of hyperbolic three-space. Let $\mathcal{V}$ denote the variety of discrete faithful representations of $\pi_1(M)$ into $\mathrm{Aut}(\widehat{\mathbb{C}})$. Then the set of hyperbolic structures on M is identified with the quotient of $\mathcal{V}$ by the action of $\mathrm{Aut}(\widehat{\mathbb{C}})$ acting by conjugation. If $\pi_1(M)$ is nonabelian, this space is Hausdorff.

A compact three-manifold M is *toroidal* if there is a map of a torus into M which is injective on the level of fundamental groups but is not homotopic into ∂M. Being atoroidal is a necessary condition for the existence of a hyperbolic structure on M, since a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of a hyperbolic three-manifold N is necessarily the fundamental group of an end of N corresponding to a rank two cusp. M is said to have *incompressible boundary* if it has no sphere boundary components and if every curve on ∂M which is contractible in M is contractible in ∂M.

A three-manifold M is said to be *Haken* if it is compact, orientable, irreducible, and contains an *incompressible surface* F. An incompressible surface F in M is an embedded connected surface not equal to a sphere or projective plane such that the inclusion map induces an injection of fundamental groups. A Haken three-manifold is one which is built from copies of the three-ball by successively gluing along incompressible boundary components. Any oriented three-manifold with a boundary component which is not a sphere is Haken.

Thurston's proof of this theorem proceeds by an inductive argument, using the fact that the manifold is Haken. At the inductive step, one has a geometrically finite (read: "nice") hyperbolic structure (ψ, N) on a compact three-manifold M with incompressible boundary and an orientation-reversing homeomorphism $h : \partial M \to \partial M$ which is regarded as gluing data. The boundary at infinity of N inherits a natural conformal structure, and the *deformation space* of the $(N, \partial N)$ then coincides with the Teichmüller space of ∂N. The gluing data h determines a map from the Teichmüller space of ∂N to itself called the *skinning map.* A fixed point for the skinning map is a hyperbolic structure N on M such that the following holds. Let Γ_N be the Kleinian group corresponding to the hyperbolic manifold N. Then there exists a finite set of elements G of $\mathrm{Aut}(\widehat{\mathbb{C}})$ such that $\mathbb{H}^3/\langle\Gamma_N, G\rangle$ is a hyperbolic structure on M/h.

A danger is that the manifold M might have a *cylinder.* A cylinder for a three-manifold M with boundary is an essential map of pairs $(I\times S^1, \partial I\times S^1) \to (M, \partial M)$ which is not homotopic into ∂M. If M has a cylinder and the homeomorphism h identifies the ends of a cylinder, the glued manifold M/h is toroidal, and hence has a topological obstruction to the existence of a hyperbolic structure. The *characteristic submanifold* of a three-manifold with incompressible boundary is the unique (up to isotopy) minimal properly embedded submanifold which contains all of the cylinders; see [Jac].

It turns out that the presence of cylinders for M is reflected in compactness properties of the deformation space of a geometrically finite hyperbolic structure (N, ψ) on M, and in the topology of the limit set of the fundamental group of N, regarded as a Kleinian group Γ_N acting on the sphere. For example, the limit set of the Kleinian group which is the fundamental group of a fixed point for the skinning map is a Sierpinski carpet where the holes are *round* discs. For simplicity, we ignore cusps. A hyperbolic three-manifold N is said to be *convex compact* if the image of the convex hull of its limit set under the projection from $\mathbb{H}^3$ to N is compact. Such a manifold has no cusps and is geometrically finite.

Theorem 1.8 (Limiting manifold theorem) *Let N be a convex compact geometrically finite hyperbolic three-manifold with nonempty incompressible boundary. Then the following are equivalent.*

1. *N is acylindrical.*

2. *Given any orientation-reversing homeomorphism $h : \partial N \to \partial N$, the quotient manifold N/h admits a hyperbolic structure.*

3. *The deformation space of N has compact closure in the space of all hyperbolic structures on M.*

4. *The limit set of the fundamental group of N, regarded as a Kleinian group, is a Sierpinski carpet.*

Alternatively, one might replace condition (2) with the following: the limit of any deformation of N corresponding to pinching a finite set of disjoint simple closed curves exists.

That (2) implies (1) is clear. If a cylinder exists, any gluing map h which identifies the ends of the cylinder yields a torus in N/h. That (3) implies (1) may be proved as follows. Pinching the ends of the cylinder (we may assume they are disjoint simple curves) yields a sequence of deformations whose limit does not exist. This is well-known; we give a new proof in Section 8.5. That (1) implies (2) is part of Thurston's geometrization theorem; it proceeds by first proving (1) implies (3) [Thu1]. An alternative proof which does not take this route may be found in [McM2]. That a cylinder in a geometrically finite hyperbolic three-manifold causes closures of the domain of discontinuity to intersect follows easily by considering the lifts of geodesics representing ends of the cylinder to the Riemann sphere under the projection map from the universal cover; we give a proof in Chapter 8. That (4) implies (1) seems to be well-known, but I have been unable to locate the original reference. It is sometimes attributed to Maskit. It also follows from work of Gromov.

1.9 Summary of chapters

2. Background from the theory of iterated rational maps. We state definitions and facts needed throughout this work.

3. Branched coverings. We define branched coverings, combinatorial equivalence, tuning, and collapsing, and prove that tuning and collapsing are inverse.

4. Combinatorial dynamics of arcs and curves. We define the pushforward relation on arcs and its well-known analog for simple closed curves, which we call the *lifting relation* on simple closed curves. Using ideas of Shishikura and Tan, we show that the presence of isotopy classes of periodic arcs strongly restricts the possible dynamics of simple closed curves, and prove their theorem in our language. We also show how the lifting and pushforward relations behave under tuning and collapsing.

5. The characteristic subcomplex. We begin by studing some motivating examples, and apply the pushforward relation to construct the combinatorial characteristic subcomplex. We apply our combinatorial analysis to the study of when and how two Fatou components touch, and when the boundary of a Fatou component is not a Jordan curve. We conclude with further examples.

6. Cylinders. We define cylinders for postcritically finite hyperbolic rational maps and prove the equivalence of our definitions of cylinders. We also define starlike polynomials. We will actually define another kind of cylinders. Combinatorial cylinders will be the cylinders defined above. Geometric cylinders will be candidates for the hypothesis of Makienko's theorem giving sufficient conditions for a hyperbolic component to have noncompact closure in moduli space. We construct a bijection between the two kinds of cylinders.

7. Existence of acylindrical starlike tunings. We show that tunings of acylindrical maps by starlike polynomials are combinatorially equivalent to rational maps.

8. Compactness properties of hyperbolic components. We re-exposit Makienko's theorem giving sufficient conditions for a hyperbolic component to be non-compact, and apply his construction to Kleinian groups. We also give some examples.

9. Rational maps whose Fatou components are Jordan domains. We prove the theorems mentioned in Subsection 1.6.1 concerning rational maps whose Fatou components are Jordan domains.

Chapter 2

Background

In this chapter we give the definitions and known results from the theory of iterated rational maps which we use throughout this work.

In Section 2.1 we list some conventions of notation.

In Section 2.2 we introduce the basic objects of study in the theory of iterated rational maps, namely the Fatou and Julia sets. We summarize known results which show that the dynamics on the Fatou set is now completely understood.

In Section 2.3 we define hyperbolic and postcritically finite rational maps. Hyperbolic postcritically finite maps have important expanding properties and combinatorial properties which allow the dynamics on their Fatou component boundaries to be described using laminations (see Section 2.6).

In Section 2.4, we discuss parameter and moduli spaces of rational maps of degree d and stability properties of hyperbolic maps.

In Section 2.5 we define "mapping scheme", which is a combinatorial tool we will use to keep track of maps from a finite set of copies of a space into itself.

In Section 2.6 we define the lamination of a postcritically finite rational map. It describes the pinching of the boundary of a Fatou component and the touching of two Fatou components.

2.1 Notation and conventions

We denote by

- $\mathbb{C}$, the complex plane,
- $\Delta = \{z \mid \| z \| < 1 \}$,
- $S^1 = \{z \mid \| z \| = 1 \}$,
- $\mathbb{C} \cup S^1_\infty$, the compactification of the complex plane by the circle at infinity,
- $\widehat{\mathbb{C}}$, the Riemann sphere,

- $\overline{X}$, the closure of a set X, and
- ∂X, the boundary of a set X.

The open unit disc Δ carries a canonical metric of constant curvature -1 which we call the *Poincaré metric*; it is given by $\frac{2dz}{1-|z|^2}$.

The Riemann sphere $\widehat{\mathbb{C}}$ carries a canonical metric of constant curvature $+1$ which we call the *spherical metric*; it is given by $\frac{2dz}{1+|z|^2}$.

The notation S^2 will refer to the Riemann sphere equipped with the spherical metric. Thus it makes sense to speak of a round circle on S^2. The space S^2 is equipped with a distinguished point which we call the *point at infinity*, and denote it by ∞.

2.2 Rational maps, the Fatou set, and the Julia set

2.2.1 Rational maps

A *rational map* $f(z)$ is a holomorphic map of the Riemann sphere $\widehat{\mathbb{C}}$ to itself. Any rational map can be written as a quotient $f(z) = p(z)/q(z)$ where $p(z)$ and $q(z)$ are relatively prime polynomials; this representation is unique up to multiplication of the pair (p, q) by nonzero complex scalars. The *degree* of f we define as the maximum of the degrees of p and q. This definition of degree can be shown to coincide, for example, with the topological degree of f as a map of the Riemann sphere to itself. It is also equal to the number of preimages of a generic point. Near any point $x \in \widehat{\mathbb{C}}$ we may choose coordinates on domain and range so that x corresponds to the origin and the map f near x is of the form $w = z^n$ for $n > 0$; the integer n is called the *local degree* of f near z. If $n > 1$ the point z is called a *critical point*; the *multiplicity* of the critical point is defined to be the integer $n - 1$. The Riemann-Hurwitz formula shows that a rational map of degree d has $2d - 2$ critical points, counted with multiplicity. Iterating the map f yields a holomorphic dynamical system. The dynamics of maps of degree less than two are completely understood, so we will assume that the degree of f is at least two.

A *conjugacy* between two rational maps f and g is a bijection $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $g \circ \phi = \phi \circ f$. The conjugacy ϕ is said to be measurable, continuous, quasiconformal, smooth, conformal, etc. according to the degree of regularity of ϕ. Degree one maps are automorphisms of the Riemann sphere, i.e. are Möbius transformations. We denote the group of automorphisms of the Riemann sphere by $\mathrm{Aut}(\widehat{\mathbb{C}})$. Two rational maps which are conjugate in $\mathrm{Aut}(\widehat{\mathbb{C}})$ have conformally isomorphic dynamics.

2.2.2 The Fatou and Julia sets

There is a basic dichotomy present in the dynamical behavior of points under the iterates of a rational map. The sphere decomposes into two complementary invariant subsets: the open *Fatou set* on which the behavior of points is stable under mild perturbations of the point, and the closed, often fractal *Julia set* where the behavior of a point is chaotic.

Definition 2.1 (Normal family) *A family of functions $\mathcal{F}$ defined on an open set $U \subset \widehat{\mathbb{C}}$ is said to be* **normal** *if every sequence $\{f_n\}$ of elements in $\mathcal{F}$ has a subsequence converging uniformly on compact subsets of U.*

Definition 2.2 (Fatou and Julia sets) *The* **Fatou set** *of a rational map $f(z)$ is defined to be the set of points $z \in \widehat{\mathbb{C}}$ such that there exists a neighborhood U of z on which the iterates of f form a normal family. The* **Julia set** *is defined to be the complement of the Fatou set. We denote the Fatou set by $F(f)$ and the Julia set by $J(f)$.*

Thus the Fatou set is open and the Julia set is closed. The Fatou and Julia sets are each fully invariant under the dynamics. This means that $f^{-1}(J(f)) = J(f), f^{-1}(F(f)) = F(f)$.

2.2.3 Periodic points

A *point of period p* of a rational map $f(z)$ is a solution of the equation $f^{\circ p}(z) = z$ which is not a solution of any similar equation for $p' < p$. A *cycle* is the forward orbit of a periodic point. Let $x_0, x_1, ..., x_{p-1}$ be a cycle of points of period p under a rational map f. The *multiplier* λ of the cycle is defined to be $\lambda = (f^{\circ p})'(x_i)$, there x_i is any point in the cycle. By the Chain Rule, it is independent of the choice of point in the cycle. Periodic points come in one of five flavors.

Definition 2.3 (Periodic points) *Let x be a periodic point with multiplier λ. The point x is called*

- **superattracting** *if $|\lambda| = 0$;*
- **attracting** *if $0 < |\lambda| < 1$;*
- **parabolic***, or* **rationally indifferent***, if $\lambda^q = 1$ for some nonzero integer q;*
- **irrationally indifferent***, if $|\lambda| = 1$ but λ is not a root of unity;*
- **repelling** *if $|\lambda| > 1$.*

The following two theorems give the local picture of the dynamics near attracting and superattracting fixed points.

Theorem 2.4 (König's theorem) *Suppose $f(z)$ is a holomorphic map such that*

$$f(z) = \lambda z + O(z^2), 0 < |\lambda| < 1.$$

Then there exists a holomorphic map ϕ defined on a neighborhood of 0 such that ϕ is injective, $\phi(0) = 0$, and ϕ gives a conjugacy between f near 0 to the map $z \mapsto \lambda z$. The germ of the map ϕ near 0 is unique up to multiplication by nonzero complex constants.

Theorem 2.5 (Böttcher's theorem) *Suppose $f(z)$ is a holomorphic map such that near the origin,*

$$f(z) = az^n + O(z^{n+1}), a \neq 0, n \geq 2.$$

Then there exists a holomorphic map ϕ defined on a neighborhood of 0 such that ϕ is injective, $\phi(0) = 0$, and ϕ gives a conjugacy between f near 0 to the map $w \mapsto w^n$ near 0. Moreover, the germ of this conjugacy near the origin is unique up to multiplication by $(n-1)$st roots of unity.

Corollary 2.6 *Let $f(z)$ be a rational map and let $\{x_i\}_{i=0}^{p-1}$ be a superattracting cycle. Let Ω_i be the Fatou component containing x_i, and let d_i be the local degree of f near x_i. If each Ω_i contains exactly one critical point of $f^{\circ p}$, then there exist Riemann mappings $\phi_i : (\Delta, 0) \to (\Omega_i, x_i)$ such that $f \circ \phi_i(z) = \phi_{i+1 \bmod p}(z^{d_i})$. The number of distinct choices for the collection of maps ϕ_i is equal to $(\prod_i d_i) - 1$.*

Proof: Since there are no other critical points in the cycle of Fatou components Ω_i, we may lift the local conjugacy given by Böttcher's theorem applied to $f^p|_{\Omega_0}$ to obtain a new local conjugacy on a larger region extending the old one. Iterating this process, we obtain a conjugacy from the open disc Δ to all of Ω_0.

■

The following theorem is due independently to Fatou and Julia.

Theorem 2.7 (Repelling points dense in J(f)) *The Julia set of a rational map $f(z)$ is equal to the closure of the set of repelling periodic points of f.*

2.2.4 The classification of stable regions

The dynamics on the Fatou set is now completely understood.

Definition 2.8 (Fatou cycles) *Let Ω be a period p Fatou component and let $h(z) = f^{\circ p}$. Then Ω is called a*

- **superattracting basin** *if every point in Ω tends to a superattracting fixed point of h;*

- **attracting basin** *if every point in* Ω *tends to an attracting fixed point of* h;
- **parabolic basin** *if every point in* Ω *tends to a parabolic fixed point of* h;
- **Siegel disc** *if* Ω *is conformally isomorphic to the unit disc* Δ *and* $h|_\Omega$ *is conformally conjugate to an irrational rotation;*
- **Arnold-Herman ring** *if* Ω *is conformally isomorphic to an annulus* $\{z : 1 < |z| < R\}$ *for some* $R > 1$, *and* $h|_\Omega$ *is conformally conjugate to an irrational rotation.*

The Schwarz lemma implies that every attracting or parabolic cycle of Fatou components contains the forward orbit of a critical point.

Theorem 2.9 (Classification of Fatou components) *Let* Ω *be a Fatou component.*
Then the image of Ω *under some iterate of* f *is periodic, and is one of the above five types. All five types can occur.*

This classification was begun by Fatou and Julia. Hermann, Arnold, and Siegel showed that Hermann-Arnold rings and Siegel discs can exist. Sullivan completed the classification of Fatou components by ruling out the existence of a so-called wandering domain, i.e. a component of $F(f)$ which is not eventually periodic.

Shishikura gave a sharp bound on the number of components of each type.

Theorem 2.10 (Number of Fatou cycles) *Let* f *be a rational map of degree* d. *Then the number of attracting and superattracting cycles, plus the number of indifferent cycles, is bounded by* $2d-2$. *The bound is sharp.*

2.2.5 The postcritical set

Notation. We let $C(f)$ denote the set of critical points of f.

Definition 2.11 (Postcritical set) *The* **postcritical set** $P(f)$ *of a rational map* $f(z)$ *is defined to be*

$$P(f) = \overline{\bigcup_{n>0, c\in C(f)} f^n(c)}.$$

The map f *is called* **postcritically finite** *if* $|P(f)| < \infty$. *We will call the preimage of the postcritical set the* **lifted postcritical set**, *and denote it by*

$$Q(f) = f^{-1}P(f).$$

We will abbreviate the phrase "postcritically finite" by PF.

The postcritical set plays an extremely important role in the theory of iterated rational maps. The lifted postcritical set will play an important role in our combinatorial analysis of PF rational maps in Chapters 4 and 5.

A rational map of degree larger than one has at least two points in its postcritical set. It will be useful to single out those maps which realize this lower bound since these will be extremely special cases to which our combinatorial techniques will not apply.

Definition 2.12 (METIS) *The Maximal Elementary Totally Invariant Subset of a rational map $f(z)$ of degree at least two is the largest set $E(f)$ consisting of at most two points such that $f^{-1}(E) = E$.*

Suppose f has degree at least two. If $|E(f)| = 1$, f is conformally conjugate to a polynomial. If $|E(f)| = 2$, f is conformally conjugate to $z \mapsto z^n$, for some integer n with $|n| \geq 2$. If $|P(f)| = 2$, then $P(f) = E(f)$, and so f is conformally conjugate to $z \mapsto z^n$.

Definition 2.13 (Postcritically elementary) *A rational map $f(z)$ we call* **postcritically elementary** *if $|P(f)| = 2$.*

The Julia set of a postcritically elementary map is a round circle in the Riemann sphere.

2.3 Hyperbolic and postcritically finite maps

2.3.1 Hyperbolic maps

Definition 2.14 (Hyperbolic maps) *A rational map $f(z)$ is said to be* **hyperbolic** *if $P(f) \cap J(f) = \emptyset$.*

We will abbreviate the phrase, "postcritically finite hyperbolic" by PFH.

A hyperbolic map cannot possess indifferent cycles or Arnold-Herman rings (see [Bea] or [Mil2]). Indifferent cycles in the Julia set attract critical points. The boundaries of Siegel discs and Arnold-Hermann rings attract critical points.

Hence hyperbolic maps have only attracting or superattracting periodic Fatou components. Also, the complement of the postcritical set of a hyperbolic map is always connected: every critical point is attracted to either an attracting or superattracting cycle.

2.3.2 Expanding properties of hyperbolic maps

Let M be a Riemannian manifold and $f : M \to M$ be a C^1 map. Let $X \subset M$ be a compact invariant subset.

Definition 2.15 (Expanding map) *The map f is said to be* **expanding** *on X if there is an integer n such that for each tangent vector v to a point $x \in X$,*

$$\| Df^{\circ n}(v) \| > \| v \|.$$

A rational map $f(z)$ is said to be *expanding* if there exists a C^1 metric ρ defined on a neighborhood U of $J(f)$ for which f is expanding on $J(f)$.

Theorem 2.16 *A rational map $f(z)$ is expanding if and only if it is hyperbolic.*

We sketch the proof. An expanding map f cannot have critical points or indifferent cycles in the Julia set. Hence if f is expanding, $|P(f) \cap J(f)| = \emptyset$, and so expanding maps are hyperbolic. Conversely, if f is hyperbolic, then $\widehat{\mathbb{C}} - P(f)$ is connected. If f is postcritically elementary, then f is conjugate to $z \mapsto z^n$ for some n, and $J(f) = S^1$. Then f is clearly expanding on $J(f)$. So suppose f is not postcritically elementary. Then by the Uniformization theorem (see e.g. [FK]), the universal cover of $\widehat{\mathbb{C}} - P(f)$ is conformally isomorphic to the unit disc. The Poincaré metric on the the unit disc descends to a metric on $\widehat{\mathbb{C}} - P(f)$. This metric pulls back to a metric on $\widehat{\mathbb{C}} - Q(f)$ such that f is an isometry on $\widehat{\mathbb{C}} - Q(f)$. Lifting to the universal cover, the Schwarz lemma shows that the inclusion map $i : \widehat{\mathbb{C}} - Q(f) \to \widehat{\mathbb{C}} - P(f)$ is a contraction. Hence f, restricted to $\widehat{\mathbb{C}} - Q(f)$, expands the Poincaré metric on $\widehat{\mathbb{C}} - P(f)$. See [Mil2], Section 14 for further details.

Since $J(f)$ is compact, if f is expanding with respect to one smooth metric, it is expanding with respect to all smooth metrics. As an important special case, a hyperbolic rational map is expanding with respect to the spherical metric on $\widehat{\mathbb{C}}$. If f is expanding with respect to some metric ρ defined on $U \supset J(f)$, then on every compact subset K such that $J(f) \subset K \subset U$, f expands the lengths of tangent vectors to points in K by a definite factor. As a consequence, we have

Theorem 2.17 *If $f(z)$ is expanding and if $L \subset J(f)$ is a connected subset such that $f|_L : L \to L$ is a homeomorphism, then L is a point.*

2.3.3 Postcritically finite maps

In this section we state a well-known property about postcritically finite maps.

Proposition 2.18 *Let $f(z)$ be a postcritically finite rational map. Then for every Fatou component Ω, $|\Omega \cap P(f)| \leq 1$ and the Julia set of f is connected.*

See [McM1], p. 35 for the proof.

If $f(z)$ is a postcritically finite rational map, then every Fatou component Ω contains exactly one point of the set $\cup_{n \geq 0} f^{\circ -n} P(f)$ which we call the *center* of Ω. The Fatou set of a postcritically finite map may be empty: this occurs if and only if there are no periodic critical points ([Mil2], Corollary 14.6).

2.4 Parameter and moduli spaces of rational maps

Since $\mathrm{Aut}(\widehat{\mathbb{C}})$ is noncompact, the orbit of any map $f \in \mathrm{Rat}_d$ under the action of $\mathrm{Aut}(\widehat{\mathbb{C}})$ by conjugation has noncompact closure in Rat_d.

Theorem 2.19 *If $d \geq 2$, the moduli space $\mathcal{M}_{\lceil}$ is Hausdorff.*

Proof: Let $f_n \to f$ be a convergent sequence in Rat_d. Let $\{A_n\}_{n=1}^{\infty} \subset \mathrm{Aut}(\widehat{\mathbb{C}})$, and suppose $A_n \circ f_n \circ A_n^{-1} \to g \in \mathrm{Rat}_d$. We must show that g is conjugate to f, i.e. that the A_n converge in $\mathrm{Aut}(\widehat{\mathbb{C}})$.

Since the number of indifferent cycles is bounded by $2d-2$, there is an integer p depending on f such that every point of period p for f is repelling. We may assume there are at least three points of period p. Let $\mathrm{Per}_p(h)$ denote the set of points of period p for a rational map $h(z)$. By Proposition 8.4, there is an integer N such that if $n \geq N$, there is a conjugacy $\phi_n : \mathrm{Per}_p(f_n) \to \mathrm{Per}_p(f)$ such that $\phi_n \to \mathrm{id}$ as $n \to \infty$. Similarly, there are conjugacies $\psi_n : \mathrm{Per}_p(A_n f_n A_n^{-1}) \to \mathrm{Per}_p(g)$ for n sufficiently large, with $\psi_n \to \mathrm{id}$ as $n \to \infty$. But then $\mathrm{Per}_p(g) = \lim_{n\to\infty} \mathrm{Per}_p(A_n f_n A_n^{-1}) = \lim_{n\to\infty} A_n(\mathrm{Per}_p(f_n)) = \lim_{n\to\infty} A_n(\mathrm{Per}_p(f))$. It follows that the A_n converge to a conjugacy between f on $\mathrm{Per}_p(f)$ and g on $\mathrm{Per}_p(g)$. Since $|\mathrm{Per}_p(f)| \geq 3$, this implies that the A_n converge in $\mathrm{Aut}(\widehat{\mathbb{C}})$.

■

Remark: Compare this with the argument in [Thu1] proving the analogous result for Kleinian groups.

The set of hyperbolic maps forms an open subset of Rat_d (see e.g. [MSS]). Hyperbolicity is invariant under conjugation by elements of $\mathrm{Aut}(\widehat{\mathbb{C}})$, and therefore it makes sense to speak of the set of hyperbolic maps in the moduli space $\mathcal{M}_{\lceil}$ of rational maps of degree d.

Definition 2.20 *A* **hyperbolic component in parameter space** *is a connected component of the space of hyperbolic maps in Rat_d. A* **hyperbolic component in moduli space** *is the image of a hyperbolic component in parameter space under the projection map from Rat_d to $\mathcal{M}_{\lceil}$.*

The next theorem says that a hyperbolic component in parameter space consists of maps which are deformations of f near $J(f)$.

Theorem 2.21 *Let $f \in Rat_d$ be a hyperbolic rational map. If g is in the hyperbolic component of f, then there exists a quasiconformal homeomorphism h sending a neighborhood U of $J(f)$ to a neighborhood V of $J(g)$ such that for all $z \in f^{-1}(U)$, $h \circ f(z) = g \circ h(z)$.*

The homeomorphism h is not canonical. The proof will use the theory of holomorphic motions. Following [McM3] and [MSS], we make the following definition.

Definition 2.22 *Let W be a connected complex manifold and $w \in W$. Let $E \subset \widehat{\mathbb{C}}$. A* **holomorphic motion of E parameterized by** (W, w) *is a family of injections $\phi_\lambda : E \to \widehat{\mathbb{C}}$, one for each $\lambda \in W$, such that $\phi_\lambda(e)$ is a holomorphic function of λ for each fixed e, and $\phi_w = id$.*

A fundamental fact about holomorphic motions is the following:

Theorem 2.23 (λ-lemma) *A holomorphic motion ϕ_λ of a set $E \subset \widehat{\mathbb{C}}$ has a unique extension to a holomorphic motion of $\overline{E}$. Given λ, the map ϕ_λ of a set E extends to a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$.*

See [MSS], [BR], [ST2] for proofs and a discussion of the general theory of extending holomorphic motions.

For convenience during the proof, we introduce the following definition.

Definition 2.24 *An* **expanding-like map** *is a triple (U_1, U_0, f), where U_0 and U_1 are open, planar, finitely connected domains with Jordan curve boundary components, $\overline{U_1} \subset U_0$, and $f : U_1 \to U_0$ is a holomorphic covering map of finite degree. The Julia set of an expanding-like map is the set $J = \cap_{n>0} f^{-n}(U_0)$.*

Note that expanding-like maps have no critical points. We allow U_1 and U_0 to be disconnected.

Lemma 2.25 *Let $f(z)$ be a hyperbolic rational map. Then there is a subset $U_0 \subset \widehat{\mathbb{C}}$ such that $(f^{-1}(U_0), U_0, f)$ is an expanding-like map with Julia set equal to $J(f)$, and such that there are no critical points in U_0.*

Proof: Let A be the set of all attracting and superattracting periodic points of f. By König's and Böttcher's theorems, near any point $a \in A$ of period p, there are local coordinates about a such that $f^{\circ p}$ is conjugate either to to $z \mapsto \lambda z$, where $|\lambda| < 1$, or to $z \mapsto z^n$ for some $n > 1$. It follows that we may find a collection $\mathcal{D}$ of closed discs in $F(f)$ with smooth boundaries about points in A such that $\overline{\widehat{\mathbb{C}} - f^{-1}(\mathcal{D})} \subset \widehat{\mathbb{C}} - \mathcal{D}$. The collection $\mathcal{D}$ contains all but finitely many postcritical points of f. Hence there is an integer n such that the interior of $f^{-n}(\mathcal{D})$ contains $P(f)$. Then setting $U_0 = \widehat{\mathbb{C}} - f^{-n}(\mathcal{D})$, $U_1 = f^{-1}(U_0)$ yields an expanding-like map. Since $U_0 \supset J(f)$ and $U_1 = f^{-1}U_0$, the Julia set of (U_1, U_0, f) is the same as $J(f)$.

∎

Proof of Theorem

It is enough to prove the following: let f be a hyperbolic map. Then there is a neighborhood W of f such that for every $f_\lambda \in W$, the conclusion of the theorem holds for $g = f_\lambda$. For a path in $H(f)$ joining f to g, by compactness, is covered by finitely many such neighborhoods W.

Let (U_1, U_0, f) be an expanding-like map given by the preceding lemma. We now claim that for all f_λ in a sufficiently small neighborhood W of f, $(f_\lambda^{-1}U_0, U_0, f_\lambda)$ is also an expanding-like map. As a subset of the sphere, the preimage $f_\lambda^{-1}U_0$ varies continuously as the map f_λ varies. Since U_0 contains no critical points, U_1 contains no critical points in its closure. Since the critical points of f_λ also vary continuously as a subset of $\widehat{\mathbb{C}}$, for all f_λ sufficiently close to f, the claim holds.

The maps $f : U_1 - f^{-1}U_1 \to U_0 - U_1$ and $f_\lambda : U_1^\lambda - f^{-1}U_1^\lambda \to U_0 - U_1^\lambda$ are holomorphic covering maps with the same combinatorics. It is therefore possible to define f_λ^{-1} so that $\phi_\lambda = f_\lambda^{-1} \circ f : U_1 - f^{-1}U_1 \to U_1^\lambda - f_\lambda^{-1}U_1^\lambda$ is a holomorphic motion of $U_1 - f^{-1}U_1$. We may then lift ϕ_λ to obtain a holomorphic motion of $U_1 - f^{-n}U_0$ for all $n > 0$.

Hence there is a holomorphic motion ϕ_λ of $U_1 - J(f)$ such that for all $z \in f^{-1}(U_1 - J(f))$, $\phi_\lambda \circ f(z) = f_\lambda \circ \phi_\lambda(z)$. By the λ-lemma, and since $J(f_\lambda)$ is nowhere dense in $\widehat{\mathbb{C}}$, this extends uniquely to a holomorphic motion of U_1 such that for all $z \in f^{-1}U_1$, $\phi_\lambda \circ f(z) = f_\lambda \circ \phi_\lambda(z)$. Setting $h = \phi_\lambda$ proves the theorem.

■

The next theorem says that if f_1 is hyperbolic and $J(f_1)$ is connected, then to understand the dynamics of f_1 on $J(f_1)$, it is enough to understand the dynamics of a postcritically finite hyperbolic map f_0 on $J(f_0)$. Combined with the preceding theorem, we have that the dynamics near the Julia set of a hyperbolic map with connected Julia set is conjugate to that of a postcritically finite hyperbolic map.

Theorem 2.26 *Let $f(z)$ be a hyperbolic rational map with connected Julia set. Then there is a unique PFH map in $H(f)$.*

This theorem is an immediate consequence of [McM1], Corollary 3.6.

2.5 Mapping schemes

The notion of a *mapping scheme* is developed in [Mil5] to study parameter spaces of holomorphic maps of finitely many copies of the complex plane into itself. This will be a convenient tool which we will use in our definition of tuning in Chapter 3 and in our definition of laminations in the following section.

Definition 2.27 (Mapping scheme) *A* **mapping scheme** *is a triple (S, τ, w) consisting of*

1. *a finite set S of points;*
2. *a function τ from S to itself;*
3. *a "weight function" w which assigns to each point $x \in S$ a nonnegative integer.*

An element $x \in S$ is called **critical** *if the weight $w(x)$ is greater than one. If $T \subset S$ is a subset which satisfies $\tau^{-1}T = T$, the triple $(T, \tau|_T, w|_T)$ will be called a* **subscheme** *of the mapping scheme (S, τ, w). If a given scheme is fixed throughout the discussion, we will refer to a subscheme by referring to its underlying set. A* **component** *is a subscheme such that for each pair $x, y \in T$, $\tau^{\circ m}(x) = \tau^{\circ n}(y)$ for some $m, n \geq 0$. A mapping scheme is said to be of* **hyperbolic type** *if for every cycle in S, the product of the weights along the cycle is larger than one.*

Example: Let $f(z)$ be a PF rational map. Then $(Q(f), f|_{Q(f)}, w_f(x))$ is a mapping scheme, where $Q(f) = f^{-1}P(f)$ and $w_f(x)$ is the local degree of f near x. If f is hyperbolic, then this mapping schema is of hyperbolic type.

Let X be an orientable manifold, possibly with boundary, and let (S, τ, w) be a mapping scheme. A **family of maps of X covering the mapping scheme** (S, τ, w) is a map $F : S \times X \to S \times X$ such that $F|_{s \times X}$ is a map of X to itself of degree $w(s)$, where degree means the cardinality of the preimage of a generic point.

Remark: In [Mil5], the weights of the mapping scheme associated to a PF polynomial $f(z)$ represent the multiplicities of points, rather than local degrees. For our purposes, however, we have found it more convenient to measure local degrees.

A mapping scheme can also be represented as a finite directed graph with weighted vertices. The vertices of the graph are the elements of S weighted by w. There is a directed edge from x to y if $\tau(x) = y$. A component of the mapping scheme is a component of this graph. A subscheme of the mapping scheme is a union of components of this graph. The disjoint union of two mapping schema is also a mapping scheme.

Definition 2.28 (Isomorphism of schemes) *Two mapping schemes* (S, τ, w) *and* (S', τ', w') *are said to be* **isomorphic** *if there is a bijection* $h : S \to S'$ *which preserves weights and for which* $h \circ \tau = \tau' \circ h$. *An* **automorphism** *of* (S, τ, w) *is an isomorphism of* (S, τ, w) *to itself.*

2.6 Laminations for postcritically finite hyperbolic rational maps

In this section, we define the lamination of a PFH rational map. This will require some results from geometric function theory which we list in the first section.

Laminations have been used to give both a combinatorial model for the global dynamics of a degree two rational map, as well as conjectural models for certain parameter spaces of rational maps (see the Notes and the end of this chapter). In this work, we define laminations with a more modest goal in mind: we seek a combinatorial description of the touching of two periodic Fatou components, and of the pinching of the boundary of a Fatou component.

2.6.1 Riemann mappings and local connectivity

Definition 2.29 *A set* $K \subset \mathbb{C}$ *is said to be* **full** *if it is compact, connected, and if its complement is nonempty and connected. A full set is said to be* **non-degenerate** *if it is not a point.*

Theorem 2.30 (Theorem of F. and M. Riesz) *Let $f(z)$ is a bounded analytic function on the unit disc. Suppose that for every $s \in E \subset S^1$,*

$$\lim_{r \to 1} f(re^{2\pi is}) = \{x\}.$$

If E has positive Lebesgue measure, then f is the constant map to $\{x\}$.

Definition 2.31 (Rays) *Let $\phi : (\Delta, 0) \to (U, z)$ be a Riemann map uniformizing an open disc U. For $t \in \mathbb{R}/\mathbb{Z}$ the* **ray of angle** *t* **for** *ϕ is the set $\{\phi(re^{2\pi it}) | r \in [0, 1)\}$, and is denoted by R_t. If $\lim_{r\to 1} \phi(re^{2\pi it})$ exists and is equal to x, the ray R_t is said to* **land** *at x.*

Remark: The set of angles for which the corresponding rays land forms has full measure with respect to Lebesgue measure on the circle (see [McM3], Theorem 6.1). We will not need this fact.

Theorem 2.32 (Carathéodory) *Let K be a full nondegenerate set in $\mathbb{C}$. Let $\phi : (\Delta, 0) \to (\widehat{\mathbb{C}} \backslash K, \infty)$ be a Riemann map uniformizing the complement of K in $\widehat{\mathbb{C}}$. Then ϕ extends to a continuous map $\overline{\phi} : \overline{\Delta} \to \widehat{\mathbb{C}}$ if and only if ∂K is locally connected, or if and only if K is locally connected.*

Thus Carathéodory's theorem says that ∂U is locally connected if and only if every rays land, and the landing point of R_t varies continuously in t. The next theorem says that two rays which land at a common point separate $\widehat{\mathbb{C}} - U$ into at least two pieces lying on different sides of their union.

Theorem 2.33 *Let K be a full nondegenerate set in $\mathbb{C}$ and $U = \widehat{\mathbb{C}} \backslash K$. Let $\phi : (\Delta, 0) \to (U, \infty)$ be a Riemann mapping. Suppose two distinct ϕ-rays R_t and $R_{t'}$ land at a common point x of ∂U. Then each component of the complement of the Jordan curve $C = R_t \cup R_{t'} \cup \{x\}$ contains a component of $K \backslash \{x\}$.*

Proof: Suppose C failed to separate K. Then some component of its complement contains no points of K. Since ϕ is a homeomorphism on U, two distinct ϕ-rays cannot intersect in U. By relabelling t and t' if necessary we may assume that for the set $W = \{re^{2\pi is} || r| < 1, s \in (t, t')\}$, $\overline{\phi(W)} \cap K = \{x\}$. Then for every $s \in (t, t')$, $\lim_{r\to 1} \phi(re^{2\pi is}) = \{x\}$. This contradicts the Theorem of F. and M. Riesz.

■

The following lemma will be useful in our discussion of laminations, and in the proofs of our theorems in Chapter 9. The first conclusion may be found in [DH1], Section 2.4.3. The remainder follows from Schonflies theorem and Jordan curve theorem.

Lemma 2.34 *Let K be a full nondegenerate subset of $\mathbb{C}$ whose boundary is locally connected. Let V be a bounded component of $\widehat{\mathbb{C}} \backslash \partial K$. Then*

1. *V is a Jordan domain,*

2. *$\overline{V}$ and $\widehat{\mathbb{C}}\backslash V$ are closed discs, and*

3. *a Jordan curve in K is contained in the closure of a unique bounded component U of $\widehat{\mathbb{C}}\backslash\partial K$.*

2.6.2 Local connectivity of Fatou component boundaries

Theorem 2.35 *Let $f(z)$ be a PF hyperbolic rational map. Then the boundary of every Fatou component is locally connected and locally path connected.*

For the case when Ω is the basin of infinity of a hyperbolic polynomial, this theorem follows from a theorem of Douady and Hubbard [DH1]. See also [Mil2], Theorem 17.5 for a proof in this case.

Proof: (Sketch) First, it suffices to prove local connectivity, since a compact, connected, locally connected metric space is locally path connected ([Mil2], Lemma 16.4).

Since every Fatou component is eventually periodic, it suffices to consider a Fatou component Ω which is fixed under f. The argument given for polynomials then applies. For completeness, we sketch the proof.

Since f is PF, there is a unique fixed critical point $x \in \Omega$. Let n be the local degree of f near x. By Corollary 2.6, there is a conjugacy $\phi : (\Delta, 0) \to (\Omega, x)$ from $z \mapsto z^n$ to $f|_\Omega$. The fact that f expands the hyperbolic metric on $S^2 - Q(f)$ implies that with respect to this metric, the length of every truncated ϕ-ray $\{\phi(re^{2\pi it})|\ r \in [1/2, 1)\}$ is uniformly bounded. Hence the continuous maps $s_r : t \to \phi(re^{2\pi it})$ converge uniformly as $r \to 1$.

∎

2.6.3 Abstract laminations

Definition 2.36 (Lamination)
- *A* **lamination** *is an equivalence relation Λ on S^1 such that the convex hulls of the equivalence classes in $\overline{\Delta}$, taken in the Euclidean metric, are disjoint in the closed unit disc. The* **support** *is the union of the equivalence classes containing more than one point.*

- *A* **chord** *of Λ is a pair of distinct points which are in the same equivalence class. A* **leaf** *is a chord such that the interior of its convex hull meets the interior of the convex hull of no other chord.*

- *A* **gap** *is the closure of a component of $\overline{\Delta} - \mathcal{L}$, where $\mathcal{L}$ is the union of the leaves.*

The definition of chord is my own. Chords will play an important role in Chapter 5. Note that a chord is not necessarily a leaf. For example, if four points form an equivalence class, the corresponding gap is a quadrilateral. The diagonals are chords which are not leaves. Our definition of support is taken from [McM3].

Remark: In [Thu4], a lamination is defined as a set of chords in the closed unit disc whose interiors are disjoint, and whose union is a closed subset of the *open* unit disc. The closure of the set of leaves of a lamination in our definition is a lamination in the sense of Thurston. Thurston calls a lamination *clean* if two chords with a common endpoint either coincide, or are the sides of a common gap. Our laminations are clean by definition.

2.6.4 Lamination associated to a polynomial

In this section, we recall the definition of the laminations associated to a polynomial as a prelude to our definition of the lamination for a rational map.

Let $p(z)$ be a monic postcritically finite degree d polynomial, and suppose its filled-in Julia set $K(p)$ is locally connected; this is true if and only if $J(p)$ is connected and locally connected (see [Mil2], Corollary 17.4). Let Ω denote the basin of infinity for $p(z)$. Since $J(p)$ is connected, Ω contains a unique critical point which is the point at infinity. By Corollary 2.6, there is a unique Riemann map $\phi : (\widehat{\mathbb{C}} - \overline{\Delta}, \infty) \to (\Omega, \infty)$ conjugating $z \mapsto z^d$ on $\widehat{\mathbb{C}} - \overline{\Delta}$ to f on Ω. We will use laminations to obtain a combinatorial model for $p(z)$ acting on $J(p)$.

By Carathéodory's theorem, the map ϕ extends continuously to S^1. The *lamination* Λ_p *of* $p(z)$ is the equivalence relation on S^1 generated by $s \sim t$ if the rays R_t and R_s land at the same point. Since a simple closed curve C in S^2 which intersects both components of the complement of a simple closed curve C' cannot intersect C' in exactly one point, this defines a lamination. Note that the equivalence classes are closed, since the map ϕ extends continuously.

Invariant laminations. The lamination associated to a monic degree d polynomial satisfies certain invariance properties, since the extension of the Riemann map over S^1 gives a semiconjugacy from z^d on S^1 to p on $J(p)$. These properties were explicitly abstracted in [Thu4]; we list them below.

Definition 2.37 (Invariant laminations) *A lamination* Λ *is said to be*

1. **forward-invariant under** $z \mapsto z^d$ *if the image of the endpoints of a leaf is either a single point, or else form the endpoints of another leaf;*
2. **backward-invariant under** $z \mapsto z^d$ *if*
   (a) *if* l *is any leaf, there is a collection of at least* d *leaves mapping onto* l *under* $z \mapsto z^d$*; and*
   (b) *for any gap* G*, the convex hull of its image is either*
      i. *a gap, or*
      ii. *a leaf, or*
      iii. *a single point;*

3. **invariant** *if it is both forward and backward invariant.*

The following proposition depends on the fact that the basin of infinity is fully invariant.

Proposition 2.38 *Let $p(z)$ be a monic degree d polynomial with locally connected Julia set. Then the lamination Λ_p is invariant under $z \mapsto z^d$.*

Proof: Forward invariance is clear. That there are at least d leafs in the preimage of a leaf follows by considering the local picture of p near a point $x \in J$ which maps onto $p(x)$: if k rays land at $p(x)$, then there are at least $n(x) \cdot k$ rays landing at x, where $n(x)$ is the local degree of f near x, since Ω is fully invariant under $p(z)$. This argument also shows that the image of a gap which is the convex hull of a single equivalence class is also a gap which is the convex hull of a single equivalence class. It remains to show that the image of a gap G which is not the convex hull of a single equivalence class is again a gap.

The boundary of such a gap R consists of leaves and portions of the circle. The image $\phi(\partial G)$ is a Jordan curve C in $J(p)$. Since f is a polynomial, $\partial K(f) = J(f)$, hence the curve C bounds a Fatou component U, by Lemma 2.34. The component U maps to a Fatou component U'. Lemma 2.34 implies that the component U' is bounded by some Jordan curve C', and so C' is the image of C under f. It then follows that the image of R must be a gap which projects to C' under ϕ.

■

Laminations as combinatorial models. The next proposition establishes that laminations form a combinatorial model for locally connected Julia sets. This fact is also stated without proof in [Thu4] and is well-known.

Proposition 2.39 *Let Λ be the lamination of a monic polynomial $p(z)$ with locally connected filled-in Julia set, and let L be the convex hull of its equivalence classes. Then there exists a map $\pi : (S^2, \overline{\Delta}) \to (\widehat{\mathbb{C}}, K(p))$ and a postcritically finite branched covering $\overline{p} : (S^2, \overline{\Delta}) \to (S^2, \overline{\Delta})$ such that*

1. *π is a semiconjugacy between $\overline{p}$ and p which is a homeomorphism off L_p, which agrees with ϕ on $\widehat{\mathbb{C}} - \Delta$, and which collapses each component of L_p to a point;*

2. *If p is postcritically finite, then $\overline{p}$ is combinatorially equivalent to p.*

The proof follows from the fact that the lamination associated to a polynomial with locally connected Julia set is invariant. The idea for producing the map $\overline{p}$ is to extend $z \mapsto z^d$ on S^1 to a map of $\overline{\Delta}$ preserving the convex hulls of equivalence classes, and then glue this extension to z^d on $\widehat{\mathbb{C}} - \Delta$. This can be done since (1) if U is a Fatou component, $\overline{U}$ is homeomorphic to a closed disc, by Lemma 2.34, and (2) the diameters of the Fatou components tend to zero, since $J(f)$ is locally connected.

Remark: In [Ree2], a generalization of invariant laminations is used to give a combinatorial model for the dynamics of a postcritically finite hyperbolic quadratic rational map on its Julia set.

2.6.5 Definition of lamination for PFH rational maps

In this section we define the lamination of a PFH rational map $f(z)$.

Definition 2.40 *Let S be a finite set. A* **lamination covering** *S is an equivalence relation on $S \times S^1$ such that the following conditions hold.*

1. *For $x = y$, the restriction of the equivalence relation to every component $(\{x\} \times S^1) \times (\{y\} \times S^1)$ is a lamination.*
2. *Suppose $x \neq y$. Let $[-1, 1] \times S^1$ be equipped with the Euclidean metric. Identify $\{1\} \times S^1$ with $\{x\} \times S^1$ and $\{-1\} \times S^1$ with $\{y\} \times S^1$. Then for each equivalence class E, there is a choice of convex hull $K(E)$ such that $K(E) \cap K(E') = \emptyset$ if $E \neq E'$. A* **chord** *in Λ is a pair of distinct points in the same equivalence class.*

This definition of lamination reduces to the usual definition in the case when $|S| = 1$.

The set of chords inherits a natural topology which is the subspace topology on the set of unordered pairs of distinct points in $S \times S^1$. We denote the space of chords by $\chi(\Lambda)$, or by χ if the lamination Λ is fixed throughout the discussion.

Definition 2.41 (Forward-invariant lamination covering a mapping scheme) *Let (S, τ, w) be a mapping scheme, and let $F_0 : S \times S^1 \to S \times S^1$ be the map covering the mapping scheme which is given by $z \mapsto z^{w(x)}$ on $\{x\} \times S^1$. A* **forward-invariant lamination Λ covering** *(S, τ, w) is a lamination covering the set S such that the image of any equivalence class under F_0 is contained in a single equivalence class.*

This reduces to the usual definition of forward-invariance for the case when $|S| = 1$. We do not require any form of backward invariance, nor do we require that the image of an equivalence class is equal to an entire equivalence class. The reason is the following. Suppose for example that Ω is a fixed Fatou component for a PFH rational map f containing a superattracting fixed point x. Let ϕ be the Riemann map produced by Corollary 2.6. Then ϕ induces a lamination on S^1 as in the case for polynomials. However, the resulting lamination can fail to satisfy the conditions of gap invariance and backward invariance. For example, see Figure 5.4.

Lamination associated to a rational map.

Let $f(z)$ be a postcritically finite hyperbolic rational map. Recall that the lifted postcritical set $f^{-1}(P(f))$ we denote by $Q(f)$.

- Let $(Q(f), f|_{Q(f)}, w_f)$ be the mapping scheme of f.

- If $f^n(x) \in P(f)$ for some n, we denote by Ω_x the Fatou component with center x.
- Let $\{\Omega_x\}_{x\in Q}$ denote the set of Fatou components with centers in $Q(f)$.
- For $x \in Q(f)$, let $\phi_x : (\Delta, 0) \to (\Omega_x, x)$ be a choice of Riemann mappings such that $f \circ \phi_x(z) = \phi_{f(x)}(z^{w_f(x)})$. Since f is postcritically finite, every Fatou component contains a unique point mapping onto $P(f)$, and so Corollary 2.6 applies. Hence these maps exist, and there are finitely many such choices.

  By Theorem 2.35, the boundary of every Fatou component of f is locally connected. Hence the maps ϕ_x may be extended to a continuous map $\overline{\phi}_x$ on $\overline{\Delta}$ by Carathéodory's theorem. We denote the collection of extensions by $\overline{\phi}_Q$.

  Thus $\overline{\phi}_Q$ gives a semiconjugacy between the map $F_0 : Q(f)\times\overline{\Delta} \to Q(f)\times\Delta$ given by $F_0(z) = z^{w_f(x)}$ on $\{x\} \times \overline{\Delta}$ and the map f on $\{\overline{\Omega}_x\}_{x\in Q(f)}$.
- Given ϕ_Q, we define the $\phi_Q - (x, s)$-ray of f by

$$R_{x,s} = \overline{\phi}_x([0, 1] \cdot \exp(2\pi i t)).$$

  Note that our definition of "ray" includes the landing point. If the choice of ϕ_Q is fixed during some discussion, we will call $R_{x,s}$ the (x, t)*-ray* of f.

