Topological Conditions for Positive Lyapunov Exponent in Unimodal Maps

Duncan James Sands
St John's College
December 1993

Submitted for the degree of Doctor of Philosophy

“This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration.”

Duncan Sands
December 1993

Summary

In this thesis we investigate the topological nature of the Collet-Eckmann condition [11] for S-unimodal maps of the interval. The Collet-Eckmann condition holds, by definition, if the Lyapunov exponent is positive along the orbit of the critical point; this is a metric condition. It implies that the Lyapunov exponent is positive Lebesgue almost everywhere [29] and that the map is chaotic [11, 45]. We relate topological and metric properties using an extended form of Hofbauer's tower construction [25]. The main results are as follows:

1. We prove that the Collet-Eckmann condition is invariant under quasi-symmetric conjugacy between S-unimodal maps. It follows that the Collet-Eckmann condition is a topological invariant of S-unimodal maps with quadratic critical points, answering a conjecture of Van Strien [50] and of Guckenheimer [18]. This uses a new result of Lyubich communicated by him to the author.

2. We show that any S-unimodal map whose kneading invariant satisfies certain simple conditions satisfies the Collet-Eckmann condition. The conditions are topological analogues of those used by Benedicks and Carleson [5] in their proof of Jakobson's theorem [26]. We also give examples of kneading invariants corresponding to failure of the Collet-Eckmann condition. This falls short of a complete classification of kneading invariants into Collet-Eckmann or non-Collet-Eckmann.

3. We prove that Lebesgue almost every value of the topological entropy has the property that any S-unimodal map with that topological entropy must satisfy the Collet-Eckmann condition. This is analogous to Lebesgue almost every rotation number having the property that any smooth circle map with that rotation number must possess an absolutely continuous invariant measure [23].

Dedicated to my family
and to Agathe

Contents

List of Figures

Introduction

In this thesis we investigate the topological nature of the Collet-Eckmann condition for S-unimodal maps of the interval. Notation and terminology are defined in chapter 1.

Pierre Collet and Jean-Pierre Eckmann introduced the Lyapunov exponent condition

$$\liminf_{i\to\infty} \frac{\log |Df^i(c_1)|}{i} > 0,$$

now called the Collet-Eckmann condition, in their 1983 paper "Positive Liapunov Exponents and Absolute Continuity for Maps of the Interval" [11]. They showed that any S-unimodal map satisfying their condition is chaotic, because it has an absolutely continuous invariant measure.[1] An S-unimodal map satisfying the Collet-Eckmann condition is called Collet-Eckmann.

The Topological Invariance of the Collet-Eckmann Condition

Our first major result is that the Collet-Eckmann condition is quasi-symmetrically invariant for S-unimodal maps and therefore topologically invariant for S-unimodal maps with quadratic critical points. The idea that the Collet-Eckmann condition might be topologically invariant is due to Van Strien [50] and Guckenheimer [18].

We prove that the Collet-Eckmann condition is invariant under quasi-symmetric conjugacy between S-unimodal maps in chapter 4. Since Lyubich has recently announced that all topological conjugacies between relevant S-unimodal maps with quadratic critical points are quasi-symmetric, it follows that the Collet-Eckmann condition is topologically invariant for such maps. By this we mean that any S-unimodal map with a quadratic critical point that is topologically conjugate to a Collet-Eckmann S-unimodal map with a quadratic critical point is also Collet-Eckmann. Lyubich's result is as yet unpublished and was communicated to me by him.

These results do not prove topological invariance when, for example, critical points

[1] They also required another condition which Nowicki has since shown to be unnecessary [45].

are non-flat rather than quadratic, or when the maps do not have negative Schwarzian derivative. A proof of Lyubich's result using only real techniques might avoid the need for a quadratic critical point.

Collet-Eckmann Kneading Invariants

Our second major result is that any S-unimodal map satisfying certain weak topological conditions is Collet-Eckmann.

We express this in terms of kneading invariants. We call a kneading invariant Collet-Eckmann if every S-unimodal map with this kneading invariant is Collet-Eckmann (not just those with quadratic critical points). We call a kneading invariant non-Collet-Eckmann if no S-unimodal map with this kneading invariant is Collet-Eckmann.

In chapter 4 we give simple conditions for a kneading invariant to be Collet-Eckmann. The kneading invariants satisfying these conditions strictly include the Misiurewicz kneading invariants, the largest class of kneading invariants previously known to be Collet-Eckmann [40]. Our conditions are analogous to the metric conditions that Benedicks and Carleson used in proving that quadratic maps are Collet-Eckmann for a positive Lebesgue measure set of parameter values [4, 5, 14].

We also describe two classes of non-Collet-Eckmann kneading invariants. This falls short of a complete classification of kneading invariants into Collet-Eckmann and non-Collet-Eckmann. Such a classification would be helpful for understanding why some sorts of combinatorial behaviour are intrinsically metrically chaotic.

The Topological Abundance of Collet-Eckmann Maps

Our last major result is that most kneading invariants are Collet-Eckmann.

We measure kneading invariants in terms of Lebesgue measure on their topological entropies. We call a value of the topological entropy Collet-Eckmann if every S-unimodal map with this topological entropy is Collet-Eckmann.

In chapter 5 we prove that Lebesgue almost every value of the topological entropy is Collet-Eckmann, so in this sense almost every kneading invariant is Collet-Eckmann. By contrast, in this same sense it is known that almost no kneading invariant is Misiurewicz [7].

Analogies with Circle Maps

Similar results are known for circle maps. Herman has shown that Lebesgue almost every rotation number has the property that any smooth circle map with that rotation number is smoothly conjugate to the corresponding rotation, and therefore has an absolutely continuous invariant measure [23, 14]. This is analogous to our result that Lebesgue almost every topological entropy is Collet-Eckmann, since every S-unimodal Collet-Eckmann map has an absolutely continuous invariant measure [11, 45].[2]

More is known for circle maps. Yoccoz has given necessary and sufficient conditions in the C^∞ case for a rotation number to have Herman's smooth conjugation property [51]. Our conditions for a value of the topological entropy to be Collet-Eckmann are only necessary.

In addition Herman has shown that rotation numbers with his smooth conjugation property are taken on by a positive Lebesgue measure set of parameter values in any reasonable one-parameter family of smooth circle maps [22, 14]; for this positive Lebesgue measure set of parameter values the circle maps therefore possess absolutely continuous invariant measures. The analogous result for S-unimodal maps would be that Collet-Eckmann values of the topological entropy are taken on by a positive Lebesgue measure set of parameter values in any reasonable one-parameter family of S-unimodal maps. A demonstration of this, presumably by analysing the topological entropy function, would give an interesting new proof of Jakobson's theorem [26, 14].

Structure of the Thesis

In chapter 1 we describe those definitions and results from the general theory of unimodal maps that will be used later. It is here that we define such terms as *S-unimodal*, *kneading invariant*, *topological entropy.*

In chapter 2 we describe a modified version of Hofbauer's tower construction [25] and an extension of his kneading invariant analysis. These are used throughout the thesis.

In chapter 3 we show how to approximate tent-map towers using parameter space information. The results are used in chapter 5.

We start our analysis of the Collet-Eckmann condition in chapter 4.

In the first section of chapter 4 we prove that the Collet-Eckmann condition is quasi-

[2] Smooth conjugacy between S-unimodal maps and tent-maps is not feasible, so a more direct analogy is unlikely (but see Nowicki and Przytycki [42]).

symmetrically invariant for S-unimodal maps and deduce that the Collet-Eckmann condition is topologically invariant for S-unimodal maps with quadratic critical points.

In the second section of chapter 4 we describe our class of Collet-Eckmann kneading invariants. In the final section of chapter 4 we describe two classes of non-Collet-Eckmann kneading invariants.

In chapter 5 we prove that Lebesgue almost every value of the topological entropy is Collet-Eckmann.

Acknowledgements

This thesis is the outcome of my studies at the Statistical Laboratory of the University of Cambridge. Everyone at the Statistical Laboratory has been very supportive. My supervisor, Colin Sparrow, deserves all my gratitude for his patience and efforts on my behalf. He has read, corrected and suggested improvements to each version. Through him I met Sebastian van Strien, whom I cannot thank enough. Without his help and encouragement this thesis could not have been written. I would like to thank Henk Bruin for teaching me about towers and sharing his ideas. My friend Roderick MacKenzie has inspired and helped me, and distracted me from my woes. Tomasz Nowicki made several useful comments on the results of chapters 3 and 5.

The Commonwealth Scholarship Commission in the United Kingdom was my main source of funding. The generosity and flexibility of the Commission enabled me to spend a year at the Ecole Normale Supérieure of Paris. The financial assistance of the Ecole Normale was unexpected and very welcome. The Statistical Laboratory of the University of Cambridge, the Technical University of Delft and St John's College, Cambridge, have also provided generous financial support.

Chapter 1

Basic Definitions and Results

Unimodal maps are interesting for both physical and mathematical reasons. They have the capacity to generate extremely complicated behaviour in spite of their apparent simplicity. They have been used to model everything from insect populations to the onset of turbulence with varying degrees of success. Since the physical aspects lie outside the scope of this thesis we refer the interested reader to the reprint collections of Cvitanović [13] and Hao Bai-Lin [21], the review paper of Eckmann and Ruelle [16], and the book of Collet and Eckmann [10].

Their mathematical study goes back to the early years of this century when Fatou [17] and Julia [28] already knew that some unimodal maps have infinitely many periodic points. Further development was slow until the seventies, since when the field has greatly expanded and matured. The books of Devaney [15], Collet and Eckmann [10], and especially de Melo and Van Strien's "One-dimensional Dynamics" [14] describe the modern theory excellently and should be considered general references for this thesis. Devaney's exposition is more elementary than the others, if more up-to-date than Collet and Eckmann's. "One-dimensional Dynamics" is the most comprehensive of the three.

In this chapter we describe the definitions and results from the theory of unimodal maps that we will use; it is not intended to be a review of the field. The material is more or less covered in each of the above three books. The section "Unimodal Maps" defines some general terminology. "The Class $\mathcal{C}$ of S-unimodal Maps" introduces the maps considered in this thesis. "Kneading Invariants and Topological Conjugacy" defines the kneading invariant and gives conditions for it to characterize a unimodal map. The relationship between a unimodal map and the tent-map with the same topological entropy is described in "Topological Entropy and Conjugacy with Tent-maps".

1.1 Unimodal Maps

In this section we define unimodal maps and terminology such as *periodic attractor*, *wandering interval*, *renormalizable map*. The definitions are standard. We simply introduce the particular forms and notations used in this thesis.

A *unimodal map* is a continuous function $f : [0;1] \to [0;1]$ [1] for which

1. $f(0) = f(1) = 0$.
2. There exists c in $(0;1)$ such that f is strictly increasing on $[0;c]$ and strictly decreasing on $[c;1]$.
3. The restrictions of f to $[0;c]$ and to $[c;1]$ are C^1.

The restriction of a function f to a set I is denoted $f\big|_I$. The last condition ensures that the derivatives of $f\big|_{[0;c]}$ and $f\big|_{[c;1]}$ have bounded magnitude. If f is differentiable at c then the derivative of f, which we write as f' or Df, is also of bounded magnitude. The point c is unique; it is called the *critical point* of f.

Example 1.1 *The* quadratic map $Q_a : x \mapsto ax(1-x)$ *is unimodal (with critical point* $c = 1/2$ *) whenever* a *is in the range* $0 < a \leq 4$. *Every unimodal quadratic map belongs to the class* $\mathcal{C}$ *of S-unimodal maps described in the next section.*

Example 1.2 *The* tent-map T_λ *is composed of two straight line segments of slope* λ *and* $-\lambda$*:*

$$T_\lambda(x) = \begin{cases} \lambda x & \text{if } x \leq \frac{1}{2} \\ \lambda(1-x) & \text{if } x \geq \frac{1}{2}. \end{cases}$$

It is unimodal (with critical point $c = 1/2$ *) if* λ *lies in* $(0;2]$. *In this thesis we only consider the range* $1 \leq \lambda \leq 2$. *No tent-map belongs to the class* $\mathcal{C}$.

[1] $[a;b]$ denotes the smallest closed interval containing both a and b regardless of the order of a and b in the real line; $(a;b)$ is its interior.

1.1.1 Orbits and Limit Sets

Repeated application of f to a point x produces its *orbit* $\mathcal{O}(x) = \{x, f(x), f(f(x)), \ldots\}$. Using f^n to indicate the n-fold composition $\underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$ and writing x_n for $f^n(x)$ allows us to write this as $\mathcal{O}(x) = \{x, x_1, x_2, \ldots\}$. By f^0 we mean the identity function. An orbit can be finite or infinite. Its *ω-limit set*

$$\omega(x) = \bigcap_{n \geq 1} \overline{\mathcal{O}(x_n)}$$

describes the region it eventually approaches. We say that a set A *attracts* x if $\omega(x) \subseteq A$. We call x *recurrent* if $x \in \omega(x)$.

1.1.2 Periodic Behaviour

A *fixed point* is one satisfying the equation $f(x) = x$. A *periodic point* corresponds to $f^n(x) = x$ for some positive integer n. The number n is called a *period* of x; the smallest positive integer p for which $f^p(x) = x$ holds is the *least period* of x. The set $P = \{x, x_1, x_2, \ldots, x_{p-1}\}$ is the *periodic orbit* generated by x, equal to the orbit of any of its elements. A fixed point generates a periodic orbit of length 1.

The *basin* of a periodic orbit P is the set of points attracted to it:

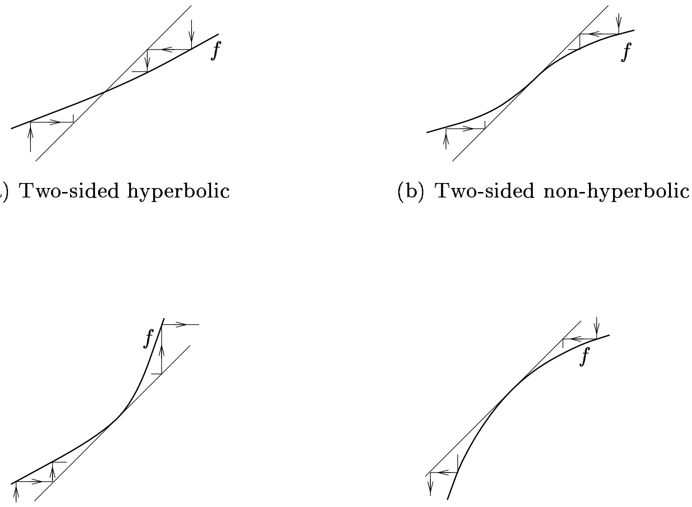
$$B_P = \{y \in [0; 1] \mid \omega(y) = P\}$$

The *immediate basin* is the union of those connected components of B_P that contain points of P. An *attracting periodic orbit*, or *periodic attractor*, has a basin with non-empty interior. It need not be attracting from both sides, as illustrated in figure 1.1. A periodic attractor is a *super-attractor* if it contains c.

1.1.3 Homtervals and Wandering Intervals

A *homterval* is a non-degenerate interval I for which $f^i|_I$ is a homeomorphism for every $i \geq 1$.

Lemma 1 *Take $[a; b]$ a subset of $[0; 1]$ and $i \geq 1$. Then $f^i\big|_{[a;b]}$ is a homeomorphism if and only if c is not in $(a_j; b_j)$ for any $0 \leq j < i$.*



(a) Two-sided hyperbolic (b) Two-sided non-hyperbolic

(c) Attracting from the left (d) Attracting from the right

Figure 1.1: Some attracting fixed points.

The images of a homterval therefore never contain c in their interior. The basin of every attracting periodic orbit contains a homterval. Not every homterval limits to a periodic attractor however: a *wandering interval* is a homterval of which none of the points are in the basin of a periodic attractor. Wandering intervals are pathological; well behaved maps do not have them (see section 1.2).

Example 1.3 *The tent-map T_λ has no homtervals for $1 < \lambda \leq 2$ and therefore no periodic attractors or wandering intervals. A tent-map for which the critical point is periodic may nonetheless show some of the dynamical characteristics of a map with a periodic attractor. For this reason we call such tent-maps* periodic.

1.1.4 Renormalization

A unimodal map is *renormalizable* if some iterate of it, when restricted to an appropriate subinterval, is itself unimodal (see figure 1.2). More exactly, f is renormalizable of degree n if f^n has a restrictive central point, defined below.

Define the map τ by $\tau(c) = c$ and, for $x \neq c$, $f(\tau(x)) = f(x)$ where $\tau(x) \neq x$. This takes x to the dynamically symmetric point on the other side of c (see figure 1.3).

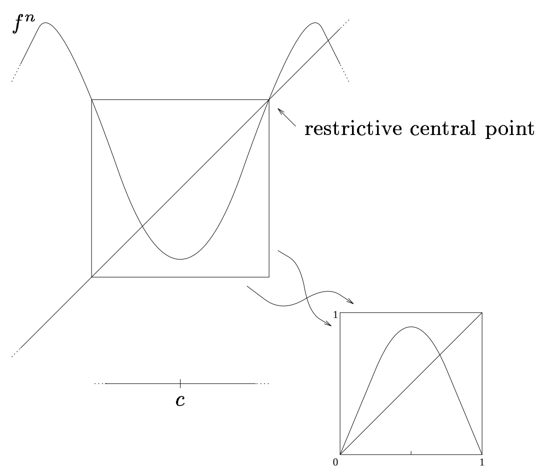


Figure 1.2: Renormalizing a unimodal map.

We call $x \neq c$ a *central point* of f^n if $f^n(x)$ equals x and f^n is increasing on $(x; c)$.[2] A central point plays the same role for f^n that the fixed point 0 plays for f.

A central point is *restrictive* if f^n maps $[x; \tau(x)]$ into itself. In this case $f^n\big|_{[x;\tau(x)]}$ can be affinely rescaled to obtain a new unimodal map, called a *renormalization* of f.

If f has no periodic attractors and is not renormalizable or is renormalizable for only finitely many values of n then we call it *finitely renormalizable.* When f has no periodic attractors and is renormalizable for infinitely many values of n we call it *infinitely renormalizable.*

Example 1.4 *For a given tent-map T_λ, $\lambda \neq 1$, Lebesgue-almost every point in $[0; 1]$ is attracted to the same set A_λ, a finite union of intervals if λ is in $(1; 2]$. Figure 1.5 plots A_λ against λ (compare figure 1.4).*

The number of intervals is equal to the maximum degree for which T_λ is renormalizable. There is only one interval for $\sqrt{2} \leq \lambda \leq 2$: these tent-maps are not renormalizable. For $1 < \lambda < \sqrt{2}$ the tent-map is renormalizable of degree 2. The point $\lambda/(\lambda+1)$ is a restrictive central point for T_λ^2. The affine function

$$\phi_\lambda : x \mapsto \frac{\lambda - (\lambda+1)x}{\lambda - 1}$$

[2]The notation $(x; c)$ does not imply that x is less than c.

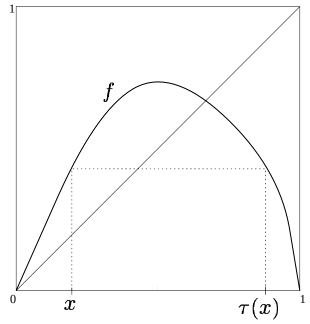


Figure 1.3: The function τ.

rescales the interval $[1/(\lambda+1); \lambda/(\lambda+1)]$ *to* $[0;1]$*; the rescaling of the restriction of* T_λ^2 *to this interval equals* T_{λ^2}*.*

If λ^2 *is less than* $\sqrt{2}$ *then* T_{λ^2} *is itself renormalizable of degree 2, and so* T_λ *is renormalizable of degree 4. By induction, for* λ *in the range* $\sqrt[2^{n+1}]{2} \le \lambda < \sqrt[2^n]{2}$ *the tent-map* T_λ *is renormalizable of degrees* $2, 4, \ldots, 2^n$ *and* A_λ *consists of* 2^n *intervals.*

1.2 The Class $\mathcal{C}$ of S-unimodal Maps

In order to prove precise results about unimodal maps it is necessary to impose regularity conditions. For this thesis they are those of the class $\mathcal{C}$. Each f in $\mathcal{C}$ is a C^3 unimodal map with

1. *Negative Schwarzian derivative*: $S(f)(x) < 0$ for every $x \neq c$ where $S(f)(x) = f'''(x)/f'(x) - 3(f''(x)/f'(x))^2/2$.
2. *Non-flat critical point*: there exist $l > 1$ and $L > 1$ such that $|x-c|^{l-1}/L \le |f'(x)| \le L|x-c|^{l-1}$ for all $x \in [0;1]$.[3]
3. $|f'(0)| > 1$.

Such an f is called *S-unimodal*.

[3] The number l is unique and is called the *order* of the critical point. Note that f' must be non-zero except at the critical point.

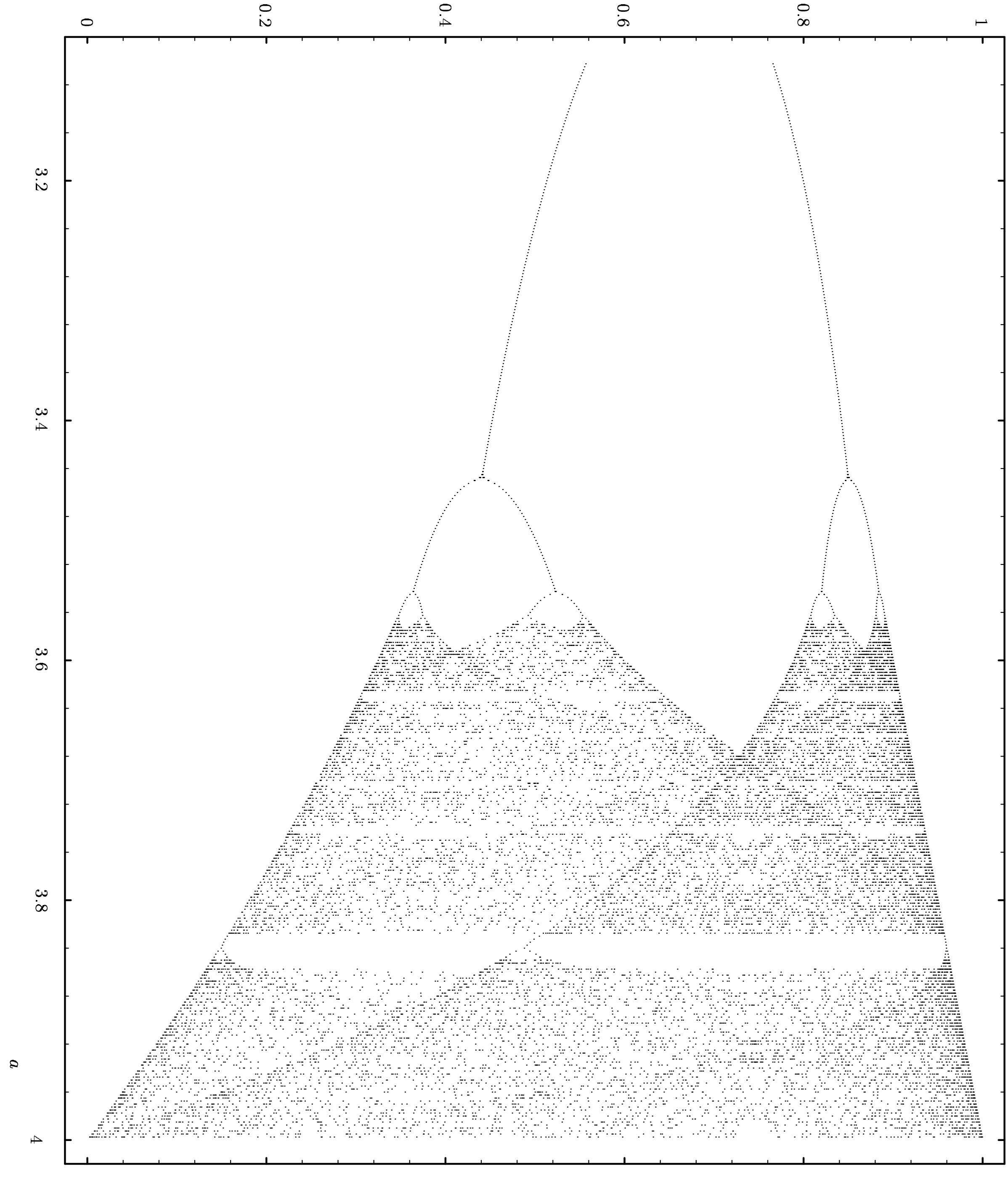


Figure 1.4: The approximate limit set of Q_a plotted against a.

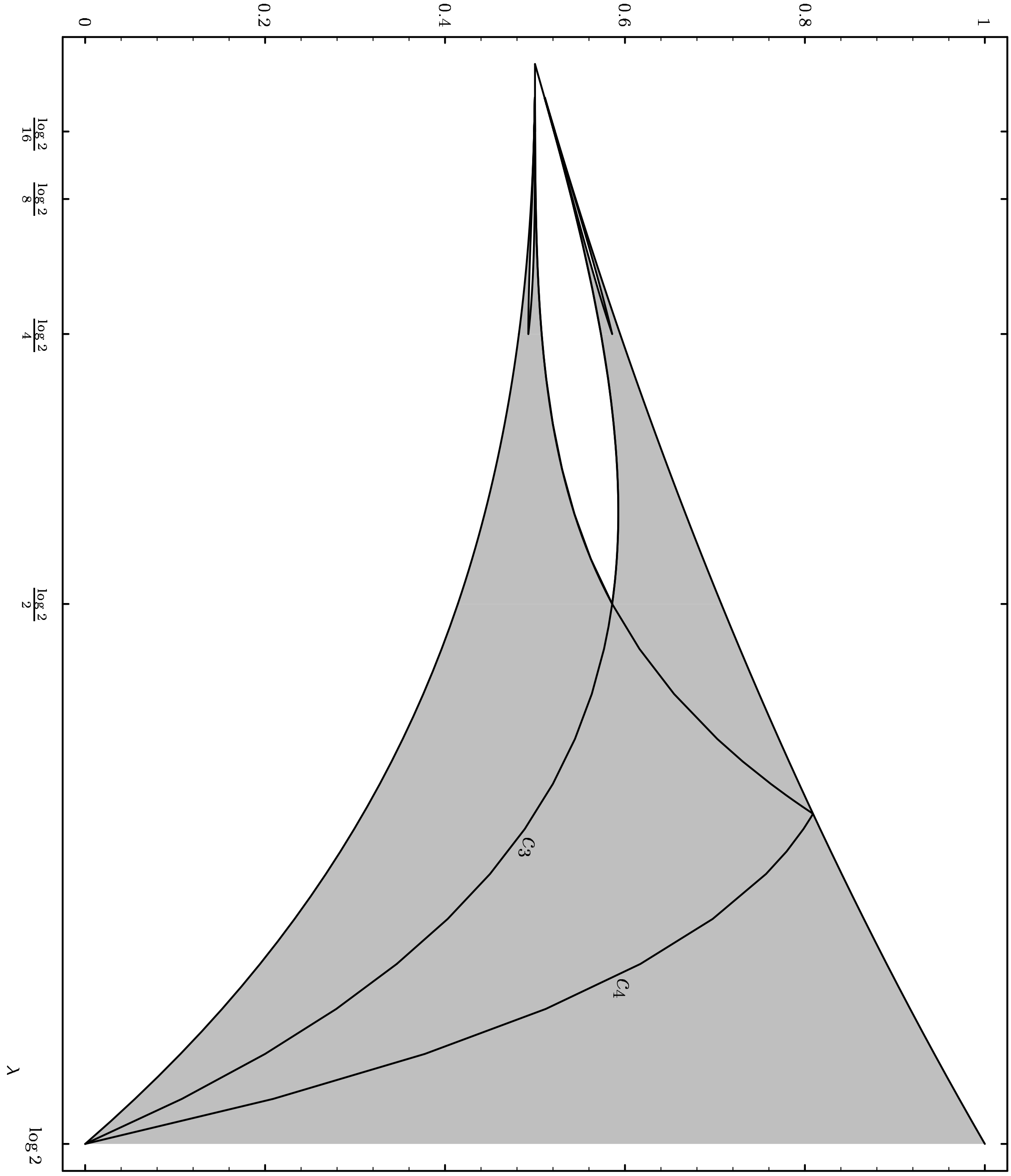


Figure 1.5: The limit set of T_λ plotted against λ; also showing c_3 and c_4.

We use $\mathcal{C}_l$ to denote the subclass with critical point of order l. Condition 3 avoids a technical problem which might otherwise arise in theorem 2.

There are only three possibilities for the behaviour of a map in $\mathcal{C}$.

Theorem 2 (Blokh and Lyubich [6]) *Each map in $\mathcal{C}$ either*

— *has a periodic attractor. This is unique and attracts almost every point. The critical point is in the immediate basin of the periodic orbit.*

or

— *is infinitely renormalizable. Almost every point is attracted to a Cantor set, the ω-limit set of the critical point.*

or

— *is finitely renormalizable. Almost every point is attracted to a finite union of intervals. The ω-limit of a dense set of points is equal to the finite union.*[4]

The proof of theorem 2 uses the non-existence of wandering intervals, a fundamental result in the theory of unimodal mappings that will also be useful to us directly:

Theorem 3 (Guckenheimer [19]) *Maps in $\mathcal{C}$ do not have wandering intervals.*

1.2.1 Consequences of Negative Schwarzian Derivative

It is easy to show that the derivative of a map with negative Schwarzian derivative has no positive local minima or negative local maxima [47]. We will use this technical result in the following easily derivable form. Call $f^i\big|_{[a;b]}$ a diffeomorphism if $Df^i(x)$ is non-zero for every $x \in (a;b)$.

Theorem 4 (Nowicki [44]) *Suppose f has negative Schwarzian derivative, $[a;b]$ is a subinterval of $[0;1]$ and $i \geq 1$. If $f^i\big|_{[a;b]}$ is a diffeomorphism then*

$$\frac{|x_i - y_i|}{|x-y|} \geq \min\{\frac{|b_i - x_i|}{|b-x|}, \frac{|x_i - a_i|}{|x-a|}\}$$

[4]For maps in $\mathcal{C}_2$ the ω-limit set of almost every point is equal to the finite union of intervals [35]. This is not true in general: according to Van Strien (personal communication) there exist finitely renormalizable maps in $\mathcal{C}$ for which almost every point is attracted to a cantor set [20] inside the union.

and in particular

$$|Df^i(x)| \geq \min\{\frac{|b_i - x_i|}{|b - x|}, \frac{|x_i - a_i|}{|x - a|}\}$$

for every x, y in $(a;b)$ with $x \neq y$.

The *distortion* of f^i restricted to an interval I is

$$\mathcal{D}(f^i|_I) = \sup_{x,y \in I} \log \frac{|Df^i(x)|}{|Df^i(y)|}.$$

The closer the distortion is to zero, the more $f^i|_I$ resembles a straight line.

We control distortion using lemma 5 below. An interval J is a *ρ-scaled neighbourhood* of a subinterval J' if the lengths of the two components of $J \setminus J'$ are both at least $\rho|J'|$.

Lemma 5 (Koebe principle,[5] Van Strien [14]) *For each $\rho > 0$ the constant $K_D(\rho) = (1+\rho)^2/\rho^2$ has the following property: for any f with negative Schwarzian derivative, any subinterval $[a;b]$ of $[0;1]$ and $i \geq 1$ such that $f^i|_{[a;b]}$ is a diffeomorphism,*

$$\frac{1}{K_D(\rho)} \leq \frac{|Df^i(x)|}{|Df^i(y)|} \leq K_D(\rho)$$

for every x, y in $(a;b)$ for which $f^i[a;b]$ is a ρ-scaled neighbourhood of $f^i[x;y]$.

1.2.2 Consequences of Non-flatness of the Critical Point

Non-flatness of the critical point limits the degree to which intervals near c are contracted by the action of f as shown by the integrated version of the non-flatness condition:

$$\frac{|x-c|^l}{lL} \leq |f(x) - f(c)| \leq L\frac{|x-c|^l}{l}. \tag{1.1}$$

Non-flatness also implies that f is reasonably symmetric about c, as shown by lemma 6 below (the lemma implies $|\tau(x) - c|$ is comparable to $|x - c|$). Recall that $\tau : [0;1] \to [0;1]$ was defined by $\tau(c) = c$ and, for $x \neq c$, $f(\tau(x)) = f(x)$ where $\tau(x) \neq x$ (see figure 1.3).

[5]In analogy with the Koebe lemma of complex analysis. The Schwarzian derivative itself plays an important role in complex analysis.

Lemma 6 *If the critical point of a unimodal map is non-flat then τ is Lipschitz.*

Proof. The function τ is clearly Lipschitz away from the critical point since it is a composition of diffeomorphisms there. It is Lipschitz at c if and only if $|\tau(x)-c|/|x-c|$ is bounded for $x \neq c$. From equation 1.1 we have $|\tau(x)-c|^l/(lL) \leq |f(\tau(x))-f(c)| = |f(x)-f(c)| \leq L|x-c|^l/l$ and so

$$\frac{|\tau(x)-c|}{|x-c|} \leq L^{2/l}$$

as required. ■

1.3 Kneading Invariants and Topological Conjugacy

The seminal papers of Metropolis et al. [37] and Milnor and Thurston [38] established the importance of symbol dynamics for the study of unimodal mappings. In particular Milnor and Thurston introduced the kneading invariant and the corresponding kneading theory with great success. The *kneading invariant* describes the critical orbit[6] of a unimodal map f in terms of an infinite one-sided symbol sequence $\theta(f) = \theta^1\theta^2\theta^3\cdots$. Each symbol is either 0, 1 or C depending on whether the corresponding image of the critical point is to the left of, to the right of or exactly equal to c:

$$\theta^i = \begin{cases} 0 & \text{if } c_i < c \\ C & \text{if } c_i = c \\ 1 & \text{if } c_i > c. \end{cases}$$

We say that f and g are *topologically equivalent* or *topologically conjugate* if there is a homeomorphism $h : [0;1] \to [0;1]$ of the interval $[0;1]$ onto itself such that $h \circ f = g \circ h$. This implies $h \circ f^n = g^n \circ h$ for every positive integer n. The map h, called the *conjugacy* between f and g, thus maps orbits of f onto orbits of g. It is automatically orientation preserving and takes the critical point of f to the critical point of g. The kneading invariants of f and g are therefore the same.

Topologically conjugate maps can be considered identical for most purposes. However there is no reason to suppose that topological conjugacy implies equality or even

[6] The *critical orbit* is the orbit of the critical point.

similarity of lengths or derivatives for the two maps since conjugacies are not normally differentiable (see chapter 4).

In 1979 Guckenheimer proved that equality of kneading invariants is often equivalent to topological conjugacy.

Theorem 7 (Guckenheimer [19]) *If f and g are both in $\mathcal{C}$ and have no periodic attractors then they are topologically conjugate if and only if $\theta(f) = \theta(g)$.*

In the presence of periodic attractors equality of kneading invariants need not imply topological conjugacy. However the kneading invariant registers the fact that there is a periodic attractor. We say that a kneading invariant is *periodic* if it consists of a finite string of symbols infinitely repeated or, equivalently, if $\theta^{i+j} = \theta^j$ for some $i \geq 1$ and all $j \geq 1$.

Lemma 8 *A map f in $\mathcal{C}$ has a periodic attractor if and only if $\theta(f)$ is periodic.*

Proof. Suppose f has no periodic attractors. Since it has no wandering intervals it has no homtervals. Therefore every subinterval I of $[0;1]$ of positive length is eventually mapped over c: there exists $j \geq 1$ such that $c \in f^j(I)$. Taking $I = [c; c_i]$ gives $\theta^{i+j} \neq \theta^j$. Since this holds for some j for all $i \geq 1$, the kneading invariant is not periodic.

For the converse, suppose first that f has a super-attractor: $c_i = c$ for some $i \geq 1$. In this case the kneading invariant is clearly periodic.

Now suppose that f has a periodic attractor of period i that is not a super-attractor: c is not in its orbit. We know from theorem 2 that c is in the immediate basin of attraction. Set $I = [c, c_i]$. Then c is not in $f^j(I)$ for all $j \geq 1$ and so i is the value we are looking for. ∎

1.4 Topological Entropy and Conjugacy with Tent-maps

Adler, Konheim and McAndrew [2] introduced the *topological entropy* in 1965 as an invariant of continuous maps. A measure of complexity, it describes the rate at which observing an orbit imperfectly gives information about its initial point. They defined the topological entropy for any continuous mapping from a compact topological space to itself. For unimodal maps there is an equivalent definition due to Misiurewicz and Szlenk [39]:

$$h_{\text{top}}(f) = \lim_{n\to\infty} \frac{\log l(f^n)}{n}$$

where $l(f^n)$ is the number of laps (monotone pieces) of f^n. This limit always exists.

The topological entropy of any unimodal map therefore lies in the range $[0; \log 2]$. Every value in this range is taken on by some unimodal map, for example $h_{\text{top}}(T_\lambda)$ equals $\log\lambda$ and this varies from 0 to $\log 2$ as λ varies from 1 to 2.

As theorem 9 below states, periodic tent-maps excepted[7] each tent-map is a standard representative of its topological entropy: every map in $\mathcal{C}$ with the same entropy is topologically conjugate to it. This is analogous to the relationship between rotations of the circle and rotation number: rational rotations excepted, a rotation is topologically conjugate to every smooth circle map with the same rotation number.

Theorem 9 (Milnor and Thurston [38, 50]; Guckenheimer [19]) *If T_λ is not periodic then any f in $\mathcal{C}$ with $h_{\text{top}}(f) = \log\lambda$ is topologically conjugate to T_λ. In particular the kneading invariants of f and T_λ are the same.*

This is a reformulation for maps in $\mathcal{C}$ of the following result of Milnor and Thurston. We say that two maps f and g are *topologically semi-conjugate* if there is a non-decreasing map $h : [0; 1] \to [0; 1]$ for which $h \circ f = g \circ h$. This is similar to topological conjugacy except that h is not required to be a homeomorphism. For h to be a homeomorphism it is enough that in addition $h^{-1}(x)$ be a single point for every x in $[0; 1]$.

