Jol H. Hubbard

ON THE BIFURCATION LOCI OF RATIONAL MAPS OF DEGREE TWO

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In 1918 Julia and Fatou proved that for any given quadratic polynomial the set of points which do not tend to infinity with repeated application of the polynomial is either connected or a Cantor set. Naturally they wondered for which quadratic polynomials would that set be connected, but no one had any real idea until 1980 when Mandelbrot began to investigate this question numerically on a computer. Mandelbrot's computer pictures indicated that the set of quadratic polynomials for which the points not attracted to infinity are connected is unlike those sets traditionally studied by mathematicians. Its boundaries seemed to be the opposite of smooth; they looked complicated at whatever level of magnification. The set in question became known as the Mandelbrot set.

The Mandelbrot set has since been besieged by mathematicians and physicists. The attackers include Benedicts, Berstein, Brolin, Branner, Carleson, Cvitanovic, Douady, Eremko, Feigenbaum, Guckenheimer, Hubbard, Lavaurs, Levy, Ljubich, Milnor, Rees, Sentenac, Sibony, Sullivan, Tan, Thurston, Yacobson, and others. To a considerable degree, the Mandelbrot set has yielded to this onslaught and is now fairly well understood.

But quadratic polynomials are such simple functions. If we study the iteration of more complicated functions, do we encounter sets which make the Mandelbrot set look tame by comparison? This work is an attempt to answer that question. In particular, we try to understand the iteration of rational functions of degree two (i.e. a quadratic polynomial divided by a quadratic polynomial) in terms of the iteration of quadratic polynomials. Despite the fact that this study is by no means complete, what we have seen so far indicates that the iteration of rational functions of degree two can be understood in terms of the iteration of quadratic polynomials, but the combinatorics are a good deal more complicated.

Chapter 1. Introduction.

§1.1. *Broad Goals.*

In 1918 Julia and Fatou proved that for any given quadratic polynomial the set of points which do not tend to infinity with repeated application of the polynomial is either connected or a Cantor set. Naturally they wondered for which quadratic polynomials would that set be connected, but no one had any real idea until 1980 when Mandelbrot began to investigate this question numerically on a computer. Mandelbrot's computer pictures indicated that the set of quadratic polynomials for which the points not attracted to infinity are connected is unlike those sets traditionally studied by mathematicians. Its boundaries seemed to be the opposite of smooth; they looked complicated at whatever level of magnification. The set in question became known as the Mandelbrot set.

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§1.2. *Introduction to the introduction.*

Sections 1.1 through 1.9 of this introduction are intended for a general audience. Section 1.10 is intended for the specialist who is already familiar with the notions of mating and capture among rational functions but who desires a specific guide to what is new mathematically in this work.

Since this paper attempts to explain the iteration of rational functions of degree two in terms of the iteration of quadratic polynomials, we cannot introduce the main ideas without a brief introduction to the dynamics of quadratic polynomials. Sections 1.3 and 1.5 do so. Sections 1.4, 1.6, 1.7, 1.8, and 1.9 introduce the ideas of this paper in an intuitive, non-rigorous way. Everything will be made precise later.

Since the subject of this paper is iteration, it is almost impossible to proceed without a notation for composition. We let $f^{\circ n}$ denote the composition of f with itself n times. So for example,

$$f^{\circ 3}(z) := f \circ f \circ f(z) = f(f(f(z))).$$

This introduction is intended for a wide audience. Experience shows that many people who might otherwise understand our terminology are confused by our view of critical points. If the reader is not confused by statements such as, "The map $f_c(z) = z^2 + c$ has a critical point at infinity," or "The critical points of a complex analytic map f are precisely those points having no neighborhood on which f is injective," then the rest of this section is of no interest.

Recall that if f is a complex analytic map, then for every z_0 in the domain of f, there is a neighborhood U of z_0, a neighborhood V of $f(z_0)$, and co-ordinates on U and V with respect to which f is $z \mapsto z^d$ for some integer $d \geq 1$. d is called the *local degree of* f *at* z_0. Those z_0 at which the local degree of f is greater than one are called *critical points* of f. So one characterization of critical points is that they are precisely those points having no neighborhood on which f is injective.

Another characterization of critical points is that they are precisely the points where the derivative of the map expressed in local co-ordinates is zero. For maps to and from the Riemann sphere one can use the local co-ordinate z for all points in $\mathbf{C}$ and $1/z$ for ∞.

For example, for e some complex number in $\mathbf{C} - \{0\}$ let

$$f_e(z) := \frac{1}{ez^2 - (e+1)z + 1}.$$

For any e in $\mathbf{C}-\{0\}$, $f_e(\infty)=0$, so to see if ∞ is a critical point of f_e we use the co-ordinate $1/z$ in the domain and z in the range. In those co-ordinates f_e is of the form

$$\frac{1}{e(1/z)^2-(e+1)(1/z)+1}$$

which has derivative 0 at 0. So ∞ is a critical point of f_e.

For another example, let

$$f_c(z):=z^2+c,$$

where c is any complex number. $f_c(\infty)=\infty$, so we use the co-ordinates $1/z$ in both domain and range. In those co-ordinates, f_c is of the form

$$1/((1/z)^2+c)$$

which has derivative 0 at 0. So ∞ is a critical point of f_c.

§**1.3.** *Quadratic polynomials.*

Suppose f and g are maps from some space X to itself. If $\phi: X\to X$ is a map with inverse map $\phi^{-1}: X\to X$ such that

$$\phi\circ f\circ\phi^{-1}=g,$$

then iteration of f and iteration of g are essentially the same because

$$\begin{aligned} g^{\circ n} &= (\phi\circ f\circ\phi^{-1})^{\circ n} \\ &= (\phi\circ f\circ\phi^{-1})\circ(\phi\circ f\circ\phi^{-1})\circ\cdots\circ(\phi\circ f\circ\phi^{-1}) \\ &= \phi\circ(f\circ f\circ\cdots\circ f)\circ\phi^{-1} \\ &= \phi\circ(f^{\circ n})\circ\phi^{-1}. \end{aligned}$$

In such a case we say that f and g are *conjugated* by ϕ.

It is not hard to see that all quadratic polynomials are conjugate to exactly one of the form

$$f_c(z) := z^2 + c$$

for some $c \in \mathbf{C}$. For any polynomial, complex numbers of sufficiently large absolute value tend to infinity under repeated applications of the polynomial. It is interesting, therefore, to consider the set K_c of points z in $\mathbf{C}$ such that $f_c^{\circ n}(z)$ does not tend to infinity as n tends to infinity. Julia and Fatou proved that K_c is connected if and only if the critical point 0 is in K_c and K_c is a Cantor set (intuitively, infinitely many separate particles of dust with an affinity for one another) if and only if 0 is not in K_c.

We look at some examples. Figures 1.1 and 1.2 show K_c in black for $c = -1$ and $c \approx -.12352 + .74290i$ respectively. The orbit of 0 is marked with white dots. In both cases K_c is connected.

In those figures, the points not in K_c have been colored in shades of red, yellow and green. The reason is the following. It can happen that K_c has no interior. (This is always the case when K_c is a Cantor set, but it can happen when K_c is connected also.) In such cases if one were to use only two colors, one for points in K_c and another for points not in K_c, then K_c would not be visible since it would be highly unlikely that any points on the grid checked by the computer would be in K_c. If, however, we shade the points not in K_c according to how many iterations of f_c were required to send them to a particular neighborhood of

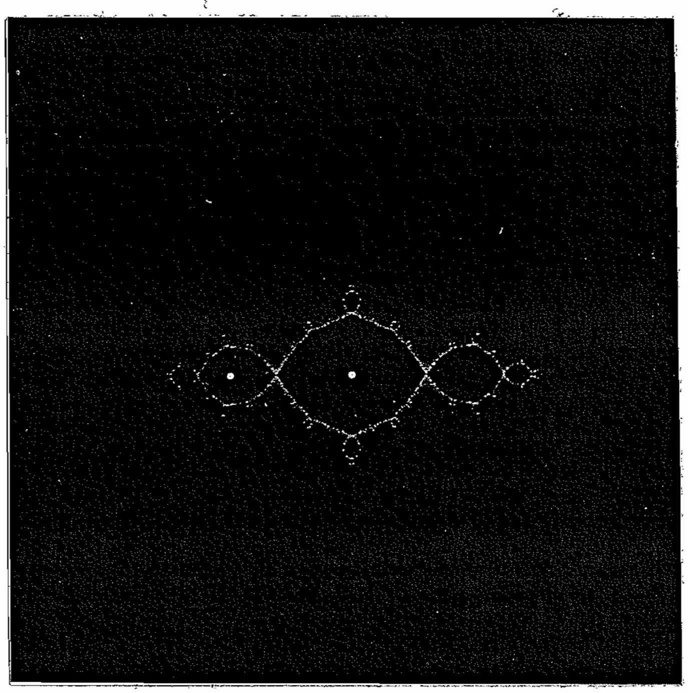

Figure 1.1. K_c for $c = -1$.

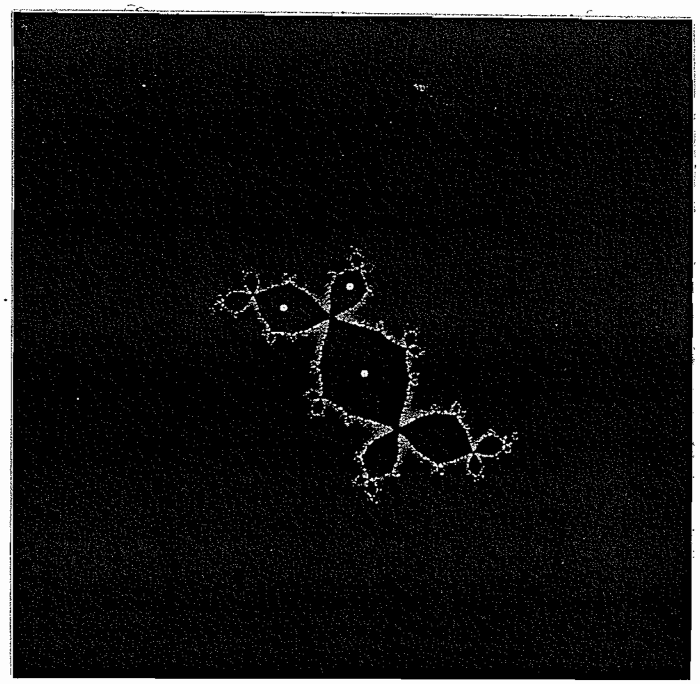

Figure 1.2. K_c for $c \approx -.12352 + .74290i$.

infinity, K_c will still be visible as in figures 1.3 and 1.4. Figure 1.3 shows K_c for $c \approx -.22815 + 1.1151i$ and figure 1.4 shows K_c for $c \approx -.28156 + .98216i$. The K_c of figure 1.3 is connected and the K_c of figure 1.4 is a Cantor set.

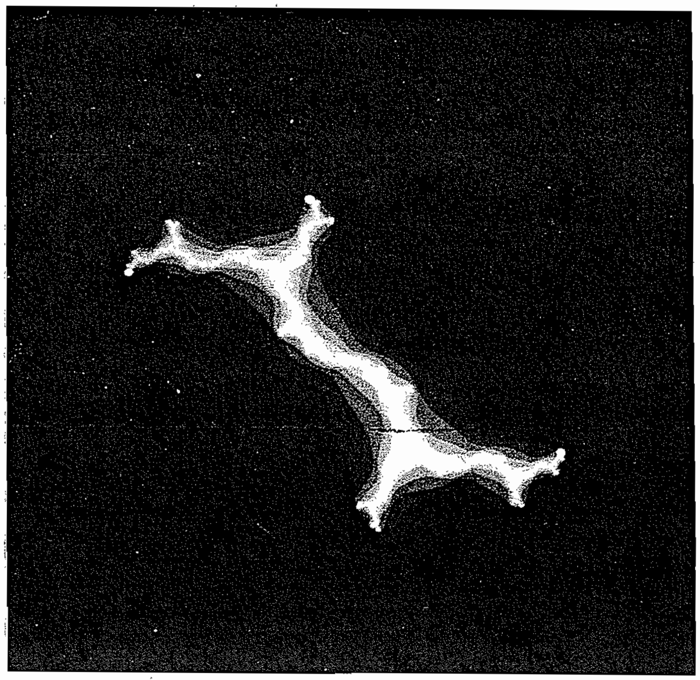

Figure 1.3. K_c for $c \approx -.22815 + 1.1151i$.

For many of the c for which K_c is connected, there is a continuous map $\hat{\gamma}_c$ mapping the unit circle onto the boundary of K_c such that

$$f_c(\hat{\gamma}_c(e^{2\pi it})) = \hat{\gamma}_c(e^{2\pi i(2t)}).$$

To lighten notation we define

$$\gamma_c(t) := \hat{\gamma}_c(e^{2\pi it}).$$

γ_c is called the *Carathéodory loop of* f_c.

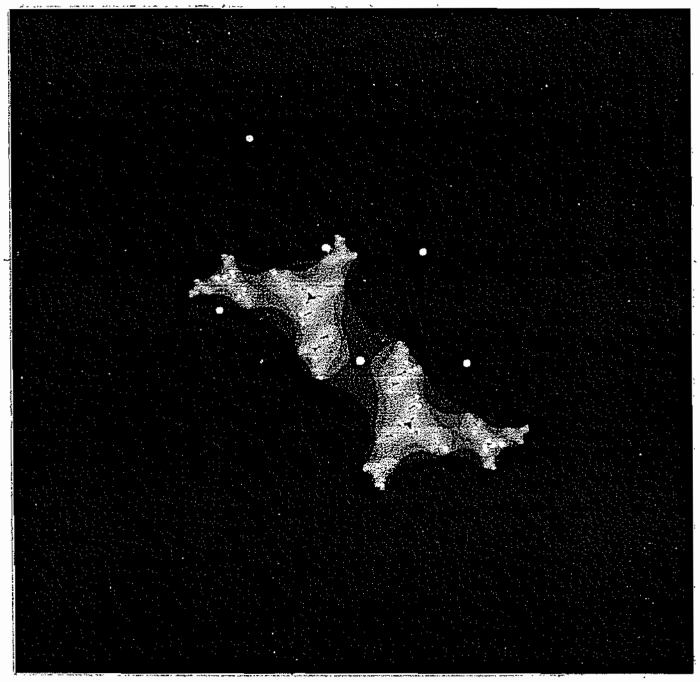

Figure 1.4. K_c for $c \approx -.28156 + .98216$.

Finally, we give a brief intuitive definition of Hubbard trees. If the orbit of 0 under f_c has finitely many points, join each pair of points in the orbit by the "shortest" curve which stays in K_c. The union of all these curves is the *Hubbard tree of* f_c.

§1.4. *Mating.*

The mating of polynomials to form rational functions of degree two was discovered by Hubbard and Douady in 1982. Suppose c_0 and c_1 are such that K_{c_0} and K_{c_1} are connected and f_{c_0} and f_{c_1} have Carathéodory loops γ_{c_0} and γ_{c_1}. The idea of mating f_{c_0} with f_{c_1} is to form a sphere out of K_{c_0} and K_{c_1} by sewing them together along their boundaries. Looking at figures 1.1, 1.2, and 1.3 one might wonder if that is possible, but in certain cases we know it is. In fact, we know that in these cases it is possible to do so in a way that $\gamma_{c_0}(t)$ sews to $\gamma_{c_1}(-t)$. In those cases we can let f_{c_0} and f_{c_1} define a map on the sphere we created by sewing.

Rational functions f are best thought of as maps from the Riemann sphere to the Riemann sphere because f might take infinity to a point in $\mathbf{C}$ and visa-versa. If the map on the sphere we formed by sewing K_{c_0} to K_{c_1} is conjugate to a rational function, we call that rational function the mating of f_{c_0} with f_{c_1}.

To make this all a bit more concrete, we illustrate how we know we can sew as described above for a particular example. Let K_{c_0} be the one pictured in figure 1.1 and let K_{c_1} be the one pictured in figure 1.2. Instead of sewing the K_c's together along their boundaries, sew them together along a big circle surrounding each K_c (see figure 1.5 where K_{c_0} is in black and K_{c_1} is in light green). We now apply a procedure devised by Thurston that allows K_{c_0} and K_{c_1} to slowly move towards each other (see figures 1.5 through 1.12). It follows from the work of Thurston, Levy, Tan and theorem 6.1.1 below that the red and blue regions separating K_{c_0} from K_{c_1} will vanish in the limit, leaving a sphere sewn in the way described above (see figure 1.12).

It also follows that the map on the sphere defined by f_{c_0} and f_{c_1} is conjugate to some rational function. We happen to know that in this case, the rational function can be expressed as

$$f_e(z) := \frac{1}{ez^2 - (e+1)z + 1}$$

for $e \approx .57735i$. It turns out that for any e in $\mathbf{C} - \{0\}$, $(\infty, 0, 1)$ forms an attracting

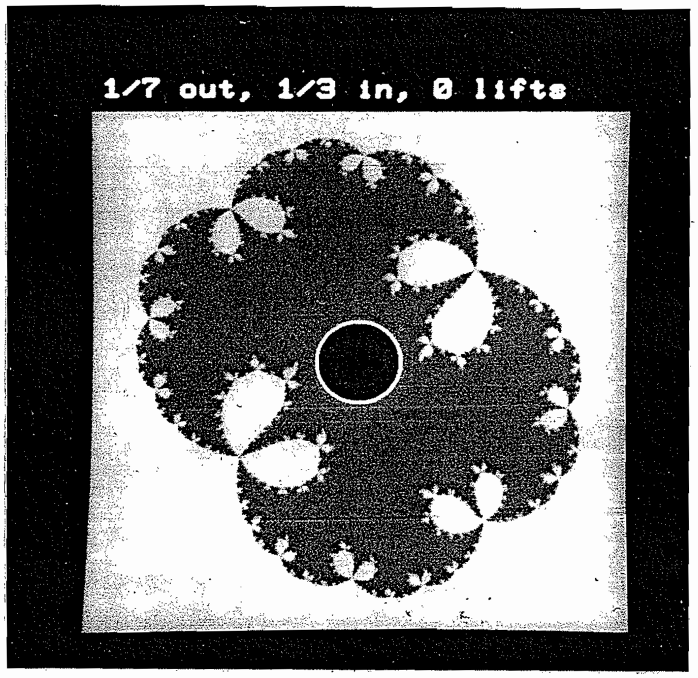

Figure 1.5. Thurston construction of 1/7 mating with 1/3, 0 lifts.

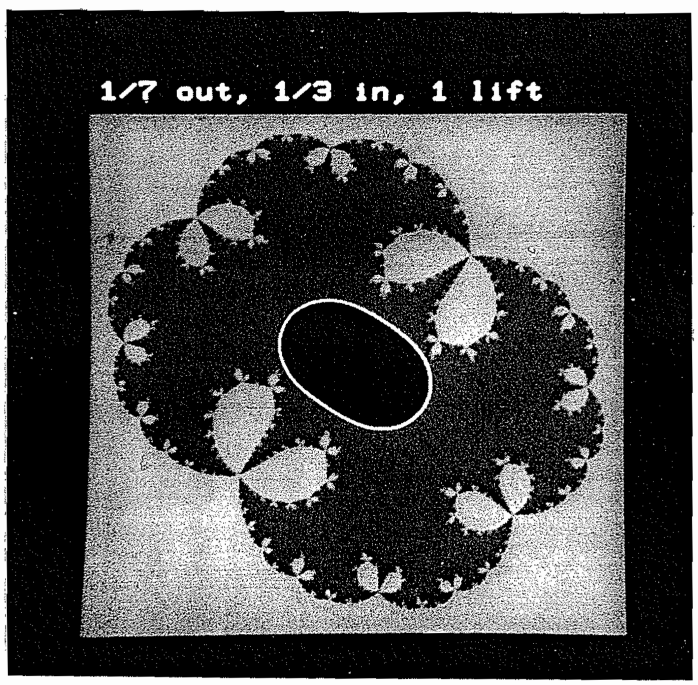

Figure 1.6. Thurston construction of 1/7 mating with 1/3, 1 lift.

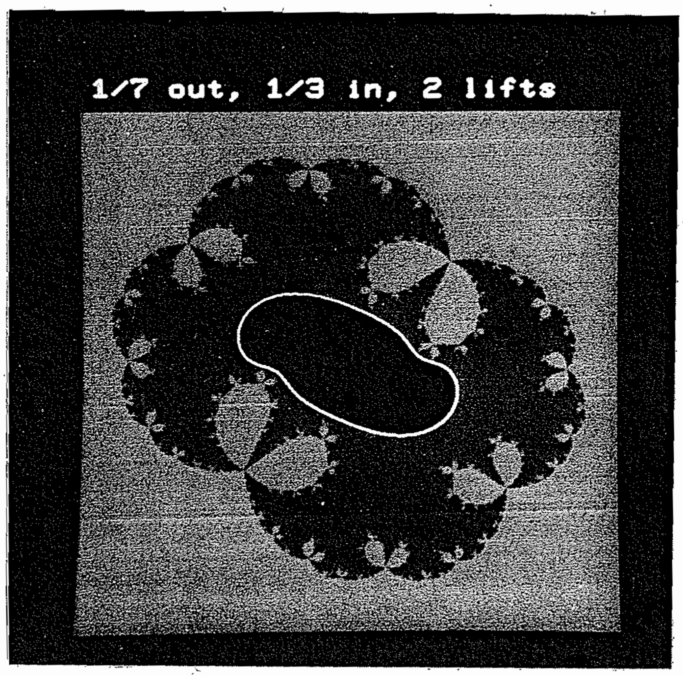

Figure 1.7. Thurston construction of 1/7 mating with 1/3, 2 lifts.

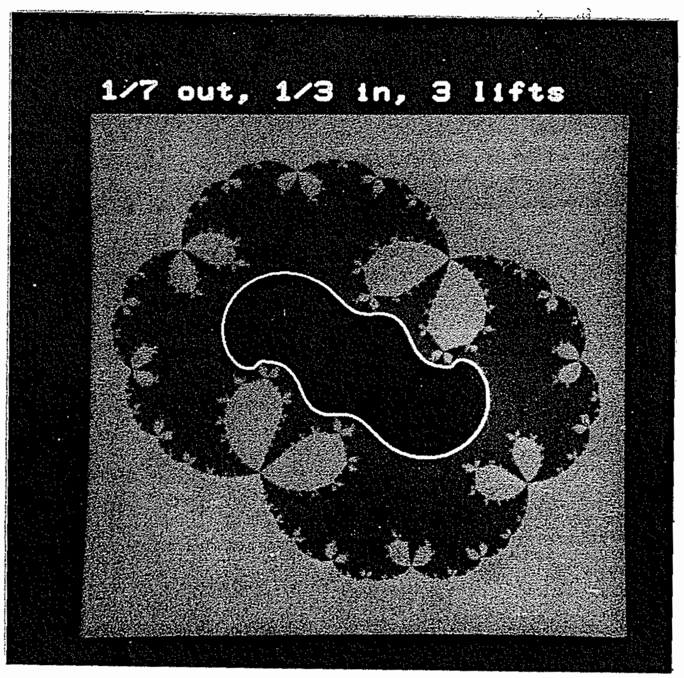

Figure 1.8. Thurston construction of 1/7 mating with 1/3, 3 lifts.

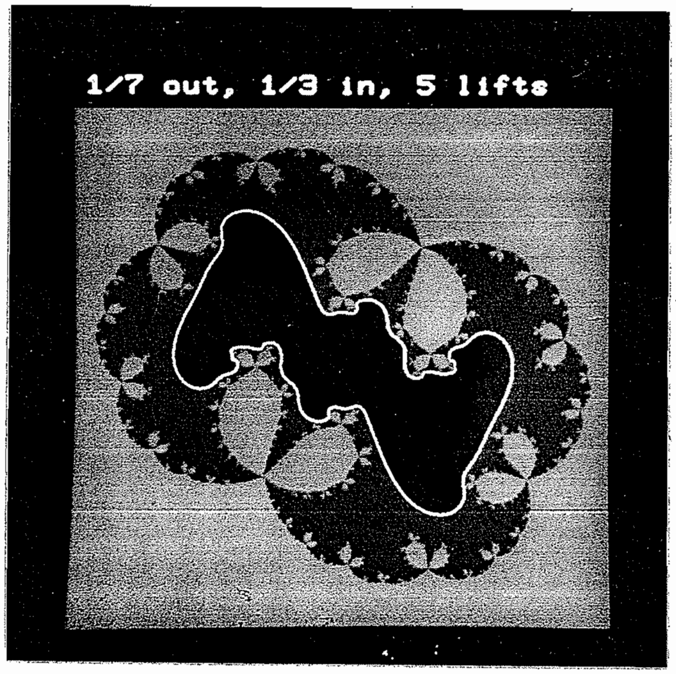

Figure 1.9. Thurston construction of 1/7 mating with 1/3, 5 lifts.