Definition 2.42 (Lamination associated to a rational map) . *The* **lamination of** f*, denoted by* Λ_f*, is the equivalence relation* $\sim$ *defined on* $Q(f)\times\partial\overline{\Delta}$ *given by* $(x, s) \sim (y, t)$ *if* $R_{x,s}$ *and* $R_{y,t}$ *have a common landing point.*

The equivalence relation defined above is indeed a lamination covering the set $Q(f)$. It is also forward-invariant under the map F_0. The image of an equivalence class under $\overline{\phi}_Q$ is a point in the boundary of $\{\Omega_x\}_{x\in Q}$, and the image of a point under f is a point. The space of chords is closed, but need not be compact. The *arc formed by the chord* $\{(x, s), (y, t)\}$ is defined as the set $\alpha((x, s), (y, t)) = \{R_{x,s} \cup R_{y,t}\}$. The interiors of arcs formed by two distinct chords intersect in at most one point, since all such intersection points are in the Julia set.

If f is hyperbolic, the lamination of f satisfies a slightly stronger version of forward invariance which we call *chord invariance.*

Proposition 2.43 (Chord invariance) *The image of a chord under* F_0 *is always a chord, and never a single point.*

Proof: If a chord (x, s) and (y, t) both mapped to the same point under F_0, the common endpoint of the rays $R_{x,s}$ and $R_{y,t}$ would be a critical point in $J(f)$.

■

2.7 Bibliographic notes

For a history of the subject up to and including the work of Fatou and Julia, see [Ale]. The original writings of Fatou and Julia are in [Fat1], [Fat2], and [Jul].

For a general introduction to the subject, see e.g. [Bea] and [Mil2].

Section 2.2.

For proofs of Böttcher's theorem, the density of repelling cycles in $J(f)$, the fact that hyperbolic maps are expanding, see e.g. [Bea] and [Mil2]. These also contain a discussion of the classification of Fatou components. For a proof of the No Wandering Domains theorem, see [Sul1] for Sullivan's original paper. A proof may also be found in [Bea]. In [MS] a modification of Sullivan's original argument is given in the spirit of the analogy with Kleinian groups. Shishikura's proof of Theorem 2.10 may be found in [Shi1].

Section 2.3. For a treatment of expanding properties of hyperbolic maps, and the fact that hyperbolic maps have no indifferent cycles, see e.g. [Bea] and [Mil2], Section 14.

Section 2.5. The theory of mapping schema is well-developed in [Mil5].

Section 2.6. A proof of Carathéodory's theorem may be found in [Mil2], Theorem 16.6. A proof of the Theorem of F. and M. Riesz may be found in [Car], Volume II, Section 313. Tan Lei and Yongchen Yin [TY] have proved the following: suppose $|P(f) \cap J(f)| < \infty$. Then every periodic component of $J(f)$ is locally connected. The proof proceeds as follows. By results of McMullen in [McM1], one reduces to the case where $J(f)$ is connected. Next, one shows that boundaries of all Fatou components are locally connected. The case of parabolic basins requires some delicate arguments. Finally, one shows that the diameters of the Fatou components tend to zero using the fact that f expands the hyperbolic metric on $\widehat{\mathbb{C}} - P(f)$.

The use of the theory of Riemann mappings and external rays to study the dynamics of iterated complex polynomials was begun in [DH1]. An introduction to this topic is given in [Mil2] and [McM3].

The theory of invariant laminations was made explicit in [Thu4]. Laminations are useful for giving combinatorial models for the dynamics of individual polynomials. Laminations have also been explicitly used to give conjectural combinatorial models for parameter spaces of special families of rational maps; for example, quadratic polynomials [Thu4], one-dimensional subspaces of degree two rational maps [Ree2], [Ahm], and cubic Newton's methods [Tan1].

Our definition of lamination is essentially different from that given in [Ahm]. Suppose two Fatou components Ω_1, Ω_2 touch along a set E. Let ϕ_i be a Riemann map to Ω_i given by Böttcher's theorem, $i = 1, 2$. In [Ahm] the associated lamination is a *pair* of equivalence relations, one for each i, defined as follows. Fix some $i \in \{1, 2\}$. Consider the set $\phi_i^{-1}(E)$. Then two points s and t are called equivalent if and only if they are the endpoints of a geodesic lying in the boundary of the Euclidean convex hull of $\phi_i^{-1}E$.

Chapter 3

Branched coverings

A **branched covering** $f : S^2 \to S^2$ is a continuous map such that for each $x \in S^2$, there exist continuous charts near x and $f(x)$ such that x and $f(x)$ correspond to the origin and f is of the form $w = z^n$ for some positive integer n. The integer n is called the **local degree** of f near n. Points of local degree greater than one are called **critical points** of f. The set of critical points of f we denote by $C(f)$. The **degree** of f is defined to be the topological degree of f as a map from the sphere to itself. For a branched covering f, we define the *postcritical set* $P(f)$ by

$$P(f) = \overline{\bigcup_{\substack{n>0 \\ c \in C(f)}} f^n(c)}.$$

A branched covering f is said to be *postcritically finite (abbreviated PF)* if $|P(f)| < \infty$.

A PF branched covering of the sphere to itself, up to a kind of isotopy, is the combinatorial analog of a PF rational map. This chapter is devoted to a discussion of decompositions and combinations of postcritically finite branched coverings of the sphere to itself.

A PF branched covering may sometimes be combined with a family of polynomials acting on copies of $\mathbb{C} \cup S^1_\infty$ to form a new branched covering of the sphere to itself. This process is called tuning; see e.g. [DH1], [Mil1], [Ree2]. In the case of quadratic complex polynomials, the term "tuning" is due to A. Douady and J.-H. Hubbard [DH1]; for maps of the interval to itself, the term is due to J. Milnor and W. Thurston [MT].

For example, let $f(z)$ be a PFH rational map, and suppose x_0 is a critical point of period $n \geq 1$ such that the first return map of x_0 has local degree two. Let $x_i = f^i(x_0)$, $i = 1, ..., n-1$. By perturbing f slightly through postcritically finite branched coverings we may assume that there exists a cycle D_i of discs centered at x_i such that $f(D_i) = D_{i+1 \bmod n}$ and such that $f^{\circ n}|_{\partial D_0}$ is topologically conjugate to $z \mapsto z^2$. Now suppose that $q(z)$ is a postcritically finite quadratic polynomial. Then q extends canonically to a map of the complex plane $\mathbb{C}$ union the circle at infinity S^1_∞. The space $\mathbb{C} \cup S^1_\infty$ is homeomorphic to a closed disc, which we identify with D_0. Then sending D_0 to D_1 via $q(z)$, D_i to $D_{i+1 \bmod n}$ via f, and the complement of the D_i's by f gives a new branched covering called the *tuning* of $f(z)$ by the polynomial $q(z)$, denoted $f * q$.

Conversely, suppose that f is a PF branched cover and $\mathcal{D} = \{D_i\}_{i=0}^{n-1}$ is a set of discs such that $f(\mathcal{D}) = \mathcal{D}$, up to isotopy. Then by collapsing each component of $\mathcal{D}$ to a points, one obtains a new branched covering g which is well-defined up to a kind of isotopy. This process we call collapsing. More formally, we say that the process of collapsing the discs $\mathcal{D}$ gives a *quotient map* $\phi : f \to g$; see [McM3], Appendix B.

We will show that tuning is inverse to collapsing in the following sense: every map R which is a tuning of another map f by a family of topological polynomials admits a quotient map to f, and every map R which admits f as a quotient is the tuning of f by some family of topological polynomials. The proof we give requires a definition of tuning for arbitrary PF branched coverings. In Section 3.4 we discuss several complications which arise when making such a definition.

The first three sections give the definitions of branched covering, combinatorial equivalence, and quotient map. Section 3.4 discusses ambiguities in the above definition of tuning, and explains how we will resolve them through the use of mapping schemes and peripherally rigid maps. Section 3.5 defines peripherally rigid maps. Section 3.6 defines matings as a warm-up to the definition of tuning in Section 3.7. The theorems in Section 3.8 show that tuning and collapsing are inverse. We conclude in Section 3.9 with a few remarks about other definitions of tuning, and the dependence of the combinatorial class of tuning on the data.

3.1 Branched coverings

Let f be a PF branched covering, and let $P(f)$ be the postcritical set of f. The *lifted postcritical set* is defined by

$$Q(f) = f^{-1}(P(f)).$$

A degree d branched covering has $2d-2$ critical points, counted with multiplicity. A branched cover is called *postcritically elementary* if $|P(f)| = 2$. A branched covering f for which $f^{-1}(\infty) = \{\infty\}$ is called a *topological polynomial.* A postcritically finite branched covering f is said to be *hyperbolic* if every cycle in $P(f)$ contains a critical point. For example, if $p(z)$ is a PFH polynomial, then p is also a PFH topological polynomial if we forget the fact that p is conformal.

Mapping scheme of a branched cover. Let f be a postcritically finite branched covering. The *mapping scheme associated to* f is the triple $(Q(f), f|_{Q(f)}, w_f)$ where $w_f(x)$ is the local degree of f near $x \in Q(f)$. The restriction of f to a subset $B \subset Q(f)$ gives a subscheme of $(Q(f), f|_{Q(f)}, w_f)$ if and only if $(f^{-1}B) \cap Q(f) = B$. The subscheme B is of hyperbolic type if and only if every cycle in B contains a critical point. This is equivalent to the product of the weights along every cycle being strictly greater than one. Hence the mapping scheme of a hyperbolic PF map is of hyperbolic type.

3.2 Combinatorial equivalence

The following definition is due to Thurston.

Definition 3.1 (Combinatorial equivalence) *Let f and g be two branched coverings of the sphere to itself. Then f and g are said to be* **combinatorially equivalent** *if there are homeomorphisms ψ_0 and ψ_1 such that*

1. $\psi_i : (S^2, P(f)) \to (S^2, P(g)), i = 0, 1;$

2. $\psi_0 \circ f = g \circ \psi_1$*; i.e. the diagram*

$$\begin{array}{ccc} (S^2, P(f)) & \xrightarrow{\psi_1} & (S^2, P(g)) \\ f \big\downarrow & & \big\downarrow g \\ (S^2, P(f)) & \xrightarrow{\psi_0} & (S^2, P(g)) \end{array}$$

   commutes;

3. ψ_0 *and* ψ_1 *are isotopic through a continuous family* $\psi_t, t \in [0, 1]$ *of homeomorphisms for which* $\psi_t|_{P(f)} = \psi_0|_{P(f)}$ *for all* t.

An isotopy class of homeomorphism $\psi : (S^2, P(f)) \to (S^2, P(g))$ is said to be a **combinatorial equivalence from f to g** if there exist $\psi_0, \psi_1 \in \psi$ making the diagram above commute. We shall denote the combinatorial class of a branched covering f by $[f]$, and the total space of all elements in $[f]$, equipped with the uniform topology, by $\mathcal{S}([\{])$.

Combinatorial equivalence defines an equivalence relation on the set of branched coverings which is much coarser than topological conjugacy.

Example: Let $f(z) = z^2$, and let g be a map defined in polar coordinates by $g(r, \theta) = (r, 2\theta)$. The map g extends uniquely to a map of the Riemann sphere. Then f and g are combinatorially equivalent, but not topologically conjugate.

A basic question is, when does a combinatorial class of postcritically finite branched covering contain a rational map? Thurston has answered this question; we state his theorem in the next chapter.

Let f and g be two combinatorially equivalent maps. A choice of combinatorial equivalence ψ between f and g determines an isomorphism between their mapping schemes which depends only on the isotopy class relative to $P(f)$ of ψ.

Proposition 3.2 *Let f be a PF branched covering. Then $|P(f)| = 2$ if and only if f is combinatorially equivalent to $z \mapsto z^n$ for some n.*

Proof: Since $|P(f)| = 2$, there are exactly two critical values of f. It follows that there are exactly two critical points of f, by counting local degrees. Again by counting local degrees, every critical point is also a critical value. We may conjugate f by a homeomorphism such that $P(f) = \{0, \infty\}$. Then the identity map gives a combinatorial equivalence between f and $z \mapsto z^n$.

∎

Remark: To specify a combinatorial class of postcritically finite branched covering of the sphere to itself, it is enough to specify a covering $f : S^2 - Q(f) \to S^2 - P(f)$ and an embedding $i : S^2 - Q(f) \to S^2 - P(f)$, up to isotopy fixing $P(f)$. The covering may be specified by choosing a basepoint $x \in S^2 - P(f)$ and choosing a subgroup of $\pi_1(S^2 - P(f), x)$. The embedding may be specified by choosing a point $y \in S^2 - Q(f)$, arranging so that $i(y) = x$, and specifying a homomorphism of $\pi_1(S^2 - Q(f)) \to \pi_1(S^2 - P(f))$ which preserves the appropriate "peripheral data". The concept of fundamental groups of surfaces with peripheral data is outlined in [Hem]. It can then be shown that this group-theoretic data is enough to pin down the combinatorial class of the map.

Remark: A useful equivalent formulation of combinatorial equivalence can be made as follows: f and g are combinatorially equivalent if there is a continuous one-parameter family f_t of branched coverings such that $f_0 = f, f_1 = g$, and $P(f_t)$ varies isotopically as t varies. Hence changing f by pre- or post- composing with a homeomorphism isotopic to the identity through maps fixing $P(f)$ yields a map which is combinatorially equivalent to f. The proof depends on results from the theory of mapping class groups; see [Bir].

3.3 Quotients

The following development is based on material found in [McM3]. Let A and B be closed subsets of S^2.

Definition 3.3 (Quotient map) *A* **quotient map** $\phi : (S^2, A) \to (S^2, B)$ *is a continuous surjective degree one map such that*

1. $\phi(A) = B$;
2. $\phi^{-1}(b)$ *is connected for all* $b \in B$;
3. $|\phi^{-1}(x)| = 1$ *for all* $x \in S^2 - B$.

Two quotient maps ϕ_0 *and* ϕ_1 *are said to be* **isotopic** *if there is a continuous one-parameter family* $\phi_t, t \in I$ *of quotient maps between them such that* $\phi_t|_A = \phi_0|_A$ *for all* $t \in I$. *The* **combinatorial class** *of a quotient map* ϕ *is the set of all quotient maps which are isotopic to* ϕ.

We now add dynamics to the picture. Intuitively, the idea is that by "collapsing" together points in the postcritical set, we may sometimes be able to get a new branched covering.

Definition 3.4 (Quotient of covering) *Let f and g be two branched coverings of the sphere, and let*

$$\phi : (S^2, P(f)) \to (S^2, P(g))$$

be a combinatorial class of quotient map. We say that g **is a quotient of** *f, and write $f \xrightarrow{\phi} g$, if there exist $\phi_0, \phi_1 \in \phi$ such that $g \circ \phi_1 = \phi_0 \circ f$, i.e. if the diagram*

$$\begin{array}{ccc} (S^2, P(f)) & \xrightarrow{\phi_1} & (S^2, P(g)) \\ f \big\downarrow & & \big\downarrow g \\ (S^2, P(f)) & \xrightarrow{\phi_0} & (S^2, P(g)) \end{array}$$

commutes. The quotient ϕ is said to be **proper** *if for some $b \in P(g)$, $|\phi^{-1}(b) \cap P(f)| \geq 2$. A point b for which this holds is referred to as a point which is* **blown up** *under ϕ^{-1}.*

Mapping scheme of a quotient map. Let $\phi : f \to g$ be a combinatorial quotient map. Let $B' \subset P(g)$ denote the set of points $x \in P(g)$ for which $|\phi^{-1}(x)| > 1$. Let B be the smallest subset containing B' such that the restriction of $(Q(g), g|_{Q(g)}, w_g)$ to B is a mapping scheme. (Thus B is the set of all points in $Q(g)$ which eventually land in blown-up points.) The *mapping scheme of ϕ* is defined to be the restriction of the mapping scheme of g to B.

Definition 3.5 (Support of quotient map) *Let $\phi : (S^2, P(f)) \to (S^2, P(g))$ be a combinatorial class of quotient map, and let $B \subset Q(g)$ be the set of all points which land on points which are blown-up under ϕ. The* **support of** *ϕ is a finite set of closed discs $\{D_x\}_{x \in Q(f)}$ contained in the domain of ϕ which is defined up to isotopy as follows. Choose a representative $\phi_0 \in \phi$. For each $x \in B$, we choose a small closed disc N_x about x such that $N_x \cap B = \{x\}$, and set $D_x = \phi_0^{-1}(N_x)$.*

Since $\phi_0^{-1}(x)$ is connected and ϕ_0 is degree one, the preimages D_x are indeed closed discs. This definition makes sense. Given a fixed choice of discs N_x, any two maps $\phi_0, \phi_1 \in \phi$ are isotopic, and so there is an ambient isotopy fixing $P(f)$ from discs D_x defined by ϕ_0 to those defined by ϕ_1. Given two sets of discs N_x, N'_x, there is an ambient isotopy fixing $P(g)$ sending N_x to N'_x. The isotopy classes of the boundaries of the D_x are independent of the choices made and form an invariant of the combinatorial quotient map. Thus the support of ϕ consists of a collection of discs in the domain which is forward-invariant, up to isotopy. Away from the support of a quotient map $\phi : f \to g$, f and g look similar, since ϕ is degree one outside of the preimages of the finite set of blown-up points.

The next proposition says that postcritical points of g which are blown up under ϕ^{-1} are forward-invariant; its proof can be found in [McM3], Appendix B.

Proposition 3.6 (Blown-up points are invariant) *Let f and g be branched coverings, let $f \xrightarrow{\phi} g$ be a combinatorial quotient map, and let $B' \subset P(g)$ denote the set of points b for which $|\phi^{-1}(b) \cap P(f)| > 1$. Then $g(B') \subset B'$.*

If D_x is a component of the support of ϕ which is the preimage of a small neighborhood $x \in B$, then the boundary of D_x maps by the local degree of g near x onto the boundary of $D_{g(x)}$, the component of the support containing $\phi^{-1}(g(x))$.

A corollary to the previous proposition is the following fact: if $\phi : f \to g$ is a combinatorial quotient map, we may take the discs D_x and N_x to be forward-invariant under f and g by suitably modifying f and g within their combinatorial classes. Moreover, one may actually achieve this modification through a family of maps which agree with f and g on $Q(f)$ and $Q(g)$, respectively.

Conversely, suppose $\mathcal{D}$ is a collection of disjoint closed discs such that $f(\mathcal{D}) \subset \mathcal{D}$, up to isotopy. Then one may form a quotient map from f to a PF branched covering g by collapsing every component of $\mathcal{D}$ to a point.

Example: Let $p(z)$ be a postcritically finite polynomial of degree d. Then z^d can be realized as a quotient of $p(z)$. To see this, note that the filled-in Julia set $K(p)$ is connected. Let $\phi_0(z)$ be a continuous map which collapses $K(p)$ to 0, sends infinity to infinity, and sends the complement of K homeomorphically to the punctured plane. Then ϕ_0 is a quotient map. Since p and z^2 are both degree two coverings of the plane off of $K(p)$ and 0 respectively, $\phi_0|_{\mathbb{C}-K(p)}$ pulls back to a map ϕ_1 which may be extended to a quotient map by sending infinity to itself and again collapsing $K(p)$ to the origin. By construction, ϕ_0 and ϕ_1 are isotopic: they agree on $K(p)$, and any two orientation-preserving homeomorphisms of the punctured plane are isotopic through maps fixing the puncture. Hence the class ϕ of ϕ_0 is a combinatorial quotient map from $p(z)$ to z^d.

Example: If $p(z)$ is an infinitely-renormalizable quadratic polynomial, McMullen [McM3] has shown that p admits infinitely many critically finite hyperbolic quotient maps p_n. Conjecturally, these maps converge to p. By work of J.-C. Yoccoz (see e.g. [Hub]), this convergence for all such p would imply the density of hyperbolic dynamics in the quadratic polynomial family.

The next proposition says that the existence of a quotient map between two coverings is a property of their classes, and not of the individual representatives.

Proposition 3.7 *Let $\phi : f \to g$ be a quotient map. quotient map. Then given any $\overline{f} \in [f]$ and $\overline{g} \in [g]$, there exists a quotient map from $\overline{f}$ to $\overline{g}$.*

Proof: Let ψ_0^f and ψ_1^f, ψ_0^g and ψ_1^g be homeomorphisms giving combinatorial equivalences from $\overline{f}$ to f and g to $\overline{g}$ as in the definition given above. Let ϕ_0 and ϕ_1 be representatives of ϕ as in the definition of quotient map. We just concatenate the diagrams on the left by the equivalence for f and on the right by the equivalence for g. By composing the three isotopies between ψ_0^f and ψ_1^f, ϕ_1 and ϕ_0, and ψ_0^g and ψ_1^g, we obtain an isotopy between the composition

$\psi_0^f \circ \phi_0 \circ \psi_0^g$ and $\psi_1^f \circ \phi_1 \circ \psi_1^g$ which is constant on the postcritical set of $\overline{f}$. Hence this composition forms a combinatorial quotient map, as required.

■

3.4 Discussion of tuning

There are three ambiguities in the definition of tuning, as sketched in the introduction to this chapter.

Let $f(z)$ be a PFH rational map. First, suppose c is a strictly preperiodic postcritical point of $f(z)$, and suppose $f(c) = x_0$. Then after gluing in the discs D_i, we no longer have a natural identification of $f(c)$ with a point in the domain of $q(z)$. Hence we must make a choice of identification of $f(c)$ with a point in D_0 which is eventually periodic under $q(z)$, if we want the tuning $f * q$ to be PF. Second, one must find and choose a perturbation of $f(z)$ which possesses an invariant set of discs $\{D_x\}$ as in the above example. Third, if the degree of q is larger than two, one must *choose* a conjugacy from q on S^1_∞ to $f^{\circ n}$ on ∂D_0 along which to identify the dynamics of q on S^1_∞ and f on ∂D_0.

We deal with the first ambiguity as follows. We first define

Family of topological polynomials covering a mapping scheme. Let (S, τ, w) be a mapping scheme. A *family $\mathcal{P}$ of topological polynomials covering* (S, τ, w) is a map

$$\mathcal{P} : S \times S^2 \to S \times S^2$$

whose restriction p_x to $\{x\} \times S^2$ maps $\{x\} \times S^2$ to $\{\tau(x)\} \times S^2$ by a degree $w(x)$ topological polynomial sending infinity to infinity with local degree $w(x)$. The family $\mathcal{P}$ is said to be **postcritically finite** if

$$|\overline{\bigcup_{\substack{n>0, x\in S\\ c\in C(p_x)}} \mathcal{P}^{\circ n}(c)}| < \infty.$$

The family $\mathcal{P}$ is said to be *a family of polynomials covering the the mapping scheme* if the map $\mathcal{P}$ is conformal, i.e. each map is given by a complex polynomial.

Example: Consider the polynomials $p_0(z) = z^2 - 1$ and $p_1(z) = z^3 - 1$. Then we may let $p_0(z)$ and $p_1(z)$ act on two copies $\widehat{\mathbb{C}}_0, \widehat{\mathbb{C}}_1$ of the sphere by sending $\widehat{\mathbb{C}}_0$ to $\widehat{\mathbb{C}}_1$ by $p_0(z)$ and $\widehat{\mathbb{C}}_1$ to $\widehat{\mathbb{C}}_0$ by $p_1(z)$. Then $\mathcal{P} = \{p_1, p_2\}$ is postcritically finite. The family $\mathcal{P}$ covers the mapping scheme (S, τ, w) where S consists of two vertices x_0, x_1 weighted by $w(x_0) = 2, w(x_1) = 3, \tau(x_0) = x_1, \tau(x_1) = x_0$.

Let $B \subset Q(f)$ be a forward-invariant subset such that $(B, f|_B, w_f|_B)$ is a subscheme of the mapping scheme of f. Let $\mathcal{P} : B \times (\mathbb{C} \cup S^1_\infty) \to B \times (\mathbb{C} \cup S^1_\infty)$ be a PF family of polynomials, extended to maps of $\mathbb{C} \cup S^1_\infty$, covering the mapping schema $B \times (\mathbb{C} \cup S^1_\infty)$. That is, the restriction of $\mathcal{P}$ to each component is a polynomial, and the forward orbits of the set of critical points of $\mathcal{P}$ is finite.

If we then carry out the tuning construction by gluing the restriction of $\mathcal{P}$ to $\{x\} \times (\mathbb{C} \cup S^1_\infty)$ into a disc D_x near x for each $x \in B$, the dynamics of the resulting tuned map $f * \mathcal{P}$ is well-defined on the postcritical set since B is a subscheme of the mapping scheme of f.

Böttcher's theorem gives a way of resolving the second problem for rational maps. That is, if $B \subset Q(f)$ is as in the previous paragraph, then there is always a *canonical* set of discs $\mathcal{D}$ such that each disc $D_x \in \mathcal{D}$ contains a unique point of B, $f(D_x) = D_{f(x)}$ for all $x \in B$, and $f : \partial D_x \to \partial D_{f(x)}$ is topologically conjugate to $z \mapsto z^{w_f(x)}$. Since for each x, the restriction $\mathcal{P}$ to $\{x\} \times (\mathbb{C} \cup S^1_\infty)$ is topologically conjugate to $z \mapsto z^{w_f(x)}$, we can indeed replace the dynamics of f on $\mathcal{D}$ with that of $\mathcal{P}$ on $B \times (\mathbb{C} \cup S^1_\infty)$. We will explain this in more detail in the next section.

If $f(z)$ is hyperbolic, then the set of conjugacies along which to glue is always finite, and so we just equip the pair $(f, \mathcal{P})$ with a choice of such a conjugacy to resolve the third ambiguity. Thus this conjugacy amounts to a choice of "gluing data" for the tuning.

However, if we attempt to define tuning for PFH branched covers in a similar fashion, we may no longer appeal to Böttcher's theorem for the second step. We will therefore restrict the definition of tuning to those PFH branched covers f which are topologically conjugate on a neighborhood of $Q(f)$ to a PFH rational map g near $Q(g)$. We call such maps f *peripherally rigid maps.*

3.5 Peripherally rigid maps

Definition 3.8 (Peripherally rigid map) *A postcritically finite branched covering f is said to be* **peripherally rigid** *if for all $x \in Q(f)$, there exist homeomorphisms $M_x : (S^2, x) \to (S^2, 0)$ such that $M_{f(x)} \circ f \circ M_x^{-1} : (\overline{\Delta}, 0) \to (\overline{\Delta}, 0)$ is given by $z \mapsto \alpha(x) z^{w_f(x)}$, where $|\alpha(x)| = 1$ and $w_f(x)$ is the local degree of f near x, and such that the discs $D_x = M_x^{-1}(\overline{\Delta})$ are disjoint.*

The set $\mathcal{D} = \cup_{x \in Q(f)} D_x$ is called the **support** *of the peripherally rigid map f. The subspace of $\mathcal{S}([\{])$ consisting of all peripherally rigid maps combinatorially equivalent to f in the subspace topology will be denoted by $\mathcal{N}([\{])$.*

The discs D_x satisfy $f(D_x) = D_{f(x)}$. As a closed subset of the sphere, the support $\mathcal{D}$ of a peripherally rigid map is uniquely determined, since points on the interior of the support converge to attracting periodic points, while points on the boundary do not.

More generally, if f is a postcritically finite branched covering, and if $B \subset Q(f)$ is any subscheme, the map f is said to be *peripherally rigid near B* if it satisfies all the conditions in the definition of peripherally rigid where we replace the set $Q(f)$ with B. The *B-support* of f is then defined to be the union of the discs D_x, $x \in B$.

We will now prove that peripherally rigid maps exist. We will not need the fact that they are dense, but it is an easy consequence of the proof.

Theorem 3.9 (Peripherally rigid maps are dense) *Let f be a PF branched covering. Then the set of peripherally rigid maps is C^0 dense in $\mathcal{S}([\{])$.*

To prove this, we will need the following notion.

Family of local homeomorphisms of S^1 covering a mapping scheme. Let (S, τ, w) be a mapping scheme. A *family F of local homeomorphisms of S^1 covering the mapping scheme* (S, τ, w) is a continuous map

$$F : S \times S^1 \to S \times S^1$$

whose restriction to the component $\{x\} \times S^1$ sends $\{x\} \times S^1$ to $\{\tau(x)\} \times S^1$ by an orientation-preserving local homeomorphism of degree $w(x)$ with respect to the counterclockwise orientation on S^1. F is said to be *rigid* if it is of the form $z \mapsto \alpha(x) z^{w(x)}$ on $\{x\} \times S^1$ for some $\alpha(x)$ with $|\alpha(x)| = 1$. The space of all such local homeomorphisms in the uniform topology will be denoted by $\mathcal{F}(S, \tau, w)$, and the space of rigid maps will be denoted by $\mathcal{R}(\mathcal{S}, \tau, \sqsupseteq)$.

The space $\mathcal{R}(S, \tau, w)$ is canonically homeomorphic to the space

$$\prod_{x \in S} S^1$$

via the homeomorphism $F \mapsto (F(x, 1))_{x \in S}$.

Theorem 3.10 *There is a canonical deformation retract of $\mathcal{F}(S, \tau, w)$ onto $\mathcal{R}(S, \tau, w)$.*

Proof: We first do the case where $|S| = 1$. The set of degree d orientation-preserving local homeomorphisms ρ of the circle to itself is a product space $S^1 \times H$. The projection maps are $\rho \mapsto \rho(1) \in S^1$ and $\rho \mapsto \rho(1)^{-1} \cdot \rho$. The fiber H_t above $t \in S^1$ is contractible to the map $z \mapsto t \cdot z^d$. To prove this, we may assume that $\rho(1) = 1$. Let $\widetilde{\rho}$ be the lift of ρ to the universal cover $(\mathbb{R}, 0)$ mapping by $x \mapsto \exp(2\pi i x)$. Let $\rho_s(x) = \exp(2\pi i[(1-s)\widetilde{\rho}(x) + s \cdot d \cdot x])$. Then ρ_s is a deformation retract of H_1 onto $z \mapsto z^d$.

To prove the general case, we apply the same argument on each factor $\{x\} \times S^1, x \in S$. The space $\mathcal{F}(S, \tau, w)$ is homeomorphic to the product space $(\prod_{x \in S} S^1) \times H$, where $H = \{F | F(S \times \{1\}) \subset S \times \{1\}\}$. The fiber H is contractible, using the same lifting argument.

■

Proof of Theorem 3.9.

We need to prove the following assertion. Let f be a PF branched covering of the sphere to itself. Then given any small ϵ-neighborhood V of $Q(f)$, there is a family of maps f_t such that $f_0 = 0$, $f_t|_{S^2 - V} = f$ for all t, and f_1 is peripherally rigid.

Suppose V is given. By the continuity of f, for each $x \in Q(f)$, there is a closed disc $D_x \subset V$ such that $D_x \cap Q(f) = \{x\}$, and such that every component of $f^{-1} D_x$ which intersects $Q(f)$ is contained in V. By precomposing f by a map isotopic to the identity through maps fixing $Q(f)$, and which are the identity off V, we may assume that $f(D_x) = D_{f(x)}$. Let $\mathcal{D} = \cup_{\S \in \mathcal{Q}(\{)} \mathcal{D}_\S$.

Hence we may assume that f has this property. The Alexander Trick ([Bir], Section 4.2) shows that we may assume f is radial. This means that for all $x \in Q(f)$, there are homeomorphisms $h_x : (D_x, x) \to (\overline{\Delta}, 0)$ such that $h_{f(x)} \circ f \circ h_x^{-1} : (\overline{\Delta}, 0) \to (\overline{\Delta}, 0)$ is given in polar coordinates by

$$(r, \theta) \to (r, \rho_x(\theta))$$

for some homeomorphism $\rho_x : S^1 \to S^1$.

The homeomorphisms $\{h_x\}_{x \in Q(f)}$ identify $f|_{\partial \mathcal{D}}$ with an element F of the space $\mathcal{F}(Q(f), f|_{Q(f)}, w_f)$. By Theorem 3.10, there is a canonical deformation of F to an element of $\mathcal{R}(Q(f), f|_{Q(f)}, w_f)$. Extending this deformation to a deformation of f through maps which are radial on the discs D_x proves the theorem.

■

We will define the "gluing data" for tuning by defining markings of peripherally rigid maps.

Definition 3.11 (Marking of peripherally rigid map) *Let f be a peripherally rigid map and $(Q(f), f|_{Q(f)}, w_f)$ its mapping scheme. Let $B \subset Q(f)$ be a subscheme of its mapping scheme. A* **marking of the rigid map f near B** *is a choice of homeomorphisms $\{M_x\}_{x \in B}$ as in the definition of peripherally rigid map. A marking of f near B is said to be* **invariant** *if in addition, $M_{f(x)} \circ f \circ M_x^{-1} : (\overline{\Delta}, 0) \to (\overline{\Delta}, 0)$ is given by $z \to 1 \cdot z^{w_f(x)}$, where $w_f(x)$ is the local degree of f near x. Two markings are said to be equivalent if they agree on the boundary of the B-support of the map f.*

Let B be a subscheme of the mapping scheme of f. The set of markings of f near B, up to equivalence, is the same as the set $\mathcal{R}(B, f|_B, w_f|_B)$: an initial choice of a marking of f near B gives an identification of $\{\partial D_x\}_{x \in B}$ with $B \times S^1$. This homeomorphism is not canonical, however, since it depends on an initial choice.

If the mapping scheme of f restricted to B is of hyperbolic type, then invariant markings always exist, since $z \mapsto \alpha z^n, n \geq 2$ is conjugate via rigid rotations of S^1 to $z \mapsto z^n$ for any α with $|\alpha| = 1$. If B is of hyperbolic type, then the set of markings of f near B is nonempty and finite, since any orientation-preserving homeomorphism $h : S^1 \to S^1$ conjugating $z \mapsto z^n, n > 1$ to itself is necessarily a rigid one, i.e. is multiplication by a root of unity.

The following examples will all be used in the sequel.

Example: Canonical peripherally rigid map associated to a monic polynomial. Let $p(z)$ be a monic degree d polynomial acting on $\widehat{\mathbb{C}}$. Let $B = \{\infty\}$. Since ∞ is a fully invariant superattracting fixed point, $p(z)$ may be extended canonically to a map on $\mathbb{C} \cup S^1_\infty$ by sending the ideal point $\exp(2\pi i t) \cdot \infty$

to the point $\exp(2\pi i d \cdot t) \cdot \infty$. We may then form a peripherally rigid map of the sphere to itself by identifying $\mathbb{C} \cup S^1_\infty$ with the lower hemisphere $\overline{\Delta}$, and setting

$$\overline{p}(z) = \begin{cases} p(z) & \text{if } z \in \overline{\Delta}, \\ z^d & \text{if } z \in \widehat{\mathbb{C}} - \Delta \end{cases}$$

We call the resulting map $\overline{p}(z)$ the *canonical map peripherally rigid near* ∞ *associated to* $p(z)$.

The map $\overline{p}(z)$ admits $d-1$ distinct invariant markings near the point at infinity (up to equivalence) which are the maps $z \mapsto \omega z$, where ω is a $(d-1)st$ root of unity. The identity map will be called the *canonical* invariant marking of $\overline{p}$ near infinity.

If p is monic quadratic polynomial, the canonical marking of $\overline{p}$ is the unique invariant marking. More generally, suppose f is a B-peripherally rigid map for some subscheme B of its mapping scheme. If every vertex of B has weight one with one exception x of weight two, then f admits a unique invariant marking near B, up to equivalence.

Example: PF maps. Let $f(z)$ be a PF rational map and $B \subset Q(f)$ a component of hyperbolic type of its mapping scheme. We can produce a B-peripherally rigid map in a canonical fashion by appealing to Böttcher's theorem.

Since B is a component of the mapping scheme of f, the set B contains a unique superattracting cycle $\{x_0, x_1, ..., x_{p-1}\}$. By Böttcher's theorem, there exist Riemann mappings $\phi_i : (\Delta, 0) \to (U_i, x_i)$ conjugating $f^{\circ p}$ on U_i to $z \mapsto z^d$, where $d = \prod_i w_f(x_i)$. Let $\Delta(r) = \{z| \, |z| < r\}$.

We now "dilate" the map f by postcomposing with a canonical homeomorphism so that the discs of radius one-quarter in the local coordinates given by Böttcher's theorem are mapped onto each other. Set $D_i = \phi_i(\Delta(1/4))$, and postcompose f by the homeomorphism g such that $g = \text{id}$ on $\widehat{\mathbb{C}} - D_i'$, and such that $\phi_{i+1} \circ g \circ \phi_i(z) : \overline{\Delta} \to \overline{\Delta}$ is given by $(r, \theta) \mapsto (\rho_i(r), \theta)$, where ρ_i is the unique piecewise-linear map chosen so that $g(f(D_i)) = D_{i+1}$. The homeomorphism g is canonical, by the uniqueness part of Böttcher's theorem.

The map $g \circ f$ may then be similarly adjusted near points in $B - \cup_i x_i$ so that the resulting map $\overline{f}$ is B-peripherally rigid. The Riemann maps $\phi_x, x \in B$ determine an invariant marking $\{M_x\}_{x \in B}$.

Example: Families of topological polynomials covering a mapping scheme. Let $\mathcal{P}$ be a family of PF topological polynomials covering a mapping scheme (S, τ, w). Then the restriction of $\mathcal{P}$ to the set of points at infinity gives a mapping scheme which is canonically isomorphism to the scheme (S, τ, w).

We say that the family $\mathcal{P}$ is *rigid near infinity* if $\mathcal{P} : S \times S^2 \to S \times S^2$ can be equipped with discs D_x about the point at infinity in $\{x\} \times S^2$ such that the following holds. There exist homeomorphisms $N_x : (S^2, \infty) \to (S^2, 0)$ such that $N_{f(x)} \circ \mathcal{P} \circ N_x^{-1} : (\overline{\Delta}, 0) \to (\overline{\Delta}, 0)$ is given by $z \mapsto \alpha(x) z^{w(x)}$, where $|\alpha(x)| = 1$. We define markings and invariant markings in a manner analogous to that for branched coverings. A family $\mathcal{P}$ of monic polynomials acting conformally on copies of the Riemann sphere and covering the mapping scheme (S, τ, w) yields a canonical family $\overline{\mathcal{P}}$ of topological polynomials which are rigid on the discs $\widehat{\mathbb{C}} - \Delta$, in the same way as for polynomials.

3.6 Mating

In this section we define the mating of two topological polynomials which are peripherally rigid near their distinguished totally invariant postcritical points.

Let f and g be two topological polynomials which are rigid near infinity. Given markings M of f and N of g near the points at infinity, a new branched covering $h = (f, M) * (g, N)$ can be constructed by removing the discs and gluing along the markings. More precisely, we set

$$h(z) = \begin{cases} M \circ f \circ M^{-1} & \text{if } z \in \widehat{\mathbb{C}} - \Delta, \\ 1/z \circ N \circ g \circ N^{-1} \circ 1/z & \text{if } z \in \overline{\Delta}. \end{cases}$$

We call the resulting map h the *mating* of f and g along M and N.

Example: matings of polynomials. Let $p(z)$ and $q(z)$ be two monic polynomials of degree d. Let $\overline{p}(z)$ and $\overline{q}(z)$ be the associated canonical peripherally rigid maps. Then by gluing along their canonical markings, we obtain a branched covering of the sphere which is called the *mating of p and q*. This coincides with the definition of mating given in [Tan2].

3.7 Tuning

Let f be a PF branched covering, and let $B \subset (Q(f), f, w_f)$ be a subscheme of the mapping scheme associated to f. Suppose that B is of hyperbolic type; this means that every cycle in B contains at least one critical point. Suppose further that f is peripherally rigid near B.

Let $\mathcal{P}$ be a PF family of topological polynomials covering the mapping scheme B, and assume that $\mathcal{P}$ is rigid near the set $\mathcal{B}$ of points at infinity. Then the mapping scheme $(\mathcal{B}, \mathcal{P}|_{\mathcal{B}}, w_{\mathcal{P}})$ is canonically isomorphic to the mapping scheme $(B, f|_B, w_f|_B)$, where $w_{\mathcal{P}}(y)$ is the local degree of $\mathcal{P}$ at the point y.

Recall that the idea is to glue in the family $\mathcal{P}$ into the B-support $\{D_x\}_{x \in B}$ of f. We now choose gluing data. Choose an invariant marking $\{M_x\}_{x \in B}$ of f. Let $\{N_{x'}\}_{x' \in \mathcal{B}}$ be an invariant marking for $\mathcal{P}$. As with matings, the *tuning of f by $\mathcal{P}$ along $\{M_x\}_{x \in B}$ and $\{N_{x'}\}_{x' \in \mathcal{B}}$* is defined by gluing $\mathcal{P}$ into the discs associated to f and B along the given markings. Suppressing the dependence on the markings, we denote the resulting map by $f * \mathcal{P}$. Since B is of hyperbolic type, the set of invariant markings is finite, therefore the set of possible choices of gluing data is also finite.

To write the formula in a convenient way, given $x \in B$, we define $m_x(z) = N_{x'}^{-1} \circ 1/z \circ M_x(z)$; given $x' \in \mathcal{B}$, we define $n_{x'}(z) = M_x^{-1} \circ 1/z \circ N_{x'}(z)$. Then

$h = f * \mathcal{P}$ is defined by

$$h(z) = \begin{cases} f(z) & \text{, if } z \text{ is not in the } B\text{-support of } f, \\ n_{y'} \circ \mathcal{P} \circ m_x(z) & \text{, if } z \in M_x^{-1}(\overline{\Delta}, \end{cases}$$

where $f(x) = y, \mathcal{P}(x') = y'$. The union of the discs D_x is called the *support* of the tuning $f * \mathcal{P}$.

Example: Mating is a special case of tuning. The mating of a pair f, g of degree d topological polynomials that are rigid near infinity is topologically conjugate to the tuning of f by g, or g by f.

Example: quadratic polynomials. Let $f(z) = z^2 + c$ be a PF hyperbolic polynomial, and let $g(z) = z^2 + c'$ be any PF quadratic polynomial. Then $f(z)$ contains a unique finite superattracting cycle of some period k. The component B of the mapping scheme containing this cycle is $Q(f) - \{\infty\}$. The set B contains a unique critical point $0 \in B$ of multiplicity one, so the mapping scheme $B \subset (Q(f), f, w_f)$ contains a unique vertex of weight larger than one which has weight equal to two. Since $f(z)$ is PF and hyperbolic, B is of hyperbolic type. The general construction given in Section 3.5 applies, and so by varying $f(z)$ within its combinatorial class we may find a representative $f_0 \in [f]$ which is B-peripherally rigid. Moreover, f_0 admits a unique invariant marking. Define a family of polynomials $\{p_x\}_{x \in B} = \mathcal{P} : B \times \widehat{\mathbb{C}} \to B \times \widehat{\mathbb{C}}$ by $p_x(z) = g(z)$ if $x = 0$ and $p_x(z) = z$ otherwise. Then $\mathcal{P}$ is PF since g is PF. Let $\overline{\mathcal{P}}$ denote the canonical peripherally rigid family of maps associated to this family of conformal monic polynomials. Mark $\overline{\mathcal{P}}$ near its points at infinity by the identity maps. Then the tuning $f_0 * g$ along these markings is canonical, up to the choice of the rigid representative f_0.

3.8 Tunings and quotients as inverses

An immediate consequence of the definition of tuning and quotient map is the existence of a canonical combinatorial class of quotient map $\phi : f * \mathcal{P} \to f$.

Theorem 3.12 (Tuning yields quotient) *Let R be the branched covering which is the tuning $f * \mathcal{P}$ of a PF branched covering by a family of topological polynomials $\mathcal{P}$. Then there is a canonical combinatorial class of quotient map $\phi : R \to f$ given by each component of the support of the tuning to a point.*

Alternatively, one may think of this as cutting out the discs $\{D_x\}$ comprising the support of the tuning and gluing in $z \mapsto z^{w(x)}$. The next theorem shows that a map which admits a quotient is a tuning of some map by a family of polynomials.

Theorem 3.13 (Quotient yields tuning) *Let $\phi : R \to f$ be a combinatorial quotient map. Suppose that the mapping scheme $B \subset (Q(f), f, w_f)$ of ϕ is of hyperbolic type. Then there are $\overline{R}$ and $\overline{f}$ combinatorially equivalent to R and f*

such that $\overline{f}$ is B-peripherally rigid and $\overline{R}$ is the tuning of $\overline{f}$ by some family $\mathcal{P}$ of topological polynomials covering B and rigid near infinity.

We call the family $\mathcal{P}$ the *family of polynomials induced by the quotient map ϕ.*

Small neighborhoods of the blown-up points in $P(f)$ under ϕ form an invariant set of discs, up to isotopy. Cutting along these discs and collapsing the boundary will yield the family of polynomials.

Proof: First, we may assume that there exist discs N_x centered at $x \in B$ such that $f(N_x) = N_{f(x)}$, and that for $D_x = \phi^{-1}(N_x)$, $R(D_x) = D_{f(x)}$, i.e. the discs N_x are mapped onto themselves by f, and their preimages under some representative of ϕ are mapped onto themselves by R.

By precomposing R by a homeomorphism isotopic to the identity through maps preserving $Q(f)$ and ∂D_x, we may assume that R leaves invariant a set of topological annuli whose interiors contain the curves ∂D_x. By conjugating R by a homeomorphism isotopic to the identity rel $Q(f)$ and ∂D_x, we may assume the annuli A_x are preserved under R, i.e. $R(A_x) = A_{f(x)}$.

By the Alexander trick and Theorem 3.10, there is a one-parameter family of maps $\overline{R}_t$ agreeing with R off of the union of the A_x and joining R to a map $\overline{R}$ which is rigid on ∂D_x. The map $\overline{R}$ is combinatorially equivalent to R, by construction.

Since the mapping scheme $(B, f|_B, w_f|_B)$ is of hyperbolic type, there exist coordinates on ∂D_x induced by homeomorphisms $h_x : \partial D_x \to S^1$ such that the map $\overline{R}|_{\partial D_x}$ in these coordinates is given by $z \mapsto z^{w(x)}$, where $w(x)$ is the local degree of f near $x \in B$. Fix one such choice of coordinates.

Let $U = S^2 - \cup D_x$. Define a branched covering $\overline{f}$ be the branched covering as follows. Set $\overline{f}|_U = \overline{R}|_U$. The homeomorphisms h_x extend to Riemann mappings $h_x : D_x \to \overline{\Delta}$. The maps $h_{f(x)} \circ \overline{f} \circ h_x^{-1}$ then extend to a map $(\overline{\Delta}, 0) \to (\overline{\Delta}, 0)$ of the form $z \mapsto z^{w_f(x)}$. We then define $\overline{f}$ on D_x by $h_{f(x)}^{-1} \circ z^{w_f(x)} \circ h_x$. Define a family $\mathcal{P}$ of topological polynomials by as follows. Set

$$\mathcal{P}|_{\{x\}\times\overline{\Delta}} = h_{f(x)} \circ f \circ h_x^{-1},$$

and

$$\mathcal{P}|_{\{x\}\times(S^2-\Delta)} = z^{w_f(x)}.$$

Then $\overline{f}$ and $\mathcal{P}$ are peripherally rigid maps equipped with invariant markings such that $\overline{f} * \mathcal{P} = \overline{R}$.

We now show that $\overline{f}$ is combinatorially equivalent to f. Since the existence of a quotient to f is an invariant of the combinatorial class of R, there is a quotient map ϕ from $\overline{R}$ to f. Let $\phi_0, \phi_1 \in \phi$ be such that $\phi_0 \circ \overline{R} = f \circ \phi_1$ with $\phi_0 \simeq \phi_1$ through maps fixing $P(\overline{R})$. The restriction of ϕ_0 and ϕ_1 to U may then be extended to give maps ψ_0, ψ_1 such that $\psi_0 \circ \overline{f} = f \circ \psi_1$. Since ϕ_0 and ϕ_1 are isotopic, so are ψ_0 and ψ_1, and so the ψ_i determine a combinatorial equivalence between $\overline{f}$ and f.

∎

3.9 Notes

Our definition of mating coincides with the *formal mating* of polynomials as defined in [Tan2]. There is another notion of mating where we are allowed to collapse pieces of the postcritical set which lie in disjoint closed discs which are permuted up to isotopy and map by degree one. We will not consider this definition; see [Tan2].

The question of how the combinatorial class of the tuning $f * \mathcal{P}$ depends on the class of f, the "class" of $\mathcal{P}$ (see Section 4.11 for the definition), and the "gluing data" is a subtle one. This will be the subject of later work.

Chapter 4

Combinatorial dynamics of arcs and curves

Let $\mathcal{C}(P(f))$ denote the set of isotopy classes of simple closed curves in $S^2 - P(f)$. A branched covering f defines a relation on $\mathcal{C}(P(f)) \times \mathcal{C}(P(f))$ which we call the *pullback relation*, and denote by f^*. The pullback relation was used by Thurston to characterize those PF branched coverings which are combinatorially equivalent to rational maps. The idea is that if there exists a finite set Γ of disjoint simple closed curves behaving in a certain way under f^*, then Γ forms a topological obstruction to the existence of a rational map which is equivalent to f. Such a set Γ is called a *Thurston obstruction*. In practice, however, it is usually difficult to decide when a given map satisfies Thurston's criterion, since there are infinitely many collections Γ to check.

In this chapter, we introduce another relation which is a natural analog of the relation f^*. Let $\mathcal{A}(P(f))$ denote the set of embedded arcs whose interiors are contained in $S^2 - P(f)$ and whose endpoints lie in $P(f)$, up to isotopy fixing $P(f)$. The map f defines a relation on $\mathcal{A}(P(f)) \times \mathcal{A}(P(f))$ which we call the *pushforward relation* and denote by f^*.