Theorem 10 (Milnor and Thurston [38, 50]) *If f is a unimodal map with positive topological entropy $h_{\text{top}}(f)$ then it is topologically semi-conjugate to the tent-map T_λ with $h_{\text{top}}(f) = \log\lambda$. The semi-conjugacy maps the critical point of f to the critical point of T_λ. If T_λ is not periodic then $h^{-1}(x)$ is either a point, contains a wandering interval or is in the basin of a periodic attractor for every x in $[0; 1]$.*

Derivation of theorem 9 from theorem 10. Since $h_{\text{top}}(f) \geq 0$ we have $\lambda \geq 1$. The hypothesis that T_λ is not periodic shows $\lambda \neq 1$ and so we will only be considering maps with positive topological entropy. In addition the semi-conjugacy h between

[7] A periodic tent-map is one for which the critical point is periodic.

f and T_λ given by theorem 10 has the property that $h^{-1}(x)$ is a point, contains a wandering interval or is in the basin of a periodic attractor for every x in $[0;1]$. Since f is in $\mathcal{C}$ it has no wandering intervals — we can immediately eliminate that possibility. We will show that f has no periodic attractors and therefore that $h^{-1}(x)$ is a point for every x in $[0;1]$; thus h is a homeomorphism and f and T_λ topologically conjugate.

So let us prove that if f has a periodic attractor then T_λ is periodic. The proof is similar to that of lemma 8. We use $c(f)$ and $c(T_\lambda)$ for the critical points of f and T_λ respectively.

Suppose first that f has a super-attractor. In this case $c_n(f) = c(f)$ for some positive integer n. The same equation $c_n(T_\lambda) = c(T_\lambda)$ holds for T_λ since h takes the critical point of f to the critical point of T_λ, and so T_λ is periodic.

Now suppose that the periodic attractor of f is not a super-attractor. We know from lemma 8 that $\theta(f)$ is periodic, say of period n. This means that $I = [c(f); c_n(f)]$ is a homterval for f. Since T_λ has no homtervals $h(I)$ must be a single point, in fact the critical point of T_λ since h maps $c(f)$ to $c(T_\lambda)$. The other endpoint $c_n(f)$ of I is mapped to $c_n(T_\lambda)$ because h is a semi-conjugacy; it is also mapped to $c(T_\lambda)$ because h collapses the entire interval I to this point. Therefore $c(T_\lambda) = c_n(T_\lambda)$ and T_λ is periodic. ■

Chapter 2

Towers and Analysis of the Kneading Invariant

In this chapter we describe a simple but effective approach to the analysis of unimodal maps using towers. Originally introduced by Hofbauer in a study of the topological entropy [24], towers were later developed by Hofbauer and Keller in order to study invariant measures [25]. They are used throughout this thesis. Our approach is based on de Melo and Van Strien's exposition [14],[1] though we use towers based at c_1 rather than at c. This gives extra combinatorial information. Some of the ideas presented here are new, especially in the later sections.

The opening sections "Definitions" and "Cutting and Co-Cutting Times" describe the tower of a unimodal map and its properties geometrically. The remaining sections "Analysis of the Kneading Invariant" and "Return Times" develop the associated combinatorial theory.

To simplify the exposition we assume that the unimodal map f has no homtervals (there are therefore no periodic attractors) and that its kneading invariant is not periodic.

2.1 Definitions

The graph of f^{n-1} consists of segments on which f^{n-1} is monotone, called *laps*. We follow the lap of f^{n-1} containing c_1 as n increases. The notation used is summarised

[1]See also the preprint of Bruin [9].

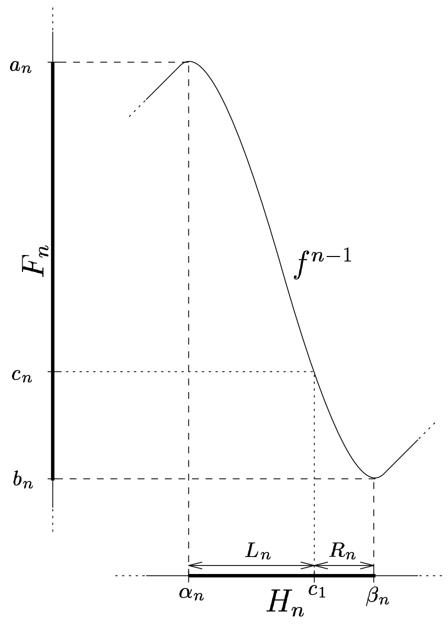


Figure 2.1: The lap of f^{n-1} containing c_1.

in figure 2.1. The left endpoint of H_n is

$$\alpha_n = \inf\{y \in [0;1] \mid f^{n-1}\big|_{[y;c_1]} \text{ is monotone}\}$$

and the right endpoint is

$$\beta_n = \sup\{y \in [0;1] \mid f^{n-1}\big|_{[c_1;y]} \text{ is monotone}\}.$$

We will sometimes refer to $L_n = [\alpha_n; c_1]$ and $R_n = [c_1; \beta_n]$ as the *left-* and *right-hand parts* of H_n respectively.

Notice that H_n is the domain, and F_n the image, of the lap of f^{n-1} containing c_1, not the lap of f^n. In particular, $F_n = f^{n-1}(H_n)$, $a_n = f^{n-1}(\alpha_n)$ and $b_n = f^{n-1}(\beta_n)$. This ensures that c_n (rather than c_{n+1}) is in F_n. Note that H_n always contains H_{n+1}.

The *tower* of f is the collection of sets $F_n \times \{n\}$ for $n = 1$, 2, etc. along with their *orientations* ($+1$ or -1 according to whether f^{n-1} is orientation preserving or reversing on H_n). These are called the *levels* of the tower; we draw them in the plane, as in figure 2.2, the arrows indicating whether the orientation is positive ($\longrightarrow$) or negative ($\longleftarrow$).

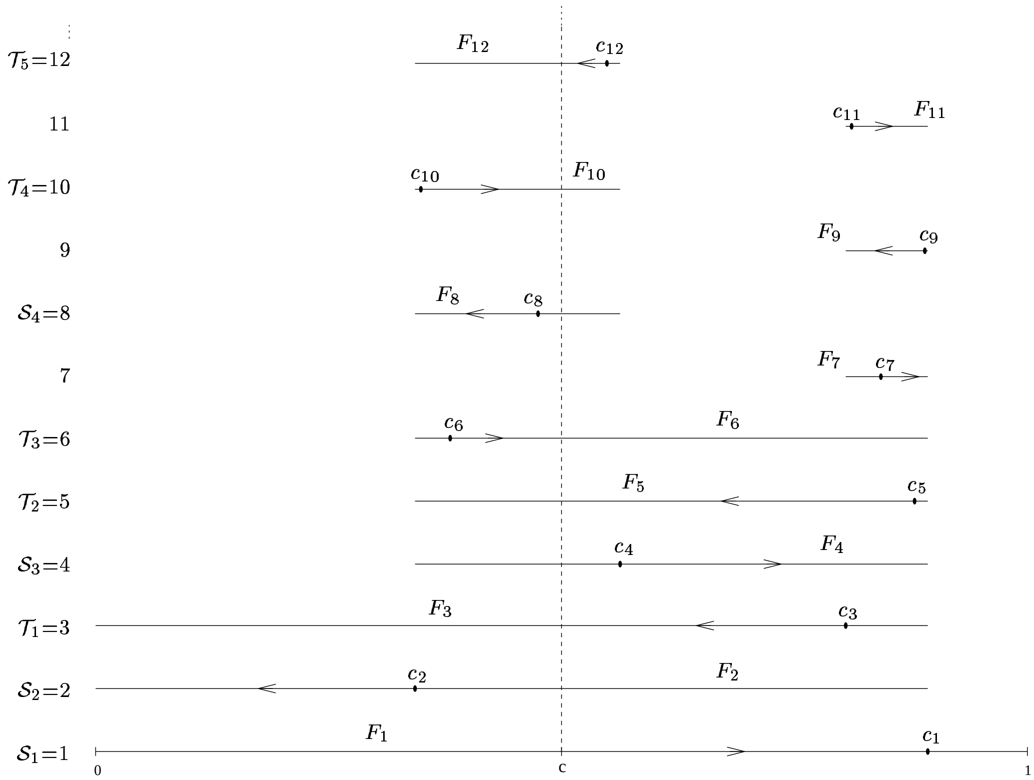


Figure 2.2: The tower of the Feigenbaum map [14].

It may seem necessary to know a great deal about laps in order to calculate a tower, but this is not at all the case. The entire tower can be generated iteratively from $F_1 = [0;1]$. The rule relating F_{n+1} to F_n depends on whether c is in F_n.

If c is not in F_n then $F_{n+1} = f(F_n)$ because $H_{n+1} = H_n$ in this case. This follows from lemma 1.

If c is in F_n then $f^n|_{H_n}$ is not monotone. There are two laps on H_n with domains $[\alpha_n; y]$ and $[y; \beta_n]$ where $y = (f^{n-1}|_{H_n})^{-1}(c)$. H_{n+1} is the domain containing c_1. Its image under f^{n-1} is the component of $F_n \setminus \{c\}$ that contains c_n (denoted E_n [2] and drawn thicker in figure 2.3). Therefore $F_{n+1} = f(E_n)$.

[2]Strictly speaking the closure of the component of $F_n \setminus \{c\}$ containing c_n. This adds the point c back in.

The recursion relationship is thus

$$F_{n+1} = \begin{cases} f(F_n) & \text{if } c \notin F_n \\ f(E_n) & \text{if } c \in F_n \end{cases} \tag{2.1}$$

where E_n is the closure of the component of $F_n \setminus \{c\}$ containing c_n, and $F_1 = [0;1]$.

The orientations of F_{n+1} and F_n are the same if c_n is to the left of c. The orientation of F_{n+1} is the reverse of that of F_n if c_n is to the right of c.

Example 2.1 *The condition that f should not have any periodic attractors and that the kneading invariant should not be periodic imply that the first two iterates of c satisfy $c_2 < c < c_1$. Therefore $F_1 = [0;1]$, $F_2 = [0;c_1]$ and $F_3 = [0;c_1]$: $a_1 = 0$ and $b_1 = 1$; $a_2 = c_1$ and $b_2 = 0$; $a_3 = c_1$ and $b_3 = 0$. The orientations are $+1$, -1 and -1 respectively.*

2.2 Cutting and Co-Cutting Times

The recursion relationship for the levels of the tower, equation 2.1, shows that if c is in F_n then F_{n+1} equals $f(E_n)$ not $f(F_n)$. In other words a piece of F_n is "cut off" when calculating F_{n+1}. If c is in F_n then we call n either a *cutting* or a *co-cutting time*, depending on the position of c in F_n. If c is in $[a_n; c_n]$ then n is a cutting time. If c is in $[b_n; c_n]$ then n is a co-cutting time. It is impossible for n to be both a cutting and a co-cutting time.

Equivalently, n is a cutting or co-cutting time when H_{n+1} is strictly smaller than H_n. A cutting time corresponds to L_{n+1} being strictly smaller than L_n, a co-cutting time to R_{n+1} being strictly smaller than R_n.

We label the cutting times $\mathcal{S}_1$, $\mathcal{S}_2$, ... and the co-cutting times $\mathcal{T}_1$, $\mathcal{T}_2$, ... in order of size, as shown in figures 2.2 and 2.3. A tower always starts with $\mathcal{S}_1 = 1$ and $\mathcal{S}_2 = 2$ (see example 2.1). The co-cutting sequence starts at $\mathcal{T}_1 = \min\{j > 1 \mid c_j > c\}$.

The importance of cutting and co-cutting times comes from their close relationship both with the structure of the tower and with the structure of the kneading invariant. The relationship with the kneading invariant is described in the next section. Here we describe their basic geometric properties.

We start by showing how to explicitly calculate tower levels. The result is stated in lemma 11.

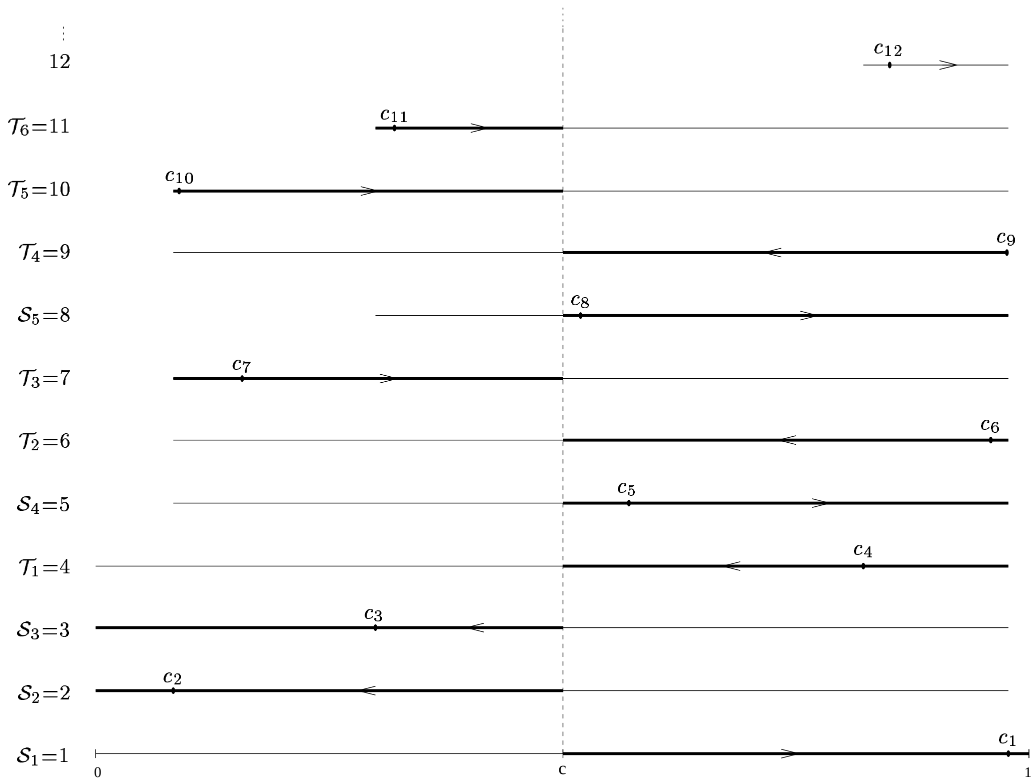


Figure 2.3: The tower of the Fibonacci map [14, 36, 31].

We derive explicit formulae for F_n and its endpoints as follows. The endpoints a_n and b_n satisfy

$$a_{n+1} = \begin{cases} f(a_n) & \text{if } c \notin [a_n; c_n] \\ c_1 & \text{if } c \in [a_n; c_n] \end{cases} \tag{2.2}$$

and

$$b_{n+1} = \begin{cases} f(b_n) & \text{if } c \notin [b_n; c_n] \\ c_1 & \text{if } c \in [b_n; c_n]. \end{cases} \tag{2.3}$$

If n is a cutting time $\mathcal{S}_j$ then $a_{n+1} = c_1$ and otherwise $a_{n+1} = f(a_n)$. Therefore $a_n = c_{n-\mathcal{S}_j}$ for $\mathcal{S}_j < n \leq \mathcal{S}_{j+1}$. In general, given n, we write $\mathcal{S}\langle n\rangle$ rather than $\mathcal{S}_j$ for the last cutting time before n:

$$\mathcal{S}\langle n\rangle = \max_{j\geq 1}\{\mathcal{S}_j \mid \mathcal{S}_j < n\}.$$

It is well defined for every $n > 1$. Clearly $a_n = c_{n-\mathcal{S}\langle n\rangle}$ for $n > 1$. Example 2.1 shows that $a_1 = 0$.

The last co-cutting time before n is

$$\mathcal{T}\langle n\rangle = \max_{j\geq 1}\{\mathcal{T}_j \mid \mathcal{T}_j < n\},$$

which is well defined for $n > \mathcal{T}_1$. Therefore $b_n = c_{n-\mathcal{T}\langle n\rangle}$ for $n > \mathcal{T}_1$. Otherwise $b_n = 0$ for $1 < n \leq \mathcal{T}_1$ and $b_1 = 1$.

In terms of F_n these calculations give

Lemma 11 *We have $F_n = [c_{n-\mathcal{S}\langle n\rangle}; c_{n-\mathcal{T}\langle n\rangle}]$ for every $n > \mathcal{T}_1$.*

Now let us derive recursion formulae for the cutting and co-cutting times.

Recall that $\mathcal{S}_{i+1} = \min\{j > \mathcal{S}_i \mid c \in [a_j; c_j]\}$. We know that $a_j = c_{j-\mathcal{S}_i}$. Therefore $\mathcal{S}_{i+1} = \min\{j > \mathcal{S}_i \mid c \in f^{j-\mathcal{S}_i}[c; c_{\mathcal{S}_i}]\} = \mathcal{S}_i + \min\{j \geq 1 \mid c \in f^j[c; c_{\mathcal{S}_i}]\}$. If we define

$$R(x) = \min\{j \geq 1 \mid c \in f^j[c; x]\}$$

then we can write

$$\mathcal{S}_{i+1} = \mathcal{S}_i + R(c_{\mathcal{S}_i})$$

and

$$\mathcal{T}_{i+1} = \mathcal{T}_i + R(c_{\mathcal{T}_i}).$$

Since f has no homtervals, $R(x)$ is finite for every $x \neq c$.

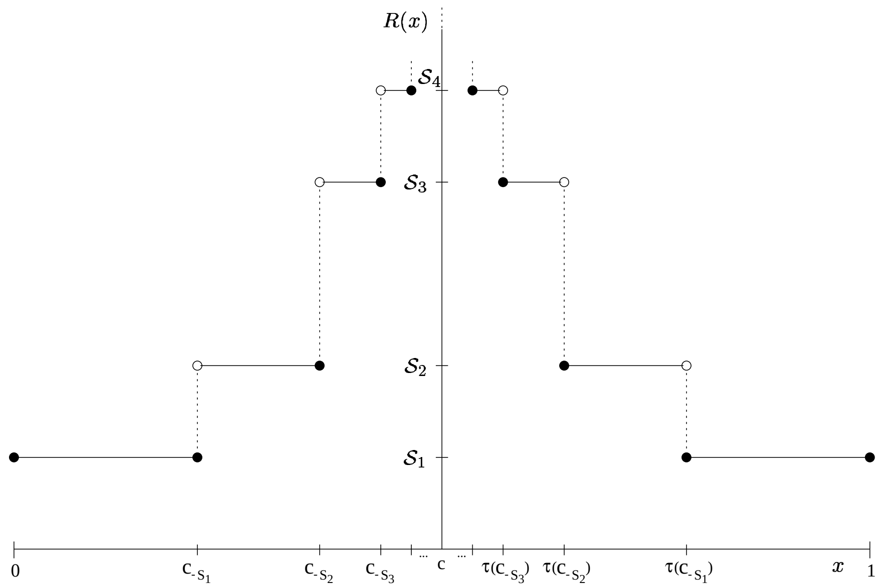


Figure 2.4: Schematic representation of the function R.

These recursion formulae and the practical calculation of cutting and co-cutting times are discussed in detail in the next section.

Now let us consider the function R.

The function R is depicted schematically in figure 2.4. The points $\{c_{-\mathcal{S}_i}\}_{i\geq 1}$ are defined below. The figure is a graphical representation of the following lemma:

Lemma 12 *We have* $R(x) = \min_{j\geq 1}\{\mathcal{S}_j \mid x \notin (c_{-\mathcal{S}_j}; \tau(c_{-\mathcal{S}_j}))\}$ *for every* $x \neq c$.

It follows from lemma 12 that every value of R is a cutting time, an important result with many combinatorial consequences. In particular, the difference between any two successive cutting times or successive co-cutting times is always a cutting time.

Before proving lemma 12 we introduce some terminology and basic results.

The following are immediate from the definition of R:

1. $R(x)$ is non-decreasing as x approaches c and becomes arbitrarily large.

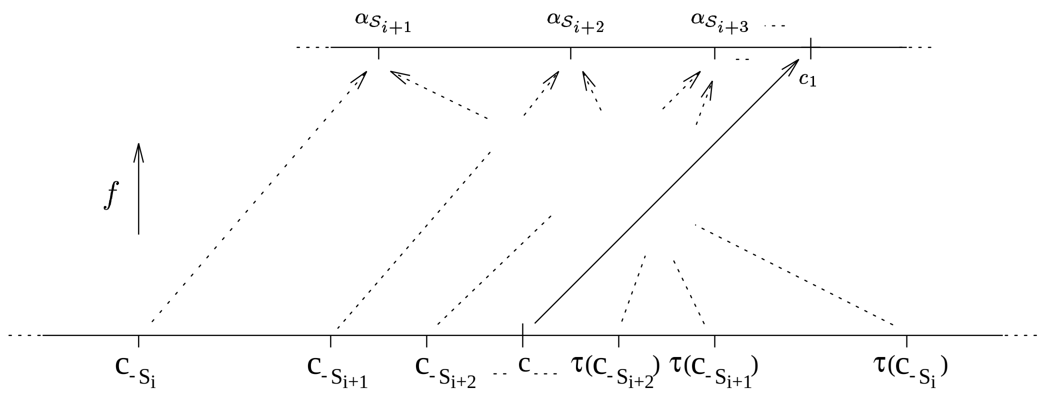


Figure 2.5: The points $\{\alpha_{\mathcal{S}_\imath+1}\}_{\imath\geq1}$; their inverses under f, the discontinuity points $\{c_{-\mathcal{S}_\imath}\}_{\imath\geq1}$ and $\{\tau(c_{-\mathcal{S}_\imath})\}_{\imath\geq1}$ of R.

2. R takes on only integer values.
3. $R(x)$ and $R(\tau(x))$ are equal.
4. Each discontinuity of R (except at c) occurs at an inverse image of c (though not every inverse image of c is a point of discontinuity).

Understanding the discontinuities of R is the key to proving lemma 12. Recall the definition of α_n, the left-hand endpoint of H_n:

$$\alpha_n = \inf\{y \in [0;1] \mid f^{n-1}\Big|_{[y;c_1]} \text{ is monotone}\}.$$

The points $\{\alpha_\imath\}_{\imath\geq1}$ are inverse images of c except for α_1 and α_2 which equal 0 and c respectively. Not all of the points $\{\alpha_\imath\}_{\imath\geq1}$ are distinct. For example $\alpha_{\mathcal{S}_i+1}$, $\alpha_{\mathcal{S}_i+2}$, $\ldots\alpha_{\mathcal{S}_{i+1}}$ are all equal; and $\alpha_{\mathcal{S}_i+1} \neq \alpha_{\mathcal{S}_i}$ from the definition of a cutting time. We choose as representative the distinct elements $\{\alpha_{\mathcal{S}_\imath+1}\}_{\imath\geq1}$. They form an increasing sequence converging to c_1 (see figure 2.5). The point $\alpha_{\mathcal{S}_i+1}$ is an inverse image of c of degree $\mathcal{S}_i - 1$, meaning $f^{\mathcal{S}_i-1}(\alpha_{\mathcal{S}_i+1}) = c$.

We denote the inverse of $\alpha_{\mathcal{S}_i+1}$ to the left of c by $c_{-\mathcal{S}_i}$; the other inverse under f is therefore $\tau(c_{-\mathcal{S}_i})$ (see figure 2.5). The notation reflects the fact that $c_{-\mathcal{S}_i}$ is an inverse image of c of degree $\mathcal{S}_i$. The points of discontinuity of R are $\{c_{-\mathcal{S}_\imath}\}_{\imath\geq1}$, $\{\tau(c_{-\mathcal{S}_\imath})\}_{\imath\geq1}$ and c. This follows from the following lemma:

Lemma 13 *The restriction* $f^{\mathcal{S}_{i+1}}\Big|_{[c_{-\mathcal{S}_i};c]}$ *is a homeomorphism and* $c_{-\mathcal{S}_i}$ *is the leftmost point with this property.*

Proof. By definition $f^{\mathcal{S}_{i+1}-1}\big|_{[\alpha_{\mathcal{S}_{i+1}};c_1]}$ is a homeomorphism and $\alpha_{\mathcal{S}_{i+1}}$ is the leftmost point with this property. We remarked above that $\alpha_{\mathcal{S}_{i+1}} = \alpha_{\mathcal{S}_i+1}$. Therefore $f^{\mathcal{S}_{i+1}-1}\big|_{[\alpha_{\mathcal{S}_i+1};c_1]}$ is a homeomorphism. It follows that $f^{\mathcal{S}_{i+1}}\big|_{[c_{-\mathcal{S}_i};c]}$ is also a homeomorphism and that $c_{-\mathcal{S}_i}$ is the leftmost point with this property. ∎

Of course the same lemma holds with $c_{-\mathcal{S}_i}$ replaced by $\tau(c_{-\mathcal{S}_i})$ and leftmost by rightmost.

We can now prove lemma 12. Take some $x \neq c$ and suppose $c_{-\mathcal{S}_{i-1}} < x < c_{-\mathcal{S}_i}$. From lemma 13 we have that $f^{\mathcal{S}_i}\big|_{[x;c]}$ is a homeomorphism, so c is not in the interior of $f^j[x;c]$ for $1 \le j < \mathcal{S}_i$. We know that c is not in $f^j[x;c]$ for $1 \le j < \mathcal{S}_i$ since (a) $x > c_{-\mathcal{S}_{i-1}}$ and (b) c is not periodic. But $f^{\mathcal{S}_i}[x;c]$ contains c because $[x;c]$ contains $c_{-\mathcal{S}_i}$. Therefore $R(x) = \mathcal{S}_i$. The other cases are similar.

2.3 Analysis of the Kneading Invariant

In this section we show how to calculate cutting and co-cutting times from the kneading invariant and describe some related results.

In the recursion formulae for the cutting and co-cutting times,

$$\mathcal{S}_{i+1} = \mathcal{S}_i + R(c_{\mathcal{S}_i})$$

and

$$\mathcal{T}_{i+1} = \mathcal{T}_i + R(c_{\mathcal{T}_i}),$$

R is only evaluated at forward images of c. Lemma 14 below shows that in this case the values of R can be calculated from the kneading invariant. The sequences of cutting and co-cutting times can therefore be calculated from the kneading invariant also.

Recall that the kneading invariant is an infinite symbol sequence $\theta(f) = \theta^1\theta^2\cdots$ where $\theta^i = 0$ if $c_i < c$, $\theta^i = C$ if $c_i = c$ and $\theta^i = 1$ if $c_i > c$. In this chapter the symbol C does not occur because the kneading invariant is assumed to be non-periodic.

Define $\mathcal{R}(n) = \min\{j \ge 1 \mid \theta^{n+j} \neq \theta^j\}$ for $n \ge 1$. This function is our main tool for analysing the kneading invariant. The relationship with R is simple:

Lemma 14 *We have $\mathcal{R}(n) = R(c_n)$ for every $n \ge 1$.*

Proof. Since c is not in $f^j[c; c_{n+j}] = [c_j; c_n]$ for $1 \le j < R(c_n)$, the points c_j and c_{n+j} have the same kneading coordinate $\theta^j = \theta^{n+j}$. But c is in $f^{R(c_n)}[c; c_n]$, so $\theta^{R(c_n)} \ne \theta^{n+R(c_n)}$. ∎

So large values of $\mathcal{R}$ occur when c_n is close to c.

Recall that all values of R are cutting times. Therefore all values of $\mathcal{R}$ are also cutting times. Writing $\mathcal{R}(n) = \mathcal{S}_j$ for some j, define $\mathcal{R}^-(n) = \mathcal{S}_{j-1}$ if $j > 1$.

The recursion formulae for the cutting and co-cutting times become

$$\mathcal{S}_{i+1} = \mathcal{S}_i + \mathcal{R}(\mathcal{S}_i) \tag{2.4}$$

and

$$\mathcal{T}_{i+1} = \mathcal{T}_i + \mathcal{R}(\mathcal{T}_i). \tag{2.5}$$

The sequences start at $\mathcal{S}_1 = 1$ and $\mathcal{T}_1 = \min\{j > 1 \mid \theta^j = 1\}$.

Example 2.2 *The kneading invariant of the Fibonacci map [14, 36, 31] starts*

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
				$\mathcal{T}_1$		$\mathcal{T}_2$	$\mathcal{T}_3$		$\mathcal{T}_4$	$\mathcal{T}_5$	$\mathcal{T}_6$			$\mathcal{T}_7$	$\mathcal{T}_8$	$\mathcal{T}_9$	
$\theta(f) =$	1	0	0	1	1	1	0	1	1	0	0	1	0	1	0	0	$\cdots$
	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$		$\mathcal{S}_4$			$\mathcal{S}_5$					$\mathcal{S}_6$				

The values of $\mathcal{R}$ are $\mathcal{R}(1) = 1$, $\mathcal{R}(2) = 1$, $\mathcal{R}(3) = 2$, $\mathcal{R}(4) = 2$, $\mathcal{R}(5) = 3$, $\mathcal{R}(6) = 1$, $\mathcal{R}(7) = 2$, $\mathcal{R}(8) = 5$, $\mathcal{R}(9) = 1$, $\mathcal{R}(10) = 1$, $\mathcal{R}(11) = 3$, $\mathcal{R}(12) = 1$, $\mathcal{R}(14) = 1$ and $\mathcal{R}(15) = 1$. These are all cutting times, for example $\mathcal{R}(8) = 5 = \mathcal{S}_4$; therefore $\mathcal{R}^-(8) = \mathcal{S}_3 = 3$. There are not enough digits to calculate $\mathcal{R}(13)$.

For example, to calculate $\mathcal{R}(1)$ compare the start of the kneading invariant to the section from position 2 onwards. Since the first coordinates compared, θ^1 and θ^2, are different, we have $\mathcal{R}(1) = 1$. To calculate $\mathcal{R}(5)$, compare θ^1 onwards to the section of the kneading invariant starting at θ^6 (figure 2.6). Since θ^6 matches θ^1 and θ^7 matches θ^2 but θ^8 does not match θ^3, we have $\mathcal{R}(5) = 3$.

To calculate the cutting times start with $\mathcal{S}_1 = 1$. Applying the recursion formula $\mathcal{S}_{i+1} = \mathcal{S}_i + \mathcal{R}(\mathcal{S}_i)$ gives $\mathcal{S}_2 = 2$, $\mathcal{S}_3 = 3$, $\mathcal{S}_4 = 5$, $\mathcal{S}_5 = 8$ and $\mathcal{S}_6 = 13$. The cutting times form the Fibonacci sequence, which is why f is called the Fibonacci map.

The co-cutting times start at $\mathcal{T}_1 = \min\{j > 1 \mid \theta^j = 1\}$ which equals 4 in this case. Applying the recursion formula $\mathcal{T}_{i+1} = \mathcal{T}_i + \mathcal{R}(\mathcal{T}_i)$ gives $\mathcal{T}_2 = 6$, $\mathcal{T}_3 = 7$, $\mathcal{T}_4 = 9$, $\mathcal{T}_5 = 10$, $\mathcal{T}_6 = 11$, $\mathcal{T}_7 = 14$, $\mathcal{T}_8 = 15$ and $\mathcal{T}_9 = 16$.

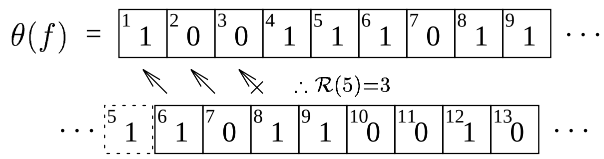


Figure 2.6: Calculating $\mathcal{R}(5)$ for the Fibonacci map.

Using lemma 11 it is now easy to determine the levels of the tower. For example, the thirteenth level is $F_{13} = [c_5; c_2]$ *since* $\mathcal{S}\langle 13\rangle = 8$ *and* $\mathcal{T}\langle 13\rangle = 11$.

Now we describe an important bound on cutting times. There is no comparable bound for co-cutting times.[3]

Lemma 15 *The inequality* $\mathcal{R}(\mathcal{S}_i) \leq \mathcal{S}_i$ *holds for every* $i \geq 1$.[4]

Proof. From figure 2.4 or lemma 12 above we know that $\mathcal{R}(\mathcal{S}_i) \leq \mathcal{S}_i$ is equivalent to $c_{\mathcal{S}_i} \notin [c_{-\mathcal{S}_i}; \tau(c_{-\mathcal{S}_i})]$.

If $c_{\mathcal{S}_i}$ was in $[c_{-\mathcal{S}_i}; \tau(c_{-\mathcal{S}_i})]$ then, because $f^{\mathcal{S}_i}[c_{-\mathcal{S}_i}; c] = f^{\mathcal{S}_i}[c; \tau(c_{-\mathcal{S}_i})] = [c; c_{\mathcal{S}_i}]$, we would have either $f^{\mathcal{S}_i}[c_{-\mathcal{S}_i}; c] \subseteq [c_{-\mathcal{S}_i}; c]$ or $f^{\mathcal{S}_i}[c; \tau(c_{-\mathcal{S}_i})] \subseteq [c; \tau(c_{-\mathcal{S}_i})]$.

But these both imply the existence of a periodic attractor since $f^{\mathcal{S}_i}\Big|_{[c_{-\mathcal{S}_i}; c]}$ and $f^{\mathcal{S}_i}\Big|_{[c; \tau(c_{-\mathcal{S}_i})]}$ are homeomorphisms [15]. ■

We end this section with a technical discussion of the relationship between $\mathcal{R}(i)$ and $\mathcal{R}(i-n)$ when $n < i < n + \mathcal{R}(n)$. Throughout the discussion n will be fixed and i will vary. We want to know when $\mathcal{R}(i) \leq \mathcal{R}(i-n)$. The results play an important role in the next section.

Sometimes $\mathcal{R}(i)$ and $\mathcal{R}(i-n)$ are equal. Taking $n = 3$, the kneading invariant shown in figure 2.7 has $\mathcal{R}(4) = \mathcal{R}(1)$, $\mathcal{R}(5) = \mathcal{R}(2)$ and $\mathcal{R}(7) = \mathcal{R}(4)$ for example. The criterion for $\mathcal{R}(i) = \mathcal{R}(i-n)$ is as follows: since $\theta^{n+1} \cdots \theta^{n+\mathcal{R}(n)-1}$ equals $\theta^1 \cdots \theta^{\mathcal{R}(n)-1}$

[3] The kneading invariant of figure 2.8(c) has $\mathcal{R}(\mathcal{T}_i) > \mathcal{T}_1$ for example.

[4] Here as in the rest of this chapter it is assumed that the kneading invariant is not periodic and that f has neither periodic attractors nor homtervals.

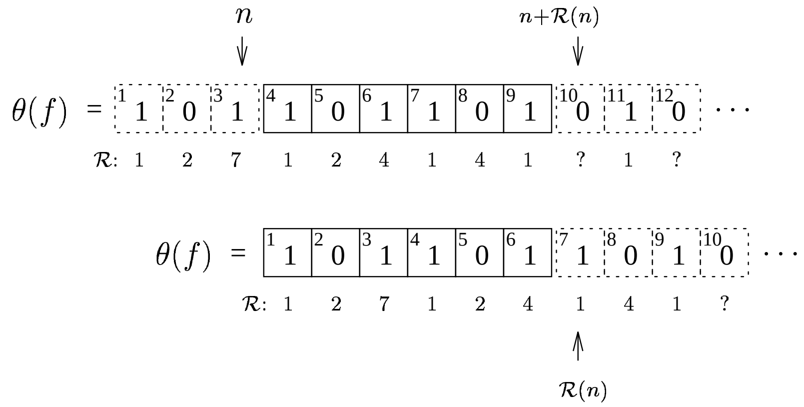


Figure 2.7: $\mathcal{R}(i)$ and $\mathcal{R}(i-n)$ for $n < i < n + \mathcal{R}(n)$. Here $n = 3$, $\mathcal{R}(n) = 7$ and $\mathcal{R}^{-}(n) = 5$.

from the definition of $\mathcal{R}$, if $\mathcal{R}(i)$ is calculated solely from $\theta^{n+1} \cdots \theta^{n+\mathcal{R}(n)-1}$ then the calculation of $\mathcal{R}(i-n)$ yields the same result. This happens exactly when $i + \mathcal{R}(i) < n + \mathcal{R}(n)$.

Sometimes $\mathcal{R}(i) > \mathcal{R}(i-n)$, for example in figure 2.7 we have $\mathcal{R}(8) > \mathcal{R}(5)$. This occurs exactly when $i + \mathcal{R}(i) > n + \mathcal{R}(n)$, in other words when $\theta^{i+1} \cdots \theta^{n+\mathcal{R}(n)} = \theta^{1} \cdots \theta^{n+\mathcal{R}(n)-i}$: recall that $\theta^{n+\mathcal{R}(n)}$ and $\theta^{\mathcal{R}(n)}$ are never equal, from the definition of $\mathcal{R}$. So while $\theta^{i-n+1} \cdots \theta^{\mathcal{R}(n)-1}$ equals $\theta^{1} \cdots \theta^{n+\mathcal{R}(n)-i-1}$ because $\theta^{i+1} \cdots \theta^{n+\mathcal{R}(n)-1}$ does, the next digit $\theta^{\mathcal{R}(n)}$ does not equal $\theta^{n+\mathcal{R}(n)-i}$ since $\theta^{n+\mathcal{R}(n)-i} = \theta^{n+\mathcal{R}(n)}$. Therefore $\mathcal{R}(i-n) = n + \mathcal{R}(n) - i$ and so $\mathcal{R}(i) > \mathcal{R}(i-n)$.

Sometimes $\mathcal{R}(i) < \mathcal{R}(i-n)$. In figure 2.7 we have $\mathcal{R}(6) < \mathcal{R}(3)$ and $\mathcal{R}(9) < \mathcal{R}(6)$ for example. The criterion is $\mathcal{R}(i) < \mathcal{R}(i-n)$ if $i + \mathcal{R}(i) = n + \mathcal{R}(n)$ since then $i + \mathcal{R}(i-n) > n + \mathcal{R}(n)$.

We summarise these results as follows:

Lemma 16 *If $n \geq 1$ and $n < i < n + \mathcal{R}(n)$ then*

1. $\mathcal{R}(i) = \mathcal{R}(i-n)$ *if and only if* $i + \mathcal{R}(i) < n + \mathcal{R}(n)$ *if and only if* $i - n + \mathcal{R}(i-n) < \mathcal{R}(n)$.
2. $\mathcal{R}(i) > \mathcal{R}(i-n)$ *if and only if* $i + \mathcal{R}(i) > n + \mathcal{R}(n)$ *if and only if* $i - n + \mathcal{R}(i-n) = \mathcal{R}(n)$.

3. $\mathcal{R}(i) < \mathcal{R}(i-n)$ *if and only if* $i + \mathcal{R}(i) = n + \mathcal{R}(n)$ *if and only if* $i - n + \mathcal{R}(i-n) > \mathcal{R}(n)$.