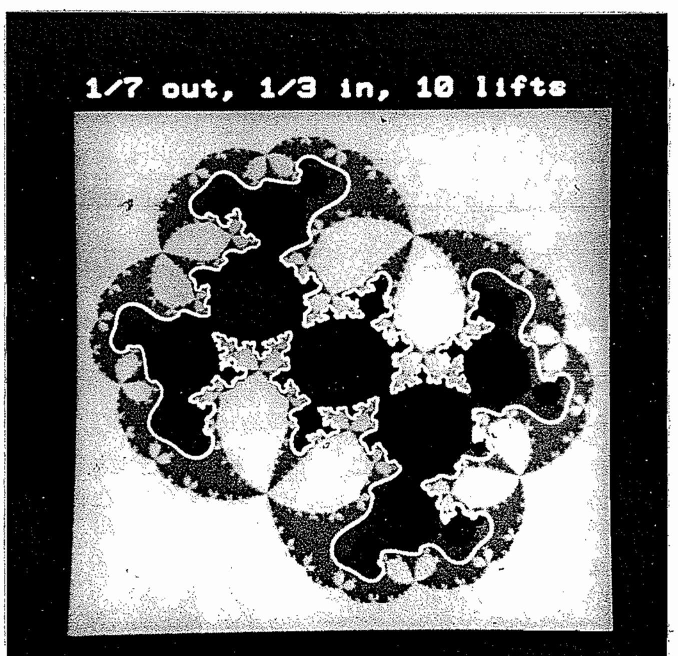

Figure 1.10. Thurston construction of 1/7 mating with 1/3, 10 lifts.

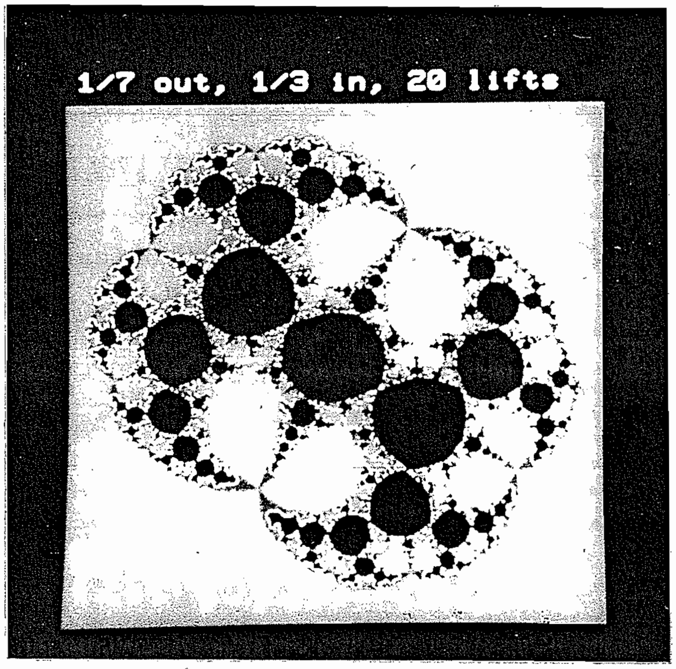

Figure 1.11. Thurston construction of 1/7 mating with 1/3, 20 lifts.

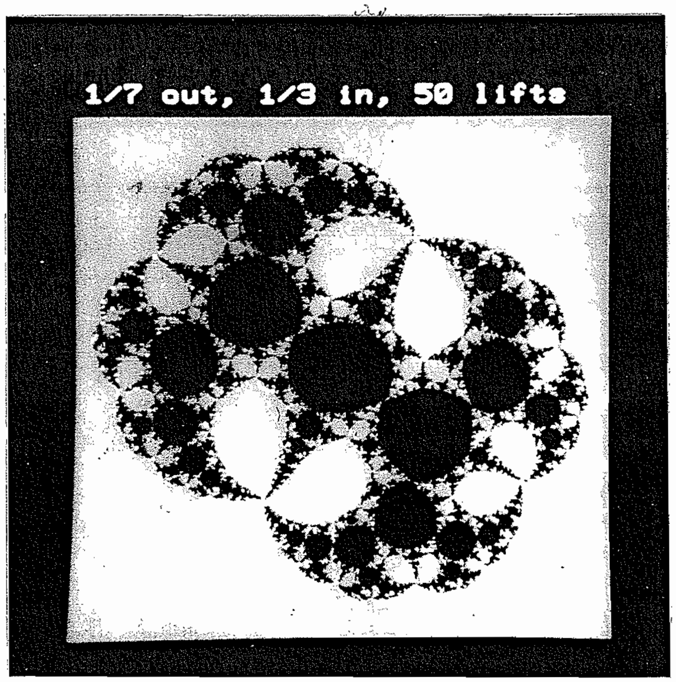

Figure 1.12. Thurston construction of 1/7 mating with 1/3, 50 lifts.

periodic cycle for f_e (i.e. $f_e(\infty) = 0$, $f_e(0) = 1$, $f_e(1) = \infty$ and for z sufficiently near ∞ but not equal to ∞, $f_e(z)$ will be near 0, $f_e^{\circ 2}(z)$ will be near 1, and $f_e^{\circ 3}(z)$ will be nearer to ∞ than was z.). In figure 1.13 we have left black those z which are not attracted to that cycle. The z which are attracted to that cycle are colored red, green, or blue depending upon what iterate (mod 3) of f_e takes z near ∞. 0 and 1 are marked by exes. Notice the similarity between figure 1.12 and figure 1.13.

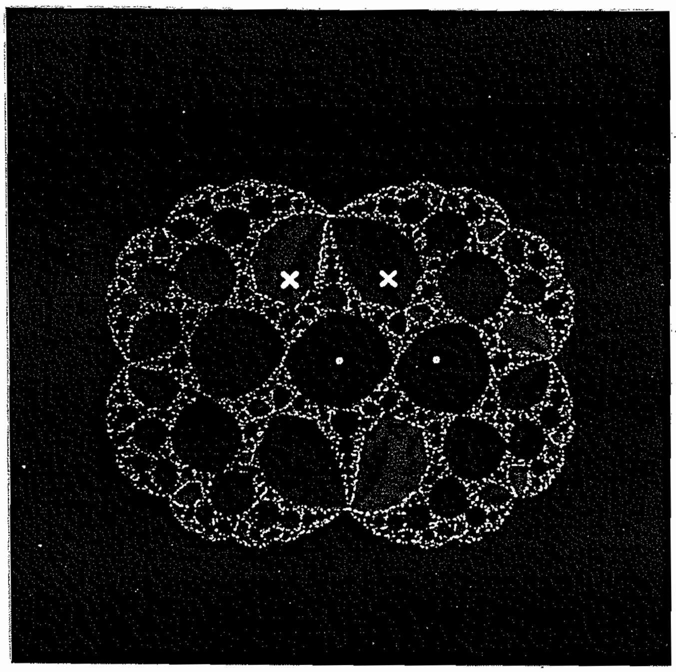

Figure 1.13. z-plane for $e \approx .57735i$.

Mating can be very exotic. For example it can be possible to mate f_{c_0} with f_{c_1} even though neither K_{c_0} nor K_{c_1} has any interior. In that case, the image of γ_{c_0} (or γ_{c_1}) in the sphere formed by sewing is the entire sphere. Not only is γ_{c_0} a Peano curve, but the map which sends $\gamma_{c_0}(t)$ to $\gamma_{c_0}(2t)$ is well defined on the sphere and conjugate to a rational function. In figures 1.14 through 1.21 we mate the f_c of figure 1.3 with itself by sewing along a big circle and letting the K_c's move towards each other. Note the Peano curve forming.

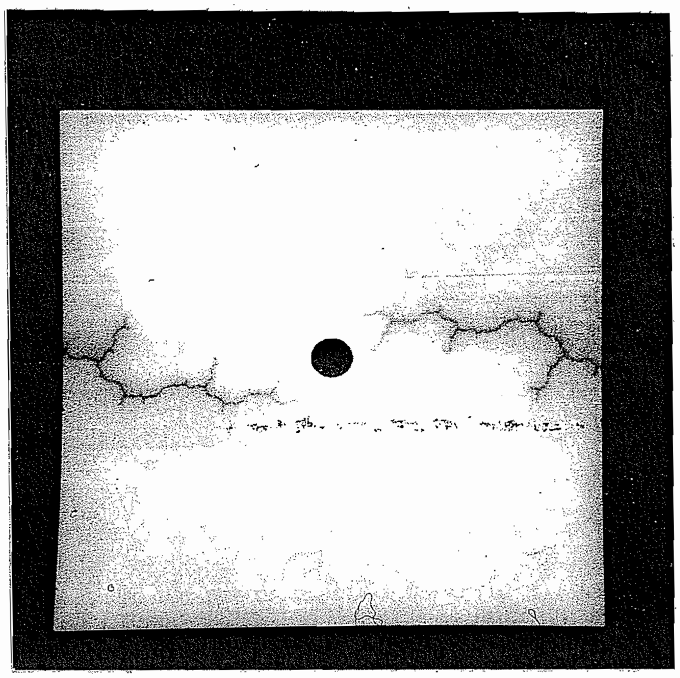

Figure 1.14. Thurston construction of 1/4 mating with 1/4, 0 lifts.

§1.5. *The Mandelbrot set.*

By definition the Mandelbrot set is

$$M := \{c \in \mathbf{C} \mid K_c \text{ is connected }\}.$$

By the work of Julia and Fatou mentioned earlier

$$M = \{c \in \mathbf{C} \mid 0 \in K_c\}.$$

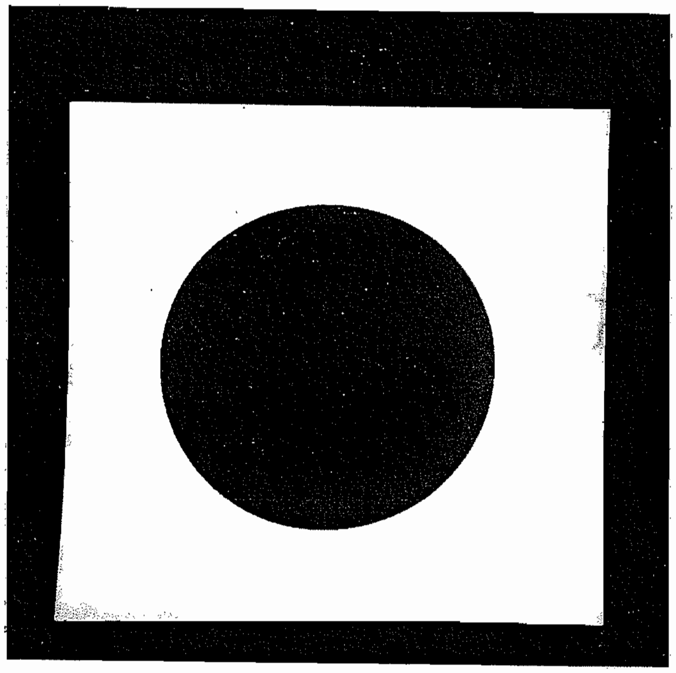

Figure 1.15. Thurston construction of 1/4 mating with 1/4, 0 lifts blown up.

Figure 1.16. Thurston construction of 1/4 mating with 1/4, 1 lift.

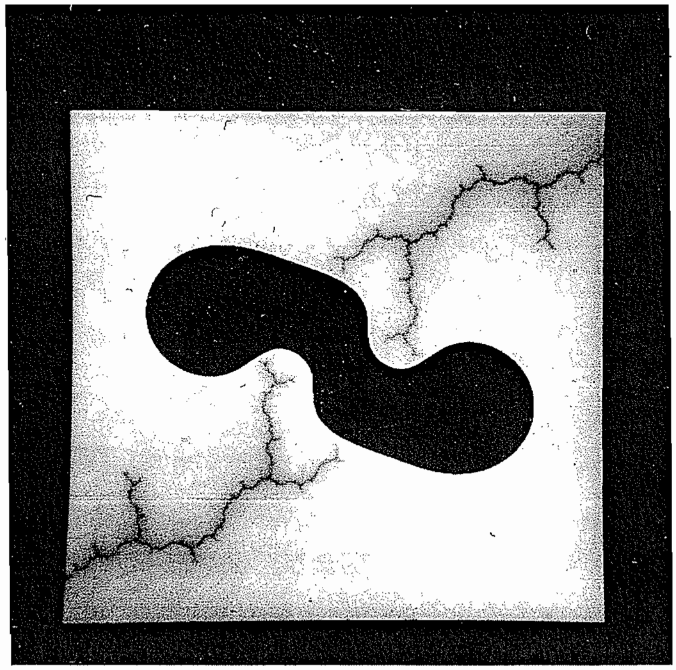

Figure 1.17. Thurston construction of 1/4 mating with 1/4, 2 lifts.

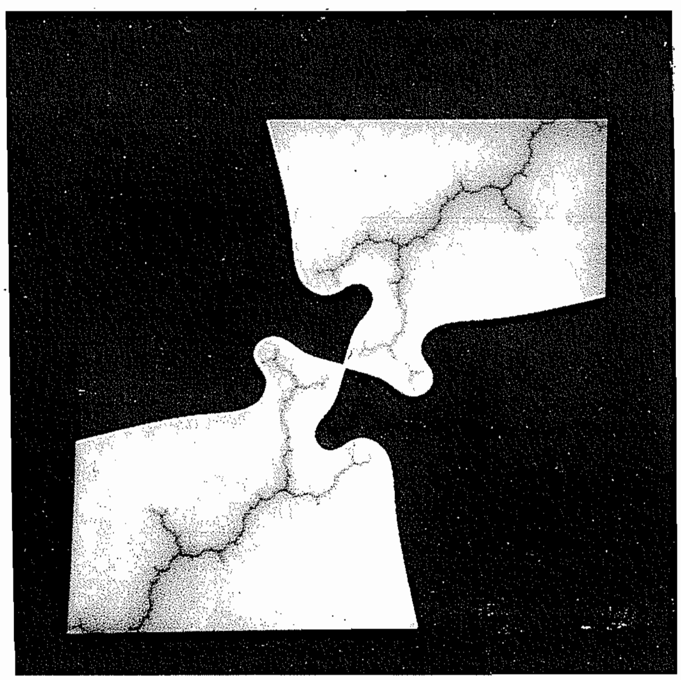

Figure 1.18. Thurston construction of 1/4 mating with 1/4, 3 lifts.

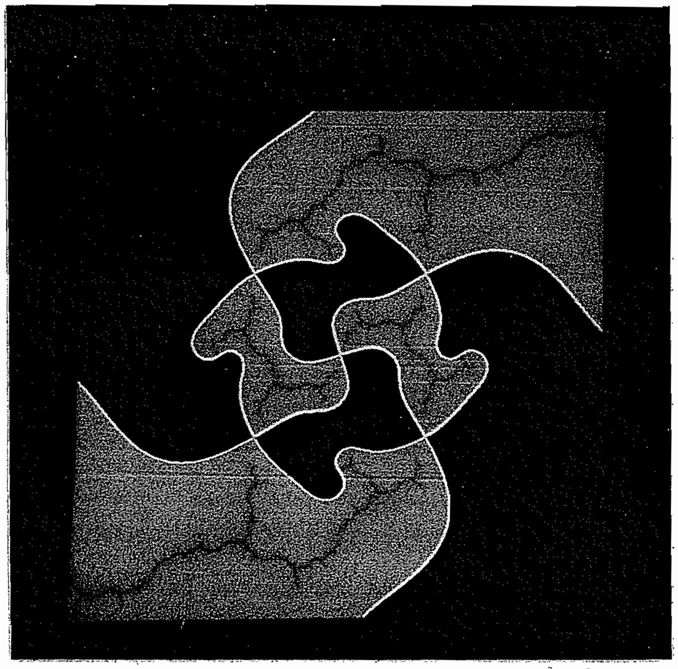

Figure 1.19. Thurston construction of 1/4 mating with 1/4, 4 lifts.

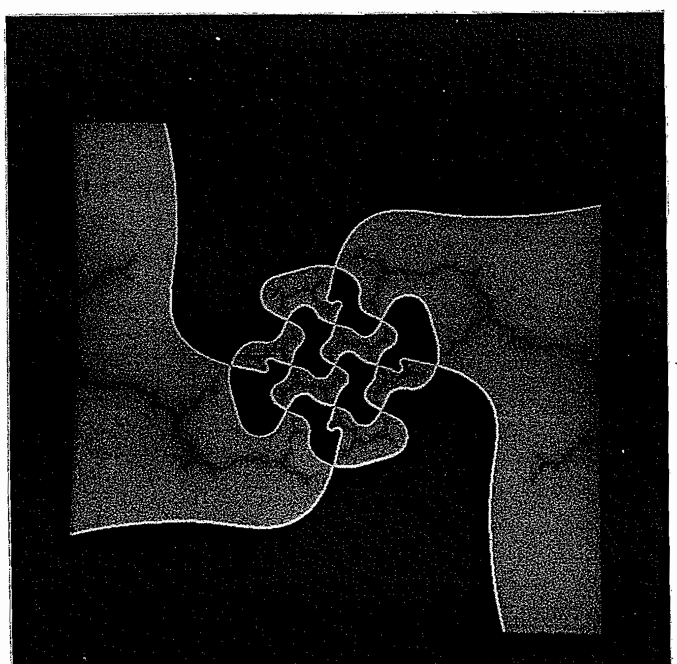

Figure 1.20. Thurston construction of 1/4 mating with 1/4, 5 lifts.

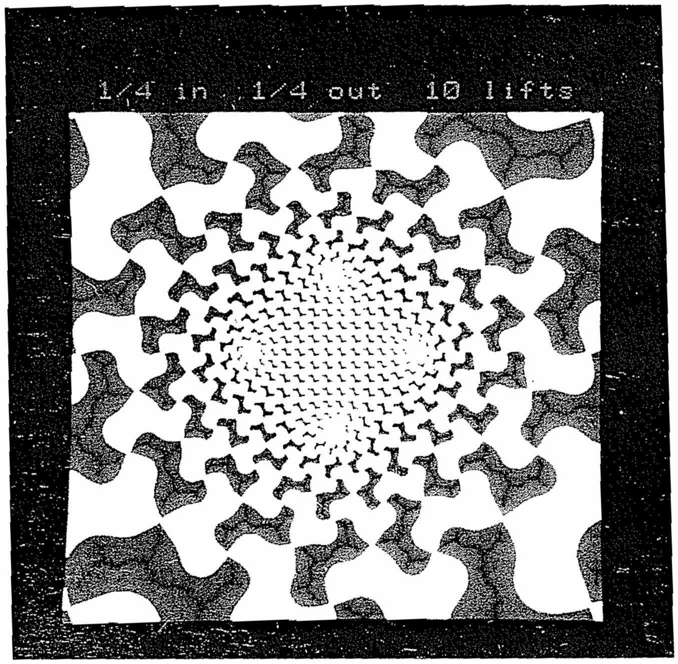

Figure 1.21. Thurston construction of 1/4 mating with 1/4, 10 lifts.

This suggests a way to make a computer picture of M. Namely, for some sampling of c in $\mathbf{C}$, see if 0 tends to infinity under repeated applications of f_c. Figure 1.22 is such a picture, with M in black and the complement of M shaded so that M will not be missed where it is thin. Outlined in figure 1.22 is figure 1.23.

Douady and Hubbard proved that M is connected. In figure 1.24 we have approximated in white how to connect a point in the interior of M to two boundary points. How to connect those points is probably obvious from figure 1.23, but in versions of M to be seen below it might not be obvious and the corresponding white lines (called veins) will be of help.

As indicated in figures 1.22 and 1.23, M is a cardioid with other parts attached. Each of those other parts is called a *limb* and is attached to the central

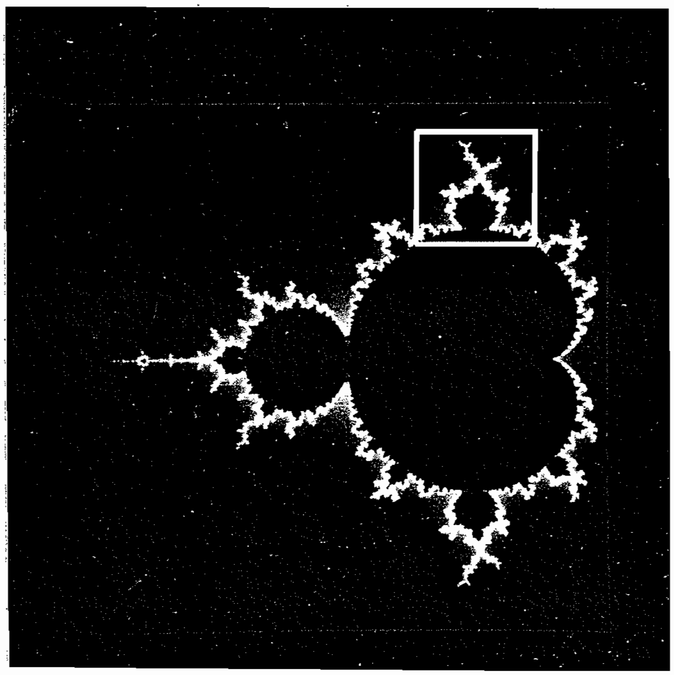

Figure 1.22. Mandelbrot set.

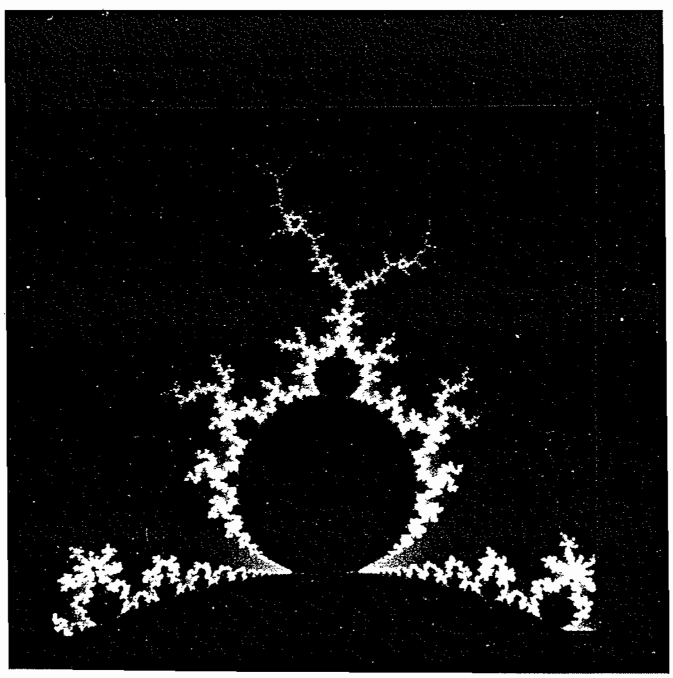

Figure 1.23. Blow up of Mandelbrot set.

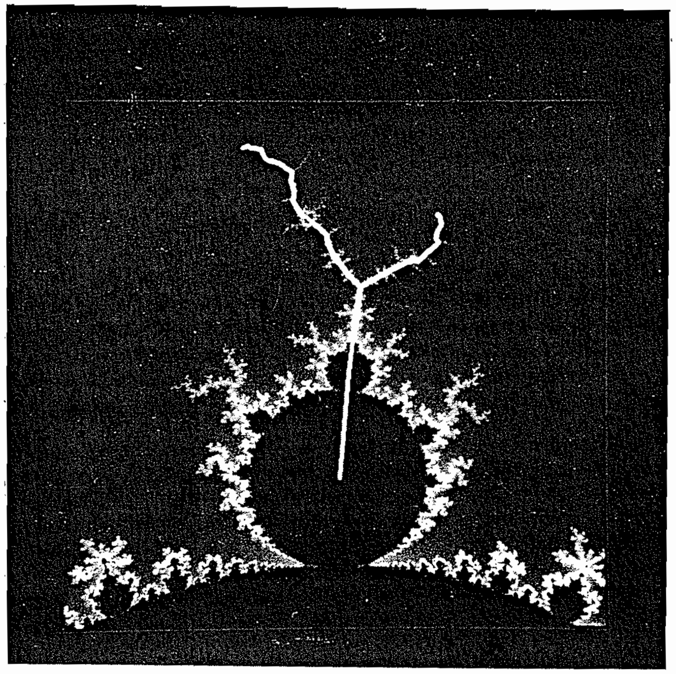

Figure 1.24. Veins of the Mandelbrot set.

cardioid at exactly one point. If the points of attachment of two limbs are complex conjugates of each other, then the limbs are called *conjugate limbs.*

Finally, we wish to describe a sort of Carathéodory loop for M. Think of M as made of a conducting piece of metal and put electric charge on M. Near infinity, the electric field lines of M will be asymptotic to rays emanating from the origin. To define $\gamma_M(t)$, find the electric field line which is asymptotic near infinity to the ray emanating from 0 and passing through $e^{2\pi i t}$. Follow that electric field line in towards M. If you approach some one particular point of the boundary of M as you follow that field line in, that point is by definition $\gamma_M(t)$. Douady and Hubbard have shown that for rational t, $\gamma_M(t)$ is well defined. We think it is well defined for all t. (γ_M actually has a lot to do with the dynamics of quadratic polynomials, but we will not discuss that until the next chapter.)