There is a natural notion of an intersection number between an isotopy class of arc and curve. Using intersection numbers and ideas of Shishikura and Tan, we show that arcs and curves which intersect have restricted dynamics under the relations f_* and f^*, respectively. We then show that topological obstructions to a branched covering being combinatorially equivalent to a rational map are severely restricted by the presence of periodic arcs under the pushforward relation. The main result is the Shishikura-Tan theorem, which says that a Thurston obstruction which intersects a periodic arc is a very special one called a *Levy cycle*.

Section 4.1 lists basic definitions. In Section 4.2 we prove an important topological fact needed for our analysis. Section 4.3 lists definitions concerning relations and their iterates. Sections 4.4 and 4.5 give the definitions and basic properties of the pullback and pushforward relations on curves and arcs.

In Section 4.6 we give a detailed analysis of how intersections of curves with arcs and arcs with arcs restrict the dynamics under the relations. In Section 4.7 we introduce linear transformations associated with invariant sets of curves and arcs and state Thurston's classification of PF rational maps. Section 4.8 applies the analysis of intersection numbers to Thurston obstructions. Section 4.9 is a digression which shows that certain linear transformations associated to pushforward and pullback relations are adjoint with respect to a pairing defined by intersection numbers.

Section 4.10 gives an analysis of how the relations behave under tuning and passing to quotients. There, we prove that a rational map R which admits a quotient is the tuning $f * \mathcal{P}$ of a rational map f by a family of polynomials $\mathcal{P}$. The fact that f is equivalent to a rational map will follow from results in [McM3]; for completeness, we also give a proof. In the appendix to this Chapter, Section 4.11, we prove a generalization of the necessity of Thurston's criterion which we will need later.

4.1 Arcs and curves

Definition 4.1 (Simple closed curves)

- *A* **simple closed curve** γ *in* (S^2, A) *is the image of a continuous embedding of the unit circle into* $S^2 - A$
- *Two simple closed curves* γ, γ' *are said to be* **isotopic relative to A**, *(or more simply,* **isotopic rel A**, *or, if the set* A *is understood, merely* **isotopic** *) if there exists a continuous one-parameter family* $h_t, t \in I$ *of embeddings of the unit circle such that* $h_0(S^1) = \gamma$ *and* $h_1(S^1) = \gamma'$.
- *Two simple closed curves* γ, γ' *will be called* **parallel** *if they are isotopic.*
- *A simple closed curve which is contractible will be called* **inessential**; *otherwise, it is* **essential**.
- *A simple closed curve* γ *is said to be* **peripheral** *if* γ *is isotopic into every neighborhood of a point* $a \in A$.
- *A* **curve system** Γ *is a collection of disjoint, essential, simple closed curves, no two of which are parallel. A curve system is called* **nonperipheral** *if none of its elements are peripheral.*
- *The set of isotopy classes of simple closed curves in* (S^2, A) *will be denoted by* $\mathcal{C}(A)$.

Remark: A nonperipheral curve system is also called a *multicurve* in [Thu4] and [DH2].

Two simple closed curves on a compact orientable surface are isotopic if and only if they are homotopic. Thus the definitions given above are equivalent to those obtained by replacing "isotopy" with "homotopy". See e.g. [Eps].

A peripheral curve is one which can be shrunk to a point in A.

Curve systems are always finite: there can be at most $2|A|-3$ elements in a curve system, and at most $|A|-3$ elements in a nonperipheral curve system. We will prove the first statement by induction; the other can be proved similarly. Let Γ be a curve system in (S^2, A). If $|A| \leq 3$, then every simple closed curve is peripheral, and the fact is obvious. So assume $|A| \geq 3$. If Γ consists entirely of peripheral elements, then $|\Gamma| \leq |A| \leq 2|A|-3$, since $|A| > 3$, so the fact is true in this case. Now suppose $\gamma \in \Gamma$ is nonperipheral. Let U and V be the closures of the two components of $S^2 - \gamma$, and suppose $|U \cap A| = m, |V \cap A| = m$. Then $m, n \geq 2$ since γ is nonperipheral. Induction now shows that there are at most $2(m+1)-3$ curves in $\Gamma \cap U$ and $2(n+1)-3$ curves in $\Gamma \cap V$. We have counted γ twice, so the total number is given by $2(m+n)-3 = 2|A|-3$.

Definition 4.2 (Arcs) • *An* **arc** α *in* (S^2, A) *is the image of an embedding of the closed unit interval* I *into* S^2 *such that the image of the interior of the interval is contained in* $S^2 - A$.

- *An* **endpoint** *of an arc* α *is the image of a point in* ∂I.
- *Two arcs* α, α' *are said to be* **isotopic relative to** A *(or more simply* **isotopic rel** A, *or* **isotopic** *) if there is a continuous one-parameter family* $h_t, t \in I$ *of embeddings of the interval such that*
  1. $h_0(I) = \alpha, h_1(I) = \alpha'$;
  2. *if* $E = h_0^{-1}(A)$, *then* $h_t|_{E \times I}$ *is constant on each component of* E, *i.e. the endpoints of* α *which lie in* A *do not move under the deformation.*
- *Two arcs will be called* **parallel** *if they are isotopic.*
- *An arc* α *is said to be* **inessential** *if it is isotopic into every neighborhood of a point* $a \in A$; *otherwise it is said to be* **essential**.
- *The set of isotopy classes of arcs in* (S^2, A) *will be denoted by* $\mathcal{A}(A)$.

Inessential arcs come in three types, depending on how many of their endpoints intersect A. An arc is inessential if and only if either one of its endpoints is not in A, or both of its endpoints are $a \in A$ and the arc can be shrunk down to the point a without crossing other points of A. We will denote the endpoints of an arc α by $e(\alpha)$. The endpoints of an essential arc, or of any arc whose endpoints are the same and lie in A, are an invariant of the isotopy class. Two arcs on a compact, orientable surface are isotopic if and only if they are homotopic; see [Eps].

Definition 4.3 (Intersection numbers) *Let* $[\eta], [\nu]$ *be two isotopy classes of arcs*
or curves. The **intersection number** *is defined by*

$$[\eta] \cdot [\nu] = \inf_{\eta \in [\eta], \nu \in [\nu]} |\ \{\ (\eta - e(\eta)) \cap (\delta - e(\delta))\ \}\ |\ .$$

Thus the intersection number between two arcs or curves is the infimum over the representatives in the classes of the number of points in their intersection, where we do not count intersecting endpoints. The intersection number of any inessential class of arc or curve with any class of arc or curve is zero. The self-intersection number $[\eta] \cdot [\eta] = 0$ for any simple closed curve or arc, since the sphere is orientable, and since we consider only embedded closed curves and arcs. If $[\gamma]$ contains elements which are peripheral and isotopic into $a \in A$, then for any essential class $[\alpha]$ of arc with an endpoint at a, $[\gamma] \cdot [\alpha] \neq 0$.

4.2 A convention and an important topological fact

In this section we define lifts of arcs. We also prove a topological proposition which says that unless $|P(f)| = 2$, no two essential lifts of a curve or arc are isotopic in $(S^2, Q(f))$.
Convention. Let f be a PF branched covering, and let α be an arc. We will call the closure of a component of the preimage of $f^{-1}(\alpha - e(\alpha))$ a *lift* of the arc α, or sometimes a *component* of the preimage of α.

Proposition 4.4 (Classes in Q encode essential lifts) *Let f be a nonelementary PF branched covering of the sphere to itself, and let η be an essential arc or simple closed curve in $(S^2, P(f))$. Then the lifts of η, as arcs or simple closed curves in $(S^2, Q(f))$, are essential, and no two lifts δ, δ' of η are isotopic rel $Q(f)$.*

Proof:
Lifts are essential. Suppose η is a curve. An inessential lift δ of η bounds a disc $D \subset S^2 - Q(f)$ which maps onto a component of $S^2 - \eta$. Thus a component of $S^2 - \eta$ contains no elements of $P(f)$, and thus η was inessential. Now suppose η is an arc with distinct endpoints. Then any lift δ of η is an arc with distinct endpoints in $Q(f)$ and is thus essential. Finally, suppose η is an arc with indistinct endpoints. If a lift δ has distinct endpoints it is essential; if its endpoints are the same, and is inessential, then similar reasoning given for the case of a curve shows that η must be inessential if δ is inessential.
Case when η is a simple closed curve. The set of lifts of η under f is a collection of *disjoint* simple closed curves. Hence if two are both essential and isotopic, they bound an open annulus $A \subset S^2 - Q(f)$. On the one hand, A contains no critical points. On the other, A is a component of the lift of a disc D which is a component of $S^2 - \eta$ under the branched covering f, a contradiction.
Case when η is an arc. There are a few sub-cases to consider.

1. **The endpoints of η, δ, δ' are distinct.** If δ and δ' are isotopic, they bound an open disc in $S^2 - Q(f)$ which maps onto the complement of η. Hence the complement of η contains no elements of $f(Q(f)) = P(f)$, which implies that f is elementary.

2. **The endpoints of η are the same.**

   (a) **The endpoints of δ, δ' are the same.** In this case if δ and δ' are isotopic then they bound an open region $A \subset S^2 - Q(f)$ which maps onto a component D of $S^2 - \eta$. Hence $D \subset S^2 - P(f)$. This implies that η is inessential.

   (b) **The endpoints of δ, δ' are different.** Similar reasoning as in the previous step applies. One concludes again that η is inessential.

■

4.3 Relations and their iterates

In this section we state definitions which are needed to discuss iterates of relations.

Defining a relation $\mathcal{R}$ on a the product of a countable set Z with itself is equivalent to specifying a directed graph whose vertices are the elements in the set Z and for which there is exactly one directed edge from x to y if $x\mathcal{R}y$. We can then *iterate* such a relation. Given a positive integer n, we form a new relation $\mathcal{R}^n$ by specifying that $x\mathcal{R}^n y$ if and only if we can get to y from x by going along exactly n directed edges.

Definition 4.5 (Invariance under relations) *Let $\mathcal{R}$ be a relation on the product of a countable set Z with itself. Let $X \subset Z$. Then X is called*

- **forward-invariant** *if for all $x \in X$, $x\mathcal{R}y$ implies $y \in X$;*

- **backward-invariant** *if for all $y \in X$, $x\mathcal{R}y$ implies $x \in X$;*

- **fully invariant** *if it is both forward- and backward-invariant;*

- **irreducible** *if given any two elements $x, x' \in X$, there exist $x = x_1, x_2, ..., x_n = x'$ such that $x_i \in X$ and $x_i\mathcal{R}x_{i+1}, i = 1, 2, ..., n-1$.*

  Furthermore, we say:

- *an* **orbit of x_0 under $\mathcal{R}$** *is a sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_n\mathcal{R}x_{n+1}$ for all n;*

- *a* **cycle** *is an orbit $\{x_n\}$ such that there exists a $p > 1$ such that $n = m \bmod p$ implies that $x_n = x_m$. The least such p is called the period of the cycle;*

- *an element $x \in X$ is* **periodic** *if it is contained in some cycle; it is* **eventually periodic** *if there is an orbit of x containing periodic elements.*

- *a* **simple cycle** *is a cycle for which the elements $\{x_0, x_1, ..., x_{p-1}\}$ are all distinct, where p is the period of the cycle.*

To say that X is irreducible means that one can get from any point of the set X to any other point using the arrows in the directed graph while staying in the set X. The set $\{ y \mid xRy \}$ is allowed to be empty. Also, if $W \subset Z$, then restricting $\mathcal{R}$ to $W \times W$ or $W \times Z$ give new relations.

Notation. We will use the notation $\mathcal{R}(x)$ to denote the set of all y for which xRy. We will often think of relations as set-valued functions.

4.4 Pullback relation on simple closed curves

In this section we define the pullback relation. Let f be a PF map with $|P(f)| > 2$. The map f will define a one-to-finite relation $f_{\mathcal{C}}^{-1}$ from $\mathcal{C}(P(f))$ to $\mathcal{C}(Q(f))$ and function $i_{\mathcal{C}}$ from $\mathcal{C}(Q(f))$ to $\mathcal{C}(P(f))$. The composition $i_{\mathcal{C}} \circ f_{\mathcal{C}}^{-1}$ will be the *pullback relation on simple closed curves* and will be denoted f^*.

The relation $f_{\mathcal{C}}^{-1}$. The map f induces a set-valued function

$$f_{\mathcal{C}}^{-1} : \mathcal{C}(P(f)) \to \mathcal{C}(Q(f))$$

defined by sending a class $[\gamma]$ to the set of classes represented by components of the preimage of some representative $\gamma \in [\gamma]$. This gives a well-defined map on classes. By Proposition 4.4,*Classes in Q encode lifts*, essential simple closed curves map to a set of essential simple closed curves, and no two essential lifts represent the same class. Moreover, this relation is injective in the sense that given $[\gamma_1] \neq [\gamma_1']$, the sets $f_{\mathcal{C}}^{-1}([\gamma_1])$ and $f_{\mathcal{C}}^{-1}([\gamma_1'])$ are disjoint, i.e. it is the inverse of a map. The image of a class has at most d elements under $f_{\mathcal{C}}^{-1}$, where d is the degree of f.

Degrees. Suppose $[\gamma] \in \mathcal{C}(P(f))$ and $[\delta] \in \mathcal{C}(Q(f))$ are classes of essential simple closed curves δ, γ with $\delta \subset f^{-1}(\gamma)$. By Proposition 4.4, δ is the unique component of the preimage of γ in $[\delta]$. Hence the degree $\deg(f : \delta \to \gamma)$ is an invariant of the pair of classes of $[\gamma] \in \mathcal{C}(P(f))$ and $[\delta] \in \mathcal{C}(Q(f))$, and will be denoted by $\deg(f : [\delta] \to [\gamma])$. The degree is an additional piece of data associated to the relation $f_{\mathcal{C}}^{-1}$.

The map $i_{\mathcal{C}}$. The inclusion $i : S^2 - Q(f) \hookrightarrow S^2 - P(f)$ (sometimes called the *erasing map* since we "erase" points in $Q(f)$ which are not in $P(f)$) induces a function $i_{\mathcal{C}} : \mathcal{C}(Q(f)) \to \mathcal{C}(P(f))$ by sending a curve in $S^2 - Q(f)$ to its class in $S^2 - P(f)$. The class of the image is independent of the representative chosen. This map is in general infinite-to-one.

The following proposition says that these relations are invariants of the combinatorial class of f.

Proposition 4.6 (Curve relations are combinatorial) *Let ψ be a combinatorial*
equivalence between PF branched coverings f and g. Then ψ induces bijections

$\psi_P : \mathcal{C}(P(f)) \to \mathcal{C}(P(g))$ and $\psi_Q : \mathcal{C}(Q(f)) \to \mathcal{C}(Q(g))$ which preserve the property of being essential, inessential, or peripheral, which preserve intersection numbers and degrees, and which the make the diagrams

$$\begin{array}{ccc} \mathcal{C}(P(f)) & \xrightarrow{f_{\mathcal{C}}^{-1}} & \mathcal{C}(Q(f)) \\ \psi_P \downarrow & & \downarrow \psi_Q \\ \mathcal{C}(P(g)) & \xrightarrow{g_{\mathcal{C}}^{-1}} & \mathcal{C}(Q(g)) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{C}(Q(f)) & \xrightarrow{i_{\mathcal{C}}(f)} & \mathcal{C}(P(f)) \\ \psi_Q \downarrow & & \downarrow \psi_P \\ \mathcal{C}(Q(g)) & \xrightarrow{i_{\mathcal{C}}(g)} & \mathcal{C}(P(g)) \end{array}$$

commute.

Proof: By the definition of combinatorial equivalence, there exist homeomorphisms ψ_0, ψ_1 such that the diagram given in the definition of combinatorial equivalence commutes. Since $\psi_0 : (S^2, P(f)) \to (S^2, P(g))$ is a homeomorphism, it induces a bijection from the set $\mathcal{C}(P(f))$ to $\mathcal{C}(P(g))$ which preserves the intersection numbers and the property of being essential, inessential, or peripheral. The map $\psi_1 : (S^2, Q(f)) \to (S^2, Q(g))$ also gives an isomorphism from $\mathcal{C}(Q(f))$ to $\mathcal{C}(Q(g))$ which preserves these properties since it is also a homeomorphism. Since $\psi_0 \circ f = g \circ \psi_1$, degrees are also preserved, as are the relations.

■

Definition 4.7 (Pullback relation on simple closed curves) *The composition $i_{\mathcal{C}} \circ f_{\mathcal{C}}^{-1}$ will be called the* **pullback relation on simple closed curves** *and will be denoted by f^*.*

Note that the pullback relation is $1 \to k$, where $k \leq d$ and d is the degree of f. Also, by the above proposition, the pullback relation is an invariant of the combinatorial class $[f]$ of branched covering.

Recall that the postcritical set of f and $f^{\circ n}$ is the same for all n. The next proposition says that the pullback relation is natural with respect to iteration.

Proposition 4.8 (Naturality of pullback) *The pullback relation commutes with iteration. That is,*

$$(f^{\circ n})^* = (f^*)^{\circ n}.$$

Proof: For any set $K \subset S^2$, the preimage $(f^{\circ n})^{-1}(K)$ can be thought of also as $(f^{-1})^{\circ n}(K)$, i.e. as taking a sequence of successive preimages. Given representatives of curve classes, we see that equality holds on the level of representatives. Since the relations are independent of representative, the proposition holds.

∎

Notation. We denote by

- $\mathcal{C}_{per}(P(f))$ the set of all *essential* classes of simple closed curves in $\mathcal{C}(P(f))$ which are periodic under the pullback relation; and
- $\mathcal{C}_{per}(Q(f))$ the set of all *essential* classes in the image $f_{\mathcal{C}}^{-1}(\, \mathcal{C}_{per}(P(f))\,)$ which map to elements of $\mathcal{C}_{per}(P(f))$ under the map $i_{\mathcal{C}}$.

We will need to single out some special kinds of periodic classes of simple closed curves.

Definition 4.9 (Levy orbit) *A* **Levy orbit** *is a sequence $\{\gamma_i\}_{i=0}^{\infty}$ of simple closed curves such that each γ_i has a lift γ'_{i+1} which is isotopic to γ_{i+1} and which maps to γ_i by degree one. A Levy orbit is called* **simple** *if for each i, there is a unique such lift γ'_{i+1}. A Levy orbit is called* **nonperipheral** *if the elements are nonperipheral. It is said to be* **periodic** *if the γ_i cycle up to isotopy. A Levy orbit whose elements are disjoint and nonperipheral and which cycle under f^* is called a* **Levy cycle**. *A Levy cycle whose elements bound disjoint discs which cycle, up to isotopy, is called a* **degenerate Levy cycle**.

In terms of the relations, a Levy orbit is an orbit $\{[\gamma_i]\}_{i=0}^{\infty}$ of essential (possibly peripheral) curve classes with the following property: for all i, there exists an element $[\gamma'_{i+1}] \in \mathcal{C}_{per}(Q(f))$ such that $[\gamma'_{i+1}] \in (f_{\mathcal{C}}^{-1}([\gamma_i]) \cap (i_{\mathcal{C}})^{-1}([\gamma_{i+1}])$ and $\deg(f : [\gamma'_{i+1}] \to [\gamma_i]) = 1$. A Levy orbit is simple if $|(f_{\mathcal{C}}^{-1})([\gamma_i]) \cap (i_{\mathcal{C}})^{-1}([\gamma_{i+1}])| = 1$.

Lemma 4.10 . *Let f be a PF branched covering of hyperbolic type. Then f has no degenerate Levy cycles.*

Proof: If g is any branched covering of a closed disc D_1 onto another closed disc D_2, then $\deg(g) = \deg(g|_{\partial D_1})$. Hence if f had a degenerate Levy cycle with permuted discs $\{D_i\}_{i=1}^{n}$, the degree of f on the interiors of the D_i must be one. Hence $D_i \cap C(f) = \emptyset$ for all i. Since the D_i are also forward-invariant up to isotopy, and f is of hyperbolic type, this implies that $D_i \cap P(f) = \emptyset$, a contradiction.

∎

4.5 Pushforward relation on arcs

In this section we define the pushforward relation on the set of classes of arcs.

The inclusion map induces a one-to-infinite relation $i_{\mathcal{A}}$ from $\mathcal{A}(P(f))$ to $\mathcal{A}(Q(f))$ and a *function* $f_{\mathcal{A}} : \mathcal{A}(Q(f)) \to \mathcal{A}(P(f))$. The composition $f_{\mathcal{A}} \circ i_{\mathcal{A}}$ will be called the *pushforward relation on arcs* and will be denoted by f_*.

We consider pushforwards, rather than pullbacks, for two reasons. The most significant one is that we will later use the pushforward relation to encode the topological dynamics of points in the Julia set which lie in the intersections of the closures of periodic Fatou components. Also, the pullback and pushforward relations induce linear transformations which are naturally adjoint.

The relation $i_{\mathcal{A}}$. The inclusion map $i : S^2 - Q(f) \hookrightarrow S^2 - P(f)$ induces a one-to-many (in general one-to-infinite) relation from $\mathcal{A}(P(f))$ to $\mathcal{A}(Q(f))$ defined by setting

$$i_{\mathcal{A}}([\alpha]) = \{ [\beta] \in \mathcal{A}(Q) \mid [i \circ \beta] = [\alpha], \text{ some } \beta \in [\beta] \} .$$

By construction, this is well-defined.

The map $f_{\mathcal{A}}$. The branched covering f induces a relation $f_{\mathcal{A}}$ from $\mathcal{A}(Q)$ to $\mathcal{A}(P)$ by setting

$$[\beta] \in f_{\mathcal{A}}([\alpha])$$

if there exist representatives $\alpha \in [\alpha], \beta \in [\beta]$ such that α is a lift of β. Again, by construction, this is well-defined on the level of classes.

Proposition 4.11 ($f_{\mathcal{A}}$ is a function) *The relation $f_{\mathcal{A}}$ is a function from $\mathcal{A}(Q(f))$ to $\mathcal{A}(P(f))$.*

Proof: Let α and α' be two arcs which represent the same class in $\mathcal{A}(Q)$. We must show that if the images of α and α' under f are arcs, then they are isotopic. By postcomposing an isotopy between α and α' with the map f, we obtain a homotopy between the images of α and α'. But homotopy and isotopy yield the same equivalence relation on the set of arcs.

■

Definition 4.12 (Multiplicity) *The* **multiplicity** *of f_* from an essential class $[\alpha_0]$ to $[\alpha_1] \in f_*([\alpha_0])$ is the nonnegative integer defined by*

$$\mathit{mult}(f_* : [\alpha_0] \to [\alpha_1]) = \left| \, i_{\mathcal{A}}([\alpha_0]) \cap (f_{\mathcal{A}})^{-1}([\alpha_1]) \, \right| .$$

The multiplicity is equal to the number of components of $f^{-1}(\alpha_1)$ which are isotopic to α_0 after including into $S^2 - P(f)$. The multiplicity is always bounded by the degree of f. Unlike degree, the multiplicity is a number which *is* encoded by the data of the relations $f_{\mathcal{A}}$ and $i_{\mathcal{A}}$.

Example: Let $f(z) = z^2 - 1$. Then $P(f) = \{\infty, 0, -1\}$ and $Q(f) = \{\infty, 0, -1, 1\}$. Let $\alpha_1 = [\infty, -1] \subset \mathbb{R} \cup \{\infty\}$. Then α_1 has two lifts α_0, α_0' which are arcs joining 0 and ∞. These arcs are nonparallel in $S^2 - Q(f)$, but their images under the inclusion map $i : S^2 - Q(f) \hookrightarrow S^2 - P(f)$ are parallel. Thus the the multiplicity of f_* from $[\alpha_0] = [\alpha_0']$ to $[\alpha_1]$ is equal to two.

Invariance properties. Let ψ_1 and ψ_0 give a combinatorial equivalence from f to g. Then ψ_0 induces a bijection $\psi_P : \mathcal{A}(P(f)) \to \mathcal{A}(P(f))$ and ψ_1 a bijection $\psi_Q : \mathcal{A}(Q(f)) \to \mathcal{A}(Q(g))$. Since ψ_0, ψ_1 form a combinatorial equivalence, $\psi_P \circ f_{\mathcal{A}} = f_{\mathcal{A}} \circ \psi_Q$ and $\psi_Q \circ i_{\mathcal{A}} = \imath_{\mathcal{A}} \circ \psi_P$. Hence the relations $i_{\mathcal{A}}$ and $f_{\mathcal{A}}$ are invariants of the combinatorial class of f.

Definition 4.13 (Pushforward relation on arcs) *The composition $f_{\mathcal{A}} \circ i_{\mathcal{A}}$ will be called the* **pushforward relation on arcs** *and will be denoted by f_*.*

By the above remark, the pushforward relation on arcs is an invariant of the combinatorial class of f. Note that the image of a class of arc under the pushforward relation may be empty, since the image under f of a complicated arc is often non-embedded.

Proposition 4.14 (Naturality of pushforward relation) *The pushforward relation on arcs is natural with respect to iteration, i.e. $(f_*)^{\circ n} = (f^{\circ n})_*$.*

The proof is analogous to the case for simple closed curves.
Notation. We will denote by

- $\mathcal{A}_{per}(P(f))$ the set of all essential classes of simple closed curves in $\mathcal{A}(P(f))$ which are periodic under the pushforward relation;
- $\mathcal{A}_{per}(Q(f))$ the set of all essential classes in the image $i_{\mathcal{A}}(\mathcal{A}_{per}(P(f)))$ which map to elements of $\mathcal{A}_{per}(P(f))$ under $f_{\mathcal{A}}$;
- $\mathcal{A}_{evp}(P(f))$, the set of all essential classes in $\mathcal{A}(P(f))$ which are eventually periodic under the pushforward orbit;
- $\mathcal{A}_{evp}(Q(f))$, the set of all essential classes in the image $i_{\mathcal{A}}(\mathcal{A}_{evp}(P(f)))$ which map to elements of $\mathcal{A}_{evp}(P(f))$ under $f_{\mathcal{A}}$.

The set $\mathcal{A}_{per}(P(f))$ may be empty. In Section 5.6 we give an explicit example of a PF hyperbolic rational map for which $\mathcal{A}_{per}(P(f))$ is empty.

4.6 Intersection numbers

In this section, we show that elements of $\mathcal{A}_{per}(P(f))$ and $\mathcal{C}_{per}(P(f))$ which have nonzero intersection number have restricted combinatorial dynamics under the pullback and pushforward relations.

The following proposition can be found in [ST1] and is the key ingredient in the proofs of the theorems in the next two subsections.

Proposition 4.15 (Degree restricts intersections) *Let $K, L \subset S^2$ be one-dimensional submanifolds, possibly with boundary, such that $|K \cap L| < \infty$. Suppose $L' \subset S^2$, and suppose that $f|_{L'} : L' \to L$ is a degree k covering map. Then*

$$|(f^{-1}K) \cap L'| = k \cdot |K \cap L|.$$

Proof: Each point in $L \cap K$ has exactly k preimages under f which lie in $L' \cap f^{-1}K$.

∎

4.6.1 Simple closed curves intersecting arcs

In this subsection, we show how the dynamics of a periodic curve class is restricted if it intersects a periodic arc class. The main result is Theorem 4.19.

In the next lemma, note that the first two parts of the conclusion involve representatives, while the last two involve isotopy classes.

Lemma 4.16 (Curve-arc lemma) *Suppose*

1. $[\alpha_0], [\alpha_1] \in \mathcal{A}_{per}(P(f))$ *with* $[\alpha_1] \in f_*[\alpha_0]$*;*

2. $[\gamma_0], [\gamma_1] \in \mathcal{C}_{per}(P(f))$ *with* $[\gamma_0] \in f^*[\gamma_1]$*;*

3. $[\gamma_1] \cdot [\alpha_1] \neq 0$*; and*

4. $\alpha_0, \alpha_1, \gamma_0,$ *and* γ_1 *are chosen such that* $f(\alpha_0) = \alpha_1, f(\gamma_0) = \gamma_1,$ *and* $|\alpha_1 \cap \gamma_1| = [\alpha_1] \cdot [\gamma_1]$.

Then

1. $|\gamma_0 \cap \alpha_0| = |\gamma_1 \cap \alpha_1|$*;*

2. γ_0 *is the unique* **component** *of* $f^{-1}\gamma_1$ *which intersects* α_0*;*

3. $[\gamma_0] \cdot [\alpha_0] = [\gamma_1] \cdot [\alpha_1]$*;*

4. $[\gamma_0]$ *is the unique* **class** *in* $f^*([\gamma_1])$ *which has nonzero intersection number with* $[\alpha_0]$.

Proof:

1. By the previous proposition, *Degree restricts intersections*, $|\gamma_0 \cap \alpha_0| \leq 1 \cdot |\gamma_1 \cap \alpha_1|$, since $f|_{\alpha_0} : \alpha_0 \to \alpha_1$ is a degree one covering map. Suppose $|\gamma_0 \cap \alpha_0| < |\gamma_1 \cap \alpha_1|$. If the periods of the given cycles containing $[\alpha_1]$ and $[\gamma_1]$ are p and q, respectively, then by pulling back along the cycles we can find lifts $\widetilde{\gamma_0}$ of γ_0 under $f^{-(p \cdot q - 1)}$ and $\widetilde{\alpha_0}$ of α_0 under $f^{-(p \cdot q-1)}$ for which $\widetilde{\gamma_0} \in [\gamma_1]$ and $\widetilde{\alpha_0} \in [\alpha_1]$. By the previous proposition, the number of points of intersection between pairs of lifts cannot increase. Hence at the final stage, we have $|\widetilde{\gamma_0} \cap \widetilde{\alpha_0}| < |\gamma_1 \cap \alpha_1|$, contradicting our hypothesis that γ_1 and α_1 minimized the number of intersections.

2. This follows immediately from the previous result, and the proposition *Degree restricts intersections*.

3. This will follow from an argument analogous to that given to prove the first statement. By the first step, $[\gamma_0] \cdot [\alpha_0] \leq [\gamma_1] \cdot [\alpha_1]$, since we have found representatives which intersect in a set of size $[\gamma_1] \cdot [\alpha_1]$. But if $[\gamma_0] \cdot [\alpha_0] < [\gamma_1] \cdot [\alpha_1]$, we can find representatives of these classes realizing this intersection number. Pulling back these representatives, the previous proposition, *Degree restricts intersections*, shows that we eventually obtain representatives for $[\gamma_1]$ and $[\alpha_1]$ which intersect in fewer than $[\gamma_1] \cdot [\alpha_1]$ points, a contradiction.

4. This follows immediately from the second step and the definition of intersection number between classes.

∎

The next proposition relates degrees and multiplicities of elements in cycles.

Proposition 4.17 *Suppose* α_0, α_1 *represent periodic classes with mult*$(f_* : [\alpha_0] \to [\alpha_1]) = m$*. Let* γ_0 *and* γ_1 *represent periodic curve classes with* $[\gamma_0] \in f^*([\gamma_1])$*. If* $[\gamma_0] \cdot [\alpha_0] \neq 0$*, then for any lift* $\gamma_0' \subset (S^2, Q(f))$ *of* γ_1 *which is isotopic to* γ_0 *under the erasing map,* $\deg([\gamma_0'] \to [\gamma_1]) \geq m$*.*

Proof: Since γ_0 and α_0 intersect, so do γ_1 and α_1. We may assume that α_1 and γ_1 are chosen so as to minimize the number of intersections. Then by Lemma 4.16, *Curve-arc lemma*, γ_0' is the unique component of the lift of γ_0 which intersects α_0. Let $\{\alpha_{01}, \alpha_{02}, ..., \alpha_{0m}\}$ be the set of periodic lifts of α_1 in $(S^2, Q(f))$ which are isotopic to α_0 after including into $(S^2, P(f))$. Then by Lemma 4.16, for all i,

$$|\gamma_0' \cap \alpha_{0i}| = |\gamma_1 \cap \alpha_0|.$$

Hence

$$\begin{aligned} m \cdot |\gamma_1 \cap \alpha_1| &= m \cdot |\gamma_0' \cap \alpha_{0i}| \\ &= |\gamma_0' \cap (\cup_i \alpha_{0i})| \\ &\leq |\gamma_0' \cap f^{-1}(\alpha_1)| \\ &= k \cdot |\gamma_1 \cap \alpha_1|, \end{aligned}$$

where $k = \deg(f : [\gamma_0'] \to [\gamma_0])$, and the last equality holds by Proposition 4.15.

∎

The next lemma says that if we consider only periodic classes, $[\gamma_1] \cdot [\alpha_1] = [\gamma_1] \cdot [\alpha_1']$, where $[\alpha_1], [\alpha_1'] \in f_*([\alpha_0])$ are two elements in the pushforward of some arc class. We will use this in Section 4.9.

Lemma 4.18 (Pushforwards intersect in same number) *Let* $[\alpha_1], [\alpha_1'], \in \mathcal{A}_{per}(P(f))$ *with* $[\alpha_1], [\alpha_1'] \in f_*[\alpha_0]$*. Suppose* $[\gamma_0], [\gamma_1] \in \mathcal{C}_{per}(P(f))$ *with* $[\gamma_0] \in f^*[\gamma_1]$*. Then* $[\gamma_0] \cdot [\alpha_0] = [\gamma_1] \cdot [\alpha_1] = [\gamma_1] \cdot [\alpha_1']$*.*

Proof:
Case 1: $[\alpha_0] \cdot [\gamma_0] = 0$. In this case, since the classes are periodic, we may pull back nonintersecting representatives of $[\alpha_0]$ and $[\gamma_0]$ by a suitable iterate to obtain representatives of $[\alpha_1], [\alpha_1']$, and $[\gamma_1]$ which do not intersect.
Case 2: $[\alpha_0] \cdot [\gamma_0] \neq 0$. In this case, both $[\alpha_1]$ and $[\alpha_1']$ satisfy the hypotheses of Lemma 4.16, *Curve-arc lemma.* So $[\gamma_1] \cdot [\alpha_1] = [\gamma_0] \cdot [\alpha_0] = [\gamma_1] \cdot [\alpha_1']$.

■

The next theorem says that an irreducible set of curves which intersects a cycle of arcs is actually a special kind of cycle: every curve in the cycle has a unique periodic lift. Moreover, all non-periodic lifts avoid the cycle of arcs. The cycle of arcs, then, controls the dynamics of the set of curves. The main reason for this is that the arcs map by degree one, which restricts the possible intersections of lifts of curves with lifts of arcs.

Theorem 4.19 (Curve intersecting arc) *Let $[\Gamma] \subset \mathcal{C}_{per}(P(f))$ be any finite set which is irreducible for f^*. If some element of $[\Gamma]$ has nonzero intersection with an element of a cycle $[\Sigma] \subset \mathcal{A}_{per}(P(f))$, then*

1. *the restriction of the relation $f_{\mathcal{C}}^{-1}$ to $[\Gamma] \times \mathcal{C}_{per}(Q(f))$ is an injective function;*
2. *the restriction of the relation f^* to $[\Gamma] \times \mathcal{C}_{per}(P(f))$ is an injective function;*
3. *the elements of $[\Gamma]$ form a cycle $\{[\gamma_i]\}_{i=0}^{p-1}$ with $[\gamma_{i+1 \bmod p}] = f^*([\gamma_i]) \cap \mathcal{C}_{per}(P(f))$;*
4. *$[\Gamma]$ is the unique cycle under the relation f^* containing any of its elements, and*
5. *Given any element $[\delta]$ in the set*

$$(\bigcup_{n>0} (f^*)^{\circ n}[\Gamma]) - [\Gamma],$$

the intersection number of $[\delta]$ with any element of $[\Sigma]$ is zero.

Proof: Let $[\gamma_1] \in [\Gamma]$ be an element which intersects an element $[\alpha_1] \in \mathcal{A}_{per}(P(f))$. Choose a cycle $\{[\alpha_k]\}_{k=0}^{q-1}$ containing $[\alpha_1]$. By Lemma 4.16, there is a unique component of the preimage of any representative of $[\gamma_1]$ which intersects a representative in $[\alpha_0]$. Hence $|f_{\mathcal{C}}^{-1}[\gamma_1]| = 1$. Denote the class in $\mathcal{C}(P(f))$ of this component by $[\gamma_0]$. Then $f^*([\gamma_1]) = [\gamma_0]$, $[\gamma_0] \in [\Gamma]$, and $[\gamma_0] \cdot [\alpha_0] \neq 0$. Repeating the argument with $[\alpha_0]$ and $[\gamma_0]$ shows that the relations defined in the first two conclusions are functions. The injectivity of the relation $f_{\mathcal{C}}^{-1}|_{[\Gamma]\times\mathcal{C}_{per}(Q(f))}$ follows from injectivity of $f_{\mathcal{C}}^{-1}$. Since $f_*|_{[\Gamma]\times\mathcal{C}_{per}(P(f))}$ is a function, and since $[\Gamma]$ is irreducible, $[\Gamma]$ must be a cycle. This also proves the injectivity statement in the second conclusion. The fourth statement follows immediately from the fact that f_* restricted to $[\Gamma] \times \mathcal{C}_{per}(P(f))$ is injective.

We now prove the last statement by contradiction. Let $[\gamma_0], [\gamma_1] \in [\Gamma]$ with $[\gamma_0] \in f^*([\gamma_1])$. Suppose that some element $[\delta] \in f^*([\gamma_1]) - [\Gamma]$ has nonzero intersection number with an element $[\alpha_0] \in [\Sigma]$. Since $[\delta] \in f^*([\gamma_1])$ intersects $[\alpha_0] \in [\Sigma]$, $[\gamma_1]$ must have nonzero intersection number with an element $[\alpha_1] \in f_*([\alpha_0]) \cap [\Sigma]$. For otherwise $[\delta]$ and $[\alpha_0]$ would be representable by elements which are the lifts of disjoint elements, and therefore $[\delta]$ and $[\alpha_0]$ would also be representable by disjoint elements, contradicting our hypothesis. By Lemma 4.16, *Curve-arc lemma*, $[\gamma_0]$ is the *unique* class in $f^*([\gamma_1])$ which intersects $[\alpha_0]$, a contradiction.

■

4.6.2 Arcs intersecting arcs

The results of this section will not be used in the sequel; we have included it for completeness. In this subsection, we show how the dynamics of a periodic arc class is restricted if it intersects another periodic arc class. The main result is Theorem 4.21.

The next lemma is the analog of Lemma 4.16, *Curve-arc lemma*, in the previous section. Its proof is completely analogous.

Lemma 4.20 (Arc-arc lemma) *Suppose*

1. $[\alpha_0], [\alpha_1] \in \mathcal{A}_{per}(P(f))$ *with* $[\alpha_1] \in f_*[\alpha_0]$;
2. $[\beta_0], [\beta_1] \in \mathcal{A}_{per}(P(f))$ *with* $[\beta_1] \in f_*[\beta_0]$;
3. $[\alpha_1] \cdot [\beta_1] \neq 0$; *and*
4. $\alpha_1 \in [\alpha_1], \beta_1 \in [\beta_1], \alpha_0 \in [\alpha_0]$, *and* $\beta_0 \in [\beta_0]$ *are chosen such that* $f(\alpha_0) = \alpha_1, f(\beta_0) = \beta_1$, *and* $|\alpha_1 \cap \beta_1| = [\alpha_1] \cdot [\beta_1]$.

Then

1. $|\alpha_0 \cap \beta_0| = |\alpha_1 \cap \beta_1|$;
2. β_0 *is the unique* **component** *of* $f^{-1}\beta_1$ *which intersects* α_0;
3. $[\alpha_0] \cdot [\beta_0] = [\alpha_1] \cdot [\beta_1]$;
4. $[\alpha_0]$ *is the unique* **class** *in* $(f_*)^{-1}([\beta_1])$ *which has nonzero intersection number with* $[\alpha_0]$.

We also obtain a similar theorem whose proof is also the same as that of Theorem 4.19, *Curve intersecting arc*, suitably modified.

Theorem 4.21 (Arc intersecting arc) *Let* $[\Sigma] \subset \mathcal{A}_{per}(P(f))$ *be any finite set which is irreducible for* f_*. *If some element of* $[\Sigma]$ *has nonzero intersection with an element of a cycle* $[\Theta] \subset \mathcal{A}_{per}(P(f))$, *then*

1. *the function* $f_{\mathcal{A}}$ *restricted to* $i_{\mathcal{A}}([\Sigma]) \cap \mathcal{A}_{per}(Q(f))$ *is an injective function;*
2. *the restriction of the relation* f_* *to* $[\Sigma] \times \mathcal{A}_{per}(P(f))$ *is an injective function;*
3. *the elements of* $[\Sigma]$ *form a cycle* $\{[\sigma_i]\}_{i=0}^{p-1}$ *with* $[\sigma_{i+1 \bmod p}] = f^*([\sigma_i]) \cap \mathcal{A}_{per}(P(f))$;

4. $[\Sigma]$ *is the unique cycle under* f_* *containing its elements, and*

5. *Given any element* $[\delta]$ *in the set*

$$(\bigcup_{n>0}(f_*)^{-n}[\Sigma]) - [\Sigma],$$

the intersection number of $[\delta]$ *with* $[\Theta]$ *is zero.*

As a consequence, we have

Theorem 4.22 *Let* $[\alpha_0], [\alpha_1] \in \mathcal{A}_{per}(P(f))$ *with* $[\alpha_1] \in f_*([\alpha_0])$, *and suppose either*

1. $|f_*([\alpha_0]) \cap \mathcal{A}_{per}(P(f))| > 1$, *or*

2. *mult*$(f_* : [\alpha_0] \to [\alpha_1]) > 1$.

Then the elements of any cycle containing $[\alpha_0]$ *or* $[\alpha_1]$ *intersect no other elements of* $\mathcal{A}_{per}(P(f))$.

Proof: The first statement follows from the second part of the conclusion of Theorem 4.21. The second follows from the first part of the conclusion of Theorem 4.21.

■

4.6.3 Arcs intersecting Levy cycles

We will need the results of this section in the sequel.

The proof of Theorems 4.19 and 4.21 rested on the fact that arcs map by degree one. Using this fact, we were able to conclude that arcs or simple closed curves which intersected periodic arcs have restricted dynamics under the relations. We can therefore prove in the same manner a similar theorem for the case of a periodic Levy orbit which intersects a cycle of arcs.

Theorem 4.23 *Suppose an element of a periodic Levy orbit* $[\Gamma] \subset \mathcal{C}_{per}(P(f))$ *and an element of a cycle* $[\Sigma] \subset \mathcal{A}_{per}(P(f))$ *have nonzero intersection. Then*

1. *the relation* $f_{\mathcal{C}}^{-1}$ *restricted to* $[\Gamma] \times \mathcal{C}_{per}(Q(f))$ *is an injective function, (and so* $[\Gamma]$ *is a simple Levy orbit), and the restriction of the relation* f^* *to* $[\Gamma] \times \mathcal{C}_{per}(P(f))$ *is an injective function;*

2. *the relation* $f_{\mathcal{A}}$ *restricted to* $i_{\mathcal{A}}([\Sigma]) \times \mathcal{A}_{per}(P(f))$ *is an injective function, (and so mult*$(f_* : [\alpha_0] \to [\alpha_1]) = 1$ *for all* $[\alpha_0], [\alpha_1] \in [\Sigma]$*) , and the restriction of the relation* f_* *to* $[\Sigma] \times \mathcal{A}_{per}(P(f))$ *is a function;*

3. $[\Gamma]$ *and* $[\Sigma]$ *are the unique cycles under* f^* *and* f_**, respectively, containing any of their elements; and*

4. *No element* $[\delta]$ *of*

$$(\bigcup_{n>0}(f^*)^{\circ n}([\Gamma])\,) - [\Gamma]$$

intersects an element of $[\Sigma]$*, and no element* $[\eta]$ *of*

$$(\bigcup_{n>0}(f_*^{-1})^{\circ n}([\Sigma])\,) - [\Sigma]$$

intersects an element of $[\Gamma]$.

4.7 Thurston's theorem

In this section we state Thurston's theorem characterizing PF rational maps as branched coverings of the sphere to itself. It is formulated in terms of the eigenvalues of linear transformations associated to multicurves with certain invariance properties.

The Perron-Frobenius theorem. We first recall some facts about square matrices with nonnegative entries; see e.g. [Gan]. A matrix with nonnegative entries is called *nonnegative.* A nonnegative square matrix (A_{ij}) is called *irreducible* if given any pair (i, j), there is an integer n such that $(A^n)_{ij} > 0$.

Theorem 4.24 (Perron-Frobenius theorem) *Let A be a nonnegative square matrix. Then*

- *there exists a positive real eigenvalue* $\lambda(A)$ *equal to the spectral radius of* A*;*
- *if in addition* A *is irreducible, there exists an eigenvector* $v(A)$ *for* $\lambda(A)$ *such that the entries of* $v(A)$ *are all strictly greater than zero.*

The eigenvalue $\lambda(A)$ is usually called the *leading eigenvalue* of A. If A and B are two nonnegative square matrices of the same dimension, and if $A_{ij} \le B_{ij}$, then $\lambda(A) \le \lambda(B)$.

Thurston linear transformation associated to a curve system. Let Γ be a curve system. The *Thurston linear transformation* is the linear map

$$A(\Gamma, f) : \mathbb{R}^{|\Gamma|} \to \mathbb{R}^{|\Gamma|}$$

given by the matrix

$$(A(\Gamma, f))_{\delta\gamma} = \sum_{\substack{\eta \in f^{-1}\gamma \\ \eta \simeq \delta}} \frac{1}{\deg(f : \eta \to \gamma)}$$

where the sum is taken over all *components* η of $f^{-1}\gamma$ which are isotopic in $S^2 - P(f)$ to δ. If there are no such elements, the sum is taken to be zero.

The leading eigenvalue of $A(\Gamma, f)$ will be denoted by $\lambda(A(\Gamma, f))$. When the discussion revolves around a fixed f we will drop reference to the map f and write $A(\Gamma)$.

Unweighted linear transformation associated to a curve system. Let Γ be a curve system. The *unweighted Thurston linear transformation* is the linear map

$$T^*(\Gamma, f) : \mathbb{R}^{|\Gamma|} \to \mathbb{R}^{|\Gamma|}$$

given by the matrix

$$(T^*(\Gamma, f))_{\delta\gamma} = \sum_{\substack{\eta \in f^{-1}\gamma \\ \eta \simeq \delta}} 1$$

where the sum is taken over all *components* η of $f^{-1}\gamma$ which are isotopic in $S^2 - P(f)$ to δ. If there are no such elements, the entry is taken to be zero. The leading eigenvalue of $T^*(\Gamma, f)$ will be denoted by $\lambda(T^*(\Gamma, f))$. Again, when the discussion revolves around a fixed map f, we will shorten the notation and denote $T^*(\Gamma, f)$ by $T^*(\Gamma)$.

Properties of $A(\Gamma)$ and $T^*(\Gamma)$. Since the entries of the weighted Thurston transformation are all less than or equal to those of the unweighted transformation, we have that $\lambda(A(\Gamma)) \leq \lambda(T^*(\Gamma))$. The linear transformation associated to a curve system is an invariant of the combinatorial class of f and the classes $[\Gamma]$ in the curve system Γ. Moreover, the data of the *degrees* plus the pullback relation $f_{\mathcal{C}}^{-1}$ and $i_{\mathcal{C}}$ suffice to determine the weighted and unweighted linear transformations associated to any curve system. The linear transformations $T^*(\Gamma)$ and $A(\Gamma)$ are irreducible if and only if the set of classes $[\Gamma]$ is irreducible under the pullback relation f^*. Finally, given a curve system Γ' and the corresponding linear transformations $A(\Gamma')$ and $T^*(\Gamma')$, there is always a subsystem $\Gamma \subset \Gamma'$ for which $[\Gamma]$ is irreducible under f^* and such that $\lambda(A(\Gamma)) = \lambda(A(\Gamma'))$ and $\lambda(T^*(\Gamma)) = \lambda(T^*(\Gamma'))$.

Invariant curve systems. Let f be a PF branched covering. An *f-invariant curve system* is a curve system $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_n\}$ such that for all i, each component of $f^{-1}(\gamma_i)$, after including into $S^2 - P(f)$, is either inessential, peripheral, or isotopic to an element of Γ.

Theorem 4.25 (Thurston) *Let f be a PF branched covering of the sphere to itself. Then f is combinatorially equivalent to a rational map g if and only if for every f-invariant nonperipheral curve system Γ, either*

1. $\lambda(A(\Gamma)) = 1$ *and the orbifold associated to f has signature* $(2, 2, 2, 2)$*, in which case g is covered by an integral endomorphism of the torus, or*
2. $\lambda(A(\Gamma)) < 1$.

In the first case, up to conjugation by elements of $Aut(\widehat{\mathbb{C}})$, there is a real one-parameter family of maps g_t combinatorially equivalent to f. In the second case, the map g is unique up to conjugation by elements of $Aut(\widehat{\mathbb{C}})$.

See [DH2] for the proof. For the definition of the orbifold associated to a rational map, see [DH2], [McM3], or [Thu4].

The first case does not arise for branched covers of hyperbolic type. In the first case of the above theorem, the maps g_t have no periodic critical points. Hence their Julia sets are all equal to the whole sphere. They are not hyperbolic maps.

Definition 4.26 (Thurston obstructions) *Let f is a PF branched covering whose orbifold does not have signature $(2,2,2,2)$. A non-peripheral f-invariant curve system Γ for which $\lambda(A(\Gamma)) \geq 1$ is called a* **Thurston obstruction**. *A non-peripheral irreducible curve system Γ for which $\lambda(A(\Gamma)) \geq 1$ is called a* **reduced Thurston obstruction**.

Any Thurston obstruction contains a reduced Thurston obstruction which is irreducible but may be non-invariant. A Levy cycle is an example of a reduced Thurston obstruction.

We will now reformulate Thurston's theorem by dropping the requirement of nonperipheral and replacing the condition of invariance with irreducibility.