Therefore $\mathcal{R}(i) \leq \mathcal{R}(i-n)$ *if and only if* $i + \mathcal{R}(i) \leq n + \mathcal{R}(n)$ *if and only if* $i - n + \mathcal{R}(i-n) \neq \mathcal{R}(n)$.

So let us prove the following lemma. This shows that $\mathcal{R}(i) \leq \mathcal{R}(i-n)$ always holds except if i is too large:

Lemma 17 (Bruin [9]) *If* $\mathcal{R}(n) > 1$ *and* $n < i < n + \mathcal{R}^-(n)$ *then* $\mathcal{R}(i) \leq \mathcal{R}(i-n)$. *On the other hand,* $\mathcal{R}(n + \mathcal{R}^-(n)) > \mathcal{R}(\mathcal{R}^-(n))$.

Proof. We will prove that if $n < i < n + \mathcal{R}^-(n)$ then $i - n + \mathcal{R}(i-n) \neq \mathcal{R}(n)$, which is equivalent to $\mathcal{R}(i) \leq \mathcal{R}(i-n)$ by lemma 16.

Take a cutting time $\mathcal{S}_i > 1$ (this represents $\mathcal{R}(n)$) and some $1 \leq j < \mathcal{S}_{i-1}$ (j represents $i-n$ so this corresponds to $n < i < n + \mathcal{R}^-(n)$). Then $i - n + \mathcal{R}(i-n) = \mathcal{R}(n)$ corresponds to $j + \mathcal{R}(j) = \mathcal{S}_i$. The proof is by contradiction so suppose $j + \mathcal{R}(j) = \mathcal{S}_i$.

Put $I = [c; c_j]$ and $J = f^{j-1}(L_{\mathcal{S}_{i-1}+1})$. Each of these intervals has c_j as an endpoint. Since c is in I but not in J, either J is contained in I or the two intervals lie side by side having only the point c_j in common.

If J is contained in I then, as c is contained in $f^{\mathcal{S}_{i-1}-j}(J)$, c is also in $f^{\mathcal{S}_{i-1}-j}(I)$. However $c \in f^{\mathcal{S}_{i-1}-j}(I)$ implies $R(c_j) \leq \mathcal{S}_{i-1} - j$. Since $R(c_j) = \mathcal{R}(j)$, this contradicts $\mathcal{R}(j) = \mathcal{S}_i - j$.

Now suppose I intersects J only at c_j. We know that $f^{\mathcal{S}_i - j}|_J$ is a homeomorphism from the definition of J, and $f^{\mathcal{S}_i - j}|_I$ is a homeomorphism from the assumption $\mathcal{R}(j) = \mathcal{S}_i - j$. It follows that $f^{\mathcal{S}_i - j}|_{I \cup J}$ is a homeomorphism. However c is in $f^{\mathcal{S}_i - j}(J)$. This means that c is not in $f^{\mathcal{S}_i - j}(I)$, contradicting $R(c_j) = \mathcal{R}(j) = \mathcal{S}_i - j$.

The inequality $\mathcal{R}(n + \mathcal{R}^-(n)) > \mathcal{R}(\mathcal{R}^-(n))$ follows from part 2 of lemma 16 and $\mathcal{R}^-(n) + \mathcal{R}(\mathcal{R}^-(n)) = \mathcal{R}(n)$ (recall that $\mathcal{R}^-(n)$ and $\mathcal{R}(n)$ are successive cutting times, so this is equation 2.4). ■

2.4 Return Times

In this section we define a new sequence, the sequence of *return times*, and describe some of its properties. It is made up of cutting and co-cutting times, but only the most important. This technical tool is used in chapters 4 and 5.

The first return time is $\mathcal{T}_1 - 1$, denoted $\mathcal{M}_1$. This is a cutting time. The subsequent return times are given by

$$\mathcal{M}_{i+1} = \mathcal{M}_i + \mathcal{R}^-(\mathcal{M}_i).$$

We prove below that $\mathcal{R}^-(\mathcal{M}_i)$ is always defined.[5]

Figure 2.8 shows the return times for several kneading invariants.

We observe

1. Each return times is either a cutting or a co-cutting time. An odd index $\mathcal{M}_{2i-1}$ corresponds to a cutting time and an even index $\mathcal{M}_{2i}$ to a co-cutting time.
2. If $\mathcal{M}_i$ is a cutting time then $\mathcal{M}_{i+1}$ is the last co-cutting time before the next cutting time. If $\mathcal{M}_i$ is a co-cutting time then $\mathcal{M}_{i+1}$ is the last cutting time before the next co-cutting time.

The last observation gives a simple way of generating return times if the cutting and co-cutting times are known.

So let us prove these observations using lemma 18 below. The cases $\mathcal{S}_4 < \mathcal{T}_2 < \mathcal{S}_5$ in figure 2.8(d) and $\mathcal{T}_1 < \mathcal{S}_3 < \mathcal{T}_2$ in figure 2.8(c) illustrate the lemma particularly well.

Lemma 18

1. If $\mathcal{S}_j < \mathcal{T}_i < \mathcal{S}_{j+1}$ then

a. $\mathcal{T}_i = \mathcal{S}_j + \mathcal{S}_k$ *for some* $k \geq 1$.

b. $\mathcal{S}_j + \mathcal{R}^-(\mathcal{S}_j) = \mathcal{T}_l$ *for some* $l \geq i$.

c. $\{\mathcal{T}_i, \mathcal{T}_{i+1}, \ldots, \mathcal{T}_l\} = \{\mathcal{S}_j + \mathcal{S}_k, \mathcal{S}_j + \mathcal{S}_{k+1}, \ldots, \mathcal{S}_j + \mathcal{R}^-(\mathcal{S}_j)\}$.

$\mathcal{T}_l$ *is the last co-cutting time before* $\mathcal{S}_{j+1}$.

[5] If we had $\mathcal{R}(\mathcal{M}_i) = 1$ then $\mathcal{R}^-(\mathcal{M}_i)$ would not be defined.

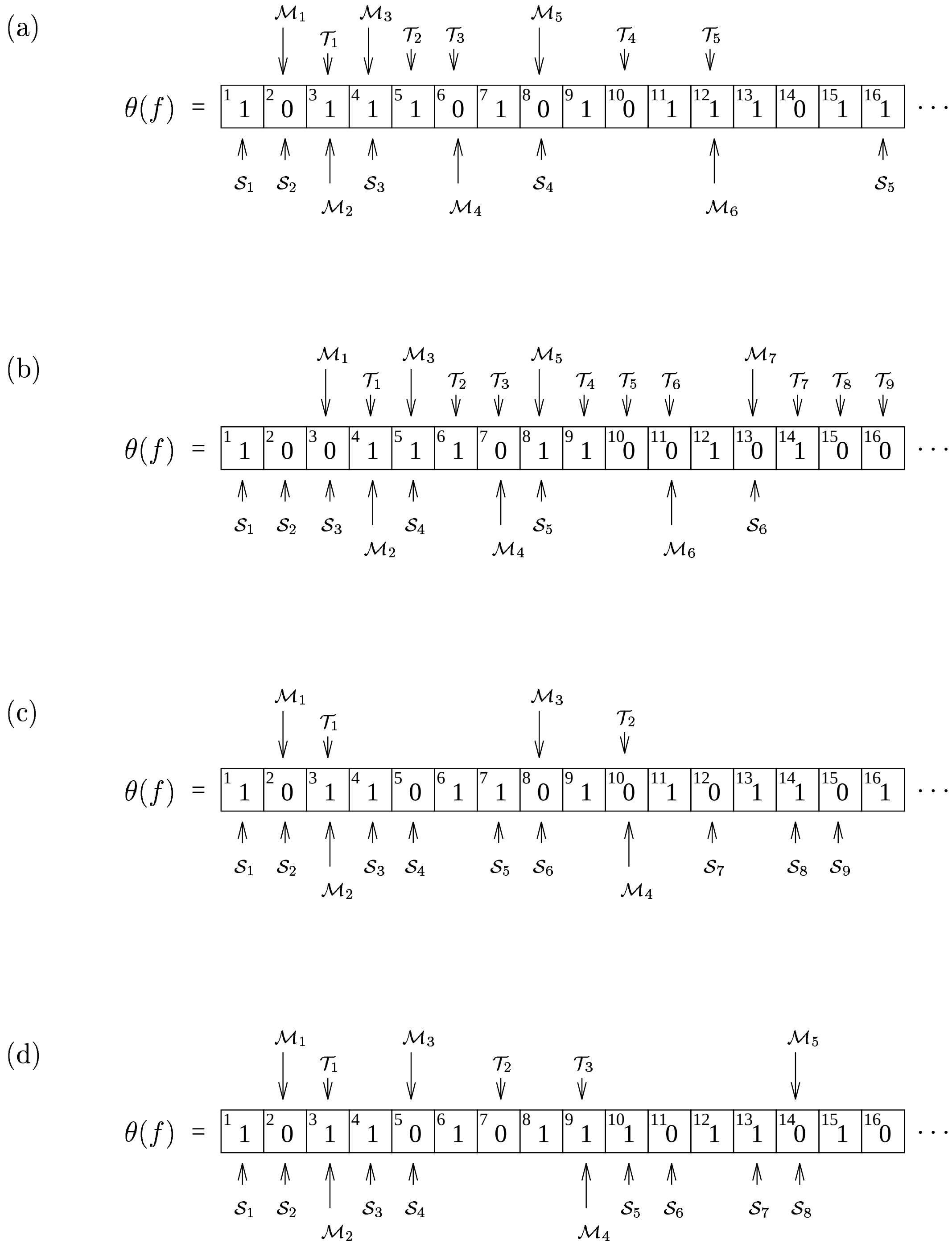


Figure 2.8: The return times, cutting times and co-cutting times of (a) the Feigenbaum map, (b) the Fibonacci map, (c) the kneading invariant of figure 2.7 and (d) a miscellaneous kneading invariant.

2. If $\mathcal{T}_j < \mathcal{S}_i < \mathcal{T}_{j+1}$ then

a. $\mathcal{S}_i = \mathcal{T}_j + \mathcal{S}_k$ *for some* $k \geq 1$.

b. $\mathcal{T}_j + \mathcal{R}^-(\mathcal{T}_j) = \mathcal{S}_l$ *for some* $l \geq i$.

c. $\{\mathcal{S}_i, \mathcal{S}_{i+1}, \ldots, \mathcal{S}_l\} = \{\mathcal{T}_j + \mathcal{S}_k, \mathcal{T}_j + \mathcal{S}_{k+1}, \ldots, \mathcal{T}_j + \mathcal{R}^-(\mathcal{T}_j)\}$.

$\mathcal{S}_l$ *is the last cutting time before* $\mathcal{T}_{j+1}$.

Proof. The proof is inductive.

Take any $\mathcal{S}_j$ and $\mathcal{T}_i$ such that $\mathcal{S}_j < \mathcal{T}_i < \mathcal{S}_{j+1}$ and suppose $\mathcal{T}_i = \mathcal{S}_j + \mathcal{S}_k$ for some $k \geq 1$. In other words, that part 1(a) of the lemma holds. The case starting the induction has $\mathcal{T}_i = \mathcal{T}_1$ and $\mathcal{S}_j = \mathcal{T}_1 - 1$; (1)(a) holds since $\mathcal{T}_i - \mathcal{S}_j = 1$, a cutting time.

Let us prove (1)(b) and (1)(c) simultaneously. If $\mathcal{T}_i = \mathcal{S}_j + \mathcal{R}^-(\mathcal{S}_j)$ then there is nothing to prove, so suppose $\mathcal{S}_k < \mathcal{R}^-(\mathcal{S}_j)$. Then $\mathcal{T}_i + \mathcal{R}(\mathcal{T}_i - \mathcal{S}_j) = \mathcal{S}_j + \mathcal{S}_k + \mathcal{R}(\mathcal{S}_k) = \mathcal{S}_j + \mathcal{S}_{k+1}$. Since $\mathcal{R}^-(\mathcal{S}_j)$ is a cutting time the assumption $\mathcal{S}_k < \mathcal{R}^-(\mathcal{S}_j)$ implies $\mathcal{S}_{k+1} \leq \mathcal{R}^-(\mathcal{S}_j)$. Therefore $\mathcal{T}_i + \mathcal{R}(\mathcal{T}_i - \mathcal{S}_j) < \mathcal{S}_j + \mathcal{R}(\mathcal{S}_j)$. The first case of lemma 16 shows that $\mathcal{R}(\mathcal{T}_i) = \mathcal{R}(\mathcal{T}_i - \mathcal{S}_j)$ and so $\mathcal{T}_{i+1} = \mathcal{S}_j + \mathcal{S}_{k+1}$.

Now repeat the argument for $\mathcal{T}_{i+1}$: either $\mathcal{T}_{i+1} = \mathcal{S}_j + \mathcal{R}^-(\mathcal{S}_j)$ or $\mathcal{T}_{i+2} = \mathcal{S}_j + \mathcal{S}_{k+2}$. Continuing, eventually we will have $\mathcal{S}_j + \mathcal{R}^-(\mathcal{S}_j) = \mathcal{T}_l$ for some $l \geq i$ and $\{\mathcal{T}_i, \mathcal{T}_{i+1}, \ldots, \mathcal{T}_l\} = \{\mathcal{S}_j + \mathcal{S}_k, \mathcal{S}_j + \mathcal{S}_{k+1}, \ldots, \mathcal{S}_j + \mathcal{R}^-(\mathcal{S}_j)\}$.

From the last statement of lemma 17 we have $\mathcal{R}(\mathcal{T}_l) = \mathcal{R}(\mathcal{S}_j + \mathcal{R}^-(\mathcal{S}_j)) > \mathcal{R}(\mathcal{R}^-(\mathcal{S}_j))$. Therefore $\mathcal{T}_{l+1} > \mathcal{S}_j + \mathcal{R}(\mathcal{S}_j) + \mathcal{R}(\mathcal{R}^-(\mathcal{S}_j)) = \mathcal{S}_{j+1}$ and so $\mathcal{T}_l$ is the last co-cutting time before $\mathcal{S}_{j+1}$.

But now (2)(a), (2)(b), and (2)(c) hold for the triple $\mathcal{T}_l < \mathcal{S}_{j+1} < \mathcal{T}_{l+1}$: $\mathcal{S}_{j+1} - \mathcal{T}_l = \mathcal{R}(\mathcal{S}_j) - \mathcal{R}^-(\mathcal{S}_j) = \mathcal{R}(\mathcal{R}^-(\mathcal{S}_j))$ is a cutting time, giving (2)(a), and parts 2(b) and 2(c) follow as for (1)(b) and (1)(c) above.

Write $\mathcal{S}_m$ for $\mathcal{T}_l + \mathcal{R}(\mathcal{T}_l)$. Then $\mathcal{S}_m$ is the last cutting time before $\mathcal{T}_{l+1}$ for the same reason that $\mathcal{T}_l$ was the last co-cutting time before $\mathcal{S}_{j+1}$.

In addition the triple $\mathcal{S}_m < \mathcal{T}_{l+1} < \mathcal{S}_{m+1}$ satisfies (1)(a). Now repeat the argument indefinitely. ∎

Observations 1 and 2 above now follow inductively. For example, if $\mathcal{M}_n = \mathcal{S}_j$, a cutting time, then the lemma shows that $\mathcal{M}_n + \mathcal{R}^-(\mathcal{M}_n)$ is a co-cutting time, the last before $\mathcal{S}_{j+1}$. These are observations 1 and 2 in this case. A similar argument proves observations 1 and 2 when $\mathcal{M}_n$ is a co-cutting time.

However to apply the lemma it is necessary to show that there is some co-cutting time[6] between $\mathcal{M}_n$ and $\mathcal{M}_n + \mathcal{R}(\mathcal{M}_n)$. But this is a simple induction using lemma 18. As a by-product we have $\mathcal{R}(\mathcal{M}_n) > 1$ and therefore $\mathcal{R}^-(\mathcal{M}_n)$ is always defined.

Now let us describe some properties of return times that will be used later.

The other results are based on the following technical lemma. It shows that iterates between return times are dominated by the return times.

Lemma 19 *If* $\mathcal{M}_n < i < \mathcal{M}_{n+1}$ *then* $\mathcal{R}(i) \leq \mathcal{R}(i - \mathcal{M}_n)$, $i + \mathcal{R}(i) \leq \mathcal{M}_n + \mathcal{R}(\mathcal{M}_n)$ *and therefore* $\mathcal{R}(i) < \mathcal{R}(\mathcal{M}_n)$.

Proof. This combines the first statement of lemma 17 and the last statement of lemma 16. ■

Let us show that every closest return time is a return time. We say that $n > 1$ is a *closest return time* if $\mathcal{R}(n)$ is larger than $\mathcal{R}(i)$ for every $1 \leq i < n$.[7]

Lemma 20 *Every closest return time is a return time.*

Proof. Since $\mathcal{R}(j) = 1$ for $j = 1, \ldots, \mathcal{M}_1 - 1$, if n is a closest return time then $n \geq \mathcal{M}_1$. If n is not a return time then $\mathcal{M}_i < n < \mathcal{M}_{i+1}$ for some $\mathcal{M}_i$, but then $\mathcal{R}(n) < \mathcal{R}(\mathcal{M}_i)$, from the last lemma, a contradiction. So n must be a return time. ■

The next result also shows that a dynamically significant event must occur at a return time, though in this case the event is more technical. This result is used in the proof of lemma 59.

Lemma 21 *If* $i > 1$ *and* $\mathcal{R}(\mathcal{S}_i) = \mathcal{S}_i$ *then* $\mathcal{S}_i$ *is a return time.*

Proof. Requiring $i > 1$ ensures $\mathcal{S}_i$ is greater than $\mathcal{M}_1$. If $\mathcal{S}_i$ is not a return time then $\mathcal{M}_n < \mathcal{S}_i < \mathcal{M}_{n+1}$ for some $\mathcal{M}_n$. Note that $\mathcal{M}_n$ must be a co-cutting time.

[6] A cutting time if $\mathcal{M}_n$ is a co-cutting time.

[7] If f is asymmetric this need not mean that $|c_n - c|$ is smaller than $|c_i - c|$ for every $1 \leq i < n$, though this is the idea behind the definition.

From lemma 18 we have $\mathcal{S}_i = \mathcal{M}_n + \mathcal{S}_k$ for some $k \geq 1$. Clearly $\mathcal{S}_k < \mathcal{S}_i$. Lemma 19 gives $\mathcal{R}(\mathcal{S}_i) \leq \mathcal{R}(\mathcal{S}_k)$. But $\mathcal{S}_k < \mathcal{S}_i$ and $\mathcal{S}_i = \mathcal{R}(\mathcal{S}_i) \leq \mathcal{R}(\mathcal{S}_k)$ show $\mathcal{R}(\mathcal{S}_k) > \mathcal{S}_k$. This contradicts the conclusion of lemma 15, namely $\mathcal{R}(\mathcal{S}_k) \leq \mathcal{S}_k$.

So $\mathcal{S}_i$ must be a return time. ■

We will also need the following lemma. A kneading invariant satisfying either condition is called slowly recurrent.

Lemma 22 *We have*

$$\lim_{l\to\infty} \limsup_{i\to\infty} \frac{\sum_{j=1}^{i} \begin{cases} \mathcal{R}(j) & \text{if } \mathcal{R}(j) \geq l \\ 0 & \text{otherwise} \end{cases}}{i} = 0$$

if and only if

$$\lim_{l\to\infty} \limsup_{i\to\infty} \frac{\sum_{j=1}^{i} \begin{cases} \mathcal{R}(\mathcal{M}_j) & \text{if } \mathcal{R}(\mathcal{M}_j) \geq l \\ 0 & \text{otherwise} \end{cases}}{\mathcal{M}_i} = 0.$$

Proof. The 'only if' part is immediate, so let us prove 'if'. Suppose

$$\lim_{l\to\infty} \limsup_{i\to\infty} \frac{\sum_{j=1}^{i} \begin{cases} \mathcal{R}(\mathcal{M}_j) & \text{if } \mathcal{R}(\mathcal{M}_j) \geq l \\ 0 & \text{otherwise} \end{cases}}{\mathcal{M}_i} = 0$$

and take any $0 < \epsilon < 1/2$. By taking l large enough we can assume

$$\frac{\sum_{j=1}^{i} \begin{cases} \mathcal{R}(\mathcal{M}_j) & \text{if } \mathcal{R}(\mathcal{M}_j) \geq l \\ 0 & \text{otherwise} \end{cases}}{\mathcal{M}_i} < \epsilon \tag{2.6}$$

for all $i \geq 1$. This implies $\mathcal{R}(\mathcal{M}_j) < \mathcal{M}_j$ if $\mathcal{R}(\mathcal{M}_j) \geq l$.

Let us prove

$$\sum_{j=1}^{n} \begin{cases} \mathcal{R}(j) & \text{if } \mathcal{R}(j) \geq l \\ 0 & \text{otherwise} \end{cases} < 2 \sum_{\mathcal{M}_j \leq n} \begin{cases} \mathcal{R}(\mathcal{M}_j) & \text{if } \mathcal{R}(\mathcal{M}_j) \geq l \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

inductively for all $n \geq 1$ and therefore

$$\frac{\sum_{j=1}^{n} \begin{cases} \mathcal{R}(j) & \text{if } \mathcal{R}(j) \geq l \\ 0 & \text{otherwise} \end{cases}}{n} < 2\epsilon \tag{2.8}$$

because of equation 2.6.

Suppose

$$\sum_{j=1}^{i} \begin{cases} \mathcal{R}(j) & \text{if } \mathcal{R}(j) \geq l \\ 0 & \text{otherwise} \end{cases} < 2 \sum_{\mathcal{M}_j \leq i} \begin{cases} \mathcal{R}(\mathcal{M}_j) & \text{if } \mathcal{R}(\mathcal{M}_j) \geq l \\ 0 & \text{otherwise} \end{cases} \tag{2.9}$$

holds for all $1 \leq i < n$. It certainly holds for $i = 1, \ldots, \mathcal{M}_1$. We verify equation 2.9 for $i = n$.

Choose $\mathcal{M}_k$ such that $\mathcal{M}_k \leq n < \mathcal{M}_{k+1}$. If $\mathcal{R}(\mathcal{M}_k) < l$ then $\mathcal{R}(n) < l$ also, from lemma 19, and there is nothing to prove.

So suppose $\mathcal{R}(\mathcal{M}_k) \geq l$. Then

$$\begin{aligned} \sum_{j=1}^{n} \begin{cases} \mathcal{R}(j) & \text{if } \mathcal{R}(j) \geq l \\ 0 & \text{otherwise} \end{cases} &= \sum_{j=1}^{\mathcal{M}_k - 1} \begin{cases} \mathcal{R}(j) & \text{if } \mathcal{R}(j) \geq l \\ 0 & \text{otherwise} \end{cases} \\ &\quad + \mathcal{R}(\mathcal{M}_k) + \sum_{j=\mathcal{M}_k+1}^{n} \begin{cases} \mathcal{R}(j) & \text{if } \mathcal{R}(j) \geq l \\ 0 & \text{otherwise} \end{cases} \\ &< 2 \sum_{\mathcal{M}_j < \mathcal{M}_k} \begin{cases} \mathcal{R}(\mathcal{M}_j) & \text{if } \mathcal{R}(\mathcal{M}_j) \geq l \\ 0 & \text{otherwise} \end{cases} \\ &\quad + \mathcal{R}(\mathcal{M}_k) + \sum_{j=\mathcal{M}_k+1}^{n} \begin{cases} \mathcal{R}(j) & \text{if } \mathcal{R}(j) \geq l \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

using the inductive hypothesis.

Lemma 19 shows that $\mathcal{R}(i) \leq \mathcal{R}(i - \mathcal{M}_k)$ for $\mathcal{M}_k < i \leq n$ and so

$$\begin{aligned} \sum_{j=\mathcal{M}_k+1}^{n} \begin{cases} \mathcal{R}(j) & \text{if } \mathcal{R}(j) \geq l \\ 0 & \text{otherwise} \end{cases} &\leq \sum_{j=1}^{n-\mathcal{M}_k} \begin{cases} \mathcal{R}(j) & \text{if } \mathcal{R}(j) \geq l \\ 0 & \text{otherwise} \end{cases} \\ &< 2\epsilon(n - \mathcal{M}_k) < 2\epsilon\mathcal{R}(\mathcal{M}_k), \end{aligned}$$

using the inductive hypothesis[8] and equation 2.6, as in equation 2.8. Therefore

$$\begin{aligned} \sum_{j=1}^{n} \begin{cases} \mathcal{R}(j) & \text{if } \mathcal{R}(j) \geq l \\ 0 & \text{otherwise} \end{cases} &< 2 \sum_{\mathcal{M}_j < \mathcal{M}_k} \begin{cases} \mathcal{R}(\mathcal{M}_j) & \text{if } \mathcal{R}(\mathcal{M}_j) \geq l \\ 0 & \text{otherwise} \end{cases} \\ &\quad + (1 + 2\epsilon)\mathcal{R}(\mathcal{M}_k) \\ &< 2 \sum_{\mathcal{M}_j \leq n} \begin{cases} \mathcal{R}(\mathcal{M}_j) & \text{if } \mathcal{R}(\mathcal{M}_j) \geq l \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

[8]Permissible since $n - \mathcal{M}_k < \mathcal{R}(\mathcal{M}_k) < \mathcal{M}_k$.

This completes the proof that equations 2.7 and 2.8 hold for all $n \geq 1$.

Since equation 2.8 holds for each $0 < \epsilon < 1/2$ if l is large enough this proves

$$\lim_{l\to\infty} \limsup_{i\to\infty} \frac{\sum_{j=1}^{i} \begin{cases} \mathcal{R}(j) & \text{if } \mathcal{R}(j) \geq l \\ 0 & \text{otherwise} \end{cases}}{i} = 0.$$

■

Chapter 3

The Parameter Dependence of Tent-Map Towers

In this chapter we describe how the tower of T_λ varies with λ. The results are used in chapter 5 to estimate the abundance of tent-maps with particular dynamical properties.

The analysis is based on a few simple observations drawn from figures 3.1–3.4. In the first section, "The Images of c", we examine $c_n(\lambda)$, the n'th image of the critical point under T_λ, in the context of figures 3.1 and 3.2. In "The Endpoints of the Tower" we describe the relationship between $c_n(\lambda)$ and the endpoints $a_n(\lambda)$ and $b_n(\lambda)$ of the n'th tower level of T_λ. In the last section, "Towers and Parameter Space", we show how to approximate tower levels using parameter space information.

Some technical details have been omitted for brevity.

In this chapter a_n, b_n and c_n mean these quantities as functions of λ. For example $c_n : [1;2] \to [0;1]$ is $\lambda \mapsto {T_\lambda}^n(c)$.

3.1 The Images of c

In this section we describe how $c_n(\lambda)$ depends on λ.

Figures 3.1 and 3.2 graph c_n for the first few values of n. We observe:

1. The graph of c_n is continuous. It is composed of smooth segments with cusps

at their end points.[1]

2. The graph of c_{n+1} has a cusp wherever the graph of c_n has a cusp; a new cusp is created for c_{n+1} wherever c_n crosses $y = 1/2$ and only there.[2]

3. The segments become steeper as n increases, except near $\lambda = 1$.

4. The segments become straighter as n increases, except near $\lambda = 1$.

Example 3.1 *Figure 3.1 shows c_3 crossing $y = 1/2$ at $\lambda = (1+\sqrt{5})/2$. The tent-map $T_{(1+\sqrt{5})/2}$ is periodic with least period 3; c_n has a cusp at $(1+\sqrt{5})/2$ for all $n > 3$. Note that if $n > 3$ is a multiple of 3 then the graph of c_n dips down and touches $y = 1/2$ at $\lambda = (1+\sqrt{5})/2$ without crossing.*

Let us now prove observations 1–4.

The proofs of the first two observations are left to the reader; it follows from observation 2 that the cusps of c_n occur at those λ values for which T_λ is periodic with period less than n. In other words, where $c_m(\lambda) = 1/2$ for some $1 \leq m < n$.

To prove observation 3 we show that slopes grow like λ^n, in other words that state and parameter space derivatives are comparable. The method of proof is taken from Benedicks and Carleson's paper [5] on the Hénon map. Similar results have been proved by Coven et al. [12]; see also the preprint of Brucks and Misiurewicz [7]. The upper bound holds even when $n < N(\epsilon)$.

Lemma 23 *For each $\epsilon > 0$ there exists $K(\epsilon) > 1$ and $N(\epsilon) > 1$ such that if λ is in $[1+\epsilon; 2]$, $n \geq N(\epsilon)$ and c_n is differentiable at λ then*

$$\frac{1}{K(\epsilon)} \leq \frac{|c_n'(\lambda)|}{\lambda^n} \leq K(\epsilon).$$

Proof. We must show that

$$\rho_n(\lambda) = \frac{|c_n'(\lambda)|}{\lambda^n}$$

is bounded from above and away from zero for n large, uniformly for $\lambda \in [1+\epsilon; 2]$.

[1]Except at the endpoints of $[1; 2]$.

[2]A careful justification of this fact needs the monotonicity in λ of the kneading invariant [8].

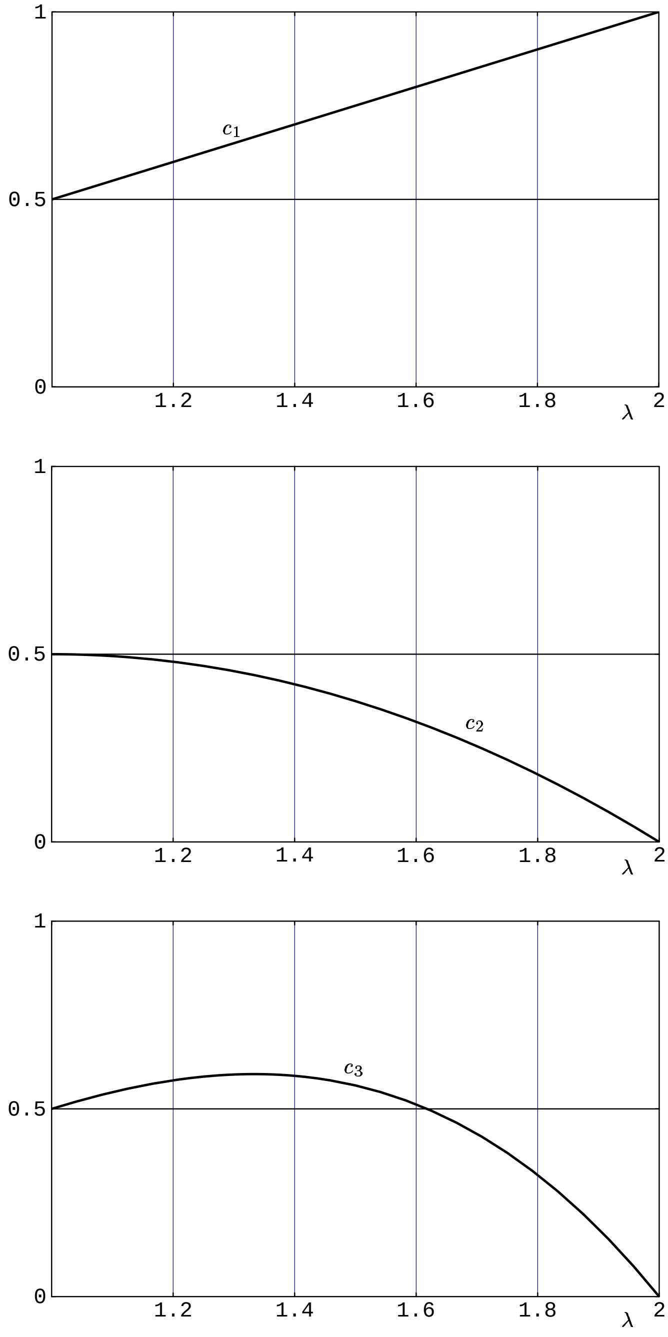


Figure 3.1: The dependence of c_1, c_2 and c_3 on λ.

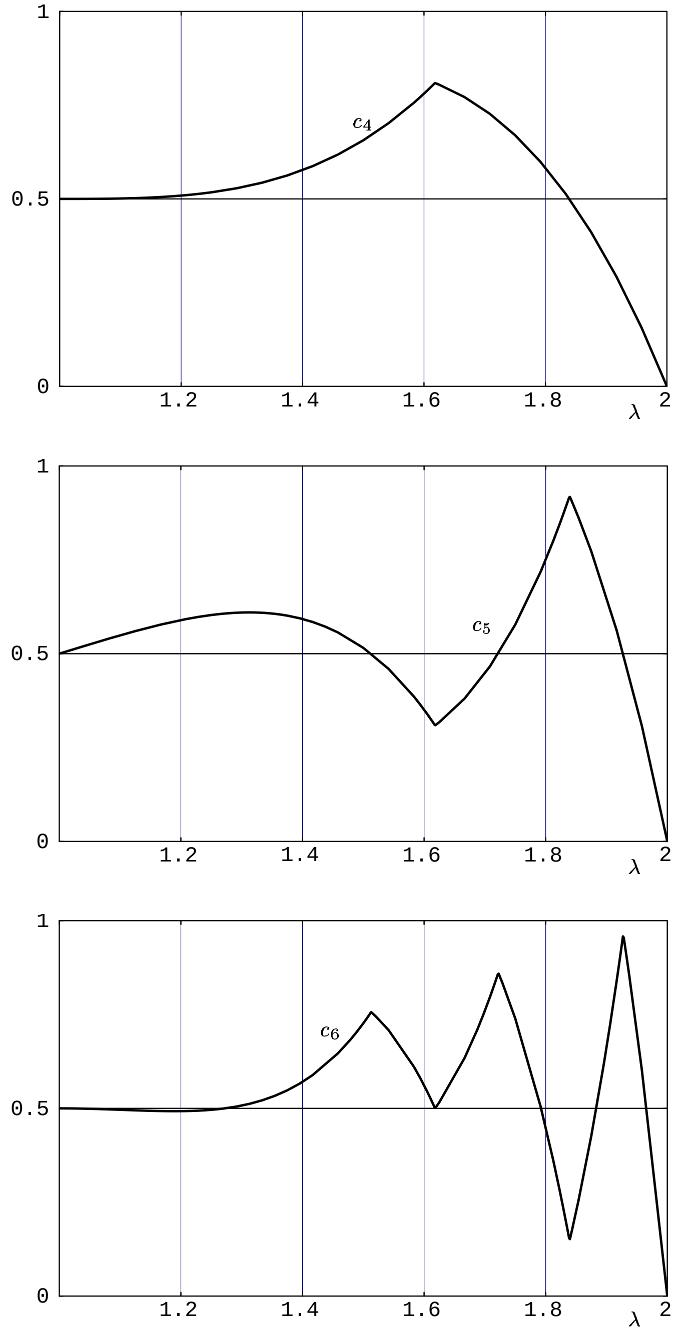


Figure 3.2: The dependence of c_4, c_5 and c_6 on λ.

The proof is in three steps. First, we show that $\rho_n(\lambda)$ converges uniformly on $[1+\epsilon; 2]$ as $n \to \infty$. The limit is uniformly bounded above on $[1+\epsilon; 2]$. It therefore suffices to show that ρ_n is uniformly bounded away from zero on $[1+\epsilon; 2]$ for n large. In the second step we observe that ρ_n is uniformly bounded away from zero on $[\sqrt{2}; 2]$ for n large. In the final step we pull the bound back to smaller λ values by renormalizing.

Differentiating $c_{n+1}(\lambda) = T_\lambda(c_n(\lambda))$ with respect to λ gives the recursion formula

$$c'_{n+1}(\lambda) = \begin{cases} \lambda c'_n(\lambda) + c_n(\lambda) & \text{if } c_n(\lambda) < \frac{1}{2} \\ \\ -\lambda c'_n(\lambda) + 1 - c_n(\lambda) & \text{if } c_n(\lambda) > \frac{1}{2} \end{cases}$$

and so $|\rho_{n+1}(\lambda) - \rho_n(\lambda)| \leq 1/(2\lambda^{n+1})$ for all $n \geq 1$. Therefore, for any $m > n \geq 1$,

$$|\rho_m(\lambda) - \rho_n(\lambda)| \leq \frac{1}{2\lambda^n(\lambda - 1)}.$$

Thus the sequence $\{\rho_\imath(\lambda)\}_{\imath \geq 1}$ is Cauchy and so converges. The convergence is clearly uniform on $[1+\epsilon; 2]$. The limit is uniformly bounded above because ρ_1 is.

It is easy to verify that $\rho_9(\lambda) > 1/(2\lambda^8(\lambda - 1))$ holds for $\sqrt{2} \leq \lambda \leq 2$. Because $|\rho_m(\lambda) - \rho_8(\lambda)| \leq 1/(2\lambda^8(\lambda - 1))$ it follows that $\rho_m(\lambda)$ is uniformly bounded away from zero for $m > 9$ and $\sqrt{2} \leq \lambda \leq 2$. There is therefore some $K_1 > 1$ such that

$$\frac{1}{K_1} \leq \frac{|c'_m(\lambda)|}{\lambda^m} \leq K_1$$

for $\sqrt{2} \leq \lambda \leq 2$ and $m \geq 10$.

Now consider $\lambda < \sqrt{2}$. We renormalize using the map $\phi_\lambda : x \mapsto (\lambda - (\lambda+1)x)/(\lambda - 1)$. This has the property

$$\phi_\lambda(T^2_\lambda(\phi^{-1}_\lambda(x))) = T_{\lambda^2}(x)$$

for all $x \in [0; 1]$. A straightforward calculation yields

$$c'_n(\lambda^2) = \frac{-(\lambda+1)}{2\lambda(\lambda-1)} c'_{2n}(\lambda) + \frac{c_{2n}(\lambda) - \frac{1}{2}}{\lambda(\lambda-1)^2}.$$

The term $(c_{2n}(\lambda) - 1/2)/(\lambda(\lambda - 1)^2)$ is bounded so can be ignored. Therefore $c'_n(\lambda^2)$ and $c'_{2n}(\lambda)$ are approximately proportional for n large. It follows that there exist $K_2 > 1$ and $N_2 > 1$ such that

$$1/K_2 \leq |c'_n(\lambda)|/\lambda^n \leq K_2$$

for $\sqrt[4]{2} \leq \lambda \leq \sqrt{2}$ and $n \geq N_2$. Continuing inductively, we conclude the existence of the required $K(\epsilon) > 1$ and $N(\epsilon) > 1$ for any $\epsilon > 0$. ■

Define $I_n(\lambda)$ to be the domain of the smooth segment of c_n containing λ. The left endpoint of $I_n(\lambda)$ is therefore

$$\max\{\hat{\lambda} < \lambda \mid T_{\hat{\lambda}} \text{ is periodic with period less than } n\},$$

and its right endpoint

$$\min\{\hat{\lambda} > \lambda \mid \hat{\lambda} = 2 \text{ or } T_{\hat{\lambda}} \text{ is periodic with period less than } n\}.$$

We do not define $I_n(\lambda)$ if λ is a cusp point of c_n or an endpoint of $[1; 2]$.