§**1.6.** *A nice rational family.*

Suppose some rational function f of degree two is the mating of f_{c_0} with f_{c_1}. So the domain of f (i.e. the Riemann sphere) can be thought of as K_{c_0} sewn to K_{c_1}. Let x_0 (resp. y_0) be the point in the domain of f corresponding to $0 \in K_{c_0}$ (resp. $0 \in K_{c_1}$). Since f_{c_0} (resp. f_{c_1}) is not injective on any neighborhood of x_0 (resp. y_0), x_0 (resp. y_0) is a critical point of f. But rational functions of degree two only have two critical points. So x_0 and y_0 are all the critical points of f.

Now we consider rational functions f which are matings of f_{c_0} with f_{c_1}, where $c_0 = -1$. We have seen K_{c_0} in figure 1.1. Since 0 is periodic of period two under f_{c_0}, by the preceding paragraph, one of the critical points of f must be periodic of period two. By conjugating f with a Möbius transformation taking that critical point to ∞, its image to 0, and the other critical point to -1, f can be written in the form

$$f_d(z) := \frac{d}{z^2 + 2z},$$

for some $d \in \mathbf{C} - \{0\}$.

Conversely, if f_d is a mating of some f_{c_3} with some f_{c_4}, then 0 must be periodic of period two for at least one of f_{c_3} and f_{c_4}. But there is only one c (namely $c_0 = -1$) for which 0 is periodic of period two under f_c. So any f_d which is a mating is the mating of f_{c_0} with some f_{c_1}.

We would like to have an algorithm which given any $d \in \mathbf{C} - \{0\}$ determines if f_d is the mating of f_{c_0} with some f_{c_1}. Unfortunately, all we have is an algorithm that can detect that f_d is not the mating of f_{c_0} with some f_{c_1}. It is based on the fact that the orbit of 0 under f_{c_0} is in the interior of K_{c_0} (see figure 1.1). That means that if f_d is a mating of f_{c_0} with some f_{c_1}, then the critical point -1 of f_d cannot be attracted to the attractive cycle $(\infty, 0)$. Figure 1.25 is based on that algorithm. If for a particular d, -1 is attracted to the cycle $(\infty, 0)$, then d is marked red or green depending upon what iterate (mod 2) of f_d brings -1 near ∞. (The yellow in the photograph is due to the photography; there was no yellow on the computer screen.) Otherwise d is left black. There is some shading in the red and green, but that can be ignored. Figure 1.26 is outlined in figure 1.25.

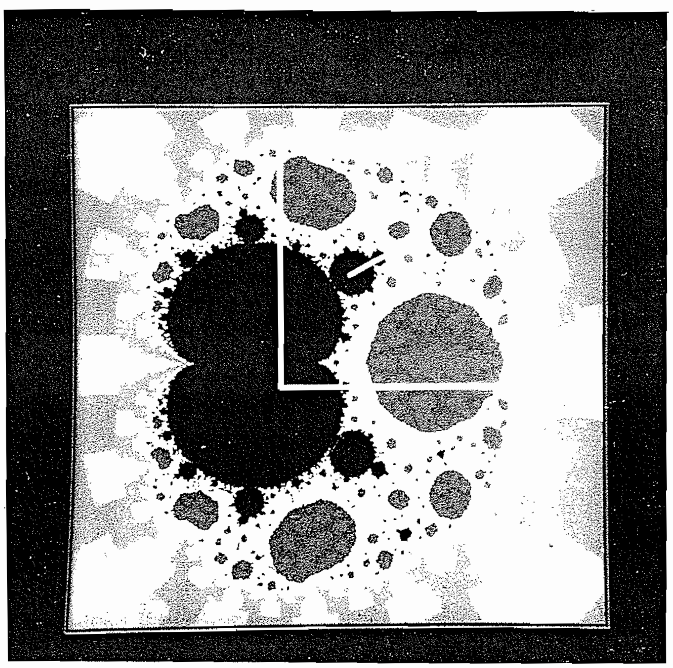

Figure 1.25. d-plane.

So the matings with f_{c_0} are all among the black. The black region resembles

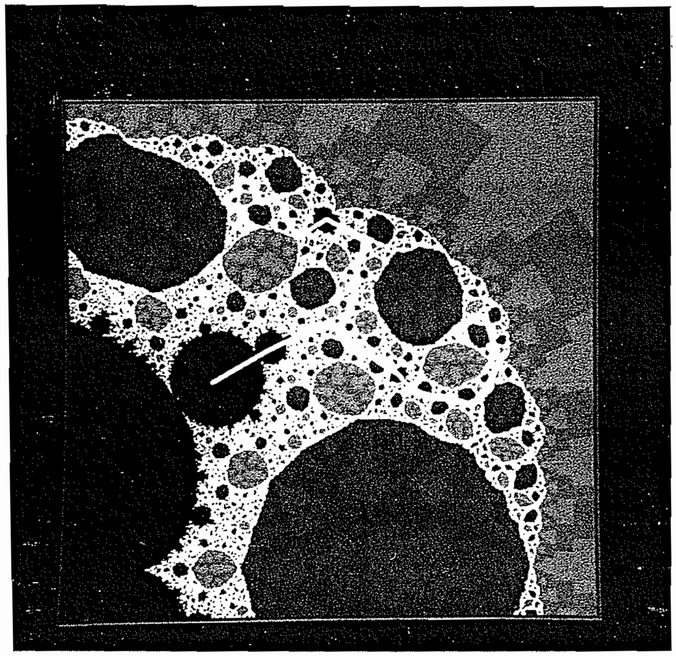

Figure 1.26. Blow up of d-plane.

M in many ways. By an argument slightly too involved for this introduction one can see that if c_0 and c_1 are in conjugate limbs of M, then the mating of f_{c_0} with f_{c_1} does not exist. Douady and Hubbard conjectured that the converse is true and that the black region in figure 1.25 is a Mandelbrot set with the limb containing $\bar{c}_0 = -1$ removed. In fact the point in M where that limb had been attached is now situated at $d = 0$, the only $d \in \mathbf{C}$ for which f_d is not a rational function of degree two. In figures 1.25 and 1.26 we have shown the veins corresponding to those in figure 1.24. We call this Mandelbrot set of matings with a limb removed a *mutilated Mandelbrot set* and the point where the removed limb had been attached is called the *amputation point*.

Actually, we believe the black set in figure 1.25 is as described above, but with some identification of boundary points of M. To see what identification there should be, we now consider the red and green regions.

It is true but not obvious that all points in the interior of K_{c_0} are attracted to the cycle $(0, -1)$ under f_{c_0} (Sullivan or Douady and Hubbard). Figure 1.27 shows K_{c_0} with different coloring than in figure 1.1. Points which are attracted to the cycle $(0, -1)$ under iteration of f_{c_0} are colored red or green depending upon which iterate (mod 2) of f_{c_0} brings the point near 0. (There is some shading in the red and green, but we can ignore that.) All other points are colored black. Figure 1.28 is figure 1.27 after it has undergone the invertible transformation $\phi(z) = -(z+1)/z$.

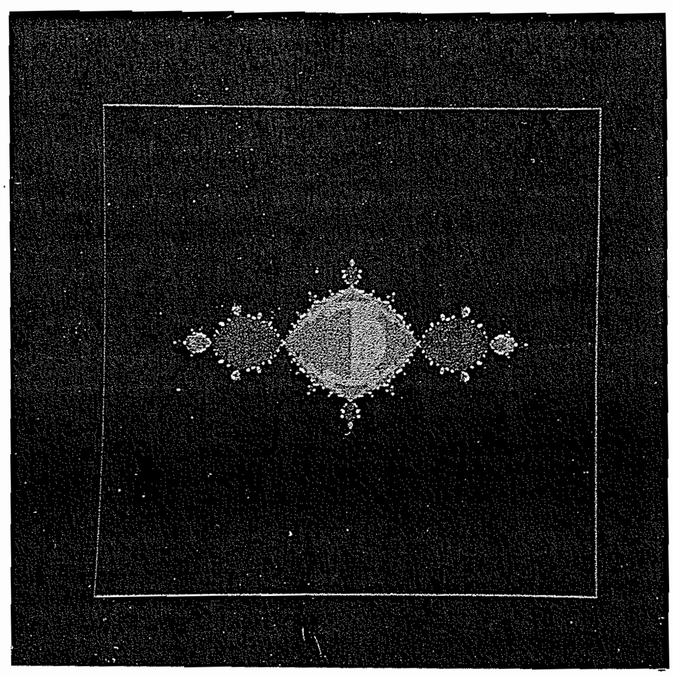

Figure 1.27. K_c for $c = -1$.

Notice the similarity between the red and green regions of the z-plane drawing in figure 1.28 and the red and green regions in the d-plane drawing in figure 1.25.

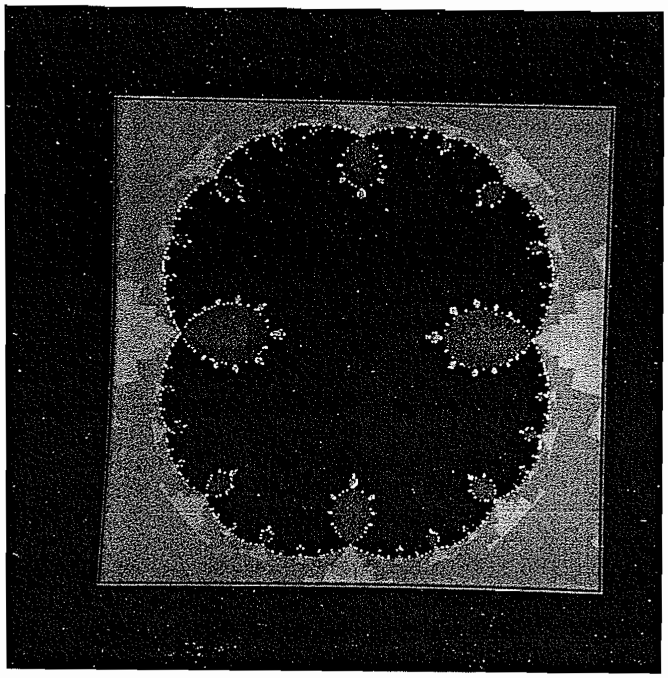

Figure 1.28. Inversion of K_c for $c = -1$.

In fact, what we believe we are seeing here is a mutilated K_{c_0} turned inside out and sewn into the mutilated M according to the rule $\gamma_M(t)$ sews to $\gamma_{c_0}(-t)$ (see figure 1.29 in which the mutilated limbs are shaded and some of the threads for sewing are shown with dotted lines). In order to show why we believe this, we have to discuss captures, the topic of the next section.

§1.7. *Captures.*

For some point y_1 in K_c, the capture at y_1 by f_c is a function built in some sense from f_c, but having y_1 as the image of a critical point other than 0. We illustrate with an example.

As in the previous section, let $c_0 = -1$ throughout. Let y_1 be the point in the interior of K_{c_0} indicated in figures 1.30 and 1.31. Figure 1.32 shows $\phi(y_1)$

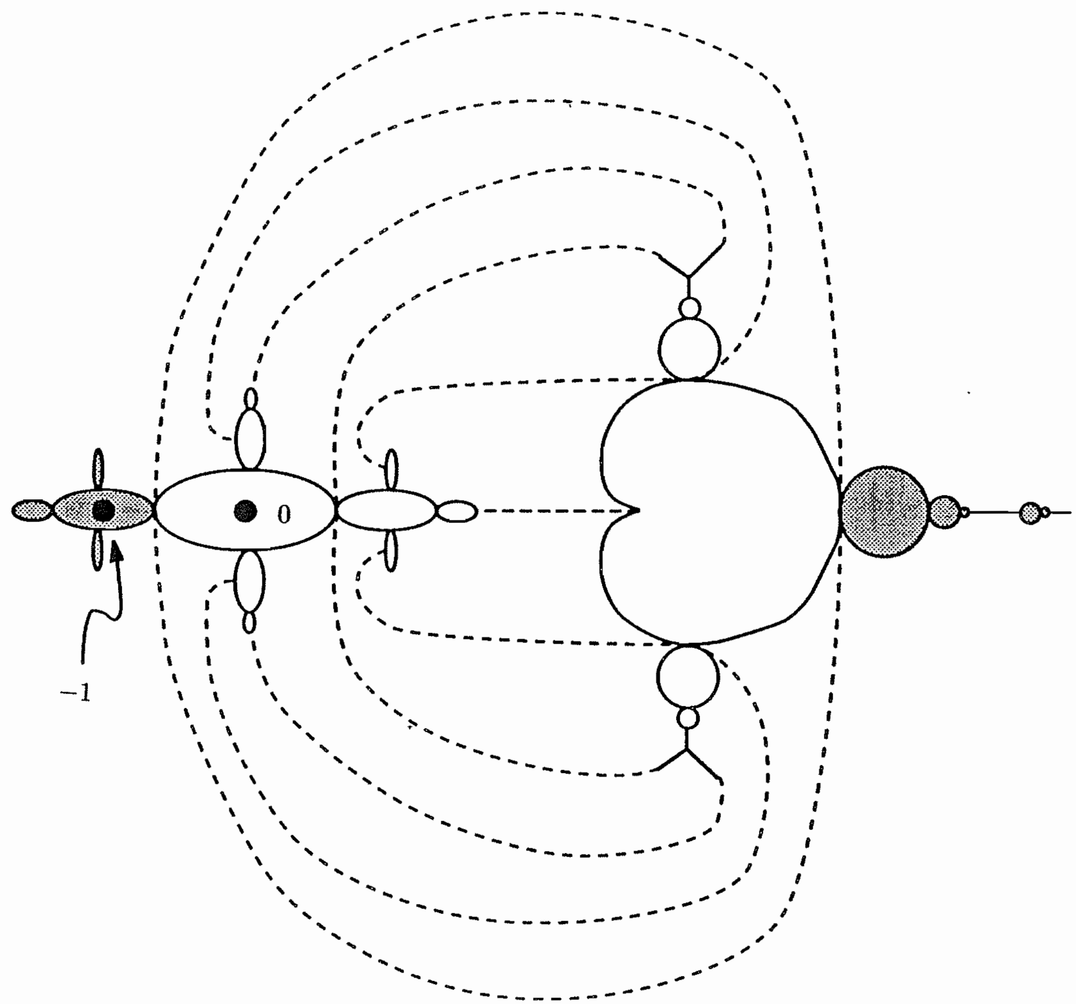

Figure 1.29. Schematic of mating of M with K_c for $c = -1$.

(where ϕ is as in the last section). We will suggest how to build f, the capture at y_1 by f_{c_0}. Unfortunately, we have not been able to make the suggested definition rigorous. We present it rather than the definition we have been able to make rigorous because it gives a better feel for captures. (To lighten notation, we no longer distinguish between $\phi(K_{c_0})$ and K_{c_0} or between $\phi \circ f_{c_0} \circ \phi^{-1}$ and f_{c_0}.)

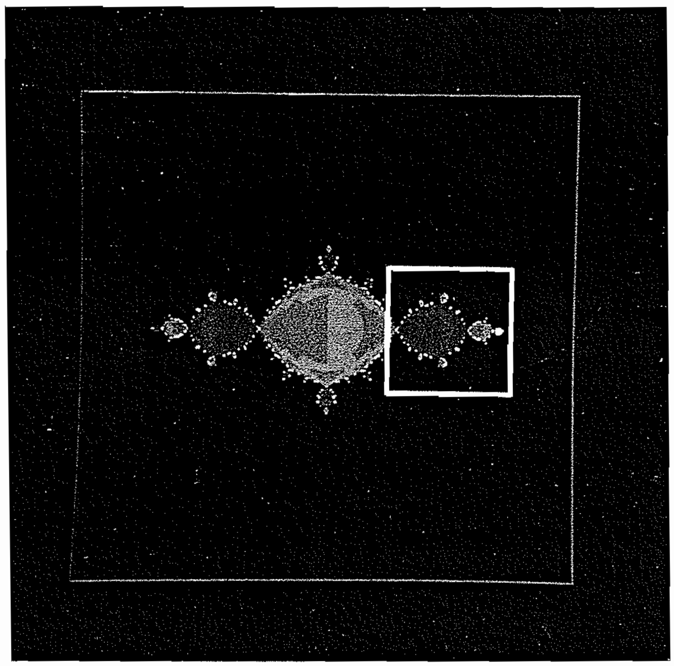

Figure 1.30. Point in K_c for $c = -1$.

Start by letting f equal f_{c_0} on K_{c_0} and let $\gamma := \gamma_{c_0}$. Since y_1 is not the image of a critical point of f, we make the following modification. Figure 1.33 is a schematic drawing of figure 1.32 with y_1 in the component labeled W. The inverse image under f of W is U' and U''. Let t_w be such that $\gamma(t_w)$ is on the boundary of W. Recall that on the boundary of K_{c_0}, f is given by

$$f(\gamma(t)) = \gamma(2t).$$

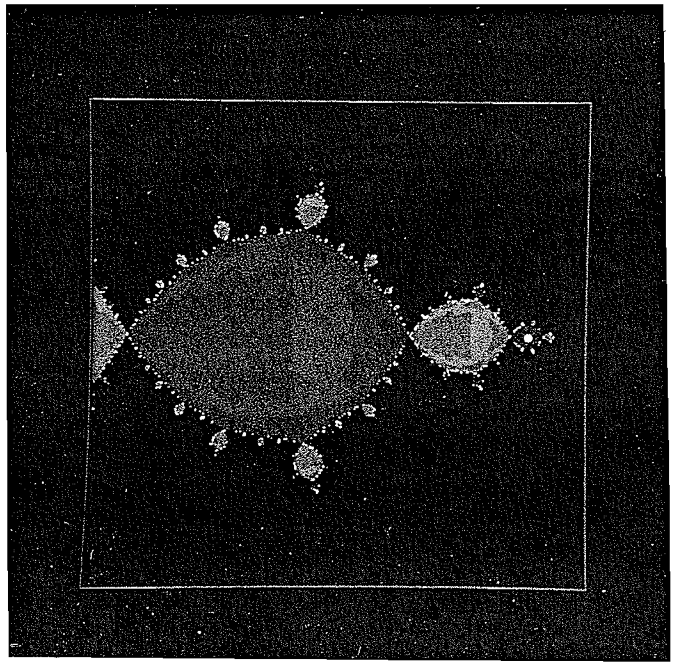

Figure 1.31. Blow up of point in K_c for $c = -1$.

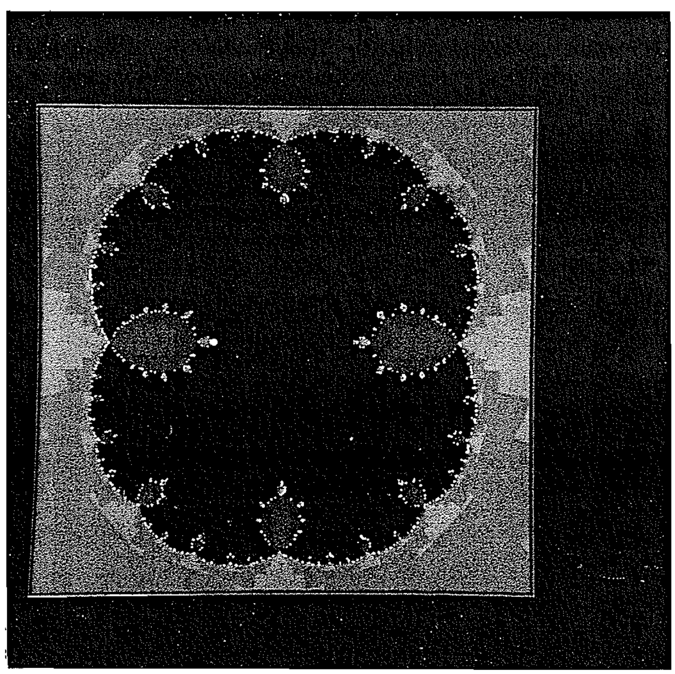

Figure 1.32. Inversion of point in K_c for $c = -1$.

So if we let $t_{u'} := t_w/2$ and $t_{u''} := t_{u'} + (1/2)$, then $\gamma(t_{u'})$ is on the boundary of U', $\gamma(t_{u''})$ is on the boundary of U'', and

$$f(\gamma(t_{u'})) = f(\gamma(t_{u''})) = \gamma(t_w).$$

Now cut γ at $t_{u'}$ and $t_{u''}$ (see figure 1.34). Reconnect (i.e. deform γ) as indicated in figures 1.35, 1.36, and 1.37. Now under the map $\gamma(t) \mapsto \gamma(2t)$, the boundary of the new component U wraps twice around the boundary of W. So we can let f map U to W so that in some coordinates on U and W f is $z \mapsto z^2$ and so that y_1 is the image of the critical point of f in U.

But now, since U' and U'' no longer exist, f is undefined on the two components which f used to map to U' and the two components which f used to map to U''. Also, f is now discontinuous on the two inverse images of $\gamma(t_{u'})$ and the two inverse images of $\gamma(t_{u''})$. So cut γ at those points (see figure 1.38) and reconnect in the only way possible (see figure 1.39). The boundary of each of the two new components V' and V'' wraps once around the boundary of U under $\gamma(t) \mapsto \gamma(2t)$. So let f map V' (resp. V'') onto U homeomorphically (i.e. so that in some coordinates on V' (resp. V'') and on U, f is the identity).

Hopefully, one can continue cutting, reconnecting, and mapping the new components homeomorphically. The new components can be formed so that the set on which f is defined is dense in the sphere. We hope that f extends continuously to the whole sphere. We also hope that the f we have formed is independent (up to conjugacy) of the choices we have made.

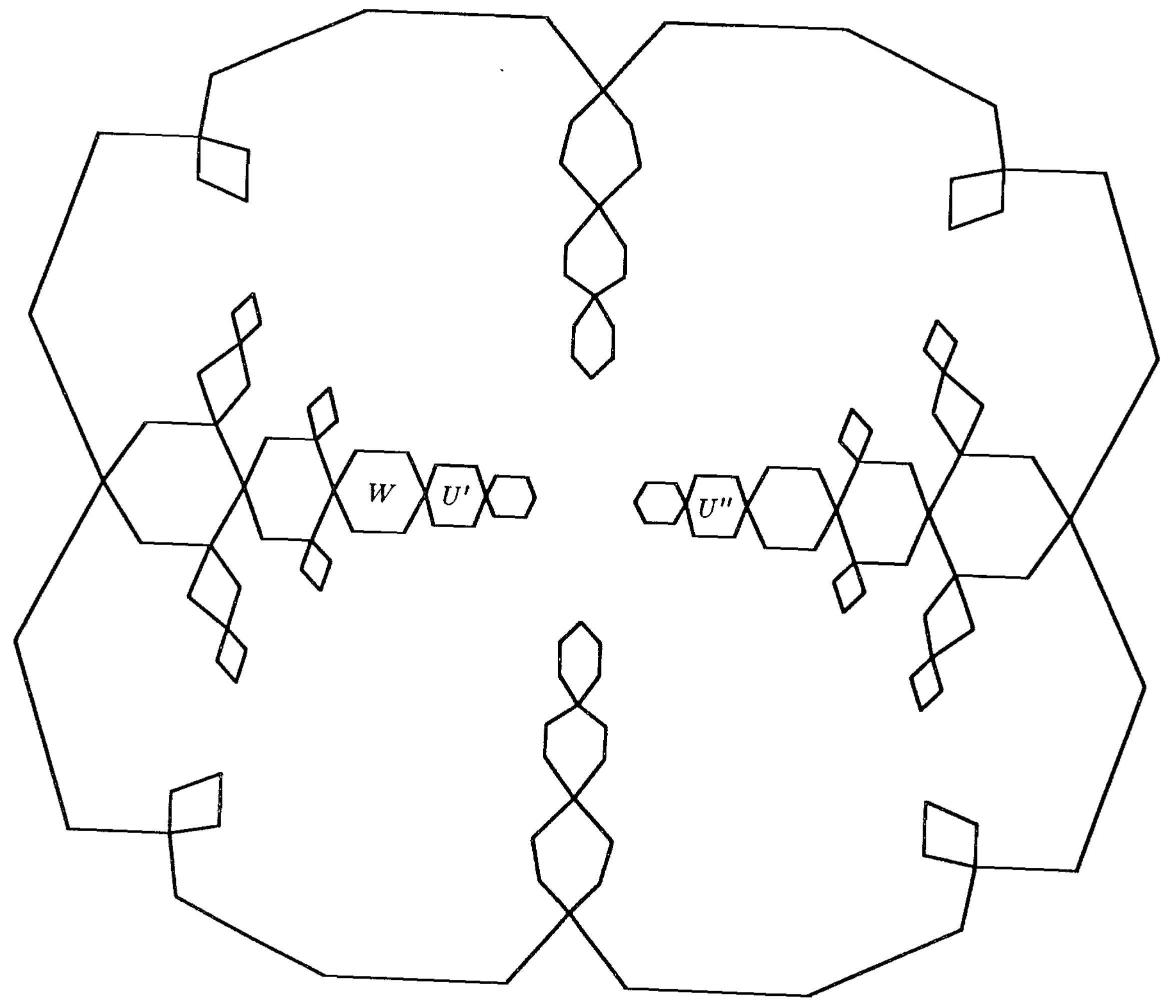

Figure 1.33. Construction of capture. Step 1.