A set of classes of disjoint simple closed curves which includes peripheral elements is irreducible if and only if the set consists only of a cycle of peripheral elements, since the preimage of a peripheral curve is either peripheral or inessential. Furthermore, if $\Gamma = \{\gamma_0, \gamma_1, ..., \gamma_{p-1}\}$ are the elements of a cycle of peripheral elements with γ_i isotopic into $x_i \in P(f)$, and if up to isotopy $f(\gamma_i) = \gamma_{i+1}(\text{mod} p)$, then $\deg f : \gamma_i \to \gamma_{i+1}$ is the local degree of f near x_i.

Theorem 4.27 *Let f be a PF branched covering of the sphere. Then f is combinatorially equivalent to a rational map g if and only if for every irreducible curve system Γ, either*

1. $\lambda(A(\Gamma)) = 1$, *which occurs if and only if either*
   (a) Γ *consists entirely of peripheral elements enclosing a cycle of point in $P(f)$ containing no critical points, or*
   (b) *the orbifold associated to f has signature $(2,2,2,2)$; or*
2. $\lambda(A(\Gamma)) < 1$.

Again, the first case does not arise for maps of hyperbolic type.

4.8 Periodic arcs restrict Thurston obstructions

We now apply the combinatorial analysis of intersecting periodic arcs and simple closed curves to the study of Thurston obstructions.

The following theorem, and the first part of the next theorem, is due to ideas of Tan Shishikura.

Theorem 4.28 *Let Γ be an irreducible set of classes for the pullback relation f^*. Suppose that an element of Γ intersects an element of $\mathcal{A}_{\surd\rceil\nabla}(\mathcal{P}(\{\}))$. Then*

1. $\lambda(T^*(\Gamma)) = 1$;
2. $\lambda(A(\Gamma)) \leq 1$;
3. *if $\lambda(A(\Gamma)) = 1$, then Γ is a simple periodic Levy orbit such that each element has a unique image under $f_{\mathcal{C}}^{-1}$ to an element of $\mathcal{C}_{per}(Q(f))$.*

Proof: By Theorem 4.19, Γ is a cycle under f^*, and each element in the cycle has a *unique* component of its preimage which is periodic. Hence the matrix giving $T^*(\Gamma)$ is a permutation matrix of zeros and ones, and so $\lambda(T^*(\Gamma)) = 1$. The second statement follows immediately from the first since the entries of the weighted Thurston transformation are less than or equal to those of the matrix for the unweighted transformation. Since Γ is a cycle, the leading eigenvalue $\lambda(A(\Gamma))$ is the product of the degrees along the cycle. Hence if $\lambda(A(\Gamma)) = 1$, then these degrees must all be equal to one, and so Γ is a periodic Levy orbit. Theorem 4.23 then applies, proving the third statement.

■

Theorem 4.29 (Shishikura-Tan Theorem) *Let Γ be a reduced Thurston obstruction.*

1. *If $[\Gamma]$ intersects an element $[\alpha_1]$ of $\mathcal{A}_{per}(P(f))$, then*
   (a) *$[\Gamma]$ is a simple nonperipheral Levy cycle;*
   (b) *the elements of $[\Gamma]$ all have zero intersection with the set*
   $$\Big(\bigcup_{n>0} (f_*)^{-n}([\Sigma]) \Big) - [\Sigma].$$
2. *$[\Gamma]$ intersects no element of $\mathcal{A}_{per}(P(f))$ which is contained in more than one cycle,*
3. *$[\Gamma]$ intersects no element of $\mathcal{A}_{per}(P(f))$ which is contained in a cycle containing elements mapping by multiplicity greater than one.*

Proof: Part 1(a) follows from Part 3 of the previous theorem and the fact that a reduced Thurston obstruction, by definition, is nonperipheral. Part 1(b) follows from Part 4 of Theorem 4.23. Part 2 follows from Theorem 4.23, Part 3. The third statement can be proved two ways. One may either use Proposition 4.17 to show that the elements of a cycle in $\mathcal{C}_{per}(P(f))$ which intersect arcs mapping by multiplicity greater than one cannot map by degree one. Alternatively, one may use Theorem 4.23, Part 2.

■

Remark: Not every PF branched covering which is not combinatorially equivalent to a rational map possesses a Levy cycle. In [ST1] an example is given of an obstructed mating of two cubic polynomials which does not possess any Levy cycles. Thus the above theorem represents a significant reduction in the kinds of Thurston obstructions which can exist. That degree two branched coverings which are not combinatorially equivalent to rational maps have Levy cycles was proved in [Lev]. Simpler proofs are given in [Ree2] and [Tan2].

4.9 Linear transformations associated to arcs: duality

In this section we show that certain linear transformations associated to the pushforward and pullback relations are adjoint with respect to the a pairing given by the intersection number.

It is hoped that this result will generalize to show that the weighted Thurston transformation is adjoint to some kind of weighted transformation of arcs. It may be necessary to expand the notion of invariant arcs to include cases of arcs which do not map by degree one to achieve this generalization.

The transformation $T_*(\Sigma)$. Let f be a branched covering. Let $\Sigma \subset \mathcal{A}_{per}(P, f)$ be any finite irreducible collection of arcs. Consider the restriction of the pushforward relation to the subset $\Sigma \times \Sigma$.

We define the *pushforward linear transformation*

$$T_*(\Sigma) : \mathbb{R}^{|\Sigma|} \to \mathbb{R}^{|\Sigma|}$$

as follows. If $\Sigma = \{[\alpha_i]\}_{i=1}^n$, then image of the basis vector $[\alpha_i]$ is defined as

$$T_*(\Sigma)([\alpha_i]) = \frac{1}{|f_*([\alpha_i]) \cap \Sigma|} \cdot \sum_{[\beta] \in f_*([\alpha_i]) \cap \Sigma} [\beta].$$

That is, we send $[\alpha_i]$ to the formal sum of the elements in the intersection of its image under the pushforward relation with $[\Sigma]$, scaled by the reciprocal of the number of such elements.

Note that the matrix associated to $T_*(\Sigma)$ has the property that the sum of the entries in any column is equal to one, by the irreducibility of $[\Sigma]$ and the definition. It then follows that the leading eigenvalue $\lambda(T_*(\Sigma))$ is equal to one.

Intersection form. The geometric intersection number between classes of arcs and simple closed curves may be extended bilinearly to give a bilinear form

$$\langle \cdot, \cdot \rangle : \mathbb{R}^{\Gamma} \times \mathbb{R}^{\Sigma} \to \mathbb{R}$$

such that $\langle [\gamma], [\alpha] \rangle = [\gamma] \cdot [\alpha]$ for all $[\gamma] \in \Gamma, [\alpha] \in \Sigma$.

Theorem 4.30 (Duality) *Let f be a PF branched covering. Let Σ be a finite irreducible set of arc classes and Γ be any irreducible curve system. Let $T^*(\Gamma)$*

denote the unweighted Thurston linear transformation associated to Γ*, and let* $T_*(\Sigma)$ *denote the pushforward linear transformation of* Σ*. Then* $T^*(\Gamma)$ *and* $T_*(\Sigma)$ *are adjoint with respect to the intersection pairing. That is, for any* $v \in \mathbb{R}^\Sigma, w \in \mathbb{R}^\Gamma$*,*

$$\langle T_*(\Sigma)v, w\rangle \ = \ \langle v, T^*(\Gamma)w\rangle.$$

Proof: It suffices to prove the theorem on basis vectors. Let $[\gamma_1] \in \Gamma, [\alpha] \in [\Sigma]$. Suppose $f_*([\alpha]) \cap \Sigma = \{[\beta_i\}_{i=1}^m$. Then

$$\langle T_*(\Sigma)[\alpha], [\gamma_1]\rangle = 1/m \cdot \langle \sum_i [\beta_i], [\gamma_1]\rangle = 1/m \cdot \sum_i [\beta_i] \cdot [\gamma_1]$$

by the definition of the transformation $T_*(\Sigma)$ and the definition of the pairing. By Lemma 4.18, since $[\gamma_1]$ and the $[\beta_i]$ are periodic, $[\beta_i] \cdot [\gamma_1]$ is the same for any element $[\beta_i] \in f_*[\alpha]$. Hence

$$1/m\langle \sum_i [\beta_i], [\gamma_1]\rangle = \langle [\beta_1], [\gamma_1]\rangle$$

for any element $[\beta_1] \in f_*([\alpha]) \cap \Sigma$. Pulling back, we apply Lemma 4.16, *Curve-arc lemma*, to conclude that

$$\langle [\beta_1], [\gamma_1]\rangle = \langle [\alpha], [\gamma_0]\rangle$$

where $[\gamma_0]$ is the *unique* class in $f^*([\gamma_1])$ which has nonzero intersection with $[\alpha]$. By this uniqueness, we have

$$\langle T_*(\Sigma)([\alpha]), [\gamma_1]\rangle = \langle [\alpha], [f]^*[\gamma_1]\rangle = \langle [\alpha], T^*(\Gamma)([\gamma_1])\rangle$$

and so the theorem holds.

■

4.10 Behavior of pullback and pushforward relations under quotients and tunings

In this section, we study how the pullback and pushforward relations change under tuning and collapsing.

The next theorem is a direct consequence of the definition of combinatorial quotient map and the definition of the relations introduced in Section 4.4. The reason is that a quotient map is degree one outside of its support, and so off its support, f looks like g, up to isotopy.

Theorem 4.31 (Quotients and simple closed curves) *Let* $\phi : f \to g$ *be a combinatorial class of quotient map. Then* ϕ *induces injections*

$$\phi_P^{-1} : \mathcal{C}(P(g)) \to \mathcal{C}(\mathcal{P}(\{))$$

and

$$\phi_Q^{-1} : \mathcal{C}(Q(g)) \to \mathcal{C}(Q(f))$$

such that

1. *ϕ_P^{-1} and ϕ_Q^{-1} map $\mathcal{C}(P(g))$ and $\mathcal{C}(Q(g))$ surjectively onto the sets of simple closed curves in $\mathcal{C}(P(f))$ and $\mathcal{C}(Q(f))$, respectively, which do not intersect the support of the ϕ;*

2. *$f_{\mathcal{C}}^{-1} \circ \phi_P^{-1} = \phi_Q^{-1} \circ g_{\mathcal{C}}^{-1}$;*

3. *$i_{\mathcal{C},f} \circ \phi_Q^{-1} = \phi_Q^{-1} \circ i_{\mathcal{C},g}$;*

4. *ϕ_P^{-1} and ϕ_Q^{-1} preserve degrees;*

5. *ϕ_P^{-1} and ϕ_Q^{-1} preserve the properties of being essential, inessential, and parallel, but not the necessarily the property of being peripheral.*

The following two theorems are due to McMullen [McM3]. For completeness, we include the proofs.

Theorem 4.32 (Quotients of rational maps are of hyperbolic type) *Let $\phi : f \to g$ be a proper combinatorial quotient map. If f is a rational map, then the mapping schema of ϕ is of hyperbolic type.*

Proof: Recall that the points of $P(g)$ which are blown-up under ϕ are forward-invariant under g. Hence if the theorem fails, there is a set of peripheral simple closed curves which cycle by local degree one under g. These simple closed curves form a peripheral Levy cycle for g. Since some point in this cycle is blown up under ϕ^{-1} to at least two points of $P(f)$, these curves form a nonperipheral Levy cycle for f, which is impossible since f is rational.

■

Theorem 4.33 (Rational quotients) *Let $f(z)$ be a PF rational map. If $\phi : f \to g$ is a quotient map, then g is combinatorially equivalent to a rational map.*

Proof: By the previous theorem, g contains a cycle of postcritical points containing a critical point. Hence the orbifold associated to g does not have signature $(2,2,2,2)$. This also implies that the orbifold of f is not the $(2,2,2,2)$ orbifold. Therefore g and f are combinatorially equivalent to rational maps if and only if they has no Thurston obstructions. By Theorem 4.31, a Thurston obstruction for g maps under ϕ^{-1} to a Thurston obstruction for f.

■

Remark: A stronger version of this theorem is proved in [McM3], which says that a PF quotient of any (not necessarily PF) rational map is combinatorially equivalent to a rational map.

The definition of combinatorial equivalence of families of branched coverings covering a mapping scheme is given in the next section. The next theorem, together with Theorem 4.33, completes the picture for decompositions of postcritically finite hyperbolic rational maps.

Theorem 4.34 *Let R be a PFH rational map and $\phi : R \to f$ be a combinatorial quotient map. Then the induced family of topological polynomials $\mathcal{P}$ is combinatorially equivalent to a family of polynomials covering the mapping scheme of ϕ.*

Corollary 4.35 *If a postcritically finite hyperbolic rational map R admits a quotient ϕ to a postcritically finite hyperbolic rational map f, then R is the tuning of f by a finite family of polynomials $\mathcal{P}$ covering the mapping scheme of ϕ.*

Proof: By the definition of induced polynomials, R is combinatorially equivalent to $f * \mathcal{P}$. Hence $\mathcal{P}$ is combinatorially equivalent to a family of maps which is obtained by first restricting the conformal map R to a set $\mathcal{D}$ of invariant discs, and then collapsing the boundary components of these discs to points.

■

The next theorem is a version of Theorem 4.31 for arcs. Its proof is also a direct consequence of the definitions of quotient map and the relations defined in Section 4.5.

Let $\phi : f \to g$ be a combinatorial class of quotient map. Let $G \subset P(g)$ denote the set of points which do not eventually land on points which are blown-up under ϕ^{-1}. Let $\mathcal{A}(P(g), G)$ denote the set of classes in $\mathcal{A}(P(g))$ represented by elements whose endpoints lie in G. Let $\mathcal{A}(Q(g), g^{-1}G)$ denote the set of classes in $\mathcal{A}(Q(g))$ represented by elements whose endpoints lie in $g^{-1}(G)$.

Theorem 4.36 (Quotients and arcs) *Let $\phi : f \to g$ be a combinatorial class of quotient map. Then ϕ induces injections*

$$\phi_P^{-1} : \mathcal{A}(P(g), G) \to \mathcal{A}(P(f))$$

and

$$\phi_Q^{-1} : \mathcal{A}(Q(g), g^{-1}G) \to \mathcal{A}(Q(f))$$

such that

1. *ϕ_P^{-1} and ϕ_Q^{-1} map $\mathcal{A}(P(g), G)$ and $\mathcal{A}(Q(g), g^{-1}G)$ surjectively onto the set of classes of arcs in $\mathcal{A}(P(f))$ and $\mathcal{A}(Q(f))$ which do not intersect the support of ϕ;*

2. *$i_{\mathcal{A},f} \circ \phi_P^{-1} = \phi_Q^{-1} \circ i_{\mathcal{A},g}$;*

3. *$f_{\mathcal{A}} \circ \phi_Q^{-1} = \phi_P^{-1} \circ g_{\mathcal{A}}$;*

4. *the maps preserve the property of being essential, inessential, and parallel.*

4.11 Appendix: PF branched covers covering a mapping scheme

In this section, we generalize PF branched covering, combinatorial equivalence, and the necessity of Thurston's theorem to PF hyperbolic families of branched coverings of the sphere to itself covering a mapping scheme.

Definitions. A *family of branched coverings of the sphere covering a mapping scheme* (S, τ, w) is a map

$$F : S \times S^2 \to S \times S^2$$

such that the restriction of F to $\{x\} \times S^2$ is a branched covering of degree $w(x)$. We denote by $C(F)$ the set of critical points of F. The *postcritical set* is the set

$$P(F) = \bigcup_{n>0, c \in C(F)} F^{\circ n}(c).$$

The family F is said to be *postcritically finite* if $|P(F)| < \infty$. We say that F is *hyperbolic* if every cycle of postcritical points contains a critical point.

We say that two PF families F, G covering a mapping scheme (S, τ, w) are *combinatorially equivalent* if there are homeomorphisms $\Psi_0, \Psi_1 : (S \times S^2, P(F)) \to (S \times S^2, P(G))$ such that $\Psi_i(\{x\} \times S^2) = \{x\} \times S^2$, $\Psi_0 \circ F = G \circ \Psi_1$, and Ψ_0 is isotopic to Ψ_1 through homeomorphisms agreeing on $P(F)$.

Generalization of the necessity of Thurston's theorem Suppose F is a hyperbolic PF family of branched covers of the sphere covering the mapping scheme (S, τ, w).

A *curve system* in $(S \times S^2, P(F))$ is a finite set of disjoint, nonparallel, essential simple closed curves.

The map F defines a pullback relation on the space of curves in the same way as for a single map. We may therefore define the Thurston linear transformation $A(\Gamma)$ associated to a curve system Γ.

Proposition 4.37 *The family F is combinatorially equivalent to a PFH family of polynomials only if for every nonperipheral F-invariant curve system Γ, the leading eigenvalue $\lambda(A(\Gamma))$ is greater than one.*

Proof: Suppose $\lambda(A(\Gamma)) \geq 1$. Then the leading eigenvalue of any iterate of A is also at least one.

We may assume that $A(\Gamma)$ is irreducible. Let n be the least common multiple of the periods of the periodic elements of S under τ. Then since the Thurston linear transformation commutes with iteration, the matrix of $A^{\circ n}$ decomposes into square blocks along the diagonal. There is one block for each periodic point of S. Since $\lambda(A(\Gamma)) \geq 1$, the leading eigenvalue of $A^{\circ n}$ coincides with the leading eigenvalue of one of the blocks. Hence there is a $(F^{\circ n})|_{\{x\} \times S^2}$-invariant nonperipheral curve system Γ' contained in $\{x\} \times S^2$ for some periodic point $x \in S$ for which $\lambda(A(\Gamma')) \geq 1$. Since F is assumed to be equivalent to a family of conformal polynomials, $(F^{\circ n})|_{\{x\}\times S^2}$ is combinatorially equivalent to a PFH rational map. Since F is hyperbolic, $(F^{\circ n})|_{\{x\}\times S^2}$ is also hyperbolic. This violates Thurston's theorem.

■

Theorem 4.38 *Let f be a PFH rational map and $B \subset Q(f)$ a subscheme of the mapping scheme of f. Let $\mathcal{P}$ denote a PFH family of polynomials covering the mapping scheme $(B, f|_B, w_f|_B)$. Let $\mathcal{D}$ denote the support of the tuning, and let $\mathcal{D}_{per}$ denote the periodic components of $\mathcal{D}$.*

If Γ is any reduced Thurston obstruction for the tuning $f * \mathcal{P}$, then every element $\gamma \in \Gamma$ has nonzero intersection number with the boundary of some component of $\mathcal{D}_{per}$.

Proof: Suppose otherwise. Since Γ is irreducible, if some element $\gamma \in \Gamma$ is contained in $\mathcal{D}_{per}$ up to isotopy, then $\Gamma \subset \mathcal{D}_{per}$, up to isotopy. But then Γ yields a reduced Thurston obstruction for $\mathcal{P}$, contradicting the previous proposition.

Conversely, if some element $\gamma \in \Gamma$ is disjoint from $\mathcal{D}_{per}$, then again by irreducibility, Γ is disjoint from $\mathcal{D}_{per}$, up to isotopy. In the latter case, by considering preimages, we may conclude that Γ is disjoint from all of $\mathcal{D}$. But this violates Theorem 4.33.

■

Chapter 5

The characteristic subcomplex

In this chapter, we modify slightly the definition of the pushforward relation on arcs. We show that this modified pushforward relation essentially determines how periodic Fatou components touch and touch themselves. The main results are:

- the construction of the combinatorial characteristic subcomplex (CCS) of a PFH rational map f, and
- the construction of the isomorphism between the CCS and the dynamics on the space of essential chords in the lamination associated to f.

The definition of a combinatorial cylinder for a PFH rational map in the next chapter will be given in terms of the combinatorial characteristic subcomplex. A chord is essential if the arc it forms is essential in $(S^2, Q(f))$.

In Section 5.1 we discuss examples which show why a modification of the pushforward relation given in Chapter 4 is necessary. Section 5.2 contains the definition of the combinatorial characteristic subcomplex. In Section 5.3 we show that dynamics on the space of essential chords is isomorphic as a topological dynamical system to the CCS. In Section 5.4 we apply the isomorphism theorem to give combinatorial characterizations of topological features of Julia sets. In Section 5.5 we discuss how the topology of the Julia set changes under tuning. We conclude in Section 5.6 with further examples, including an informal discussion of Tan Lei's construction of "blowing up an arc".

5.1 Examples

The semibasilica. The proof that the lamination of a polynomial is backward invariant depends on the fact that for polynomials, the basin of infinity is totally invariant.

The following example shows that this need not be the case in general. We begin by comparing it with a well-known quadratic polynomial.

Figure 5.2 is the filled-in Julia set of the quadratic polynomial $p(z) = z^2 - 1$. The unique finite critical point at 0 is periodic of period two. The 1/3 and 2/3 rays in the basin of infinity meet at the α-fixed point of $p(z)$. Let $\eta = R_{1/3} \cup R_{2/3}$. Then η is an arc with endpoints at infinity mapping homeomorphically onto itself under $p(z)$ so that the 1/3 and 2/3 rays are interchanged, and so has period one under the pushforward relation. There are two preimages of η: η itself, and ν, the union of the 1/6 and 5/6 rays, together with the point at infinity. Note that ν is inessential in $(\widehat{\mathbb{C}}, P(p))$, but is essential in $(\widehat{\mathbb{C}}, Q(p))$. Figure 5.1 represents the polynomial $z \mapsto z^2 - 1$ as a branched covering of the sphere. Arcs with a given Roman number map to the arc with the same Roman number. Arcs with arrows have the point at infinity as an endpoint.

The lamination for $p(z)$ consists of the segment joining 1/3 and 2/3, together with the unique set of backward images such that the resulting set of leaves is disjoint. A few preimages are shown in Figure 5.3. The set of leaves is countable: every leaf eventually lands on the unique fixed leaf.

Figure 5.4 is a picture of the Julia set of the degree three rational map given by

$$f(z) = \frac{2(z-1)^2(z+2)}{3z-2},$$

which we call the degree three semibasilica. The point at infinity is a fixed simple critical point, the point 0 is a critical point of multiplicity two and period two, and 1 is a simple critical point mapping onto 0. The Riemann map to basin of infinity given by Böttcher's theorem allows us to define rays for this basin, determines a lamination corresponding to the identification of the landing points of these rays. As with the basilica, the 1/3 and 2/3 rays land at a common repelling fixed point for f, and are interchanged under f. Let $\eta' = R_{1/3} \cup R_{2/3} \cup$. Then η' is a period one arc under the pushforward relation. However, η' has no essential preimages in $(\widehat{\mathbb{C}}, Q(f))$ joining the point at infinity to itself. The arc η' has three lifts, one of which is itself; the other two "unwind" under f^{-1} to form a simple closed curve in the sphere mapping by degree two onto η. See Figure 5.5.

The lamination for the basin of infinity for the map giving the semibasilica consists of a single leaf joining the 1/3 and 2/3 points. Hence this lamination is not backward-invariant, since the $1/3 - 2/3$ leaf has only a single preimage. Also, the condition of gap invariance fails. The Jordan curve bounding the right-hand half of the Figure 8 comprising the boundary of the basin of infinity does not map to a Jordan curve under f.

There are actually many postcritically finite hyperbolic rational maps for which the lamination produced by the Riemann map to a periodic superattracting basin does not yield an invariant lamination. For example, the family

$$g_{r,d}(z) = \frac{r}{d-1} N_d \circ p_d \circ M,$$

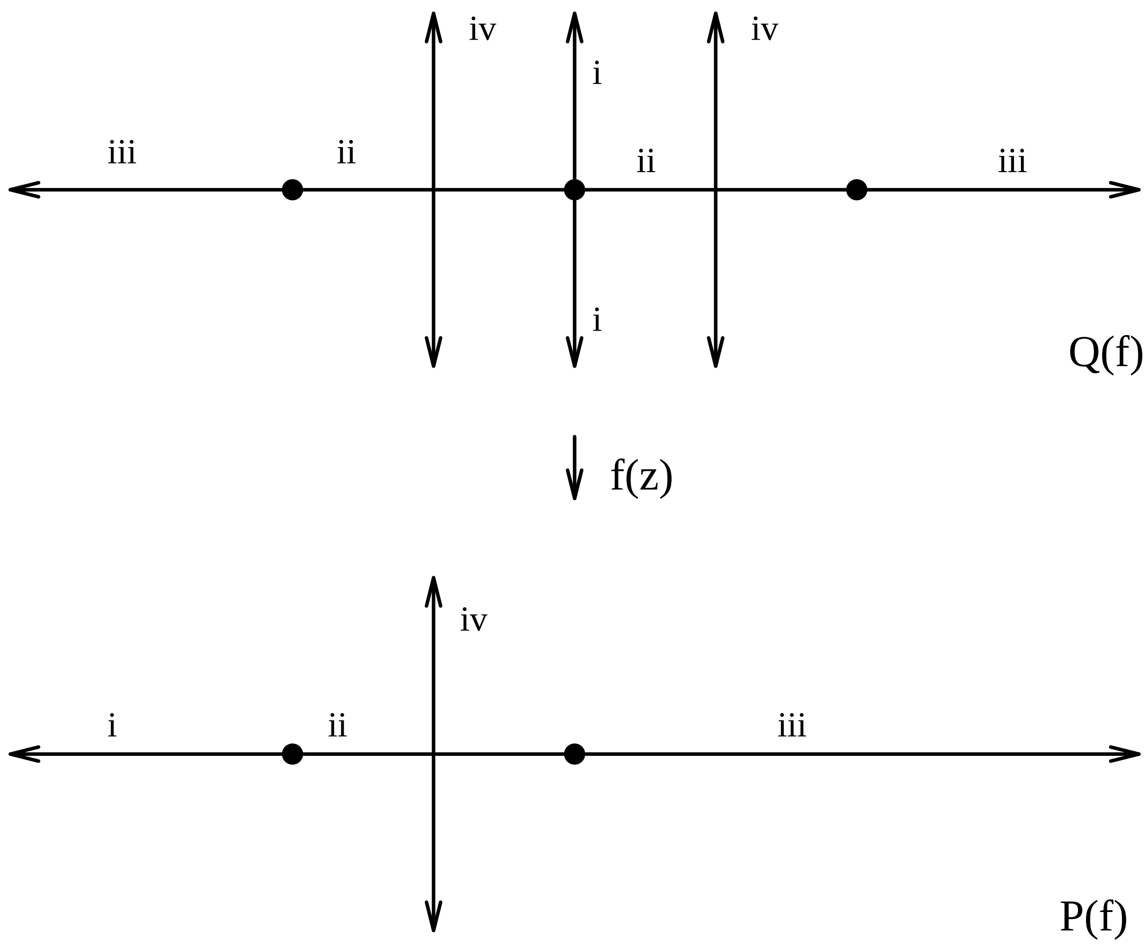

Figure 5.1: The map $z \mapsto z^2 - 1$ as a branched covering of the sphere.

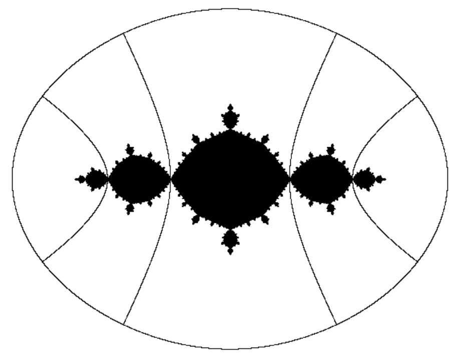

Figure 5.2: The filled-in Julia set of the degree two basilica.

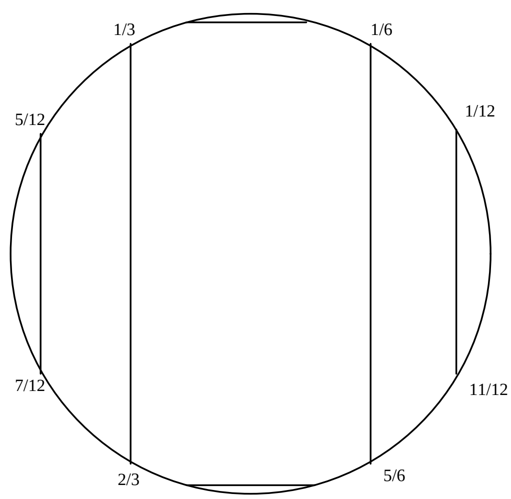

Figure 5.3: A portion of the lamination for the degree two basilica.

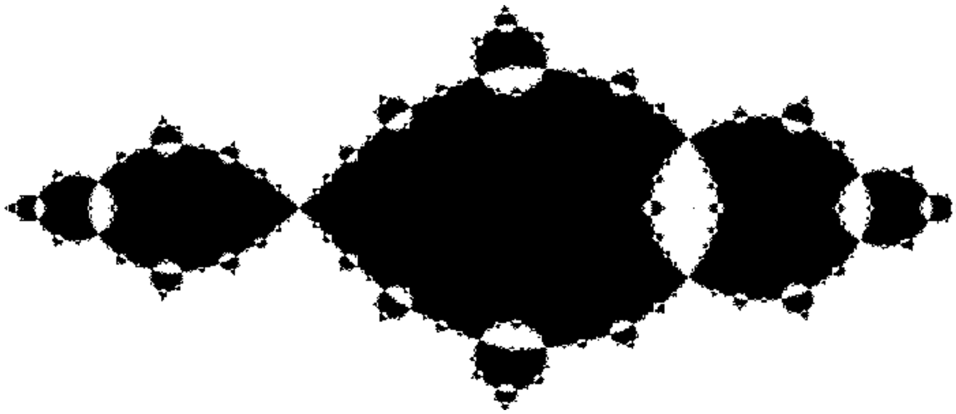

Figure 5.4: The degree three semibasilica.

where $M(z) = \frac{z-1}{z}$, $p_d(z) = (d-1)z^d - dz^{d-1} + 1$, and $N_d = (1-d)\frac{z-1}{z}$, yields many such examples. For maps in this family, the point at infinity is a fixed simple critical point, 1 is a critical point of multiplicity $d-2$ mapping onto 0, and 0 is a critical point of multiplicity $d-1$ mapping onto $-r$. When $d = 3$ and $r = 2$, we obtain the degree three semibasilica. When $d = 3$ and r is chosen so that 0 is periodic of period three and $g_{2,3}(0)$ is complex, the result is a "semirabbit", shown in Figure 5.7. In comparison, the Julia set of $f(z) = z^2 - 0.1226... + 0.7449...i$ is commonly called *Douady's rabbit.* The 1/7, 2/7, and 4/7 rays meet at a common repelling fixed point; see Figure 5.6.

For the degree three semirabbit, however, there are only two rays in the immediate basin of infinity meeting the preimage of the common landing point of the 1/7-2/7-4/7 rays, as opposed to three in the polynomial case. This also shows that the image of a finite-sided gap need not be an entire finite-sided gap.

When $r = d-1$, the result for all degrees is a variation of the semibasilica; the degree four example is shown in Figure 5.8, found with assistance from J. Kahn and C. McMullen.

In Subsection 5.6.2 we outline a construction, due to Tan Lei, which we will show always yield rational maps with this property.

These examples suggest that the dynamics of classes of arcs joining periodic points in the postcritical set play an important role in determining invariance properties of the set of points where the boundary of a Fatou component is pinched.

The real period three quadratic polynomial.

Figure 5.9 shows the filled-in Julia set of the quadratic polynomial $p(z) = z^2 - 1.7548877...$ for which the critical point 0 is periodic of period three.

The lamination has a period three cycle of leaves given by $1/7{-}6/7$, $2/7{-}5/7$, and $4/7 - 3/7$. It also has a period one cycle of leaves given by $1/3 - 2/3$. Let $\eta = R_{1/3} \cup R_{2/3}$. Let $\alpha_0 = R_{2/7} \cup R_{5/7}$, $\alpha_1 = R_{4/7} \cup R_{3/7}$, and $\alpha_2 = R_{1/7} \cup R_{6/7}$.

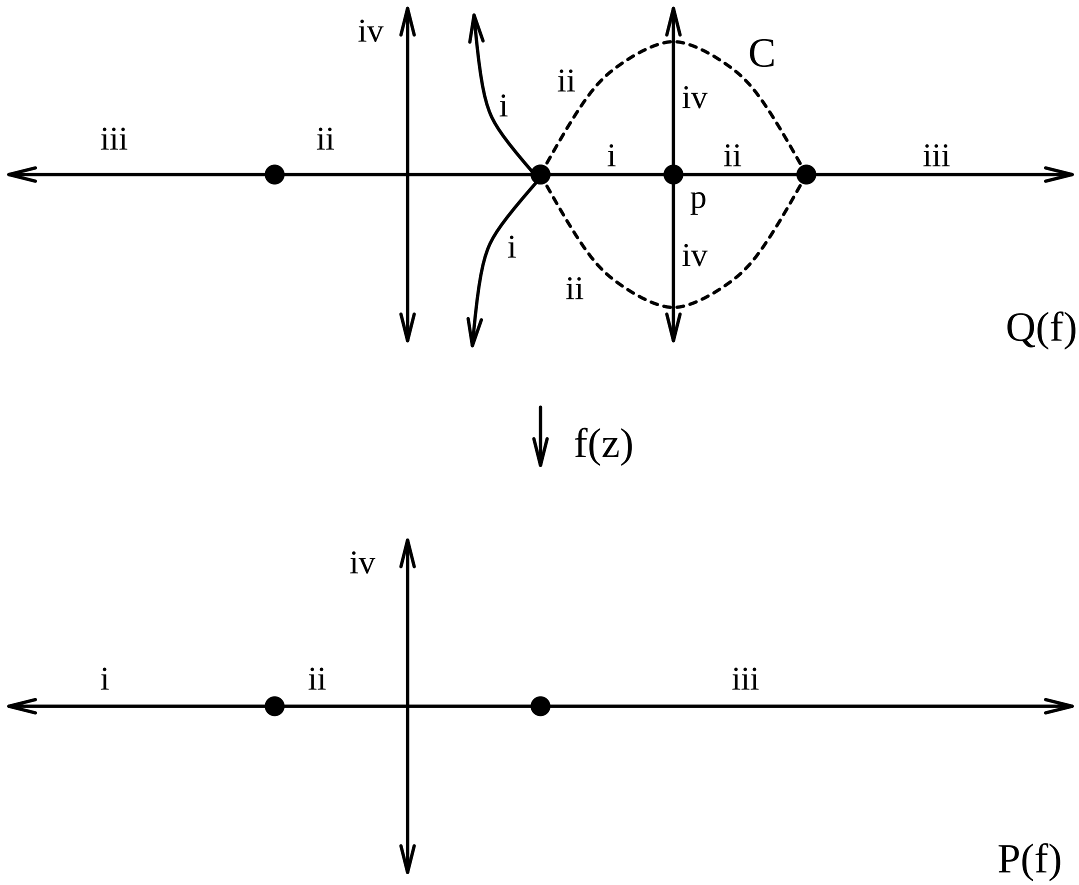

Figure 5.5: Arcs for the semibasilica. The bounded region formed by the dashed curve C maps onto the complement of the arc numbered by (ii). The point p maps to the point at infinity.

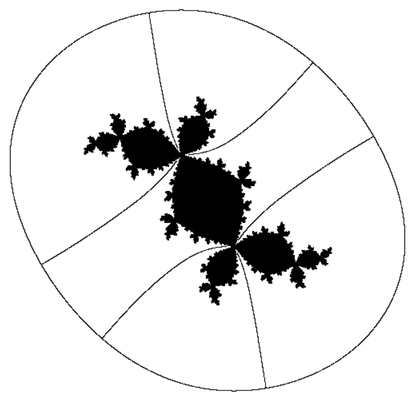

Figure 5.6: Douady's rabbit. The critical point at zero is periodic of period three.

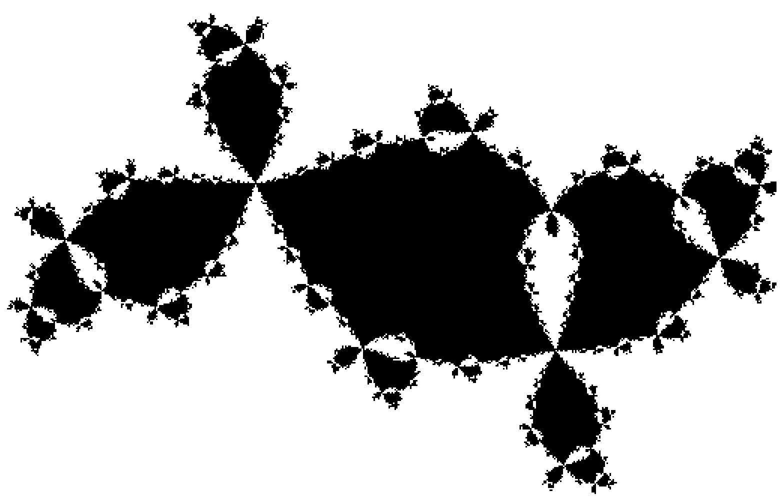

Figure 5.7: The degree three semirabbit.

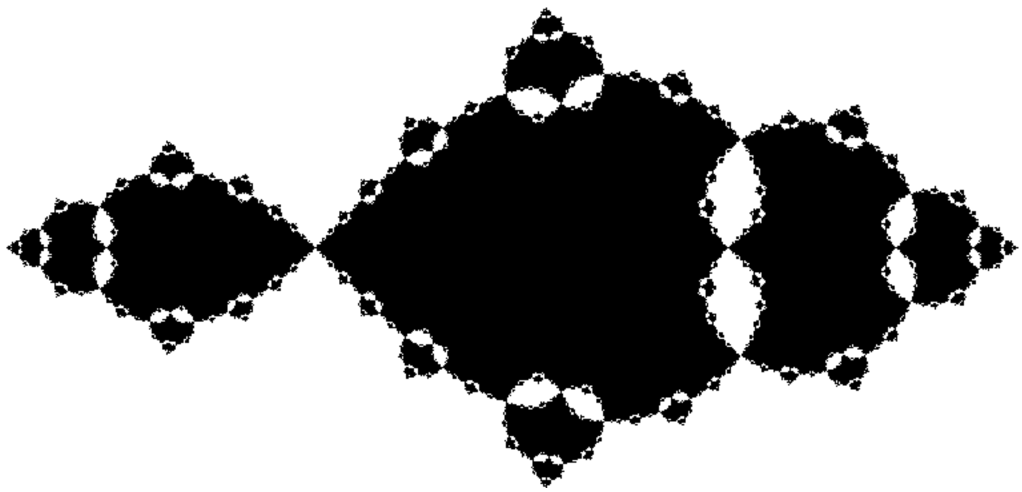

Figure 5.8: The degree four semibasilica.

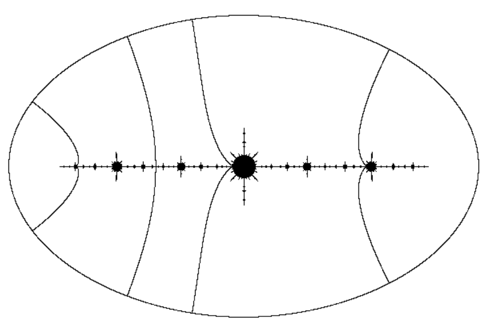

Figure 5.9: The filled-in Julia set of $z^2 - 1.754877....$ The critical point 0 is periodic of period three.

Note that $[\alpha_0] = [\alpha_1] = [\eta]$ in $\mathcal{A}(P(p))$.

Under the sixth iterate of this polynomial, each of the arcs is fixed as an oriented arc, up to isotopy. However, the three classes $[\alpha_0], [\alpha_1]$, and $[\eta]$ are all the same as elements of $\mathcal{A}(P(p))$, since the postcritical set does not change by taking iterates. Hence the orbit $\{[\beta_i]\}_{i=1}^{\infty}$ given by $\beta_i = \alpha_0$ for all i does not determine the ray α_0 uniquely, since there are at least three other rays with the same itinerary as isotopy classes of arcs in $\mathcal{A}_{per}(P(p))$.

To get around this difficulty, we will consider isotopy classes of arcs in $\mathcal{A}(Q(f))$.

5.2 The combinatorial characteristic subcomplex

Let f be a postcritically finite branched covering. Recall that pushforward relation f_* on $\mathcal{A}(P(f))$ is defined by the composition $f_{\mathcal{A}} \circ i_{\mathcal{A}}$. We will now define an analogous relation on $\mathcal{A}(Q(f))$.

5.2.1 The Q-pushforward relation

Let $f_*^Q : \mathcal{A}(Q(f)) \to \mathcal{A}(Q(f))$ be given by $f_*^Q = i_{\mathcal{A}} \circ f_{\mathcal{A}}$. Then f_*^Q defines a relation from $\mathcal{A}(Q(f))$ to itself which we call the *Q-pushforward relation.* This relation is an invariant of the combinatorial class of f, by the results proved in Section 4.5. However, naturality under iteration no longer makes sense, since the sets $Q(f^{\circ n}) = (f^{\circ n})^{-1}(P(f))$ and $Q(f) = f^{-1}(P(f))$ are distinct unless f is elementary.

The following proposition says that given an essential arc α and a finite sequence of essential classes of successive backward images under the pushforward relation, there is a unique sequence of preimages of α representing the sequence of classes.

Proposition 5.1 *Let $\{[\alpha_i]\}_{i=1}^{n} \subset \mathcal{A}(Q(f))$ be a set of essential classes satisfying $[\alpha_{i+1}] \in f_*^Q([\alpha_i]), i = 1, 2, ..., n-1$. Suppose $\beta_n \in [\alpha_n]$. Then there exists a unique arc $\beta_1 \in [\alpha_1]$ such that $[f^{\circ i}(\beta_1)] = [\alpha_i]$, $i = 1, ..., n-1$, and $f^{\circ n}(\beta_1) = \beta_n$.*

Proof: This follows immediately from Proposition 4.4 and induction on n.

∎

5.2.2 Finiteness of eventually periodic arc classes for rational maps

Theorem 5.2 *Let $f(z)$ be a PFH rational map. Then $\mathcal{A}_{evp}(Q(f))$ is finite.*

Proof: We will use the expanding properties of f. The proof will not give an explicit bound. The idea is to pull back arcs and show that a suitably defined "length" shrinks by a definite multiplicative factor, up to an additive constant.

Let ϕ_Q be a family of Riemann mappings as defined in Section 2.6. Let $U = \phi_Q(Q \times \{z : |z| < 1/2\})$. Then each component of U is an open disc meeting exactly one point of $Q(f)$. Let $K = \widehat{\mathbb{C}} - U$. Then $f^{-1}(K) \subset K$, and every component of $K - f^{-1}(K)$ is a half-open annulus.

Given $[\alpha] \in \mathcal{A}(Q(f))$, define

$$L([\alpha]) = \inf\{l(\alpha' \cap K) | \alpha' \in [\alpha] \text{ and } \alpha' \cap K \text{ is connected }\},$$

where l denotes length in the spherical metric. Note that a sequence $\{[\alpha_n]\}_{n=1}^{\infty}$ tends to infinity in $\mathcal{A}(Q(f))$ only if $L([\alpha_n]) \to \infty$.

Set $\lambda = (\sup_{z \in f^{-1}(K)} \| f'(z) \|)^{-1}$, where the norm of the derivative is measured in the spherical metric. Set $C = 2 \cdot \sup \{\text{diam}(C) | C \text{ is a component of } K - f^{-1}(K)\}$.

Lemma 5.3 *If* $[\alpha_1] \in f_*^Q([\alpha_0])$, *then* $L([\alpha_0]) \leq \lambda \cdot L([\alpha_1]) + C$.

To see this, fix $\epsilon > 0$, and choose $\eta_1 \in [\alpha_1]$ such that $\eta_1 \cap K$ is connected and $l(\eta_1 \cap K) < L([\alpha_1]) + \epsilon$. By Proposition 4.4, there is a unique lift η_0 of η_1 representing $[\alpha_0]$.

The set $\eta_0 \cap f^{-1}(K)$ is connected, so η_0 intersects $\partial f^{-1}(K)$ in two points, say a and b. Let x and y be the endpoints of η_0, and let C_x and C_y be the components of $K - f^{-1}(K)$ separating x and y from the remainder of $Q(f)$. Call the boundary component of C_x nearest x the *inner boundary component* of C_x, and similarly call the boundary component of C_y nearest x the *inner boundary component* of C_y.

Modify η_0 as follows. Remove $\eta_0 \cap f^{-1}(U)$. Glue in a connected spherical geodesic segment in C_x joining a and any point a' on the inner boundary component of C_x. Join a' to x by an arbitrary curve in U. Repeat this construction with the other end of η_0. Call the resulting arc η_0'. Since we have only modified η_0 on U, η_0' is isotopic to η_0.

Then $\eta_0' \cap K$ is connected, $l(\eta_0 \cap f^{-1}(K)) \leq \lambda \cdot l(\eta_1 \cap K)$, and $l(\eta_0 \cap (K - f^{-1}(K))) \leq C$. Hence $l(\eta_0') \leq \lambda \cdot l(\eta_0) + C$, and so

$$L([\alpha_0]) \leq l(\eta_0' \cap K) \leq \lambda \cdot l(\eta_1 \cap K) + C < \lambda \cdot (L([\alpha_1]) + \epsilon) + C.$$

Since ϵ is arbitrary, this proves the lemma.

The function $h(x) = \lambda \cdot x + C$ has a unique attracting fixed point x_0. If $[\alpha]$ is any period p class in $\mathcal{A}(Q(f))$, by pulling back $[\alpha]$, we have that $L([\alpha]) \leq h^{\circ p}(L([\alpha]))$. Hence $L([\alpha]) \leq x_0$, so the set of periodic classes of arcs is bounded. If $[\alpha]$ is a preperiodic class which maps onto a periodic class $[\beta]$ after n iterates, then $L([\alpha]) \leq h^{\circ n}(L([\beta])$. Since $L([\beta]) \leq x_0$, $L([\alpha]) \leq x_0$. Hence the length of every eventually periodic arc class is uniformly bounded, and so the set of eventually periodic arcs is finite.

■

Remark: This theorem may fail for maps with Levy cycles. Let $p(z) = z^2 - 1$ and let $f(z) = p * p$ be the mating of $p(z)$ with itself. Let X_- denote the union of the segment $[-1,1]$ with the $1/3$ and $2/3$ rays in the extended complex plane, regarded as the lower hemisphere of the Riemann sphere. Then X_- is topologically a figure "X". Let X_+ denote its reflection through the unit circle by the map $z \mapsto 1/\overline{z}$. Then $X = X_- \cup X_+$ is a graph in the sphere which is forward-invariant under f and contains a Levy cycle γ which is the union of the $1/3$ and $2/3$ rays. By altering f slightly within its combinatorial class, we may assume $f^2 = \mathrm{id}$ on an annulus which is a regular neighborhood of this Levy cycle. We may find infinitely many essential arcs which are periodic of period one obtained by performing a Dehn twist about this annulus.

Remark: If f is a PF branched covering, the set of elements of the mapping class group of $S^2 - P(f)$ which commute with f, up to isotopy, forms a group which we call the set of isotopy automorphism of f. If f is a rational map, this group is the same as the group of conformal automorphisms of f. If f has a Levy cycle, however, the group of isotopy automorphisms is infinite: conjugating by Dehn twists about the curves in the cycle yield an infinite set of elements.

Using Theorem 4.21, it is possible to give a slightly stronger proof of the above theorem. The idea is that two periodic classes of arcs which intersect in more than one point of their interiors yield a Levy cycle for f. This proof also allows one to calculate a bound on $\mathcal{A}(Q(f))$ in terms of $|Q(f)|$.

5.2.3 Construction of the CCS

By Theorem 5.2, if f is a PFH rational map, the set $\mathcal{A}_{evp}(Q(f))$ of eventually periodic classes in $\mathcal{A}(Q(f))$ under the relation f_Q^* is finite. Set $A = \mathcal{A}_{evp}(Q(f))$ and $f_A = f_Q^*|_{A\times A}$.

The pair (A, f_A) may be described by a matrix of zeros and ones. Suppose $A = \{[\alpha_i]\}_{i=1}^n$. We encode the relation (A, f_A) by the matrix $(A_{ij})_{i,j=1}^n$ where $A_{ij} = 1$ if $[\alpha_i] \in f_A([\alpha_j])$ and is zero otherwise.

Let Σ denote the set of one-sided infinite sequences $\{[\beta_i]\}_{i=0}^\infty$ of elements of A, subject to the constraint that for all i, $[\beta_{i+1}] \in f_A([\beta_i])$. Equivalently, the set Σ is the set of all infinite trips through the directed graph associated to (A, f_A). The set Σ has a natural metric: the distance between two sequences $\{[\alpha_i]\}_{i=1}^\infty$ and $\{[\beta_i]\}_{i=1}^\infty$ is given by

$$d([\alpha_i],[\beta_i]) = \sum_{i>0} \frac{\delta([\alpha_i],[\beta_i])}{2^i},$$

where $\delta([\alpha_i],[\beta_i]) = 0$ if the classes are the same and is equal to one otherwise.

Equipped with this metric, Σ is a compact metric space which is either a a point, a countable set, or a Cantor set union a countable set, since any totally disconnected compact metric space has of one of these forms. (Section 2.15, [HY]). The *shift map* σ is the map of Σ to itself which forgets the first term in each sequence.

Definition 5.4 (Combinatorial characteristic subcomplex) *The* **combinatorial**

characteristic subcomplex (CCS for short) *of a PFH* $f(z)$ *the subshift of finite type* (Σ, σ).