Let us prove observation 4. This was that segments of c_n become straighter as n increases, in other words, that the *distortion*

$$\mathcal{D}(c_n\big|_{I_n(\lambda)}) = \sup_{\lambda_1, \lambda_2 \in I_n(\lambda)} \log \frac{|c_n'(\lambda_1)|}{|c_n'(\lambda_2)|}$$

decreases as n approaches infinity. We prove in lemma 24 below that the distortion limits to zero. Brucks and Misiurewicz [7] have a similar result.

Lemma 24 *If T_λ is not periodic then $\lim_{n\to\infty} \mathcal{D}(c_n\big|_{I_n(\lambda)}) = 0$. The convergence is uniform on $[1+\epsilon; 2]$ for every $\epsilon > 0$.*

Proof. Let us first show that the distortion is bounded for n large. We know that $\lambda \neq 1$ since T_λ is not periodic. If n is large enough and $\epsilon > 0$ small enough then $I_n(\lambda) \subseteq [1+\epsilon; 2]$. Thus from the last lemma there is a $K > 1$ such that

$$\frac{1}{K} \leq \frac{|c_n'(\hat{\lambda})|}{\hat{\lambda}^n} \leq K$$

for n large enough and $\hat{\lambda} \in I_n(\lambda)$. Bounding $\mathcal{D}(c_n\big|_{I_n(\lambda)})$ on $I_n(\lambda)$ is therefore equivalent to bounding ${\lambda_1}^n/{\lambda_2}^n$ in a way which is independent of n and holds for any $\lambda_1, \lambda_2 \in I_n(\lambda)$. Since

$$\frac{{\lambda_1}^n}{{\lambda_2}^n} = (1 + \frac{\lambda_1 - \lambda_2}{\lambda_2})^n \leq e^{\frac{n|\lambda_1 - \lambda_2|}{\lambda_2}}$$

this amounts to bounding $n|\lambda_1 - \lambda_2|$. But $|\lambda_1 - \lambda_2| \leq K/{\lambda_3}^n$ for some λ_3 in $I_n(\lambda)$ since

$$\begin{aligned} 1 &\geq |c_n(\lambda_1) - c_n(\lambda_2)| \\ &= |c_n'(\lambda_3)||\lambda_1 - \lambda_2| \geq \frac{{\lambda_3}^n|\lambda_1 - \lambda_2|}{K}. \end{aligned}$$

Since $n/\lambda_3{}^n \to 0$ as $n \to \infty$ the distortion is bounded. We will use this fact freely in the remainder of the proof.

So now let us prove that the distortion limits to zero. Take $\lambda_1, \lambda_2 \in I_n(\lambda)$. Then

$$\begin{aligned}\left|\frac{|c_n'(\lambda_1)|}{|c_n'(\lambda_2)|} - 1\right| &\leq \frac{|c_n'(\lambda_1) - c_n'(\lambda_2)|}{|c_n'(\lambda_2)|} \\ &\leq \frac{|c_n''(\lambda_3)||\lambda_1 - \lambda_2|}{|c_n'(\lambda_2)|},\end{aligned}$$

for some $\lambda_3 \in (\lambda_1; \lambda_2)$. Substituting $|\lambda_1 - \lambda_2| \leq K/\lambda_3{}^n$ and $|c_n'(\lambda_2)| \geq \lambda_3{}^n/K$ gives

$$\left|\frac{|c_n'(\lambda_1)|}{|c_n'(\lambda_2)|} - 1\right| \leq \frac{|c_n''(\lambda_3)|K^2}{\lambda_3{}^{2n}}.$$

We need to show that $|c_n''(\lambda_3)|/\lambda_3{}^{2n} \to 0$ as $n \to \infty$ no matter how we choose $\lambda_3 \in I_n(\lambda)$. From the recursion formula

$$c_{m+1}''(\lambda_3) = \begin{cases} 2c_m'(\lambda_3) + \lambda_3 c_m''(\lambda_3) & \text{if } c_m(\lambda) < \frac{1}{2} \\ \\ -2c_m'(\lambda_3) - \lambda_3 c_m''(\lambda_3) & \text{if } c_m(\lambda) > \frac{1}{2} \end{cases}$$

we deduce

$$\frac{|c_{m+1}''(\lambda_3)|}{\lambda_3{}^{1.5(m+1)}} \leq \frac{|c_m''(\lambda_3)|}{\lambda_3{}^{1.5m}} + \frac{2}{\lambda_3{}^{0.5m}} \frac{|c_m'(\lambda_3)|}{\lambda_3{}^m}$$

by dividing through by $\lambda^{1.5m+1}$, taking magnitudes and adjusting some indices. It was shown in the last lemma that $|c_m'(\lambda_3)|/\lambda_3{}^m$ has an upper bound independent of m and uniform on $[1+\epsilon; 2]$ for any $\epsilon > 0$. It follows that $|c_n''(\lambda_3)|/\lambda_3{}^{1.5n}$ is uniformly bounded above on $[1+\epsilon; 2]$ and so $|c_n''(\lambda_3)|/\lambda_3{}^{2n} \to 0$ as $n \to \infty$, as required. ∎

3.2 The Endpoints of the Tower

In this section we describe how the endpoints $a_n(\lambda)$ and $b_n(\lambda)$ of the n'th tower level vary with λ.

It is important to realise that, unlike $c_n(\lambda)$, the endpoints do not vary continuously with λ. However figures 3.3 and 3.4 show that a_n and b_n are piecewise continuous, with their discontinuities occurring at points where c_n has cusps.

Denote by $I_n{}^o$ the interior of I_n. We observe:

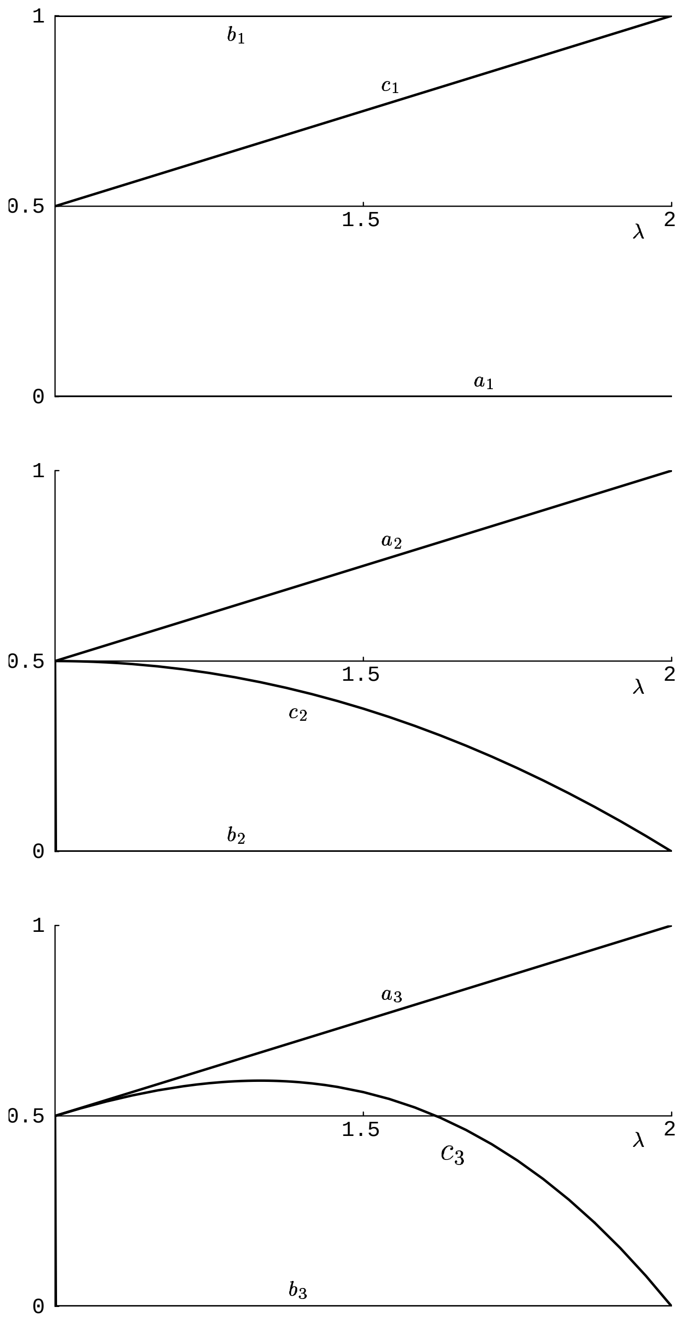


Figure 3.3: The dependence of a_n, b_n and c_n on λ for $n = 1$, 2 and 3.

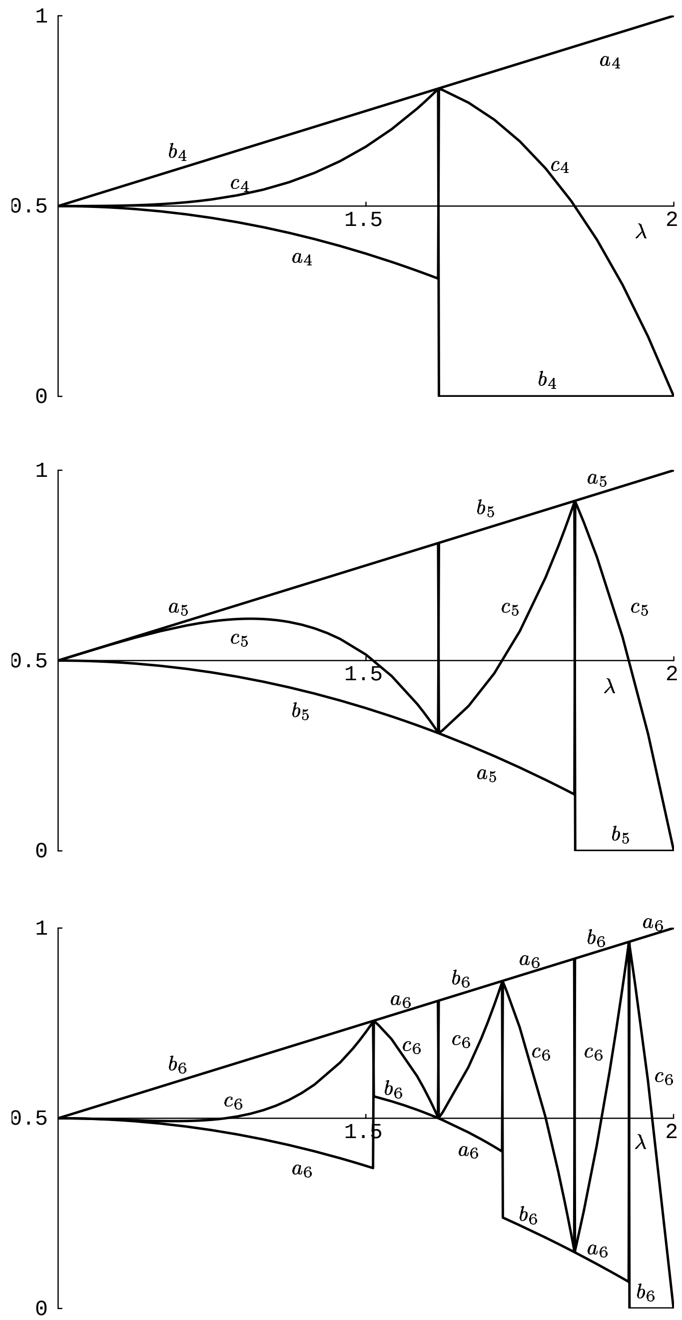


Figure 3.4: The dependence of a_n, b_n and c_n on λ for $n = 4$, 5 and 6.

Lemma 25 *For each smooth segment I_n of c_n,*

1. *a_n and b_n are continuous and differentiable on $I_n(\lambda)^o$.*
2. *Neither a_n nor b_n intersects $y = 1/2$ on $I_n{}^o$.*
3. *$c_n(\lambda)$ lies strictly between $a_n(\lambda)$ and $b_n(\lambda)$ for every $\lambda \in I_n{}^o$.*

Proof. Recall from chapter 2 that $a_n(\lambda) = c_{n-\mathcal{S}\langle n\rangle(\lambda)}(\lambda)$ and $b_n(\lambda) = c_{n-\mathcal{T}\langle n\rangle(\lambda)}(\lambda)$. Though both $\mathcal{S}\langle n\rangle(\lambda)$ and $\mathcal{T}\langle n\rangle(\lambda)$ depend on λ they are constant on $I_n(\lambda)^o$ because they only depend on the first $n-1$ digits of the kneading invariant and these digits are constant on $I_n(\lambda)^o$. Otherwise there would be some $\hat{\lambda} \in I_n(\lambda)^o$ with $c_m(\hat{\lambda}) = 1/2$ for some $1 \leq m < n$. But then $\hat{\lambda}$ would be a cusp point of c_{m+1}, and therefore of c_n, in the interior of $I_n(\lambda)$, a contradiction.

Let us write $a_n(\lambda) = c_{m_1}(\lambda)$ and $b_n(\lambda) = c_{m_2}(\lambda)$ for $\lambda \in I_n{}^o$. Since m_1 and m_2 are both strictly less than n the curves c_{m_1} and c_{m_2} are smooth on $I_n{}^o$. This proves the first part of the lemma.

If a_n or b_n intersected $y = 1/2$ on $I_n{}^o$ then c_{m_1+1} or c_{m_2+1} and so c_n would have a cusp on $I_n{}^o$, a contradiction. This proves the second part of the lemma.

The last part of the lemma holds because $c_n(\lambda)$ is always in $F_n(\lambda) = [a_n(\lambda); b_n(\lambda)]$, and in the interior unless T_λ is periodic of period less than n. That is, unless λ is a cusp point of c_n. ∎

Now let us prove that a_n and c_n 'join up' at left-hand endpoints of segments, and that b_n and c_n 'join up' at right-hand endpoints of segments:[3]

Lemma 26 *For every $n > 1$ and smooth segment $I_n = [z_1; z_2]$ of c_n, with $z_1 < z_2$,*

$$\lim_{\lambda \downarrow z_1} a_n(\lambda) = c_n(z_1)$$

and

$$\lim_{\lambda \uparrow z_2} b_n(\lambda) = c_n(z_2).$$

[3]This is the reason for the alternating high and low positions of the pieces of a_n and b_n in figure 3.4.

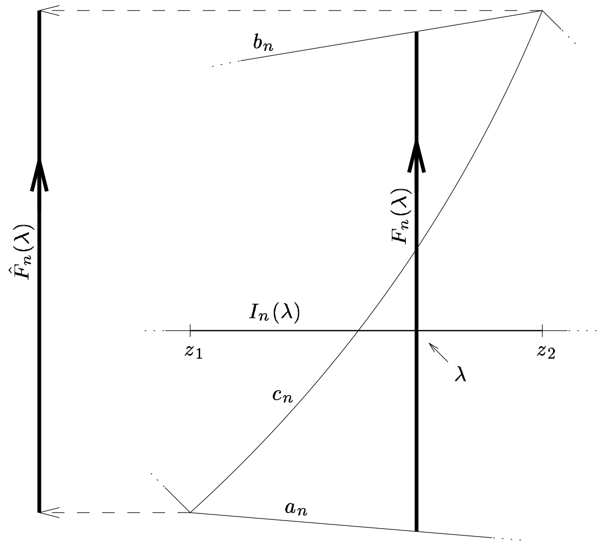


Figure 3.5: The definition of $\hat{F}_n(\lambda)$.

Proof. This is a simple induction, using the recursion relationships (2.2) and (2.3) for a_n and b_n. The monotonicity of the kneading invariant prevents exotic behaviour by c_n. ■

Our final result shows that the slope of c_n becomes increasingly steeper than the slopes of a_n and b_n as n increases:

Lemma 27 *If T_λ is not periodic then*

$$\lim_{n\to\infty} \frac{a_n'(\lambda)}{c_n'(\lambda)} = 0$$

and

$$\lim_{n\to\infty} \frac{b_n'(\lambda)}{c_n'(\lambda)} = 0.$$

Proof. From lemma 23 we have $|a_n'(\lambda)| = |c_{m_1}'(\lambda)| \leq K\lambda^{m_1}$ and $|c_n'(\lambda)| \geq \lambda^n/K$ if n is large enough. Recall that $m_1 = n - \mathcal{S}\langle n\rangle(\lambda)$. Therefore

$$\frac{|a_n'(\lambda)|}{|c_n'(\lambda)|} \leq \frac{K^2}{\lambda^{\mathcal{S}\langle n\rangle(\lambda)}}.$$

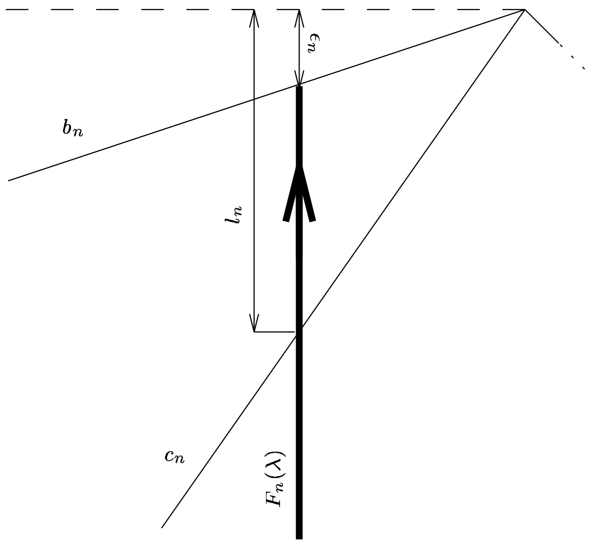


Figure 3.6: Upper right corner of figure 3.5.

But $\mathcal{S}\langle n\rangle(\lambda) \to \infty$ as $n \to \infty$ and so $|a_n'(\lambda)|/|c_n'(\lambda)| \to 0$. The other limit is proved similarly. ■

3.3 Towers and Parameter Space

In this section we illustrate the above results by describing a simple way of approximating the tower levels of a tent-map using parameter space information.

Recall that the n'th tower level of T_λ is $F_n(\lambda) = [a_n(\lambda); b_n(\lambda)]$. We define $\hat{F}_n(\lambda)$, our approximation to $F_n(\lambda)$, to be $[c_n(z_1); c_n(z_2)]$ where $I_n(\lambda) = [z_1; z_2]$ (see figure 3.5).

Let us prove that this approximation is good.

The *Hausdorff distance* $\mathcal{H}(I, J)$ between two intervals I and J is the size of the largest of the components of $I \setminus J$ and $J \setminus I$.

Proposition 28 *If T_λ is not periodic then*

$$\limsup_{n\to\infty} \frac{\mathcal{H}(F_n(\lambda), \hat{F}_n(\lambda))}{|\hat{F}_n(\lambda)|} = 0.$$

Proof. We use the notation of figure 3.6. We can treat the curves as straight lines because of lemma 24. The ratio ϵ_n/l_n thus equals the ratio of the slopes of b_n and c_n. Lemma 27 states that this goes to zero as $n \to \infty$. Therefore so does $\epsilon_n/|\hat{F}_n(\lambda)|$. A similar argument works for the other end of $F_n(\lambda)$. Therefore

$$\frac{\mathcal{H}(F_n(\lambda), \hat{F}_n(\lambda))}{|\hat{F}_n(\lambda)|} \to 0.$$

■

Chapter 4

The Collet-Eckmann Condition

We say that f in $\mathcal{C}$ is *Collet-Eckmann*, or satisfies *the Collet-Eckmann condition*, if

$$\liminf_{i\to\infty} \frac{\log |Df^i(c_1)|}{i} > 0$$

or equivalently, if there exist $\kappa > 0$ and $\lambda > 1$ such that

$$|Df^n(c_1)| > \kappa\lambda^n$$

for all $n \geq 1$. The metric properties of such maps have been intensively studied [30, 32]. For example it is known that they have invariant measures absolutely continuous with respect to Lebesgue measure [11, 45] and positive Lyapunov exponent Lebesgue almost everywhere [29]. In this chapter we consider the Collet-Eckmann condition from a topological point of view

In the first section, "Topological Invariance of the Collet-Eckmann Condition", we show that the Collet-Eckmann condition is topologically invariant for maps in $\mathcal{C}_2$. We do this by proving the quasi-symmetric invariance of the Collet-Eckmann condition and then applying a new result of Lyubich.

In "Collet-Eckmann Kneading Invariants" we describe a large class of kneading invariants for which any map in $\mathcal{C}$ must be Collet-Eckmann. These kneading invariants satisfy conditions analogous to the metric conditions of Benedicks and Carleson [4, 5, 14].

In "Non-Collet-Eckmann Kneading Invariants", the final section, we describe some kneading invariants for which no map in $\mathcal{C}$ is Collet-Eckmann. This falls short of classifying every kneading invariant as Collet-Eckmann or non-Collet-Eckmann.

4.1 Topological Invariance of the Collet-Eckmann Condition

In this section we show that the Collet-Eckmann condition is topologically invariant in $\mathcal{C}_2$.[1]

Theorem 29 *If $f \in \mathcal{C}_2$ is Collet-Eckmann and topologically conjugate to $g \in \mathcal{C}_2$ then g is Collet-Eckmann.*

Theorem 29 is a consequence of the following two results. The first result, that conjugating homeomorphisms are in general quasi-symmetric, was communicated to me by Mikhail Lyubich; the proof has not yet been circulated.[2] The second result, that the Collet-Eckmann condition is invariant under quasi-symmetric conjugacy, is proved in this section.

We say that a homeomorphism $h : [0;1] \to [0;1]$ is M-quasi-symmetric [3, 33] if

$$\frac{1}{M} \leq \frac{|h(x+\epsilon) - h(x)|}{|h(x-\epsilon) - h(x)|} \leq M$$

for any x in $[0;1]$ and $\epsilon > 0$ for which $x + \epsilon$ and $x - \epsilon$ are both in $[0;1]$. It is quasi-symmetric if it is M-quasi-symmetric for some $M \geq 1$.

Theorem 30 (Lyubich) *If $f \in \mathcal{C}_2$ is finitely renormalizable, without periodic attractors and topologically conjugate to $g \in \mathcal{C}_2$ then the conjugating homeomorphism is quasi-symmetric.*

Theorem 30 is a personal communication from Lyubich.

Theorem 31 *If $f \in \mathcal{C}$ is Collet-Eckmann and topologically conjugate to $g \in \mathcal{C}$ and if the conjugating homeomorphism is quasi-symmetric in a neighbourhood of c_1 then g is Collet-Eckmann.*

Theorem 31 is proved below.

[1] Recall that $\mathcal{C}_2$ is the subclass of $\mathcal{C}$ for which critical points are of order 2.

[2] Results along these lines by Jakobson and Swiątek [27] are available in preprint form.

To prove theorem 29 from theorems 30 and 31, suppose $f \in \mathcal{C}_2$ is Collet-Eckmann and topologically conjugate to $g \in \mathcal{C}_2$. Since f is Collet-Eckmann it has no periodic attractors and is finitely renormalizable [43, 14]. From theorem 30 the homeomorphism conjugating f to g is quasi-symmetric. Theorem 31 shows that g is therefore Collet-Eckmann, completing the proof of theorem 29.

In the remainder of this section we prove theorem 31. The proof uses the following lemma, a useful characterization of the Collet-Eckmann condition for maps in $\mathcal{C}$. This result is also used in the next section.

Lemma 32 *A map f in $\mathcal{C}$ satisfies the Collet-Eckmann condition if and only if it has no periodic attractors and there exist $\kappa > 0$ and $\lambda > 1$ such that*

$$\frac{|c_{\mathcal{S}_i+1} - c_1|}{|\alpha_{\mathcal{S}_i+1} - c_1|} > \kappa\lambda^{\mathcal{S}_i} \tag{4.1}$$

and

$$\frac{|c_{\mathcal{T}_i+1} - c_1|}{|\beta_{\mathcal{T}_i+1} - c_1|} > \kappa\lambda^{\mathcal{T}_i} \tag{4.2}$$

for all $i \geq 1$.

Proof. The 'only if' part of the present lemma has been proved by Nowicki [45, 46].

So suppose f has no periodic attractors and that there exist $\kappa > 0$ and $\lambda > 1$ such that equations 4.1 and 4.2 hold for all $i \geq 1$. Define

$$\rho_i = \min\{\frac{|a_{i+1} - c_{i+1}|}{|\alpha_{i+1} - c_1|}, \frac{|b_{i+1} - c_{i+1}|}{|\beta_{i+1} - c_1|}\}.$$

Note that $[a_{i+1}; c_{i+1}] = f^i[\alpha_{i+1}; c_1]$ and $[b_{i+1}; c_{i+1}] = f^i[\beta_{i+1}; c_1]$.

It follows from lemma 4 that f is Collet-Eckmann if there exist $\kappa > 0$ and $\lambda > 1$ such that $\rho_i > \kappa\lambda^i$ for all $i \geq 1$. We will prove $\rho_i > \kappa_1\lambda_1^i$ for all $i \geq 1$ inductively using the values of κ_1 and λ_1 chosen below.

We choose $\kappa_1 > 0$ and $\lambda_1 > 1$ as follows: there is some $N \geq 1$ for which $\kappa\lambda^N > 1$. Take $\lambda_1 > 1$ such that ${\lambda_1}^N < \kappa\lambda^N$. Since $\mathcal{S}\langle i\rangle \to \infty$ as $i \to \infty$ and likewise for $\mathcal{T}\langle i\rangle$, we can choose N_1 large enough that $\mathcal{S}\langle N_1\rangle \geq N$ and $\mathcal{T}\langle N_1\rangle \geq N$. Take $\kappa_1 = \min\{1, \min_{1\leq j\leq N_1} \rho_j\}$. This gives $\rho_i \geq \kappa_1{\lambda_1}^i$ automatically for $1 \leq i \leq N_1$.

Take $n > N_1$ and suppose $\rho_i \geq \kappa_1{\lambda_1}^i$ for all $i < n$. Let us show $\rho_n \geq \kappa_1\lambda_1^n$. First we show

$$\frac{|a_{n+1} - c_{n+1}|}{|\alpha_{n+1} - c_1|} \geq \kappa_1\lambda_1^n.$$

Rewrite $|a_{n+1} - c_{n+1}|/|\alpha_{n+1} - c_1|$ as

$$\frac{|a_{n+1} - c_{n+1}|}{|\alpha_{n+1} - c_1|} = \frac{|c_{\mathcal{S}\langle n+1\rangle+1} - c_1|}{|\alpha_{\mathcal{S}\langle n+1\rangle+1} - c_1|}\frac{|a_{n+1} - c_{n+1}|}{|c_{\mathcal{S}\langle n+1\rangle+1} - c_1|}, \tag{4.3}$$

using $\alpha_{n+1} = \alpha_{\mathcal{S}\langle n+1\rangle+1}$.

The factor $|c_{\mathcal{S}\langle n+1\rangle+1} - c_1|/|\alpha_{\mathcal{S}\langle n+1\rangle+1} - c_1|$ can be estimated directly from equation 4.1:

$$\frac{|c_{\mathcal{S}\langle n+1\rangle+1} - c_1|}{|\alpha_{\mathcal{S}\langle n+1\rangle+1} - c_1|} > \kappa\lambda^{\mathcal{S}\langle n+1\rangle}.$$

The other factor is $|a_{n+1} - c_{n+1}|/|c_{\mathcal{S}\langle n+1\rangle+1} - c_1|$.

If $n = \mathcal{S}\langle n+1\rangle$ then $|a_{n+1} - c_{n+1}|/|c_{\mathcal{S}\langle n+1\rangle+1} - c_1| = 1 \geq \kappa_1$.

When $n > \mathcal{S}\langle n+1\rangle$ we show $|a_{n+1} - c_{n+1}|/|c_{\mathcal{S}\langle n+1\rangle+1} - c_1| \geq \kappa_1\lambda_1^{n-\mathcal{S}\langle n+1\rangle}$ using $\rho_{n-\mathcal{S}\langle n+1\rangle} \geq \kappa_1\lambda_1^{n-\mathcal{S}\langle n+1\rangle}$: since $f^{n-\mathcal{S}\langle n+1\rangle}\big|_{[c_{\mathcal{S}\langle n+1\rangle+1};c_1]} = f^{n-\mathcal{S}\langle n+1\rangle}\big|_{f^{\mathcal{S}\langle n+1\rangle}[\alpha_{n+1};c_1]}$ is a homeomorphism, by definition $H_{n-\mathcal{S}\langle n+1\rangle+1}$ contains $[c_{\mathcal{S}\langle n+1\rangle+1}; c_1]$. Applying theorem 4 to $f^{n-\mathcal{S}\langle n+1\rangle}$, with $[a;b] = H_{n-\mathcal{S}\langle n+1\rangle+1}$, $x = c_1$ and $y = c_{\mathcal{S}\langle n+1\rangle+1}$, we have

$$\begin{aligned}\frac{|a_{n+1} - c_{n+1}|}{|c_{\mathcal{S}\langle n+1\rangle+1} - c_1|} &= \frac{|x_{n-\mathcal{S}\langle n+1\rangle} - y_{n-\mathcal{S}\langle n+1\rangle}|}{|x-y|} \\ &\geq \rho_{n-\mathcal{S}\langle n+1\rangle} \geq \kappa_1{\lambda_1}^{n-\mathcal{S}\langle n+1\rangle}.\end{aligned}$$

Substituting back into equation 4.3,

$$\frac{|a_{n+1} - c_{n+1}|}{|\alpha_{n+1} - c_1|} \geq \kappa\lambda^{\mathcal{S}\langle n+1\rangle}\kappa_1{\lambda_1}^{n-\mathcal{S}\langle n+1\rangle} \geq \kappa_1{\lambda_1}^{n}, \tag{4.4}$$

where the last inequality uses the choice of λ_1.

The same methods give

$$\frac{|b_{n+1} - c_{n+1}|}{|\beta_{n+1} - c_1|} \geq \kappa_1{\lambda_1}^{n}. \tag{4.5}$$

Equations 4.4 and 4.5 together show $\rho_n \geq \kappa_1{\lambda_1}^n$.

Therefore by induction $\rho_i \geq \kappa_1\lambda_1^i$ for all $i \geq 1$ and so f is Collet-Eckmann. ■

The following is a standard result [33]:

Lemma 33 *For each $M \geq 1$ there exists $K > 0$ and $\eta > 0$ such that*

$$|h(x) - h(y)| \leq K|x - y|^{\eta}$$

for every x, $y \in [0;1]$ and M-quasi-symmetric homeomorphism $h : [0;1] \to [0;1]$.

So let us prove theorem 31:

Theorem 31 *If $f \in \mathcal{C}$ is Collet-Eckmann and topologically conjugate to $g \in \mathcal{C}$ and if the conjugating homeomorphism is quasi-symmetric in a neighbourhood of c_1 then g is Collet-Eckmann.*

Proof. Suppose f is Collet-Eckmann, so neither f nor g has a periodic attractor. Lemma 32 shows that there exist $\kappa_1 > 0$ and $\lambda_1 > 1$ such that

$$\frac{|c_{\mathcal{S}_i+1}(f) - c_1(f)|}{|\alpha_{\mathcal{S}_i+1}(f) - c_1(f)|} > \kappa_1 \lambda_1^{\mathcal{S}_i} \tag{4.6}$$

and

$$\frac{|c_{\mathcal{T}_i+1}(f) - c_1(f)|}{|\beta_{\mathcal{T}_i+1}(f) - c_1(f)|} > \kappa_1 \lambda_1^{\mathcal{T}_i} \tag{4.7}$$

for all $i \geq 1$. Note that equations 4.6 and 4.7 show that $|\alpha_{\mathcal{S}_i+1}(f) - c_1(f)| \ll |c_{\mathcal{S}_i+1}(f) - c_1(f)|$ and $|\beta_{\mathcal{T}_i+1}(f) - c_1(f)| \ll |c_{\mathcal{T}_i+1}(f) - c_1(f)|$ if i is large.

Lemma 32 shows that g is Collet-Eckmann if there exist $\kappa_2 > 0$ and $\lambda_2 > 0$ such that

$$\frac{|c_{\mathcal{S}_i+1}(g) - c_1(g)|}{|\alpha_{\mathcal{S}_i+1}(g) - c_1(g)|} > \kappa_2 \lambda_2^{\mathcal{S}_i} \tag{4.8}$$

and

$$\frac{|c_{\mathcal{T}_i+1}(g) - c_1(g)|}{|\beta_{\mathcal{T}_i+1}(g) - c_1(g)|} > \kappa_2 \lambda_2^{\mathcal{T}_i} \tag{4.9}$$

for all $i \geq 1$.

So let us show the existence of $\kappa_2 > 0$ and $\lambda_2 > 1$ such that equations 4.8 and 4.9 hold for all $i \geq 1$. We first rescale equations 4.6 and 4.8 using

$$\begin{aligned} \pi_i : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \frac{x - c_{\mathcal{S}_i+1}(f)}{|c_{\mathcal{S}_i+1}(f) - c_1(f)|} \end{aligned}$$

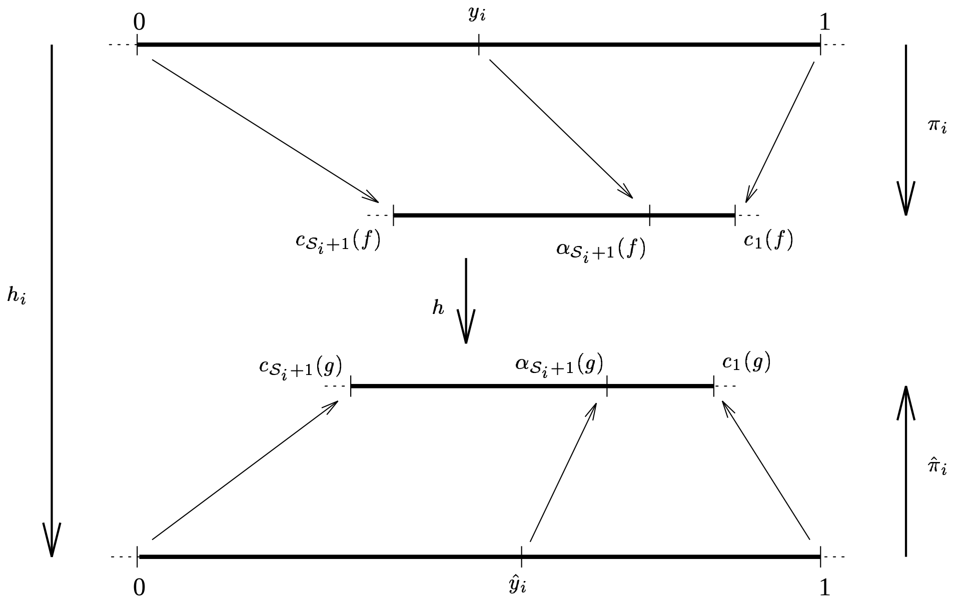


Figure 4.1: Rescaling the conjugating homeomorphism.

and

$$\begin{aligned}\hat{\pi}_i : \mathbb{R} &\to \mathbb{R}\\ x &\mapsto \frac{x - c_{\mathcal{S}_i+1}(g)}{|c_{\mathcal{S}_i+1}(g) - c_1(g)|}.\end{aligned}$$

Define $h_i = \hat{\pi}_i \circ h \circ \pi_i^{-1}$, $y_i = \pi_i(\alpha_{\mathcal{S}_i+1}(f))$ and $\hat{y}_i = \hat{\pi}_i(\alpha_{\mathcal{S}_i+1}(g))$. We have $h_i(y_i) = \hat{y}_i$, $h_i(1) = 1$ and $h_i : [0;1] \to [0;1]$ a homeomorphism for all $i \geq 1$ (see figure 4.1). In this notation equation 4.6 is

$$|y_i - 1| < \lambda_1^{-\mathcal{S}_i}/\kappa_1 \tag{4.10}$$

for all $i \geq 1$.

To show the existence of $\kappa_2 > 0$ and $\lambda_2 > 1$ such that equation 4.8 holds for all $i \geq 1$, in this notation

$$|\hat{y}_i - 1| < \lambda_2^{-\mathcal{S}_i}/\kappa_2 \tag{4.11}$$

for all $i \geq 1$, take $M \geq 1$ such that h is M-quasi-symmetric in a neighbourhood of c_1 and note that h_i is M-quasi-symmetric since it is a linear rescaling of h. From lemma 33 there exist $K > 0$ and $\eta > 0$ depending only on M such that

$$|h_i(x) - h_i(y)| \leq K|x - y|^\eta$$

for every $x, y \in [0;1]$ and $i \geq 1$. In particular, $|\hat{y}_i - 1| \leq K|y_i - 1|^\eta$ for all $i \geq 1$. Therefore $|y_i - 1| < \lambda^{-\mathcal{S}_i}/\kappa_1$, equation 4.10, implies $|\hat{y}_i - 1| < K\lambda_1^{-\eta\mathcal{S}_i}/\kappa_1^\eta$, and this holds for all $i \geq 1$.[3]

Taking $\kappa_2 = K/\kappa_2^\eta$ and $\lambda_2 = \lambda_1^\eta$, we therefore have equation 4.11 and thus equation 4.8 for all $i \geq 1$.

A similar argument shows that equation 4.7 implies equation 4.9, again with $\lambda_2 = \lambda_1^\eta$. The rescaling needs to be modified since $c_{\mathcal{T}_n}$ and $\beta_{\mathcal{T}_n}$ are on opposite sides of c_1. This is not difficult, though it does result in a slightly different value of κ_2.

Therefore g is Collet-Eckmann since equations 4.8 and 4.9 hold for all $i \geq 1$. ■

4.2 Collet-Eckmann Kneading Invariants

We call a kneading invariant *Collet-Eckmann* if every f in $\mathcal{C}$ with this kneading invariant is Collet-Eckmann. In this section we give simple conditions for a kneading invariant to be Collet-Eckmann.

Before stating the main result of this section we give some definitions. Unless stated otherwise, Δ is a fixed positive integer. To simplify notation, dependence on Δ is only indicated where necessary.