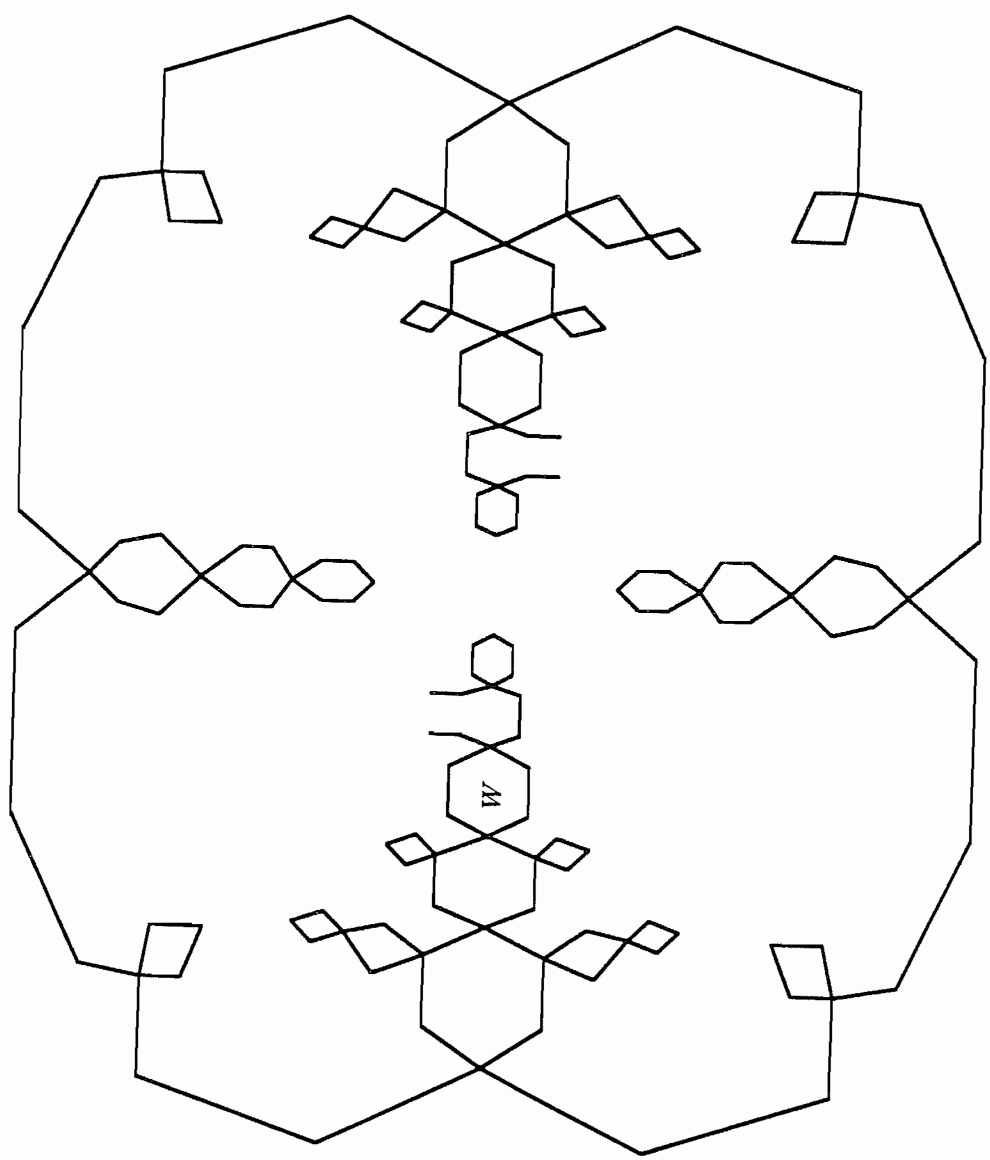

Figure 1.34. Construction of capture. Step 2.

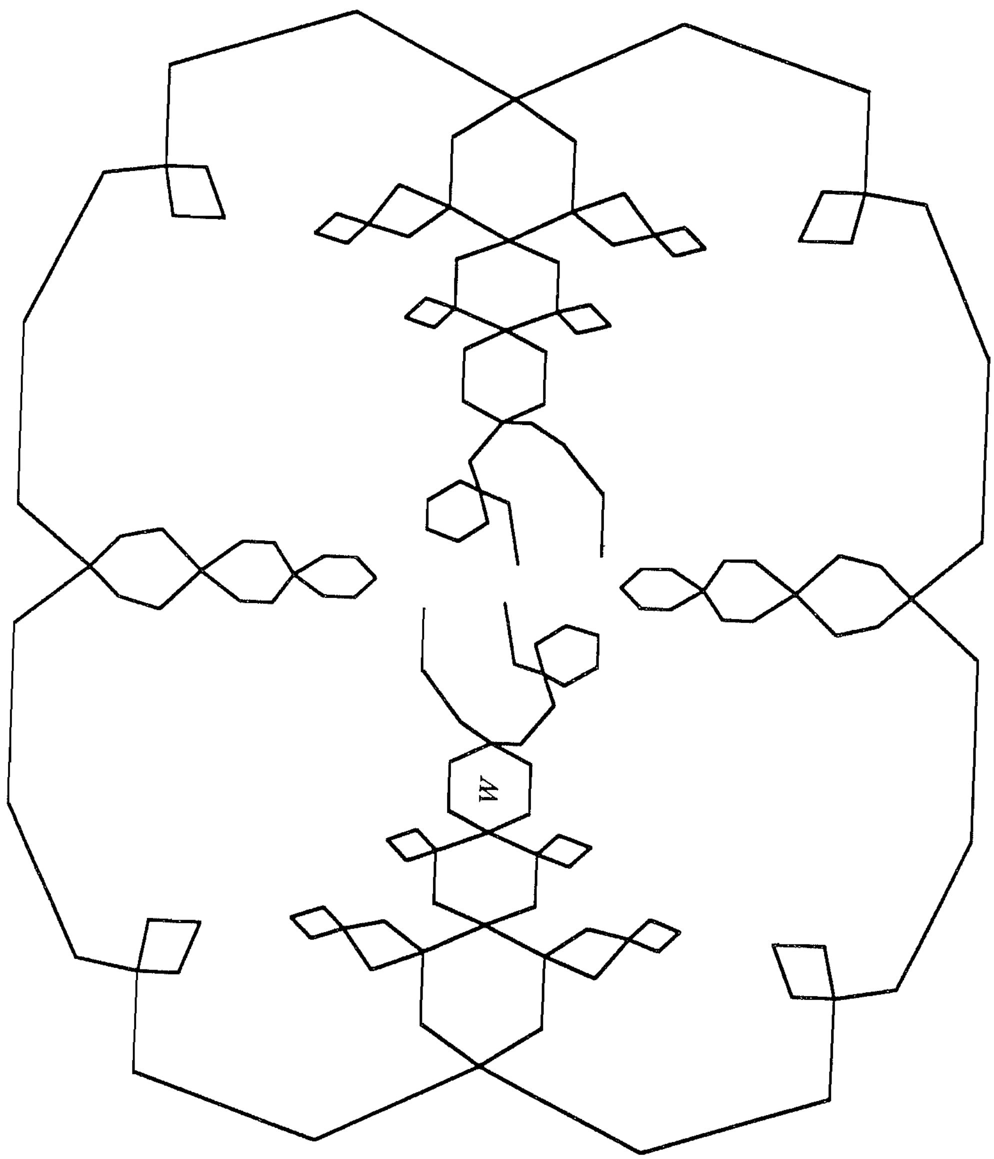

Figure 1.35. Construction of capture. Step 3.

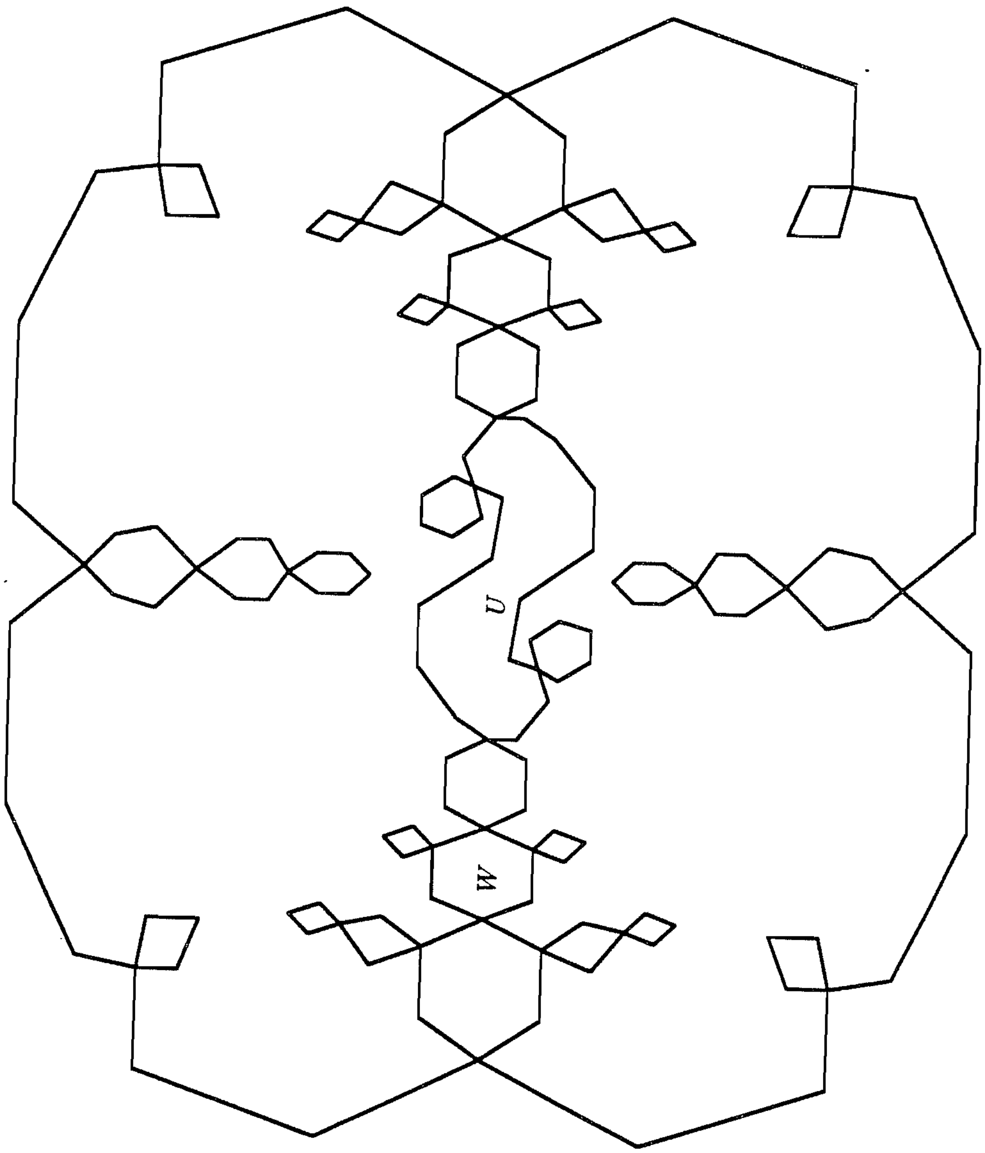

Figure 1.36. Construction of capture. Step 4.

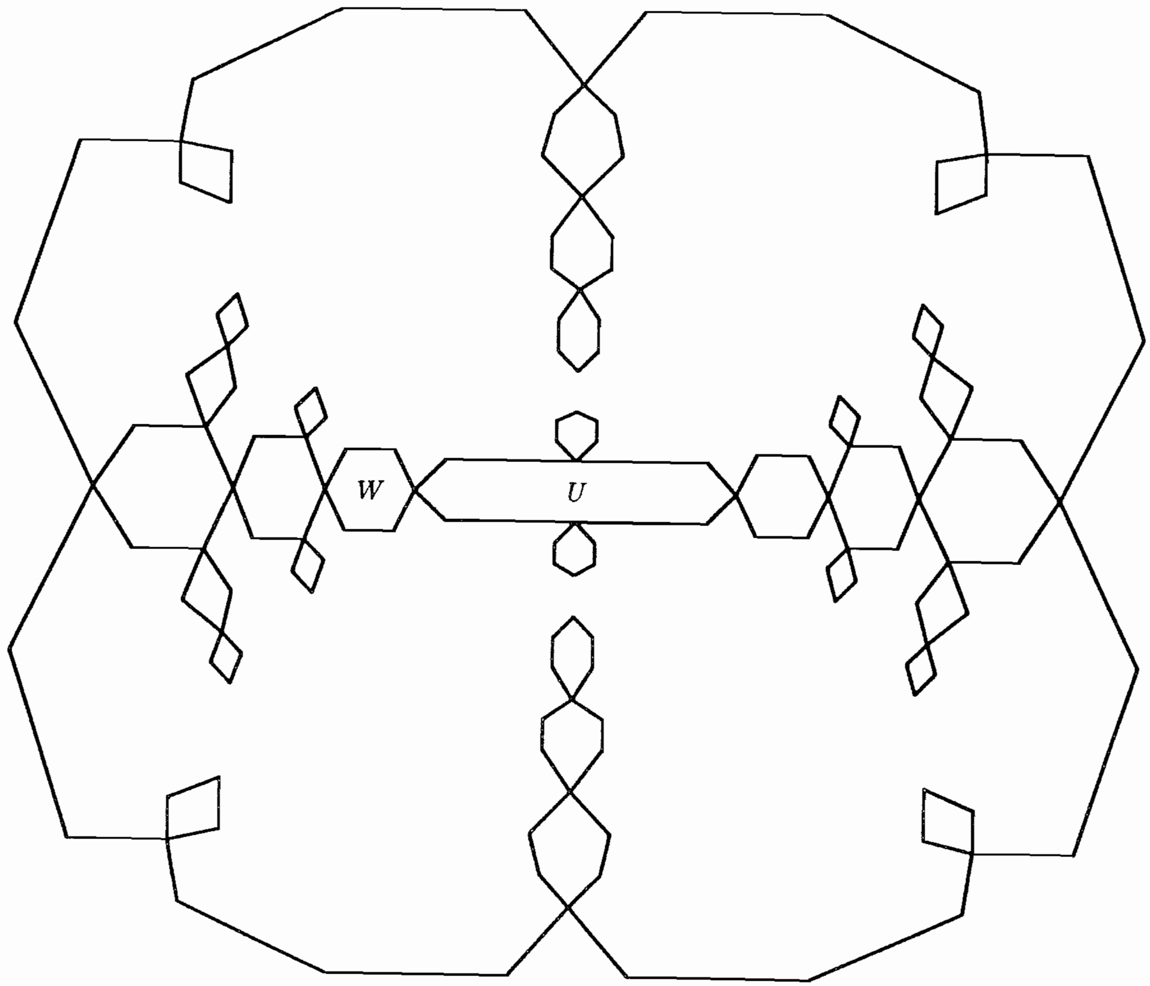

Figure 1.37. Construction of capture. Step 5.

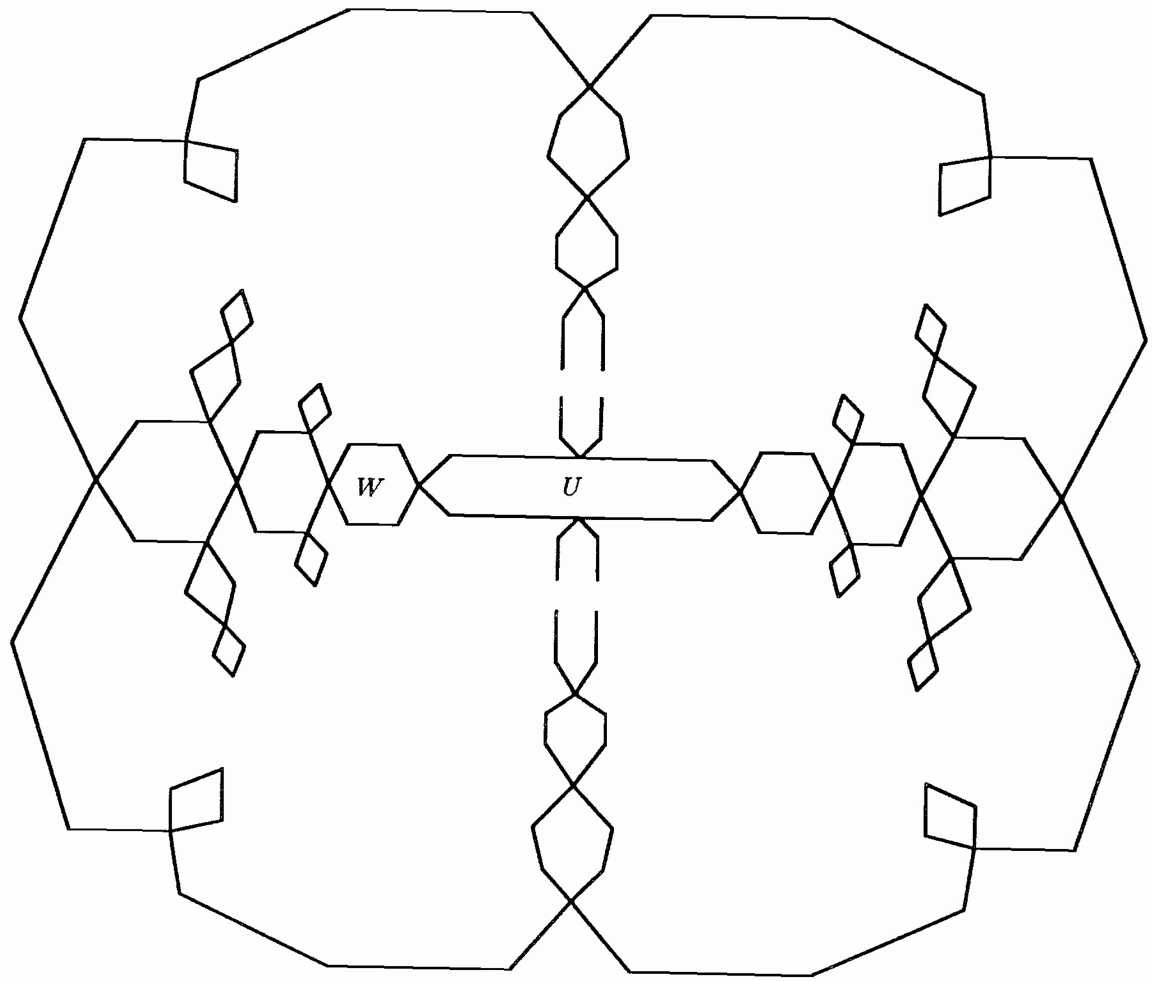

Figure 1.38. Construction of capture. Step 6.

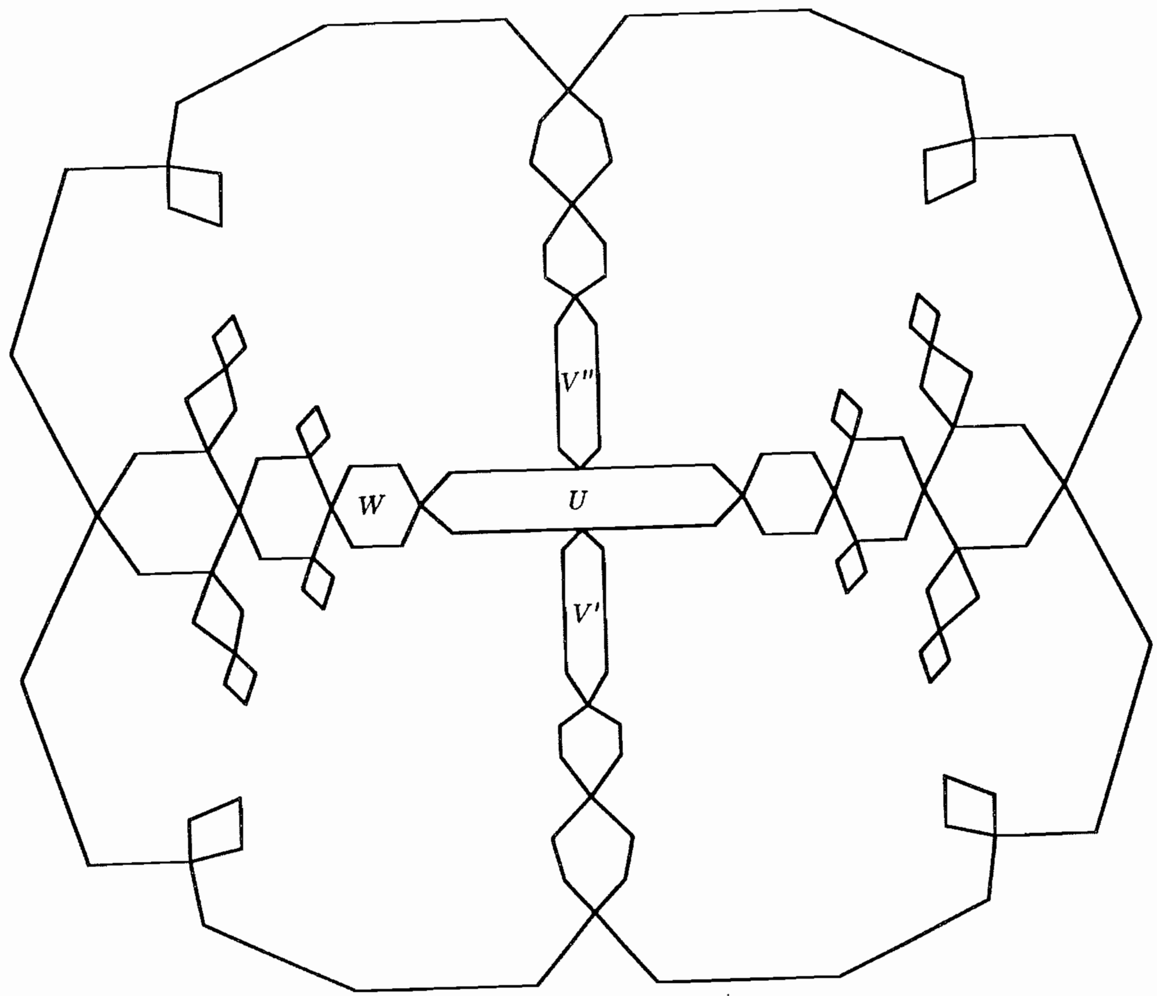

Figure 1.39. Construction of capture. Step 7.

As mentioned above, we have not been able to make this definition rigorous, but our computer experiments suggest it is correct. For example, to form figure 1.40 we chose a particular d in $\mathbf{C} - \{0\}$ and colored points black if they were not attracted to the cycle $(\infty, 0)$ and otherwise red or green, depending upon which iterate (mod 2) of f_d took the point near ∞. (Again, the yellow is due to the photography.) The critical point of f_d not equal to ∞ is -1. We have marked $f_d(-1)$ with a white dot and $f_d(\infty) = 0$ with a white ex. Figure 1.41 is figure 1.40 with everything blacked out except the components containing ∞ and 0. Figures 1.42 through 1.49 show successive inverse images under f_d of the colored regions in figure 1.41.

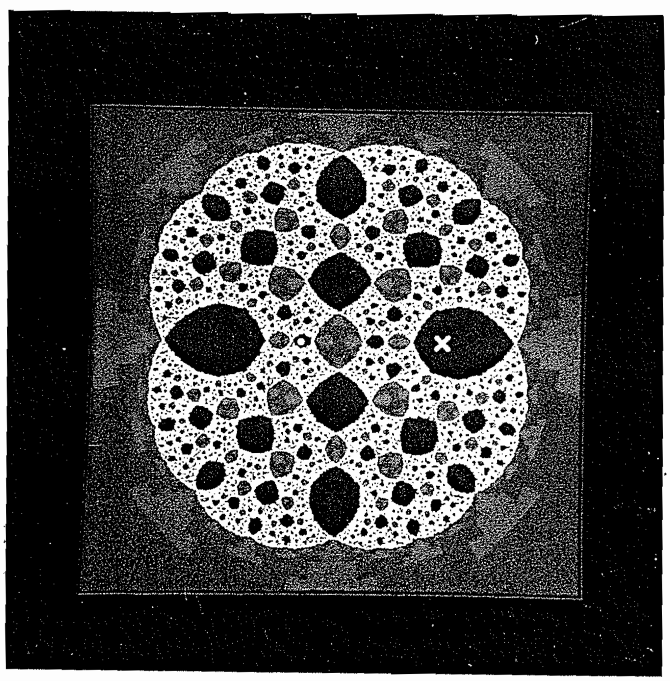

Figure 1.40. A capture.