5.3 The isomorphism $G : (\Sigma, \sigma) \to (\chi_{ess}, F_0)$

Let $f(z)$ be a PFH rational map. We let

- (A, f_A) denote the restriction of the Q-pushforward relation to the set of eventually periodic classes of arcs in $A_{evp}(Q(f))$. Then A consists of a finite set of classes $\{[\alpha_n]\}$, and f_A is given by a matrix $(A_{k,l})$ such that $A_{k,l} = 1$ if $[\alpha_k] \in f_A([\alpha_l])$ and $A_{k,l} = 0$ otherwise.
- (Σ, σ) denote the CCS of f;
- ϕ_Q be a choice of Riemann maps;
- Λ be the lamination of f with respect to ϕ_Q, which is invariant with respect to the map $F_0 : Q(f) \times S^1 \to Q(f) \times S^1$ covering the mapping scheme of f, and
- χ, the space of chords of Λ_f.

Essential and inessential chords.

Definition 5.5 (Essential chord) *A chord* $\{(x, s), (y, t)\}$ *of the lamination* Λ *is called* **essential** *if the arc* $\alpha = R_{x,s} \cup R_{y,t}$ *that it forms is essential in* $(S^2, Q(f))$.

A chord is inessential if and only if its endpoints are the same, and the arc it forms bounds a unique disc in $S^2 - Q(f)$.

Let χ_{ess} denote the space of essential chords in the subspace topology of chords on Λ. Since any lift of an inessential arc is again inessential, $F_0(\chi_{ess}) \subset \chi_{ess}$.

Theorem 5.6 (Chords are eventually essential) *Let* f *be a PFH rational map and* Λ *the lamination of* f. *Then every chord of* Λ *eventually maps onto an essential chord under* F_0.

Proof: We argue by contradiction, using expansion of the spherical metric. Suppose $\{(x, s), (y, t)\}$ is an inessential chord, and let α_0 be the arc it forms. Set $\alpha_n = f^{\circ n}(\alpha)$. Suppose that α_n is inessential for every n.

Without loss of generality, we may assume that the endpoint of α_0 is periodic. Let x_n denote the endpoint of α_n, and let Ω_{x_n} denote the Fatou component containing x_n. Then the Ω_n are periodic Fatou components with non-Jordan curve boundary. The arcs α_n separate $\widehat{\mathbb{C}}$ into two discs, exactly one of which is contained in $\widehat{\mathbb{C}} - Q(f)$. Let U_n denote this disc. Then U_n contains components of $J(f)$, by Theorem 2.33. Let $K_n = \overline{J(f) \cap U_n}$. Since α_n is inessential for all

n, U_{n-1} is a component of $f^{-1}(U_n)$, and $f : \overline{U_{n-1}} \to \overline{U_n}$ is a homeomorphism. Hence $f : K_{n-1} \to K_n$ is a homeomorphism. Since f is hyperbolic, some iterate of f expands the spherical metric by a definite factor on $J(f)$. The previous two facts imply that there exist an integer $N > 0$ and $\lambda < 1$ such that for every $n \geq N$, $\operatorname{diam}(K_{n-N}) \leq \lambda \operatorname{diam}(K_n)$. Since the diameters of the K_n are uniformly bounded, the diameter of K_0 must be zero. But this violates the conclusion of Theorem 2.33.

■

The preimage of an inessential chord under F_0 is always a canonically determined collection of chords. To see this, we may assume $F_0 : \{x\} \times S^1 \to \{x\} \times S^1$. Let s and t are the angles of an inessential chord. Let L be a component of the complement $S^1 - \{s, t\}$. Then there are exactly $w_f(x)$ components of the preimage of L under F_0, where $w_f(x)$ is the degree of F_0 on $\{x\} \times S^1$. Two points in the preimage of the set $\{s, t\}$ form a chord if they are the endpoints of one of these components. Hence the space of chords is determined by the space of essential chords. Consequently, the lamination is uniquely determined by the set of essential chords.

Proposition 5.7 *Let $\{(x, s), (y, t)\}$ be any essential chord. Then $\alpha((x, s), (y, t)) \in A$.*

Proof: A sequence of essential classes $\{[\alpha_n]\}_{n=1}^{\infty}$ tends to infinity in $\mathcal{A}(Q(f))$ only if there is a pair of arcs with intersection number at least two.

Let $\alpha_n = f^{\circ n}(\alpha), n \geq 0$. Then the interiors of the α_n may intersect in at most one point. Since the α_n are all essential, by the above remark, there must exist some $m > n$ such that $[\alpha_n] = [\alpha_n]$, and so $[\alpha] \in A$.

■

Hence the arc α formed by an essential chord determines an itinerary $\{[f^{\circ n}(\alpha)]\}_{n=0}^{\infty} \subset \Sigma$.

The isomorphism theorem.

The previous theorem implies that the map sending an essential chord to the itinerary of the arc it forms (as a class in A) induces a map $H : \chi_{ess} \to \Sigma$ conjugating F_0 to σ. This map is clearly continuous: if two chords are close, the arcs they form are close as closed subsets of the sphere, and so the classes of these arcs must also be close since there are no postcritical points in the Julia set.

The goal of this section is to prove

Theorem 5.8 *The map $H : \Sigma \to \chi_{ess}$ is a homeomorphism.*

As an immediate corollary, we have

Corollary 5.9 *The space of essential chords is compact.*

To prove this theorem, we need some preliminary results and notation.

Hausdorff metric on closed subsets of S^2. Let K and L be two closed subsets of S^2. Let $d(x, y)$ denote the chordal distance between two points. Then the *Hausdorff distance* between K and L is defined by

$$D(K,L) = \max\left\{\sup_{x\in K}\{d(x,L)\}, \sup_{y\in L}\{d(K,y)\}\right\}.$$

The Hausdorff metric turns the set of all closed subsets of S^2 into a compact Hausdorff metric space.

Peripherally rigid arcs. Any arc α in $(S^2, Q(f))$ with endpoints in $Q(f)$ is isotopic to an arc which coincides with a ϕ_Q-ray near its endpoints. We will refer to such an arc as a *peripherally rigid* arc.

If α is a peripherally rigid arc with endpoints x and y, we set

$$E_x(\alpha) = \sup\{r|\alpha \text{ coincides with } \phi_x([0,r)\cdot\exp(2\pi is))\},$$

and

$$E_y(\alpha) = \sup\{r|\alpha \text{ coincides with } \phi_y([0,r)\cdot\exp(2\pi it))\},$$

and

$$E(\alpha) = \min\{E_x(\alpha), E_y(\alpha)\},$$

where ϕ_x and ϕ_y are the restrictions of ϕ to $\{x\}\times\overline{\Delta}$ and $\{y\}\times\overline{\Delta}$. Thus for peripherally rigid arcs α, $1 \geq E(\alpha) > 0$. We will call the part of a peripherally rigid arc coinciding with a part of a pair of rays near the center of Fatou components the *rigid part* and its complement the *non-rigid part*. We denote the non-rigid part by α^{nr}. We define the *angles* of α by $\angle_x(\alpha) = s$, $\angle_y(\alpha) = t$, and $\angle(\alpha) = \{s,t\}$.

Good isotopies. Let α and β be two peripherally rigid arcs isotopic in $(S^2, Q(f))$ with endpoints x and y. Suppose $E(\alpha), E(\beta) < 1$. Then there exists an isotopy h_t from $\alpha = h_0(I)$ to $\beta = h_1(I)$ through peripherally rigid arcs $h_t(I)$ such that $E(h_t(I)) \geq \min(E(\alpha), E(\beta))$. If $E(\alpha) = E(\beta) = 1$, we may choose the isotopy h_t to satisfy $E(h_t(I)) \geq 1/2$. We call such an isotopy h a *good isotopy.*

The *length* of h is defined as follows. Set

$$l_x(h) = \int_0^1 |\angle_x(h_t(I))|dt$$

and

$$l_y(h) = \int_0^1 |\angle_y(h_t(I))|dt.$$

The length $l(h)$ we define by

$$l(h) = \max(l_x(h), l_y(h)).$$

Lemma 5.10 *1. If β is a lift of α and the endpoints of β are x and y , then $E(\beta) = (E(\alpha))^{1/d}$, where $d = \min(w_f(x), w_f(y))$.*

2. *If β_i is a lift of α under $f^{\circ i}$, then*

$$diam(\beta_i^{nr}) \leq \lambda^i \cdot diam(\alpha^{nr}),$$

where $\lambda < 1$ is a constant depending only on α and f.

3. *Let h be a good isotopy from α_1 to β_1. Let α_0, β_0 be isotopic lifts of α_1 and β_1 with endpoints x and y. Then h lifts to a good isotopy $\widetilde{h}$ from α_0 to β_0 such that*

$$l(\widetilde{h}) \leq \max(1/w_f(x), 1/w_f(y)) \cdot l(h).$$

Proof: Part (1) follows immediately from the definition of $E(\alpha)$. Part (2) follows from the fact that f expands the spherical metric by a definite factor on every compact subset of $\widehat{\mathbb{C}} - Q(f)$. Proposition 4.4implies that the isotopy h in Part (3) lifts to an isotopy from α_0 to β_0; it is good by Part (1). The length satisfies the given inequality since the map F_0 is a degree $w_f(z)$ covering map on each component $\{z\} \times S^1$.

■

Proof of Theorem 5.8.

We will construct the inverse G of H.

Choose a fixed set of representatives $\{\alpha_n\}$ for the set A which are peripherally rigid. Given $\{[\alpha_{n(i)}]\}_{i=0}^{\infty} \in \Sigma$, let β_i be the unique lift of $\alpha_{n(i)}$ under f^{-i} such that for all $0 \leq j \leq i$, $[f^{\circ j}(\beta_i)] = [\alpha_{n(j)}]$. The lift β_i exists and is unique by Proposition 5.1, and is peripherally rigid by 5.10.

Define

$$G(\{[\alpha_{n(i)}]\}) = \lim_{i \to \infty} \beta_i,$$

where the limit is the Hausdorff limit. We claim

1. The limit exists, is an arc α formed by a chord, and satisfies $[f^{\circ i}(\alpha)] = [\alpha_{n(i)}]$ for all i.

2. The definition of G is independent of the chosen arcs $\{\alpha_n\}$, so long as they are peripherally rigid.

3. $G \circ H = \mathrm{id}_{\chi_{ess}}$.

Proof of 1: First, for each ordered pair (k, l) such that $A_{k,l} \neq 0$, choose a good isotopy $h_{l,k}$ from α_l to the unique lift $\widetilde{\alpha}_k$ of α_k that is isotopic to α_l. The idea of the proof of this step is to lift a collection of these isotopies along the given orbit and then concatenate to obtain an isotopy of finite length from α_0 to β_i.

Next, for $0 \leq j \leq i$, let h_j^i denote the unique lift of $h_{n(j)n(j+1)}$ such that for all $k \leq j$, $f^{\circ k}(h_j^i)$ is an isotopy between a pair of arcs in $[\alpha_{n(k)}]$. (That is, we just lift $h_{n(j)n(j+1)}$ along the given orbit. Uniqueness of this lift is guaranteed by Proposition 5.1.) The map h^i_j is then an isotopy between a pair of arcs in $[\alpha_{n(0)}]$.

Let $h^i = h^i_0 * h^i_1 * ... * h^i_i$ denote the concatenation of the isotopies h^i_j. Then h^i is a good isotopy from $\alpha_{n(0)}$ to β_i.

We now claim that the lengths of the good isotopies h^i are uniformly bounded. Since f is hyperbolic, every postcritical point eventually lands on a periodic critical point. By Lemma 5.10, $l(h^i_j) \leq C_1 \cdot 2^{-C_2 i}$, where C_1 and C_2 are positive integers depending only on the mapping schema of f. Hence the lengths $l(h^i)$ are all uniformly bounded independent of i and the chosen orbit. Hence the limit $\lim_{i\to\infty} \angle(\beta_i)$ exists. Since $E(\beta_i) \to 1$ and $\mathrm{diam}(\beta_i^{nr}) \to 1$ by Lemma 5.10, the β_i converge in the Hausdorff topology to an arc α formed by a chord. By construction, $f^{\circ i}(\alpha) = [\alpha_{n(i)}]$ for all i, so α is essential since $[\alpha] = [\alpha_{n(0)}] \in A$.

Proof of 2: Suppose $\{\alpha'_n\}$ is any other initial choice of peripherally rigid representatives of A. Let β'_i be the corresponding lifts as in the definition of G. Let ρ_n be a good isotopy from α_n to α'_n. Then for all i, $\rho_{n(i)}$ lifts (by Proposition 5.1) along the orbit to a good isotopy ρ^i from β_i to β'_i. By Lemma 5.10, and the fact that f is hyperbolic, $l(\rho^i) \to 0$ as $i \to \infty$. Since $\mathrm{diam}(\beta_i^{nr}) \to 0$ and $\mathrm{diam}((\beta'_i)^{nr}) \to 0$ as $i \to \infty$, we have that $\lim_{i\to\infty}(\beta_i) = \lim_{i\to\infty}(\beta'_i)$ in the Hausdorff topology.

Proof of 3:

The previous two steps show that H is surjective. We now show H is injective.

Let α and β be two arcs formed by a pair of distinct chords in χ_{ess}. Suppose the itineraries of these arcs were the same, measured as isotopy classes in A. Let $\alpha_i = f^{\circ i}(\alpha), \beta_i = f^{\circ i}(\beta)$. Fix some large i. Choose a good isotopy ρ between α_i and β_i. Then by Proposition 5.1, and Lemma 5.10, ρ lifts under $f^{\circ i}$ to a good isotopy ρ^i between α and β. Since f is hyperbolic, by Lemma 5.10, $l(\rho^i) \leq C_1 \cdot 2^{-C_2 i}$, for the constants C_1 and C_2 as in Step 1. Since i is arbitrary, it follows that we can make the length of ρ^i arbitrarily small, and hence that $\alpha = \beta$.

∎

Remark: In [McM3], a similar argument is given to give a topological criterion for a set of periodic external rays to land at a common point for a quadratic polynomial p with connected Julia set. The analog of an arc is a topological tree which coincides with a set of external rays outside of some given equipotential, and which is periodic up to isotopy fixing $P(p)$ through such trees. No other assumptions are placed on p. The argument proceeds by pulling back the tree under the dynamics and using a form of expansion to show that the lifts converge to a set of periodic external rays landing at a common point.

5.4 The topology of $J(f)$

We now list some corollaries relating the combinatorial characteristic subcomplex (Σ, σ) and the topology of $J(f)$. We assume the same notation as in the previous two sections.

The starting point of our discussion is the fact that the map $\overline{\phi}_Q$ defines a continuous map from the space χ of chords to $\{\partial\Omega_x\}_{x\in Q(f)}$.

5.4.1 Finitely many rays land

In this subsection, we prove

Theorem 5.11 *Let Ω be a Fatou component of a PFH rational map f. Let $z \in \partial\Omega$. Then the number of internal rays of f landing at z from Ω is bounded by $|Q(f)| - 1$.*

The bound is almost sharp: there are 2 rays landing at the α-fixed point of the quadratic polynomial $f(z) = z^2 - 1$, and $|Q(f)| = 4$.
Proof: The proof is outlined as follows.

1. **Step 1.** Show that we may assume Ω is periodic.

2. **Step 2.** Show that we may assume that z is periodic. To prove this, we will use the fact that every chord is eventually essential and the compactness of the space of essential chords.

3. **Step 3.** There are at most finitely many rays landing at z.

   This step follows immediately from a well-known fact: if X is a compact metric space and $f : X \to X$ is a homeomorphism which expands distances by a definite factor, then X is finite. See [Mil4], Lemma 18.8 for the proof.

4. **Step 4.** The chords formed by the rays landing at the periodic point z are all periodic, hence they are all essential. This will prove the bound.

Proof:

Let $\mathcal{S}$ be the angles corresponding to the set of rays landing at z. Let $\mathcal{R}$ be the set of rays in Ω with angles in $\mathcal{S}$.

1. **Step 1.** Since there are no critical points in $J(f)$, $f^{\circ n}(\mathcal{R})$ is a homeomorphism for every $n > 0$. Hence, the number of rays landing at $f^{\circ n}(z)$ is at least as great as the number landing at z. So pushing z forward under f cannot decrease $|\mathcal{S}|$.

   Every Fatou component is eventually periodic, by the classical case of the No Wandering Domains theorem. Hence we may assume that $\Omega = \Omega_x$, where $x \in P(f)$ is periodic.

2. **Step 2.**

   We begin by proving a lemma.

   Let $\{(x, s), (x, t)\}$ be any chord in $\{x\} \times S^1$. Define the *length* $l_x(s, t)$ to be the distance between s and t along the circle S^1 of circumference 1. Thus the length of a chord is always at most one-half. The length function is continuous on the space of chords. Since the space of essential chords is compact, every essential chord has length at least $L > 0$.

   Lemma 5.12 *Let $\mathcal{T}_0$ be a set of three essential chords $\{(x, s_i), (x, t_i)\}, i = 1, 2, 3$, such that*

   (a) $s_i, t_i \in \mathcal{S}$, where $\mathcal{S}$ is the set of angles of rays landing at some point $z \in \partial\Omega$;

   (b) with respect to the cyclic ordering on the unit circle, $s_1 < t_1 \leq s_2 < t_2 \leq s_3 < t_3$;

   (c) the arcs α_i formed by $\{(x, s_i), (x, t_i)\}$ are distinct essential classes in A.

   Then the Euclidean area of the convex hull of $\mathcal{S}$ is bounded from below by a positive constant depending only on f.

   Condition (2) implies that the rays formed by the angles s_i, t_i and landing at z are also cyclically ordered near z, but in the opposite fashion.

   Proof: The length function is continuous. The space of essential chords is compact by Corollary 5.9. Hence there is a lower bound to the length of each chord in $\mathcal{T}_0$ which depends only on f. By condition (2), the Euclidean area of the convex hull of the angles of the chords in $\mathcal{T}_0$ is bounded from below by a positive continuous function in the lengths of the chords in $\mathcal{T}_0$. The convex hull of the angles in $\mathcal{T}_0$ is contained in the convex hull of $\mathcal{S}$.

   ∎

   We now prove Step 2.

   We may assume $|\mathcal{S}| \geq 3$. Then there exists a collection $\mathcal{T}$ of three chords $\{(x, s_i), (x, t_i)\}, i = 1, 2, 3$ satisfying (1) and (2) above. Let α_i be the arc formed by the ith chord. For all $n \geq 0$, the forward images $\mathcal{T}_n = F_0^{\circ n}(\mathcal{T})$ also satisfy (1) and (2). For since f is hyperbolic, f is a local homeomorphism near each $z \in J(f)$. Hence the cyclic ordering of the rays landing at z is the same as the cyclic ordering of the images of these rays at the point $f(z)$.

   We will now prove that there is some $N > 0$ such that $n \geq N$ implies $\mathcal{T}_n$ satisfies (3). By Theorem 5.6, each chord in $\mathcal{T}$ is eventually essential. Moreover, we cannot decrease $|\mathcal{S}|$ by pushing z forward under f. Hence there exists an N such that $n \geq N$ implies that $\mathcal{T}_n$ consists of essential chords. Let $\alpha_i^n = f^{\circ n}(\alpha_i)$. Then the α_i^n are essential for all n and i. By condition (2), for a given n, the classes $[\alpha_i^n], i = 1, 2, 3$ are all distinct. Hence $\mathcal{T}_n$ satisfies (3).

Let $\mathcal{S}_n = f^{\circ n}(\mathcal{S})$. By the Lemma and the previous paragraph, the Euclidean area of the convex hulls of the $\mathcal{S}_n$ are bounded from below. Since the total area of the disc is finite, for some $n > m$, $\mathcal{S}_n \cap \mathcal{S}_m \neq \emptyset$. The sets $\mathcal{S}_n$ either coincide, or are disjoint. Hence $\mathcal{S}_n$ is eventually periodic, and so z is eventually periodic. Hence we may assume z is periodic.

3. **Step 3.** Let p be the period of Ω_x. The map $F_0^{\circ p} : \{x\} \times S^1 \to \{x\} \times S^1$ is expanding. Since $\overline{\phi}_x : \{x\} \times S^1 \to \overline{\Omega_x}$ is continuous, $\mathcal{S} \subset S^1$ is closed. The point z is periodic of some period k, and f is hyperbolic, hence $F_0^{kp} : \mathcal{S} \to \mathcal{S}$ is injective. By Lemma 18.8 of [Mil2], $\mathcal{S}$ is finite.

4. **Step 4.** By Theorem 5.6, every periodic chord is essential. Hence every pair of angles in $\mathcal{S}$ determines an essential chord. Let $\mathcal{R} = \cup_{s \in \mathcal{S}}(R_{x,s})$. Then every component of $\widehat{\mathbb{C}} - \mathcal{R}$ is an open disc containing points in $Q(f)$. The center x of Ω_x lies in $\mathcal{R}$, hence there are at most $|Q(f)| - 1$ components of $\widehat{\mathbb{C}} - \mathcal{R}$. This proves the theorem.

■

5.4.2 Touching of Fatou components

By Theorem 5.11, the composition $\overline{\phi}_Q \circ G : \chi_{ess} \to \{\partial\Omega_x\}_{x \in Q(f)}$ is a continuous semiconjugacy for which the size of a preimage of a point is uniformly bounded by $|Q(f)| - 1$. We will use this fact to prove several theorems which are steps toward a classification of the possible dynamics of $f(z)$ on the set of touching points. We first establish some notation.

Notation. We denote by

- x and y, two distinct periodic points of $Q(f)$;
- p, the least common multiple of the periods of x and y;
- d_x and d_y the local degrees of $f^{\circ p}$ near x and y respectively;
- Ω_x and Ω_y, the Fatou components containing x and y;
- $A_{x,y} \subset A$, the set of all classes with distinct endpoints x and y;
- $\Sigma_{x,y}$, the subspace of Σ consisting of all sequences beginning with elements of $A_{x,y}$.

The subspace $\Sigma_{x,y}$ is closed and forward- invariant under the shift σ.

The following two theorems are immediate consequences of Theorem 5.11, Theorem 5.8, and the continuity of $\overline{\phi_Q}$.

Theorem 5.13 (Two distinct components touching) *The intersection* $\overline{\Omega_x} \cap \overline{\Omega_y}$ *either*

1. *is empty, if and only if* $\Sigma_{x,y}$ *is empty;*
2. *is finite, if and only if* $\Sigma_{x,y}$ *is finite;*
3. *is countably infinite, if and only if* $\Sigma_{x,y}$ *is countably infinite.*
4. *contains a Cantor set, if and only if* $\Sigma_{x,y}$ *contains a Cantor set.*

Theorem 5.14 *Let* $\partial_{ess}(\Omega_x)$ *denote the image of* $\Sigma_{x,x}$ *under* $\overline{\phi}_Q \circ G$*. Then* $\partial_{ess}(\Omega_x)$

1. *is empty, if and only if* $\partial\Omega$ *is a Jordan curve, if and only if* $\Sigma_{x,x}$ *is empty;*
2. *is finite, if and only if* $\Sigma_{x,x}$ *is finite;*
3. *is countable, if and only if* $\Sigma_{x,x}$ *is countable;*
4. *contains a Cantor set, if and only if* $\Sigma_{x,x}$ *contains a Cantor set.*

Theorem 5.15 *The set* $\partial\Omega_x$ *is a Jordan curve and* $\Omega_x \cap \Omega_y = \partial\Omega_x$ *if and only if every sequence* $\{[\alpha_i]\}_{i=1}^{\infty} \in \Sigma_{x,y}$ *has at least* d_x *preimages under* $\sigma^p|_{\Sigma_{x,y}}$.

Proof: We first prove the necessity. By hypothesis, since the endpoints are distinct, every chord $\{(x,s),(y,t)\}$ is essential. By Theorem 5.8, an infinite sequence in $\Sigma_{x,y}$ corresponds to a unique chord $\{(x,s),(y,t)\}$. Since $\Omega_x \cap \Omega_y = \partial\Omega_x$ is a Jordan curve, there is a lift of the arc α formed by $\{(x,s),(y,t)\}$ under $(f^{-1})^{\circ p}$ to d_x distinct essential arcs joining x to y. Applying the isomorphism theorem again, we conclude that there are d_x preimages of the given element in $\Sigma_{x,y}$ under $\sigma^p|_{\Sigma_{x,y}}$.

We now prove the sufficiency. Let $\mathcal{T} \subset \{x\} \times S^1$ be the set of angles for chords of the form $\{(x,s),(y,t)\}$. The set $\mathcal{T}$ is closed by the continuity of $\overline{\phi}_x$. We will show $\mathcal{T}$ is dense in $\{x\} \times S^1$. By hypothesis and Theorem 5.8, $\mathcal{T}$ is nonempty and every chord $\{(x,s),(y,t)\}$ has d_x preimages of the form $\{(x,s'),(y,t')\}$. Since d_x is also the degree of $F_0^p|_{x\times S^1}$, every point in $\mathcal{T}$ has d_x preimages in $\mathcal{T}$, and so $\mathcal{T}$ is dense in the circle.

■

We now relate the touching of arbitrary Fatou components to the touching of periodic components. The proof of the following theorem follows immediately from the fact that there are no critical points in the Julia set.

Theorem 5.16 *Let* $f(z)$ *be a PFH rational map. Let* x, y *be two distinct points in the grand orbit of* $P(f)$*. Then*

1. *If* $\partial\Omega_x$ *is not a Jordan curve, then* $\partial\Omega_x$ *is also not a Jordan curve.*
2. *If* $\overline{\Omega_x} \cap \overline{\Omega_y} \neq \emptyset$*, then*

(a) If $f(x) \neq f(y)$, then $\overline{\Omega_{f(x)}} \cap \overline{\Omega_{f(y)}} \neq \emptyset$.

(b) If $f(x) = f(y)$, then $\partial\Omega_{f(x)}$ is not a Jordan curve.

Case 2(b) can occur for hyperbolic postcritically finite maps. For the semibasilica, the basin of the pole $p = 2/3$ touches the basin of infinity in two points, each of which map onto the unique separating point in the boundary of the basin of infinity.

Definition 5.17 (Sierpinski carpet) *A* **Sierpinski carpet** *is a closed subset of S^2 which is the complement of a countable dense family of open discs whose diameters tend to zero and whose closures are pairwise disjoint closed discs.*

Corollary 5.18 *The Julia set $J(f)$ is a Sierpinski carpet if and only if $A = \emptyset$, if and only if $\mathcal{A}_{evp}(P(f)) = \emptyset$, if and only if $\mathcal{A}_{per}(P(f)) = \emptyset$.*

Proof: Since f is expanding there are at most finitely many Fatou components with a given spherical diameter. Hence it suffices to show (1) no two Fatou components touch, and (2) every Fatou component has Jordan curve boundary. By repeated applications of Theorem 5.16, two Fatou components touch if and only if either two periodic Fatou components touch, or there exists a periodic Fatou component with non-Jordan curve boundary. Hence it suffices to prove (1) and (2) for the case of periodic Fatou components. Since $A = \emptyset$ by hypothesis, $\Sigma_{x,y} = \emptyset$ for every $x, y \in Q(f)$. Theorems 5.13 and 5.14 then imply that (1) and (2) hold for periodic components. The last two statements are clear.

■

In practice, it is easiest to verify the last condition.

Polynomials. In the special case where f is a polynomial, the CCS takes a special form. First, $\Sigma_{x,\infty}$ is nonempty for every x. For since ∞ is totally invariant, the preimage of any arc joining $x \in P(f)$ to infinity pulled back to a point $y \in Q(f)$ is also an arc joining y to ∞. Moreover, by Theorem 5.14, $\Sigma_{\infty,\infty}$ is empty if and only if the basin of infinity has Jordan curve boundary. The basin of infinity of a PFH polynomial is a Jordan curve if and only if it is conjugate to an elementary map. If $x \neq \infty$, $\Sigma_{x,x} = \emptyset$ since every bounded Fatou component has Jordan curve boundary. If $x, y \neq \infty$, then $|\Sigma_{x,y}| = 1$ by Theorem 5.8 and the fact that two bounded Fatou components can meet in at most one point.

Definition 5.19 (Starlike polynomial) *A PFH polynomial $f(z)$ is said to be starlike if there exists a finite connected graph $G \subset \mathbb{C}$ such that the following holds:*

- *$P(f) - \{\infty\} = V(G) = G \cap P(f)$, where $V(G)$ is the set of edges of G;*
- *$f : G \to G$ maps edges homeomorphically to edges, up to isotopy through maps fixing $P(f)$.*

Given G, we denote by G_{per} the periodic part of G.

Thus the edges of G form representatives of elements of $\mathcal{A}_{evp}(P(f))$. There is a natural generalization of the definition of starlike to families $\mathcal{P}$ of polynomials covering a mapping schema.

Definition 5.20 (PFH family of starlike polynomials) *Let $\mathcal{P}$ be a PFH family of polynomials covering the mapping schema (S, τ, w). $\mathcal{P}$ is said to be starlike if there is a finite graph $G \subset S \times \mathbb{C}$ with vertices $V(G)$ such that for all x, $G \cap (\{x\} \times \widehat{\mathbb{C}})$ is connected, and such that the following two conditions hold:*

- $P(\mathcal{P}) - (S \times \{\infty\}) = V(G) = G \cap P(\mathcal{P})$;
- *$\mathcal{P} : G \to G$ maps edges homeomorphically to edges, up to isotopy through maps fixing $P(\mathcal{P})$.*

Given G, we denote by G_{per} the periodic part of G.

In light of the discussion in the previous paragraph, the following proposition is immediate:

Proposition 5.21 *A PFH polynomial $f(z)$ is starlike if and only if for each pair of distinct points $x, y \in P(f) - \{\infty\}$, there exists a finite sequence $x = x_1, x_2, ..., x_n = y$ of points in $P(f)$ such that $\Sigma_{x_i, x_{i+1}} \neq \emptyset, i = 1, 2, ..., n-1$.*

Note that the restriction is on $\Sigma_{x,y}$ for pairs of points in $P(f)$, not $Q(f)$.

The notion of a *Hubbard tree* of a postcritically finite polynomial f was introduced in [DH1] as a way of combinatorially encoding the tree-like structure and combinatorial dynamics of its Julia set. It can be shown that a PFH polynomial f is starlike if and only if its Hubbard tree intersects the Julia set of f in a finite number of points, iff $\Sigma_{\infty,\infty}$ is finite.

Example: Let $f(z) = z^2 + c$ be any PFH quadratic polynomial which is the center of a hyperbolic component tangent to the main cardioid. Then $f(z)$ is starlike, and these are all of the starlike PFH quadratic polynomials of this form.

5.5 How tuning affects the Julia set

By Theorems 4.36 and 3.12, if $R = f * \mathcal{P}$, then there is an injection from the set of elements of A_f which do not intersect the support of the tuning into A_R which preserves the pushforward relation f_*^Q. Hence there is a continuous conjugacy from an invariant subspace of (Σ_f, σ_f) into (Σ_R, σ_R).

Conjecturally, the Julia set for $f * \mathcal{P}$ is equal to a quotient space of the Julia set for f. The idea is to transport the lamination for $\mathcal{P}$, regarded as a disjoint union of convex hulls of equivalence classes in the open unit disc, into the Fatou components $\{\Omega_x\}_{x \in Q(f)}$. By taking preimages of f, one transports these leaves and gaps to a fully invariant subset of $F(f)$. The Julia set for $f * \mathcal{P}$ is then conjecturally the quotient space obtained by collapsing the gaps and leaves to points.

Thus the above remark may be viewed as a first step in the verification of this conjectural picture.

Remark: I am grateful to Curt McMullen for suggesting this approach. See [McM3] for related arguments in the case of quadratic polynomials.

5.6 Examples

In this section we discuss several more examples illustrating the phenomena discussed in this chapter.

5.6.1 A Julia set which is a Sierpinski carpet

In [Mil4], an explicit example is given of a Julia set of a quadratic postcritically finite hyperbolic quadratic rational map whose Julia set is a Sierpinski carpet. The proof uses the theory of polynomial-like mappings and the symmetry of the map; the map is conjugate to a map which commutes with conjugation. The postcritical set of this map has nine points.

We now give an explicit example of a Julia set for a cubic postcritically finite hyperbolic rational map whose Julia set we shall prove to be a Sierpinski carpet using Corollary 5.18. Though of higher degree, the postcritical set is much smaller, consisting of only four points, and this makes the analysis much easier. The example we give is also real. Let $f_c(z) = c \cdot \frac{(z-1)^2(z+2)}{3z-2}$. This is equal to $g_{c,3}$ for the family $g_{r,d}$ defined in Section 5.1. There is a unique real value for c for which the following hold: (1) the origin is periodic of period three; (2) -2 is in the forward orbit of 0; (3) the image of 0 is strictly between 0 and 1; this occurs when $c \approx -0.695620....$ Let $f(z)$ denote the corresponding map. Then $P(f)$ consists of a cycle of period three and a fixed point. The Julia set of f is shown in Figure 5.10.

We first express this map as a branched covering of the sphere to itself which has the same mapping schema and ordering of postcritical set. We will then prove that there are no Thurston obstructions for this covering. By the uniqueness of the parameter value, this will imply that the candidate branched covering is combinatorially equivalent to $f(z)$. Figure 5.11 describes a covering $g(z)$ of the sphere. Laying the top picture over the bottom defines a branched covering of the sphere to itself.

Proof that g is combinatorially equivalent to a rational map. Since g has exactly four postcritical points, there is at most one element of any Thurston obstruction. Since g commutes with conjugation, and all of the postcritical points lie on the extended real axis, any Thurston obstruction must be symmetric with respect to the real axis, up to isotopy. But there are exactly two such curves: one enclosing the points -2 and 0, the other enclosing 0 and $-c$, and by examining the diagram in Figure 5.11 one can rule these out.

Proof that $J(f)$ is a Sierpinski carpet. By Corollary 5.18, it suffices to prove that $A = A_g = \emptyset$.

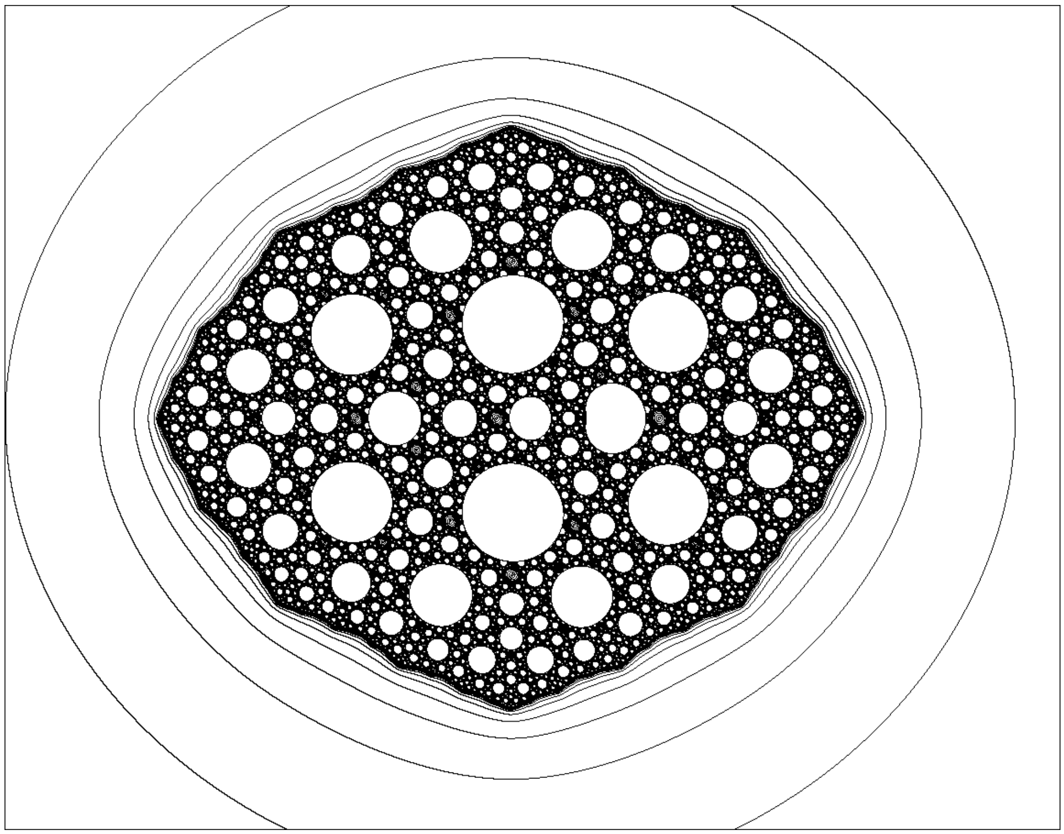

Figure 5.10: The Julia set of $f(z) = -0.695620\frac{(z-1)^2(z+2)}{3z-2}$ is a Sierpinski carpet. The Fatou component containing the pole $z = 2/3$ is too small to be visible at this scale. Equipotentials for the immediate basin of infinity are shown.

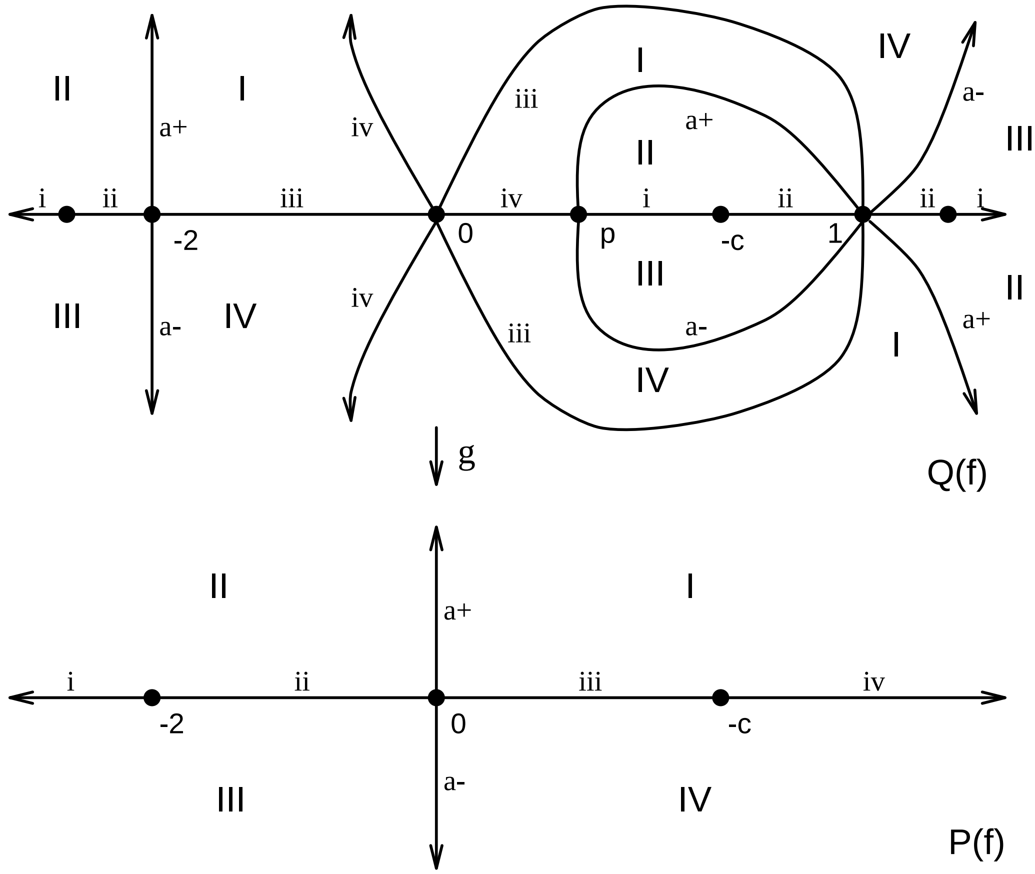

Figure 5.11: A triangulation expressing $g(z)$ as a branched covering of the sphere. Edges marked by small Roman numerals or signed letters map to edges marked by the same numeral. Faces marked by capital Roman numerals map to faces marked by the same capital Roman numeral. The pole of the map is denoted p. The top picture is to be overlaid the bottom to obtain a covering of the sphere to itself.

Since the map is real, the reflection of any periodic arc through the real axis is also a periodic arc. By Theorem 5.8, a pair of classes of periodic arcs is always representable by a pair of arcs associated to chords. The interiors of these arcs intersect in at most one point. It follows that any periodic class of arc is representable by an arc which is symmetric with respect to the real axis.

The proof now proceeds in several steps.

1. $\Sigma_{\infty,\infty} = \emptyset$. In this case, the possible arcs which may arise are determined by where they intersect the real axis. But an examination of Figure 5.11 shows that any essential arc joining infinity to itself is eventually inessential, hence cannot be periodic. Thus the basin of infinity has Jordan curve boundary.

2. $\Sigma_{\infty,x} = \emptyset, x \in P(f) - \{\infty\}$. Again, by the symmetry, any arc joining the point at infinity to a finite point in $P(f)$ is represented by an arc lying entirely in the upper or lower half-planes. But any arc joining the point $-c$ to ∞ and lying entirely in the upper half plane must pass through faces which map onto faces I, II, and IV, and hence its image cannot be symmetric with respect to the real axis. Thus the basin of infinity touches no other periodic Fatou components.

3. $\Sigma_{x,x} = \emptyset$, $x \in P(f) - \{\infty\}$. Any arc joining the point -2 with itself, symmetric with the real axis, and essential as an arc in $(S^2, P(f))$ maps to an arc which intersects the real axis at least twice, since it must intersect at least two edges which map to a common edge in the real axis. This cannot occur, hence every bounded Fatou component has Jordan curve boundary.

4. $\Sigma_{x,y} = \emptyset, x, y \in P(f) - \{\infty\}$. By the symmetry of the map, any periodic arc joining a pair of bounded points of the postcritical set must intersect the real axis in at most one point. If it intersects the real axis essentially (i.e. the point of intersection cannot be removed through an isotopy of the arc), the union of this arc with its reflection yields a pair of essential arcs meeting in a point. But this can occur if and only if a bounded Fatou component did not have Jordan curve boundary. Thus no two bounded Fatou components touch.

■

5.6.2 Blowing up an arc (with Tan Lei)

In this section, we present an informal discussion of Tan Lei's construction of "blowing up an arc". A more thorough treatment will be presented in a co-authored preprint in the near future.

Tan Lei has shown that examples like the semibasilica and semirabbit may be obtained topologically by a construction which she has called "blowing up an arc". This process is a topological operation on certain kinds of branched

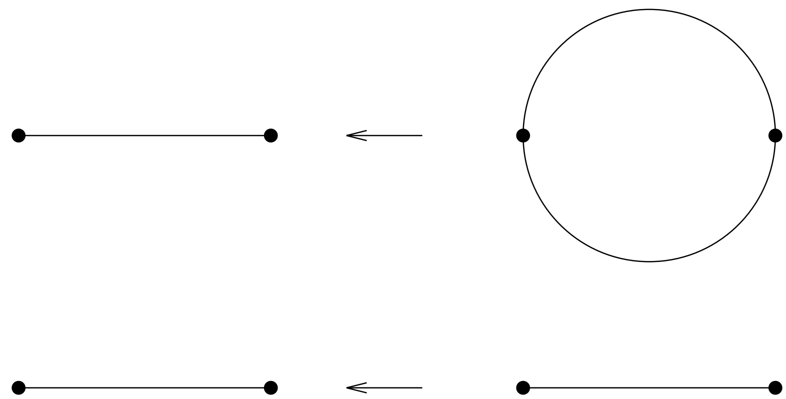

Figure 5.12: The interior of the disc maps homeomorphically to the complement of the arc β.

coverings which yield new branched coverings of one or several degrees higher. Using Theorem 4.29, we have been able to prove that intelligently blowing up an arc for a branched covering which is combinatorially equivalent to a rational map yields a new map which is combinatorially equivalent to a rational map.

This is a perhaps remarkable theorem since it is a general way of constructing new rational maps out of old ones, and is not limited to maps of a certain degree or to maps which possess some global combinatorial property. Moreover, the construction seems to be robust: empirically, it works for non-critically finite maps as well.

Let f be a postcritically finite branched covering of the sphere to itself. We do not assume that the map is nonelementary. It will be convenient to think of f as a covering from one copy of the sphere to another copy, followed by the inverse of an identification of the domain with the range. We will change the covering to change the map, and leave the identification alone. Let α be an essential arc in $(S^2, P(f))$ with *distinct* endpoints, and suppose that $\beta = f(\alpha)$ is an embedded arc which also has distinct endpoints.

We may form a new branched covering $\text{Blow}(f, \alpha, 2)$ as follows (the reason for the "2" will become clear momentarily). Cut the sphere along α. Sew in a copy D of the closed unit disc, and map D to the *complement* of $f(\alpha) = \beta$ in such a way that the interior maps homeomorphically and the boundary is a degree two branched covering. See Figure 5.12.

For example, one can realize this conformally via the map $z \mapsto z + 1/z$ which takes Δ to $\widehat{\mathbb{C}} - [-2, 2]$ and S^1 onto $[-2, 2]$. The resulting map can be easily shown to yield a branched covering of the sphere. The degree is increased by one. The postcritical set remains the same. The local degree near each endpoint of α is increased by one. The multiplicity $\text{mult}(\text{Blow}(f, \alpha, 2)) : [\alpha] \to [\beta])$ is increased by one.

Now, the map $z \mapsto z + 1/z$ is holomorphically conjugate to the map $z \mapsto z^2$.

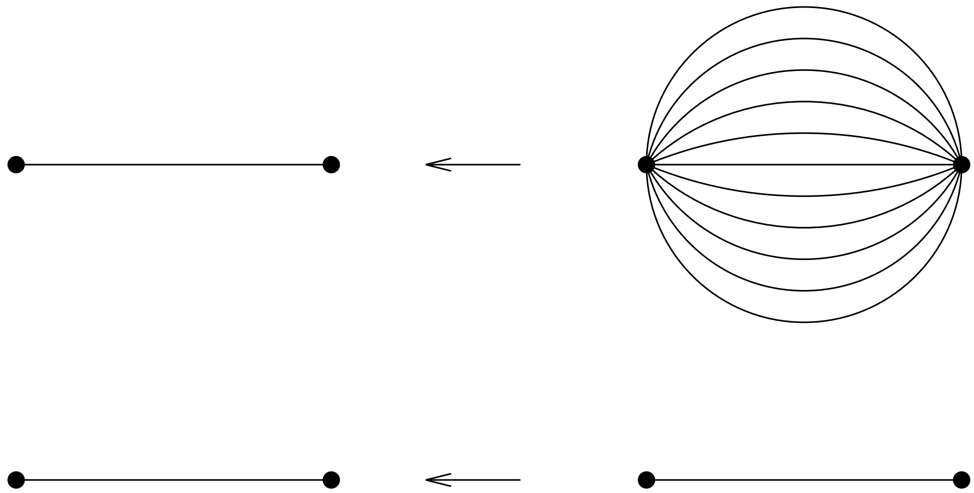

Figure 5.13: Gluing for Blow(f, α, d).

There are therefore variants of this construction. Instead of the open disc D mapping homeomorphically onto the complement of β, we may map D to the sphere punctured at the endpoints of β by a d-to-one map. See Figure 5.13.

There is a generalization of this construction where we cut not along the entire arc, but only along part of it. The postcritical set, however, now changes under this operation. For example, if α is fixed pointwise under f, we may blow up a sub-arc of α to obtain a map with two extra fixed critical points. We will refer to this more general construction as blowing up an arc as well.

To define this construction more precisely, it will be convenient to consider postcritically finite branched coverings with some additional associated data. This idea was suggested to me by A. Poirier.

Postcritically finite branched covering with additional marked preperiodic points. Let $f : S^2 \to S^2$ be a postcritically finite branched covering. Let $X \subset S^2$ be a finite subset containing $P(f)$. A map g combinatorially equivalent to f and satisfying $g(X) \subset X$ is called *f marked by the set X*.

Example: Let $f(z)$ be a postcritically finite rational map, and let X be any finite forward-invariant set of periodic and preperiodic points containing the postcritical set. Then $f(z)$ may be regarded as a postcritically finite branched covering marked by the set X.

Example: Let $f(z)$ be a postcritically finite rational map, and let R_t be a periodic or preperiodic ray for the map ϕ. Choose a point $x \neq 0$ on this ray. Then by postcomposing f by a suitable homeomorphism isotopic to the identity, we may assume that x is periodic of the same period as R_t if R_t is periodic, and similarly for the preperiodic case. Let g_1 denote this map. Then marking g_1 by the forward orbit of x together with the postcritical set produces an example of a branched covering marked by additional points.

There is a natural notion of combinatorial equivalence of postcritically finite branched coverings with additional marked points: two maps f, g marked by $X(f), X(g)$ respectively are combinatorially equivalent if they satisfy the definition of combinatorial equivalence given in Section 3.2 with the sets $P(f), P(g)$ replaced by the sets $X(f), X(g)$. Thus this definition of combinatorial equivalence reduces to the usual one in the case when $X = P(f)$. Given a branched covering f marked by X, one may define the pushforward relation on arcs with endpoints in X as for ordinary branched coverings. Thus it makes sense to speak of periodic and preperiodic arcs. We let $\mathcal{A}_{evp}(X(f))$ denote the set of eventually periodic arcs for f under the pushforward relation.

Blowing up an arc is a combinatorial construction on postcritically finite branched coverings marked by additional points. Note that when α and β are periodic, the combinatorial dynamics of the blown-up map on the complement of the forward orbit of α is essentially the same as the original map. One can make this precise by proving an analog of Theorems 4.31 and 4.36 for blowing up an arc.

The next theorem says when blowing up an arc in a rational map viewed as a postcritically finite branched covering marked by a set of points again yields a rational map.

Theorem 5.22 (Blowing up arcs in rational maps) *Let $f_1(z)$ be a postcritically finite rational map. Let $X \subset \widehat{\mathbb{C}}$ be a finite subset containing $P(f)$, and let f denote a map which is combinatorially equivalent to f_1 and satisfies $f(X) \subset X$. Regard f as a branched covering marked by X. Suppose α and β are chosen so that $[\alpha] \in \mathcal{A}_{evp}(X(f))$, and let $[\beta] \in f_*([\alpha])$. Let $g = \mathit{Blow}(f, \alpha, n)$.*

1. *If α is periodic, g is combinatorially equivalent to a rational map if and only if at least one endpoint of α is contained in a superattracting cycle of f_1.*

2. *If α is strictly preperiodic, g is combinatorially equivalent to a rational map if and only if at least one endpoint of α lands in a superattracting cycle of f_1.*

Proof: The necessity of the conditions is clear: otherwise, small neighborhoods of the periodic arcs in the forward orbit of α lift univalently under g along the orbit, forming a Levy cycle for g.