Given a kneading invariant we define its *Δ-return times* $\nu_1, \nu_2, \ldots$ as follows:

$$\nu_1 = \min\{j \geq 1 \mid \mathcal{R}(j) > \Delta\}$$

and

$$\nu_{i+1} = \min\{j \geq \nu_i + p_i \mid \mathcal{R}(j) > \Delta\},$$

where $p_i \equiv \mathcal{R}^-(\nu_i)$ is called the *binding period* of ν_i. Lemma 43 below shows that $\nu_1, \nu_2, \ldots$ are just those return times $\mathcal{M}_i$ with $\mathcal{R}(\mathcal{M}_i) > \Delta$. There can be either a finite or infinite number of Δ-return times. We will assume for simplicity that there are infinitely many Δ-return times. The results hold in either case.

[3]There are two minor problems which may occur. First, y_i may not be in $[0;1]$. This occurs when $|c_{\mathcal{S}_i+1}(f) - c_1(f)|$ is less than $|\alpha_{\mathcal{S}_i+1}(f) - c_1(f)|$. This can only happen for a finite number of i so can be ignored.

Second, h_i may not be quasi-symmetric on the whole of $[0;1]$. This occurs when $c_{\mathcal{S}_i+1}(f)$ is outside the domain of quasi-symmetry of h. A variation of the argument easily deals with this case.

We say that an iterate n is *bound* if $\nu_i < n \leq \nu_i + p_i$ for some $i \geq 1$. Otherwise n is *free*. We use $B(n)$ to denote the number of bound iterates occurring before time n: if $n > \nu_1$ then

$$B(n) = \begin{cases} \sum_{j=1}^{i} p_j & \text{if } n \geq \nu_i + p_i \\ n - \nu_i + \sum_{j=1}^{i-1} p_j & \text{if } \nu_i < n < \nu_i + p_i \end{cases}$$

where $\nu_i = \max\{\nu_j \mid \nu_j < n\}$. If $n \leq \nu_1$ then $B(n) = 0$.

Similar definitions were used by Benedicks and Carleson [4, 5, 14] in their proof that Q_a is Collet-Eckmann for a positive Lebesgue measure set of a. However they define return times and binding periods using metric rather than topological criteria.

Benedicks and Carleson's metric conditions (BA) and (FA) for Q_a to be Collet-Eckmann are analogous to our topological conditions, equations 4.12 and 4.13 respectively in theorem 34 below.[4]

While there are maps satisfying our conditions that do not satisfy Benedicks and Carleson's and vice versa,[5] it follows from work of Tsujii [49] that Lebesgue almost every quadratic map close to a Misiurewicz map satisfying Benedicks and Carleson's (BA) and (FA) satisfies our equations 4.12 and 4.13.

The main result of this section is

Theorem 34 *Any kneading invariant satisfying*

$$\limsup_{i\to\infty} \frac{\mathcal{R}(i)}{i} = 0 \tag{4.12}$$

and

$$\limsup_{i\to\infty} \frac{B(i)}{i} < 1 \tag{4.13}$$

for some $\Delta \geq 1$ *is Collet-Eckmann.*

Before giving the proof of theorem 34 we state some corollaries.

We say that f is *Misiurewicz* if it has no periodic attractors and the critical point is not recurrent. The following result was originally proved by Misiurewicz [40, 48]:

[4]Benedicks and Carleson also require Q_a to be close to a Misiurewicz map. This was an artifact of their method and has been eliminated in theorem 34.

[5]Equation 4.12 is stronger than (BA) because it implies $\limsup_{i\to\infty} -\log|c_i - c|/i = 0$. This follows from lemma 39.

Corollary 35 *Every Misiurewicz map in $\mathcal{C}$ is Collet-Eckmann.*

Proof. We claim that $f \in \mathcal{C}$ is Misiurewicz if and only if the function $\mathcal{R}$ is bounded. Indeed, f has no periodic attractors if and only if its kneading invariant is not periodic, which is equivalent to $\mathcal{R}(n)$ always being finite. Moreover, since c is not recurrent, the orbit of c_1 always stays a definite distance away from c. This implies that $R(c_n)$ is bounded independently of n, and so $\mathcal{R}$ is bounded (see chapter 2).

If Δ is larger than the bound on $\mathcal{R}$ then there are no Δ-return times, so $B(i) = 0$ for all i. Clearly $\limsup_{i\to\infty} \mathcal{R}(i)/i = 0$.

Since the hypotheses of theorem 34 are satisfied, f is Collet-Eckmann. ■

We say that a kneading invariant is *slowly recurrent* if

$$\lim_{l\to\infty} \limsup_{i\to\infty} \frac{\sum_{j=1}^{i} \begin{cases} \mathcal{R}(j) & \text{if } \mathcal{R}(j) \geq l \\ 0 & \text{otherwise} \end{cases}}{i} = 0.$$ [6]

In chapter 5 we show that, in a sense, almost every kneading invariant is slowly recurrent.

Corollary 36 *Every slowly recurrent kneading invariant is Collet-Eckmann.*

Proof. Clearly $\limsup_{i\to\infty} \mathcal{R}(i)/i = 0$. In addition $\limsup_{i\to\infty} B(i)/i \to 0$ as $\Delta \to \infty$, from the definition and the definition of slow recurrence. Therefore the hypotheses of theorem 34 are satisfied if Δ is sufficiently large. ■

There are two main steps in the proof of theorem 34.

First we prove theorem 37 below. We say that n is close to i if $i < n \leq i + \mathcal{R}(i)$ and $\mathcal{R}(i) \leq \Delta$. Denote the cardinality of A by $\#(A)$.

[6] This definition was inspired by Tsujii's metric notion of *weak regularity* [49]. It can be shown that every f in $\mathcal{C}$ with a slowly recurrent kneading invariant is weakly regular. The converse does not seem to hold.

Theorem 37 *If f is in $\mathcal{C}$ and satisfies*

$$\limsup_{i\to\infty} \frac{\mathcal{R}(i)}{i} = 0,$$

$$\liminf_{i\to\infty} \frac{\#(\{1 < \mathcal{S}_j \le i \mid \mathcal{S}_j \textit{ is close to } \mathcal{S}\langle\mathcal{S}_j\rangle \textit{ and } \mathcal{T}\langle\mathcal{S}_j\rangle\})}{i} > 0$$

and

$$\liminf_{i\to\infty} \frac{\#(\{\mathcal{T}_1 < \mathcal{T}_j \le i \mid \mathcal{T}_j \textit{ is close to } \mathcal{S}\langle\mathcal{T}_j\rangle \textit{ and } \mathcal{T}\langle\mathcal{T}_j\rangle\})}{i} > 0$$

for some $\Delta \ge 1$ then f is Collet-Eckmann.

Theorem 37 is proved below in "Conditions for being Collet-Eckmann".

Next we show that the hypotheses of theorem 34 imply those of theorem 37, and therefore that any f satisfying the hypotheses of theorem 34 is Collet-Eckmann. In fact we prove that the hypotheses of theorems 34 and 37 are equivalent:

Theorem 38 *If f satisfies $\limsup_{i\to\infty} \mathcal{R}(i)/i = 0$ then*

$$\limsup_{i\to\infty} \frac{B(i)}{i} < 1$$

for some $\Delta \ge 1$ if and only if

$$\liminf_{i\to\infty} \frac{\#(\{1 < \mathcal{S}_j \le i \mid \mathcal{S}_j \textit{ is close to } \mathcal{S}\langle\mathcal{S}_j\rangle \textit{ and } \mathcal{T}\langle\mathcal{S}_j\rangle\})}{i} > 0$$

and

$$\liminf_{i\to\infty} \frac{\#(\{\mathcal{T}_1 < \mathcal{T}_j \le i \mid \mathcal{T}_j \textit{ is close to } \mathcal{S}\langle\mathcal{T}_j\rangle \textit{ and } \mathcal{T}\langle\mathcal{T}_j\rangle\})}{i} > 0$$

for some $\Delta \ge 1$.

Theorem 34 therefore follows from theorems 37 and 38. We prove theorem 38 in "Equivalent Conditions" below.

4.2.1 Conditions for being Collet-Eckmann

Here we prove theorem 37:

Theorem 37 *If f is in $\mathcal{C}$ and satisfies*

$$\limsup_{i\to\infty} \frac{\mathcal{R}(i)}{i} = 0,$$

$$\liminf_{i\to\infty} \frac{\#(\{1 < \mathcal{S}_j \leq i \mid \mathcal{S}_j \textit{ is close to } \mathcal{S}\langle\mathcal{S}_j\rangle \textit{ and } \mathcal{T}\langle\mathcal{S}_j\rangle\})}{i} > 0$$

and

$$\liminf_{i\to\infty} \frac{\#(\{\mathcal{T}_1 < \mathcal{T}_j \leq i \mid \mathcal{T}_j \textit{ is close to } \mathcal{S}\langle\mathcal{T}_j\rangle \textit{ and } \mathcal{T}\langle\mathcal{T}_j\rangle\})}{i} > 0$$

for some $\Delta \geq 1$ then f is Collet-Eckmann.

We need the following technical result:

Lemma 39 *If f in $\mathcal{C}$ has no periodic attractors then there exists $K > 0$ such that*

$$-\log|c_n - c| \leq K\mathcal{R}(n)$$

for every $n \geq 1$.

Proof. Let us show there exists a $K_1 > 1$ such that

$$|c_{-\mathcal{S}_i} - c| \geq K_1^{-\mathcal{S}_i}$$

for every $i \geq 1$.

First note that $f^{\mathcal{S}_i}[c_{-\mathcal{S}_i}; c]$ cannot be significantly smaller than $[c_{-\mathcal{S}_i}; c]$ itself. After all, $f^{\mathcal{S}_i}$ maps $[c_{-\mathcal{S}_i}; c]$ diffeomorphically to $[c; c_{\mathcal{S}_i}]$. Since $\mathcal{R}(\mathcal{S}_i) \leq \mathcal{S}_i$ (lemma 15), lemma 12 shows that $[c; c_{\mathcal{S}_i}]$ contains either $[c_{-\mathcal{S}_i}; c]$ or $[c; \tau(c_{-\mathcal{S}_i})]$. Take $K_2 > 1$ such that $|\tau(c_{-\mathcal{S}_i}) - c| \geq |c_{-\mathcal{S}_i} - c|/K_2$. Such a $K_2 > 1$ exists because τ is Lipschitz, from lemma 6. We thus have

$$|f^{\mathcal{S}_i}[c_{-\mathcal{S}_i}; c]| \geq \frac{|c_{-\mathcal{S}_i} - c|}{K_2}. \tag{4.14}$$

Now note that f shrinks $[c_{-\mathcal{S}_i}; c]$ substantially if $c_{-\mathcal{S}_i}$ is close to c. Indeed, from equation 1.1,

$$|f[c_{-\mathcal{S}_i}; c]| \leq \frac{L|c_{-\mathcal{S}_i} - c|^l}{l} \tag{4.15}$$

where $L > 1$ and $l > 1$ measure the non-flatness of the critical point.

Finally, $f^{\mathcal{S}_i - 1}$ can only expand $f[c_{-\mathcal{S}_i}; c]$ by a limited amount:

$$|f^{\mathcal{S}_i}[c_{-\mathcal{S}_i}; c]| \leq K_3^{\mathcal{S}_i - 1}|f[c_{-\mathcal{S}_i}; c]| \tag{4.16}$$

where K_3 is the maximum size of the derivative of f.

Combining equations 4.14, 4.15 and 4.16,

$$\frac{|c_{-\mathcal{S}_i} - c|}{K_2} \leq |f^{\mathcal{S}_i}[c_{-\mathcal{S}_i}; c]| \leq K_3^{\mathcal{S}_i - 1}|f[c_{-\mathcal{S}_i}; c]| \leq \frac{L|c_{-\mathcal{S}_i} - c|^l}{l} K_3^{\mathcal{S}_i - 1}.$$

Rearranging gives $|c_{-\mathcal{S}_i} - c| \geq K_1^{-\mathcal{S}_i}$, if we absorb the constants into one.

So let us show the result follows. Take any $n \geq 1$. We know that $\mathcal{R}(n) = \mathcal{S}_i$ for some $i \geq 1$ and from lemma 12 we have $c_n \notin (c_{-\mathcal{S}_i}; \tau(c_{-\mathcal{S}_i}))$. Since $|\tau(c_{-\mathcal{S}_i}) - c| \geq |c_{-\mathcal{S}_i} - c|/K_2$ we have $|c_n - c| \geq |c_{-\mathcal{S}_i} - c|/K_2$. Therefore

$$|c_n - c| \geq K_1^{-\mathcal{R}(n)}/K_2.$$

Rearranging and taking $K \geq 2 \log K_1$ gives $-\log |c_n - c| \leq K\mathcal{R}(n)$. ■

There are two steps in the proof of theorem 37.

First we characterize the Collet-Eckmann condition when $\limsup_{i \to \infty} \mathcal{R}(i)/i = 0$ using the following variation of lemma 32:

Lemma 40 *If f is in $\mathcal{C}$ and $\limsup_{i\to\infty} \mathcal{R}(i)/i = 0$ then f is Collet-Eckmann if and only if there exist $K > 0$ and $\lambda > 1$ such that*

$$|\alpha_{\mathcal{S}_i + 1} - c_1| < K\lambda^{-\mathcal{S}_i}$$

and

$$|\beta_{\mathcal{T}_i + 1} - c_1| < K\lambda^{-\mathcal{T}_i}$$

for all $i \geq 1$.

Proof. First note that if $\limsup_{i\to\infty} \mathcal{R}(i)/i = 0$ then f has no periodic attractors: in lemma 8 we showed that if f has a periodic attractor then its kneading invariant is periodic. But a periodic kneading invariant has $\limsup_{i\to\infty} \mathcal{R}(i)/i = \infty$.

Lemma 32 shows that f is Collet-Eckmann if and only if there exist $\kappa > 0$ and $\lambda > 1$ such that

$$\frac{|c_{\mathcal{S}_i+1} - c_1|}{|\alpha_{\mathcal{S}_i+1} - c_1|} > \kappa\lambda^{\mathcal{S}_i}$$

and

$$\frac{|c_{\mathcal{T}_i+1} - c_1|}{|\beta_{\mathcal{T}_i+1} - c_1|} > \kappa\lambda^{\mathcal{T}_i}$$

for all $i \geq 1$.

In other words, f is Collet-Eckmann if and only if

$$\liminf_{i\to\infty} \frac{\log|c_{\mathcal{S}_i+1} - c_1| - \log|\alpha_{\mathcal{S}_i+1} - c_1|}{\mathcal{S}_i} > 0$$

and

$$\liminf_{i\to\infty} \frac{\log|c_{\mathcal{T}_i+1} - c_1| - \log|\beta_{\mathcal{T}_i+1} - c_1|}{\mathcal{T}_i} > 0.$$

Let us show

$$\liminf_{i\to\infty} \frac{\log|c_{\mathcal{S}_i+1} - c_1|}{\mathcal{S}_i} = 0.$$

Indeed, lemma 39 gives a $K > 0$ such that $-\log|c_n - c| \leq K\mathcal{R}(n)$ for all $n \geq 1$ and in particular $-\log|c_{\mathcal{S}_i} - c| \leq K\mathcal{R}(\mathcal{S}_i)$. Therefore

$$-K\frac{\mathcal{R}(\mathcal{S}_i)}{\mathcal{S}_i} \leq \frac{\log|c_{\mathcal{S}_i} - c|}{\mathcal{S}_i} < 0$$

for all $i \geq 1$. Since $\limsup_{i\to\infty} \mathcal{R}(\mathcal{S}_i)/\mathcal{S}_i = 0$ we have $\liminf_{i\to\infty} \log|c_{\mathcal{S}_i} - c|/\mathcal{S}_i = 0$. It follows that $\liminf_{i\to\infty} \log|c_{\mathcal{S}_i+1} - c_1|/\mathcal{S}_i = 0$.

A similar argument shows

$$\liminf_{i\to\infty} \frac{\log|c_{\mathcal{T}_i+1} - c_1|}{\mathcal{T}_i} = 0.$$

Therefore f is Collet-Eckmann if and only if

$$\limsup_{i\to\infty} \frac{\log|\alpha_{\mathcal{S}_i+1} - c_1|}{\mathcal{S}_i} < 0$$

and

$$\limsup_{i\to\infty} \frac{\log|\beta_{\mathcal{T}_i+1} - c_1|}{\mathcal{T}_i} < 0$$

for all $i \geq 1$. In other words, if there exist $K > 0$ and $\lambda > 1$ such that

$$|\alpha_{\mathcal{S}_i+1} - c_1| < K\lambda^{-\mathcal{S}_i}$$

and

$$|\beta_{\mathcal{T}_i+1} - c_1| < K\lambda^{-\mathcal{T}_i}$$

for all $i \geq 1$. ■

Now we prove theorem 41 below. Lemma 40 shows that theorem 37 follows from theorem 41.

Theorem 41 *If $f \in \mathcal{C}$ satisfies*

$$\liminf_{i\to\infty} \frac{\#(\{1 < \mathcal{S}_j \le i \mid \mathcal{S}_j \text{ is close to } \mathcal{S}\langle\mathcal{S}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{S}_j\rangle\})}{i} > 0$$

and

$$\liminf_{i\to\infty} \frac{\#(\{\mathcal{T}_1 < \mathcal{T}_j \le i \mid \mathcal{T}_j \text{ is close to } \mathcal{S}\langle\mathcal{T}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{T}_j\rangle\}}{i} > 0$$

for some $\Delta \ge 1$ then there exist $K > 0$ and $\lambda > 1$ such that

$$|\alpha_i - c_1| < K\lambda^{-i}$$

and

$$|\beta_i - c_1| < K\lambda^{-i}$$

for all $i \ge 1$.

Most of the work in proving theorem 41 is done in the following lemma. Recall that $|\alpha_{i+1} - c_1| < |\alpha_i - c_1|$ if and only if i is a cutting time and $|\beta_{i+1} - c_1| < |\beta_i - c_1|$ if and only if i is a co-cutting time.

Lemma 42 *For each $\Delta \ge 1$ there exists a $0 < \rho < 1$ such that*

$$\frac{|\alpha_{\mathcal{S}_i+1} - c_1|}{|\alpha_{\mathcal{S}_i} - c_1|} < \rho$$

whenever $\mathcal{S}_i$ is close to $\mathcal{S}\langle\mathcal{S}_i\rangle$ and $\mathcal{T}\langle\mathcal{S}_i\rangle$ and

$$\frac{|\beta_{\mathcal{T}_i+1} - c_1|}{|\beta_{\mathcal{T}_i} - c_1|} < \rho$$

whenever $\mathcal{T}_i$ is close to $\mathcal{S}\langle\mathcal{T}_i\rangle$ and $\mathcal{T}\langle\mathcal{T}_i\rangle$.

Proof. Suppose $\mathcal{S}_i$ is close to $\mathcal{S}\langle\mathcal{S}_i\rangle$ and $\mathcal{T}\langle\mathcal{S}_i\rangle$. First we will prove that there exists an $\epsilon > 0$ such that $|c_{\mathcal{S}_i - \mathcal{S}\langle\mathcal{S}_i\rangle} - c| > \epsilon$ and $|c_{\mathcal{S}_i - \mathcal{T}\langle\mathcal{S}_i\rangle} - c_{\mathcal{S}_i}| > \epsilon$ (see figure 4.2). Recall that $|c_{\mathcal{S}_i - \mathcal{S}\langle\mathcal{S}_i\rangle} - c|$ and $|c_{\mathcal{S}_i - \mathcal{T}\langle\mathcal{S}_i\rangle} - c_{\mathcal{S}_i}|$ are the endpoints of the $\mathcal{S}_i$'th tower level.

For the first inequality note that $\mathcal{S}_i - \mathcal{S}\langle\mathcal{S}_i\rangle \le \Delta$ from the definition of close. Therefore $c_{\mathcal{S}_i - \mathcal{S}\langle\mathcal{S}_i\rangle}$ is one of $c_1, c_2, \ldots c_\Delta$ and taking $\epsilon < \min\{|c_1 - c|, \ldots, |c_\Delta - c|\}$ shows $|c_{\mathcal{S}_i - \mathcal{S}\langle\mathcal{S}_i\rangle} - c| > \epsilon$.

Take $\alpha \in [\alpha_{\mathcal{S}_i}; c_1]$ to be the point mapping between $c_{\mathcal{S}_i - \mathcal{S}\langle\mathcal{S}_i\rangle}$ and c, and distance $\epsilon/2$ from c, under $f^{\mathcal{S}_i - 1}$, as in figure 4.2. The above inequalities show that $[c_{\mathcal{S}_i - \mathcal{S}\langle\mathcal{S}_i\rangle}; c_{\mathcal{S}_i - \mathcal{T}\langle\mathcal{S}_i\rangle}]$

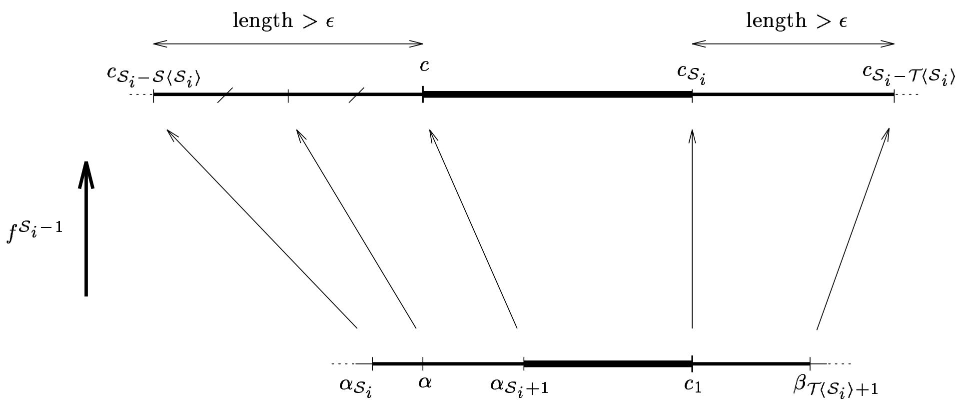


Figure 4.2: Cutting the $\mathcal{S}_i$'th tower level.

is a $\epsilon/2$-scaled neighbourhood of $[f^{\mathcal{S}_i-1}(\alpha); c_{\mathcal{S}_i}]$. Lemma 5 then shows that for every $x, y \in [\alpha; c_1]$ we have

$$\frac{1}{K_D(\epsilon/2)} \leq \frac{|Df^{\mathcal{S}_i-1}(x)|}{|Df^{\mathcal{S}_i-1}(y)|} \leq K_D(\epsilon/2) \tag{4.17}$$

where $K_D(\epsilon/2)$ was defined in lemma 5.

We can now estimate

$$\frac{|\alpha_{\mathcal{S}_i+1} - c_1|}{|\alpha_{\mathcal{S}_i} - c_1|}.$$

Take $x \in [\alpha; \alpha_{\mathcal{S}_i+1}]$ such that $|Df^{\mathcal{S}_i-1}(x)| = |f^{\mathcal{S}_i-1}(\alpha) - c|/|\alpha - \alpha_{\mathcal{S}_i+1}|$. Take $y \in [\alpha_{\mathcal{S}_i+1}; c_1]$ such that $|Df^{\mathcal{S}_i-1}(y)| = |c_{\mathcal{S}_i} - c|/|\alpha_{\mathcal{S}_i+1} - c_1|$. Equation 4.17 gives

$$\frac{\epsilon/2}{|\alpha - \alpha_{\mathcal{S}_i+1}|} = |Df^{\mathcal{S}_i-1}(x)| \leq K_D(\epsilon/2)|Df^{\mathcal{S}_i-1}(y)| = K_D(\epsilon/2)\frac{|c_{\mathcal{S}_i} - c|}{|\alpha_{\mathcal{S}_i+1} - c_1|}.$$

Replacing $|c_{\mathcal{S}_i} - c|$ by 1, its upper bound, and rearranging,

$$|\alpha - \alpha_{\mathcal{S}_i+1}| \geq \frac{\epsilon}{2K_D(\epsilon/2)}|\alpha_{\mathcal{S}_i+1} - c_1|.$$

Using $|\alpha_{\mathcal{S}_i} - c_1| > |\alpha - \alpha_{\mathcal{S}_i+1}| + |\alpha_{\mathcal{S}_i+1} - c_1|$, a simple calculation now gives

$$\frac{|\alpha_{\mathcal{S}_i+1} - c_1|}{|\alpha_{\mathcal{S}_i} - c_1|} < \frac{1}{1 + \frac{\epsilon}{2K_D(\epsilon/2)}}.$$

Taking $\rho = 1/(1+\epsilon/(2K_D(\epsilon/2))) < 1$ we therefore have

$$\frac{|\alpha_{\mathcal{S}_i+1} - c_1|}{|\alpha_{\mathcal{S}_i} - c_1|} < \rho.$$

A similar calculation shows

$$\frac{|\beta_{\mathcal{T}_i+1} - c_1|}{|\beta_{\mathcal{T}_i} - c_1|} < \rho$$

whenever $\mathcal{T}_i$ is close to $\mathcal{S}\langle\mathcal{T}_i\rangle$ and $\mathcal{T}\langle\mathcal{T}_i\rangle$. ■

So let us prove theorem 41.

The last lemma gives

$$|\alpha_i - c_1| \leq \rho^{\#(\{1<\mathcal{S}_j\leq i|\mathcal{S}_j \text{ is close to } \mathcal{S}\langle\mathcal{S}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{S}_j\rangle\})}$$

and

$$|\beta_i - c_1| \leq \rho^{\#(\{\mathcal{T}_1<\mathcal{T}_j\leq i|\mathcal{T}_j \text{ is close to } \mathcal{S}\langle\mathcal{T}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{T}_j\rangle\})}$$

for all $i \geq 1$.

Since

$$\liminf_{i\to\infty} \frac{\#(\{1 < \mathcal{S}_j \leq i \mid \mathcal{S}_j \text{ is close to } \mathcal{S}\langle\mathcal{S}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{S}_j\rangle\})}{i} > 0$$

and

$$\liminf_{i\to\infty} \frac{\#(\{\mathcal{T}_1 < \mathcal{T}_j \leq i \mid \mathcal{T}_j \text{ is close to } \mathcal{S}\langle\mathcal{T}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{T}_j\rangle\}}{i} > 0$$

this shows the existence of $K > 0$ and $\lambda > 1$ such that

$$|\alpha_i - c_1| < K\lambda^{-i}$$

and

$$|\beta_i - c_1| < K\lambda^{-i}$$

for all $i \geq 1$, proving theorem 41.

Theorem 34 now follows.

4.2.2 Equivalent Conditions

Here we prove theorem 38:

Theorem 38 *If f satisfies $\limsup_{i\to\infty} \mathcal{R}(i)/i = 0$ then*

$$\limsup_{i\to\infty} \frac{B(i)}{i} < 1$$

for some $\Delta \geq 1$ if and only if

$$\liminf_{i\to\infty} \frac{\#(\{1 < \mathcal{S}_j \leq i \mid \mathcal{S}_j \text{ is close to } \mathcal{S}\langle\mathcal{S}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{S}_j\rangle\})}{i} > 0$$

and

$$\liminf_{i\to\infty} \frac{\#(\{\mathcal{T}_1 < \mathcal{T}_j \leq i \mid \mathcal{T}_j \text{ is close to } \mathcal{S}\langle\mathcal{T}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{T}_j\rangle\})}{i} > 0$$

for some $\Delta \geq 1$.

First we show that Δ-return times are return times. The 'if' part of theorem 38 will then follow as a corollary.

Lemma 43 *The Δ-return times $\nu_1, \nu_2, \ldots$ are exactly those return times $\mathcal{M}_i$ with $\mathcal{R}(\mathcal{M}_i) > \Delta$.*

Proof. Let us show that every Δ-return time is a return time. Clearly $\nu_1 = \min\{j \geq 1 \mid \mathcal{R}(j) > \Delta\}$ is a closest return time, so a return time by lemma 20.

Suppose ν_i is a return time, so we can write $\nu_i = \mathcal{M}_j$ for some $j \geq 1$. Let us show ν_{i+1} is a return time. If ν_{i+1} is not a return time then $\mathcal{M}_k < \nu_{i+1} < \mathcal{M}_{k+1}$ for some $k \geq j$. In fact $k > j$ because $\nu_{i+1} \geq \nu_i + \mathcal{R}^-(\nu_i)$. This implies $\mathcal{R}(\mathcal{M}_k) \leq \Delta$; but lemma 19 gives $\mathcal{R}(\nu_{i+1}) < \mathcal{R}(\mathcal{M}_k) \leq \Delta$, contradicting $\mathcal{R}(\nu_{i+1}) > \Delta$. Therefore ν_{i+1} is a return time. By induction every Δ-return time is a return time.

The proof that every $\mathcal{M}_i$ with $\mathcal{R}(\mathcal{M}_i) > \Delta$ is a Δ-return time is left to the reader.
■

Corollary 44 *We have that $\limsup_{i\to\infty} B(i)/i$ is a non-increasing function of Δ.*

Proof. It follows from the last lemma that a free iterate cannot change into a bound iterate as Δ increases, though a bound iterate may become free. Therefore $B(i)$ is a non-increasing function of Δ for each fixed $i \geq 1$. The result follows. ■

Corollary 45 *If*

$$\liminf_{i\to\infty} \frac{\#(\{1 < \mathcal{S}_j \leq i \mid \mathcal{S}_j \textit{ is close to } \mathcal{S}\langle\mathcal{S}_j\rangle \textit{ and } \mathcal{T}\langle\mathcal{S}_j\rangle\})}{i} > 0$$

and

$$\liminf_{i\to\infty} \frac{\#(\{\mathcal{T}_1 < \mathcal{T}_j \leq i \mid \mathcal{T}_j \textit{ is close to } \mathcal{S}\langle\mathcal{T}_j\rangle \textit{ and } \mathcal{T}\langle\mathcal{T}_j\rangle\})}{i} > 0$$

then

$$\limsup_{i\to\infty} \frac{B(i)}{i} < 1.$$

Proof. Let us show that every $n > \mathcal{M}_1$ which is close to $\mathcal{S}\langle n\rangle$ and $\mathcal{T}\langle n\rangle$ is free. If $n \leq \mathcal{M}_1$ then it is free trivially because $\nu_1 \geq \mathcal{M}_1$.

So take some $n > \mathcal{M}_1$ which is close to $\mathcal{S}\langle n\rangle$ and $\mathcal{T}\langle n\rangle$ and write $\mathcal{M}_i < n \leq \mathcal{M}_{i+1}$ for some $i \geq 1$. Since either $\mathcal{S}\langle n\rangle = \mathcal{M}_i$ or $\mathcal{T}\langle n\rangle = \mathcal{M}_i$ we have $\mathcal{R}(\mathcal{M}_i) \leq \Delta$ from the definition of close. So $\mathcal{M}_i$ is not a Δ-return time and therefore n is free.

Thus

$$\begin{aligned} \frac{B(i)}{i} &= \frac{\#(\{1 \leq n < i \mid n \text{ is bound}\})}{i} \\ &\leq 1 - \frac{\#(\{1 < \mathcal{S}_j \leq i \mid \mathcal{S}_j \text{ is close to } \mathcal{S}\langle\mathcal{S}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{S}_j\rangle\})}{i}. \end{aligned}$$

Taking limits, the result follows. ■

So let us prove the 'only if' part of theorem 38.

Lemma 46 *If f satisfies*

$$\limsup_{i\to\infty} \frac{\mathcal{R}(i)}{i} = 0$$

and

$$\limsup_{i\to\infty} \frac{B(i)}{i} < 1$$

for some $\Delta \geq 1$ *then*

$$\liminf_{i\to\infty} \frac{\#(\{1 < \mathcal{S}_j \leq i \mid \mathcal{S}_j \textit{ is close to } \mathcal{S}\langle\mathcal{S}_j\rangle \textit{ and } \mathcal{T}\langle\mathcal{S}_j\rangle\})}{i} > 0$$

and

$$\liminf_{i\to\infty} \frac{\#(\{\mathcal{T}_1 < \mathcal{T}_j \leq i \mid \mathcal{T}_j \textit{ is close to } \mathcal{S}\langle\mathcal{T}_j\rangle \textit{ and } \mathcal{T}\langle\mathcal{T}_j\rangle\})}{i} > 0$$

for some $\Delta \geq 1$.[7]

First we give some definitions.

We call n *strongly free* if $n \leq \nu_1$ or $\nu_j + \mathcal{R}(\nu_j) + \Delta < n \leq \nu_{j+1}$ for some $j \geq 1$.

Put $C(n) = \#(\{1 \leq i \leq n \mid i \text{ is strongly free}\})$ and $\hat{C}(n) = \#(\{\nu_1 < i \leq n \mid i \text{ is strongly free}\})$, so $C(n) = \hat{C}(n) + \min\{n, \nu_1\}$.

These definitions are motivated by the following result:

Lemma 47 *We have*

$$\#(\{1 < \mathcal{S}_j \leq n \mid \mathcal{S}_j \textit{ is close to } \mathcal{S}\langle\mathcal{S}_j\rangle \textit{ and } \mathcal{T}\langle\mathcal{S}_j\rangle\}) \geq \frac{\hat{C}(n)}{\Delta}$$

and

$$\#(\{\mathcal{T}_1 < \mathcal{T}_j \leq n \mid \mathcal{T}_j \textit{ is close to } \mathcal{S}\langle\mathcal{T}_j\rangle \textit{ and } \mathcal{T}\langle\mathcal{T}_j\rangle\}) \geq \frac{\hat{C}(n)}{\Delta}$$

for every $n \geq 1$.

Proof. Let us show

$$\#(\{\nu_i < \mathcal{S}_j \leq \nu_{i+1} \mid \mathcal{S}_j \text{ is close to } \mathcal{S}\langle\mathcal{S}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{S}_j\rangle\}) \geq \frac{\#(\{\nu_i < j \leq \nu_{i+1} \mid j \text{ is strongly free}\})}{\Delta}$$

[7] Note that the value of Δ for which we will prove

$$\liminf_{i\to\infty} \frac{\#(\{1 < \mathcal{S}_j \leq i \mid \mathcal{S}_j \text{ is close to } \mathcal{S}\langle\mathcal{S}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{S}_j\rangle\})}{i} > 0$$

and

$$\liminf_{i\to\infty} \frac{\#(\{\mathcal{T}_1 < \mathcal{T}_j \leq i \mid \mathcal{T}_j \text{ is close to } \mathcal{S}\langle\mathcal{T}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{T}_j\rangle\})}{i} > 0$$

will in general be large, even if $\limsup_{i\to\infty} B(i)/i < 1$ holds for Δ small.

and

$$\begin{aligned}&\#(\{\nu_i < \mathcal{T}_j \leq \nu_{i+1} \mid \mathcal{T}_j \text{ is close to } \mathcal{S}\langle\mathcal{T}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{T}_j\rangle\})\\ &\qquad \geq \frac{\#(\{\nu_i < j \leq \nu_{i+1} \mid j \text{ is strongly free}\})}{\Delta}\end{aligned}$$

for all $i \geq 1$. Summing over i then gives the result.

So take some $i \geq 1$. We can suppose

$$\frac{\#(\{\nu_i < j \leq \nu_{i+1} \mid j \text{ is strongly free}\})}{\Delta} > 0$$

and this implies $\nu_{i+1} > \nu_i + \mathcal{R}(\nu_i) + \Delta$. Note that one of $\{\nu_i + p_i, \nu_i + \mathcal{R}(\nu_i)\}$ is a cutting time, and the other is a co-cutting time, because ν_i is a return time. Now note that if $\nu_i + p_i \leq \mathcal{S}_j < \nu_{i+1}$ then $\mathcal{R}(\mathcal{S}_j) \leq \Delta$ and therefore $\mathcal{S}_{j+1} - \mathcal{S}_j \leq \Delta$. Likewise if $\nu_i + p_i \leq \mathcal{T}_j < \nu_{i+1}$ then $\mathcal{R}(\mathcal{T}_j) \leq \Delta$ and $\mathcal{T}_{j+1} - \mathcal{T}_j \leq \Delta$. Therefore

$$\#(\{\mathcal{S}_j \mid \nu_i + \mathcal{R}(\nu_i) < \mathcal{S}_j \leq \nu_{i+1}\}) \geq \frac{\nu_{i+1} - \nu_i - \mathcal{R}(\nu_i) - \Delta}{\Delta}$$

and

$$\#(\{\mathcal{T}_j \mid \nu_i + \mathcal{R}(\nu_i) < \mathcal{T}_j \leq \nu_{i+1}\}) \geq \frac{\nu_{i+1} - \nu_i - \mathcal{R}(\nu_i) - \Delta}{\Delta}.$$

Since these $\mathcal{S}_j$ and $\mathcal{T}_j$ are all close to their preceding cutting and co-cutting times, this gives

$$\begin{aligned}&\#(\{\nu_i < \mathcal{S}_j \leq \nu_{i+1} \mid \mathcal{S}_j \text{ is close to } \mathcal{S}\langle\mathcal{S}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{S}_j\rangle\})\\ &\qquad \geq \frac{\#(\{\nu_i < j \leq \nu_{i+1} \mid j \text{ is strongly free}\})}{\Delta}\end{aligned}$$

and

$$\begin{aligned}&\#(\{\nu_i < \mathcal{T}_j \leq \nu_{i+1} \mid \mathcal{T}_j \text{ is close to } \mathcal{S}\langle\mathcal{T}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{T}_j\rangle\})\\ &\qquad \geq \frac{\#(\{\nu_i < j \leq \nu_{i+1} \mid j \text{ is strongly free}\})}{\Delta}\end{aligned}$$

for all $i \geq 1$. ■

We need the following technical lemma. We write $B(i;\Delta)$, $C(i;\Delta)$ and so forth to show dependence on Δ explicitly.