In defining f we did not alter the component of K_{c_0} containing 0 and $f = f_{c_0}$

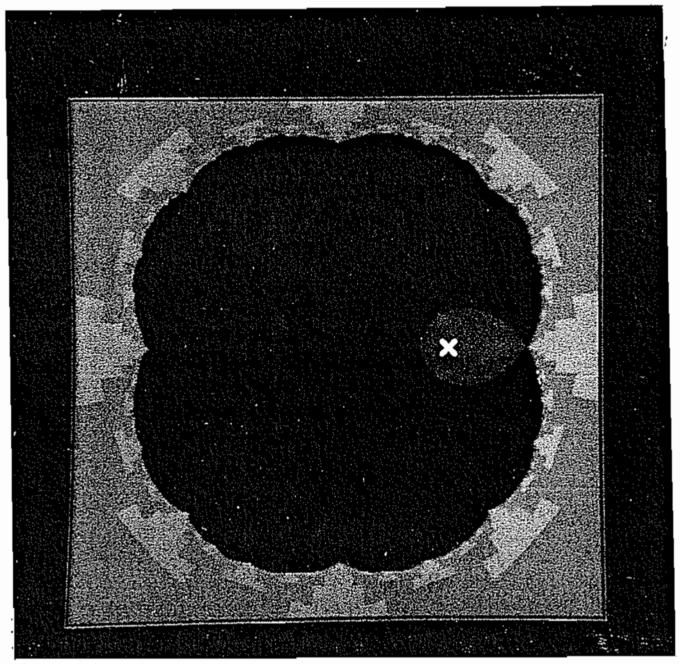

Figure 1.41. Blackened capture, 0 lifts.

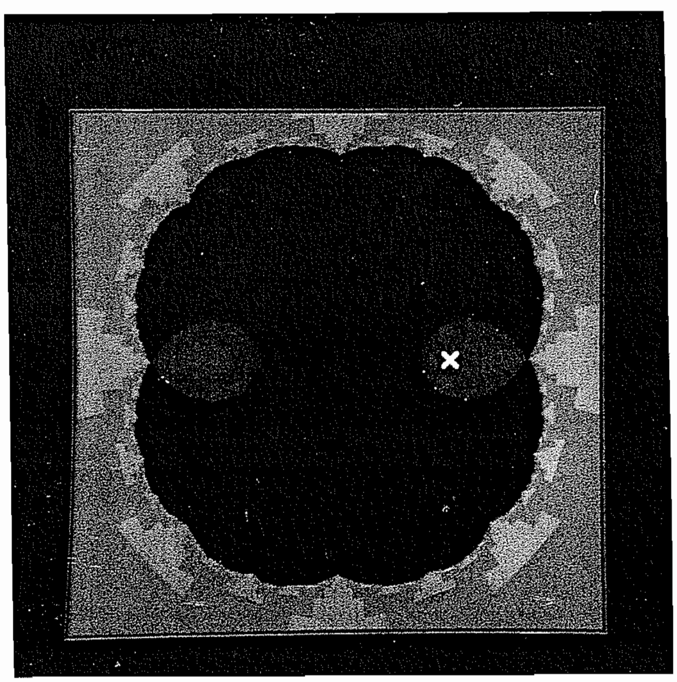

Figure 1.42. Blackened capture, 1 lift.

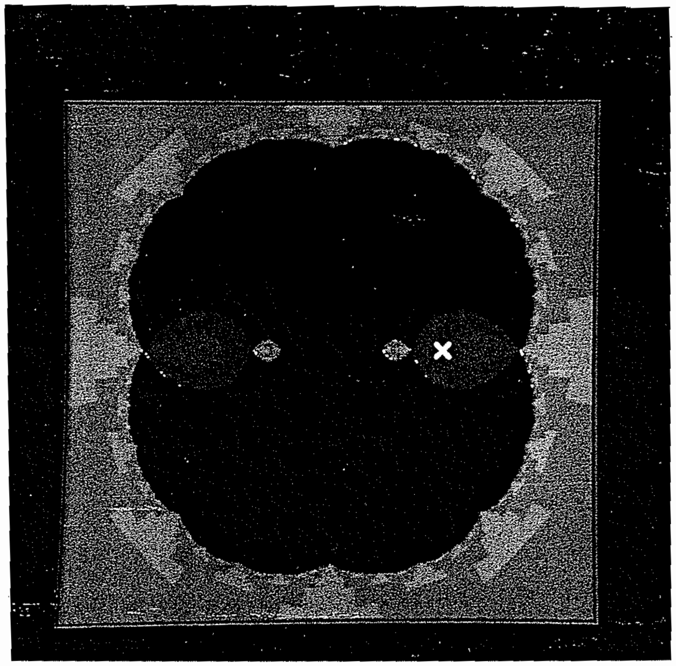

Figure 1.43. Blackened capture, 2 lifts.

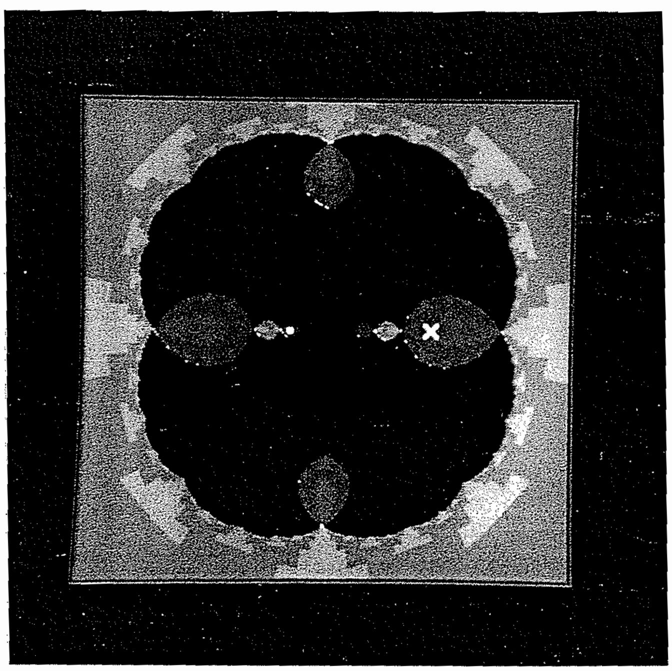

Figure 1.44. Blackened capture, 3 lifts.

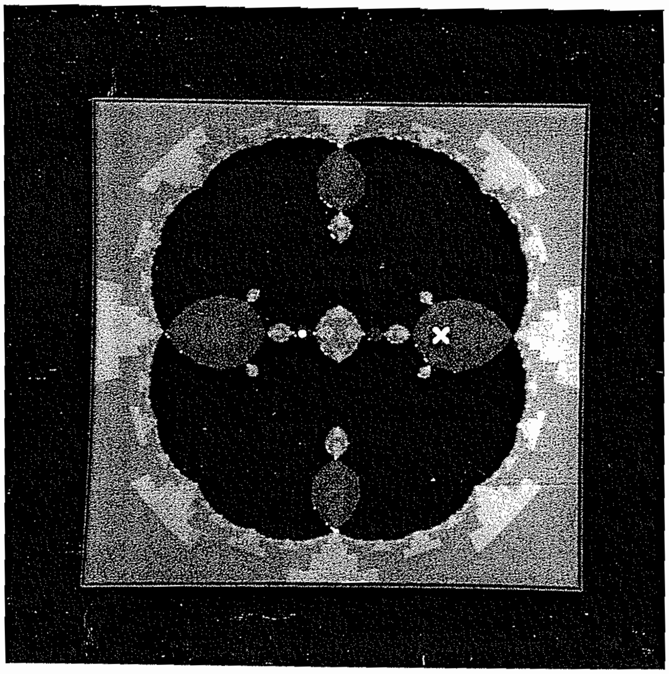

Figure 1.45. Blackened capture, 4 lifts.

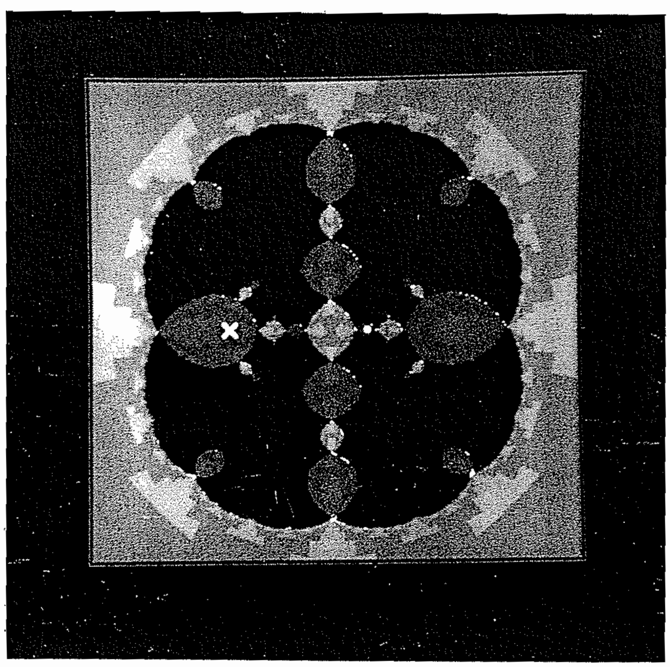

Figure 1.46. Blackened capture, 5 lifts.

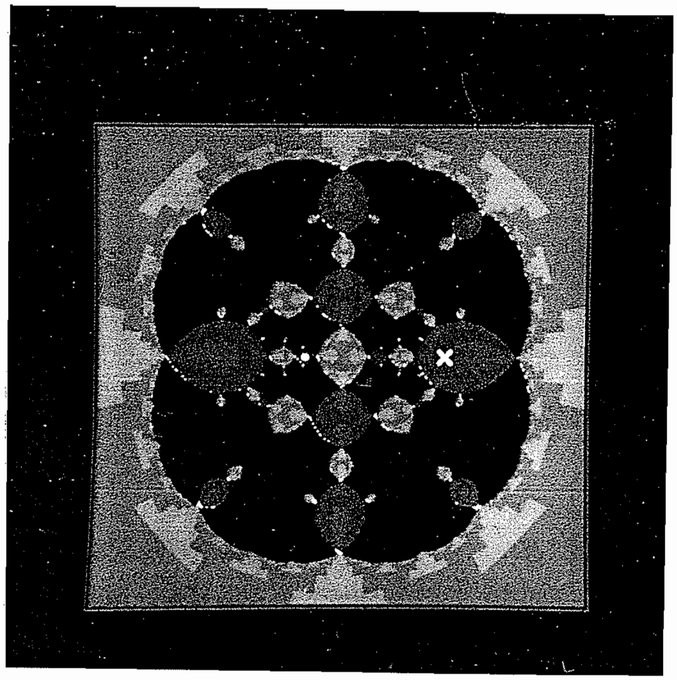

Figure 1.47. Blackened capture, 6 lifts.

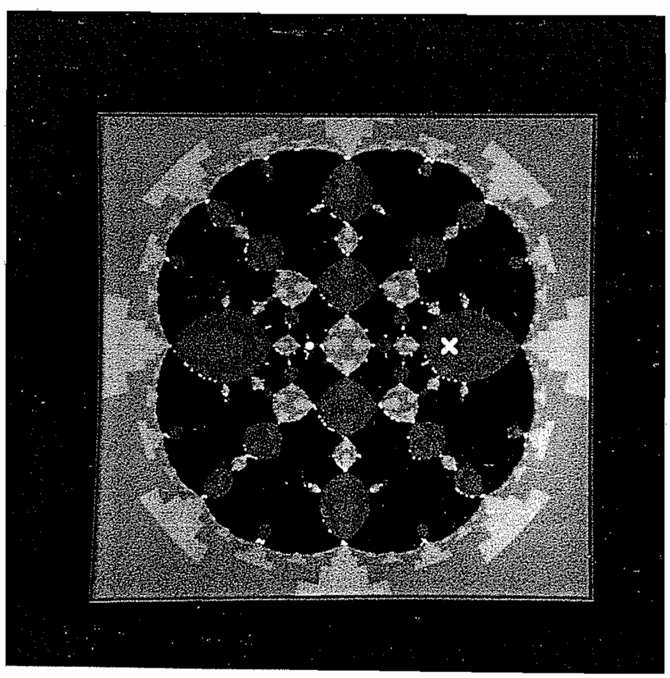

Figure 1.48. Blackened capture, 7 lifts.

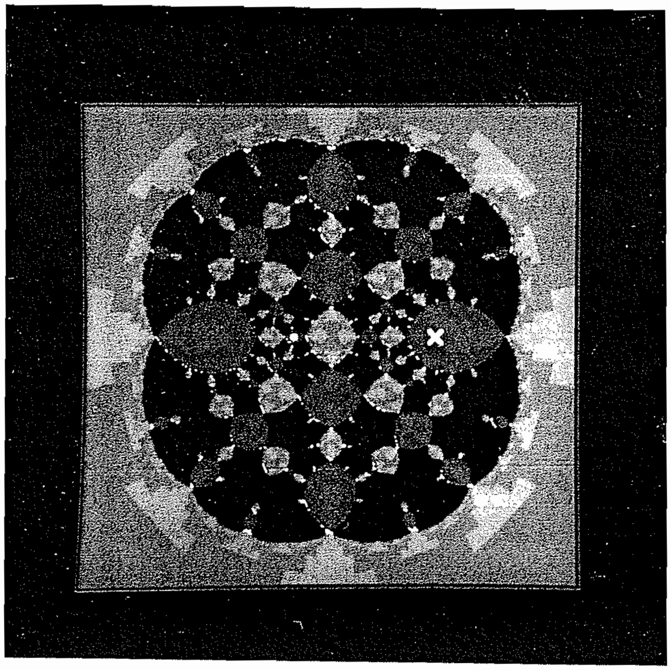

Figure 1.49. Blackened capture, 8 lifts.

on that component. So f has a critical point x_0 corresponding to the critical point 0 of f_{c_0}. f also has a critical point, which we shall call y_0, which we created when we let U map to the component containing y_1 like $z \mapsto z^2$. Also, since we did not alter the component of K_{c_0} containing $f_{c_0}(0) = -1$, and $f = f_{c_0}$ on that component, $f(f(x_0)) = x_0$.

It is important to note further that in defining f we did not alter in any way the components of K_{c_0} containing the orbit of y_1 under f_{c_0} and we left $f = f_{c_0}$ on those components. So since every point in the interior of K_{c_0} is attracted under iteration of f_{c_0} to the cycle $(0, -1)$, y_1 will be attracted to the cycle $(x_0, f(x_0))$ under iteration of f.

We have suggested a definition of the capture at y_1 by f_{c_0} for a particular y_1

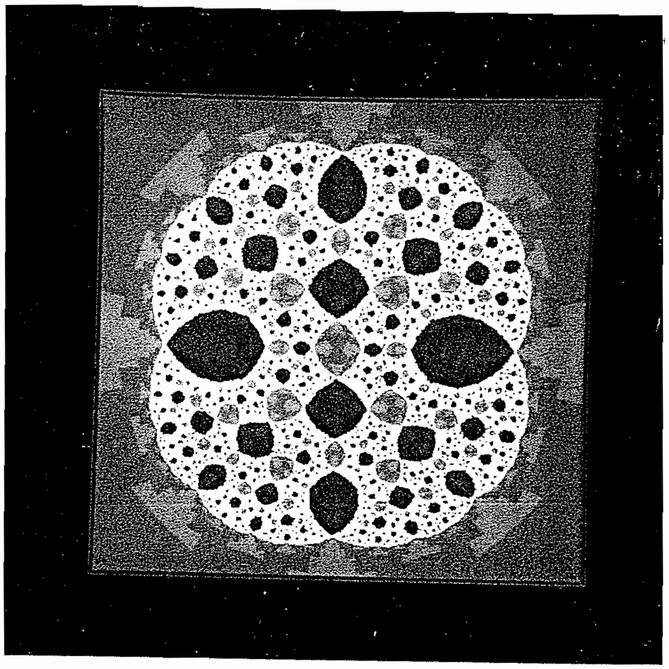

Figure 1.50. Moving capture, step 0.

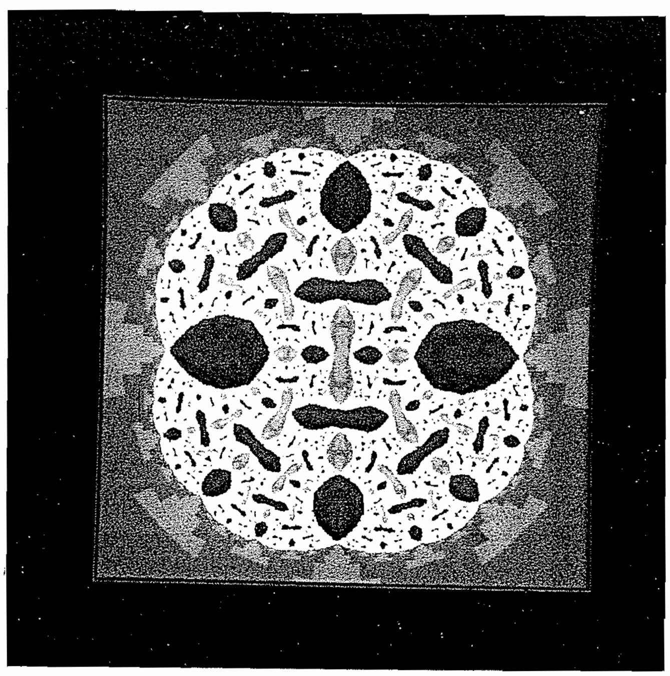

Figure 1.51. Moving capture, step 1.

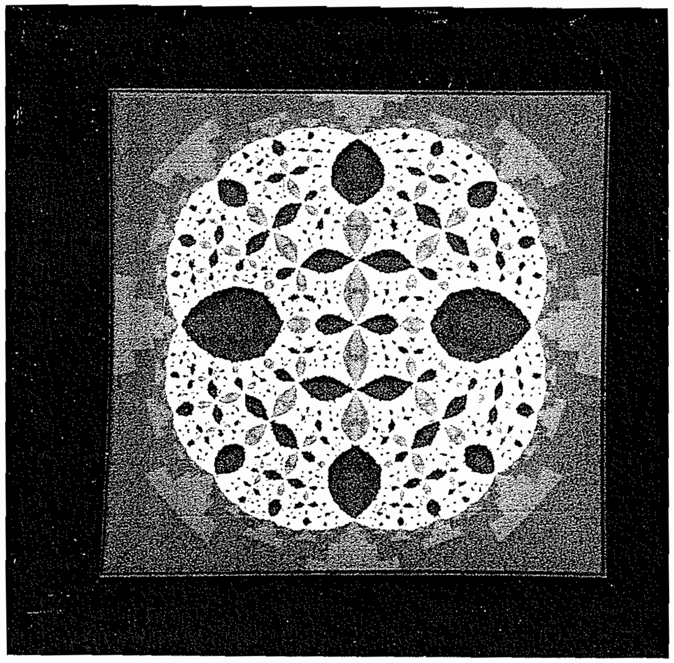

Figure 1.52. Moving capture, step 2.

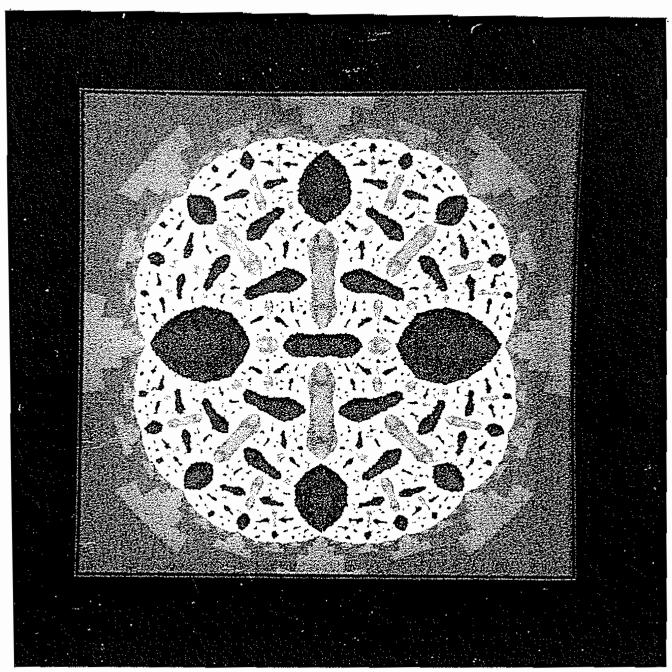

Figure 1.53. Moving capture, step 3.

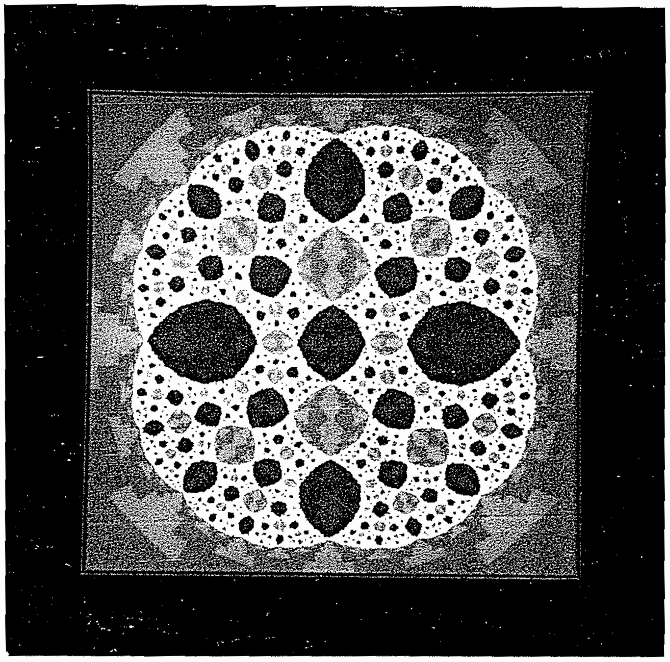

Figure 1.54. Moving capture, step 4.

pairs of components of K_{c_0} with single components. In deforming this capture to a capture at y_1' for y_1' on the boundary, we have made each single component back into a pair of components. We claim that the components have all the necessary connections to reform K_{c_0}. In fact, the connections between components of K_{c_0} were never destroyed; the pairs of components were just merged to single components having connections to twice as many components as either component in the pair. But whereas each pair was well separated in K_{c_0}, in the capture at y_1', they are touching at the pinch point. So the capture at y_1' by f_{c_0} can be viewed as formed by taking f_{c_0} acting on K_{c_0}, then pulling appropriate pairs of points in

$$\bigcup_{n=1}^{\infty} f_{c_0}^{-n}(\{y_1'\})$$

together. (Since

$$\bigcup_{n=1}^{\infty} f_{c_0}^{-n}(\{y_1'\})$$

is dense in the boundary of K_{c_0}, this causes some further identification of points on the boundary of K_{c_0}.)

Not all points in the boundary of K_{c_0} are on the boundary of a component of the interior of K_{c_0}, but the view of capture at a boundary point presented in the preceding paragraph makes sense for those points also. We have some reason to believe that the capture at such a point y_1' so defined would be in some sense the limit of captures at points in the centers of a sequence of components of the interior of K_{c_0} approaching y_1'.

§**1.8.** *A nice rational family revisited.*

We now can explain why we believe that figure 1.25 is a mutilated Mandelbrot set sewn to a mutilated K_{c_0} according to the rule $\gamma_M(t)$ sews to $\gamma_{c_0}(-t)$. As in the previous two sections, let $c_0 = -1$ throughout.

Recall that the points d in figure 1.25 colored red or green are the d for which the critical point -1 of f_d is attracted to the cycle $(\infty, 0)$ and the choice of red and green depends upon what iterate (mod 2) of f_d carries -1 near ∞. Now suppose f_{y_1} is a rational function of degree two which is conjugate to the capture at y_1 by f_{c_0}. Except for the special case where y_1 equals the critical point x_0 of f_{y_1}, f_{y_1} will be conjugate to f_d for some d by a Möbius transformation taking x_0 to ∞, $f_{y_1}(x_0)$ to 0, and y_0 to -1. Call that d, $d(y_1)$. If y_1 is in the interior of K_{c_0}, then $d(y_1)$

will have to be colored red or green as we mentioned in the previous section. It is reasonable to believe that $d(y_1)$ is continuous in y_1. (See figures 1.55 through 1.60. Figure 1.55 shows the same portion of the d-plane as does figure 1.26. Figure 1.56 is outlined in figure 1.55 and has four d marked with exes and numbered 0 through 3. Figures 1.57 through 1.60 show the corresponding z-plane of f_d with $f_d(-1)$ marked with a white dot and 0 with an ex.) So the closure of the red and green regions in figure 1.25 contains the continuous image of the set of y_1 in K_{c_0} such that the capture at y_1 by f_{c_0} is defined and conjugate to a rational function.