We now prove the sufficiency. It suffices to show there exist no reduced Thurston obstructions.

Suppose $[\alpha]$ is periodic for f. The class $[\alpha]$ is then periodic for g as well, by construction of g. Hence any Thurston obstruction intersecting the orbit of $[\alpha]$ must be a simple Levy cycle, by Theorem 4.29, Part 1(a). Since blowing up increases the multiplicity, by Theorem 4.29, Part 3, no Thurston obstruction can intersect the orbit of $[\alpha]$. But on the complement of the orbit of $[\alpha]$, the combinatorial dynamics of g is the same as that of f. Hence if g has a Thurston obstruction, so does f, a contradiction.

Now suppose that $[\alpha]$ is strictly preperiodic. Any reduced Thurston obstruction intersecting the orbit of α must be a Levy cycle. For an element of a Levy cycle may be pushed forward until it intersects a periodic cycle Σ in the forward orbit of $[\alpha]$, in which case Theorem 4.29 applies. By Part 1(b) of this theorem, such a Levy cycle must have zero intersection with the set of strict preimages of the elements of Σ. We may now reason similarly as in the previous case: a Levy cycle for g avoiding the orbit of α must yield a Levy cycle for f, a contradiction.

■

As a consequence of Theorems 5.13 and Theorems 5.14, we have

Corollary 5.23 *Let f and g be as above. Suppose the arc $[\alpha]$ which is blown up is periodic and has one endpoint x which is not in $P(f_1)$. Let y denote the other endpoint of $[\alpha]$. Then for the blown-up map g,*

1. *x is a periodic critical point;*
2. *$\partial\Omega_x$ Jordan curve and $\overline{\Omega}_x \cap \overline{\Omega}_y$ is a Jordan curve;*
3. *the basin of y for the blown-up map does not have Jordan curve boundary unless g is elementary.*

Proof: The first part follows from the definition of the construction. To see the second part, note that The local degree of the first return map of x under $\text{Blow}(f, \alpha, n)$ is equal to n. But n is also the multiplicity of α under $\text{Blow}(f, \alpha, n)$. By Theorem 5.15, the second part of the corollary follows. The third part then follows immediately: two open discs in the sphere with locally connected boundary cannot intersect in a Jordan curve unless either one disc is not a Jordan domain, or the both are Jordan domains meeting in a common boundary. The latter cannot occur unless the map g is elementary.

■

Example: The degree three semibasilica and semirabbit. By examining Figure 5.5, it can be shown that map yielding the degree three semibasilica is $\text{Blow}(z^2 - 1, [0, 1], 2)$, i.e. obtained from the map yielding the usual basilica by simple blowing up of the preperiodic arc $[0, 1]$. Similarly, the degree three semirabbit is obtained by blowing up a preperiodic segment contained in the filled Julia set and joining the point 0 to the unique non-periodic preimage of 0.

Example: The degree four semibasilica. This map is the same as $\text{Blow}(z^2 - 1, [0, 1], 3)$.

Example: Cubic polynomials. If $f(z) = z^2$, we may blow up an arc joining a periodic or strictly preperiodic point on S^1 to the point at infinity. The resulting cubic map has a totally invariant critical point at infinity, and so is a cubic polynomial.

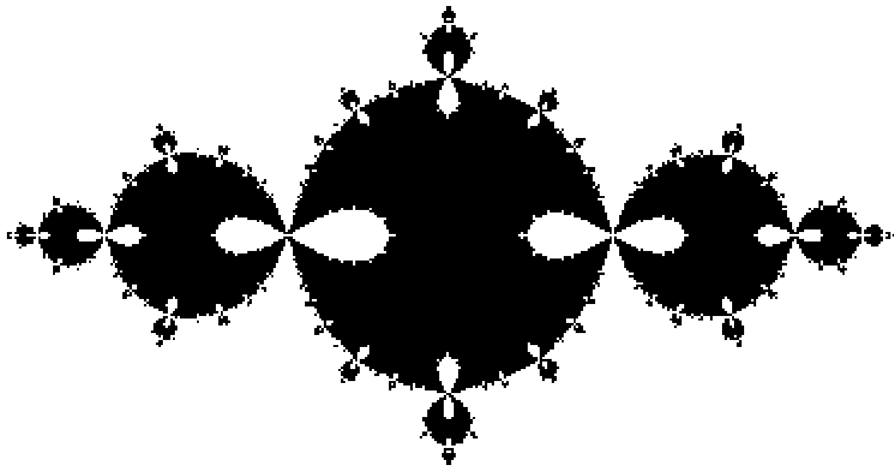

Figure 5.14: Blowing up a periodic arc.

Example: The family $g_c(z) = \frac{(z-1)^3}{\frac{3z}{2c+1}-1}$ **.** This family consists of the set of cubic rational maps for which infinity is a fixed simple critical point, the point one is a critical point of local degree three mapping onto zero, and zero maps onto one. There is one other simple critical point at the point c. This family contains all maps which are obtained by simple blowing up of the basilica along *internal* rays joining 0 to a point on the boundary of its immediate basin. For example, there is a unique value for $c \approx 0.768...$ for which c is real, periodic of period two, and lying strictly between zero and one; see Figure 5.14. This example is obtained as follows. The period two cycle of Fatou components for the basilica admit a unique parameterization by Riemann mappings given by Böttcher's theorem. Let R_0 denote the zero ray in the basin of zero and R_{-1} denote its image. By postcomposing $z^2 - 1$ with an isotopy we may assume that R_0 and R_1 are fixed pointwise under the second iterate of a map f combinatorially equivalent to $z \mapsto z^2 - 1$. Choose a point $x \in R_0$ which is not an endpoint. Choose a homeomorphism h such that for $f_1 = h \circ f$, $f_1^{\circ 2}(x) = x$. Mark f_1 by $\{x, f_1(x)\} \cup P(f_1)$. Let α denote the arc which is the segment in R_0 joining 0 to x. Then f_c for c as above is combinatorially equivalent to $\text{Blow}(f, \alpha, 2)$.

If we allow x to be the landing point of R_0, then x is fixed under the map f. The resulting map is combinatorially equivalent to g_c, where $c = 1/2$; see Figure 5.15.

A portion of the parameter space for this family is shown in Figure 5.16. The picture is centered at the point $c = 1$ and the distance from the center to the left edge is about $1/3$. The maps g_c corresponding to blowing up rays in the immediate basin of zero of angle p/q for q even are contained in the white outward-sticking bulges; those for q odd correspond to copies of the Mandelbrot set sticking inward. These copies have very small diameter and do not appear

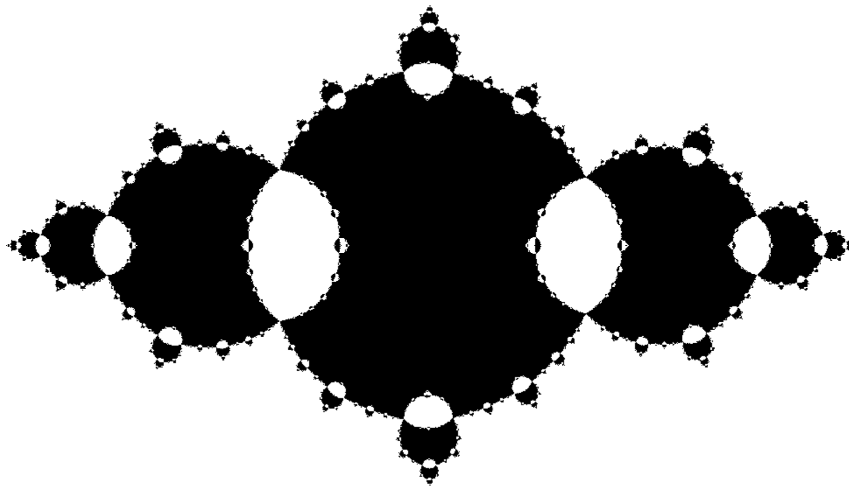

Figure 5.15: Blowing up R_0.

clearly at this scale. The semibasilica is contained in the center of the white region at the far right. The maps in the white region in the center of the picture have disconnected Julia sets.

Tan Lei has done a combinatorial study of this family, and has found it to be intimately related to the mating of cubic polynomials. Let S be the set of cubic polynomials with connected Julia set and one critical point fixed. This set was studied by D. Faught [Fau], J. Milnor in [Mil3], and by Tan Lei in [Tan1]. Tan Lei (unpublished) has shown that any element map $q \in S$ which is matable with the map $p_0(z) = -(z-1)^3$ occurs in this family.

Remark: Tan Lei (personal communication) has also observed that some maps with disconnected Julia set in this family arise by quasiconformal surgery on quadratic maps with connected Julia set.

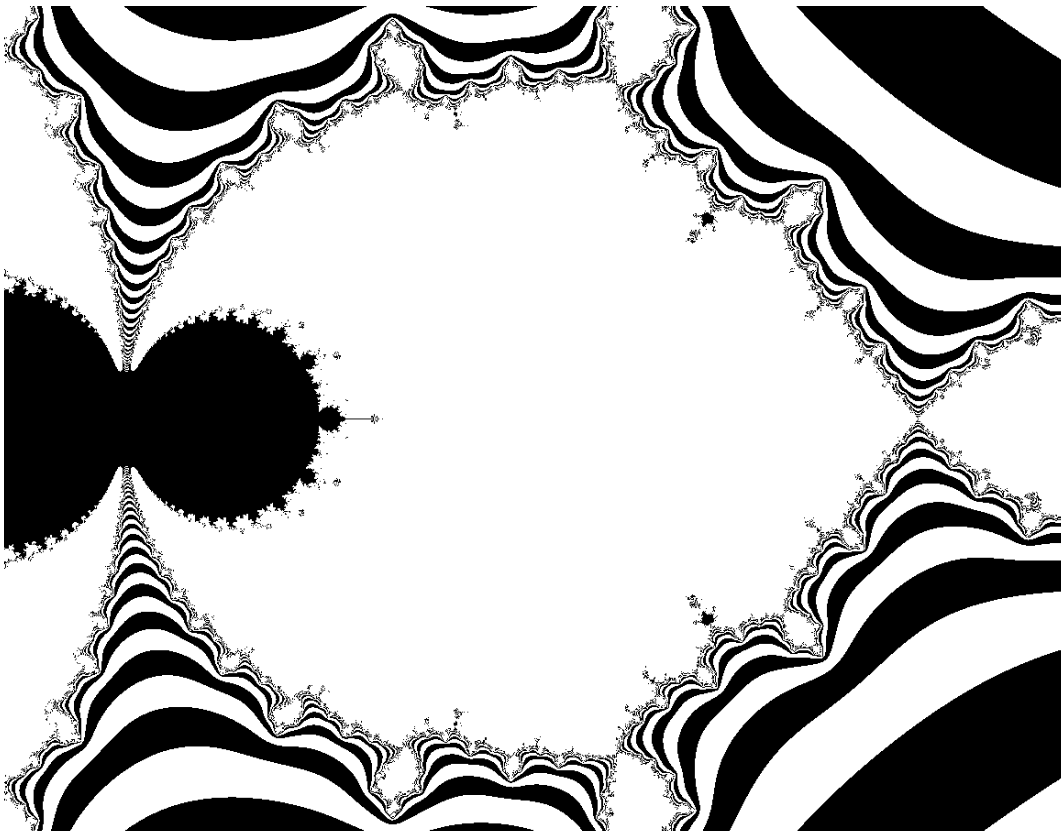

Figure 5.16: Parameter space for g_c

Chapter 6

Cylinders

In this chapter we define cylinders for a rational map, in analogy with cylinders for hyperbolic three-manifolds.

We will actually define two kinds of cylinders. Let f_0 be a postcritically finite hyperbolic rational map. Combinatorial cylinders will be defined for f_0 using the combinatorial characteristic subcomplex. We prove

Theorem 6.1 (Characterization of cylindrical) *A PFH rational map $f_0(z)$ is combinatorially cylindrical if and only if there exists a finite set $\mathcal{R}$ of internal rays of f such that*

1. *$f(\mathcal{R}) = \mathcal{R}$ (periodicity),*
2. *the union of the rays in $\mathcal{R}$ separates the sphere (separation), and*
3. *no proper subset of $\mathcal{R}$ satisfies (1) and (2) above (minimality).*

Geometric cylinders will be defined for maps f_1 in the hyperbolic component $H(f_0)$ in parameter space which contain no periodic critical points. We prove

Theorem 6.2 *There is a bijection between geometric cylinders for f_1 and combinatorial cylinders for f_0.*

For compact three-manifolds, the Annulus Theorem [Jac] asserts that a three-manifold which has a cylinder has an embedded cylinder. We define embedded combinatorial and embedded geometric cylinders as well, and formulate several conjectures relating the kinds of cylinders.

Section 6.1 contains the definition of combinatorial cylinder and proves the first theorem. Section 6.2 contains the definition of geometric cylinder. Section 6.3 proves the second theorem. Sections 6.4 and 6.5 define embedded combinatorial and geometric cylinders. We conclude in Section 6.6 with a list of conjectures.

6.1 Combinatorial cylinders

In this section, we define combinatorial cylinders for a PFH rational map f and relate them to the topology and dynamics of $J(f)$.

Definition 6.3 *A finite set W of arcs in $(S^2, Q(f))$ is said to* **separate** $Q(f)$ *if the set*

$$\mathcal{T} = \bigcup_{\alpha \in W} \alpha$$

contains a simple closed curve γ such that each component of $\widehat{\mathbb{C}} - \gamma$ contains a point of $Q(f)$. A finite set of classes of arcs $[\Sigma] \subset \mathcal{A}(Q(f))$ is said to **separate Q(f)** *if every set of representatives of $[\Sigma]$ separates $Q(f)$.*

Let f be a postcritically finite hyperbolic rational map. Let A be the set of eventually periodic classes of arcs in $\mathcal{A}(Q(f))$ under the Q-pushforward relation. Let (Σ_f, σ_f) be the combinatorial characteristic subcomplex associated to f.

Definition 6.4 *A* **combinatorial cylinder** *is a finite subset $C \subset \Sigma_f$ satisfying*

1. **Periodicity.** $\sigma_f(C) = C$.
2. **Separation.** *Let $A(C)$ denote the union of all classes in A which appear as a term in any element of C. Then $A(C)$ separates $Q(f)$.*
3. **Minimality.** *No proper subset of C satisfies the previous two conditions.*

The **length** *of a cylinder is the number of distinct endpoints in the classes in $A(C)$. An n-***cylinder** *is a cylinder of length n. A map is said to be* **combinatorially n-cylindrical** *if it has a cylinder of length n. The* **period** *of a cylinder is the least common multiple of the periods of elements of C under σ_f.*

Recall that the map $G : (\Sigma_f, \sigma_f) \to (\chi_{ess}, F_0)$ is an isomorphism, by Theorem 5.8.

Theorem 6.1 follows immediately from

Theorem 6.5 (Chords for cylinders separate sphere) *Let $f(z)$ be PFH rational map which is not postcritically elementary. Let $C \subset \Sigma_f$ be a finite subset. Then C is a combinatorial cylinder if and only if*

1. *$G(C)$ satisfies $F_0(G(C)) = G(C)$.*
2. *Let W be the collection of arcs formed by the chords in $G(C)$, and let $\mathcal{T}$ be the union of the set of arcs in W. Then $\widehat{\mathbb{C}} - \mathcal{T}$ is disconnected.*
3. *no proper sub-collection of chords in $G(C)$ satisfies the previous two conditions.*

Our proof of this theorem will depend on the following lemma.

Lemma 6.6 (**Q-separation lemma**) *Let f be a PFH branched covering of the sphere to itself which is not postcritically elementary. Let W be a finite set of arcs in $(S^2, Q(f))$, and let $\mathcal{T} = \cup_{\alpha \in W} \alpha$. Suppose $f|_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ is a homeomorphism. If W does not separate $Q(f)$, then either*

1. *$\mathcal{T}$ is a finite union of disjoint trees in S^2, or*
2. *There is a collection $\mathcal{D}$ of closed discs in S^2 whose interiors are contained in $S^2 - (\mathcal{T} \cup Q(f))$, whose boundaries are simple closed curves in $\mathcal{T}$, and which cycle under f.*

Note that W is a collection of arcs, not isotopy classes of arcs. The hypotheses imply that every arc is periodic as a subset of the sphere.

Proof: Suppose $\mathcal{T}$ is not a disjoint union of trees. Since W does not separate $Q(f)$, there exist simple closed curves contained in $\mathcal{T}$ which bound open discs in $\widehat{\mathbb{C}} - (\mathcal{T} \cup Q(f))$. Any such curve γ bounds a unique disc D in $\widehat{\mathbb{C}} - (\mathcal{T} \cup Q(f))$, since f is not postcritically elementary.

Hence there exist simple closed curves $\gamma_1, \gamma_0 \subset \mathcal{T}$ such that the following hold:

- γ_i bounds a unique discs D_i in $\widehat{\mathbb{C}} - (\mathcal{T} \cup Q(f)), i = 1, 2$;
- γ_0 is a lift of γ_1;
- there is a component D_0' of $f^{-1}D_1$ such that $D_0' \subset D_0$.

Condition (3) holds since $\widehat{\mathbb{C}} - \overline{D_0}$ contains points of $Q(f)$. For if no component of $f^{-1}(D_1)$ is contained in D_0, then $\widehat{\mathbb{C}} - \overline{D_0}$ is a component of $f^{-1}(D_0)$. The open disc D_0 contains no points of $Q(f)$ by hypothesis, hence if condition (3) fails, then $\widehat{\mathbb{C}} - D_0$ contains no points of $f^{-1}Q(f) \supset Q(f)$, a contradiction to the fact that f is not postcritically elementary.

We will show that $D_0' = D_0$; this will prove the lemma. Let $\gamma_0' = \partial D_0'$. Then γ_0' is a Jordan curve in $\widehat{\mathbb{C}}$. Let us call $\gamma_i \cap Q(f)$ the *vertices* of $D_i, i = 1, 2$. Call the set of points $\gamma_0' \cap Q(f)$ the *vertices* of D_0'. Note that the vertices of D_1 are points in $P(f)$, since they are periodic. Hence the vertices of D_0' are points of $Q(f)$. A vertex of D_0' cannot be contained in the interior of D_0, since by hypothesis D_0 contains no points of $Q(f)$ in its interior. Since $\gamma_0 \subset \mathcal{T}$, γ_0 is periodic under f. Hence the set of vertices of D_0' is equal to the set of vertices of D_0. Let $\{\alpha_i\}_{i=1}^n$ be the subset of W consisting of arcs that are contained in γ_1. Let $\{\eta_i\}_{i=1}^n$ denote the subset of arcs in W that are contained in γ_0. Then $f(\eta_i) = \alpha_i, i = 1, 2, \ldots n$. Let $\{\eta_i'\}_{i=1}^n$ denote the set of arcs in $(S^2, Q(f))$ formed by γ_0', labelled so that $f(\eta_i') = \alpha_i, i = 1, 2, \ldots, n$. Each η_i is either isotopic in $(S^2, Q(f))$ to a unique arc η_i' which is contained in γ_0', since D_0 contains no points of $Q(f)$, or coincides with some η_i'. The former case cannot occur if f is not postcritically elementary, by Proposition 4.4. Hence $D_0 = D_0'$.

∎

Proof of Theorem 6.5.

We prove sufficiency by contradiction, using the expansion of the Poincaré metric and the Q-separation lemma. Since G is a conjugacy, if $G(C)$ is and finite satisfies $F_0(G(C)) = G(C)$, then C is finite and satisfies $\sigma_f(C) = C$. Let W be the set of arcs formed by the chords $G(C)$. Let $\mathcal{T} = \cup_{\alpha \in W} \alpha$. Then $\mathcal{T}$ is the union of an invariant set of periodic internal rays for f. Suppose that $\mathcal{T}$ separates the sphere. We will show that $A(C)$ separates $Q(f)$.

Two arcs $\alpha, \beta \in W$ intersect only in their endpoints, or in a single point in $J(f)$. Therefore if W separates points of $Q(f)$, then there exists a collection of arcs $W' \subset W$ whose elements represent distinct isotopy classes in $(S^2, Q(f))$, and which separates $Q(f)$. Moreover, since the elements of W' intersect minimally, the classes $[W']$ must separate $Q(f)$. Hence it suffices to prove that W separates $Q(f)$.

Suppose that W did not separate $Q(f)$. Since the chords $G(C)$ satisfy $F_0(G(C)) = G(C)$, $f(W) = W$. The set $\mathcal{T}$ is not a union of disjoint trees, since $\mathcal{T}$ separates the sphere. Hence by the Q-separation lemma, the collection $\mathcal{D}$ of components of $\widehat{\mathbb{C}} - \mathcal{T}$ that bound discs in $Q(f)$ cycle under f.

We will show this is impossible, using the fact that f expands the Poincaré metric.

Suppose $\{D_i\}_{i=0}^{p-1}$ is one such cycle of open discs in $\widehat{\mathbb{C}} - (\mathcal{T} \cup Q(f))$.

Case 1: *For some i, ∂D_i meets the Julia set in at least two points a, b.* Since the D_i are invariant, there exists a topological arc L joining a to b in $\overline{D_i} - P(f)$ which is invariant under $f^{\circ p}$, up to isotopy in D_i through maps fixing the endpoints of L. Let $L_n = (f^{\circ p}|_{D_i})^{-n}(L)$. Then L_n is homotopic to L for all n. However, since f expands the Poincaré metric on $\widehat{\mathbb{C}} - P(f)$ uniformly off of a neighborhood of $Q(f)$, the diameters of the L_n must tend to zero. It follows that the endpoints of L cannot be distinct, a contradiction.

Case 2: *For every i, ∂D_i meets the Julia set in exactly one point.* Then ∂D_i consists of the union of a pair of rays R_{x_i,s_i}, R_{x_i,t_i} from a point $x_i \in P(f)$ to itself. But by Theorem 5.6, the chord $\{(x_i, s_i), (x_i, t_i)\}$ is eventually essential under iteration. Hence the arc formed by this chord is essential in $(S^2, Q(f))$. Since the endpoints of this arc are the same, this implies that this arc forms a simple closed curve in $(S^2, Q(f))$ separating points of $Q(f)$.

We now show the minimality of the set of classes $A(C)$. Given any proper subset $C' \subset C$, we have $G(C') \subset G(C)$. Hence by condition (3), the union of the arcs forming the chords $G(C')$ cannot separate the sphere. But these arcs represent the classes $A(C')$, by Theorem 5.8, hence $A(C')$ does not separate $Q(f)$.

We now show necessity. If C is a cylinder, the set of classes $A(C)$ separate $Q(f)$. $G(C)$ is a collection of chords whose arcs W represent the classes $A(C)$, by Theorem 5.8. Hence W separates $Q(f)$, and so the set $\cup_{\alpha \in W} \alpha$ separates the sphere. It remains to show the minimality of the set W. If some sub-collection W' satisfied conditions (1) and (2). By Theorem 5.8, there is a sub-collection C' such that the arcs formed by $G(C')$ is the collection W'. By the sufficiency,

C' is a cylinder. By the minimality condition, $C' = C$, hence W is minimal with respect to conditions (1) and (2).

■

6.2 Geometric cylinders for rational maps.

In this section we define geometric cylinders for rational maps. Let f_1 be a hyperbolic rational map without periodic critical points. Geometric cylinders are finite collections of isotopy classes of simple closed curves in the quotient Riemann surface of f_1, which we define below. These surfaces are analogous to the Riemann surfaces at infinity for a geometrically finite hyperbolic manifold with infinite volume.

Quotient Riemann surface of a rational map. Our definition is taken from [MS]. Let $f(z)$ be an arbitrary rational map. The *grand orbit equivalence relation* of f is the equivalence relation on $\widehat{\mathbb{C}}$ defined by $x \sim y$ if there exist positive integers m and n such that $f^{\circ m}(x) = f^{\circ n}(y)$. Let $\widehat{J}(f)$ denote the closure of the grand orbit equivalence classes of all periodic and postcritical points. Let $\widehat{F}(f)$ denote the complement of $\widehat{J}(f)$. The map f acts on $\widehat{F}(f)$ as a covering transformation. Let $\widehat{F}_{dis}(f)$ denote the subspace of $\widehat{F}(f)$ on which the iterates of f act discretely; this subspace is equal to the grand orbit of all attracting and parabolic Fatou components minus the set $\widehat{J}(f)$. The quotient $X(f) = \widehat{F}_{dis}(f)/ < f >$ will be called the *quotient Riemann surface* of f. This surface is sometimes empty. We let $\pi : \widehat{F}_{dis}(f) \to X(f)$ denote the quotient map.

The components of $X(f)$ are finitely punctured tori (one for each cycle of attracting, but not superattracting, Fatou components), or punctured copies of $\mathbb{C}^*$ (one for each cycle of parabolic Fatou components). Each component of this surface is hyperbolic, i.e. admits a conformal metric of constant curvature -1. The number of such components is bounded by $2d-2$ where d is the degree of f, since every attracting or parabolic cycle of Fatou components contains a critical point. The number of punctures can vary, however, since two critical points can have the same grand orbit.

Geometric cylinders. Let $f_1(z)$ be a hyperbolic rational map without periodic critical points. Then $X(f_1)$ is a finite union of punctured tori.

Lifts of curves. Suppose $\gamma \subset X(f_1)$ is a closed curve, not necessarily simple, such that there is a cycle $W(\gamma)$ of components of $\pi^{-1}(\gamma)$. We call the set W, if it exists, the *lift* of γ. Each component of $W(\gamma)$ is a curve in $F(f)$ which cycles under f and limits, at one end, at an attracting periodic point of f_1. If the lift exists, it is unique.

Note that the only closed curves for which lifts do not exist are closed curves which lift under π^{-1} to closed curves which separate a point in the grand orbit of an attracting cycle from the Julia set. The property of a closed curve having a lift is an invariant of its homotopy class. A homotopy class of closed curve which has a lift is called *liftable*.

Definition 6.7 (Geometric cylinder) *A* **geometric cylinder** *for f_1 is a finite collection $[\Theta] = \{[\theta_i]\}_{i=1}^n$ of distinct liftable isotopy classes of closed curves on $X(f_1)$ such that the following holds. Given any collection $\Gamma = \{\gamma_i\}_{i=1}^n$ such that $[\gamma_i] = [\theta_i], i = 1, 2, ...n$, the set $W = \bigcup_{\gamma in \Gamma} W(\gamma)$ satisfies:*

1. **Periodicity.** $f_1(W) = W$.
2. **Separation.** $\overline{W}$ *separates the sphere.*
3. **Minimality.** *No proper subset of Γ has the previous two properties.*

The set W is called the lift of Γ. The **length** *of $[\Theta]$ is the number of Fatou components intersecting W. The* **period** *of $[\Theta]$ is the least common multiple of the periods of the components of W.*

The period and length are invariants of $[\Theta]$.

We do not require that the set each class in $[\Theta]$ is representable by a simple closed curve, nor do we require that pairs of distinct classes in $[\Theta]$ have zero intersection number.

6.3 A bijection between combinatorial and geometric cylinders

In this section we prove

Theorem 6.8 (Bijection of cylinders) *Let $f_0(z)$ be a postcritically finite hyperbolic rational map and let $f_1(z)$ be any map in $H(f_0)$ without periodic critical points. Then there is a bijection Ψ from the set of geometric cylinders of f_1 to the set of combinatorial cylinders of f_0 which preserve length and periods.*

To prove Theorem 6.8 we will need to some facts about the topology of plane sets. Let K be a nondegenerate full set in $\mathbb{C}$.

Definition 6.9 (Access) *Let $x \in \partial K$. An* **access** α **to** x **from** $\widehat{\mathbb{C}} - K$ *is the image of $[0,1]$ under an embedding ρ into the sphere such that $[0,1) \subset \widehat{\mathbb{C}} - K$ and $\rho(1) = \{x\}$. A point $x \in \partial K$ is said to be* **accessible** *if there exists an access to x from $\widehat{\mathbb{C}} - K$. Two accesses α, β to x are said to be* **homotopic** *if there is a continuous one-parameter family of accesses to x joining them.*

Proposition 6.10 *Let K be a nondegenerate full locally connected set in $\mathbb{C}$. Let $\phi : (\Delta, 0) \to (\widehat{\mathbb{C}} - K, \infty)$ be a Riemann map. Then a homotopy class of access to x determines a unique ϕ-ray landing at x in the given homotopy class.*

The proof depends on the following construction, due to Carathéodory, which we take from [Mil2], Chapter 15. This proof was sketched to me by McMullen.

Let $U = \widehat{\mathbb{C}} - K$. A *transverse arc* is a set $A \subset \overline{U} - \{\infty\}$ which is homeomorphic to $[0, 1]$ and which intersects ∂U only at its endpoints. The *neighborhood* $N(A)$ of a transverse arc is the component of $U - A$ not containing ∞. A *fundamental chain* is an infinite sequence $A_1, A_2, ...$ of disjoint transverse arcs such that the corresponding neighborhoods are nested, i.e. $N(A_1) \supset N(A_2) \supset ....$ Two fundamental chains $\{A_i\}_{i=1}^{\infty}, \{B_i\}_{i=1}^{\infty}$ are *equivalent* if each $N(A_i)$ contains some $N(B_j)$ and each $N(B_j)$ contains some $N(A_i)$. An equivalence class of fundamental chains is called a *prime end* of U. Any two equivalent chains are either equivalent, or are disjoint.

The *Carathéodory completion* of $\widehat{U}$ of U is the union of U and the space of prime ends of U, topologized as follows. For any transverse arc A, define a neighborhood $\mathcal{N}(A)$ which is the union of $N(A)$ and the set of prime ends containing a representative fundamental chain which is contained in $N(A)$. If $U = \Delta \subset \widehat{\mathbb{C}}$, then the identity map determines a homeomorphism of $\widehat{U}$ with $\overline{\Delta}$. More generally,

Proposition 6.11 *The map* $\phi^{-1} : (U, \infty) \to (\Delta, 0)$ *extends uniquely to a homeomorphism between* $\widehat{U}$ *and* $\overline{\Delta}$

See e.g. [Mil2], Theorem 15.9 for the proof.

We now prove the theorem. Let $\alpha_t, t \in [0, 1]$ be a homotopy between accesses α_0, α_1 to a point $x \in \partial K$. Since K is locally connected, ϕ is continuous on $\overline{\Delta}$, hence there are unique arcs $\widetilde{\alpha}_i \subset \overline{\Delta}$ such that $\phi(\widetilde{\alpha}_i) = \alpha_i, i = 0, 1$. By the Theorem of F. and M. Riesz ([Car], Section 313), each arc $\widetilde{\alpha}_i$ has a unique limit point $t_i \in S^1$. Let R_{t_i} be the ϕ-ray of angle t_i.

It is enough to prove that $t_0 = t_1$. By the above theorem, it is enough to show that there is a single fundamental chain $\{A_i\}_{i=1}^{\infty}$ such that for for any i, there is an $\epsilon(i)$ such that for all t, the set $\alpha_t((1-\epsilon, 1)) \subset N(A_i)$. But this is clear since the homotopy α_t is continuous and fixes $\alpha_t(1) = \{x\}$.

∎

Invariant accesses. Let $f(z)$ be hyperbolic rational map. Suppose that Ω is a periodic Fatou component and $x \in \partial\Omega$. An **invariant access to x from** Ω is an access α of x contained in $\overline{\Omega}$ such that $f^{\circ n}(\alpha) \supset \alpha$ for some $n > 0$. The point x is called the **landing point** of α. Two invariant accesses are said to be **homotopic** if they are joined by a continuous one-parameter family of invariant accesses. A set W' of homotopy classes of invariant accesses is called a *cycle* if f acts transitively on W' and $f(W') = W'$, up to homotopy. If W is a finite collection of cycles of invariant accesses, we say that W *separates* $J(f_1)$ if $J(f_1) - \overline{W}$ is disconnected.

We will use the following proposition in the proof of the next theorem. Its proof follows immediately from the definition of homotopy of accesses.

Proposition 6.12 *Let f_1 be a hyperbolic rational map without periodic critical points. Let $\gamma \subset X(f_1)$ be a simple closed curve. Then the lift $W(\gamma)$ is a cycle of invariant accesses whose homotopy classes depend only on the homotopy class of $[\gamma]$.*

Proof of Theorem 6.8.
Let $f_1 \in \mathrm{Rat}_d$ be a hyperbolic rational map contained in the hyperbolic component of a postcritically finite hyperbolic map f_0 in parameter space.

Let us denote by

- $\mathcal{GC}(f_1)$, the set of geometric cylinders for f_1,
- $\mathcal{SA}(f_1)$, the set of finite collections W of homotopy classes of cycles of invariant access which separate $J(f_1)$
- $\mathcal{SA}(f_0)$, the set of finite collections W of homotopy classes of cycles of invariant accesses which separate $J(f_0)$
- $\mathcal{CC}(f_0)$, the set of combinatorial cylinders of f_0.

By Theorem 6.5 and Proposition 6.10, there is a bijection between $\mathcal{CC}(f_0)$ and $\mathcal{SA}(f_0)$. By Theorem 2.21, there is a conjugacy between f_0 and f_1 on a neighborhood of $J(f_0)$ and $J(f_1)$ respectively. This conjugacy transports homotopy classes of invariant accesses to points in $J(f_0)$ to homotopy classes of invariant accesses to points in $J(f_1)$. A collection of accesses which separates $J(f_0)$ is mapped to a collection of accesses which separates $J(f_1)$, since h is a homeomorphism on a neighborhood of $J(f_0$. By the previous proposition, there is a bijection between the set of cycles of homotopy classes of closed curves on $X(f_1)$ and the set of cycles of invariant accesses to points in $J(f_1)$. This gives a bijection between $\mathcal{SA}(f_1)$ and $\mathcal{GC}(f_1)$, by the definition of geometric cylinder. Composing these bijections gives the desired bijection Ψ.

∎

6.4 Embedded combinatorial cylinders.

In this section, we define embedded combinatorial cylinders, and give a nontrivial example of a cubic polynomial with no embedded combinatorial 1-cylinders.
Rotation numbers. Let F be a family of local homeomorphisms of S^1 covering the mapping schema (S, τ, w), and let $E \subset S \times S^1$ be any finite subset for which $F(E) = E$. We say that E has a *generalized rotation number* if there exists an orientation-preserving homeomorphism H of $S \times S^1$ such that $H|_{E_{per}} = F|_{E_{per}}$.

We now define embedded combinatorial cylinders. Recall that the map $G : (\Sigma_f, \sigma_f) \to (\chi_{ess}(f), F_0)$ gives a conjugacy from the combinatorial characteristic subcomplex to the action of f on the space of essential chords. A chord is a pair of points in $Q(f) \times S^1$ which are equivalent for the lamination of f.

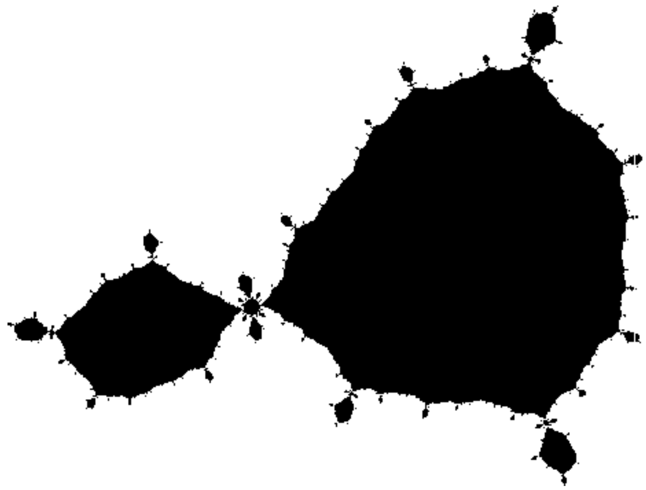

Figure 6.1: A cubic polynomial without an embedded combinatorial 1-cylinder.

Definition 6.13 (Embedded combinatorial cylinder.) *Let C be a combinatorial cylinder for a postcritically finite hyperbolic rational map $f(z)$. The cylinder C is said to be* **embedded** *if $G(C) \subset Q(f) \times S^1$ has a generalized rotation number.*

Example: Let $f(z) = z^2 - 1$. Let η denote the union of the 1/3 and 2/3 rays, together with the point at infinity. Recall that η is forward-invariant, and so its class is also forward-invariant. Then the orbit $\{[\eta]\}_{i=1}^{\infty}$ is a combinatorial 1-cylinder of period one. The image $G(\{[\eta]\})$ consists of the points $\exp(2\pi i 1/3)$ and $\exp(2\pi i 2/3)$, which are interchanged under $F_0 = z^2$. Thus this cylinder is also embedded.

Example: A cubic polynomial with no embedded 1-cylinders.

Using Tan Lei's blowing up construction, we may obtain examples of polynomials with 1-cylinders which are not embedded.

First, note that since the boundaries of bounded Fatou components for polynomials are all Jordan domains, there are no 1-cylinders whose endpoints are not the point at infinity.

To construct our example, let R_t denote the ray of external angle t for the basin of infinity of the map $f(z) \mapsto z^2$. Consider $f(z)$ marked by the forward orbit of the landing point x_0 of the ray $R_{3/5}$. Note that the ray $R_{3/5}$ does not have a well-defined rotation number as seen from infinity, since $1/5 \mapsto 3/5 \mapsto 4/5 \mapsto 2/5 \mapsto 1/5$ under angle tripling. Let $f_0(z) = \mathrm{Blow}(f, R_{3/5}, 2)$. The result is the cubic polynomial $f_0(z) = z^3 + c \cdot z^2$ for $c \approx 1.151613988... + 0.435445479...i$, up to affine conjugacy; see Figure 6.1.

The high precision is necessary since the diameters of the period four Fatou components are quite small. The lamination for this polynomial has exactly four periodic leaves $l_0 = 29/80 - 56/80$, $l_1 = 7/80 - 8/80$, $l_2 = 21/80 - 24/80$, and $l_3 = 63/80 - 72/80$. The map $f_0(z)$ sends l_i to l_{i+1} mod 4. Since the rays defining the endpoints of these leaves do not have a combinatorial rotation number, but do separate the sphere, the union of these rays form a combinatorial cylinder which is not embedded. Moreover, by construction, there are no periodic rays joining infinity to itself other than the ones determined by the angles for the l_i. Hence there are no other 1-cylinders, and so there are no embedded 1-cylinders.

Remark: One may also construct this example by using Hubbard trees; see e.g. [Poi].

6.5 Embedded geometric cylinders

In this section, we define embedded geometric cylinders.

Let $f_1(z)$ be a hyperbolic rational map without periodic critical points. Let $X(f_1)$ be the quotient surface of f_1.

Definition 6.14 (Embedded geometric cylinder.) *An* **embedded geometric**
cylinder *for f_1 is a geometric cylinder $[\Theta]$ which is representable by a collection of disjoint simple closed curves.*

Thus an invariant lift of an embedded geometric cylinder consists of disjoint arcs in the Fatou set.

Example: It is possible to show that the example of Figure 6.1 has no perturbations in its hyperbolic component which have embedded geometrically 1-cylinders.

If we perturb $f_0(z)$ slightly near infinity so that the resulting map $f_1(z)$ has an attracting fixed point at infinity and a single critical point in the basin of infinity of multiplicity 2, the quotient surface $X(f_1)$ becomes a once-punctured torus. Since f_0 is hyperbolic, the maps f_0 and f_1 are quasiconformally conjugate on some neighborhood of their Julia sets, by Theorem 2.21. It follows that the map $f_1(z)$ has a geometric cylinder consisting of two classes of curves on a punctured torus.

We claim these geodesics are not disjoint and simple, hence that the cylinder formed by them is not embedded. The map f_1 has a cycle of invariant accesses without a rotation number. It follows that these invariant accesses cannot be disjoint. Lifting f_1 on the basin of infinity to the unit disc via a Riemann map, one obtains a degree three Blaschke product with a single critical point and a cycle of arcs joining an attracting fixed point to a cycle of points on the circle without a rotation number. It follows that these arcs must intersect. Hence the images of these invariant accesses under the projection map π to $X(f_1)$ must intersect.

6.6 Conjectures about cylinders

In this section we discuss several conjectures about cylinders.

Conjecture 6.15 *Let $f_0(z)$ be a postcritically finite hyperbolic rational map and let C_0 be a combinatorial embedded cylinder. Then there exists a perturbation of f_0 to a map f_1 without periodic critical points, and a choice of conjugacy h from f_0 near $J(f_0)$ to f_1 near $J(f_1)$, such that the bijection Ψ in Theorem 6.8 sends the cylinder C_0 to an embedded geometric cylinder Γ_1*

A weaker version of this conjecture is

Conjecture 6.16 *Let $f_0(z)$ be a postcritically finite hyperbolic rational map. If f_0 has an embedded combinatorial cylinder, then there exists an embedded geometric cylinder for a perturbation f_1 of f_0.*

The analog of the Annulus Theorem of Jaco and Shalen is then

Conjecture 6.17 (Cylinder conjecture) *A combinatorially cylindrical PFH rational map $f_0(z)$ has an embedded combinatorial cylinder.*

P. Makienko ([Mak]) proves a related theorem, which we recast into the language of cylinders.

Theorem 6.18 (Makienko) *Let $f_0(z)$ be a PFH polynomial with connected Julia set.*

If $f_0(z)$ has a repelling fixed point x with combinatorial rotation number $p/q, p \neq 0$, then

1. *There is a cycle $\{R_{t_i}\}_{i=1}^{q-1}$ of q rays landing at x such that the angles t_i are ordered by $t_{i(0)} < t_{i(1)} < ... < t_{i(q-1)}$. The cycle of chords $\{ \{(\infty, t_{n(i)}), (\infty, t_{n(i)+1 \bmod q})\} \}_{i=1}^{q-1}$ corresponds to an embedded combinatorial cylinder under the isomorphism G between essential chords and the combinatorial characteristic subcomplex.*

2. *There is a perturbation near infinity of f_0 to a map f_1 and a conjugacy h such that the above combinatorial cylinder corresponds under the bijection Ψ induced by h to an embedded geometric 1-cylinder for f_1.*

Otherwise, f_0 has a superattracting fixed point x.

1. *There is a combinatorial 2-cylinder of period 2 formed by two chords $\{(x, s_1), (\infty, t_1)\}$ and $\{(x, s_2), (\infty, t_2)\}$.*

2. *There is a perturbation f_1 of f_0 near infinity and near x, and a conjugacy h from f_0 near $J(f_0)$ to f_1 near $J(f_1)$, such that the above combinatorial cylinder corresponds under the induced bijection Ψ to an embedded geometric 2-cylinder.*

Since the theorem covers all PFH polynomials, the Cylinder Conjecture is thus established for all PFH polynomials.

Chapter 7

Existence of acylindrical starlike tunings

In this chapter, we prove

Theorem 7.1 *The tuning of an acylindrical PFH rational map $f(z)$ by a PFH family of starlike polynomials $\mathcal{P}$ is combinatorially equivalent to a rational map.*

Conjecturally, the map $f * \mathcal{P}$ is also acylindrical.

First, we fix some notation. Next, we outline the proof. We then give the proofs of the steps in the outline.

Comparison with matings of quadratic polynomials.

This will be an informal discussion; the details will be postponed to a later, more thorough discussion.

In [Tan2], it is proved that the mating of two PFH quadratic polynomials p and q exists if and only if p and q are not in complex conjugate limbs of the Mandelbrot set. The argument proceeds as follows. A Thurston obstruction is reduced to a very special kind of Levy cycle. This Levy cycle is pulled back under the dynamics. The mated map is not quite expanding, but it does have the property that as the Levy cycle is pulled back under the dynamics, the limit is a graph consisting of unions of external rays for p and q ([Tan2], Proposition 2.7). A possible complication is that this graph may not separate the sphere. However, the fact that the Levy cycle is quite special implies that they do indeed separate the sphere. Moreover, the rays forming this graph must actually be the union of the set of rays landing at the α-fixed points of p and q, implying that p and q are in complex conjugate limbs.

The same argument, mildly generalized to the case of tunings rather than matings, shows the following theorem. It says that an obstruction to a tuning $f * \mathcal{P}$ of a PFH rational map by a family of starlike polynomials is an obvious one.

Theorem 7.2 *Let f be a PFH rational map with $|P(f)| > 2$. If the tuning $f * \mathcal{P}$ of f by a starlike family of polynomials is not combinatorially equivalent* *to a rational map, then there is a finite set $\mathcal{R}_f$ of internal rays of f and a finite set $\mathcal{R}_{\mathcal{P}}$ of external rays for $\mathcal{P}$ such that $f(\mathcal{R}_f) = W_f$, $\mathcal{P}(\mathcal{R}_{\mathcal{P}}) = \mathcal{R}_{\mathcal{P}}$, and such that the union $\mathcal{R}_f \cup \mathcal{R}_{\mathcal{P}}$ forms a Levy cycle for the tuning $f * \mathcal{P}$.*

7.1 Notation

Let $f(z)$ be an acylindrical PFH rational map, let B be a subschema of the mapping schema $(Q(f), f|_{Q(f)}, w_f)$, and let B_{per} be the subset of points in B which are periodic under f. We will use the notation $f(z)$ to also denote the canonical peripherally rigid map associated to f, as constructed in Section 3.5. Choose a fixed marking for f near B.

Let $\mathcal{P}$ be a PFH family of monic starlike polynomials covering the mapping schema B. Let G be a graph as in the definition of PFH family of starlike polynomials. Then the edges of G form eventually periodic arcs for $\mathcal{P}$. By definition of starlike, $\mathcal{P}|_{G_{per}}$ is a homeomorphism from G to itself, up to isotopy through maps fixing $P(\mathcal{P})$. We will also use the notation $\mathcal{P}$ to refer to the canonical family of associated maps which are peripherally rigid near infinity.

Let $R = f * \mathcal{P}$ denote the tuning of f by $\mathcal{P}$, glued along the chosen marking for f and the canonical marking of $\mathcal{P}$ near infinity.

We will denote the support of the tuning by $\mathcal{D} = \cup_{x \in B} D_x$. D_x is a round disc in $\widehat{\mathbb{C}}$ which is identified with the disc $\overline{\Delta}$ in the domain of $\mathcal{P}|_{\{x\}\times\widehat{\mathbb{C}}}$, by the definition of tuning. Thus $R(D_x) = D_{f(x)}$ for all $x \in B$. We set $\mathcal{D}_{per} = \cup_{x \in B_{per}} D_x$. Thus components of $\mathcal{D}_{per}$ cycle under R.

7.2 Outline of the proof

By Thurston's theorem, if $f * \mathcal{P}$ is not combinatorially equivalent to a rational map, there is a Thurston obstruction Γ'. Let Γ be any reduced Thurston obstruction contained in Γ'.

1. By Theorem 4.38, every $\gamma \in \Gamma$ has nonzero intersection number with ∂D_x for some $D_x \in \mathcal{D}_{per}$. We may assume that $\Gamma \cap (\mathcal{D} - \mathcal{D}_{per}) = \emptyset$.

2. By the definition of intersection number, if γ has nonzero intersection number with ∂D, then γ must separate points of $P(R) \cap \mathcal{D}$. Since $\mathcal{P}$ is starlike and Γ is irreducible, there exists $\gamma \in \Gamma$ and $\alpha \in E(G_{per})$ such that $[\gamma] \cdot [\alpha] \neq 0$. Thus $[\alpha] \in \mathcal{A}_{per}(P(R))$.

3. By Theorem 4.29, since Γ is irreducible, Γ must be a simple Levy cycle.

4. We may assume that elements of Γ minimize the number of intersections with components of $\partial\mathcal{D}_{per}$. Since Γ is a simple Levy cycle, there exists a homeomorphism h isotopic to the identity through maps fixing $Q(R)$ and $\mathcal{D}$ such that for $R_1 = R \circ h$, $R_1|_\Gamma : \Gamma \to \Gamma$ is a homeomorphism.

5. Let f_1 denote the map which is obtained from R_1 as follows. Cut out $\mathcal{D}$. Glue in maps ρ_x such that $\rho_x|_{\partial D_x} = R|_{\partial D_x}$, ρ_x carries the center of ∂D_x to the center of $\partial D_{f(x)}$, and such that ρ_x carries radial segments isometrically to radial segments. The resulting map R_1 is combinatorially equivalent to f. Let W be the union of $\Gamma \cap (\widehat{\mathbb{C}} - \mathcal{D})$ together with radial segments joining the centers of the discs D_x to the points of $\Gamma \cap \partial D_x$. Then W is a finite set of essential arcs joining points of $P(f_1)$ such that $f_1|_W$ is a homeomorphism. Note that W is not homeomorphic to a tree since Γ consists of simple closed curves.
6. If W does not separate $Q(f)$, we will show by using Lemma 6.6 that the Levy cycle Γ must be degenerate. This, by Lemma 4.10, is impossible since R is of hyperbolic type.
7. Therefore the arcs W separate the sphere in a Q-essential fashion, and so f_1 has a cylinder. Since f_1 is combinatorially equivalent to f, f has a cylinder.

7.3 Proof of the theorem

1. This step follows immediately from Theorem 4.38.
2. Suppose $[\gamma] \cdot [\partial D_x] \neq 0$, $D_x \in \mathcal{D}_{per}$. If γ intersects no edge of G_{per}, then for all n, no preimage of γ under $(f^{\circ n})^{-1}$ intersects a preimage of a periodic edge in G_{per}. Since Γ is irreducible, γ cannot intersect any element in G, since the edges of G are all eventually periodic. But then γ is isotopic to a curve which does not intersect ∂D, a contradiction.