Lemma 48 *If*

$$\limsup_{i\to\infty} \frac{\mathcal{R}(i)}{i} = 0$$

and

$$\limsup_{i\to\infty} \frac{B(i;\Delta)}{i} < 1$$

for some $\Delta \geq 1$ *then there exist* $\hat{\Delta} \geq 1$, $\Theta > 1$ *and* $\epsilon > 0$ *with the following property: for every* $\Delta_1 \geq \hat{\Delta}$, $\Delta_2 \geq \Theta\Delta_1$ *and* $i \geq 1$, *either*

$$1 - \frac{B(i;\Delta_2)}{i} \geq \frac{3}{2}\left(1 - \frac{B(i;\Delta_1)}{i}\right)$$

or

$$\frac{C(i;\Delta_1)}{i} \geq \epsilon.$$

Proof. First let us choose the values of $\hat{\Delta} \geq 1$, $\Theta > 1$ and $\epsilon > 0$: put $\rho = \lim_{\Delta\to\infty} \limsup_{i\to\infty} B(i;\Delta)/i < 1$ and take $\hat{\Delta}_1$ and N_1 large enough that $B(i;\hat{\Delta}_1)/i < \sqrt{\rho}$ for all $i \geq N_1$. Take $\hat{\Delta}_2$ large enough that $p_i(\Delta) < \mathcal{R}(\nu_i(\Delta)) \leq (1+\epsilon)p_i(\Delta)$ for every $\Delta \geq \hat{\Delta}_2$ and $i \geq 1$. The existence of such a $\hat{\Delta}_2$ uses $\limsup_{i\to\infty} \mathcal{R}(i)/i = 0$ and is left to the reader. Set $\epsilon = (1-\sqrt{\rho})/6$, $\Theta = 1/\epsilon$ and $\hat{\Delta} = \max\{\hat{\Delta}_1, \hat{\Delta}_2, \max_{1\leq j<N_1} \mathcal{R}(j)\}$.

Now take any $\Delta_1 \geq \Delta$, $\Delta_2 \geq \Theta\Delta_1$ and $i \geq 1$, and suppose

$$1 - \frac{B(i;\Delta_2)}{i} < \frac{3}{2}\left(1 - \frac{B(i;\Delta_1)}{i}\right).$$

Let us prove $C(i;\Delta_1)/i \geq \epsilon$.

If $i \leq \nu_1(\Delta_1)$ then $C(i;\Delta_1)/i = 1$, giving the result, so suppose $i > \nu_1(\Delta_1)$. Take $k_1 \geq 1$ and $k_2 \geq 1$ such that $\nu_{k_1}(\Delta_1) < i \leq \nu_{k_1}(\Delta_1)$ and $\nu_{k_2}(\Delta_2) < i \leq \nu_{k_2}(\Delta_2)$. Then

$$\begin{aligned} i - C(i;\Delta_1) &\leq \sum_{j=1}^{k_1}(\mathcal{R}(\nu_j(\Delta_1)) + \Delta_1) \\ &\leq (1+\epsilon)B(i;\Delta_1) + k_1\Delta_1 \\ &= (1+\epsilon)B(i;\Delta_1) + (k_1 - k_2 + k_2)\Delta_1, \end{aligned}$$

where we have assumed $i > \nu_k(\Delta_1) + \Delta_1$ in the first step for simplicity; this does not alter the final result. Now we use $k_2 \leq B(i;\Delta_2)/\Delta_2$ and $k_1 - k_2 \leq (B(i;\Delta_1) - B(i;\Delta_2))/\Delta_1$. Recall that $\Delta_2 \geq \Theta\Delta_1$. Thus

$$i - C(i;\Delta_1) \leq 2B(i;\Delta_1) - B(i;\Delta_2) + \epsilon B(i;\Delta_1) + \frac{B(i;\Delta_2)}{\Theta}.$$

Note that $\Theta = 1/\epsilon$ and $1 - B(i;\Delta_2)/i < 3(1 - B(i;\Delta_1)/i)/2$, and that $B(i;\Delta) \leq i$ for all $\Delta \geq 1$. Therefore

$$i - C(i;\Delta_1) \leq \frac{i + B(i;\Delta_1)}{2} + 2\epsilon i.$$

We have $B(i;\Delta_1)/i < \sqrt{\rho}$ using $i > N_1$ and the definition of N_1.[8] Recall that $\epsilon = (1-\sqrt{\rho})/6$. Therefore

$$\begin{aligned} i - C(i;\Delta_1) &\leq i(\frac{1+\sqrt{\rho}}{2} + \frac{1-\sqrt{\rho}}{3}) \\ &= i(1-\epsilon). \end{aligned}$$

Thus

$$\frac{C(i;\Delta_1)}{i} \geq \epsilon$$

and the lemma is proved. ■

So let us prove lemma 46:

Lemma 46 *If f satisfies*

$$\limsup_{i\to\infty} \frac{\mathcal{R}(i)}{i} = 0$$

and

$$\limsup_{i\to\infty} \frac{B(i)}{i} < 1$$

for some $\Delta \geq 1$ then

$$\liminf_{i\to\infty} \frac{\#(\{1 < \mathcal{S}_j \leq i \mid \mathcal{S}_j \textit{ is close to } \mathcal{S}\langle\mathcal{S}_j\rangle \textit{ and } \mathcal{T}\langle\mathcal{S}_j\rangle\})}{i} > 0$$

and

$$\liminf_{i\to\infty} \frac{\#(\{\mathcal{T}_1 < \mathcal{T}_j \leq i \mid \mathcal{T}_j \textit{ is close to } \mathcal{S}\langle\mathcal{T}_j\rangle \textit{ and } \mathcal{T}\langle\mathcal{T}_j\rangle\})}{i} > 0$$

for some $\Delta \geq 1$.

Proof. Take $\hat{\Delta} \geq 1$, $\Theta > 1$ and $\epsilon > 0$ as in the last lemma, put $\rho = \lim_{\hat{\Delta}\to\infty} \limsup_{i\to\infty} B(i)/i$ and $N = [-\log(1-\sqrt{\rho})/\log 1.5]$, where $[x]$ denotes the greatest integer less than or equal to x.

Set $\Delta = \Theta^N\hat{\Delta}$ and take $i \geq 2\nu_1(\Delta)/\epsilon$. Put

$$M = \min\{j \geq 0 \mid 1 - \frac{B(i;\Theta^{j+1}\hat{\Delta})}{i} < \frac{3}{2}\left(1 - \frac{B(i;\Theta^j\hat{\Delta})}{i}\right)\}.$$

Since $B(i;\hat{\Delta})/i \leq \sqrt{\rho}$, as was shown in the proof of lemma 48, we have $M \leq N$.

[8] We have $i > N_1$ because $i > \nu_1(\Delta_1) > N_1$; and $\nu_1(\Delta_1) > N_1$ because $\mathcal{R}(\nu_1(\Delta_1)) > \hat{\Delta} \geq \max_{1\leq j<N_1} \mathcal{R}(j)$.

Because $1 - B(i;\Theta^{M+1}\hat{\Delta})/i < 3(1 - B(i;\Theta^M\hat{\Delta})/i)/2$, lemma 48 shows

$$\frac{C(i;\Theta^M\hat{\Delta})}{i} \geq \epsilon.$$

Using $i \geq 2\nu_1(\Delta)/\epsilon$ we therefore have $\hat{C}(i;\Theta^M\hat{\Delta})/i \geq \epsilon/2$. Lemma 47 gives

$$\frac{\#(\{1 < \mathcal{S}_j \leq i \mid \mathcal{S}_j \text{ is close to } \mathcal{S}\langle\mathcal{S}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{S}_j\rangle\})}{i} \geq \frac{\epsilon}{2\Theta^M\hat{\Delta}} \geq \frac{\epsilon}{2\Delta}$$

and

$$\frac{\#(\{\mathcal{T}_1 < \mathcal{T}_j \leq i \mid \mathcal{T}_j \text{ is close to } \mathcal{S}\langle\mathcal{T}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{T}_j\rangle\})}{i} \geq \frac{\epsilon}{2\Theta^M\hat{\Delta}} \geq \frac{\epsilon}{2\Delta},$$

independently of i. Therefore

$$\liminf_{i\to\infty} \frac{\#(\{1 < \mathcal{S}_j \leq i \mid \mathcal{S}_j \text{ is close to } \mathcal{S}\langle\mathcal{S}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{S}_j\rangle\})}{i} > 0$$

and

$$\liminf_{i\to\infty} \frac{\#(\{\mathcal{T}_1 < \mathcal{T}_j \leq i \mid \mathcal{T}_j \text{ is close to } \mathcal{S}\langle\mathcal{T}_j\rangle \text{ and } \mathcal{T}\langle\mathcal{T}_j\rangle\})}{i} > 0.$$

■

This completes the proof of theorem 38.

4.3 Non-Collet-Eckmann Kneading Invariants

We call a kneading invariant non-Collet-Eckmann if no f in $\mathcal{C}$ with this kneading invariant is Collet-Eckmann. For example periodic kneading invariants are non-Collet-Eckmann since no map in $\mathcal{C}$ with a periodic attractor is Collet-Eckmann; likewise infinitely renormalizable kneading invariants are non-Collet-Eckmann because all Collet-Eckmann maps are known to be finitely renormalizable [43, 14].

In this section we describe two classes of non-Collet-Eckmann kneading invariants. The results complement those of the last section though fall short of classifying all kneading invariants as Collet-Eckmann or non-Collet-Eckmann.

The first class we consider consists of those kneading invariants for which

$$\limsup_{i\to\infty} \frac{\mathcal{R}(i)}{i} = 0$$

and

$$\lim_{i\to\infty} \frac{B(i)}{i} = 1$$

for every $\Delta \geq 1$. In theorem 49 below we show that every such kneading invariant is non-Collet-Eckmann.

In theorem 34 we showed that any kneading invariant satisfying

$$\limsup_{i\to\infty} \frac{\mathcal{R}(i)}{i} = 0$$

and

$$\limsup_{i\to\infty} \frac{B(i)}{i} < 1$$

for some $\Delta \geq 1$ is Collet-Eckmann.

Neither theorem 34 nor theorem 49 applies to those kneading invariants with

$$\limsup_{i\to\infty} \frac{\mathcal{R}(i)}{i} = 0$$

for which

$$\limsup_{i\to\infty} \frac{B(i)}{i} = 1$$

for all $\Delta \geq 1$ but

$$\liminf_{i\to\infty} \frac{B(i)}{i} < 1$$

for some $\Delta \geq 1$. There seem to be both Collet-Eckmann and non-Collet-Eckmann kneading invariants of this type.

Theorem 49 *Any kneading invariant satisfying*

$$\limsup_{i\to\infty} \frac{\mathcal{R}(i)}{i} = 0$$

and

$$\lim_{i\to\infty} \frac{B(i)}{i} = 1$$

for every $\Delta \geq 1$ *is non-Collet-Eckmann.*

To prove theorem 49 we need the following technical lemma. This result is analogous to lemma 6.1 in de Melo and Van Strien's exposition [14, 41] of Benedicks and Carleson's work. Recall that every value of R is a cutting time. If $R(x) = \mathcal{S}_i$ then we define $R^-(x) = \mathcal{S}_{i-1}$ if $i > 1$.

Lemma 50 *Suppose* $\limsup_{i\to\infty} \mathcal{R}(i)/i = 0$. *For every* $\Delta \geq 1$ *there exists* $\epsilon(\Delta) > 0$ *with the following property: for any* $x \in [0;1]$ *with* $R(x) > \Delta$ *we have*

1. *A distortion estimate on* $[x_1; c_1]$*:*

$$e^{-\epsilon(\Delta)R^-(x)} \leq \frac{|Df^i(y)|}{|Df^i(z)|} \leq e^{\epsilon(\Delta)R^-(x)}$$

for every $y, z \in [x_1; c_1]$ *and* $0 \leq i < R^-(x)$.

2. *A bound on derivative growth at* x*:*

$$|Df^i(x)| \leq e^{\epsilon(\Delta)R^-(x)}|Df^{i-1}(c_1)|^{1/l}$$

for every $1 \leq i \leq R^-(x)$. *Here* $l > 1$ *is the order of the critical point.*

In addition $\epsilon(\Delta) \to 0$ *as* $\Delta \to \infty$.

Proof. Let us show that the second part of the lemma follows from the first part, so take $\epsilon(\Delta) > 0$ such that the first part holds and any $x \in [0;1]$ with $R(x) > \Delta$. Take $y \in [x_1; c_1]$ such that $|x_i - c_i| = |Df^{i-1}(y)||x_1 - c_1|$.

From part 1 we have $|Df^{i-1}(x_1)| \leq |Df^{i-1}(y)|e^{\epsilon(\Delta)R^-(x)}$, so the definition of y and $|x_i - c_i| \leq 1$ give

$$|Df^{i-1}(x_1)| \leq \frac{e^{\epsilon(\Delta)R^-(x)}}{|x_1 - c_1|}. \tag{4.18}$$

From the definition of non-flatness of the critical point there exist $L > 1$ and $l > 1$ such that $|x-c|^{l-1}/L \leq |Df(x)| \leq L|x-c|^{l-1}$. Equation 1.1 gives $|x-c|^l \leq lL|x_1-c_1|$, and combining the two inequalities yields $|Df(x)| \leq K|x_1 - c_1|^{(l-1)/l}$ for some $K > 0$. Using this to eliminate $|x_1 - c_1|$ in equation 4.18, a simple rearrangement gives

$$|Df(x)| \leq \frac{e^{\epsilon(\Delta)R^-(x)}}{|Df^{i-1}(x_1)|^{(l-1)/l}},$$

where we have absorbed unwanted constants into $\epsilon(\Delta)$. Therefore

$$\begin{aligned} |Df^i(x)| &= |Df(x)||Df^{i-1}(x_1)| \\ &\leq e^{\epsilon(\Delta)R^-(x)}\frac{|Df^{i-1}(x_1)|}{|Df^{i-1}(x_1)|^{(l-1)/l}} = e^{\epsilon(\Delta)R^-(x)}|Df^{i-1}(x_1)|^{1/l}. \end{aligned}$$

Part 1 gives $|Df^{i-1}(x_1)| \leq |Df^{i-1}(c_1)| e^{\epsilon(\Delta) R^-(x)}$ and thus $|Df^i(x)| \leq e^{2\epsilon(\Delta)R^-(x)}$ $|Df^{i-1}(x_1)|^{1/l}$. This proves the second part of the lemma.

Let us prove the first part of the lemma. We start by bounding the distortion of $f^{R^-(x)-1}\big|_{[x_1;c_1]}$ using lemma 5.

We know that $f^{R^-(x)-1}\big|_{[x_1;c_1]}$ is a diffeomorphism since $R^-(x) < R(x)$. Recall that the largest interval around c_1 on which $f^{R^-(x)-1}$ is a diffeomorphism is $H_{R^-(x)} = [\alpha_{R^-x}; \beta_{R^-x}]$. Clearly $[x_1; c_1] \subseteq [\alpha_{R^-(x)}; c_1]$. Let us show that $f^{R^-(x)-1}(H_{R^-(x)}) = [a_{R^-(x)}; b_{R^-(x)}]$ is a ρ-scaled neighbourhood of $f^{R^-(x)-1}[x_1; c_1] = [x_{R^-(x)}; c_{R^-(x)}]$ where $\rho = e^{-\hat{\epsilon}(\Delta)R^-(x)}$ and $\hat{\epsilon}(\Delta) \to 0$ as $\Delta \to \infty$. A *ρ-scaled neighbourhood* was defined just before the statement of lemma 5.

First we estimate $|a_{R^-(x)} - x_{R^-(x)}|$. Because $R^-(x)$ is a cutting time, the interval $[a_{R^-(x)}; b_{R^-(x)}]$ contains c. Also $x_{R^-(x)}$ and $c_{R^-(x)}$, both in this interval, are on the same side of c since $R^-(x) < R(x)$, and this is the same side as $b_{R^-(x)}$ because $R^-(x)$ is a cutting time. Therefore $|a_{R^-(x)} - x_{R^-(x)}| \geq |a_{R^-(x)} - c|$. Recall that $a_{R^-(x)} = c_{R^-(x)-\mathcal{S}\langle R^-(x)\rangle}$. Lemma 39 shows there exists a $K > 1$ such that $|c_n - c| \geq K^{-\mathcal{R}(n)}$ for all $n \geq 1$, and this gives us $|a_{R^-(x)} - c| \geq K^{-\mathcal{R}(R^-(x)-\mathcal{S}\langle R^-(x)\rangle)}$, and so

$$|a_{R^-(x)} - x_{R^-(x)}| \geq K^{-\mathcal{R}(R^-(x)-\mathcal{S}\langle R^-(x)\rangle)}.$$

We leave it to the reader to show $\mathcal{R}(R^-(x) - \mathcal{S}\langle R^-(x)\rangle)/R^-(x) \to 0$ as $\Delta \to \infty$. Note that $R^-(x) \to \infty$ as $\Delta \to \infty$, and that there exists a $K_1 > 0$ such that $\mathcal{R}(n) \leq K_1 n$ for all $n \geq 1$, because $\limsup_{n\to\infty} \mathcal{R}(n)/n = 0$.

Taking $\epsilon_1(\Delta) = \log K \; \mathcal{R}(R^-(x) - \mathcal{S}\langle R^-(x)\rangle)/R^-(x)$ we thus have $|a_{R^-(x)} - x_{R^-(x)}| \geq e^{-\epsilon_1(\Delta)R^-(x)}$ and $\epsilon_1(\Delta) \to 0$ as $\Delta \to \infty$.

Now we estimate $|b_{R^-(x)} - c_{R^-(x)}|$. Note that $[b_{R^-(x)}; c_{R^-(x)}]$ does not contain c because $R^-(x)$ is a cutting time, not a co-cutting time. Denote by $\mathcal{T}^+\langle R^-(x)\rangle$ the first co-cutting time after $R^-(x)$. We therefore have $c \in f^{\mathcal{T}^+\langle R^-(x)\rangle - R^-(x)}[b_{R^-(x)}; c_{R^-(x)}]$ and this is the first time that c is in an image of $[b_{R^-(x)}; c_{R^-(x)}]$. We have

$$|f^{\mathcal{T}^+\langle R^-(x)\rangle - R^-(x)}[b_{R^-(x)}; c_{R^-(x)}]| \geq |c_{\mathcal{T}^+\langle R^-(x)\rangle} - c| \geq K^{-\mathcal{R}(\mathcal{T}^+\langle R^-(x)\rangle)}.$$

Take K_2 to be the maximum value of $|Df|$, so

$$|f^{\mathcal{T}^+\langle R^-(x)\rangle - R^-(x)}[b_{R^-(x)}; c_{R^-(x)}]| \leq K_2^{\mathcal{T}^+\langle R^-(x)\rangle - R^-(x)} |b_{R^-(x)} - c_{R^-(x)}|.$$

Therefore

$$|b_{R^-(x)} - c_{R^-(x)}| \geq K^{-(\mathcal{T}^+\langle R^-(x)\rangle - R^-(x) + \mathcal{R}(\mathcal{T}^+\langle R^-(x)\rangle))},$$

if necessary taking K bigger than K_2.

Note that $\mathcal{T}^+\langle R^-(x)\rangle - R^-(x) < \mathcal{R}(\mathcal{T}\langle R^-(x)\rangle)$. We leave it to the reader to show $(\mathcal{T}^+\langle R^-(x)\rangle - R^-(x))/R^-(x) \to 0$ and $\mathcal{R}(\mathcal{T}^+\langle R^-(x)\rangle)/R^-(x) \to 0$ as $\Delta \to \infty$.

Taking $\epsilon_2(\Delta) = \log K\ (\mathcal{T}^+\langle R^-(x)\rangle - R^-(x) + \mathcal{R}(\mathcal{T}^+\langle R^-(x)\rangle))/R^-(x)$ we therefore have $|b_{R^-(x)} - c_{R^-(x)}| \geq e^{-\epsilon_2(\Delta)R^-(x)}$ and $\epsilon_2(\Delta) \to 0$ as $\Delta \to \infty$.

Putting $\hat{\epsilon}(\Delta) = \max\{\epsilon_1(\Delta), \epsilon_2(\Delta)\}$ and $\rho = e^{-\hat{\epsilon}(\Delta)R^-(x)}$, the inequalities $|a_{R^-(x)} - x_{R^-(x)}| \geq e^{-\hat{\epsilon}(\Delta)R^-(x)}$ and $|b_{R^-(x)} - c_{R^-(x)}| \geq e^{-\hat{\epsilon}(\Delta)R^-(x)}$ show that $f^{R^-(x)-1}(H_{R^-(x)})$ is a ρ-scaled neighbourhood of $f^{R^-(x)-1}[x_1; c_1]$. The reader has shown $\hat{\epsilon}(\Delta) \to 0$ as $\Delta \to \infty$.

Lemma 5 now gives

$$\frac{\rho^2}{(1+\rho)^2} \leq \frac{|Df^{R^-(x)-1}(y)|}{|Df^{R^-(x)-1}(z)|} \leq \frac{(1+\rho)^2}{\rho^2}$$

for every $y, z \in [x_1; c_1]$. Writing $(1+\rho)^2/\rho^2 = e^{\epsilon(\Delta)R^-(x)}$ we clearly have $\epsilon(\Delta) \to 0$ as $\Delta \to \infty$.

We use a trick to bound $|Df^i(y)|/|Df^i(z)|$ when $1 \leq i < R^-(x) - 1$: we can conclude from lemma 5 that the above bounds on the distortion hold equally well for $f^{R^-(x)-1-i}\big|_{f^i[x_1;c_1]}$ because $f^{R^-(x)-1}(H_{R^-(x)})$ is equally well a ρ-scaled neighbourhood of $f^{R^-(x)-1-i}(f^i[x_1; c_1])$.

Since $Df^i(y) = Df^{R^-(x)-1}(y)/Df^{R^-(x)-1-i}(f^i(y))$ and likewise with y replaced by z, we have

$$e^{-2\epsilon(\Delta)R^-(x)} = \frac{\rho^4}{(1+\rho)^4} \leq \frac{|Df^i(y)|}{|Df^i(z)|} \leq \frac{(1+\rho)^4}{\rho^4} = e^{2\epsilon(\Delta)R^-(x)}$$

for every $y, z \in [x_1; c_1]$ and $0 \leq i < R^-(x)$. This completes the proof of the lemma. ■

So let us prove theorem 49:

Theorem 49 *Any kneading invariant satisfying*

$$\limsup_{i\to\infty} \frac{\mathcal{R}(i)}{i} = 0$$

and

$$\lim_{i\to\infty} \frac{B(i)}{i} = 1$$

for every $\Delta \geq 1$ *is non-Collet-Eckmann.*

Proof. Set

$$\lambda^+ = \limsup_{i\to\infty} \frac{\log |Df^i(c_1)|}{i}$$

and

$$\lambda^- = \liminf_{i\to\infty} \frac{\log |Df^i(c_1)|}{i}.$$

We will argue by contradiction by supposing $\lambda^- > 0$. Let us first show that if Δ is large enough then any $x \in [0;1]$ with $R(x) > \Delta$ has

$$|Df^i(x)| \leq e^{i\lambda^+/\sqrt[4]{l}}$$

for every $1 \leq i \leq R^-(x)$.

From part 2 of the last lemma we have $|Df^i(x)| \leq e^{\epsilon(\Delta)R^-(x)}|Df^{i-1}(c_1)|^{1/l}$ for all $1 \leq i \leq R^-(x)$. Since $\epsilon(\Delta) \to 0$ as $\Delta \to \infty$ we can take Δ large enough that

$$|Df^i(x)| \leq |Df^{i-1}(c_1)|^{1/\sqrt{l}} \tag{4.19}$$

for every $1 \leq i \leq R^-(x)$. This uses $\lambda^- > 0$.

Now take N large enough that $|Df^i(c_1)| < e^{i\lambda^+\sqrt[4]{l}}$ for every $i \geq N$. This uses $l > 1$ and the definition of λ^+. From equation 4.19 we therefore have

$$|Df^i(x)| \leq e^{i\lambda^+/\sqrt[4]{l}}$$

for every $N < i \leq R^-(x)$. We can extend this inequality to all $1 \leq i \leq R^-(x)$ if we take Δ large enough. After all, by taking Δ large enough we can force $\max\{|Df(x)|, \ldots, |Df^N(x)|\}$ as small as we like. We therefore have

$$|Df^i(x)| \leq e^{i\lambda^+/\sqrt[4]{l}}$$

for every $1 \leq i \leq R^-(x)$.

Let us apply this to the orbit of the critical point. Recall that $\mathcal{R}(\nu_j) > \Delta$ for every $j \geq 1$ and so

$$|Df^i(c_{\nu_j})| \leq e^{i\lambda^+/\sqrt[4]{l}}$$

for every $1 \leq i \leq p_j$.

Take any $n \geq 1$ and note that $|Df^n(c_1)| = \prod_{i=1}^n |Df(c_i)|$. The terms in this product are of two sorts: if i is bound, by ν_j say, then $|Df(c_i)|$ is a term of $|Df^{p_j}(c_{\nu_j})|$; if i is free then $|Df(c_i)| \leq K$ where K is the maximum value of $|Df|$. Therefore

$$\begin{aligned} |Df^n(c_1)| &\leq K^{n-B(n)} \prod_{\nu_j<n} |Df^{p_j}(c_{\nu_j})| \\ &\leq e^{(n-B(n))\log K + \lambda^+ B(n)/\sqrt[4]{l}} \end{aligned}$$

if n itself is free. If n is bound then the first inequality is not quite correct, but the conclusion still holds: we have

$$|Df^n(c_1)| \leq e^{[(1-B(n)/n)\log K + \lambda^+ B(n)/(n\sqrt[4]{l})]n}$$

for all $n \geq 1$. Since $\lim_{n\to\infty} B(n)/n = 1$, taking logarithms and limsups gives the contradiction $\lambda^+ \leq \lambda^+/\sqrt[4]{l}$. ■

The second class of kneading invariants we consider are those with

$$\lim_{k\to\infty} \limsup_{i\to\infty} \frac{\sum_{j=1}^i \begin{cases} \mathcal{R}(j) & \text{if } \mathcal{R}(j) \geq k \\ 0 & \text{otherwise} \end{cases}}{i} = \infty.$$

We show in theorem 51 below that any such kneading invariant is non-Collet-Eckmann.

In the last section we showed that any kneading invariant with

$$\lim_{k\to\infty} \limsup_{i\to\infty} \frac{\sum_{j=1}^i \begin{cases} \mathcal{R}(j) & \text{if } \mathcal{R}(j) \geq k \\ 0 & \text{otherwise} \end{cases}}{i} = 0$$

is Collet-Eckmann. Again there are both Collet-Eckmann and non-Collet-Eckmann kneading invariants that satisfy neither of these conditions.

Theorem 51 *If a kneading invariant satisfies*

$$\lim_{k\to\infty} \limsup_{i\to\infty} \frac{\sum_{j=1}^i \begin{cases} \mathcal{R}(j) & \textit{if } \mathcal{R}(j) \geq k \\ 0 & \textit{otherwise} \end{cases}}{i} = \infty$$

then it is non-Collet-Eckmann.

Proof. We argue by contradiction, so suppose f is Collet-Eckmann and

$$\lim_{k\to\infty} \limsup_{i\to\infty} \frac{\sum_{j=1}^i \begin{cases} \mathcal{R}(j) & \text{if } \mathcal{R}(j) \geq k \\ 0 & \text{otherwise} \end{cases}}{i} = \infty.$$

Using the fact that f is Collet-Eckmann it is not hard to show the existence of a $K > 1$ such that $|c_n - c| < K^{-\mathcal{R}(n)}$ for all $n \geq 1$, at least if $\mathcal{R}(n) \geq 2$. From non-flatness of the critical point we have $|Df(c_n)| \leq L|c_n - c|^{l-1}$ for some $L > 1$ and $l > 1$ and therefore we can write

$$|Df(c_n)| \leq LK^{-(l-1)\mathcal{R}(n)}$$

for all $n \geq 1$. Note that L is also a bound for the maximum size of Df.

Take k large enough that $LK^{-(l-1)\mathcal{R}(n)} \leq K^{-(l-1)\mathcal{R}(n)/2}$ whenever $\mathcal{R}(n) \geq k$. Then

$$\begin{aligned} |Df^i(c_1)| &= \prod_{j=1}^{i} |Df(c_j)| \\ &= \prod_{j \leq i \text{ and } \mathcal{R}(j) < k} |Df(c_j)| \prod_{j \leq i \text{ and } \mathcal{R}(j) \geq k} |Df(c_j)| \\ &\leq L^i K^{-(l-1)\sum_{j \leq i \text{ and } \mathcal{R}(j) \geq k} \mathcal{R}(j)/2}. \end{aligned}$$

Since

$$\frac{\sum_{j \leq i \text{ and } \mathcal{R}(j) \geq k} \mathcal{R}(j)}{i} \to \infty$$

as $i \to \infty$, this gives the contradiction $\liminf_{i \to \infty} |Df^i(c_1)| = 0$. ∎

Corollary 52 *If a kneading invariant satisfies* $\limsup_{i \to \infty} \mathcal{R}(i)/i = \infty$ *then it is non-Collet-Eckmann.*

There are Collet-Eckmann kneading invariants with $\limsup_{i \to \infty} \mathcal{R}(i)/i > 0$.

Chapter 5

The Topological Abundance of Collet-Eckmann Maps

Recall that the topological entropy takes its values in $[0; \log 2]$. We call a value of the topological entropy *Collet-Eckmann* if every f in $\mathcal{C}$ with this topological entropy is Collet-Eckmann. The main result of this chapter is

Theorem 53 *Lebesgue almost every value of the topological entropy is Collet-Eckmann.*

Therefore, in this sense, almost every kneading invariant is Collet-Eckmann.[1]

Theorem 53 follows from

Theorem 54 *The kneading invariant of T_λ is slowly recurrent for Lebesgue almost every $1 \le \lambda \le 2$.*

Let us explain why theorem 53 follows from theorem 54. Slow recurrence was defined in chapter 4.

If the kneading invariant of T_λ is slowly recurrent then T_λ is not periodic. If $f \in \mathcal{C}$ has topological entropy $\log \lambda$ then lemma 9 shows that f is topologically conjugate to T_λ. In particular f and T_λ have the same slowly recurrent kneading invariant. In corollary 36 we showed that any map in $\mathcal{C}$ with a slowly recurrent kneading invariant is Collet-Eckmann, so f is Collet-Eckmann. This shows that $\log \lambda$ is a Collet-Eckmann value of the topological entropy.

[1]By contrast, almost no kneading invariant is Misiurewicz [7].

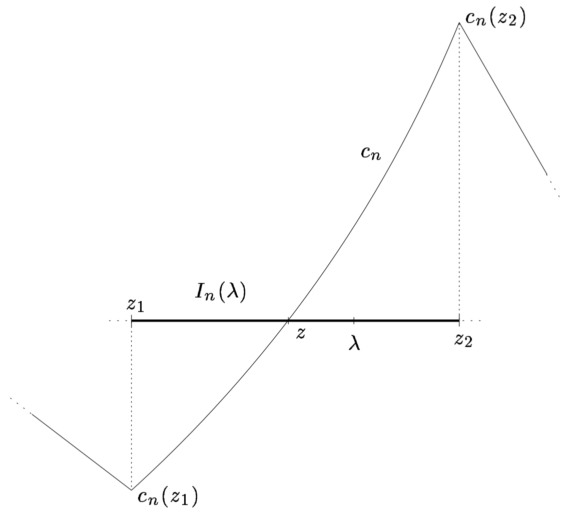


Figure 5.1: The smooth segment of c_n containing λ.

Theorem 54 therefore shows that Lebesgue almost every value of the topological entropy is Collet-Eckmann.

There are two main steps in the proof of theorem 54.

The first step characterizes slow recurrence in terms of parameter space properties as follows. Suppose T_λ is not periodic and denote by $I_n(\lambda) = [z_1; z_2]$ the smooth segment of c_n containing λ (see figure 5.1). Define $\Diamond_\lambda(n) = 0$ if $c \notin (c_n(z_1); c_n(z_2))$. Otherwise there is some unique $z \in (z_1; z_2)$ such that $c_n(z) = c$ and we define

$$\Diamond_\lambda(n) = \log \min\{\frac{|z_1 - z|}{|\lambda - z|}, \frac{|z_2 - z|}{|\lambda - z|}\}.$$

Only large positive value of $\Diamond_\lambda(n)$ will be relevant.

Theorem 55 *For each $1 \le \lambda \le 2$ the kneading invariant of T_λ is slowly recurrent if and only if T_λ is not periodic and*

$$\lim_{l\to\infty} \limsup_{i\to\infty} \frac{\sum_{j=1}^{i} \begin{cases} \Diamond_\lambda(j) & \text{if } \Diamond_\lambda(j) \ge l \\ 0 & \text{otherwise} \end{cases}}{i} = 0.$$

We prove theorem 55 in the first section, "The Parameter Space Characterization of Slow Recurrence".

The second step performs the measure estimates for theorem 54. The result is due to Tsujii [49], with modifications.

Theorem 56 (Tsujii) *We have that T_λ is not periodic and*

$$\lim_{l\to\infty}\limsup_{i\to\infty}\frac{\sum_{j=1}^{i}\begin{cases}\Diamond_\lambda(j) & \text{if } \Diamond_\lambda(j)\ge l\\ 0 & \text{otherwise}\end{cases}}{i}=0$$

for Lebesgue almost every $1\le\lambda\le 2$.

Theorem 54 follows from theorems 55 and 56. Theorem 56 is proved in the second section, "Measure Estimates".

If we only consider λ values for which T_λ is close to a fixed Misiurewicz map then our method of proof essentially reduces to the method by which Benedicks and Carleson proved that many quadratic maps are Collet-Eckmann [4, 5, 14]. However the method used here continues to work even without the support of a Misiurewicz map. This lets us conclude that T_λ has a slowly recurrent kneading invariant for Lebesgue almost every λ and not just for a positive measure set.

5.1 The Parameter Space Characterization of Slow Recurrence

In this section we prove theorem 55:

Theorem 55 *For each $1\le\lambda\le 2$ the kneading invariant of T_λ is slowly recurrent if and only if T_λ is not periodic and*

$$\lim_{l\to\infty}\limsup_{i\to\infty}\frac{\sum_{j=1}^{i}\begin{cases}\Diamond_\lambda(j) & \text{if } \Diamond_\lambda(j)\ge l\\ 0 & \text{otherwise}\end{cases}}{i}=0.$$

First we use the techniques of chapter 3 to move the problem to the state space of T_λ. Recall that the endpoints of $F_n(\lambda)$ are $a_n(\lambda)=c_{n-\mathcal{S}\langle n\rangle}(\lambda)$ and $b_n(\lambda)=c_{n-\mathcal{T}\langle n\rangle}(\lambda)$. Define $\hat\Diamond_\lambda(n)$, the state space analogue of $\Diamond_\lambda(n)$, by

$$\hat\Diamond_\lambda(n)=\min\{\frac{|a_n(\lambda)-c|}{|c_n(\lambda)-c|},\frac{|b_n(\lambda)-c|}{|c_n(\lambda)-c|}\}.$$

Only large positive values of $\hat{\Diamond}_\lambda(n)$ will be relevant. If $\hat{\Diamond}_\lambda(n)$ is positive then $F_n(\lambda)$ contains c.

Lemma 57 *If T_λ is not periodic then*

$$\lim_{l\to\infty}\limsup_{i\to\infty}\frac{\sum_{j=1}^{i}\begin{cases}\Diamond_\lambda(j) & \text{if } \Diamond_\lambda(j)\geq l\\ 0 & \text{otherwise}\end{cases}}{i}=0$$

if and only if

$$\lim_{l\to\infty}\limsup_{i\to\infty}\frac{\sum_{j=1}^{i}\begin{cases}\hat{\Diamond}_\lambda(j) & \text{if } \hat{\Diamond}_\lambda(j)\geq l\\ 0 & \text{otherwise}\end{cases}}{i}=0.$$

Proof. Let us show $|\hat{\Diamond}_\lambda(n)-\Diamond_\lambda(n)|<1/2$ holds whenever n is large and $\hat{\Diamond}_\lambda(n)>1$ or $\Diamond_\lambda(n)>1$. The result clearly follows from this.

Suppose $\hat{\Diamond}_\lambda(n)>1$ or $\Diamond_\lambda(n)>1$ and write z_1 and z_2 for the endpoints of $I_n(\lambda)$.

If $\Diamond_\lambda(n)>1$ then there exists, from the definition of $\Diamond_\lambda(n)$, a unique $z\in(z_1;z_2)$ such that $c_n(z)=c$ (see figure 5.2(a)).

Such a z also exists when $\hat{\Diamond}_\lambda(n)>1$ if n is large enough. Indeed, if $\hat{\Diamond}_\lambda(n)>0$ then $c\in F_n(\lambda)$ so the situation is either as in figure 5.2(a) or figure 5.2(b). We eliminate figure 5.2(b) by noting that the slope of c_n is much greater than the slope of a_n if n is large. In figure 5.2(b) this gives $|a_n(\lambda)-c|<|c_n(\lambda)-c|$ and the contradiction $\hat{\Diamond}_\lambda(n)<0$.

From lemma 24 we can treat the curves in figure 5.2(a) as straight lines. Recall that

$$\Diamond_\lambda(n)=\log\min\{\frac{|z_1-z|}{|\lambda-z|},\frac{|z_2-z|}{|\lambda-z|}\}.$$

Using "similar triangles" we have

$$\Diamond_\lambda(n)=\log\min\{\frac{|c_n(z_1)-c|}{|c_n(\lambda)-c|},\frac{|c_n(z_2)-c|}{|c_n(\lambda)-c|}$$

and we wish to compare this with

$$\hat{\Diamond}_\lambda(n)=\log\min\{\frac{|a_n(\lambda)-c|}{|c_n(\lambda)-c|},\frac{|b_n(\lambda)-c|}{|c_n(\lambda)-c|}\}.$$

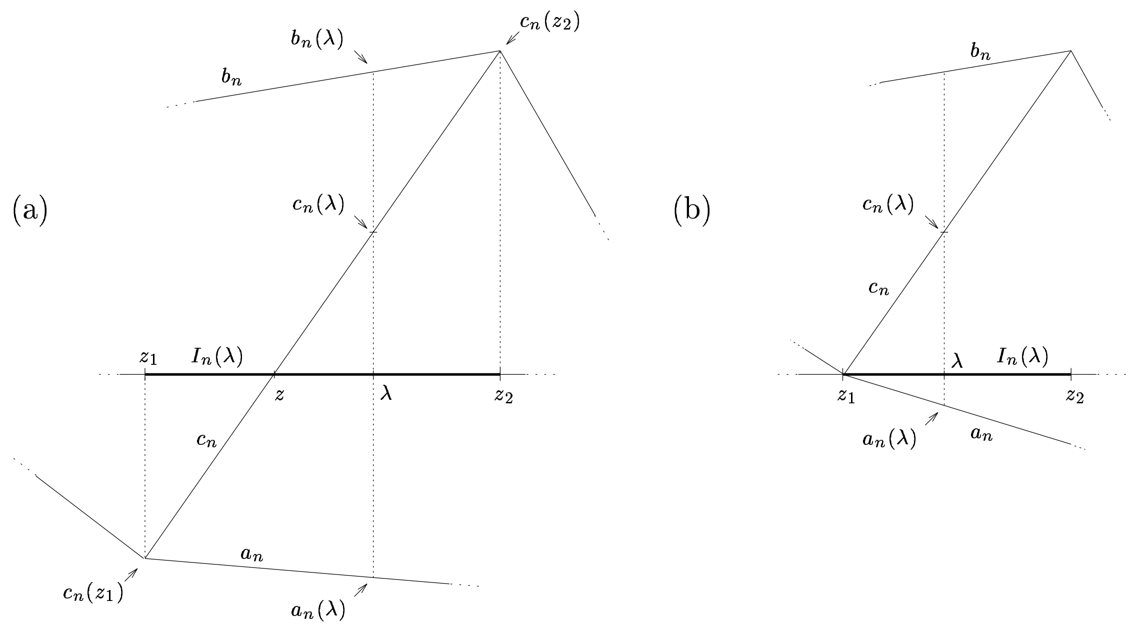


Figure 5.2: (a) When $\Diamond_\lambda(n) > 0$. (b) Here $\Diamond_\lambda(n) = 0$.