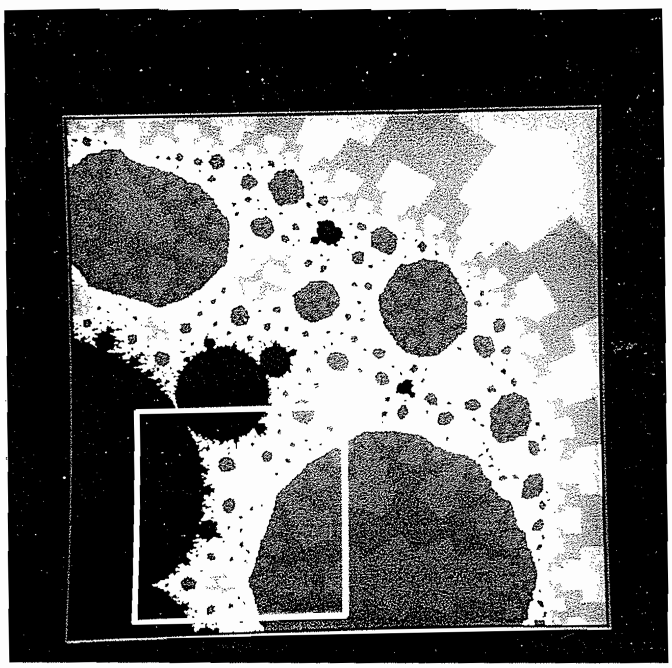

Figure 1.55. d-plane.

We have explained the mutilated K_{c_0} in figure 1.25; now we should explain the sewing of $\gamma_M(t)$ to $\gamma_{c_0}(-t)$. Even though we have proved such a sewing for a dense set of t in chapter 8 (for a somewhat different definition of captures), the

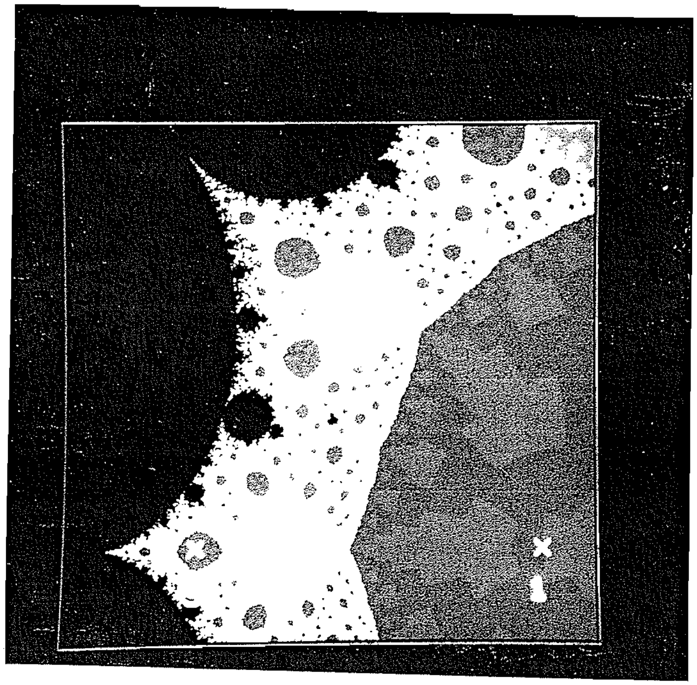

Figure 1.56. Blow up of d-plane.

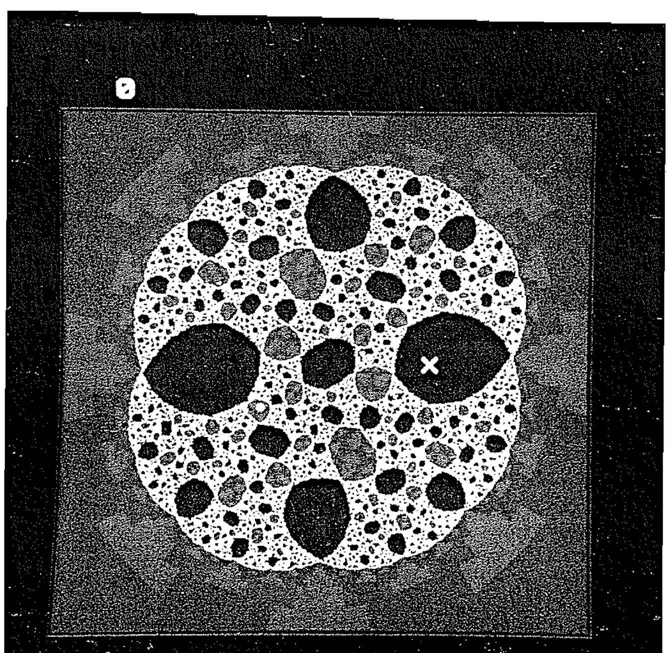

Figure 1.57. Capture 0.

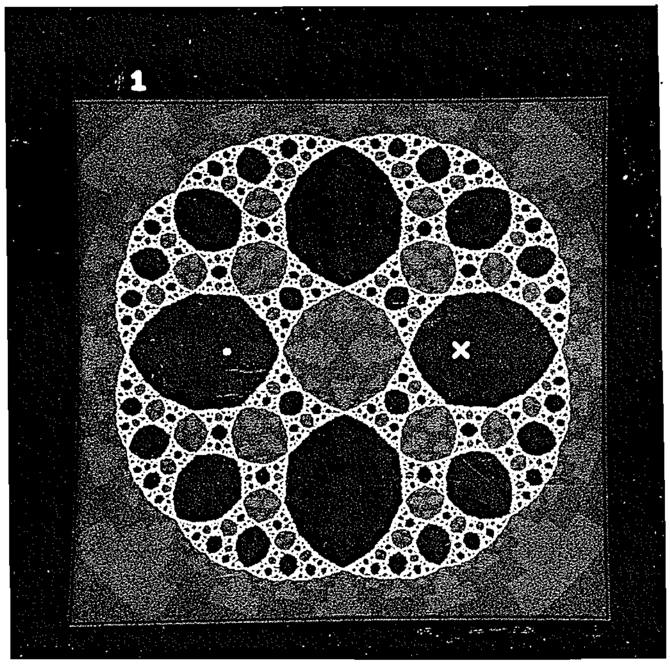

Figure 1.58. Capture 1.

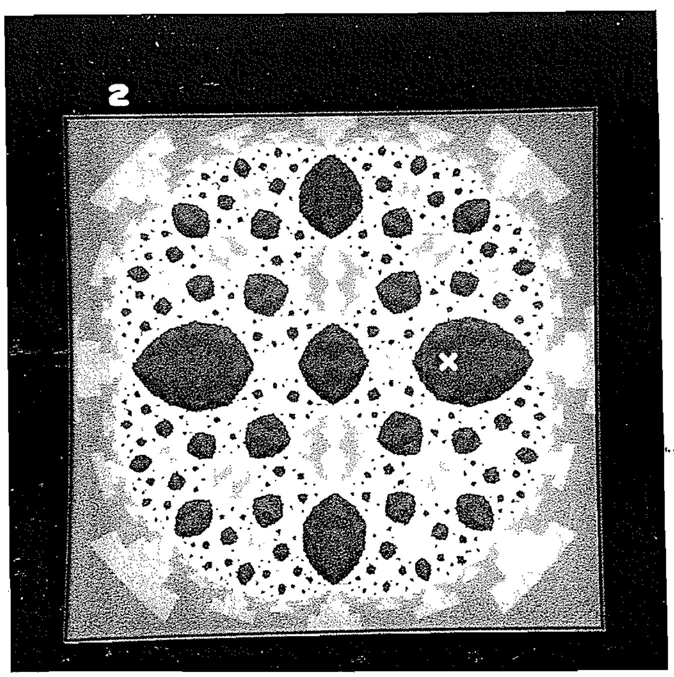

Figure 1.59. Capture 2.

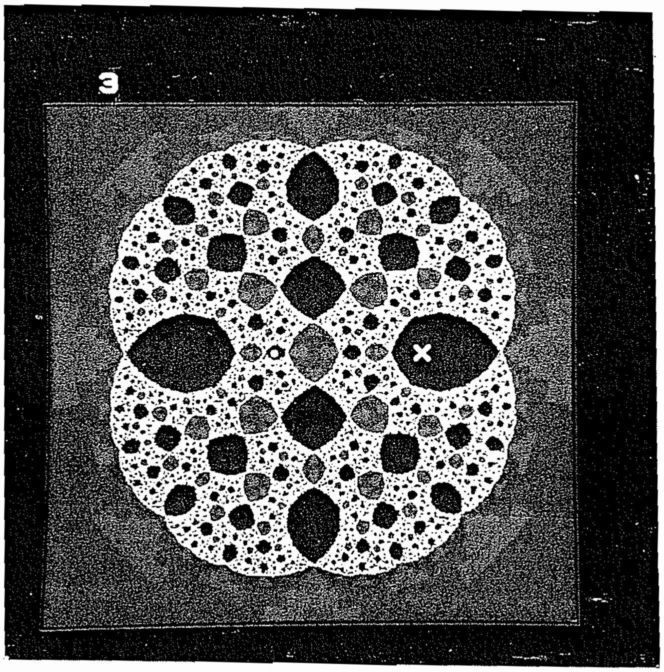

Figure 1.60. Capture 3.

proof rests on arguments due to Thurston which we have never been able to make intuitive in this context. We will, therefore, only offer a plausibility argument.

Let t_1 be a rational number in lowest terms with even denominator, and let $c_1 := \gamma_M(t_1)$. Douady and Hubbard proved that K_{c_1} has empty interior and that

$$\gamma_{c_1}(t_1) = c_1 = f_{c_1}(0).$$

So the mating of f_{c_0} with f_{c_1} can be viewed as formed by sewing various points on the boundary of K_{c_0} to each other and making

$$\gamma_{c_0}(-t_1) \sim \gamma_{c_1}(t_1)$$

into the image of a critical point. Recall the last view presented in the previous section of the capture at y_1 by f_{c_0} for y_1 on the boundary of K_{c_0}. If

$$y_1 = \gamma_{c_0}(-t_1),$$

we said that the capture at y_1 by f_{c_0} is formed by pulling various points on the boundary of K_{c_0} together in such a way that $y_1 = \gamma_{c_0}(-t_1)$ becomes the image of a critical point.

So the mating of f_{c_0} with $\gamma_M(t_1)$ and the capture at $\gamma_{c_0}(-t_1)$ by f_{c_0} are formed in roughly the same way. The part we have not been able to make intuitive is why the identification of points on the boundary of K_{c_0} is the same in both cases.

§1.9. *A not-so-nice rational family.*

Due to what we have seen in the d-plane, the reader might be feeling optimistic about understanding all rational functions of degree two in terms of matings and captures. We know matings and captures are not enough, but we do not know whether or not we can understand all rational functions of degree two in terms of matings, captures and things called anti-matings and anti-captures. In this section we present another family of rational functions of degree two with the purpose of showing that even just matings and captures can be rather complicated. For e in $\mathbf{C} - \{0\}$, let

$$f_e(z) = \frac{1}{ez^2 - (e+1)z + 1}.$$

One critical point of f_e is ∞, and $f_e(\infty) = 0$, $f_e(0) = 1$, and $f_e(1) = \infty$. The other critical point of f_e is $(e+1)/2e$.

Since one critical point is periodic of period three, if f_e is a mating of f_{c_2} with f_{c_1}, 0 must be periodic of period three for one of f_{c_2} and f_{c_1}. There are only three c for which 0 is periodic of period three for f_c. They are

1) $c_0' \approx -1.754877$,

2) $c_0'' \approx -.12352 + .74291i$, and

3) $\bar{c}_0''$ = the complex conjugate of c_0''.

We have seen $K_{c_0''}$ in figure 1.2. $K_{\bar{c}_0''}$ is just the complex conjugate of $K_{c_0''}$. Figure 1.61 shows $K_{c_0'}$ in black with the orbit of 0 marked with white dots. Figure 1.62 shows schematically the location in M of c_0', c_0'' and $\bar{c}_0''$.

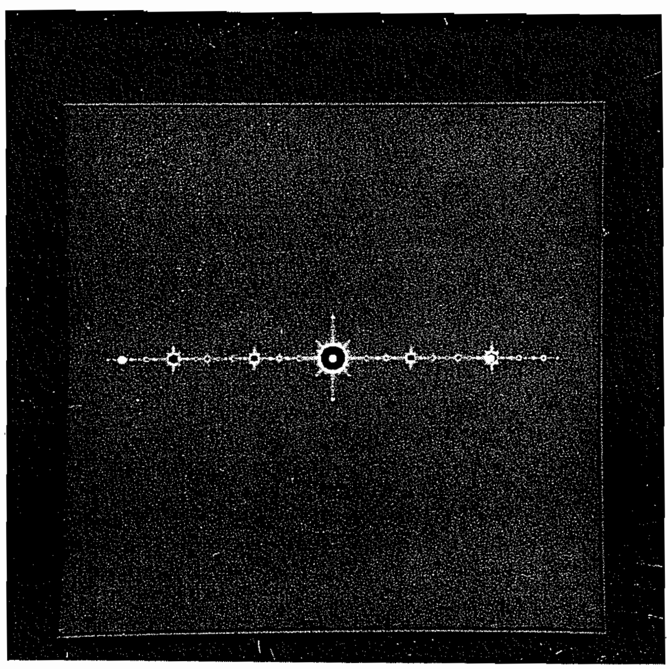

Figure 1.61. K_c for $c \approx -1.754877$.

Conversely, any mating with c_0', c_0'' or $\bar{c}_0''$ is conjugate by a Möbius transformation to f_e for some e. So among the f_e we expect to see three mutilated

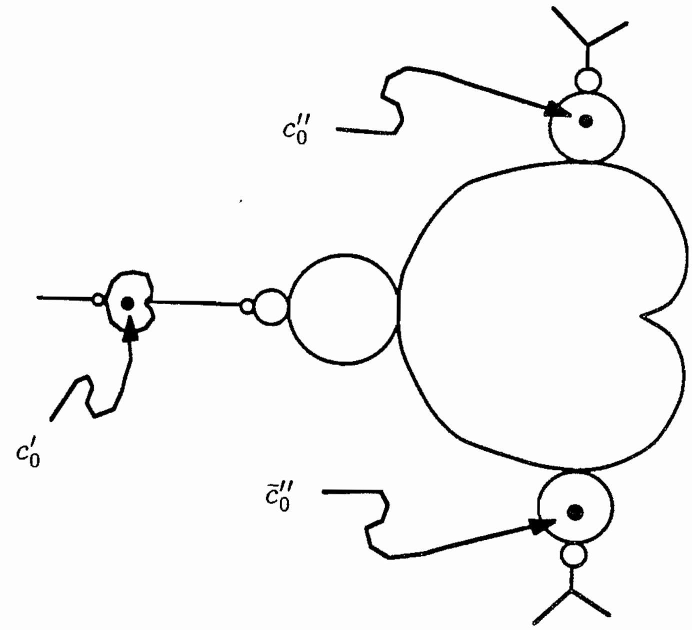

Figure 1.62. Location of c_0', c_0'', and $\bar{c}_0''$ in M.

Mandelbrot sets of matings. The mutilated Mandelbrot set of matings with c_0' (which we shall call M') should contain the unshaded portion of M in figure 1.63. Figure 1.64 shows the expected mutilated M for matings with c_0'' (denoted by M''), and figure 1.65 shows that for $\bar{c}_0''$ (denoted by $\bar{M}''$).

Figure 1.66 shows an e-plane picture analogous to the d-plane picture shown in figure 1.25. M', M'' and $\bar{M}''$ are in fact to be found in the black. M' has been turned inside out. The cusp of its central cardioid is near the left edge of figure 1.66, and its amputation point is at ∞. To see the others, we look at some blow-ups. Figure 1.67 is outlined in figure 1.66, and figure 1.68 is outlined in figure 1.67. (We should mention that the round red region and the round green region are mistakes. They should be blue and red respectively. Also the yellow

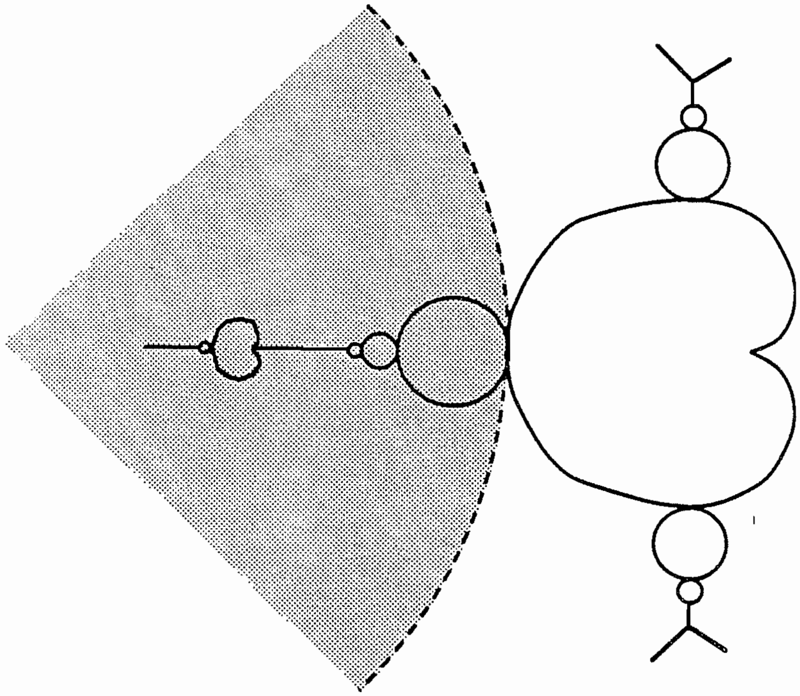

Figure 1.63. Mutilated Mandelbrot set of matings with c_0'.

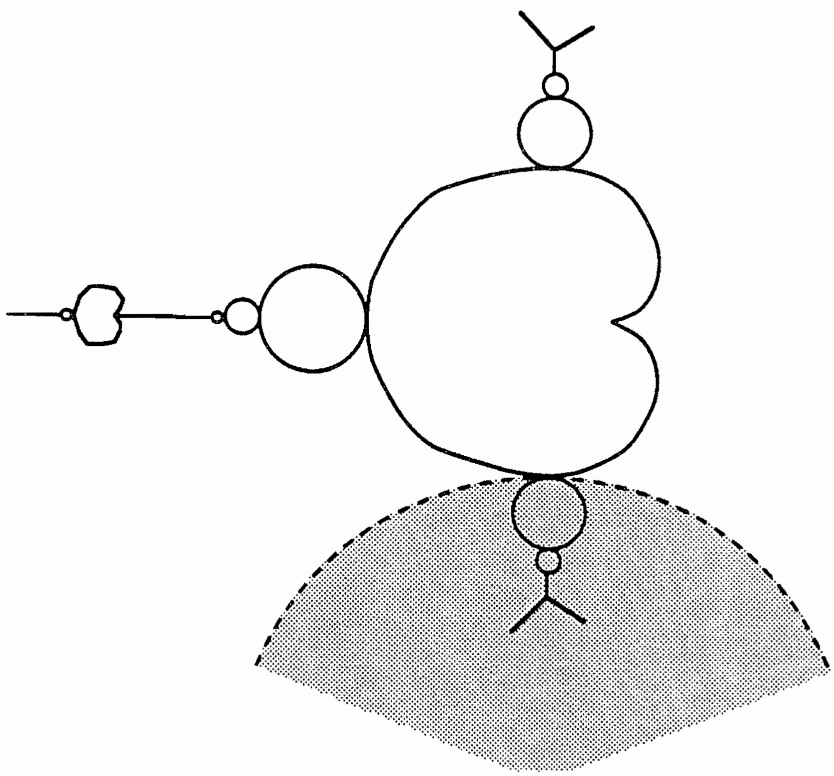

Figure 1.64. Mutilated Mandelbrot set of matings with c_0''.

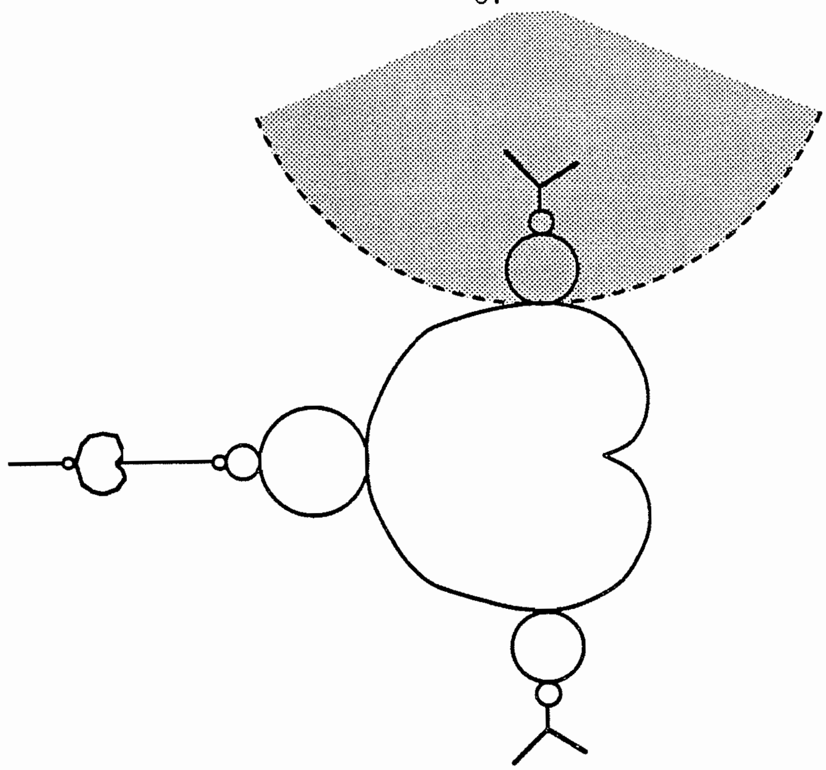

Figure 1.65. Mutilated Mandelbrot set of matings with $\bar{c}_0'$.

circle around the round green region is due to the photography.)

In figure 1.68 we have marked the same veins we marked in figures 1.24, 1.25, and 1.26. The larger ones are the veins in M', the smaller ones are in M''. Figure 1.69 shows the smaller ones in greater detail. M'' is in figure 1.68, but it is somewhat distorted. Figure 1.70 shows M'' undistorted with some components labeled and figure 1.71 shows how M'' sits in figure 1.68 with the components labeled as in figure 1.70. $\bar{M}''$ is just the complex conjugate of M''. They both have their amputation point at $e = 0$.

Notice that the region of M'' labeled e in figure 1.71 is also part of M' as indicated by the big veins in figure 1.68. Similarly, the region labeled c in figure 1.71 is part of both M' and M''. We call this phenomenon shared mating.

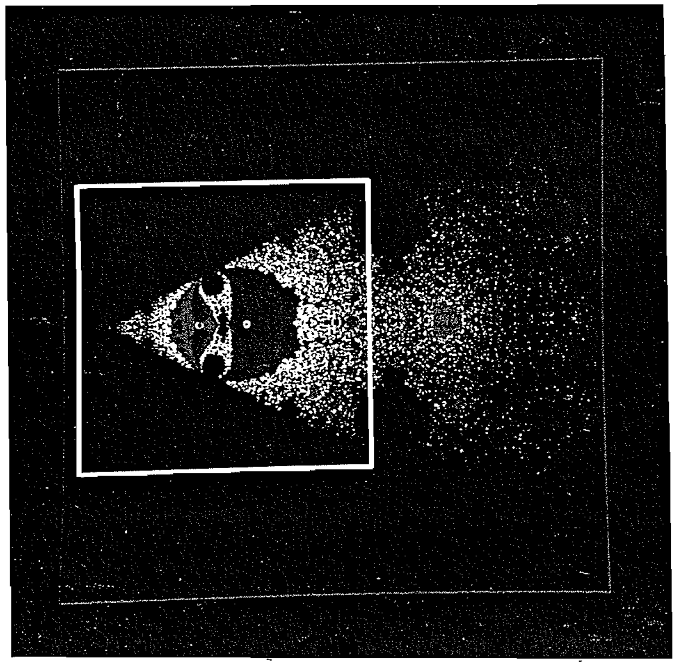

Figure 1.66. e-plane.

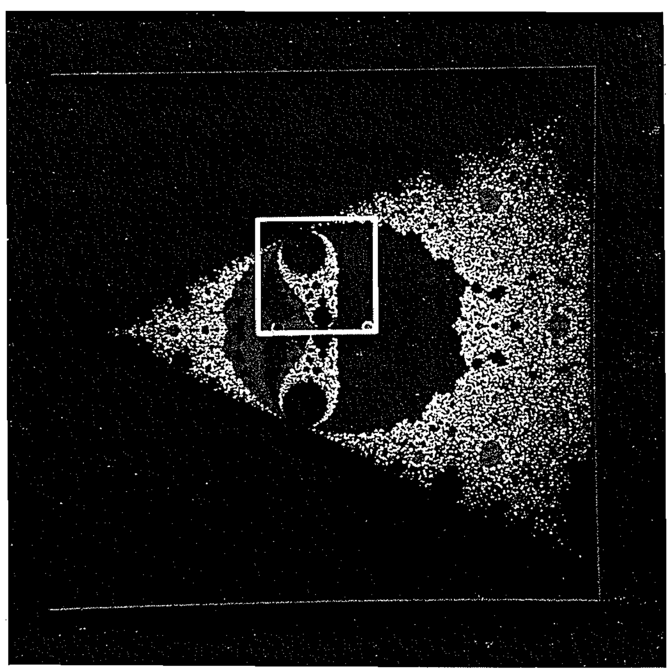

Figure 1.67. Blow up of e-plane.