   Hence γ intersects a periodic edge of G, and so γ intersects an element of $\mathcal{A}_{per}(P(f))$.
3. By Theorem 4.29, the Shishikura-Tan theorem, Γ is a simple Levy cycle.
4. We now show that we may adjust R so that Γ is invariant.

   First, we may choose representatives of Γ so that for every $\gamma \in \Gamma$, and for every $D_x \in \mathcal{D}_{per}$, $|\gamma \cap \partial D_x| = [\gamma] \cdot [\partial D_x]$.

   Next, since $\Gamma = \{\gamma_i\}_{i=1}^n$ is a simple Levy cycle, for each $\gamma_{i+1} \in \Gamma$, there is a unique lift γ_i' of γ_{i+1} isotopic to an element $\gamma_i \in \Gamma$. Let $h_i : S_i^1 \times I$ be an isotopy from $\gamma_i \in \Gamma$ to the lift γ_i'. We may assume that $h_i(S_i^1 \times I) \cap (\mathcal{D} \cup Q(R) - \mathcal{D}_{per}) = \emptyset$, since Γ is irreducible and simple.

   By approximating the γ_i with smooth arcs, and the isotopies h_i with smooth maps, we may assume that each isotopy h_i is transverse to the boundaries of the elements of $\mathcal{D}_{per}$.We may also assume that the collection $\{h_i\}_{i=1}^n$ gives an isotopy H of the disjoint union of the elements of Γ whose trace does not intersect $Q(R)$ or $\mathcal{D} - \mathcal{D}_{per}$. We wish to show that the isotopy H may be extended to an isotopy on $\Gamma \cup \{\partial D\}_{D \in \mathcal{D}_{per}}$. Let H_t be the map at time t, and let $\Gamma_t = H_t(\cup_i S_i^1)$.

By transversality, $H_t^{-1}(\Gamma_t \cap \partial\mathcal{D}_{per})$ forms a submanifold in the disjoint union of annuli $\cup_{i=1}^{n}(S_i^1 \times I)$. The only components of such submanifolds are arcs joining $S_i^1 \times \{0\}$ to $S_i^1 \times \{1\}$.

It follows that the set intersection points $\Gamma_t \cap \partial\mathcal{D}_{per}$ moves isotopically under H_t. From this it follows that we may extend H to an isotopy of $\Gamma \cup \{\partial D\}_{x\in\mathcal{D}_{per}}$. We may assume that the trace of this isotopy is disjoint from $Q(R) \cup \mathcal{D} - \mathcal{D}_{per}$. Extend H to a an isotopy of a regular neighborhood of $\Gamma \cup \{\partial D\}_{x\in\mathcal{D}_{per}}$ which fixes $Q(R)$ and $\mathcal{D} - \mathcal{D}_{per}$. By the Ambient Isotopy Theorem [Hir] there is an extension to an isotopy H_t of all of $\widehat{\mathbb{C}}$ that preserves $Q(R)$, $\mathcal{D} - \mathcal{D}_{per}$, and $\mathcal{D}_{per}$. Let $h = H_1$ be the end map of the isotopy. Then for $R_1 = R \circ h$, $R_1|_\Gamma : \Gamma \to \Gamma$ is a homeomorphism. Moreover, R_1 preserves $Q(R)$ and $\mathcal{D}$.

5. Let $\{\overline{\Delta_x}\}_{x\in\mathcal{D}}$ be a collection of disjoint discs indexed by B. The marking of f gives an identification of D_x with $\overline{\Delta_x}$ which sends the center of D_x to 0. The map $R_1|_{\partial D_x}$ induces a map sending $\partial\overline{\Delta_x}$ to $\partial\overline{\Delta}_{f(x)}$ by a degree $w_f(x)$ local homeomorphism. Extend each of these maps radially to a map ρ_x on all of $\overline{\Delta_x}$. Denote by f_1 the map which is given by $f_1 = R_1$ on $\widehat{\mathbb{C}} - \mathcal{D}$ and by $f_1 = \rho_x$ on D_x. Then f_1 is combinatorially equivalent to f. Note that $f_1|_{\widehat{\mathbb{C}}-\mathcal{D}_{per}} = R_1|_{\widehat{\mathbb{C}}-\mathcal{D}_{per}}$.

   Let W denote the set $\Gamma \cap (\widehat{\mathbb{C}} - \mathcal{D}_{per})$, together with radial segments joining $0 \in \overline{\Delta_x}$ to $\Gamma \cap \partial D_x$ for each $x \in B_{per}$. Then f_1 sends W homeomorphically into itself. W cannot be homeomorphic to a tree since Γ consists of simple closed curves.

6. Suppose W did not separate $Q(f_1)$. We will show that Γ is a degenerate Levy cycle. This is impossible, by Lemma 4.10.

   Define a *piece* to be a component of $\widehat{\mathbb{C}} - (\mathcal{D}_{per} \cup \Gamma)$, regarded as a subset of the domain of R. Call a piece *essential* if it contains elements of $Q(R)$. The assumption that W does not separate the sphere in a $Q(f_1)$-essential fashion implies that there is a unique essential piece; call it U_0. By Lemma 6.6, the set of inessential pieces are permuted under the map R_1.

   Each curve $\gamma \in \Gamma$ has a preferred component of its complement, namely, the one which does not intersect U_0. Call this component the *inside* of γ; we denote it by $\mathrm{Ins}(\gamma)$. We let $\mathrm{Ins}(\Gamma)$ denote the union of the insides of elements of Γ. The inside of a curve $\gamma \in \Gamma$, intersected with $\widehat{\mathbb{C}} - \mathcal{D}_{per}$, is a union of inessential pieces. Since $f_1 = R_1$ on $\widehat{\mathbb{C}} - \mathcal{D}_{per}$, the inessential pieces are permuted homeomorphically under R_1. Hence the components of $\mathrm{Ins}(\Gamma) \cap (\widehat{\mathbb{C}} - \mathcal{D}_{per})$ are permuted under R_1.

   We now claim that the components of $\mathrm{Ins}(\gamma) \cap \mathcal{D}_{per}$ are also permuted under R_1. This will show that Γ is a degenerate Levy cycle.

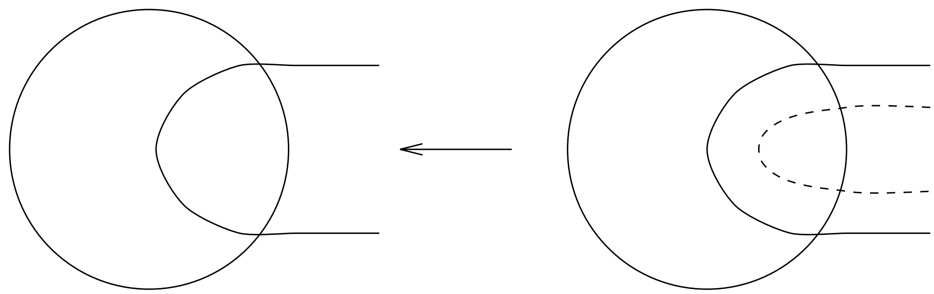

Figure 7.1:

Let $x_1 \in B$ and let D_{x_1} be a component of $\mathcal{D}_{per}$. Let $\gamma_1 \in \Gamma$. Let K_1 be a component of the intersection of $\mathrm{Ins}(\gamma_1)$ with D_{x_1}. Let $\gamma_0 = (R_1|_\Gamma)^{-1}(\gamma_1)$. Then $\eta_1 = \partial K_1 \cap \gamma_1$ lifts to a segment η_0 which is a component of $\gamma_0 \cap D_{x_0}$, for some uniquely determined $x_0 \in B$. Let K_0 be the component of $D_{x_0} - \eta_0$ which intersects the inside of γ_0. See Figure 7.1.

If K_0 is not the lift of K_1, then there is a strictly preperiodic component η_0' in the preimage of η_1 which is contained in K_0. But this implies that there is a strictly preperiodic element γ_0' of $R_1^{-1}(\gamma_1)$ which intersects the inside of γ_0. Since the intersection of $\mathrm{Ins}(\gamma_0)$ with $\widehat{\mathbb{C}} - \mathcal{D}_{per}$ is a union of pieces which are all permuted homeomorphically under f, this is impossible.

■

Chapter 8

Noncompactness in moduli space

In this chapter we discuss compactness properties of hyperbolic components in the *moduli space* of rational maps.

We prove the following two theorems, which are reformulations of special cases of theorems of Makienko [Mak], and which we prove using his techniques.

Theorem 8.1 *Let $f(z)$ be a hyperbolic map with connected Julia set and without periodic critical points. If $f(z)$ has an embedded geometrically cylinder, then $H(f)$ does not have compact closure in $\mathcal{M}_{\lceil}$.*

Theorem 8.2 *Let f be a hyperbolic rational map without periodic critical points. If $J(f)$ is disconnected, then $H(f)$ does not have compact closure in $\mathcal{M}_{\lceil}$.*

A hyperbolic rational map with disconnected Julia set is the analog of a geometrically finite three-manifold with compressible boundary.

We will give a new application of Makienko's techniques to the study of Kleinian groups. We show that the deformation space of a geometrically finite Kleinian group without parabolics arising from a cylindrical or boundary-compressible three-manifold is noncompact. While this result is now well known, the proof we present is similar to Makienko's proof, and relies almost exclusively on two-dimensional methods.

Section 8.1 is a survey of known compactness and noncompactness results for hyperbolic components of rational maps. Section 8.2 contains a brief discussion of the Teichmüller space of a rational map and Kleinian group. A general definition of the Teichmüller space of a holomorphic dynamical system may be found in [MS]. Section 8.3 gives the proof of Theorems 8.1 and 8.2. Section 8.4 contains a brief discussion of some examples. Section 8.5 gives a proof of the analogous theorem for Kleinian groups. Sections 8.6 and 8.7 contain some technical results used in the proofs.

8.1 Other known compactness and noncompactness results

Rees [Ree1] and Ahmadi [Ahm] have obtained partial compactness results for certain subspaces of degree two hyperbolic components. Their proofs use Thurston's characterization of postcritically finite rational maps as branched coverings.

C. Peterson [Pet] has obtained noncompactness results for hyperbolic components in Rat_2 containing quadratic polynomials f. His approach is to obtain estimates for the multiplier of the β-fixed point for a perturbation f_λ of f which has an attracting fixed point of multiplier λ at infinity. He shows that if f has combinatorial rotation number p/q, then as $\lambda \to \exp(-2\pi ip/q)$, the multiplier at the α-fixed point of f_λ tends to $\exp(2\pi ip/q)$. For any rational map with multipliers $\eta_i =\neq 1$ at its fixed points, we have the equality

$$\sum_i \frac{1}{1-\eta_i} = 1,$$

(see [Mil2], Theorem 9.2). Hence the multipliers of the β-fixed points of f_λ must tend to infinity, so there is no limit of the f_λ. His techniques may also be used to show that the hyperbolic components containing PFH quadratic rational maps which have obstructed tunings all have noncompact closures, using the classification by M. Rees and D. Ahmadi.

Milnor in [Mil4], asks when a hyperbolic component in the moduli space of degree two maps has compact closure. The Quadratic Mating Conjecture asserts that mating can be defined continuously for any pair of points not in complex conjugate limbs of the Mandelbrot set. He asserts that this conjecture implies the following: if f_a and f_b are PFH quadratic polynomials which are matable, then the hyperbolic component containing the mated map has compact closure in the moduli space of degree two maps. He also presents computer pictures which illustrate the results found in [Pet].

However, there is no known example of a hyperbolic component which is known to have compact closure in M_d for some d. McMullen [McM1] poses a conjecture which reduces to the following for hyperbolic maps: if $J(f)$ is a Sierpinski carpet and f is hyperbolic, then the hyperbolic component in moduli space containing f has compact closure in $\mathcal{M}_\lceil$.

8.2 The Teichmüller space of a rational map

Our definition of the Teichmüller space of a rational map is taken from [MS], where a general notion of the Teichmüller space of a holomorphic dynamical system is defined.

Definitions. Let $f(z)$ be a rational map. We let

$$M_1(\widehat{\mathbb{C}}, f) = \{\mu\frac{d\bar{z}}{dz} | \parallel \mu \parallel_\infty < 1, f^*(\mu) = \mu\}$$

denote the space of f-invariant Beltrami differentials. Let $QC(\widehat{\mathbb{C}}, f)$ denote group of all quasiconformal homeomorphisms of $\widehat{\mathbb{C}}$ conjugating f to itself, and let $QC_0(\widehat{\mathbb{C}}, f)$ denote the subgroup consisting of maps isotopic to the identity. The group $QC(\widehat{\mathbb{C}}, f)$ acts on $M_1(\widehat{\mathbb{C}}, f)$ by pullback.

Definition 8.3 *The* **Teichmüller space** *of a rational map $f(z)$ is defined as the quotient space $M_1(\widehat{\mathbb{C}}, f)/QC_0(\widehat{\mathbb{C}}, f)$ and is denoted by Teich$(\widehat{\mathbb{C}}, f)$.*

Teich$(\widehat{\mathbb{C}}, f)$ may be equal to a point. If it is not equal to a point, it is a connected complex manifold equipped with a natural metric (see [MS]).

There is a natural map Φ : Teich$(\widehat{\mathbb{C}}, f) \to \mathcal{M}_\lceil$ obtained by using the Measurable Riemann Mapping Theorem (MRMT); see e.g. [Gar]. Given an f-invariant Beltrami differential, the MRMT yields a quasiconformal map $h_\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, well-defined only up to postcomposition with Möbius transformations, which transports the conformal structure given by μ to the standard one. Thus $h_\mu \circ f \circ h_\mu^{-1} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a conformal map of the Riemann sphere which is quasiconformally conjugate to f. Moreover, this new map is the same for any μ' which is the pullback of μ by any quasiconformal self-conjugacy of f which is isotopic to the identity. This gives the desired map. The fibers of the map Φ : Teich$(\widehat{\mathbb{C}}, f) \to H(f)$ are discrete and coincide with the orbits of the modular group Mod$(\widehat{\mathbb{C}}, f)$ (Theorem 2.3, [MS]).

Now suppose f is hyperbolic and has superattracting cycles or critical relations, i.e. no two critical points are identified under the grand orbit equivalence relation and no critical points are periodic. Then by Theorem 2.2, [MS], Teich$(\widehat{\mathbb{C}}, f)$ is naturally identified with the Teichmüller space of the quotient surface $X(f)$, as defined in Chapter 6.

We will assume this for the remainder of this chapter.

Since f is hyperbolic, the quotient space $X(f) = \hat{\Omega}(f)/f$ is then a union of punctured tori., Teich$(\widehat{\mathbb{C}}, f)$ is naturally isomorphic to Teich$(X(f))$, and Φ : Teich$(\widehat{\mathbb{C}}, f) \to H(f) \subset \mathcal{M}_\lceil$. In general, this map is not onto. Two maps in the image of Φ are conjugate on all of $\widehat{\mathbb{C}}$. However, the two maps in the same hyperbolic component may not be conjugate on all of $\widehat{\mathbb{C}}$, since they need not be conjugate on their postcritical sets.

8.3 Hyperbolic components in moduli space with noncompact closure, after P. Makienko

8.3.1 Tending to infinity in moduli space

How can we tell if a sequence of rational maps does not converge in $\mathcal{M}_\lceil$? Makienko's idea is to argue by contradiction: the assumption of convergence in $\mathcal{M}_\lceil$implies that there is a convergent sequence in Rat$_d$, and this implies that all periodic points of a given sufficiently high period converge without collision. More precisely,

Proposition 8.4 (Convergence of periodic points) *Suppose $f_n \to f$ in Rat_d . Let $N(f)$ denote the length of the largest nonrepelling cycle of f. Then for every $p > N(f)$, there is an integer $N(p, f)$ such that for all $n \geq N(p, f)$, there is a conjugacy ϕ_n, isotopic to the identity, from the set of p-periodic points of f to those of f_n. Moreover, these conjugacies converge to the identity as n tends to infinity.*

Proof: Repelling cycles of a map g are simple periodic points of g, and are thus determined by polynomial equations in the coefficients of g with only simple roots. Hence the location and multipliers of repelling cycles of maps near g vary continuously as a function defined on an open neighborhood of g in Rat_d. Moreover, the condition of being a repelling cycle is an open condition. The claim then follows.

∎

In particular, repelling periodic points of f_∞ cannot undergo wild bifurcations under arbitrarily small perturbations of f_∞.

Corollary 8.5 (Geometric limit contains algebraic limit) *Suppose $f_n \to f_\infty$ in Rat_d . Then the Julia set $J(f)$ is contained in the Hausdorff (geometric) limit of $J(f_n)$.*

Proof: Choose a finite set of long periodic cycles which approximate the Julia set of f in the Hausdorff topology. These cycles do not disappear under perturbations, and moreover they do not move very far in the spherical metric.

∎

If the convergent sequence $\{f_n\}$ happens to be constructed by deformations which are conformal off some open set, this has strong consequences.

Proposition 8.6 (Convergent deformations off an open set) *Suppose $\{h_n\}$ is a sequence of quasiconformal conjugacies between f and f_n whose complex dilatations all vanish on an open connected set D. Suppose $f_n \to f_\infty$ in Rat_d. Then*

1. *if $f_n \neq f$ (up to conjugacy) for some n, then $D \subset F(f)$;*
2. *the restrictions $\{h_n|_D\}$ form a normal family of holomorphic functions;*
3. *if h_∞ is any nonconstant limiting map, then $h_\infty(D) \subset F(f_\infty)$.*

Proof: If D intersects $J(f)$, then since f is locally eventually onto near $J(f)$, the complex dilatation of each h_n is identically zero. Otherwise, $h_n(D) \subset F(f_n)$, for all n. A sequence of holomorphic embeddings of an open set $U \neq \widehat{\mathbb{C}}, \mathbb{C}^*$, or $\mathbb{C}$ into $\widehat{\mathbb{C}}$is normal if the Hausdorff limit of the complement of their images consists of three or more points; this limit exists, since the set of closed subsets of $\widehat{\mathbb{C}}$is compact in the Hausdorff topology. But the Hausdorff limit of the Julia sets of the f_n contains the Julia set $J(f_\infty)$. Since $J(f_\infty)$ contains at least three points, and since $h_n(D) \subset F(f_n$, the second result holds, by Corollary 8.5. The images of a closed disc in D under h_n eventually nearly coincide if there is a nonconstant limit h_∞. It follows easily that the image of any such disc under h_N for N sufficiently large is in the intersection of the Fatou sets for infinitely many f_m, and so the third result follows, by Corollary 8.5.

■

Any nonconstant limit of holomorphic embeddings is again an embedding, so h_∞ is an embedding of D into $\widehat{\mathbb{C}}$. Note that when there exists a nonconstant limit h_∞ on D, by induction we can assume that there exist nonconstant limits on the grand orbit of D. So if a nonconstant limit exists, it prolongs to a holomorphic conjugacy (which we also denote by h_∞ between f and f_∞ on the grand orbit of D. Moreover, limits which are constant can often be ruled out if D separates nontrivial pieces of the dynamics.

Proposition 8.7 (Limit is nonconstant) *With the hypotheses of the previous proposition, suppose that either*

1. *D separates points in $J(f)$, or*
2. *D contains infinitely many periodic points on its boundary.*

Then any limiting map h_∞ is nonconstant.

Proof: If f has one of these properties, then so does every f_n. Now choose p sufficiently large and for which there are points of period p which are either separated by D or on ∂D. Then period p points of f_∞ must be very close to those for f_n for large n, and so it follows that the limiting map cannot be constant.

■

8.3.2 Pinching deformations

Any doubly-connected Riemann surface is isomorphic to either the punctured plane, punctured disc, or to a Euclidean cylinder of height H and circumference C for some finite H, C. The ratio H/C is called the *modulus* of the annulus. A basic fact in conformal geometry is that if X is a Riemann surface and $A \subset X$ is an essential embedded annulus of large modulus, then the hyperbolic length of the geodesic in the free homotopy class of A (if it exists) is short. The converse is also true. We will also need the fact that if A is an annulus contained in the Riemann sphere of definite diameter and large modulus, then the diameter of at least one of the complementary components is small.

The basic idea is to consider shrinking a geodesic on the quotient surface $X(f)$. We will carry out this shrinking by defining a sequence of Beltrami coefficients which vanish on most of $X(f)$ so that we may apply the results in the previous section.

If $A = \{z|1/r < |z| < r\}$, the modulus of A is $\pi/\log(r)$. Let $F(z) = z|z|$; this defines a quasiconformal map of A into $\mathbb{C}$ which is an affine stretch of constant dilatation equal to two in the Euclidean coordinates for A. Let η_n be the dilatation of F^n. Let $C_n = F^{-n}A$, and define a Beltrami differential μ_n on A by

$$\mu_n = \begin{cases} 0 & \text{on} A\backslash C_1 \\ \eta_i & \text{on} C_i\backslash C_{i+1} \\ \eta_n & \text{on} C_n \end{cases}$$

See Figure 8.1. Note that the pairs of annuli $C_i\backslash C_{i+1}$ are all conformally isomorphic, so that the moduli of (A, μ_n) tend to infinity. Let h_μ be the straightening map for μ. If $m > n$, the composition $h_m \circ h_n^{-1}$ is conformal on $A\backslash C_n$, which is a pair of annuli of modulus $n/2$ mod(A).

Now let γ be a simple closed geodesic on a Riemann surface X and let $A(\gamma)$ be an embedded annulus homotopic to γ. The *pinching deformation* associated to $A(\gamma)$ is the sequence in Teich(X) defined by Beltrami differentials μ_n as above which are supported on $A(\gamma)$. Since the μ_n are rotationally symmetric the Beltrami differential is independent of the map used to uniformize $A(\gamma)$ and so this map is well-defined. We may apply this construction to a disjoint union of geodesics as well. By construction, the μ_n tend to one in norm, and since the moduli of the straightened annuli tend to infinity, the length of the corresponding geodesic tends to zero. So X tends to infinity both in Teichmüller and moduli space. See Figure 8.1.

Now let f be a hyperbolic rational map without critical relations and $X(f)$ the quotient surface. Let $\gamma \subset X(f)$ be a simple closed geodesic, and let $A(\gamma)$ be an embedded annulus about γ; one always exists, by the well-known Collar Theorem. Denote by D_X the complement of A in X and D the lift of D_X to $\widehat{\mathbb{C}}$.

Consider a sequence of pinching deformations supported on this annulus. By lifting this differential to the Riemann sphere, we then obtain a sequence f_n of rational maps and quasiconformal conjugacies h_n from f to f_n which are conformal on D. Suppose $f_n \to f_\infty$ for some lift to Rat$_d$. If a nonconstant limiting map h_∞ on D exists, it is a conformal conjugacy of f on D to f_∞ on a subset $D_\infty \subset F(f_\infty)$. The quotient of this subset by f_∞ is an open subset of the limiting Riemann surface X_∞, and so h_∞ descends to an embedding ψ of D into $X(f_\infty)$ so that the diagram in Figure 8.2 commutes on D.

8.3.3 Proofs of the theorems

Compressible case

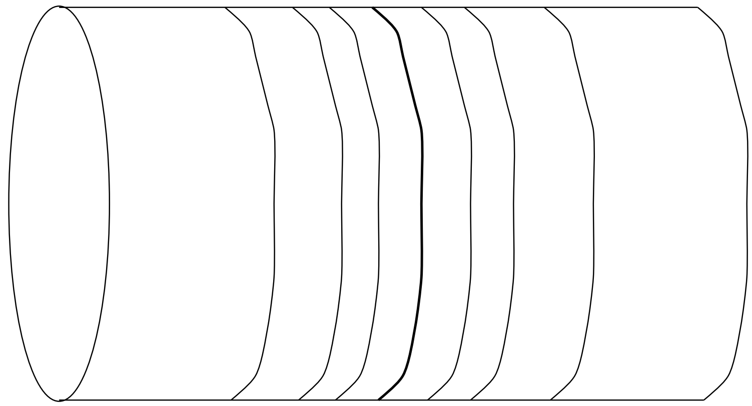

Figure 8.1: The deformation is a horizontal affine stretch along rings. Here the dilatation for μ_3 is shown. The rings are all conformally isomorphic with respect to the conformal structure induced by μ_3.

Since $J(f)$ is disconnected, there exists a simple closed geodesic $\gamma \subset X(f)$ which lifts to simple closed curves in $\widehat{\mathbb{C}}$. Let $A(\gamma)$ be a collar about γ, and let D be as in the previous section. Then since $J(f)$ is disconnected, some component U of D is a cover of D_X which separates components of $J(f)$. Consider a sequence of pinching deformations supported on $A(\gamma)$, and suppose the corresponding rational maps converged in Rat_d . By Propositions 8.6 and 8.7 , there is a limiting nonconstant conjugacy h_∞ from f to f_∞ on D. Let $U_\infty = h_\infty U$. By the pinching construction, there are annuli $B_{n,i}$ disjoint from U_∞ with $\partial B_{n,i} \supset \partial U_\infty$ of arbitrarily large modulus. Since U_∞ has definite diameter, this implies that components of the complement of U_∞ must be collapsed to points, and this is impossible, by Proposition 8.4.

■

More generally, the image of $\mathrm{Teich}(\widehat{\mathbb{C}}, f)$ under Φ for any f with disconnected Julia set does not have compact closure in $\mathcal{M}_{\lceil}$. The proof is essentially the same.

Remark: A similar proof also applies to show that geometrically finite hyperbolic three-manifolds with compressible boundary do not have precompact deformation spaces in the algebraic topology.

Incompressible with embedded geometric cylinder case

Let $([\gamma], p)$ be an invariant access for $p \in \partial\Omega$ and suppose γ is the corresponding geodesic in $X(f)$. Let $A(\gamma)$ be an annular collar, D_X its complement in $X(f)$, and D the lift of D_X to $\widehat{\mathbb{C}}$. Consider the pinching deformation supported on A, and assume that a limiting map $f_\infty \in \mathrm{Rat}_d$ exists. Any limiting

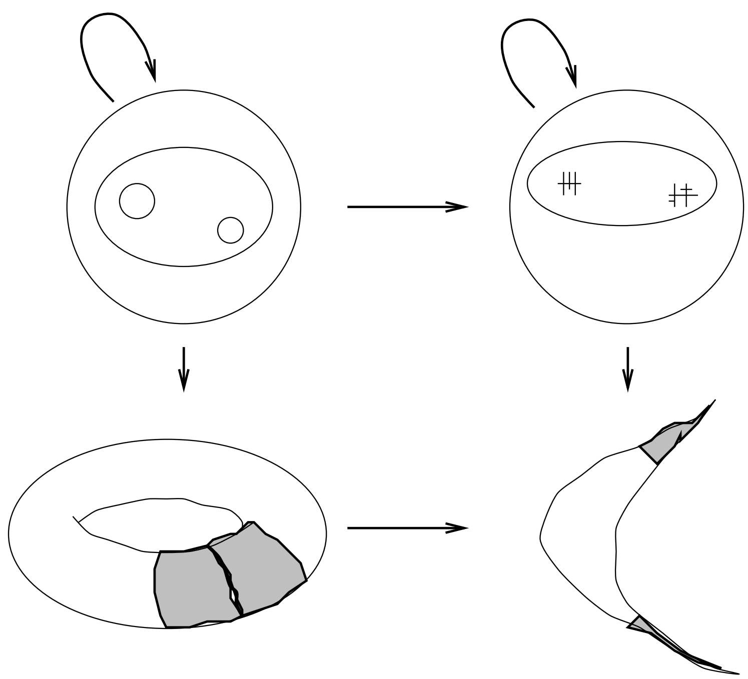

Figure 8.2: A nonconstant limit h_∞ gives a holomorphic conjugacy from f on D to f_∞ on $D_\infty = h_\infty D$. The diagram commutes on D. Marked points are not shown.

map h_∞ is necessarily nonconstant. For if U is a component of D whose closure intersects the attractor $a \in \Omega$, the intersection of $\overline{U}$ with $J(f)$ is a Cantor set containing infinitely many periodic points, and so Proposition 8.7 applies. Let U_i be the components of D whose closures contain a; they cycle of some period k. By Proposition 8.6, $U_i^\infty = h_\infty U_i$ is also a periodic subset of $F(f_\infty)$. The points $h_n(a)$ also converge to a periodic point a_∞ of f_∞. The Maskit Inequality implies that a_∞ is parabolic (see Appendix I), and hence that every point in U_i^∞ tends to a_∞ under iteration of f_∞. Hence if Ω_i^∞ is the Fatou component of f_∞ containing U_i^∞, Ω_i^∞ is an immediate attracting parabolic basin for the parabolic point a_∞. Let $X(f_\infty)$ denote the corresponding quotient surface for f_∞; the component coming from the Ω_i^∞ is isomorphic to $\mathbb{C}^*$ with added marked points. The boundary components of $\psi(D)$ lift to curves in Ω_i^∞ which are invariant under f_∞^k, and hence the endpoints of this lift terminate at a_∞, since Ω_i^∞ is parabolic. It follows that we can always extend h_∞ continuously to the boundary of the U_i.

We first claim

Proposition 8.8 (Limit is literally pinched) *The map h_∞ can be extended to give a smooth semiconjugacy $H : \Omega \to \widehat{\mathbb{C}}$ from f^k to f_∞^k which is a conjugacy on the complement of $\pi^{-1}\gamma$ and which collapses each component of $\pi^{-1}\gamma$ to a point.*

See Figure 8.3.

Proof: It is enough to prove this for the pinching deformation associated to a single geodesic. First, note that the complement of U_i^∞ in Ω_i^∞ cannot contain any marked points. Otherwise, for large n, the complement of $h_n U_i$ in its corresponding Fatou component contains marked points, which is impossible since $A(\gamma)$ contains no marked points and marked points move continuously under small perturbations. The complement of $\psi(D)$ thus consists of a union of punctured discs. Hence there is a smooth extension of ψ to a diffeomorphism of $X(f)\backslash\gamma$ onto $X(f_\infty)$ which is unique up to isotopy. Since the complement of $\psi(D)$ contains no marked points, each component of the complement lifts to a topological disc. Hence there is a lifting this extension to $\widehat{\mathbb{C}}$, and this gives our map H. That H collapses components of $\pi^{-1}\gamma$ follows easily from the local picture of the dynamics near parabolic points.

∎

Proof of Theorem

Applying the pinching construction to disjoint annuli about the geodesics representing independent accesses, we conclude by the previous proposition that there is a continuous semiconjugacy from f^k on Ω (or $f^{k_1 k_2}$ on $\Omega_1 \cup \Omega_2$ in the second case) into $\widehat{\mathbb{C}}$which collapses the lifts of the geodesics to a point. But by hypotheses, some of these lifts piece together to form a simple closed curve separating the U_i, which is impossible.

∎

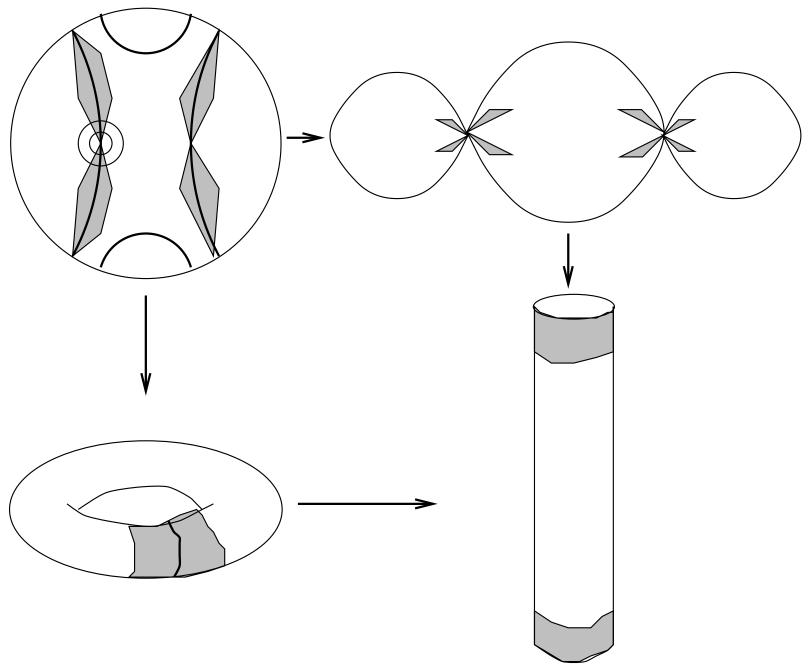

Figure 8.3: The limit is parabolic and literally pinched along preimages of γ. We have suppressed most lifts of $A(\gamma)$ and marked points.

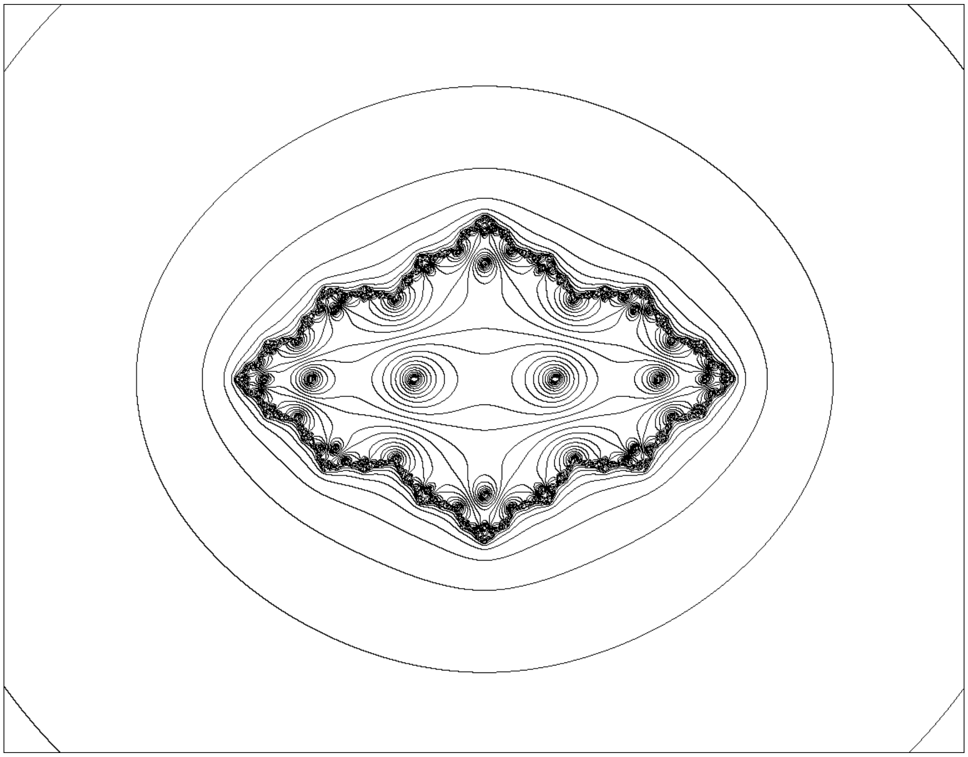

Figure 8.4: The pinching deformations have a limiting map. The curves drawn are the preimages of circles near the attractors a and ∞. The lifts of γ are shown as transversals to these curves.

8.4 Examples

The assumption that f have no critical relations is not absolutely necessary. In the examples below, we keep one cycle superattracting.

Consider the map $f_c(z) = z^2 + c$, c small and negative real, regarded as a polynomial map of $\mathbb{C}$ to $\mathbb{C}$. The quotient $X(f)$ of the basin of zero is a once-punctured torus. If a denotes the attracting fixed point of f_c, the multiplier of a is real and negative. The quotient surface $X(f)$ is isomorphic to the quotient of $\mathbb{C}^*$ by $< z \mapsto \lambda z >$, via the local linearization of f near a, where we also transport marked points by the linearization map. The map $z \mapsto \lambda z$ preserves any line through the origin which avoids the marked point, and flips direction. It follows that this line projects to a geodesic in the quotient torus, and this geodesic lifts in $\widehat{\mathbb{C}}$to a pair of arcs joining a to a two-cycle of repelling periodic points. As we perform the pinching deformation, the number c decreases towards -1. The limit exists, as a polynomial. See Figure 8.4.

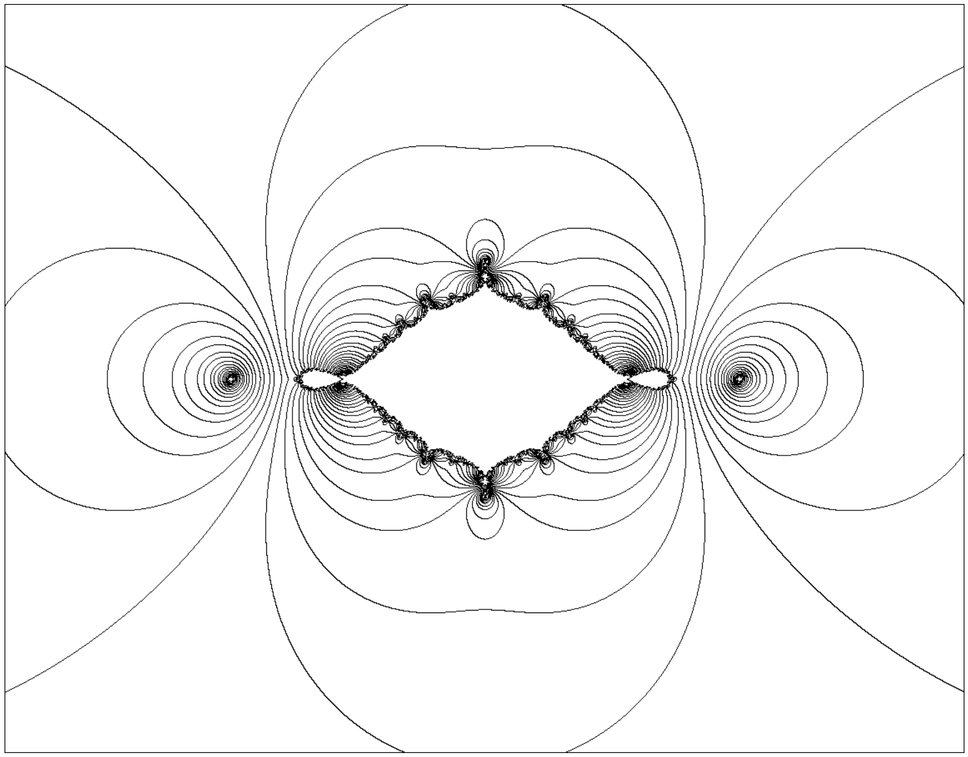

Figure 8.5: The pinching deformations have no limiting map.

Next consider the map $g_c(z) = \frac{z^2-1}{cz^2+1}$, c small and negative real. For this family, 0 and -1 form a period two superattracting cycle. The quotient surface is a punctured torus lifting to a fully invariant basin containing infinity. Applying the analysis in the previous section to the basin of infinity, we see that there is a geodesic in the quotient surface which this time lifts to a pair of accesses which terminate at a common repelling fixed point. As we carry out the pinching deformation, c tends to -1, and we are forced to decide what portion of the dynamics we wish to keep through the use of normalization. If we normalize to keep a superattracting period two cycle at zero and -1, we are forced to collapse other portions of the dynamics. See Figure 8.5. It appears that in the limit, the Fatou component containing -1 disappears.

Next, we consider the family $f_c(z) = \frac{z^2-1}{z^2+tz}$ has a period three cycle of multiplier $2t^2/(1+t)$ at $0, 1, \infty$. If $|t| < 1$ this cycle contains both critical points, and when $t = 0$ the map f_t is critically finite with both critical points in the same period three cycle. As t stays real and increases from 0 to 1, the quotient surface is a twice-punctured torus. The analysis of Blaschke products given in [Mak] shows that there are two disjoint curves in this punctured torus which lift

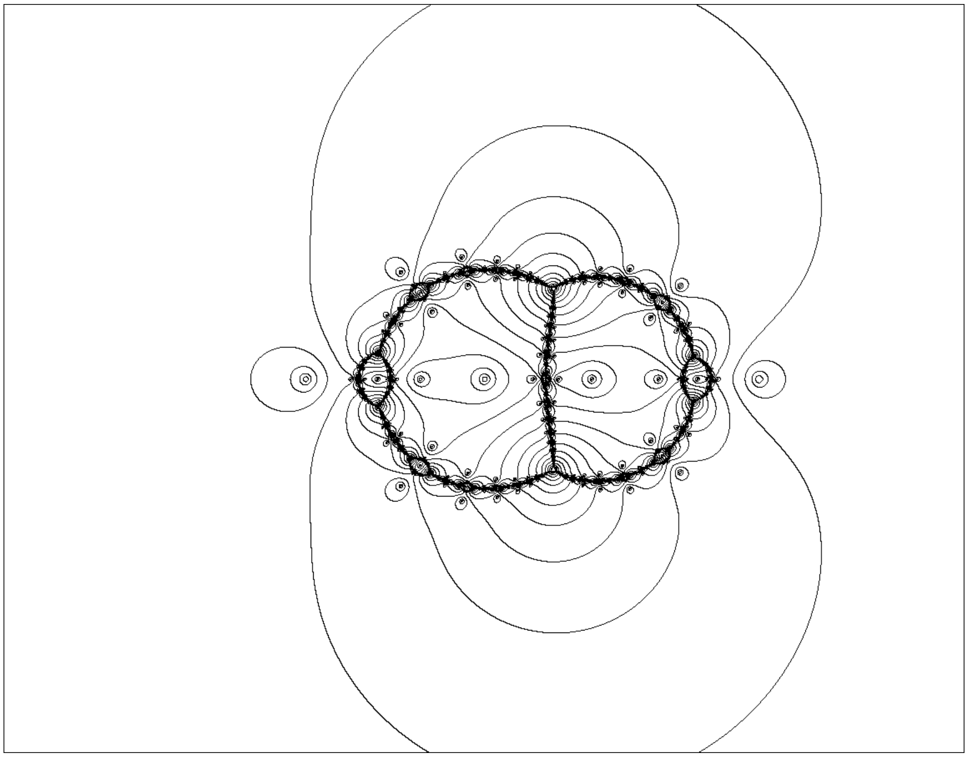

Figure 8.6: The pinching deformations have no limiting map. Four invariant accesses meet at a common fixed point to form a simple closed curve in the sphere which must be collapsed if a limit exists.

to two independent invariant accesses $([\alpha_i], p), ([\alpha_i], q)$ in each attracting basin Ω_i which join the attractors to repelling fixed points p and q. See Figure 8.6.

It appears as if we are losing a portion of the dynamics in the limit. However, it is not obvious from the picture that no limit in $\mathcal{M}_\lceil$exists, since our picture may reflect the wrong choice of normalization: we may be artificially preventing the attracting fixed points from colliding and becoming parabolic.

8.5 Analogs for Kleinian groups

In this section we formulate and prove analogous theorems for Kleinian groups. We concentrate on the two-dimensional, rather than three-dimensional, aspects of the theory.

Let G be a torsion-free finitely generated group, and denote by $R_G$the set of discrete faithful representations of G into $\mathrm{Aut}(\widehat{\mathbb{C}})$. Endow G with the compact-open topology; this gives the same topology as that of pointwise convergence on some fixed set of generators for G. This topology we refer to as the *algebraic topology* and denote the space by R_G. This space is naturally a subspace of the space of all representations of G. The image of G under some such representation we call a *Kleinian group* and denote by Γ. We will usually refer to the image of G under the representation, rather than the representation itself. The group of Möbius transformations acts on this space by conjugation; the quotient we denote by $\mathcal{M}_G$. If G is *nonelementary*, that is, contains no abelian subgroup of finite index, $\mathcal{M}_G$ is Hausdorff [Thu1] , and any limiting representation in $R_G$is also discrete and faithful, by Chuckrow's Theorem (see [Mas]). $\mathcal{M}_G$is naturally a subspace of a larger, typically non-Hausdorff space which is the space of all representations, modulo conjugation.

Definition 8.9 *We say that a Kleinian group Γ is* **geometrically finite** *if for all points x in its limit set Λ, there exists an element $\gamma \in \Gamma$ such that $|\gamma' x| > 1$, where the norm of the derivative is measured with respect to the spherical metric.*

This is usually defined by requiring that a unit neighborhood of the convex core of the quotient three-manifold has finite volume. The definitions coincide, by a theorem of Sullivan [Sul2]. For our purposes, geometrically finite groups play the role of hyperbolic rational maps.

In what follows, we shall restrict our attention to nonelementary geometrically finite Kleinian groups without parabolic elements. For such groups, the domain of discontinuity $\Omega(\Gamma)$ is always nonempty, and the quotient surface $X(\Gamma)$ is a finite union of compact Riemann surfaces, by the Ahlfors Finite Area Theorem. As for rational maps, we have that the Teichmüller space $\text{Teich}(\widehat{\mathbb{C}}, \Gamma)$ is naturally isomorphic to $\text{Teich}(X(\Gamma))$, and the MRMT gives a map from $\text{Teich}(X(\Gamma))$ into $\mathcal{M}_G$which factors through the set $GF(\Gamma)$ of geometrically finite representations of Γ.

Definition 8.10 *Let $\Omega_0 \subset \Omega(\Gamma)$ be a component of the domain of discontinuity of a Kleinian group Γ. Let $p \in \partial\Omega$. An* **access** *$([\alpha], p)$ of p in Ω_0 is a closed topological arc with interior contained in Ω_0 and with an endpoint equal to p, up to isotopy in Ω_0 fixing endpoints. An access $([\alpha], p)$ is said to be* **invariant** *if it isotopic to the lift of a simple closed curve in $X(\Gamma)$. Two invariant accesses are said to be* **independent** *if the corresponding simple closed curves can be isotoped to be disjoint.*

Note that the endpoints of an invariant access are always fixed points for some element $\gamma \in \Gamma$.

Theorem 8.11 *Let Γ be a geometrically finite Kleinian group without parabolics. Suppose $\Lambda(\Gamma)$ is disconnected. Then $GF(\Gamma)$ does not have compact closure in $\mathcal{M}_G$.*

Theorem 8.12 *Let Γ be a geometrically finite Kleinian group without parabolics. Suppose that there are two components Ω_1, Ω_2 of $\Omega(\Gamma)$, not necessarily distinct, and two points $p, q \in \partial\Omega_1 \cap \partial\Omega_2$ for which there exist two independent* *invariant accesses* $[\alpha_i] \subset \Omega_i, i = 1, 2$ *whose endpoints are* p *and* q. *Then* $GF(\Gamma)$ *does not have compact closure in* $\mathcal{M}_G$.

It turns out that precisely the same proof works as that given for rational maps.

A basic fact which we need is the following: given any finitely generated subgroup of $\mathrm{Aut}(\widehat{\mathbb{C}})$, there are only finitely many conjugacy classes of parabolic or elliptic elements; see [Mas]. So any nonelementary group has lots of hyperbolic elements.

Proposition 8.13 (Convergence of hyperbolic elements) *Suppose* $\Gamma_n \to \Gamma_\infty$ *in* R_G. *Then given any hyperbolic element* $\gamma_\infty \in \Gamma_\infty$, *there exists an integer* $N(\gamma)$ *such that for all* $n \geq N(\gamma)$, γ *is close to* γ_n *in* $Aut(\widehat{\mathbb{C}})$, *and* $\gamma_n \to \gamma_\infty$, *where* γ_n *is the corresponding element in* Γ_n.

Proof: This follows immediately from the definition of $R_G$as a topological space.

∎

Corollary 8.14 (Geometric limit contains algebraic limit) *Suppose* $\Gamma_n \to \Gamma_\infty$ *in* R_G. *Then the limit set of* Γ_∞ *is contained in the Hausdorff limit of the limit sets for* Γ_n.

Proof: The fixed points of hyperbolic elements are dense in the limit set for any nonelementary group.

∎

The analogs of Propositions 8.6 and 8.7 are proved in exactly the same way as for rational maps; one replaces the Fatou set with the domain of discontinuity. For brevity, we omit the statements.

Proof of Theorem 8.11.

Lemma 8.15 *Suppose* Ω_0 *is a non-simply connected component of* $\Omega(\Gamma)$. *Then there exists a simple closed curve* γ *in* $X(\Gamma)$ *which lifts to a simple closed curve in* Ω_0 *separating components of* $\Lambda(\Gamma)$.

Proof: I don't know of a strictly two-dimensional argument. In Appendix II, we give a three-dimensional argument.

∎

Now let $A(\gamma)$ be an annular collar about γ, and consider the pinching deformations supported on $A(\gamma)$. Assume that there is a limiting group in $\mathcal{M}_G$. Then as before, this implies that there is a sequence of lifts to $R_G$which limit on some group $\Gamma_\infty \in R_G$. The remainder of the proof is precisely the same as that given for rational maps.

■

Proof of Theorem 8.12

Let $\tilde{\gamma}$ be an invariant access to p in a component Ω_0 of $\Omega(\Gamma)$, and suppose γ is the corresponding simple closed curve in $X(\Gamma)$. Again, consider the pinching deformations supported on an annular collar about γ, and suppose that there is a limiting group. The same discussion given for rational maps applies: the limiting element $\gamma_\infty \in \Gamma_\infty$ is necessarily parabolic, by the Maskit Inequality (see Appendix). Moreover, there is an extension H of the nonconstant limiting map h_∞ to a semiconjugacy of Γ to Γ_∞ on all of Ω_0 which collapses lifts of γ to points, since γ_∞ is parabolic. The remainder of the proof goes through just as for rational maps.

■

8.6 Appendix I: The Maskit inequality

Suppose $\lambda \in \mathbb{C}^*$, $|\lambda| > 1$, and let $\mathbf{T}_\lambda$be the quotient of $\mathbf{C}^*$ by the group generated by $z \mapsto \lambda z$. Then $\mathbf{T}_\lambda$is a torus. The family of circles about the origin descend to give a well-defined longitude on $\mathbf{T}_\lambda$, which we denote by l. There is no well-defined meridian; fixing one is equivalent to choosing a branch of $\log \lambda$. Let us suppose we have fixed a meridian; call it m.