The difference between the first terms is

$$\log \frac{|a_n(\lambda) - c|}{|c_n(\lambda) - c|} - \log \frac{|c_n(z_1) - c|}{|c_n(\lambda) - c|} = \log \frac{|a_n(\lambda) - c|}{|c_n(z_1) - c|}.$$

Let us estimate

$$\begin{aligned} \left| \frac{|a_n(\lambda) - c|}{|c_n(z_1) - c|} - 1 \right| &\leq \frac{|a_n(\lambda) - c_n(z_1)|}{|c_n(z_1) - c|} \\ &= \frac{|a_n(\lambda) - c_n(z_1)|}{|c_n(\lambda) - c_n(z_1)|} \frac{|c_n(\lambda) - c_n(z_1)|}{|c_n(z_1) - c|}. \end{aligned}$$

The first term $|a_n(\lambda) - c_n(z_1)|/|c_n(\lambda) - c_n(z_1)|$ is the ratio of the slopes of a_n and c_n. Lemma 23 shows that this converges to zero as $n \to \infty$.

The second term $|c_n(\lambda) - c_n(z_1)|/|c_n(z_1) - c|$ is bounded: if $\Diamond_\lambda(n) > 1$ then $|c_n(\lambda) - c_n(z_1)|/|c_n(z_1) - c| < 1$ from the definition of $\Diamond_\lambda(n)$. If $\hat{\Diamond}_\lambda(n) > 1$ and n is large then $|c_n(\lambda) - c_n(z_1)|/|c_n(z_1) - c| < 2$ (otherwise the contradiction $\hat{\Diamond}_\lambda(n) < 1$ can be derived; we leave this to the reader).

Therefore, if n is large enough,

$$\left| \frac{|a_n(\lambda) - c|}{|c_n(z_1) - c|} - 1 \right| < 1/4$$

and so

$$\left| \log \frac{|a_n(\lambda) - c|}{|c_n(\lambda) - c|} - \log \frac{|c_n(z_1) - c|}{|c_n(\lambda) - c|} \right| < -\frac{1}{2}.$$

A similar calculation shows

$$\left| \log \frac{|b_n(\lambda) - c|}{|c_n(\lambda) - c|} - \log \frac{|c_n(z_2) - c|}{|c_n(\lambda) - c|} \right| < -\frac{1}{2}.$$

Therefore $|\hat{\Diamond}_\lambda(n) - \Diamond_\lambda(n)| < 1/2$. This proves the lemma. ■

From here on we drop the explicit dependence of $\hat{\Diamond}_\lambda(n)$ on λ.

Call T_λ *good* if T_λ is not periodic and

$$\lim_{l \to \infty} \limsup_{i \to \infty} \frac{\sum_{j=1}^{i} \begin{cases} \hat{\Diamond}(j) & \text{if } \hat{\Diamond}(j) \geq l \\ 0 & \text{otherwise} \end{cases}}{i} = 0.$$

The notation *good* is only used in this section.

Lemma 57 shows that theorem 58 below is equivalent to theorem 55. We will prove theorem 58. The proof of theorem 58 takes up the remainder of this section.

Theorem 58 *A tent-map has a slowly recurrent kneading invariant if and only if it is good.*

The idea is that $\hat{\Diamond}(n)$ and $\mathcal{R}(n)$ are comparable often enough to deduce the equivalence of goodness and slow recurrence. Recall that the kneading invariant of T_λ is slowly recurrent if

$$\lim_{l \to \infty} \limsup_{i \to \infty} \frac{\sum_{j=1}^{i} \begin{cases} \mathcal{R}(j) & \text{if } \mathcal{R}(j) \geq l \\ 0 & \text{otherwise} \end{cases}}{i} = 0.$$

We say that $\hat{\Diamond}(n)$ and $\mathcal{R}(n)$ are *comparable* if

$$\frac{1}{K} \leq \frac{\mathcal{R}(n)}{\hat{\Diamond}(n)} \leq K,$$

where $K > 1$ will be fixed independent of n. If $\hat{\Diamond}(n)$ and $\mathcal{R}(n)$ were always comparable, which they are not, the equivalence of goodness and slow recurrence would be immediate.

We start with the technical lemma on which our comparability estimates are based (compare lemma 39).

Lemma 59 *If T_λ is either good or has a slowly recurrent kneading invariant*[2] *then there exists $K > 1$ such that*

$$\frac{1}{K} \leq \frac{\mathcal{R}(n)}{-\log|c_n - c|} \leq K$$

for all $n \geq 1$.

Proof. We will prove that there exist $K_1 > 1$ and $K_2 > 1$ such that

$$K_1^{-\mathcal{S}_i} \leq |c_{-\mathcal{S}_i} - c| \leq K_2^{-\mathcal{S}_i} \tag{5.1}$$

for all $i \geq 1$. Let us show that the result follows. From lemma 12 we have $|c_{-\mathcal{R}(n)} - c| \leq |c_n - c| \leq |c_{-\mathcal{R}^-(n)} - c|$ and therefore, if equation 5.1 holds for all $i \geq 1$,

$$K_1^{-\mathcal{R}(n)} \leq |c_n - c| \leq K_{2.}^{\mathcal{R}^-(n)}$$

Lemma 15 and the definition of $\mathcal{R}^-$ show $\mathcal{R}(n) \leq 2\mathcal{R}^-(n)$ (all cutting times satisfy $\mathcal{S}_{i+1} \leq 2\mathcal{S}_i$). Using this and taking logarithms,

$$\frac{1}{\log K_1} \leq \frac{\mathcal{R}(n)}{-\log|c_n - c|} \leq \frac{2}{\log K_2}.$$

Taking $K > \max\{\log K_1, 2/\log K_2\}$ gives the lemma.

The existence of K_2 is easy: lemma 13 shows that $T_\lambda^{\mathcal{S}_i}\Big|_{[c_{-\mathcal{S}_i};c]}$ is a homeomorphism. As T_λ is a tent-map this means that $T_\lambda^{\mathcal{S}_i}\Big|_{[c_{-\mathcal{S}_i};c]}$ is linear with slope $\pm\lambda^{\mathcal{S}_i}$. But $T_\lambda^{\mathcal{S}_i}[c_{-\mathcal{S}_i}; c] \subseteq [0; 1]$ and so $|c_{-\mathcal{S}_i} - c| = |T_\lambda^{\mathcal{S}_i}[c_{-\mathcal{S}_i}; c]|/\lambda^{\mathcal{S}_i} \leq \lambda^{-\mathcal{S}_i}$ for all $i \geq 1$. We can therefore take $K_2 = \lambda$.

[2] Some such hypothesis is necessary. Using the techniques of chapter 3 it is possible to show that there are uncountably many tent-maps for which the conclusion of lemma 59 fails to hold.

The existence of K_1 is more difficult. The proof is divided into two steps.

Step 1. Define $D(\mathcal{S}_i) = \log(|c_{-\mathcal{S}_{i-1}} - c|/|c_{\mathcal{S}_i} - c|)$ if $\mathcal{R}(\mathcal{S}_i) = \mathcal{S}_i$ and otherwise define $D(\mathcal{S}_i) = 0$. Define $D(n) = 0$ if n is not a cutting time.

In this step we prove that if

$$\frac{\sum_{j=1}^{n} D(j)}{n}$$

is bounded independently of n then there exists $K_1 > 1$ such that $|c_{-\mathcal{S}_i} - c| \geq K_1^{-\mathcal{S}_i}$ for all $i \geq 1$, as required.

Let us estimate $|c_{-\mathcal{S}_i} - c|$ in terms of $|c_{-\mathcal{R}(\mathcal{S}_i)} - c|$. Since $T_\lambda^{\mathcal{S}_i}\big|_{[c_{-\mathcal{S}_i};c]}$ is a homeomorphism and therefore linear we have $|c_{-\mathcal{S}_i} - c| = |c_{\mathcal{S}_i} - c|/\lambda^{\mathcal{S}_i}$, as above. Lemma 12 gives $|c_i - c| \geq |c_{-\mathcal{R}(\mathcal{S}_i)} - c|$ and so

$$|c_{-\mathcal{S}_i} - c| \geq \frac{|c_{-\mathcal{R}(\mathcal{S}_i)} - c|}{\lambda^{\mathcal{S}_i}}. \tag{5.2}$$

Equation 5.2 is uninformative when $\mathcal{R}(\mathcal{S}_i) = \mathcal{S}_i$ since $|c_{-\mathcal{S}_i} - c|$ occurs on both sides. In this case we modify the equation to include the factor $D(\mathcal{S}_i)$: define $\tilde{\mathcal{R}}(\mathcal{S}_i) = \mathcal{S}_{i-1}$ if $\mathcal{R}(\mathcal{S}_i) = \mathcal{S}_i$ and $\tilde{\mathcal{R}}(\mathcal{S}_i) = \mathcal{R}(\mathcal{S}_i)$ if $\mathcal{R}(\mathcal{S}_i) < \mathcal{S}_i$, so $\tilde{\mathcal{R}}(\mathcal{S}_i) < \mathcal{S}_i$ for all $i > 1$. Then

$$|c_{-\mathcal{S}_i} - c| \geq \frac{|c_{-\tilde{\mathcal{R}}(\mathcal{S}_i)} - c|}{e^{\mathcal{S}_i \log\lambda + D(\mathcal{S}_i)}}. \tag{5.3}$$

This is the same estimate as that of equation 5.2 except when $\mathcal{R}(\mathcal{S}_i) = \mathcal{S}_i$, in which case the estimate is trivial.

Using equation 5.3 repeatedly,

$$\begin{aligned} |c_{-\mathcal{S}_i} - c| &\geq \frac{|c_{-\tilde{\mathcal{R}}(\mathcal{S}_i)} - c|}{e^{\mathcal{S}_i \log\lambda + D(\mathcal{S}_i)}} \\ &\geq \frac{|c_{-\tilde{\mathcal{R}}(\tilde{\mathcal{R}}(\mathcal{S}_i))} - c|}{e^{(\mathcal{S}_i + \tilde{\mathcal{R}}(\mathcal{S}_i))\log\lambda + D(\mathcal{S}_i) + D(\tilde{\mathcal{R}}(\mathcal{S}_i))}} \\ &\geq \dots \\ &\geq \frac{|c_{-1} - c|}{e^{\sum_{j=0}^{\mathcal{N}(\mathcal{S}_i)} (\tilde{\mathcal{R}}^j(\mathcal{S}_i)\log\lambda + D(\tilde{\mathcal{R}}^j(\mathcal{S}_i)))}} \end{aligned}$$

where $\mathcal{N}(\mathcal{S}_i) = \min\{j \geq 0 \mid \tilde{\mathcal{R}}^j(\mathcal{S}_i) = 1\}$.

Let us show that there exists $K_3 > 0$ such that

$$\sum_{j=0}^{\mathcal{N}(\mathcal{S}_i)} (\tilde{\mathcal{R}}^j(\mathcal{S}_i)\log\lambda + D(\tilde{\mathcal{R}}^j(\mathcal{S}_i))) \leq K_3 \mathcal{S}_i$$

for every $i \geq 1$. We can then take $K_1 = e^{K_3}$.

The hypothesis that $\sum_{j=1}^{n} D(j)/n$ is bounded shows that $\sum_{j=0}^{\mathcal{N}(\mathcal{S}_i)} D(\tilde{\mathcal{R}}^j(\mathcal{S}_i))/\mathcal{S}_i$ is bounded, so we can ignore the second term in the sum.

So it is enough to find $K_3 > 0$ such that $\sum_{j=0}^{\mathcal{N}(\mathcal{S}_i)} \tilde{\mathcal{R}}^j(\mathcal{S}_i) \leq K_3\mathcal{S}_i$ for all $i > 1$. Let us prove inductively that

$$\sum_{j=0}^{\mathcal{N}(\mathcal{S}_i)} \tilde{\mathcal{R}}^j(\mathcal{S}_i) \leq 2\mathcal{S}_{i+1} \tag{5.4}$$

holds for all $i \geq 1$. Since $\mathcal{S}_{i+1}$ never exceeds $2\mathcal{S}_i$, this trick gives $\sum_{j=0}^{\mathcal{N}(\mathcal{S}_i)} \tilde{\mathcal{R}}^j(\mathcal{S}_i) \leq 4\mathcal{S}_i$ and we take $K_3 = 4$.

Equation 5.4 holds if $i = 1$. Take $i > 1$ and write $\tilde{\mathcal{R}}(\mathcal{S}_i) = \mathcal{S}_k$. Since $k < i$ we can suppose $\sum_{j=0}^{\mathcal{N}(\mathcal{S}_k)} \tilde{\mathcal{R}}^j(\mathcal{S}_k) \leq 2\mathcal{S}_{k+1}$. Then

$$\begin{aligned} \sum_{j=0}^{\mathcal{N}(\mathcal{S}_i)} \tilde{\mathcal{R}}^j(\mathcal{S}_i) &= \mathcal{S}_i + \sum_{j=0}^{\mathcal{N}(\tilde{\mathcal{R}}(\mathcal{S}_i))} \tilde{\mathcal{R}}^j(\tilde{\mathcal{R}}(\mathcal{S}_i)) \\ &\leq \mathcal{S}_i + 2\mathcal{S}_{k+1}. \end{aligned}$$

We must show $\mathcal{S}_i + 2\mathcal{S}_{k+1} \leq 2\mathcal{S}_{i+1}$.

If $\mathcal{R}(\mathcal{S}_i) = \mathcal{S}_i$ then $\mathcal{S}_{k+1} = \mathcal{S}_i$, so we need to verify $3\mathcal{S}_i \leq 2\mathcal{S}_{i+1}$; but this is clear since $\mathcal{S}_{i+1} = \mathcal{S}_i + \mathcal{R}(\mathcal{S}_i) = 2\mathcal{S}_i$.

Otherwise $\mathcal{S}_k = \mathcal{R}(\mathcal{S}_i) = \mathcal{S}_{i+1} - \mathcal{S}_i$ and so $\mathcal{S}_i + 2\mathcal{S}_{k+1} = \mathcal{S}_i + 2\mathcal{S}_k + 2(\mathcal{S}_{k+1} - \mathcal{S}_k) = 2\mathcal{S}_{i+1} + 2(\mathcal{S}_{k+1} - \mathcal{S}_k) - \mathcal{S}_i$. The problem is therefore reduced to showing that $2(\mathcal{S}_{k+1} - \mathcal{S}_k) - \mathcal{S}_i$ is non-positive. Because $\mathcal{S}_{k+1}$ is no greater than $2\mathcal{S}_k$ we have $\mathcal{S}_{k+1} - 2\mathcal{S}_k \leq 0$ and it suffices to show that $\mathcal{S}_{k+1} - \mathcal{S}_i$ is non-negative. This is clear since k is less than i.

Therefore $\sum_{j=0}^{\mathcal{N}(\mathcal{S}_i)} \tilde{\mathcal{R}}^j(\mathcal{S}_i) \leq 4\mathcal{S}_i$ for all $i \geq 1$.

Unravelling the argument, we have shown that if

$$\frac{\sum_{j=1}^{n} D(j)}{n}$$

is bounded independently of n then there exists some $K_1 \geq 1$ such that $|c_{-\mathcal{S}_i} - c| \geq K_1^{-\mathcal{S}_i}$ for all $i \geq 1$.

Step 2. In this final step we show that

$$\frac{\sum_{j=1}^{n} D(j)}{n}$$

is bounded independently of n if T_λ is either good or has a slowly recurrent kneading invariant.

Combined with the last step this will complete the proof of the lemma.

Suppose T_λ has a slowly recurrent kneading invariant. Then $\limsup_{i\to\infty} \mathcal{R}(i)/i = 0$, so $\mathcal{R}(\mathcal{S}_i) = \mathcal{S}_i$ can only occur finitely often. This means that $D(n) = 0$ except for finitely many values of n, and so $\sum_{j=1}^n D(j)/n$ is bounded.

Suppose T_λ is good. Let us prove that if $D(n) > 0$ then $D(n) \le \hat{\Diamond}(n)$. We specify $D(n) > 0$ because $\hat{\Diamond}(n)$ may be negative when $D(n) = 0$. It can be shown from the definition of goodness that

$$\frac{\sum_{j=1}^n \begin{cases} \hat{\Diamond}(j) & \text{if } \hat{\Diamond}(j) > 0 \\ 0 & \text{otherwise} \end{cases}}{n}$$

is bounded, so $D(n) \le \hat{\Diamond}(n)$ will prove that $\sum_{j=1}^n D(j)/n$ is bounded.

If $D(n) > 0$ then $n = \mathcal{S}_i$ for some $i \ge 1$, $\mathcal{R}(\mathcal{S}_i) = \mathcal{S}_i$ and $D(n) = \log(|c_{-\mathcal{S}_{i-1}} - c|/|c_{\mathcal{S}_i} - c|)$.

We have

$$\hat{\Diamond}(n) = \log\min\{\frac{|c_{n-\mathcal{S}\langle n\rangle} - c|}{|c_n - c|}, \frac{|c_{n-\mathcal{T}\langle n\rangle} - c|}{|c_n - c|}\}.$$

Therefore $D(n) \le \hat{\Diamond}(n)$ if

$$\min\{|c_{n-\mathcal{S}\langle n\rangle} - c|, |c_{n-\mathcal{T}\langle n\rangle} - c|\} > |c_{-\mathcal{S}_{i-1}} - c|.$$

Lemma 12 shows that this is equivalent to $\mathcal{R}(n - \mathcal{S}\langle n\rangle) \le \mathcal{S}_{i-1}$ and $\mathcal{R}(n - \mathcal{T}\langle n\rangle) \le \mathcal{S}_{i-1}$.

Let us show $\mathcal{R}(n - \mathcal{S}\langle n\rangle) \le \mathcal{S}_{i-1}$. Since $n = \mathcal{S}_i$ we have $\mathcal{S}\langle n\rangle = \mathcal{S}_{i-1}$. Therefore $n - \mathcal{S}\langle n\rangle = \mathcal{R}(\mathcal{S}_{i-1})$ and $\mathcal{R}(n - \mathcal{S}\langle n\rangle) \le n - \mathcal{S}\langle n\rangle = \mathcal{R}(\mathcal{S}_{i-1}) \le \mathcal{S}_{i-1}$.

Let us show $\mathcal{R}(n - \mathcal{T}\langle n\rangle) \le \mathcal{S}_{i-1}$. Since $\mathcal{R}(\mathcal{S}_i) = \mathcal{S}_i$ we know from lemma 21 that $\mathcal{S}_i$ is a return time, so write $n = \mathcal{M}_j$. Then $\mathcal{T}\langle n\rangle = \mathcal{M}_{j-1}$ and $n - \mathcal{T}\langle n\rangle = \mathcal{R}^-(\mathcal{M}_{j-1})$. In lemma 17 we proved $\mathcal{R}(\mathcal{R}^-(\mathcal{M}_{j-1})) < \mathcal{R}(\mathcal{M}_j)$. With our definitions this gives $\mathcal{R}(n - \mathcal{T}\langle n\rangle) \le \mathcal{S}_{i-1}$.

Therefore $D(n) \le \hat{\Diamond}(n)$ whenever $D(n) > 0$, proving

$$\frac{\sum_{j=1}^n D(j)}{n}$$

is bounded independently of n if T_λ is good. ■

Corollary 60 *Every tent-map with a slowly recurrent kneading invariant is good.*

Proof. Suppose T_λ has a slowly recurrent kneading invariant. From lemma 59 there is a $K > 1$ such that $-\log|c_n - c| \leq K\mathcal{R}(n)$ for all $n \geq 1$. However $\hat{\Diamond}(n) \leq -\log|c_n - c|$ is immediate from the definition of $\hat{\Diamond}(n)$. Therefore $\hat{\Diamond}(n) \leq K\mathcal{R}(n)$ for all $n \geq 1$. This is enough to conclude that T_λ is good. ■

It remains to be shown that every good tent-map has a slowly recurrent kneading invariant. We will use the following technical lemma to identify times when $\mathcal{R}(n)$ and $\hat{\Diamond}(n)$ are comparable.

Lemma 61 *If T_λ is good then there exists $K > 1$ and $l > 1$ such that*

$$\frac{1}{K} \leq \frac{\mathcal{R}(\mathcal{M}_n)}{\hat{\Diamond}(\mathcal{M}_n)} \leq K$$

whenever

1. $\mathcal{R}(\mathcal{M}_n) > \mathcal{M}_n - \mathcal{S}\langle\mathcal{M}_n\rangle$
   or both $\mathcal{R}(\mathcal{M}_n) \geq (\mathcal{M}_n - \mathcal{S}\langle\mathcal{M}_n\rangle)/4$ *and* $\mathcal{M}_n - \mathcal{S}\langle\mathcal{M}_n\rangle \geq l$

and

2. $\mathcal{R}(\mathcal{M}_n) > \mathcal{M}_n - \mathcal{T}\langle\mathcal{M}_n\rangle$
   or both $\mathcal{R}(\mathcal{M}_n) \geq (\mathcal{M}_n - \mathcal{T}\langle\mathcal{M}_n\rangle)/4$ *and* $\mathcal{M}_n - \mathcal{T}\langle\mathcal{M}_n\rangle \geq l$.

Proof. In the proof of corollary 60 we showed that $\hat{\Diamond}(\mathcal{M}_n) \leq K_1\mathcal{R}(\mathcal{M}_n)$ holds for all $n \geq 1$, where $K_1 > 1$ is the constant given by lemma 59. Therefore we only need to find $K > 1$ such that $\mathcal{R}(\mathcal{M}_n) \leq K\hat{\Diamond}(\mathcal{M}_n)$ holds, or equivalently that

$$\mathcal{R}(\mathcal{M}_n) \leq K \log \frac{|c_{\mathcal{M}_n - \mathcal{S}\langle\mathcal{M}_n\rangle} - c|}{|c_{\mathcal{M}_n} - c|} \tag{5.5}$$

and

$$\mathcal{R}(\mathcal{M}_n) \leq K \log \frac{|c_{\mathcal{M}_n - \mathcal{T}\langle\mathcal{M}_n\rangle} - c|}{|c_{\mathcal{M}_n} - c|} \tag{5.6}$$

hold, whenever the hypotheses of the lemma are satisfied.

Let us first show that equations 5.5 and 5.6 hold if $\mathcal{R}(\mathcal{M}_n) > \mathcal{M}_n - \mathcal{S}\langle\mathcal{M}_n\rangle$ and $\mathcal{R}(\mathcal{M}_n) > \mathcal{M}_n - \mathcal{T}\langle\mathcal{M}_n\rangle$. We will use this fact below when proving the general case.

From lemma 59 we have

$$\frac{1}{K_1} \leq \frac{\mathcal{R}(\mathcal{M}_n)}{-\log|c_{\mathcal{M}_n} - c|} \leq K_1$$

and

$$\frac{1}{K_1} \leq \frac{\mathcal{R}(\mathcal{M}_n - \mathcal{S}\langle\mathcal{M}_n\rangle)}{-\log|c_{\mathcal{M}_n - \mathcal{S}\langle\mathcal{M}_n\rangle} - c|} \leq K_1.$$

Combining the two equations, a simple calculation gives

$$\mathcal{R}(\mathcal{M}_n) \leq K_1 \log \frac{|c_{\mathcal{M}_n - \mathcal{S}\langle\mathcal{M}_n\rangle} - c|}{|c_{\mathcal{M}_n} - c|} + K_1^2 \mathcal{R}(\mathcal{M}_n - \mathcal{S}\langle\mathcal{M}_n\rangle). \tag{5.7}$$

To estimate the last term we use the fact that $\mathcal{M}_n - \mathcal{S}\langle\mathcal{M}_n\rangle$ is always a cutting time. Indeed, if $\mathcal{M}_n$ is a cutting time then $\mathcal{S}\langle\mathcal{M}_n\rangle$ is the preceding cutting time and therefore $\mathcal{M}_n - \mathcal{S}\langle\mathcal{M}_n\rangle = \mathcal{R}(\mathcal{S}\langle\mathcal{M}_n\rangle)$, a cutting time. If $\mathcal{M}_n$ is a co-cutting time then $\mathcal{S}\langle\mathcal{M}_n\rangle = \mathcal{M}_{n-1}$ and so $\mathcal{M}_n - \mathcal{S}\langle\mathcal{M}_n\rangle = \mathcal{R}^-(\mathcal{S}\langle\mathcal{M}_n\rangle)$, a cutting time.

Therefore we can write $\mathcal{M}_n - \mathcal{S}\langle\mathcal{M}_n\rangle = \mathcal{S}_i$ and using lemma 15 rewrite equation 5.7 as

$$\mathcal{R}(\mathcal{M}_n) \leq K_1 \log \frac{|c_{\mathcal{M}_n - \mathcal{S}\langle\mathcal{M}_n\rangle} - c|}{|c_{\mathcal{M}_n} - c|} + K_1^2 \mathcal{S}_i. \tag{5.8}$$

To estimate $\mathcal{S}_i$ we use the hypothesis $\mathcal{R}(\mathcal{M}_n) > \mathcal{S}_i$. From lemma 12 this implies $c_{\mathcal{M}_n} \in (c_{-\mathcal{S}_i}; \tau(c_{-\mathcal{S}_i}))$ and so, since T_λ is symmetric, $|c_{\mathcal{M}_n} - c| < |c_{-\mathcal{S}_i} - c|$. We showed $|c_{-\mathcal{S}_i} - c| = |c_{\mathcal{S}_i} - c|/\lambda^{\mathcal{S}_i}$ in the proof of lemma 59, near the start. Therefore $|c_{\mathcal{M}_n} - c| < |c_{\mathcal{S}_i} - c|/\lambda^{\mathcal{S}_i}$ which we write as

$$\mathcal{S}_i \log\lambda < \log \frac{|c_{\mathcal{M}_n - \mathcal{S}\langle\mathcal{M}_n\rangle} - c|}{|c_{\mathcal{M}_n} - c|}. \tag{5.9}$$

Combining equations 5.8 and 5.9,

$$\mathcal{R}(\mathcal{M}_n) \leq K_1 \left(1 + \frac{K_1}{\log\lambda}\right) \log \frac{|c_{\mathcal{M}_n - \mathcal{S}\langle\mathcal{M}_n\rangle} - c|}{|c_{\mathcal{M}_n} - c|}. \tag{5.10}$$

Putting $K = K_1(1 + \frac{K_1}{\log\lambda})$, this is equation 5.5.

A similar argument shows

$$\mathcal{R}(\mathcal{M}_n) \leq K_1 \left(1 + \frac{K_1}{\log \lambda}\right) \log \frac{|c_{\mathcal{M}_n - \mathcal{T}\langle \mathcal{M}_n \rangle} - c|}{|c_{\mathcal{M}_n} - c|},$$

which is equation 5.6, again with $K = K_1(1 + \frac{K_1}{\log \lambda})$.

This shows that $\mathcal{R}(\mathcal{M}_n) \leq K\hat{\lozenge}(\mathcal{M}_n)$ holds for some $K > 1$ independent of n whenever $\mathcal{R}(\mathcal{M}_n) > \mathcal{M}_n - \mathcal{S}\langle \mathcal{M}_n \rangle$ and $\mathcal{R}(\mathcal{M}_n) > \mathcal{M}_n - \mathcal{T}\langle \mathcal{M}_n \rangle$.

Now let us use this fact to show that if T_λ is good then $\limsup_{i\to\infty} \mathcal{R}(i)/i = 0$.

It is clear from the definition of goodness that $\limsup_{i\to\infty} \hat{\lozenge}(i)/i = 0$. To prove $\limsup_{i\to\infty} \mathcal{R}(i)/i = 0$ it is enough to prove the existence of a $K > 1$ such that $\mathcal{R}(i) \leq K\hat{\lozenge}(i)$ whenever i is a closest return time, closest return times corresponding to the largest values of $\mathcal{R}$. From lemma 20 we know that closest return times are return times, so we can write $i = \mathcal{M}_j$. If we can show $\mathcal{R}(\mathcal{M}_j) > \mathcal{M}_j - \mathcal{S}\langle \mathcal{M}_j \rangle$ and $\mathcal{R}(\mathcal{M}_j) > \mathcal{M}_j - \mathcal{T}\langle \mathcal{M}_j \rangle$ then what we proved above will give us $\mathcal{R}(\mathcal{M}_j) \leq K\hat{\lozenge}(\mathcal{M}_j)$ as required.

Let us show $\mathcal{R}(\mathcal{M}_j) > \mathcal{M}_j - \mathcal{S}\langle \mathcal{M}_j \rangle$. We know $\mathcal{M}_j - \mathcal{S}\langle \mathcal{M}_j \rangle = \mathcal{R}(\mathcal{S}\langle \mathcal{M}_j \rangle)$ if $\mathcal{M}_j$ is a cutting time and $\mathcal{M}_j - \mathcal{S}\langle \mathcal{M}_j \rangle = \mathcal{R}^-(\mathcal{S}\langle \mathcal{M}_j \rangle) < \mathcal{R}(\mathcal{S}\langle \mathcal{M}_j \rangle)$ if $\mathcal{M}_j$ is a co-cutting time. Since $\mathcal{M}_j$ is a closest return time, $\mathcal{R}(\mathcal{S}\langle \mathcal{M}_j \rangle) < \mathcal{R}(\mathcal{M}_j)$ and thus $\mathcal{R}(\mathcal{M}_j) > \mathcal{M}_j - \mathcal{S}\langle \mathcal{M}_j \rangle$.

A similar argument shows $\mathcal{R}(\mathcal{M}_j) > \mathcal{M}_j - \mathcal{T}\langle \mathcal{M}_j \rangle$. Therefore $\mathcal{R}(i) \leq K\hat{\lozenge}(i)$ whenever i is a closest return time, proving that $\limsup_{i\to\infty} \mathcal{R}(i)/i = 0$.

Finally, suppose we have

1. $\mathcal{R}(\mathcal{M}_n) > \mathcal{M}_n - \mathcal{S}\langle \mathcal{M}_n \rangle$
or both $\mathcal{R}(\mathcal{M}_n) \geq (\mathcal{M}_n - \mathcal{S}\langle \mathcal{M}_n \rangle)/4$ and $\mathcal{M}_n - \mathcal{S}\langle \mathcal{M}_n \rangle \geq l$

and

2. $\mathcal{R}(\mathcal{M}_n) > \mathcal{M}_n - \mathcal{T}\langle \mathcal{M}_n \rangle$
or both $\mathcal{R}(\mathcal{M}_n) \geq (\mathcal{M}_n - \mathcal{T}\langle \mathcal{M}_n \rangle)/4$ and $\mathcal{M}_n - \mathcal{T}\langle \mathcal{M}_n \rangle \geq l$.

Since T_λ is good we have $\limsup_{i\to\infty} \mathcal{R}(i)/i = 0$. We require l to be large enough that $\mathcal{R}(i)/i < 1/(8K_1^2)$ for all $i \geq l$.

Reconsider equation 5.7:

$$\mathcal{R}(\mathcal{M}_n) \leq K_1 \log \frac{|c_{\mathcal{M}_n - \mathcal{S}\langle \mathcal{M}_n \rangle} - c|}{|c_{\mathcal{M}_n} - c|} + K_1^2 \mathcal{R}(\mathcal{M}_n - \mathcal{S}\langle \mathcal{M}_n \rangle). \tag{5.11}$$

If $\mathcal{R}(\mathcal{M}_n) > \mathcal{M}_n - \mathcal{S}\langle \mathcal{M}_n \rangle$ then we proceed as before, obtaining equation 5.10.

If $\mathcal{R}(\mathcal{M}_n) \geq (\mathcal{M}_n - \mathcal{S}\langle \mathcal{M}_n \rangle)/4$ and $\mathcal{M}_n - \mathcal{S}\langle \mathcal{M}_n \rangle \geq l$ then $\mathcal{R}(\mathcal{M}_n - \mathcal{S}\langle \mathcal{M}_n \rangle) < (\mathcal{M}_n - \mathcal{S}\langle \mathcal{M}_n \rangle)/(8K_1^2) < \mathcal{R}(\mathcal{M}_n)/(2K_1^2)$ using the choice of l. Substituting back into equation 5.11 gives

$$\mathcal{R}(\mathcal{M}_n) < 2K_1 \log \frac{|c_{\mathcal{M}_n - \mathcal{S}\langle \mathcal{M}_n \rangle} - c|}{|c_{\mathcal{M}_n} - c|}. \tag{5.12}$$

Therefore, since we have either equation 5.10 or equation 5.12,

$$\mathcal{R}(\mathcal{M}_n) \leq \max\{2K_1, K_1 \left(1 + \frac{K_1}{\log \lambda}\right)\} \log \frac{|c_{\mathcal{M}_n - \mathcal{S}\langle \mathcal{M}_n \rangle} - c|}{|c_{\mathcal{M}_n} - c|}.$$

A similar argument shows

$$\mathcal{R}(\mathcal{M}_n) \leq \max\{2K_1, K_1 \left(1 + \frac{K_1}{\log \lambda}\right)\} \log \frac{|c_{\mathcal{M}_n - \mathcal{T}\langle \mathcal{M}_n \rangle} - c|}{|c_{\mathcal{M}_n} - c|}.$$

Taking $K = \max\{2K_1, K_1(1 + \frac{K_1}{\log \lambda})\}$ we therefore have $\mathcal{R}(\mathcal{M}_n) \leq K\hat{\Diamond}(\mathcal{M}_n)$. This completes the proof. ■

So let us complete the proof of theorem 58.

Lemma 62 *Every good tent-map has a slowly recurrent kneading invariant.*

Proof. This means showing that

$$\lim_{k\to\infty} \limsup_{i\to\infty} \frac{\sum_{j=1}^{i} \begin{cases} \hat{\Diamond}(j) & \text{if } \hat{\Diamond}(j) \geq k \\ 0 & \text{otherwise} \end{cases}}{i} = 0 \tag{5.13}$$

implies

$$\lim_{k\to\infty} \limsup_{i\to\infty} \frac{\sum_{j=1}^{i} \begin{cases} \mathcal{R}(j) & \text{if } \mathcal{R}(j) \geq k \\ 0 & \text{otherwise} \end{cases}}{i} = 0$$

or equivalently, from lemma 22,

$$\lim_{k\to\infty}\limsup_{i\to\infty}\frac{\sum_{j=1}^{i}\begin{cases}\mathcal{R}(\mathcal{M}_j) & \text{if } \mathcal{R}(\mathcal{M}_j)\geq k\\ 0 & \text{otherwise}\end{cases}}{\mathcal{M}_i}=0. \tag{5.14}$$

We will show equation 5.14.

Take $K>1$ and $l>1$ as given by lemma 61. Let us first show that

$$\frac{1}{K}\leq\frac{\mathcal{R}(\mathcal{M}_n)}{\mathcal{M}_n}\leq K$$

holds whenever $n>2$ and

(1) $\mathcal{R}(\mathcal{M}_n)\geq l$

(2) $\mathcal{R}(\mathcal{M}_n)\geq\mathcal{R}(\mathcal{M}_{n-1})/4$

and

(3) $\mathcal{R}(\mathcal{M}_n)\geq\mathcal{R}(\mathcal{M}_{n-2})/4$.

We do this by verifying the hypotheses of lemma 61. So take $n>2$ for which $\mathcal{M}_n$ satisfies (1)–(3) above and suppose $\mathcal{M}_n$ is a cutting time. The method works equally well if $\mathcal{M}_n$ is a co-cutting time.

Hypothesis 2 of lemma 61 is that either $\mathcal{R}(\mathcal{M}_n)>\mathcal{M}_n-\mathcal{T}\langle\mathcal{M}_n\rangle$ or both $\mathcal{R}(\mathcal{M}_n)\geq(\mathcal{M}_n-\mathcal{T}\langle\mathcal{M}_n\rangle)/4$ and $\mathcal{M}_n-\mathcal{T}\langle\mathcal{M}_n\rangle\geq l$.

Let us verify the hypothesis. Because $\mathcal{M}_n$ is a cutting time we have $\mathcal{T}\langle\mathcal{M}_n\rangle=\mathcal{M}_{n-1}$ and so $\mathcal{M}_n-\mathcal{T}\langle\mathcal{M}_n\rangle=\mathcal{R}^-(\mathcal{M}_{n-1})$. If $\mathcal{R}^-(\mathcal{M}_{n-1})<l$ then $\mathcal{R}(\mathcal{M}_n)>\mathcal{M}_n-\mathcal{T}\langle\mathcal{M}_n\rangle$, from (1). Otherwise $\mathcal{M}_n-\mathcal{T}\langle\mathcal{M}_n\rangle\geq l$ and $\mathcal{R}(\mathcal{M}_n)\geq(\mathcal{M}_n-\mathcal{T}\langle\mathcal{M}_n\rangle)/4$ follows from (2) since $\mathcal{R}(\mathcal{M}_{n-1})>\mathcal{R}^-(\mathcal{M}_{n-1})=\mathcal{M}_n-\mathcal{T}\langle\mathcal{M}_n\rangle$. Therefore hypothesis 2 holds in either case.

Hypothesis 1 of lemma 61 is that either $\mathcal{R}(\mathcal{M}_n)>\mathcal{M}_n-\mathcal{S}\langle\mathcal{M}_n\rangle$ or both $\mathcal{R}(\mathcal{M}_n)\geq(\mathcal{M}_n-\mathcal{S}\langle\mathcal{M}_n\rangle)/4$ and $\mathcal{M}_n-\mathcal{S}\langle\mathcal{M}_n\rangle\geq l$.