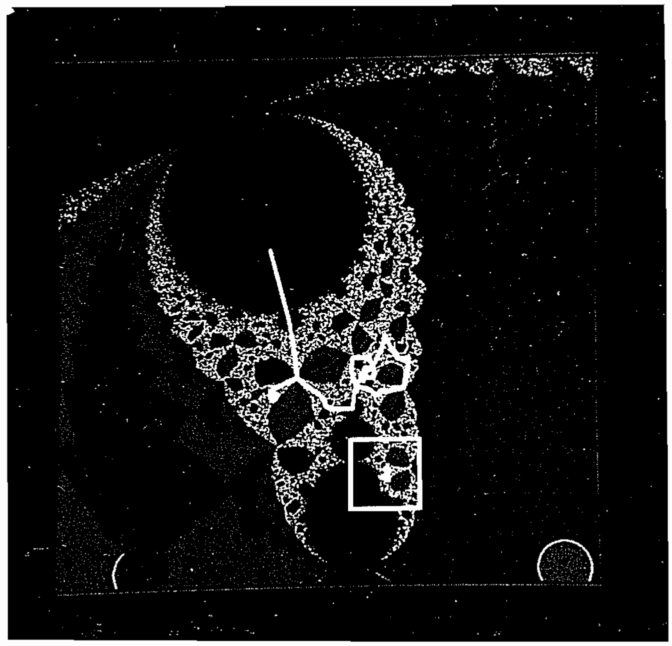

Figure 1.68. Further blow up of e-plane with veins.

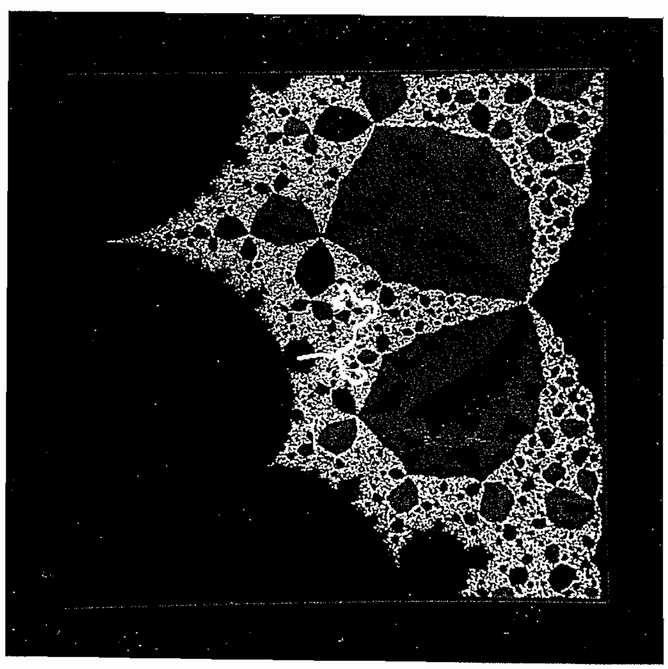

Figure 1.69. Blow up of smaller veins.

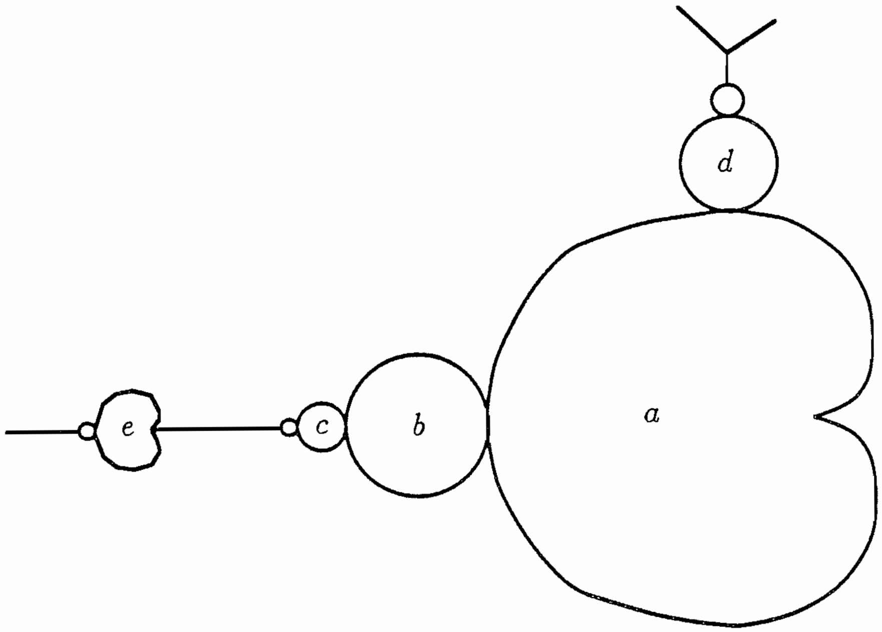

Figure 1.70. M'' undistorted.

In shared mating, a single rational function can be interpreted as a mating in two different ways.

We have actually seen shared mating before. We saw in the d-plane a mutilated Mandelbrot set of matings with f_{c_0} for $c_0 = -1$. That mutilated Mandelbrot set was sewn into a mutilated K_{c_0} according to the rule $\gamma_M(t)$ sews to $\gamma_{c_0}(-t)$. There are many pairs t_0, t_1 for which

$$\gamma_{c_0}(-t_0) = \gamma_{c_0}(-t_1)$$

but

$$\gamma_M(t_0) \neq \gamma_M(t_1).$$

So the f_d which is the capture at $\gamma_{c_0}(-t_0)$ by f_{c_0} is the mating of f_{c_0} with $\gamma_M(t_0)$

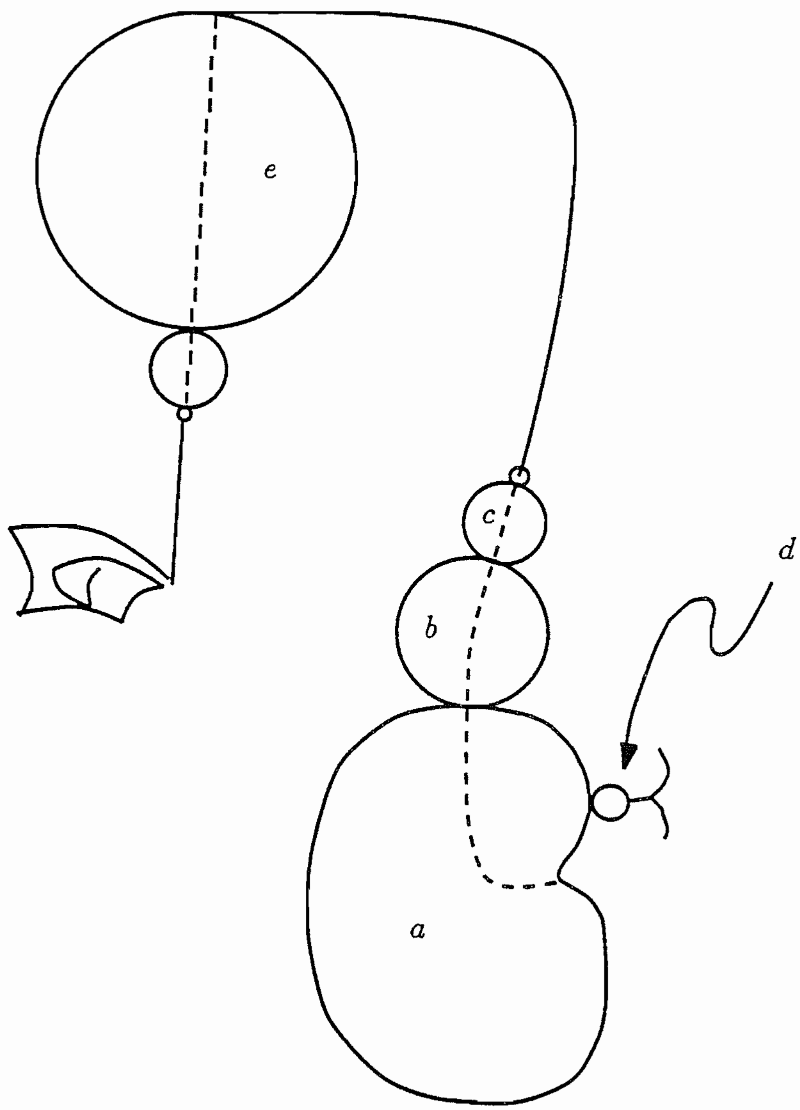

Figure 1.71. How M'' sits in the e-plane.

and the mating of f_{c_0} with $\gamma_M(t_1)$. We will call shared matings arising in this way degenerate.

The shared matings we see in regions c and e of figure 1.71 are different from degenerate matings in two ways. First, they are part of different mutilated Mandelbrot sets of matings. Second, they are in the interior of mutilated Mandelbrot sets of matings. We look at an example.

Figure 1.72 shows the z-plane for f_e, where e is a point in the interior of the region labeled c in figure 1.71. The coloring is exactly the same as that of figure 1.13. The points 0 and 1 have been marked with a white ex and the orbit of the critical point $(e+1)/2e$ is marked with white dots. Figures 1.74 through 1.86 show the K_c of figure 1.2 and the K_c of figure 1.73 slowly mating to form f_e. Figures 1.88 through 1.100 show the K_c of figure 1.61 and the K_c of figure 1.87 slowly mating to also form f_e.

In chapter 11 we prove a theorem which has as a consequence that all the points of M' in figure 1.68 between the big blue region and the big red region are also in M''. The converse is not true, as shown by the following figures. Figure 1.101 shows the same portion of the e-plane as does figure 1.67, and figure 1.102 is outlined in figure 1.101. In figure 1.102 we have approximated half the boundary of M' by drawing white lines between successive points of the form

$$\gamma_{M'}\left(\frac{p}{2^{12}}\right)$$

for $0 < p < 2^{11}$. Figure 1.103 is figure 1.101 with this approximation drawn in.

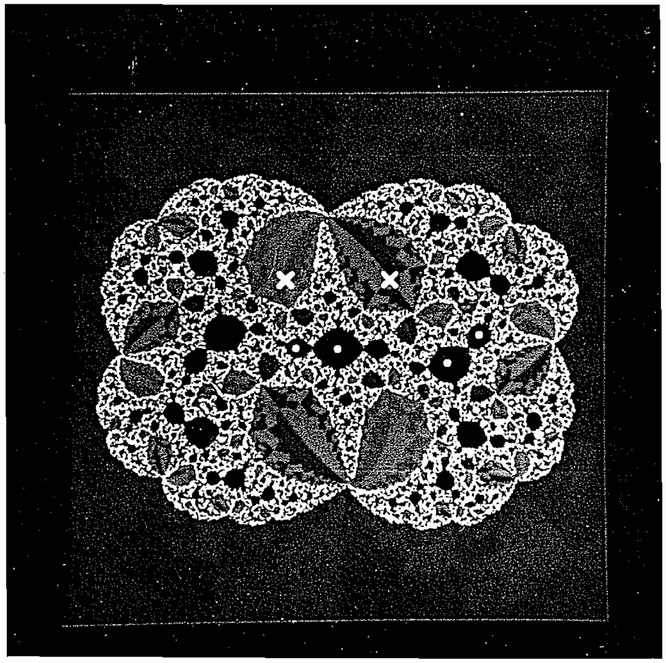

Figure 1.72. z-plane for e in center of region C.

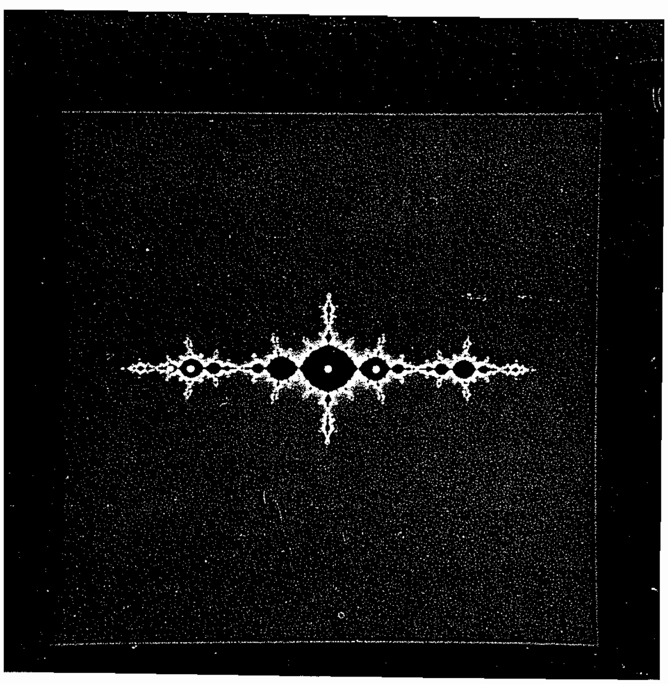

Figure 1.73. K_c for bifurcation of bifurcation.

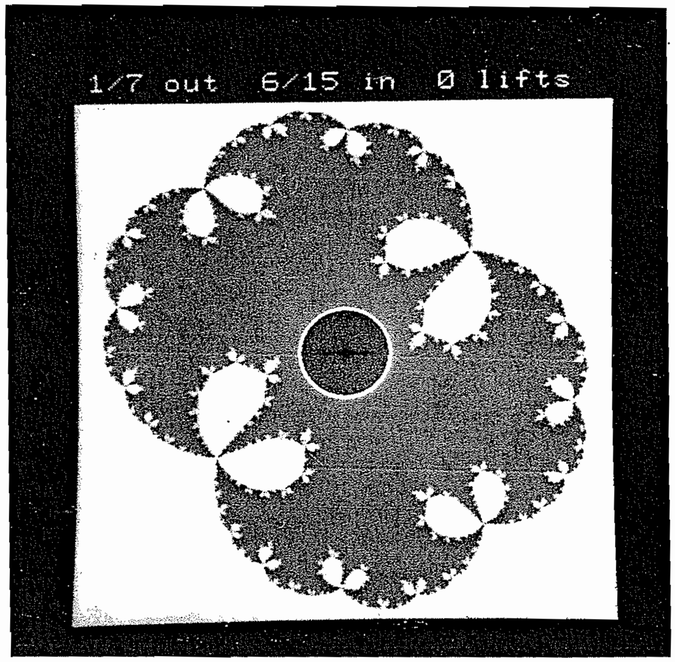

Figure 1.74. Thurston construction of 1/7 mating with 6/15, 0 lifts.

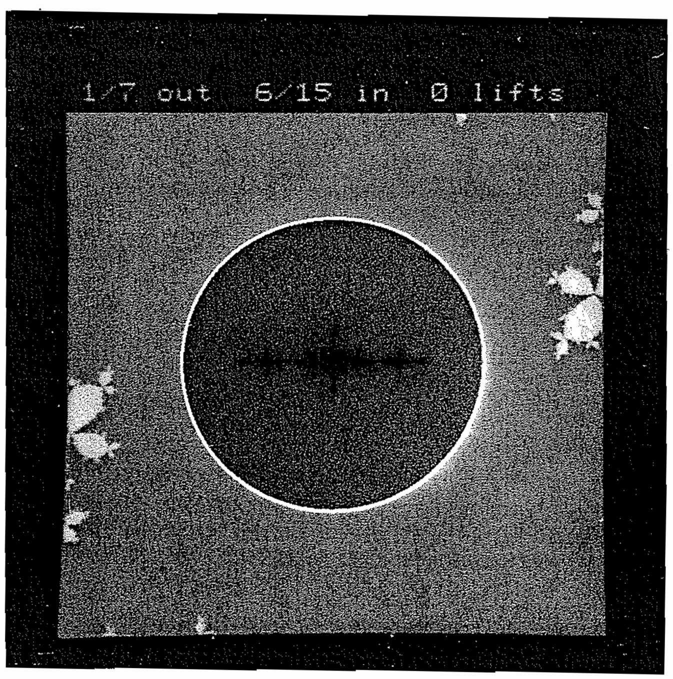

Figure 1.75. Thurston construction of 1/7 mating with 6/15, 0 lifts blown up.

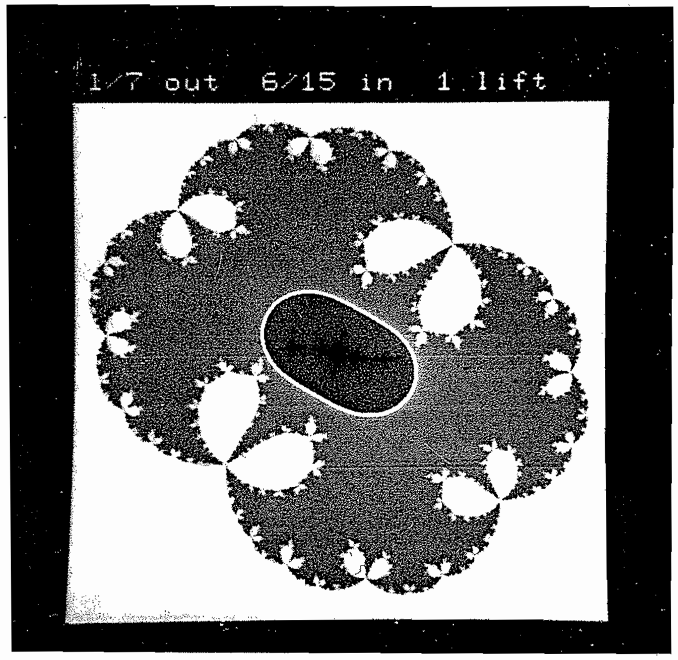

Figure 1.76. Thurston construction of 1/7 mating with 6/15, 1 lift.

Figure 1.77. Thurston construction of 1/7 mating with 6/15, 2 lifts.

Figure 1.78. Thurston construction of 1/7 mating with 6/15, 3 lifts.

Figure 1.79. Thurston construction of 1/7 mating with 6/15, 5 lifts.

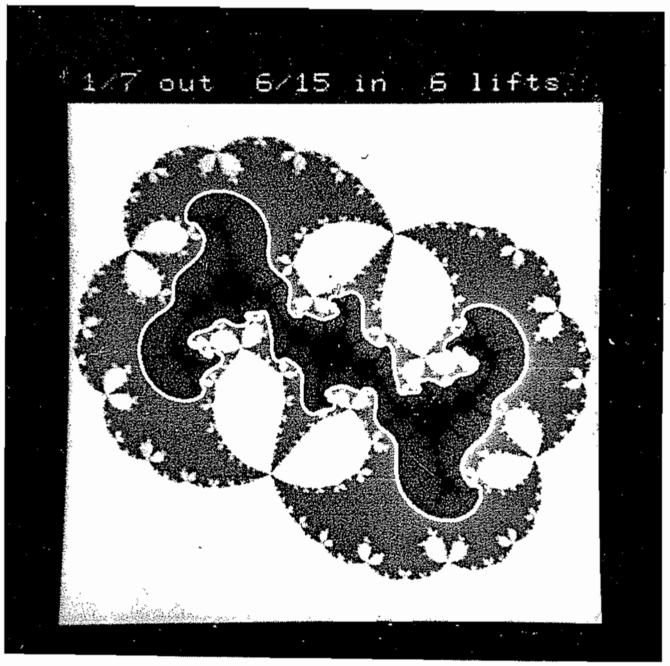

Figure 1.80. Thurston construction of 1/7 mating with 6/15, 6 lifts.

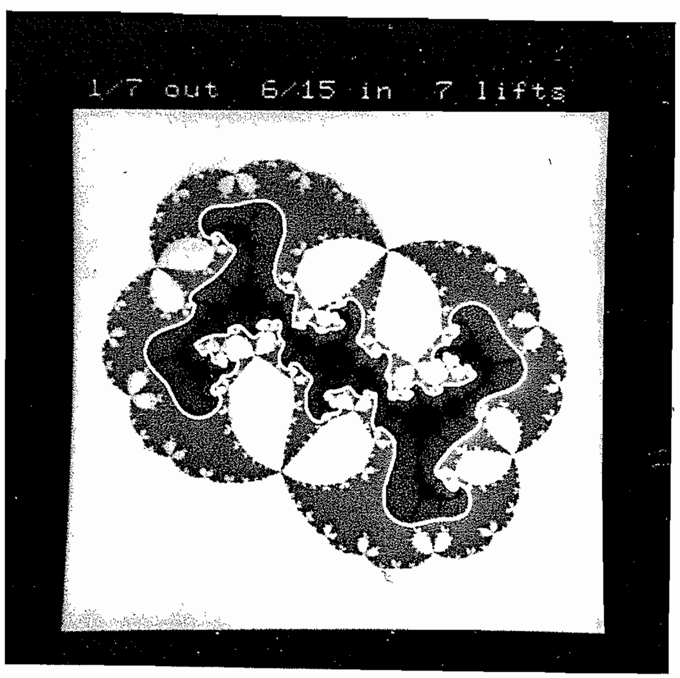

Figure 1.81. Thurston construction of 1/7 mating with 6/15, 7 lifts.

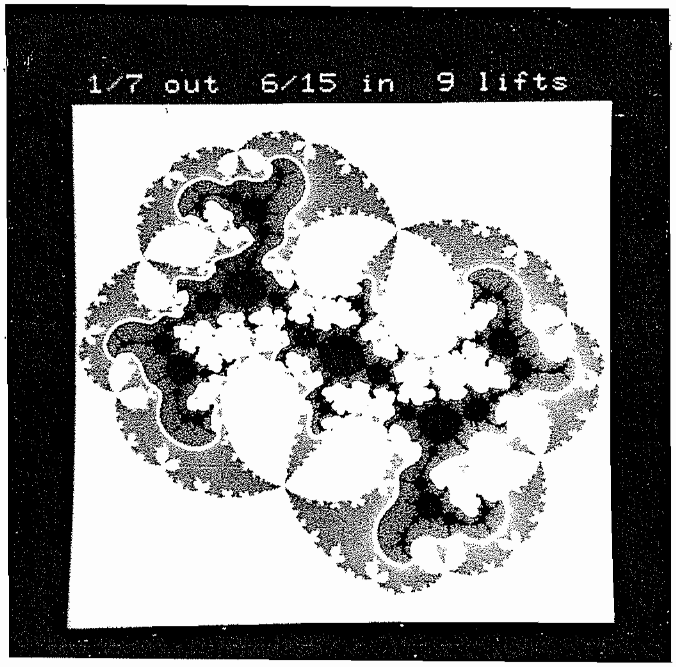

Figure 1.82. Thurston construction of 1/7 mating with 6/15, 9 lifts.

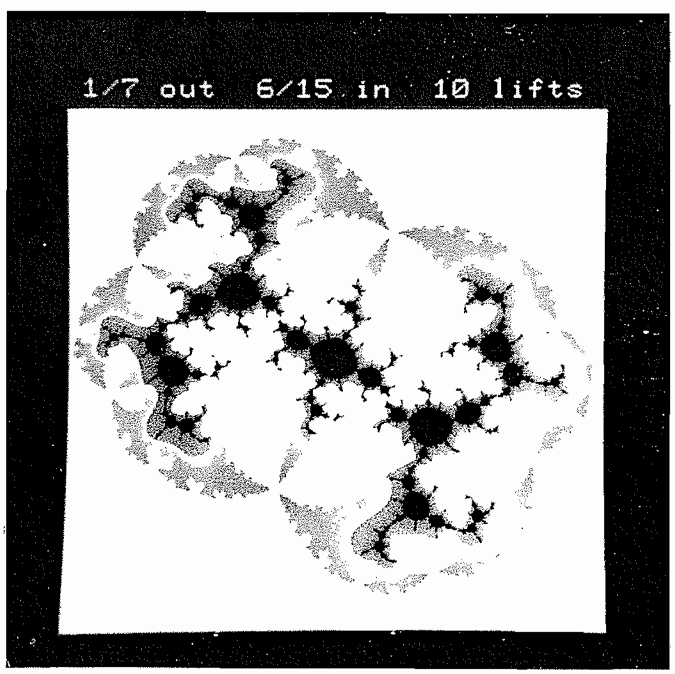

Figure 1.83. Thurston construction of 1/7 mating with 6/15, 10 lifts.