Theorem 8.16 (Maskit inequality) *Suppose $\gamma \subset \mathbf{T}_\lambda$ is freely homotopic to $l^{-p}m^q$. Let $A(\gamma)$ be an annular collar about γ. Then the modulus of $A(\gamma)$ satisfies*

$$\frac{\log^2|\lambda| + (\arg(\lambda) - 2\pi p/q)^2}{2\pi \log|\lambda|} \leq 1/Mod(A(\gamma))$$

In particular, if the modulus of $A(\gamma)$ tends to infinity, then the norm of λ must tend to one. The proof of this follows easily from the extremal length definition of modulus. See [Mas].

8.7 Appendix II: Three-dimensional topology

$\mathrm{Aut}(\widehat{\mathbf{C}})$ is naturally isomorphic to the orientation-preserving isometries of hyperbolic three-space. If M is a compact three-manifold with boundary, the *set of all hyperbolic structures on* M, denoted by $H(M)$, is the set of all pairs (h, N), where $h : M \to N$ is a homotopy equivalence preserving peripheral structure, modulo the equivalence relation $(h, N) \sim (h', N')$ if there is an orientation-preserving isometry $i : N \to N'$ homotopic to $h' \circ h^{-1}$. Any point in this set determines uniquely a representation of the fundamental group of M, up to conjugation, and conversely. So the sets $H(M)$ and $\mathcal{M}_G$ are naturally isomorphic. $H(M)$ equipped with the algebraic topology is usually denoted by $AH(M)$ and is called the *algebraic topology* on the set of all hyperbolic structures on M. If

N is a geometrically finite hyperbolic 3-manfifold without cusps, and has fundamental group Γ, there is a natural compactification $\overline{N}$ of N by the boundary at infinity, which is the quotient of $\Omega(\Gamma)$ by Γ. With this compactification, the homotopy equivalence between M and N can be extended to one of pairs $(M, \partial M)$, and by a theorem of Waldhausen [Jac], may be assumed to be a homeomorphism, since M is necessarily Haken (see below).

Theorems 8.11 and 8.12 are better known via the three-dimensional picture.

Definition 8.17 *Let M be an oriented compact three-manifold with boundary, and suppose that no boundary components are spheres.*

1. *M is said to have* **incompressible boundary** *if for every component $S \subset \partial M$, the map $i_* : \pi_1 S \to \pi_1 M$ induced by inclusion is an injection.*

2. *M is said to be* **cylindrical** *if there is a map of pairs $(S^1 \times I, S^1 \times \partial I) \to (M, \partial M)$ which is essential and not homotopic into ∂M.*

Theorem 8.18 (Compressible implies AH(M) noncompact) *Suppose M has compressible boundary. Suppose $N \in AH(M)$ with GFWP fundamental group Γ. Then $GF(\Gamma)$ has noncompact closure in $\mathcal{M}_G$.*

Theorem 8.19 (Cylindrical implies AH(M) noncompact) *Suppose M is cylindrical, and suppose $N \in AH(M)$ with GFWP fundamental group Γ. Then $GF(\Gamma)$ has noncompact closure in $\mathcal{M}_G$.*

These theorems will follow immediately from Theorems 8.11 and 8.12, once we know how to extract *independent* accesses.

A technical point is that a compact oriented three-manifold with non-sphere boundary is Haken; see [Hem].

In the first case, this follows from the Loop Theorem ([Jac], p.2) , which implies that if some component S of ∂M does not inject on the level of π_1, then one can find a *simple* closed curve in S which bounds an *embedded* disc in M. In the second case, this follows from the Annulus Theorem of for Haken three-manifolds of Jaco and Shalen (see [Jac], p. 154), which similarly implies that if M is cylindrical, then one can find an *embedded* cylinder in M. By Waldhausen's theorem, there are embedded compressing discs or cylinders in $\overline{N}$. Their boundaries give the required curves representing invariant accesses.

Chapter 9

Jordan domain Fatou components

In this chapter, we prove two theorems which give information about the topology of the Julia set for a class of maps defined by conditions on its mapping schema.

We prove

Theorem 9.1 *Let f be a critically finite rational map with exactly two critical points, not counting with multiplicity. Then exactly one of the following possibilities holds:*

- *f is conjugate to z^d and the Julia set is a Jordan curve, or*
- *f is conjugate to a polynomial of the form $z^d + c, c \neq 0$, and the Fatou component containing infinity is the unique Fatou component which is not a Jordan domain, or*
- *f is not conjugate to a polynomial, and every Fatou component is a Jordan domain.*

Theorem 9.2 *Let f be a hyperbolic critically finite rational map for which every postcritical point is periodic. Then there is at least one cycle of Fatou components with Jordan curve boundary.*

Corollary 9.3 *Let f be a critically finite map for which every postcritical point lies in the same cycle. Then every Fatou component has Jordan curve boundary.*

Our proof of these theorems is based on an analysis of how Jordan curves in the Julia set of a rational map behave under taking preimages.

Remarks:

1. Any rational map with exactly two critical points is conjugate to a map of the form $M \circ z^d$, where M is a Möbius transformation determined up

to nonzero scalar multiples. Any critically finite quadratic rational map has two critical points and so is in this family. Thus Theorem 9.1 covers all critically finite quadratic maps.

2. Shishikura [Shi2] has proved a related result. Let $p(z)$ be a cubic polynomial with three distinct roots. Let $N(z) = z - p'(z)/p(z)$ be the rational map which is Newton's method applied to p. For these maps, it is known that the roots of p are fixed critical points lying in simply-connected basins $A_i, i = 1, 2, 3$. Suppose the Julia set for $N(z)$ is locally connected. Then Shishikura proves that the A_i and their first preimages all have Jordan curve boundary. See [Tan1].

3. F. v. Haeseler [vH] has also proved a related theorem in the topological category. We state the theorem as it appears in [Tan1].

   Theorem 9.4 *Let $U \subset S^2$ be open and simply connected. Suppose $F : \overline{U} \to \overline{U}$ and $H : \overline{\Delta} \to \overline{U}$ are continuous maps such that*

   (a) $|\{F^{-1}(x)\}| \leq 2$ for all x in ∂U,

   (b) $H : \Delta \to U$ is a homeomorphism,

   (c) $F(H(z)) = H(z^2)$ for all z in $\overline{\Delta}$, and

   (d) F extends to an injective map in a neighborhood of $H(1)$ in S^2.

   Then $H : \partial\Delta \to \partial U$ is a homeomorphism iff there exists an open connected set $V \subset S^2 - U$ such that $H(1) \in \partial V$.

   As Tan Lei remarks, condition (d) is not explicitly stated in [vH] and is automatically satisfied if F is a rational map, U is a Fatou component fixed under F, and $F|_U$ is conjugate via H to $z \mapsto z^2$ on Δ. An example due to Shishikura shows it is necessary. Tan Lei also notes that condition (a) is difficult to check, and that Shishikura does not use this result in proving his result about Newton's method Julia sets. The examples we give in section 5.1 shows that condition (a) need not always be satisfied.

4. Since there are usually many rational maps with isomorphic mapping schema, there are probably very few theorems which give topological information about the Julia set from the postcritical data alone.

Section 9.1 develops techniques used in the proof. Section 9.2 gives a sketch of the proof of Theorem 9.1. Section 9.3 collects notation and a few technical facts used in the proof. Section 9.4 contains the proofs of the theorems.

9.1 Dynamics of Jordan curves in $J(f)$

In this section we study the dynamics of Jordan curves in the Julia set. Since the forward image of a Jordan curve is not usually a Jordan curve, we consider the dynamics under preimages of the map.

Let f be a critically finite hyperbolic rational map and Ω be a Fatou component of f. A choice of point $x \in \Omega$ determines a preferred orientation on the set of Jordan curves in the Julia set: a curve γ is positively oriented if x lies in the component of the complement of γ lying to the right of γ. Since Ω is connected, this orientation depends only on Ω and not on the chosen point. Given an oriented Jordan curve γ in $J(f)$, this orientation determines the *sign* $\operatorname{sign}(\gamma)$ which is defined to be *positive* if the curve is positively oriented and *negative* otherwise.

Given an oriented Jordan curve γ in the sphere, let the *inside* $\operatorname{Ins}(\gamma)$ and *outside* $\operatorname{Out}(\gamma)$ of the curve denote the components of its complement lying to its left and right, respectively. Let Γ_0 be the set of positively oriented (with respect to Ω) Jordan curves in $\partial\Omega$. With respect to this orientation, $\Omega \subset \operatorname{Out}(\gamma)$ for all $\gamma \in \Gamma_0$. If we write $\widehat{\mathbf{C}}\backslash\overline{\Omega}$ as a union of connected components, each of these components is a Jordan domain (see Lemma 2.34) with boundary in the Julia set. Let A denote the set of closures of such components, plus the choice of either a positive or a negative sign for each component . Then $\Gamma_0 = \cup_{a\in A^+}\partial a$, where A^+ is the set of positive elements of A. It is useful to imagine conjugating the map so $x \in \Omega$ becomes the point at infinity. Then $\widehat{\mathbf{C}}\backslash\Omega$ is a full set in the plane, and the orientation induced by Ω then gives every $\gamma \in \Gamma_0$ the usual planar counterclockwise orientation.

The idea is to study how a sequence of backward images of curves in Γ_0 intersects elements of A by writing the itinerary of such a sequence with respect to the "alphabet" A, equipped with signs to record orientation. We do this in order to deduce information about how the discs $\operatorname{Ins}(\gamma)$ and $\operatorname{Out}(\gamma)$ behave under inverse branches of f. There are two basic ingredients: Montel's Theorem, which implies that a forward-invariant open set omitting at least three points is in the Fatou set, and the fact that preimages of disjoint sets are disjoint.

Given Ω, let Γ_0 be as above, and denote by $\Sigma\Gamma_0$ the set of *sequences* of consecutive preimages of curves $\gamma_0 \in \Gamma_0$, i.e. the set of *sequences* $\{\gamma_0, \gamma_1, \gamma_2, ...\}$ where $\gamma_n \in f^{-n}\Gamma_0$, is a Jordan curve in the Julia set equipped with an orientation so that the map $f : \gamma_{n+1} \to \gamma_n$ is orientation-preserving. Each term of the sequence has a well-defined sign. If two consecutive terms have different signs, we say that there is a *sign change* in the sequence; this occurs when the map f reverses orientation with respect to the orientation defined by Ω.

A Jordan curve in the Julia set is contained in the closure of a unique component of $\widehat{\mathbf{C}}\backslash\overline{\Omega}$. This gives a well-defined *projection map* $p_A : \Gamma_0 \to A$ which sends a Jordan curve in the Julia set to the signed element of A which contains it. Thus for an oriented curve $\gamma \in J(f), p_A(\gamma) = a$ if and only if the signs of γ and a agree, and $\gamma \subset a$. This map extends to a map $p_A : \Sigma\Gamma_0 \to A^{\mathbf{N}}$ by sending the sequence $\{\gamma_n\}_{n\in\mathbf{N}}$ to the sequence $\{p_A(\gamma_n)\}_{n\in\mathbf{N}}$. Denote the image of this map by ΣA.

The space $\Sigma\Gamma_0$ is too large to be analyzed by studying its image under p_A, so we introduce a smaller space which captures the features in which we are interested. For any finite collection of disjoint Jordan curves in a component $a \in A$, there is at least one *outermost* one, i.e. one which is not separated from

Ω by any other curve in the collection. It is not unique, in general. We want to choose a subset of $\Sigma\Gamma_0$ which consists of such "outermost" curves. Given $\Sigma\Gamma_0$, let $\Sigma_{Out}\Gamma_0$ denote the subset of sequences which have the following property: given any two consecutive terms γ_{n+1}, γ_n regarded as unoriented curves, γ_n is outermost among the collection $f^{-1}(\gamma_n) \cap a$, where $a = p_\Omega(\gamma_{n+1})$ is the component containing γ_{n+1}.

It follows easily that all outermost preimages of γ_n have the same sign. For the space $\Sigma_{Out}\Gamma_0$, in a sequence $\{\gamma_n\}$ the nth term is positive (negative) and the $n+1$st term is negative (positive), if and only if the preimage V of $\text{Ins}(\gamma_n)$ (respectively $\text{Out}(\gamma)$) contains Ω, and in this case ∂V forms the set of outermost preimages of γ_n. As a consequence, we have the following basic fact:

Proposition 9.5 (Ω fixed iff no sign changes) *The Fatou component Ω is forward-invariant if and only if there are no sign changes in elements of $\Sigma_{Out}\Gamma_0$.*

Also, if the nth term is positive (negative), and there are no sign changes after this term, then the full backward preimage of $\text{Ins}(\gamma_n)$ (respectively $\text{Out}(\gamma_n)$) under all iterates of f avoids Ω.

Examples:

1. Let $f(z) = 1/z^2$, $x = \infty$, and Ω be the basin of infinity. Then Γ_0 consists of a single element and so Γ_0 consists of a single element $\{+\}$. The sequence space $\Sigma\Gamma_0$ then consists of one element $\{+, -, +, ...\}$. There is a single component of the complement of $\overline{\Omega}$, and so $\Sigma\Gamma_0$ and ΣA coincide. Since all curves are outermost, the set $\Sigma_{Out}\Gamma_0$ is the same as $\Sigma\Gamma_0$.

2. Let f be any critically finite hyperbolic quadratic polynomial different from z^2. Let Ω be the basin of infinity. Then Γ_0 consists of Jordan curves which form the boundaries of the bounded Fatou components, and so Γ_0 is the set of oriented Jordan curves in the Julia set. This implies that every curve is outermost. Moreover, these are boundaries of the bounded Fatou components. Hence A and Γ_0 are naturally isomorphic via p_A. The map f respects the orientations on these curves, hence the sequence spaces contain no sequences with sign changes. It follows that the two sets $\Sigma\Gamma_0$ and ΣA are isomorphic via p_A. Note that in the above two cases, since the image of a Jordan curve in the Julia set is again a Jordan curve in the Julia set, we can actually extend the above constructions to sets of bi-infinite sequences. This is not always possible; see the examples in Section 5.1.

3. Let $f(z) = z^2 - 1$ but now let $x = -1$ and Ω be the Fatou component containing zero. Then Γ_0 consists of a single element $+\gamma_{-1}$, where we use the plus sign to emphasize its positive orientation with respect to Ω. Note that the orientation of $+\gamma_{-1}$ is different from the usual planar orientation in which planar Jordan curves go counterclockwise: the boundary of Ω goes clockwise (with respect to the usual planar orientation). All other curves in $J(f)$ go counterclockwise with respect to the usual planar orientation. If $+\gamma_1$ and $+\gamma_0$ denote the boundaries of the Fatou components

containing one and zero, respectively, together with orientations induced by Ω, then the preimage of $+\gamma_{-1}$ consists of a single curve $-\gamma_0$. The oriented curve $-\gamma_0$ has two preimages, one of which is $+\gamma_{-1}$, and the other is $-\gamma_1$.

There are two periodic sequences in this space: $\{+\gamma_{-1}, -\gamma_0, +\gamma_{-1}, ...\}$ and $\{+\gamma_0, -\gamma_{-1}, +\gamma_0, ...\}$. For any other sequence, if the curve $+\gamma_1$ or $-\gamma_1$ appears as the nth term, there are no sign changes after the nth term. Again, all curves are outermost.

Since Ω is a Jordan domain, A consists of a single letter a, and so the map p_Ω just records the sequence of signs.

For example, in the three examples above, all curves are outermost. In the last example, there is not a unique outermost curve.

We now list the basic facts concerning the structure of the set $\Sigma_{Out}\Gamma_0$.

Proposition 9.6 (Ω not fixed) *Suppose Ω is not forward-invariant under f. Let $a_\Omega \in A$ contain $f(\Omega)$. Then*

1. *for $+\eta_0 = \partial a_\Omega$, with positive orientation, there exists an outermost preimage η_1 of η_0 such that $p_A(\eta_1) = -a_\Omega$.*
2. *If $\{\gamma_n\}$ has a sign change between γ_N and γ_{N+1}, then $p_A(\gamma_N) = \pm a_\Omega$, and there is some other outermost preimage γ'_{N+1} of γ_N such that $p_A(\gamma'_{N+1}) = \pm a_\Omega$. The signs of γ_{N+1} and γ'_{N+1} are both the opposite of the sign of γ_N.*

Proof: By definition $f(\Omega) \subset \text{Ins}(\eta_0)$, hence there is a preimage V of $\text{Ins}(\eta_0)$ containing Ω. It follows that there is a preimage η_1 of η_0 of negative orientation. The set of outermost preimages of η_0 are all negatively oriented and form the boundary of V. If no such preimage is contained in a_Ω, then $V \supset \text{Ins}(\eta_0)$. But then Montel's theorem implies that V is in the Fatou set, which is impossible. This proves the first statement.

To prove the second statement, we may suppose that γ_N is positively oriented; the argument in the other case proceeds similarly. A sign change at γ_N implies that a component of the preimage of $\text{Ins}(\gamma_N)$ contains Ω. Hence $\text{Ins}(\gamma_N) \supset f(\Omega)$. Since γ_N is assumed positively oriented, $\text{Ins}(\gamma)$ is a Jordan domain in the complement of Ω, and since it contains $f(\Omega)$ it must itself be contained in the component a_Ω containing $f(\Omega)$. The proof of the last half of the second statement is completely analogous to the proof of the first step.

∎

Denote the interior of a set X by $\text{Int}(X)$. The next proposition relates the dynamics of curves in the boundary of Ω to dynamics in the complement of $\overline{\Omega}$.

Proposition 9.7 (Ω fixed) *Suppose Ω is forward-invariant under f. Let $E \subset Int(a)$ be a nonempty subset, and suppose $f^{-1}E \subset \cup_{i=1}^{i=r} b_i$, where the b_i's are* *distinct elements of A and each intersects the preimage of E. Let γ_0 be the positively oriented boundary of a. Then for any positively oriented outermost γ_1 such that $f(\gamma_1) = \gamma_0$, $p_A(\gamma_1) = b_i$ for some b_i.*

Proof: This follows directly from the fact that preimages of disjoint sets are disjoint. Since $\mathrm{Out}(\gamma_0)$ is disjoint from $\mathrm{Int}(a)$, the component V of the preimage of $\mathrm{Out}(\gamma_0)$ containing Ω_0 is disjoint from the full preimage of $\mathrm{Int}(a)$. The set V must be disjoint from the full preimage of E. The boundary components of V comprise the entire set of outermost preimages of γ_0. Hence the boundary components of V must lie in the union of the b_is.

■

We next refine the conclusions in the previous two propositions in cases where the topology is simpler.

Proposition 9.8 (Ω not fixed plus disc preimages) *Suppose for all Jordan curves $\gamma \in J(f)$, every component of Ins(γ) is a Jordan domain. Then*

1. *If sign$(\gamma_N) \neq$ sign(γ_{N+1}), then $p_A(\gamma_N) = -p_A(\gamma_{n+1}) = \pm a_\Omega$, i.e. any sign changes are concentrated in a_Ω.*
2. *For any sequence $\{\gamma_n\} \in \Sigma_{Out}\Gamma_0$, if $p_A(\gamma_0) \neq \pm a_\Omega$, there are no sign changes;*
3. *$(\cup_{n>0} f^n\Omega)\backslash\Omega \subset a_\Omega$;*
4. *$f^{-1}\Omega \subset \Omega$.*

Proof: The first statement follows from Proposition 9.6, Ω *not fixed*, and the fact that there is a unique outermost preimage if there is a sign change, since the preimage V of $\mathrm{Ins}(\gamma)$ containing Ω is a Jordan domain, by hypothesis.

To see the next statement, let $\{\gamma_n\}$ be any subsequence containing a sign change and suppose $p_A(\gamma_0) \neq a_\Omega$. Let N be the point at which the first sign change occurs. We may assume that γ_N is positively oriented and γ_{N+1} negatively oriented. Since every preimage of $\mathrm{Ins}(\gamma)$ is a Jordan domain, by hypothesis, and since there are no sign changes until the Nth step, $\mathrm{Ins}(\gamma_N)$ avoids Ω and maps onto $\mathrm{Ins}(\gamma_0)$. The component V of the preimage of $\mathrm{Ins}(\gamma_N)$ containing Ω is a Jordan domain containing Ω and whose boundary is γ_{N+1}. Hence there is a unique outermost preimage. By Proposition 9.6, Ω *not fixed,* $p(\gamma_{N+1}) = a_\Omega$. If $p(\gamma_N) \neq a_\Omega$, then $V \subset \mathrm{Ins}(\gamma_0)$, so $f^N(V) \subset V$. But then Montel's theorem implies that V is in the Fatou set, which is impossible since by hypothesis there are at least two Jordan curves in $\partial\Omega$.

That (2) implies (3) is clear, since the absence of sign changes means the backward orbits of the insides avoid Ω.

Finally, by the first part, it follows that the preimage of $\mathrm{Out}(\partial a_\Omega)$ must be contained in a_Ω. Since $\Omega \subset \mathrm{Out}(\partial a_\Omega)$, the conclusion holds.

■

Proposition 9.9 (**Ω fixed plus disc preimage**) *Suppose E, a, and γ_0 are as in Proposition 9.7. Suppose the component V of the preimage of $Out(\gamma_0)$ containing Ω is a Jordan domain. Then $f^{-1}E \subset b$ for a unique $b \in A$, and Ω is a Jordan domain if and only if $a = b$.*

Proof: The boundary of V is the unique outermost preimage of γ_0, since V is a Jordan domain. Thus $f^{-1}E$ is contained in a unique $b \in A$, by Proposition 9.7, Ω *fixed*. If Ω is already a Jordan domain the statement is trivially satisfied; the other direction follows from the fact that if $a = b$, then $\partial V \subset a$ and so $V \supset \mathrm{Out}(\gamma_0)$. But then V is in the Fatou set. Since $\partial V \subset J(f)$, Ω is a Jordan domain.

■

9.2 Sketch of proof of Theorem 9.1

Theorem 9.2 is essentially a straightforward application of Proposition 9.9, Ω *fixed plus disc preimage*.

Fact 9.10 *If f is a rational map and U is a Jordan domain whose closure contains at most one critical value, then every component of the preimage of U is also a Jordan domain.*

Proof: Since $\overline{U}$ is simply-connected, such a critical value cannot be a critical value for the map restricted to $\overline{V}$ in the sense that the restriction is locally non-injective near the critical point. Hence f restricted to the closure of V is a homeomorphism.

■

There is another basic topological fact about rational maps with exactly two critical points which makes the analysis given in the preceding section much easier, since there are very limited possibilities for the preimage of a Jordan domain. See Figure 9.1.

Fact 9.11 *Let f be a rational map with exactly two critical values. Let U be a Jordan domain and V a component of its preimage. Then exactly one of the following holds:*

1. *if U contains no critical values, $\overline{V}$ maps homeomorphically to $\overline{U}$;*

2. *if $\overline{U}$ contains exactly one critical value in its interior, then V is a Jordan domain mapping as a degree d cover branched over exactly one point in U;*

3. *if U contains two critical values, then V is topologically a sphere minus d disjoint closed discs, the boundaries of which map homeomorphically to their images;*

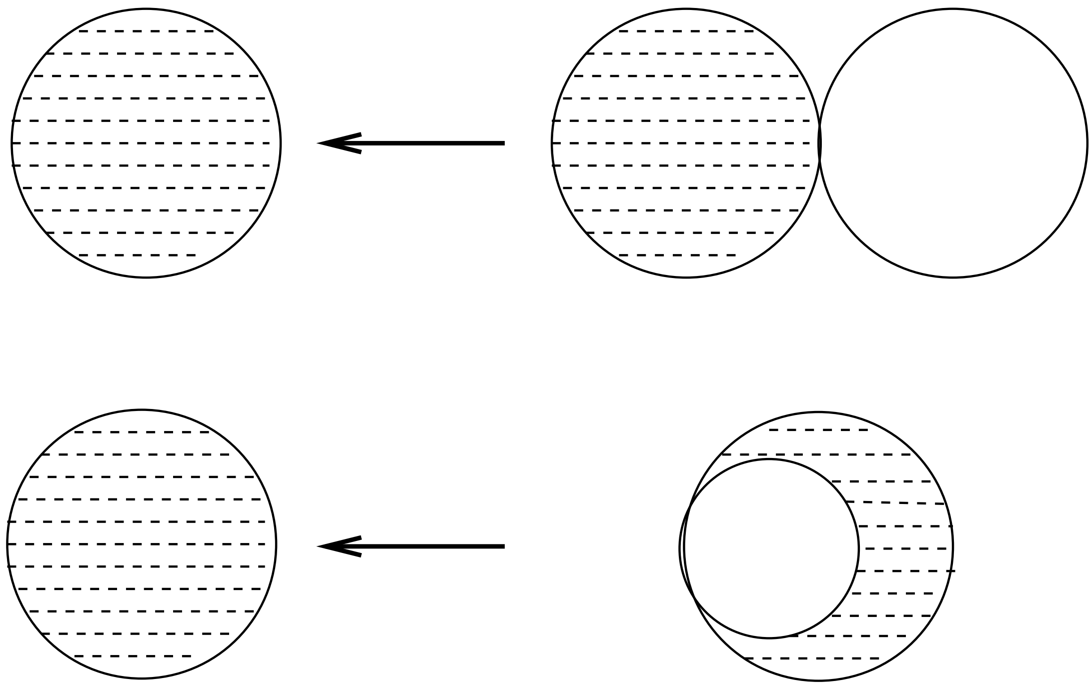

Figure 9.1: In the top figure, U contains a critical value on its boundary and none in its interior. In the bottom figure, U contains a critical value in its interior and on its boundary.

4. *if $\overline{U}$ contains one critical value in its interior and one in its boundary, then V is topologically the sphere minus a union of D closed discs meeting in one common point on their boundaries, and the boundaries of these discs map homeomorphically to their images;*

5. *if $\overline{U}$ contains two critical values on its boundary, then V is a Jordan domain mapping homeomorphically to its image, and this homeomorphism extends to a homeomorphism of $\overline{V}$.*

Proof: This is a topological assertion, not a dynamical one. Choose coordinates on domain and range so that the map is $z \mapsto z^d$ with respect to these coordinates. The fact is then clear.

∎

Note that a component of the preimage of a Jordan domain is either a Jordan domain, or is the full preimage.

Suppose that f is a critically finite hyperbolic quadratic rational map. Then f has exactly two critical points. To prove Theorem 9.1 we use the analysis in the previous section. It is easy to reduce to the following: let c_1 be a period

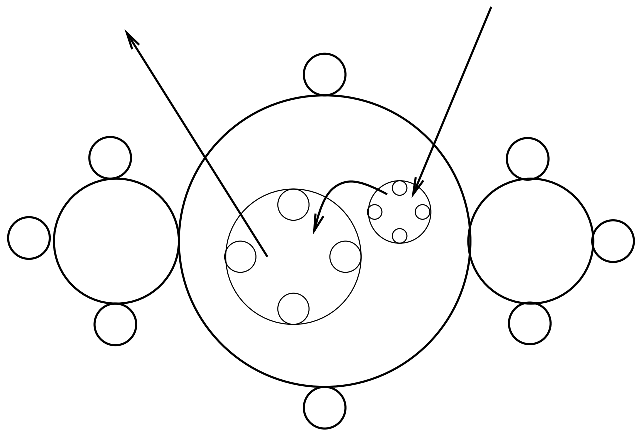

Figure 9.2: The $\Omega_i, i = 1, ..., p-1$ are contained in V.

$p \geq 2$ critical point and let v_1 be the image of c_1. Let Ω_0 be the Fatou component containing v_1. We will show that Ω_0 is a Jordan domain. It is useful to visualize the map conjugated so that v_1 is at the point at infinity. Let $\Omega_i = f^{p-i}\Omega_0, i = 1, ..., p$; note that $\Omega_p = \Omega_0$. Since the Ω_i are Fatou components, they are contained in unique components of $\widehat{\mathbf{C}} \backslash \overline{\Omega_0}, i = 1, ..., p-1$. Since Ω is not fixed, there are sign changes in the set of sequences $\Sigma_{Out}\Gamma_0$. Let a_{Ω_0} be the component of $\widehat{\mathbf{C}} \backslash \overline{\Omega_0}$ containing f_{Ω_0}, and let γ_0 be the positively oriented boundary of a_{Ω_0}.

Step 1

Lemma 9.12 *The set* $\cup_{i=1}^{p-1}\Omega_i \subset a_{\Omega_0}$.

Proof: This follows immediately from Proposition 9.8, Ω *not fixed plus disc preimages.*

■

We now have a basic picture of part of the dynamics; see Figure 9.2.

Step 2

Lemma 9.13 *Let* f, Ω_0, *and* γ_0 *be as above. Let* v_2 *be the other critical value of the map* f. *If* $v_2 \in Int(a_{\Omega_0})$, *then* Ω_0 *is a Jordan domain.*

Proof:

Let $D_0 = \mathrm{Out}(\gamma_0)$. The pth iterate of f fixes $\Omega_1 \subset a_{\Omega_0}$, so by Proposition 9.9, Ω *fixed plus disc preimage*, it suffices to prove that the component D_p of the preimage of D_0 under f^p containing Ω_0 is a Jordan domain. We prove this by pulling back D_0 along the orbit of Ω_0 and using induction.

Let D_i be the component of the preimage of D_0 under f^p containing $\Omega_i, i = 0, ..., p$. We first claim that $D_1 \subset a_\Omega$. Since $v_2 \in a_{\Omega_0}$, D_0 contains exactly critical value in its closure, so D_1 is a Jordan domain. The boundary γ_1 of D_1 is the full preimage of the boundary of D_1, which is γ_0. By Proposition 9.6, Ω *not fixed*, we must have $\gamma_1 \subset a_{\Omega_0}$ and its sign must be negative. Since the sign of $gamma_1$ is negative, $\text{Out}(\gamma_1) = D_1$ avoids Ω_0 and so $D_1 \subset a_{\Omega_0}$.

We now use induction. Assume D_i is a Jordan domain contained in $a_{\Omega_0}, i = 1, ..., n$. Then D_{n+1} is also a Jordan domain since D_n contains at most one critical value in its closure. A sign change between γ_n and γ_{n+1} implies that $\gamma_{n+1} \subset a_{\Omega_0}$, by Proposition 9.8, Ω ***not fixed plus disc preimages***, and hence that $D_{n+1} \supset D_0$. But this implies that Ω_0 is a Jordan domain fixed under the $(n + 1)$st iterate of f, which is impossible if $n + 1 < p$. The absence of a sign change then implies that $D_{n+1} \subset a_{\Omega_0}$, and so the induction proceeds. At the $(n+1)$st stage, we must see a sign change since $D_p \supset \Omega_0$. This proves the step.

∎

Step 3

So to prove the theorem, it suffices to show that $v_2 \in \text{Ins}(\gamma)$.

The proof of this step requires more information than used in the preceding steps. In particular, we need to know that if D_p is the component of the preimage of $D_0 = \text{Out}(\gamma)$ under f^p containing Ω_0, then $\partial D_p \subset \partial \Omega_0$. We also need the fact that a hyperbolic critically finite rational map cannot send a Jordan curve in the Julia set homeomorphically to itself.

We argue by contradiction. Again, let $D_0 = \text{Out}(\gamma)$. If $v_2 \notin \text{Ins}(\gamma)$, the preimage of D_0 is topologically an annulus D_1 whose boundary components map univalently to ∂D_0. The fact that there are only two critical points, plus the picture obtained in the first step, implies that as we pull back along the orbit of c_1, we obtain annuli $D_i \subset a_{\Omega_0}$ such that D_{i+1} maps homeomorphically to $D_i, i = 2, ..., p - 1$, and $\partial D_i \subset \partial \Omega_i, i = 0, ..., p$; see Lemma 9.14. In Figure 9.3 we have drawn the situation for degree three hyperbolic maps.

We now apply Proposition 9.7, Ω *fixed*, to the pth iterate of f to conclude that there is a sequence of the form $\{a_{\Omega_0}, a_{\Omega_0}, *\}$. But then the discussion above implies that γ_0 maps homeomorphically to its image under f^p, which is impossible for a Jordan curve in the Julia set of a hyperbolic map.

∎

Extending to higher-degree maps and to non-hyperbolic maps

The proofs of the first two steps do not make reference to the degree of the map. The proof of the third step also goes through: the only difference is that D_1 is homeomorphic to a sphere minus d disjoint closed discs, where d is the degree of the map.

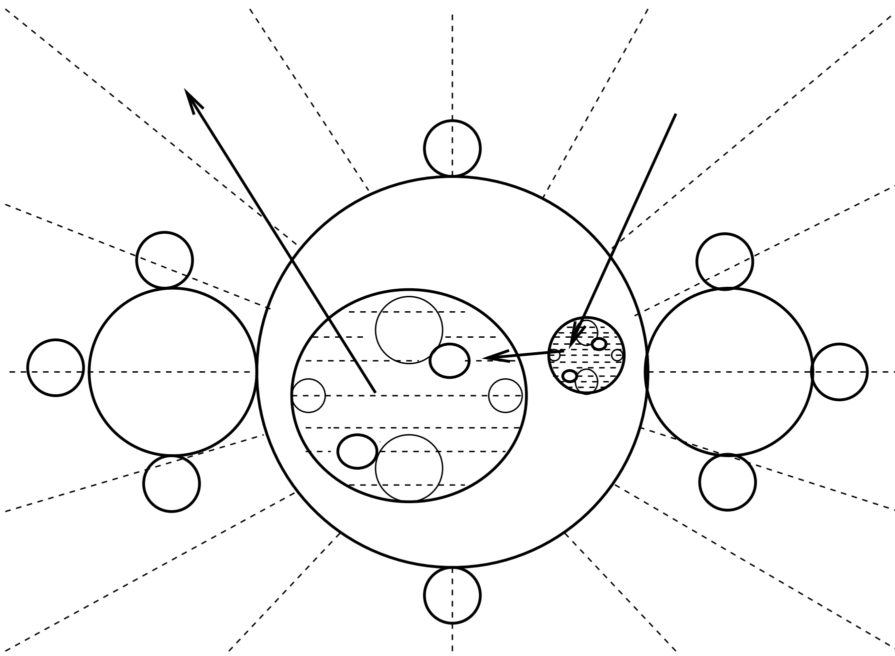

Figure 9.3: The dashed regions $D_i \supset \Omega_i, \partial D_i \subset \partial\Omega_i$, and $D_i \subset V, i = 1, ..., p-1$.

The results stated in Section 9.1 remain true even when there is a single critical points in the Julia set: if a curve in the Julia set passes through the critical value, components of its preimage are topologically one-point unions of Jordan curves, and we may take these curves to be the "preimages" of the curve.

So the first two steps may be proved in precisely the same manner. Moreover, the third step also goes through with minimal modifications. Instead of D_1 being homeomorphic to a sphere minus d disjoint closed discs, it is homeomorphic to a sphere minus d closed discs which meet in exactly one point. Also, critically finite maps are expanding on their Julia sets with respect to the canonical orbifold metric; this rules out the existence of Jordan curves in the Julia set mapping homeomorphically to themselves. The remainder of the proof of this step also applies in this case.

9.3 Background

We will denote the two critical *points* of a critically finite map f with only two critical points by c_1 and c_2 , and the corresponding critical *values* by v_1 and v_2 (they are necessarily distinct). We will only consider maps with at least one periodic critical point which we will denote by c_1, for otherwise the Fatou set of f is empty. Throughout, U and V, possibly with subscripts, will denote open discs, where "disc" means a simply connected subset of $\widehat{\mathbf{C}}$ whose complement contains at least two points. We will reserve the notation Ω, possibly with subscripts, for components of the Fatou set.

9.3.1 Topology

The following lemma is somewhat technical. The idea is to control how the boundaries of two nested sets behave under taking preimages. We will only need the case where X_0 is a Fatou component homeomorphic to an open disc and Y_0 is homeomorphic to the sphere minus a finite union of closed discs whose boundaries meet in at most one point. See Figure 9.4.

Lemma 9.14 *Let $f : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be a rational map. Let X_0 and Y_0 be proper open subsets of $\widehat{\mathbf{C}}$with $X_0 \subset Y_0$. Suppose $\partial Y_0 \subset \partial X_0$.*

1. *If $Y_1 = f^{-1}Y_0$ and $X_1 = f^{-1}X_0$, then $\partial Y_0 \subset \partial X_0$.*
2. *If Y_1 is a component of $f^{-1}Y_0$, if $f|_{\overline{Y_1}} : \overline{Y_1} \to \overline{Y_0}$ is a homeomorphism, and if $X_1 = (f|_{\overline{Y_1}})^{-1}(X_0)$, then $\partial Y_1 \subset \partial X_1$.*

Proof:

1. Since f is a nonconstant rational map, it is an open map, and so for any proper open subset $Z \subset \widehat{\mathbf{C}}$, $f^{-1}\partial Z = \partial f^{-1}Z$. So $\partial Y_1 = \partial f^{-1}Y_0 = f^{-1}\partial Y_0 \subset f^{-1}\partial X_0 = \partial f^{-1}X_0 = \partial X_1$.

2. Since Y_1 is a component of $f^{-1}Y_0$, $f(\partial Y_1) \subset \partial Y_0$. Since $f : \overline{Y_1} \to \overline{Y_0}$ is a homeomorphism and $X_1 = (f|_{\overline{Y_1}})^{-1}(X_0), \partial X_1 = (f|\overline{Y_1})^{-1}(\partial X_0)$. Hence $\partial Y_1 \subset \partial X_1$.

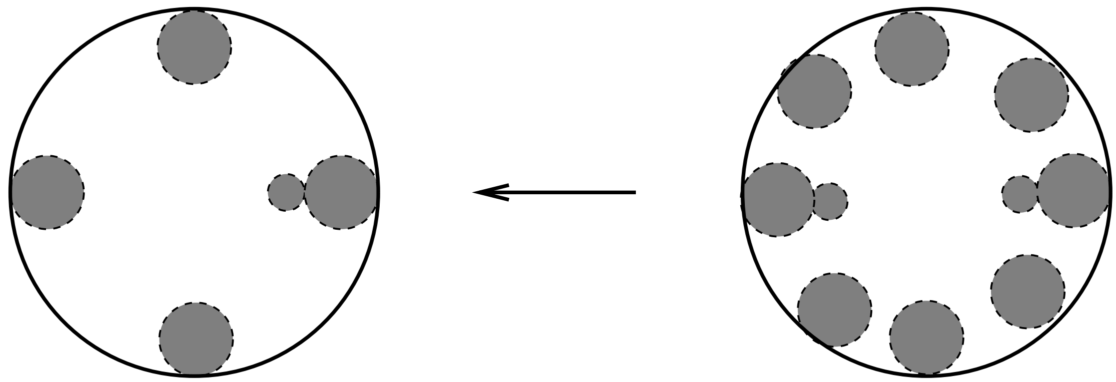

Figure 9.4: Y_0 and Y_1 are the large discs. X_i is the complement of the shaded discs in $Y_i, i = 1, 2$.

■

Remark: The second statement is no longer true if we drop the requirement that $\overline{Y_1}$ maps homeomorphically to $\overline{Y_0}$. For example, let Y_0 be the open unit disc, ,let X_0 be the open disc minus the interval [0,1), and let $f(z) = z^2$ map the Riemann sphere to itself. Let X_1 be the intersection of the upper half-plane $\{z|Im(z) > 0\}$ with the unit disc, and let Y_1 be the unit disc again. Then all other hypotheses of the lemma are satisfied but $\partial Y_1 \not\subset \partial X_1$.

9.3.2 Expanding nature of critically finite maps

For reference, we state the corollary to Montel's theorem which we have been using.

Lemma 9.15 *Let f be a rational map such that some iterate maps an open set U into itself. Suppose the complement of U contains at least three points. Then U is in the Fatou set.*

Proof: This follows immediately from Montel's Theorem: the iterates of f, when restricted to U, must all avoid three values, and hence form a normal family of holomorphic functions. So U is in the Fatou set.

■

Critically finite maps have important expanding properties analogous to those of hyperbolic maps. For the definition of orbifold, the canonical orbifold associated to a critically finite map, and the definition of the orbifold Poincaré or Euclidean metric associated to this orbifold, see [Mil2]. This metric is constructed so as to behave very much like the usual Poincaré or Euclidean metric. on $\widehat{\mathbf{C}}\backslash P(f)$. Let $P_{hyp}(f)$ denotes the set of postcritical points which eventually land on cycles containing critical points. Then the orbifold metric is supported on $\widehat{\mathbf{C}}\backslash P_{hyp}(f)$ and lifts under f to a metric on $\widehat{\mathbf{C}}\backslash f^{-1}P_{hyp}(f)$. With respect to this metric on $\widehat{\mathbf{C}}\backslash f^{-1}P_{hyp}(f)$ and the orbifold metric on $\widehat{\mathbf{C}}\backslash P_{hyp}(f)$, the inclusion is a contraction. Hence

Proposition 9.16 *Let f be a critically finite map. Then f is uniformly expanding with respect to the canonical orbifold metric on the complement of a neighborhood of $f^{-1}P_{hyp}(f)$. In particular, f is uniformly expanding on $J(f)$.*

Proof: See [Mil2], Theorem 14.4.

The next two facts may then be proved in exactly the same manner as for hyperbolic maps.

Lemma 9.17 *Let f be a critically finite map and K a compact connected set in $J(f)$. If $f(K) = K$ and f is injective on K, then K is a point.*

Proposition 9.18 *Let f be a critically finite rational map and Ω a period p Fatou component. Then $\partial\Omega$ and $\overline{\Omega}$ are locally connected and locally path connected.*

9.4 Proof of Theorems

For this section, f will denote a critically finite rational map with exactly two critical points, at least one of which is periodic of period $p \geq 1$. We denote by

- c_1, a periodic critical *point* of period $p \geq 1$;
- v_1, the image of c_1;
- v_2, the other critical *value*;
- Ω_0, the Fatou component containing v_1;

  If $p \geq 2$, denote by

- $\Omega_i = f^{p-i}\Omega_0$, i.e. the Fatou component containing $f^{p-i}(v_1)$, $i = 1, ..., p$;
- γ_0, the boundary of a_{Ω_0}, oriented so $\mathrm{Out}(\gamma_0) \supset \Omega_0$;
- V, the inside $\mathrm{Ins}(\gamma_0)$. This is the same as the interior of a_{Ω_0}.
- D_0 the outside $\mathrm{Out}(\gamma_0)$. This is the same as the complement of a_{Ω_0}.
- D_i, the component of $f^{-i}D_0$ containing $f^{p-i}(v_1)$, $i = 1, ..., p$.

We further assume by conjugating f that the point v_1 is at infinity. We let f denote this new map as well. Then $\widehat{\mathbf{C}}\backslash\Omega_0$ is a full set in the plane with locally connected boundary, by Lemma 9.18. If $p = 1$ the map f is conjugate to a polynomial, and Ω_0 corresponds to the basin at infinity. By Lemma 2.34, V and D_0 are *open* discs with Jordan curve boundaries, and their closures are closed discs. Since $\Omega_0 \subset D_0$, we have that $\Omega_i \subset D_i, i = 1, ..., p$. Note also that $\partial D_0 \subset \partial\Omega_0$, and that $\Omega_0 = \Omega_p$. Also, note that $\partial D_i \subset J(f)$.

9.4.1 Proof of Theorem 9.1

We first prove that Ω_0 is a Jordan domain if $p \geq 2$. As mentioned in Section 9.2, it suffices to prove that $v_2 \in V$.

Lemma 9.19 *Suppose Ω is not fixed under f. Then $v_2 \in V$.*

Proof:

1. We first show that $\partial D_p \subset \partial\Omega_0$.

   Since $c_1 \in \Omega_0 \subset D_1$, the first case of Lemma 9.14 applies, and so $\partial D_1 \subset \partial\Omega_1 \subset \overline{V}$. It follows that D_1 must be contained in V. For otherwise, $D_1 \supset D_0$, and so $D_1 \subset F(f)$, by Lemma 9.15. But then $p = 1$ and so Ω is fixed.

   We now argue by induction. Assume that D_i is contained in V, and that $\partial D_i \subset \partial\Omega_i$. We show that this implies $\partial D_{i+1} \subset \partial\Omega_{i+1}, i = 1, ..., p-1$, and that D_{i+1} is contained in V if $i = 1, ..., p-2$. Since $\Omega_i \subset V$, and $D_i \subset V$ with $\partial D_i \subset \partial\Omega_i$, D_i is contained in V, for otherwise D_i contains D_0, implying that Ω_0 is fixed under f^{i+1}. The set V is an open topological disc with Jordan curve boundary in $J(f)$ containing at most one critical value in its closure. Let V' be the component of $f^{-1}V$ containing D_{i+1}. Since V contains no critical values, by Fact 9.11, the basic fact for maps with two critical points, $f|_{\overline{V'}} : \overline{V'} \to \overline{V}$ is a homeomorphism. By restriction, $f|_{\overline{D_{i+1}}} : \overline{D_{i+1}} \to \overline{D_i}$ is also a homeomorphism. We may now apply the second case of Lemma 9.14 to conclude that $\partial D_{i+1} \subset \partial\Omega_{i+1}$. Moreover, if $i + 1 < p$, then $\Omega_{i+1} \subset V$, and hence $\partial\Omega_{i+1} \subset \overline{V}$. An application of Lemma 9.15 again shows that $D_{i+1} \subset V$. Hence $\partial D_p \subset \partial\Omega_p = \Omega_0$, and $D_p \supset \Omega_0$.

2. We next claim that the boundary of every component of the complement of D_{i+1} (for convenience, let us call these *boundary pieces* of D_i), $i = 0, ..., p-1$ maps injectively onto its image under f, and so that every boundary piece of D_p maps injectively onto its image ∂D_0 under f^p.

   This follows easily: the map $f : D_1 \to D_0$ has this property, by the basic fact for maps with two critical point, and for $1 \leq i \leq p-1$, the map $f : \overline{D_{i+1}} \to \overline{D_i}$ is a homeomorphism, by the argument given above.

   The next step provides a contradiction.

3. We now claim that f^p maps γ_0 homeomorphically to itself, which violates Lemma 9.17.

   By Proposition 9.7, Ω *fixed*, applied to f^p, there must be some boundary piece of D_p contained in a_{Ω_0}. But since $\partial D_p \subset \partial\Omega$, this implies that some boundary piece of D_p is actually equal to ∂a_{Ω_0}. The map f restricted to a single boundary component of $D_i, i = 1, ..., p$ is one-to-one, by the previous step, and so the map f^p sends γ_0 homeomorphically to itself.

∎

Proof of Theorem 9.1

Suppose f is conjugate to a polynomial which is not z^d. A critically finite polynomial which is not conjugate to $z \mapsto z^d$ cannot have a Jordan domain for its basin of infinity. For such a polynomial has at least two bounded Fatou components, and the boundaries of these components are distinct Jordan curves, by lemma 2.34. These Jordan curves are contained in the connected Julia set, which is the boundary of the basin of infinity. It follows that the basin of infinity cannot be a Jordan domain.

So we may assume that f is not conjugate to a polynomial, and so $p \geq 2$.

By the previous argument, Ω_0 is a Jordan domain. Moreover, $v_2 \notin \overline{\Omega_0}$, by Lemma 9.19. Hence Ω_1 is a Jordan domain, since it is a covering of a closed disc branched over exactly one point in its interior. Moreover, Ω_1 is the unique preimage of Ω_0 so any Fatou component mapping to Ω_0 maps first onto Ω_1. Any other Fatou component in the grand orbit of Ω_0 can contain at most one critical value in its closure, since $v_1 \in \Omega_0$. Induction now shows that any Fatou component in the grand orbit of Ω_0 is a Jordan domain. Since our choice of periodic critical point was arbitrary, every Fatou component in the grand orbit of a periodic Fatou component is a Jordan domain. Since every Fatou component of f is eventually periodic, these account for all Fatou components, and so every Fatou component is a Jordan domain.

∎

9.4.2 Proof of Theorem 9.2

Given an element $x \in P(f)$, there is a partial ordering on $P(f)$ defined as follows: for two elements p and q of $P(f)$, $p < q$ if the boundary of the Fatou component containing q separates p from x.

Let y be any minimal element with respect to this ordering. Then y is periodic of period $p \geq 1$, since f is postcritically periodic. Let Ω be the Fatou component containing y and let Γ_0 be the set of oriented Jordan curves in the boundary of Ω. Since y is minimal and f is hyperbolic, $P(f)\backslash\{y\} \subset \text{Ins}(\gamma_0)$, for some unique $\gamma_0 \in \Gamma_0$. Moreover, since f is postcritically periodic, $(f^{-p}(P(f)\backslash\{y\})) \cap (P(f)\backslash\{y\}) \neq \emptyset$. (If $p \geq 1$ this is obvious, since f^p must fix every point in the orbit of y; if $p = 1$, this follows since not all points in the postcritical set can land on y if f is postcritically periodic). Hence $f^p \mathrm{Ins}(\gamma_0) \cap \mathrm{Ins}(\gamma_0) \neq \emptyset$. Moreover, $\mathrm{Out}(\gamma_0)$ contains a unique critical value of f^p in its closure, since y is minimal, hence every component of the preimage of $\mathrm{Out}(\gamma_0)$ under f^p is a Jordan domain. Proposition 9.9, Ω *fixed plus disc preimage*, applied to the pth iterate of f now shows that Ω_0 is a Jordan domain. Since f is hyperbolic, there are no elements of $P(f)$ in $\partial\Omega$. So every preimage Ω' of Ω is also a Jordan domain, since $\overline{\Omega'}$ is a branched cover of $\overline{\Omega}$ branched over at most one point.

■

Proof of Corollary 3 By the above Theorem, the periodic cycle of Fatou components consists of Jordan domains. Since there are no critical points in the Julia set, there are no critical values for iterates of f in the boundaries of these Jordan domains. Hence they all pull back to Jordan domains under iterates of f.

■

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