Let us verify the hypothesis. The argument is not quite the same as for hypothesis 2. We know that $\mathcal{M}_n-\mathcal{S}\langle\mathcal{M}_n\rangle=\mathcal{R}(\mathcal{S}\langle\mathcal{M}_n\rangle)$ since $\mathcal{M}_n$ is a cutting time. As before, if $\mathcal{R}(\mathcal{S}\langle\mathcal{M}_n\rangle)<l$ then (1) gives $\mathcal{M}_n-\mathcal{S}\langle\mathcal{M}_n\rangle=\mathcal{R}(\mathcal{S}\langle\mathcal{M}_n\rangle)<\mathcal{R}(\mathcal{M}_n)$. The other possibility is $\mathcal{M}_n-\mathcal{S}\langle\mathcal{M}_n\rangle\geq l$ and we need to show $\mathcal{R}(\mathcal{M}_n)\geq(\mathcal{M}_n-\mathcal{S}\langle\mathcal{M}_n\rangle)/4$, which is the same as $\mathcal{R}(\mathcal{M}_n) \geq \mathcal{R}(\mathcal{S}\langle\mathcal{M}_n\rangle)$. This splits into two cases according to whether $\mathcal{S}\langle\mathcal{M}_n\rangle = \mathcal{M}_{n-2}$ or $\mathcal{M}_{n-1} < \mathcal{S}\langle\mathcal{M}_n\rangle < \mathcal{M}_{n-1} + \mathcal{R}^-(\mathcal{M}_{n-1})$.

In the first case $\mathcal{R}(\mathcal{M}_n) \geq \mathcal{R}(\mathcal{S}\langle\mathcal{M}_n\rangle)/4$ because of (3). Otherwise $\mathcal{R}(\mathcal{S}\langle\mathcal{M}_n\rangle) < \mathcal{R}(\mathcal{M}_{n-1})$ from lemma 19 and so $\mathcal{R}(\mathcal{M}_n) \geq \mathcal{R}(\mathcal{S}\langle\mathcal{M}_n\rangle)/4$ from (2). Therefore hypothesis 1 holds in all cases.

Applying lemma 61 therefore shows that

$$\frac{1}{K} \leq \frac{\mathcal{R}(\mathcal{M}_n)}{\mathcal{M}_n} \leq K$$

whenever we have (1)–(3) for some $n > 2$.

Denote by E the set of $n > 2$ for which $\mathcal{M}_n$ satisfies (2) and (3), and add $n = 1$ and $n = 2$ to simplify matters later:

$$E = \{1, 2\} \cup \{n > 2 \mid \mathcal{R}(\mathcal{M}_n) \geq \mathcal{R}(\mathcal{M}_{n-1})/4 \text{ and } \mathcal{R}(\mathcal{M}_n) \geq \mathcal{R}(\mathcal{M}_{n-2})/4\}.$$

From the above argument and equation 5.13 we have

$$\lim_{k\to\infty} \limsup_{i\to\infty} \frac{\sum_{j\in E \text{ and } j\leq i} \begin{cases} \mathcal{R}(\mathcal{M}_j) & \text{if } \mathcal{R}(\mathcal{M}_j) \geq k \\ 0 & \text{otherwise} \end{cases}}{\mathcal{M}_i} = 0. \tag{5.15}$$

We claim that

$$\lim_{k\to\infty} \limsup_{i\to\infty} \frac{\sum_{j=1}^{i} \begin{cases} \mathcal{R}(\mathcal{M}_j) & \text{if } \mathcal{R}(\mathcal{M}_j) \geq k \\ 0 & \text{otherwise} \end{cases}}{\mathcal{M}_i} = 0 \tag{5.16}$$

follows from equation (5.15) and the definition of E.

Fix $k \geq l$ and define $r_j = \mathcal{R}(\mathcal{M}_j)$ if $\mathcal{R}(\mathcal{M}_j) \geq k$ and $r_j = 0$ otherwise. With this notation

$$\sum_{j=1}^{i} \begin{cases} \mathcal{R}(\mathcal{M}_j) & \text{if } \mathcal{R}(\mathcal{M}_j) \geq k \\ 0 & \text{otherwise} \end{cases} = \sum_{j=1}^{i} r_j.$$

If $j \notin E$ then, from the definition of E, either $r_j < r_{j-1}/4$ or $r_j < r_{j-2}/4$. Putting

$$q(j) = \begin{cases} j-1 & \text{if } j \notin E \text{ and } r_j < r_{j-1}/4 \\ j-2 & \text{if } j \notin E \text{ and } r_j \geq r_{j-1}/4.^3 \\ j & \text{if } j \in E \end{cases}$$

we have $r_j < r_{q(j)}/4$ for all $j \notin E$. Defining $Q(j) = \{i \geq 1 \mid q^n(i) = j \text{ for some } n \geq 0\}$ it is clear that $\{1, 2, 3, \ldots\} = \bigcup_{j\in E} Q(j)$, a disjoint union.

[3] In this case $r_j < r_{j-2}/4$.

Let us prove that

$$\sum_{j=1}^{i} r_j \leq 2 \sum_{j \in E \text{ and } j \leq i} r_j$$

for all $i \geq 1$.

Take j in E and $m \in Q(j)$. Then $q^n(m) = j$ for some $n \geq 0$ and we can suppose n is minimal. Clearly $r_m \leq r_j/4^n$. Since q never decreases by more than 2 we have $n \geq (m-j)/2$. Therefore $r_m \leq r_j/2^{m-j}$ and so

$$\sum_{m \in Q(j)} r_m \leq r_j \sum_{m=j}^{\infty} \frac{1}{2^{m-j}} = 2r_j.$$

Thus

$$\sum_{j=1}^{i} r_j = \sum_{j \in E \text{ and } j \leq i} \; \sum_{m \in Q(j)} r_m \leq 2 \sum_{j \in E \text{ and } j \leq i} r_j$$

for all $i \geq 1$. In other words,

$$\sum_{j=1}^{i} \begin{cases} \mathcal{R}(\mathcal{M}_j) & \text{if } \mathcal{R}(\mathcal{M}_j) \geq k \\ 0 & \text{otherwise} \end{cases} \leq 2 \sum_{j \in E \text{ and } j \leq i} \begin{cases} \mathcal{R}(\mathcal{M}_j) & \text{if } \mathcal{R}(\mathcal{M}_j) \geq k \\ 0 & \text{otherwise.} \end{cases}$$

Therefore equation 5.16 follows from equation 5.15 and T_λ has a slowly recurrent kneading invariant. ■

5.2 Measure Estimates

In this section we prove theorem 56:

Theorem 56 (Tsujii [49]) *We have that T_λ is not periodic and*

$$\lim_{l \to \infty} \limsup_{i \to \infty} \frac{\sum_{j=1}^{i} \begin{cases} \Diamond_\lambda(j) & \textit{if } \Diamond_\lambda(j) \geq l \\ 0 & \textit{otherwise} \end{cases}}{i} = 0$$

for Lebesgue almost every $1 \leq \lambda \leq 2$.

First we give some definitions.

We use P_n to denote the set consisting of 1, 2 and the λ values corresponding to cusp points of c_n. The smooth segment $I_n(\lambda)$ of c_n containing λ therefore equals $[z_1; z_2]$ where $z_1 = \max\{z \in P_n \mid z < \lambda\}$ and $z_2 = \min\{z \in P_n \mid z > \lambda\}$.

Write Δ_n for $P_{n+1} \setminus P_n$, so each $z \in \Delta_n$ has $c_n(z) = c$. Recall that $\Diamond_\lambda(n) = 0$ if $\lambda \in P_n$ or if $\Delta_n \cap I_n(\lambda)$ is empty. If $\lambda \notin P_n$ and there is some $z \in \Delta_n \cap I_n(\lambda)$ then we defined

$$\Diamond_\lambda(n) = \log \min\{\frac{|z_1 - z|}{|\lambda - z|}, \frac{|z_2 - z|}{|\lambda - z|}\}$$

if $\lambda \neq z$, where $[z_1; z_2] = I_n(\lambda)$. The point z is unique. We define $\Diamond_z(n) = \infty$ for convenience.[4]

Put

$$E_\delta = \{\lambda \in [1; 2] \mid \lim_{l \to \infty} \limsup_{i \to \infty} \frac{\sum_{j=1}^{i} \begin{cases} \Diamond_\lambda(j) & \text{if } \Diamond_\lambda(j) \geq l \\ 0 & \text{otherwise} \end{cases}}{i} > \delta\}.$$

This set contains every $\lambda > 1$ for which T_λ is periodic.

We use $|\ |$ to denote Lebesgue measure. To prove theorem 56 it is clearly enough to prove $|E_\delta| = 0$ for each $\delta > 0$.

To prove $|E_\delta| = 0$ we use the following result. Define $\limsup_{i \to \infty} A_i = \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} A_j$ for $\{A_\imath\}_{\imath \geq 1}$ a sequence of sets.

Lemma 63 (Borel-Cantelli [34]) *If $\{A_\imath\}_{\imath \geq 1}$ is a sequence of Lebesgue measurable subsets of an interval and*

$$\sum_{i=1}^{\infty} |A_i| < \infty$$

then

$$|\limsup_{i \to \infty} A_i| = 0.$$

Proof. Clearly $|\limsup_{i \to \infty} A_i| = \lim_{i \to \infty} |\bigcup_{j=i}^{\infty} A_i| \leq \lim_{i \to \infty} \sum_{j=i}^{\infty} |A_i|$. As $\sum_{i=1}^{\infty} |A_i|$ is a convergent series, this last term is equal to zero. ■

Fix $\delta > 0$ small. First we express E_δ in such a way that lemma 63 is directly applicable to showing $|E_\delta| = 0$.

[4] This differs from earlier in this chapter where $\Diamond_\lambda(n)$ was left undefined for periodic T_λ.

Define Ω_n^l to be the set of all $(m_1, \ldots, m_s; i_1, \ldots, i_s) \in \bigcup_{s=1}^{\infty} \mathbb{N}^{2s}$ such that

1. $m_1 \geq l, \ldots, m_s \geq l$,
2. $1 \leq i_1 < i_2 < \ldots < i_s \leq n$, and
3. $\sum_{j=1}^{s} m_j = [\delta n]$.

By $[x]$ we mean the greatest integer less than or equal to x.

Put $Z_i^m = \{\lambda \in [1;2] \mid \Diamond_\lambda(i) \geq m\}$ and $A_n^l = \bigcup_{(m_1,\ldots,i_s)\in\Omega_n^l} \bigcap_{j=1}^{s} Z_{i_j}^{m_j}$.

Note that $\Diamond_\lambda(i_j) \geq m_j$ for each $\lambda \in \bigcap_{j=1}^{s} Z_{i_j}^{m_j}$ and $1 \leq j \leq s$.

Lemma 64 *We have*[5]

$$E_\delta \subseteq \limsup_{l\to\infty} \limsup_{n\to\infty} A_n^l.$$

Proof. Take any $\lambda \in E_\delta$ and choose a $\delta_1 > \delta$ such that

$$\lim_{l\to\infty} \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} \begin{cases} \Diamond_\lambda(i) & \text{if } \Diamond_\lambda(i) \geq l \\ 0 & \text{otherwise} \end{cases}}{n} > \delta_1.$$

Then there are l arbitrarily large with

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} \begin{cases} \Diamond_\lambda(i) & \text{if } \Diamond_\lambda(i) \geq l \\ 0 & \text{otherwise} \end{cases}}{n} > \delta_1,$$

and for each such l there are n arbitrarily large such that

$$\frac{\sum_{i=1}^{n} \begin{cases} \Diamond_\lambda(i) & \text{if } \Diamond_\lambda(i) \geq l \\ 0 & \text{otherwise} \end{cases}}{n} > \delta_1.$$

[5] An inequality because $\limsup_{l\to\infty} \limsup_{n\to\infty} A_n^l$ may contain points for which

$$\lim_{l\to\infty} \limsup_{i\to\infty} \frac{\sum_{j=1}^{i} \begin{cases} \Diamond_\lambda(j) & \text{if } \Diamond_\lambda(j) \geq l \\ 0 & \text{otherwise} \end{cases}}{i} = \delta.$$

Take such an n, denote those $i \leq n$ which satisfy $\Diamond_\lambda(i) \geq l$ by $i_1, i_2, \ldots, i_s$ and put $m_j = [\Diamond_\lambda(i_j)]$ for each $1 \leq j \leq s$. Because $\sum_{j=1}^{s} \Diamond_\lambda(i_j) > n\delta_1$ and each $\Diamond_\lambda(i_j) \geq l$, we can take l large enough that $\sum_{j=1}^{s} m_j > \delta n \geq [\delta n]$. This is the case if $l \geq \delta_1/(\delta_1 - \delta)$ for example.

We claim that there exists $\{i'_1, \ldots, i'_{s'}\} \subseteq \{i_1, \ldots, i_s\}$ and integers $m'_1 \geq [l/2], \ldots, m'_{s'} \geq [l/2]$ with the following properties: if $i'_{j'} = i_j$ then $m'_{j'} \leq m_j$, and $\sum_{j=1}^{s} m'_j = [\delta n]$. The simple proof is left to the reader.

These two properties give $\lambda \in \bigcap_{j=1}^{s'} Z_{i'_j}^{m'_j}$ and $(m'_1, \ldots, m'_{s'}; i'_1, \ldots, i'_{s'}) \in \Omega_n^{[l/2]}$ respectively; therefore $\lambda \in A_n^{[l/2]}$ and, n being arbitrarily large, $\lambda \in \limsup_{n\to\infty} A_n^{[l/2]}$. Since l is arbitrarily large we have $\lambda \in \limsup_{l\to\infty} \limsup_{n\to\infty} A_n^l$.

Therefore $E_\delta \subseteq \limsup_{l\to\infty} \limsup_{n\to\infty} A_n^l$. ■

So if we can show $\sum_{n=1}^{\infty} |A_n^l| < \infty$ then we can conclude $|E_\delta| = 0$ from lemma 63.

We use the next two lemmas to estimate $|A_n^l|$. Lemma 65 below counts the number of elements of Ω_n^l. We use # to indicate cardinality.

Lemma 65 *There exists $N \geq 1$ such that*

$$\#(\Omega_n^l) \leq e^{[\delta n]/4}$$

for all $n \geq 1$ whenever $l \geq N$.

Proof. Define the projection π of Ω_n^l onto $\bigcup_{s=1}^{\infty} \mathbb{N}^s$ by $\pi(m_1, \ldots, m_s; i_1, \ldots, i_s) = (i_1, \ldots, i_s)$, put $\Omega = \pi(\Omega_n^l)$ and $\Omega_{i_1,\ldots,i_s} = \pi^{-1}(i_1, \ldots, i_s)$. This decomposes Ω_n^l as $\bigcup_{(i_1,\ldots,i_s)\in\Omega} \Omega_{i_1,\ldots,i_s}$ and therefore

$$\#(\Omega_n^l) \leq \#(\Omega_{i_1,\ldots,i_s})\, \#(\Omega).$$

Let us show

$$\#(\Omega_{i_1,\ldots,i_s}) \leq \mathrm{C}_{[\delta n/l].}^{[\delta n]+[\delta n/l]} \tag{5.17}$$

By C_b^a we mean $\frac{a!}{b!(a-b)!}$.

Think of $\mathrm{C}_{[\delta n/l]}^{[\delta n]+[\delta n/l]}$ as representing the number of ways of placing $[\delta n/l]$ black pebbles and $[\delta n]$ white pebbles in a line, in any order. If we can associate a distinct such pebble sequence to each $(m_1, \ldots, m_s; i_1, \ldots, i_s) \in \Omega_{i_1,\ldots,i_s}$ then we will have shown equation 5.17. Note that the values of $i_1, \ldots, i_s$ are fixed.

So take $(m_1, \ldots, m_s; i_1, \ldots, i_s) \in \Omega_{i_1,\ldots,i_s}$ and imagine forming a line by putting down m_1 white pebbles, followed by a black pebble, m_2 white pebbles, again a black pebble, and so on, finishing with m_s white pebbles and a black pebble. Now add $[\delta n/l] - s$ black pebbles on to the end; note that $s \leq [\delta n/l]$ due to the restrictions on the m's.

A total of $[\delta n] + [\delta n/l]$ pebbles have been laid down, $[\delta n/l]$ of them black and $[\delta n]$ of them white. The values of $m_1, \ldots, m_s$ can clearly be reconstructed from the line, proving that each such pebble sequence is distinct. This shows equation 5.17.

Now let us show

$$\#(\Omega) \leq C^{n+[\delta n/l]}_{[\delta n/l]}. \tag{5.18}$$

If we had a line of $n + [\delta n/l]$ white pebbles, we could colour black the pebbles at positions $i_1, i_2, \ldots, i_s$ and a further $[\delta n/l] - s$ pebbles chosen from the pebbles in positions $n+1$ to $n + [\delta n/l]$. Each distinct $(i_1, \ldots, i_s) \in \Omega$ gives a distinct way of colouring black $[\delta n/l]$ pebbles chosen from a total of $n + [\delta n/l]$ white pebbles. Equation 5.18 follows.

Now let us estimate

$$\#(\Omega^l_n) \leq \#(\Omega_{i_1,\ldots,i_s})\ \#(\Omega) \leq C^{[\delta n]+[\delta n/l]}_{[\delta n/l]}\ C^{n+[\delta n/l]}_{[\delta n/l]} \tag{5.19}$$

using Stirling's formula [1], written in the form

$$C^{a+b}_b \leq \frac{(1+b/a)^a(1+b/a)^b}{(b/a)^b},$$

which holds for all positive integers a and b. This gives

$$C^{n+[\delta n/l]}_{[\delta n/l]} \leq \left(\frac{(1+\delta/l)^{1/\delta}(1+\delta/l)^{1/l}}{(\delta/l)^{1/l}}\right)^{[\delta n]}$$

so if we choose N large enough — recall $l \geq N$ — then

$$C^{n+[\delta n/l]}_{[\delta n/l]} \leq e^{[\delta n]/8}.$$

Similarly,

$$C^{[\delta n]+[\delta n/l]}_{[\delta n/l]} \leq \left(\frac{(1+1/l)(1+1/l)^{1/l}}{(1/l)^{1/l}}\right)^{[\delta n]}$$

and

$$C^{[\delta n]+[\delta n/l]}_{[\delta n/l]} \leq e^{[\delta n]/8}$$

if N is large enough. Therefore

$$\#(\Omega^l_n) \leq e^{[\delta n]/4}.$$

■

Continuing the calculation of $|A^l_n|$, we now estimate the size of $\bigcap_{j=1}^s Z^{m_j}_{i_j}$:

Lemma 66 *If $l \geq 4$ then*

$$|\bigcap_{j=1}^{s} Z_{i_j}^{m_j}| \leq e^{-[\delta n]/2}$$

for every $(m_1, \ldots, m_s; i_1, \ldots, i_s) \in \Omega_n^l$ and $n \geq 1$.

Proof. We will use the following notation. For $z \in \Delta_i$ we put $d_i(z) = \min_{\hat{z} \in P_i} |z - \hat{z}|$. Then $D_i(z) \equiv (z - d_i(z); z + d_i(z))$ is the subset of $I_i(z)$ on which the function $\lambda \mapsto \lozenge_\lambda(i)$ is positive. We denote by $D_i \equiv \bigcup_{z \in \Delta_i} D_i(z)$ the subset of $[1; 2]$ on which $\lambda \mapsto \lozenge_\lambda(i)$ is positive. This is a disjoint union. Define $Z_i^m(z) = Z_i^m \cap D_i(z) = [z - e^{-m} d_i(z); z + e^{-m} d_i(z)]$ and note that $Z_i^m = \bigcup_{z \in \Delta_i} Z_i^m(z)$, a disjoint union, and $|Z_i^m(z)| = e^{-m}|D_i(z)|$.

We prove by induction on $s \geq 2$ that

$$|Z_{i_1}^{m_1}(z) \cap \bigcap_{j=2}^{s} Z_{i_j}^{m_j}| \leq |Z_{i_1}^{m_1}(z)|\, e^{-\sum_{j=2}^{s} m_j/2} \tag{5.20}$$

for any $z \in \Delta_{i_1}$ and choice of integers $1 \leq i_1 < i_2 < \ldots < i_s$ and $m_1 \geq 4, \ldots, m_s \geq 4$.

To see that the result follows, take any $(m_1, \ldots, m_s; i_1, \ldots, i_s) \in \Omega_n^l$. From disjointness we have

$$|\bigcap_{j=1}^{s} Z_{i_j}^{m_j}| = \sum_{z \in \Delta_{i_1}} |Z_{i_1}^{m_1}(z) \cap \bigcap_{j=2}^{s} Z_{i_j}^{m_j}|.$$

Using equation 5.20 to estimate each term in the sum,

$$|\bigcap_{j=1}^{s} Z_{i_j}^{m_j}| \leq e^{-\sum_{j=2}^{s} m_j/2} \sum_{z \in \Delta_{i_1}} |Z_{i_1}^{m_1}(z)|.$$

Now we estimate $\sum_{z \in \Delta_{i_1}} |Z_{i_1}^{m_1}(z)|$. To do this we use $|\bigcup_{z \in \Delta_{i_1}} D_{i_1}(z)| \leq 1$, which comes from $\bigcup_{z \in \Delta_{i_1}} D_{i_1}(z) \subseteq [1; 2]$. The union is disjoint so $|\bigcup_{z \in \Delta_{i_1}} D_{i_1}(z)| = \sum_{z \in \Delta_{i_1}} |D_{i_1}(z)|$. Therefore $\sum_{z \in \Delta_{i_1}} |Z_{i_1}^{m_1}(z)| = \sum_{z \in \Delta_{i_1}} e^{-m_1}|D_{i_1}(z)| \leq e^{-m_1}$. This shows

$$|\bigcap_{j=1}^{s} Z_{i_j}^{m_j}| \leq e^{-\sum_{j=1}^{s} m_j/2}.$$

From the definition of Ω_n^l we have $\sum_{j=1}^{s} m_j = [\delta n]$ and therefore

$$|\bigcap_{j=1}^{s} Z_{i_j}^{m_j}| \leq e^{-[\delta n]/2},$$

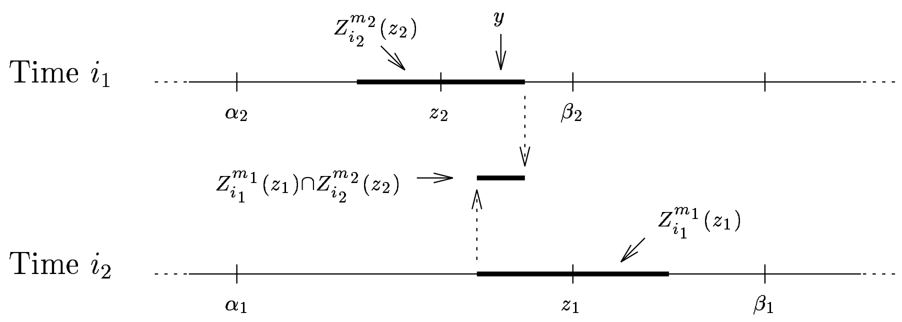


Figure 5.3: The intersection of $Z_{i_1}^{m_1}(z_1)$ and $Z_{i_2}^{m_2}(z_2)$.

as required.

So let us now prove by induction on $s \geq 2$ that

$$|Z_{i_1}^{m_1}(z) \cap \bigcap_{j=2}^{s} Z_{i_j}^{m_j}| \leq |Z_{i_1}^{m_1}(z)|\, e^{-\sum_{j=2}^{s} m_j/2} \tag{5.21}$$

for any $z \in \Delta_{i_1}$ and choice of integers $1 \leq i_1 < i_2 < \ldots < i_s$ and $m_1 \geq 4, \ldots, m_s \geq 4$.

First let us show that equation 5.21 holds when $s = 2$. In other words that

$$|Z_{i_1}^{m_1}(z_1) \cap Z_{i_2}^{m_2}| \leq |Z_{i_1}^{m_1}(z_1)|\, e^{-(m_1+m_2)/2} \tag{5.22}$$

holds for any $z_1 \in \Delta_{i_1}$ and integers $1 \leq i_1 < i_2$ and $m_1 \geq 4$ and $m_2 \geq 4$.

Take $z_1 \in \Delta_{i_1}$ and $z_2 \in \Delta_{i_2}$ and suppose $Z_{i_1}^{m_1}(z_1) \cap Z_{i_2}^{m_2}(z_2) \neq \emptyset$. We first show $D_{i_2}(z_2) \subseteq Z_{i_1}^{m_1-2}(z_1)$.

Put $\alpha_j = \max\{z \in P_{i_j} \mid z < z_j\}$ and $\beta_j = \min\{z \in P_{i_j} \mid z > z_j\}$ for $j = 1, 2$. Because $i_2 > i_1$ we have $z_1 \in P_{i_2}$ and therefore either $[\alpha_2; \beta_2] \subseteq [\alpha_1; z_1]$ or $[\alpha_2; \beta_2] \subseteq [z_1; \beta_1]$ (see figure 5.3). Suppose without loss of generality that $[\alpha_2; \beta_2] \subseteq [\alpha_1; z_1]$ and take $y \in Z_{i_1}^{m_1}(z_1) \cap Z_{i_2}^{m_2}(z_2)$.

Clearly we can choose $z_2 < y < z_1$ and so we have $\alpha_1 \leq \alpha_2 < z_2 < y < \beta_2 \leq z_1$, as in figure 5.3. The definition of $Z_{i_2}^{m_2}(z_2)$ gives $|y - z_2| < |\beta_2 - z_2|e^{-m_2} \leq |z_1 - z_2|e^{-m_2}$. Using this equation we can estimate $|z_1 - z_2|$ in terms of $|y - z_1|$:

$$\begin{aligned} |z_1 - z_2| &= |y - z_1| \frac{|z_1 - z_2|}{|z_1 - z_2| - |y - z_2|} \\ &< |y - z_1| \frac{1}{1 - e^{-m_2}} \leq \frac{e^2}{2}|y - z_1|, \end{aligned}$$

where we have used $m_2 \geq 4$.

Now $D_{i_2}(z_2) \subseteq [z_2 - |z_1 - z_2|; z_1]$ so if λ is in $D_{i_2}(z_2)$ then $|\lambda - z_1| \leq 2|z_1 - z_2| \leq e^2|y - z_1|$. Since $y \in Z_{i_1}^{m_1}(z_1)$, this gives $\lambda \in Z_{i_1}^{m_1-2}(z_2)$ and so $D_{i_2}(z_2) \subseteq Z_{i_1}^{m_1-2}(z_1)$.

Now let us show equation 5.22. Put $\mathcal{Z} = \{z \in \Delta_{i_2} \mid Z_{i_2}^{m_2}(z) \cap Z_{i_1}^{m_1}(z_1) \neq \emptyset\}$. Then $\bigcup_{z\in\mathcal{Z}} D_{i_2}(z) \subseteq Z_{i_1}^{m_1-2}(z_1)$, and this is a disjoint union. The disjointness implies $|\bigcup_{z\in\mathcal{Z}} Z_{i_2}^{m_2}(z)| = e^{-m_2}|\bigcup_{z\in\mathcal{Z}} D_{i_2}(z)| \leq e^{-m_2}|Z_{i_1}^{m_1-2}(z_1)| = e^{-(m_2-2)}|Z_{i_1}^{m_1}(z_1)|$. However $e^{-(m_2-2)} \leq e^{-m_2/2}$ because $m_2 \geq 4$. Therefore

$$|Z_{i_1}^{m_1}(z_1) \cap Z_{i_2}^{m_2}| = |Z_{i_1}^{m_1}(z_1) \cap (\bigcup_{z\in\mathcal{Z}} Z_{i_2}^{m_2}(z))| \leq |Z_{i_1}^{m_1}(z)|\, e^{-m_2/2},$$

which is equation 5.22.

The demonstration of equation 5.21 when $s > 2$ is left to the reader. ■

Combining lemmas 65 and 66 shows that

$$|A_n^l| \leq e^{[\delta n]/4} e^{-[\delta n]/2} \leq e^{-[\delta n]/4}$$

whenever $l \geq \max\{N, 4\}$. Using lemma 63 we therefore have

$$|\limsup_{n\to\infty} A_n^l| = 0$$

and using lemma 63 once more,

$$|\limsup_{l\to\infty} \limsup_{n\to\infty} A_n^l| = 0.$$

We conclude from lemma 64 that $|E_\delta| = 0$. Since $\delta > 0$ was arbitrary this proves theorem 56.

Bibliography

[1] Milton Abramowitz and Irene A. Stegun, editors. *Handbook of Mathematical Functions.* Dover, 1965.

[2] R. L. Adler, A. G. Konheim, and M. H. McAndrew. Topological entropy. *Transactions of the American Mathematical Society*, 114:309–319, 1965.

[3] Lars V. Ahlfors. *Lectures on Quasiconformal Mappings.* Wadsworth & Brooks/Cole, 1987. Reprint of van Nostrand 1966.

[4] Michael Benedicks and Lennart Carleson. On iterations of $1 - ax^2$ on $(-1, 1)$. *Annals of Mathematics*, 122:1–25, 1985.

[5] Michael Benedicks and Lennart Carleson. The dynamics of the Hénon map. *Annals of Mathematics*, 133:73–169, 1991.

[6] A. M. Blokh and M. Yu. Lyubich. Measurable dynamics of S-unimodal maps of the interval. *Annales Scientifiques de l'École Normale Supérieure*, 24(5):545–573, 1991.

[7] K. Brucks and M. Misiurewicz. Trajectory of the turning point is dense for almost all tent maps. Pre-print; Karen Brucks: kmbrucks@csd4.csd.uwm.edu; Michał Misiurewicz: mmisiure@indyvax.iupui.edu, June 1993.

[8] K. M. Brucks and B. Diamond. Monotonicity of auto-expansions. *Physica D*, 51(1–3):39–42, 1991.

[9] H. Bruin. Notes on the kneading map. The Faculty of Technical Mathematics and Informatics, Julianalaan 132, 2628 BL Delft, the Netherlands, 1992.

[10] P. Collet and J.-P. Eckmann. *Iterated Maps on the Interval as Dynamical Systems.* Birkhäuser, 1980.

[11] P. Collet and J.-P. Eckmann. Positive Liapunov exponents and absolute continuity for maps of the interval. *Ergodic Theory and Dynamical Systems*, 3:13–46, 1983.

[12] Ethan M. Coven, Ittai Kan, and James A. Yorke. Pseudo-orbit shadowing in the family of tent maps. *Transactions of the American Mathematical Society*, 308(1):227–241, July 1988.

[13] Pedrag Cvitanović. *Universality in Chaos*. Adam Hilger Ltd, Techno House, Redcliffe Way, Bristol BS1 6NX,ffUnited Kingdom, 1984. Reprint collection.

[14] Welington de Melo and Sebastian van Strien. *One-dimensional Dynamics*. Springer-Verlag, 1993.

[15] Robert L. Devaney. *An Introduction to Chaotic Dynamical Systems*. Addison-Wesley, second edition, 1989.

[16] J.-P. Eckmann and D. Ruelle. Ergodic theory of chaos and strange attractors. *Reviews of Modern Physics*, 57(3):617–656, 1985.

[17] P. Fatou. Sur les équations fonctionnelles. Premier mémiore. *Bulletin de la Société Mathématique de France*, 47:161–271, 1919. Parts 2 and 3 are in volume 48, 1920.

[18] J. Guckenheimer, 1989. Talk at the low dimensional dynamics conference in College Park. Cited by Jakobson and Swiątek [27].

[19] John Guckenheimer. Sensitive dependence to initial conditions for one dimensional maps. *Communications in Mathematical Physics*, (70):133–160, 1979.

[20] John Guckenheimer and Stewart Johnson. Distortion of S-unimodal maps. *Annals of Mathematics*, 132(1):71–130, 1990.

[21] Hao Bai-Lin. *Chaos*. World Scientific, 1984. Reprint collection.

[22] Michael R. Herman. Mesure de Lebesgue et nombre de rotation. In Jacob Palis and Manfredo do Carmo, editors, *Geometry and Topology, III Latin American School of Mathematics, July 1976*, pages 271–293, Rio de Janeiro, 1977. Springer-Verlag. Lecture Notes in Mathematics 597.

[23] Michael R. Herman. Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. *Publications Mathématiques de l'IHES*, (49):5–233, 1979.

[24] Franz Hofbauer. The topological entropy of the transformation $x \mapsto ax(1-x)$. *Monatschefte für Mathematik*, 90:117–141, 1980.

[25] Franz Hofbauer and Gerhard Keller. Quadratic maps without asymptotic measure. *Communications in Mathematical Physics*, 127(2):319–337, 1990.

[26] M. Jakobson. Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. *Communications in Mathematical Physics*, 81(1):39–88, 1981.

[27] M. Jakobson and G. Świątek. Quasisymmetric conjugacies between unimodal maps. SUNY Stony Brook preprint ims91-16, September 1991.

[28] Gaston Julia. Mémoire sur l'itération desfffonctions rationnelles. *Journal de Mathématiques Pures et Appliquées*, 1:47–245, 1918. Incorrectly described as seventh series, volume 4 on inside front page.

[29] Gerhard Keller. Exponents, attractors and Hopf decompositions for interval maps. *Ergodic Theory and Dynamical Systems*, 10:717–744, 1990.

[30] Gerhard Keller. Lyapunov exponents and complexity for interval maps. In L. Arnold, H. Crauel, and J. P. Eckmann, editors, *Lyapunov Exponents*, Lecture Notes in Mathematics, pages 216–231. Springer-Verlag, 1991.

[31] Gerhard Keller and Tomasz Nowicki. Fibonacci maps re(al)visited. Preprint, September 1992.

[32] Gerhard Keller and Tomasz Nowicki. Spectral theory, zeta functions and the distribution of periodic points for Collet-Eckmann maps. *Communications in Mathematical Physics*, 149(1):31–69, 1992.

[33] O. Lehto and K. I. Virtanen. *Quasiconformal Mappings in the Plane.* Springer-Verlag, second edition, 1973. Translation from the German.

[34] M. Loève. *Probability Theory.* University Series in Higher Mathematics. Van Nostrand, second edition, 1960.

[35] Mikhail Lyubich. Combinatorics, geometry and attractors of quasi-quadratic maps. SUNY Stony Brook preprint ims92-18, revised Oct. 1992, February 1992.

[36] Mikhail Lyubich and John Milnor. The Fibonacci unimodal map. SUNY Stony Brook preprint ims91-15.

[37] N. Metropolis, M. L. Stein, and P. R. Stein. On finite limit sets for transformations on the unit interval. *Journal of Combinatorial Theory (A)*, (15):25–44, 1973.

[38] John Milnor and William Thurston. On iterated maps of the interval. In *Dynamical Systems: Proceedings 1986-87*, pages 465–563. Springer-Verlag, 1988. Lecture Notes in Mathematics 1342.

[39] M. Misiurewicz and W. Szlenk. Entropy of piecewise monotone mappings. *Studia Mathematica*, 67:45–63, 1980. Results announced in Astérisque 1977, vol. 50.

[40] Michał Misiurewicz. Absolutely continuous measures for certain maps of an interval. *Publications Mathématiques de l'I.H.E.S.*, (53):17–51, 1981.

[41] Leonardo Mora and Marcelo Viana. Abundance of strange attractors. Preprint, to appear in Acta Mathematica, 1990.

[42] T. Nowicki and F. Przytycki. The conjugacy of Collet-Eckmann's map of the interval with the tent map is Hölder continuous. *Ergodic Theory and Dynamical Systems*, 9:379–388, 1989.

[43] Tomasz Nowicki. On some dynamical properties of S-unimodal maps on an interval. *Fundamenta Mathematicae*, 126(1):27–43, 1985.

[44] Tomasz Nowicki. Symmetric S-unimodal mappings and positive Liapunov exponents. *Ergodic Theory and Dynamical Systems*, 5:611–616, 1985.

[45] Tomasz Nowicki. A positive Liapunov exponent for the critical value of an S-unimodal map implies uniform hyperbolicity. *Ergodic Theory and Dynamical Systems*, (8):425–435, 1988.

[46] Tomasz Nowicki. Some dynamical properties of S-unimodal maps. *Fundamenta Mathematicae*, 142(1):45–57, 1993.

[47] David Singer. Stable orbits and bifurcation of maps of the interval. *SIAM Journal of Applied Mathematics*, 35(2):260–267, 1978.

[48] W. Szlenk. Some dynamical properties of certain differentiable mappings of an interval. *Boletin de la Sociedad Matemática Mexicana*, 24(2):57–82, 1979.

[49] Masato Tsujii. Positive Lyapunov exponents in families of one-dimensional systems. *Inventiones Mathematicae*, 111(1):113–137, 1993.

[50] Sebastian van Strien. Smooth dynamics on the interval (with an emphasis on quadratic-like maps). In T. Bedford and J. Swift, editors, *New Directions in Differentiable Dynamical Systems*, pages 57–119. Cambridge University Press, 1988. London Mathematical Society Lecture Note Series 127.

[51] J.-C. Yoccoz. Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne. *Annales Scientifiques de l'École Normale Supérieure*, 17(3):333–359, 1984.

She comes! she comes! the sable Throne behold
Of Night Primaeval, and of Chaos old!
Before her, Fancy's gilded clouds decay,
And all its varying Rainbows die away,
Wit shoots in vain its momentary fires,
The meteor drops, and in a flash expires.
As one by one, at dread Medea's strain,
The sick'ning stars fade off th' ethereal plain;
As Argus' eyes, by Hermes' wand opprest,
Clos'd one by one to everlasting rest;
Thus at her felt approach, and secret might,
Art after Art goes out, and all is Night.
See skulking Truth to her old cavern fled,
Mountains of Casuistry heap'd o'er her head!
Philosophy, that lean'd on Heav'n before,
Shrinks to her second cause, and is no more.
Physic of Metaphysic begs defence,
And Metaphysic calls for aid on Sense!
See Mystery to Mathematics fly!
In vain! they gaze, turn giddy, rave, and die.
Religion blushing veils her sacred fires,
And unawares Morality expires.
Nor public Flame, nor private, dares to shine;
Nor human Spark is left, nor Glimpse divine!
Lo! thy dread Empire, CHAOS! is restor'd;
Light dies before thy uncreating word:
Thy hand, great Anarch! lets the curtain fall;
And Universal Darkness buries All.

Alexander Pope, *from* The Dunciad.