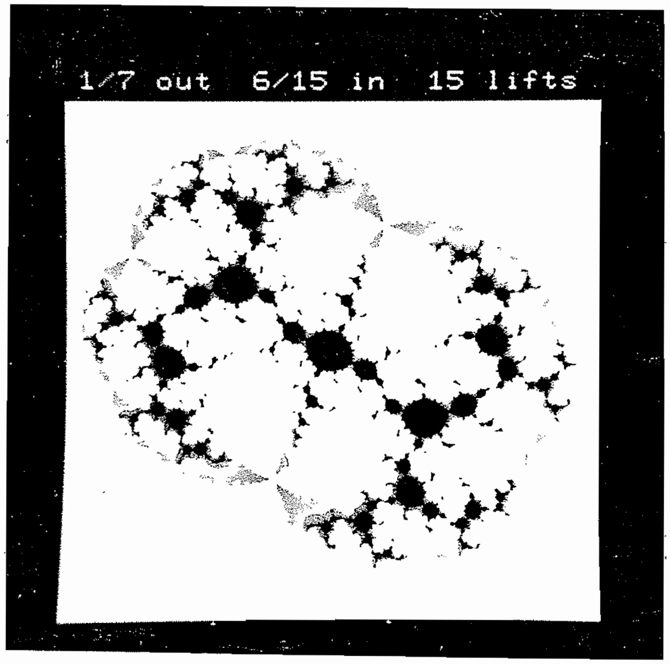

Figure 1.84. Thurston construction of 1/7 mating with 6/15, 15 lifts.

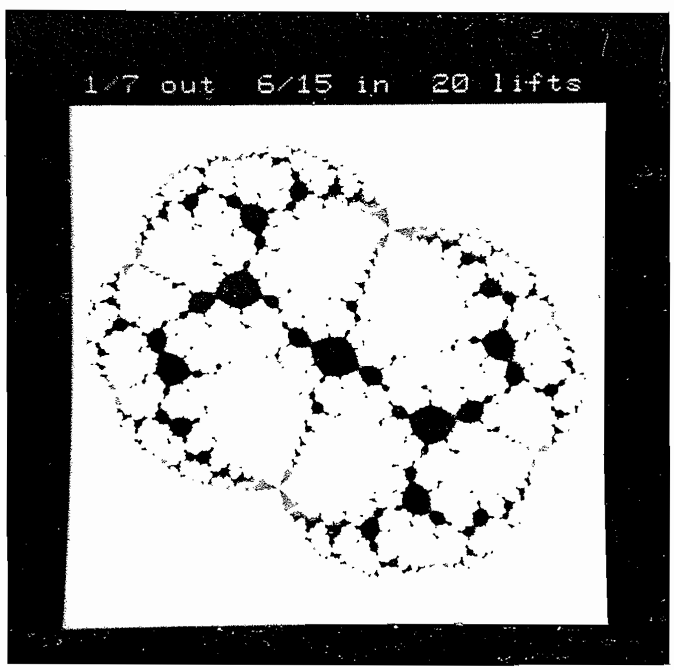

Figure 1.85. Thurston construction of 1/7 mating with 6/15, 20 lifts.

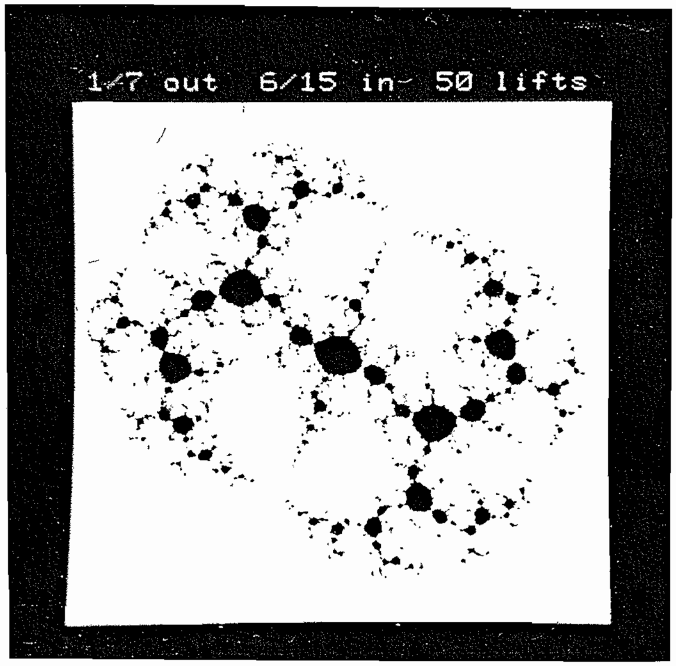

Figure 1.86. Thurston construction of 1/7 mating with 6/15, 50 lifts.

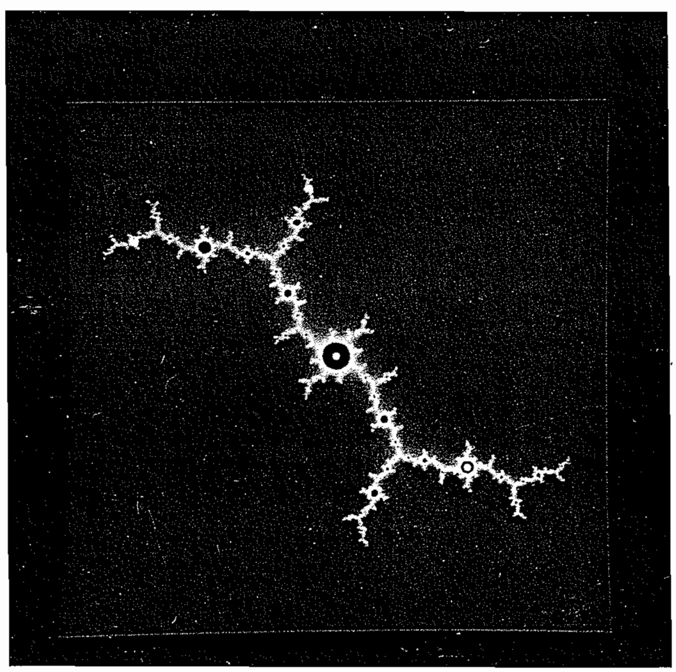

Figure 1.87. K_c.

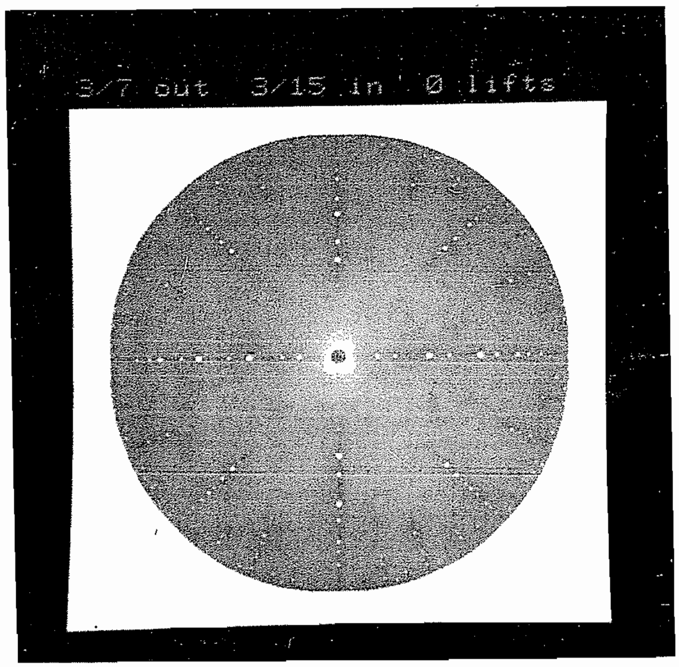

Figure 1.88. Thurston construction of 3/7 mating with 3/15, 0 lifts.

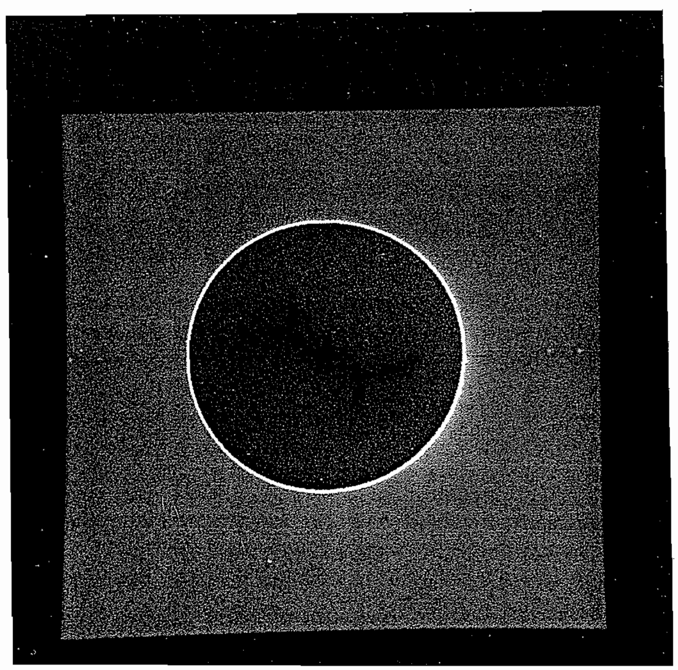

Figure 1.89. Thurston construction of 3/7 mating with 3/15, 0 lifts blown up.

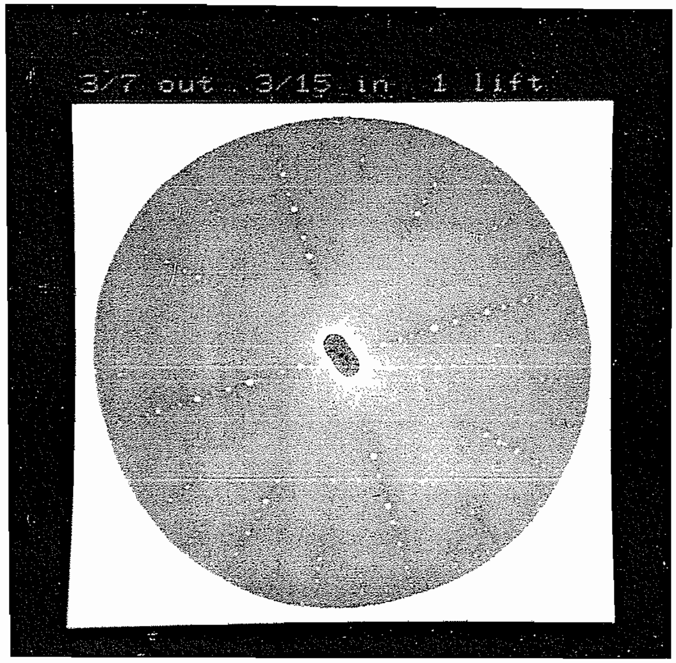

Figure 1.90. Thurston construction of 3/7 mating with 3/15, 1 lift.

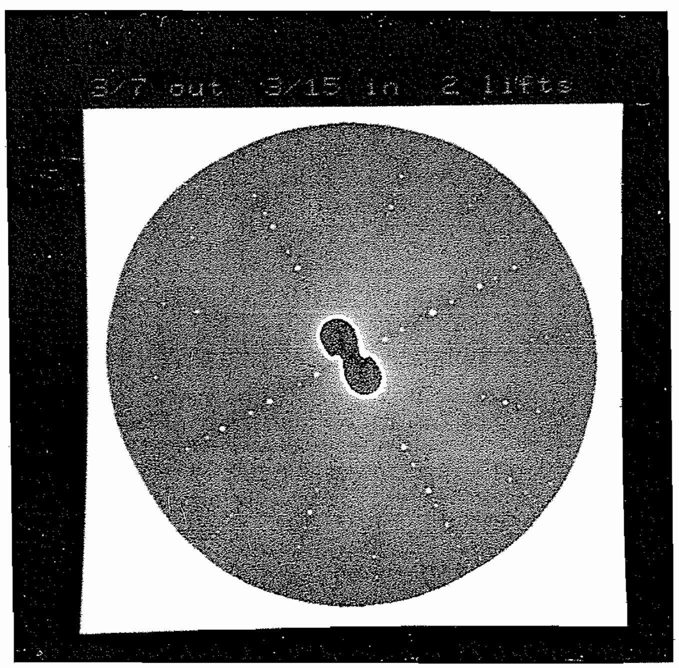

Figure 1.91. Thurston construction of 3/7 mating with 3/15, 2 lifts.

Figure 1.92. Thurston construction of 3/7 mating with 3/15, 3 lifts.

Figure 1.93. Thurston construction of 3/7 mating with 3/15, 5 lifts.

Figure 1.94. Thurston construction of 3/7 mating with 3/15, 6 lifts.

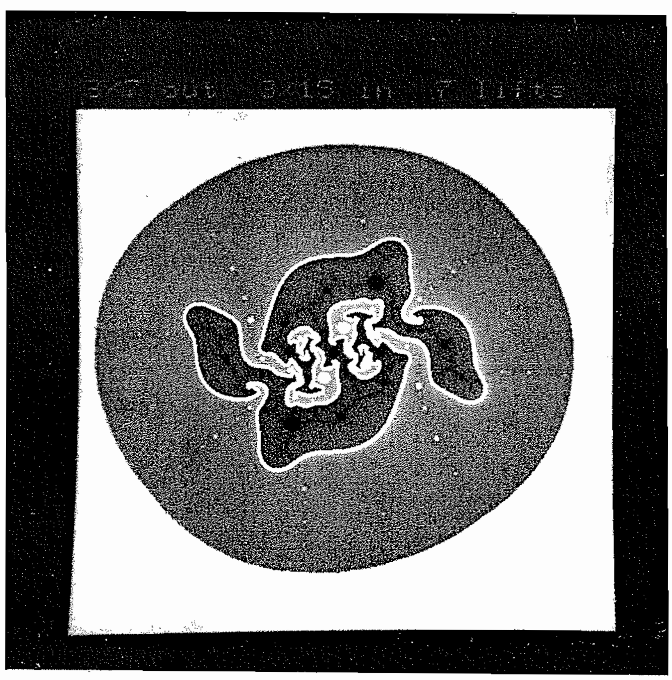

Figure 1.95. Thurston construction of 3/7 mating with 3/15, 7 lifts.

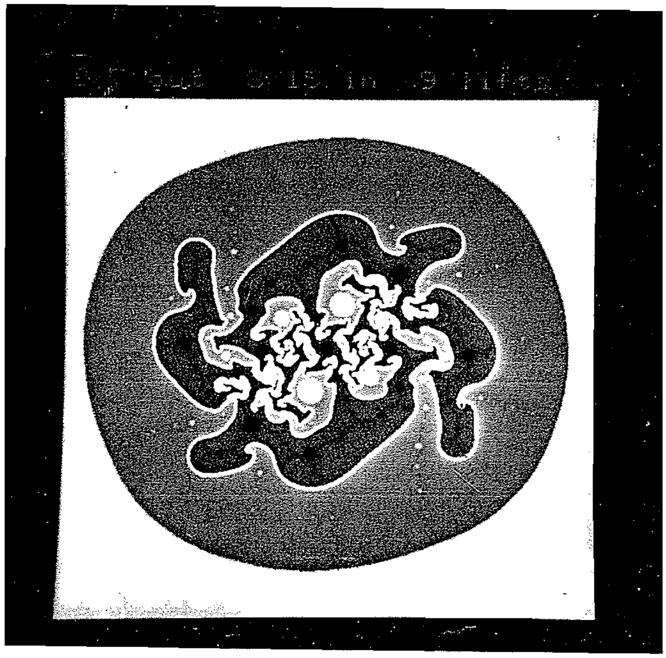

Figure 1.96. Thurston construction of 3/7 mating with 3/15, 9 lifts.

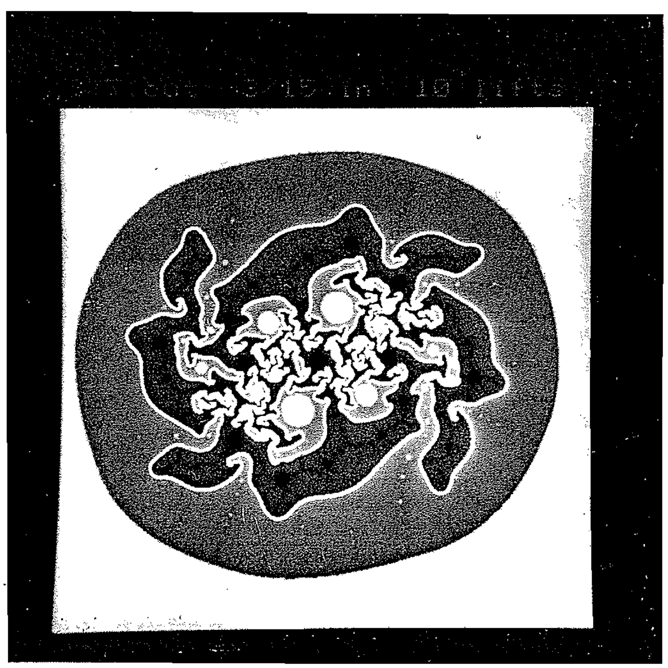

Figure 1.97. Thurston construction of 3/7 mating with 3/15, 10 lifts.

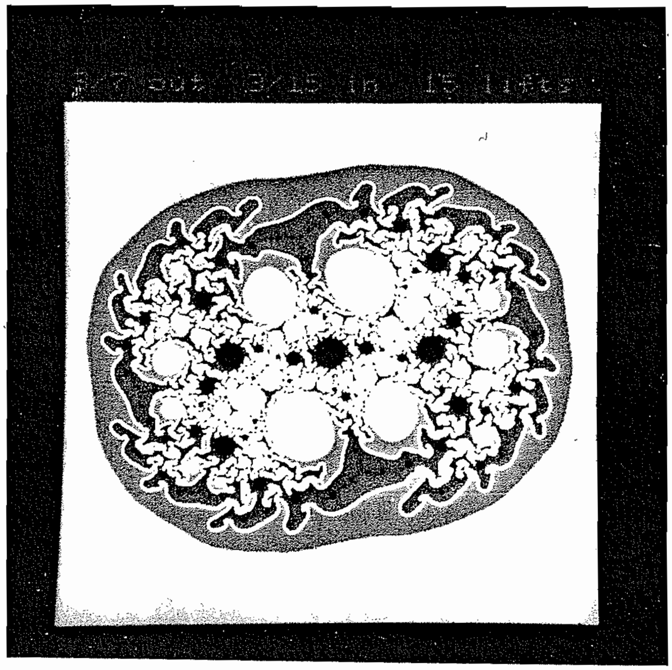

Figure 1.98. Thurston construction of 3/7 mating with 3/15, 15 lifts.

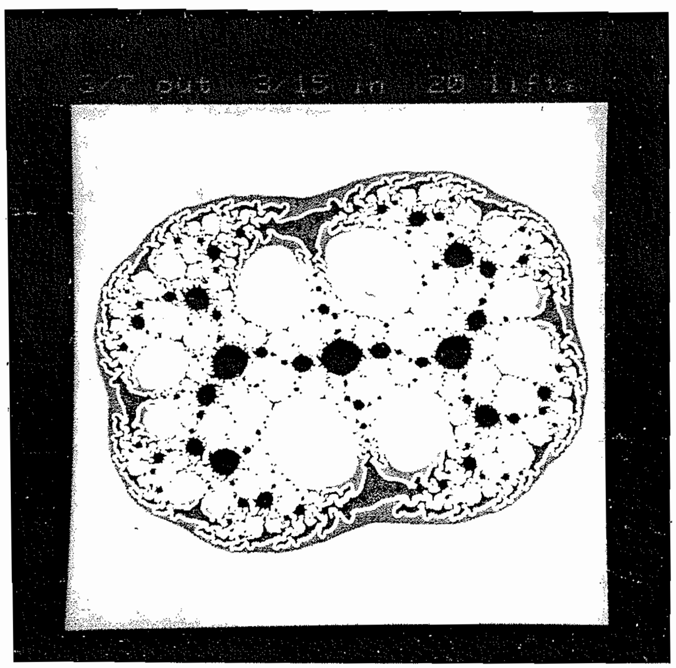

Figure 1.99. Thurston construction of 3/7 mating with 3/15, 20 lifts.

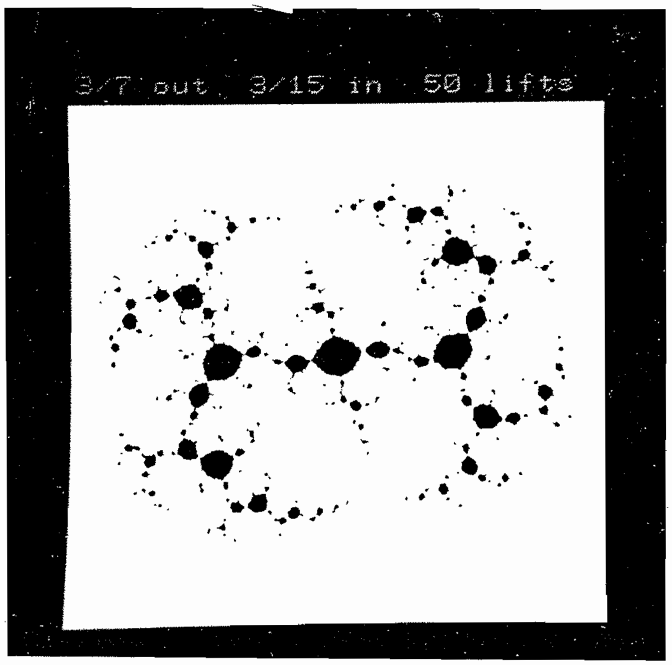

Figure 1.100. Thurston construction of 3/7 mating with 3/15, 50 lifts.

Figure 1.104 is outlined in figure 1.103. Figure 1.105 is figure 1.104 with that portion of the approximated boundary of M' between

$$\gamma_{M'}\left(\frac{293}{2^{11}}\right) \quad \text{and} \quad \gamma_{M'}\left(\frac{585}{2^{11}}\right)$$

greatly refined. Figure 1.106 shows a similar greatly refined approximation to the boundary of M''.

It is fun to look at close-ups of these approximations. Figures 1.108 and 1.109 are outlined in figure 1.107. Figure 1.108 shows our approximation of the boundary of M' and figure 1.109 shows our approximation of the boundary of M''. In chapter 11 we give an algorithm (based on a conjecture) to determine which points in the boundary of M'' are also in the boundary of M'. The algorithm also shows that those points in the boundary of M'' which are also in the boundary of M'

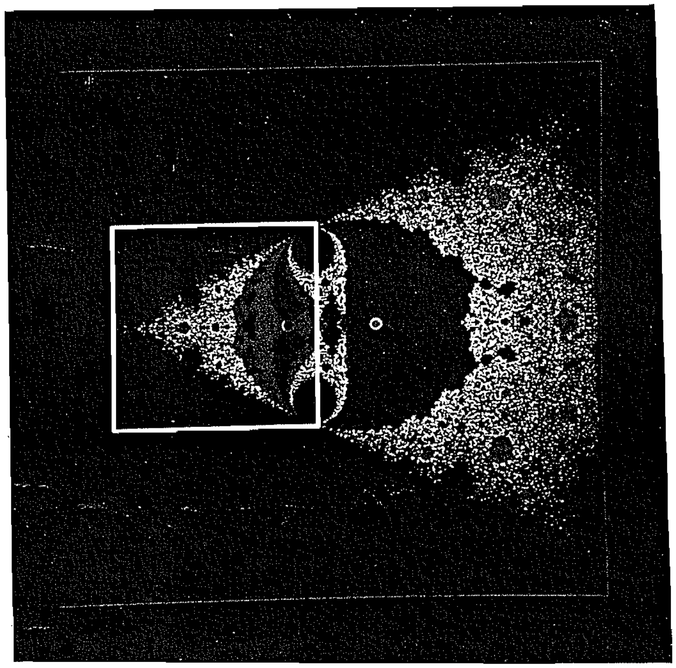

Figure 1.101. e-plane.

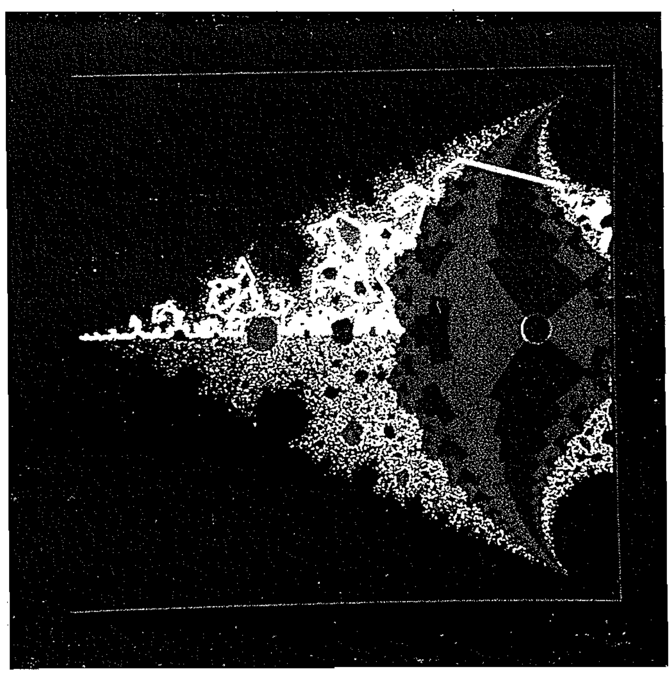

Figure 1.102. e-plane with half of boundary of M'.

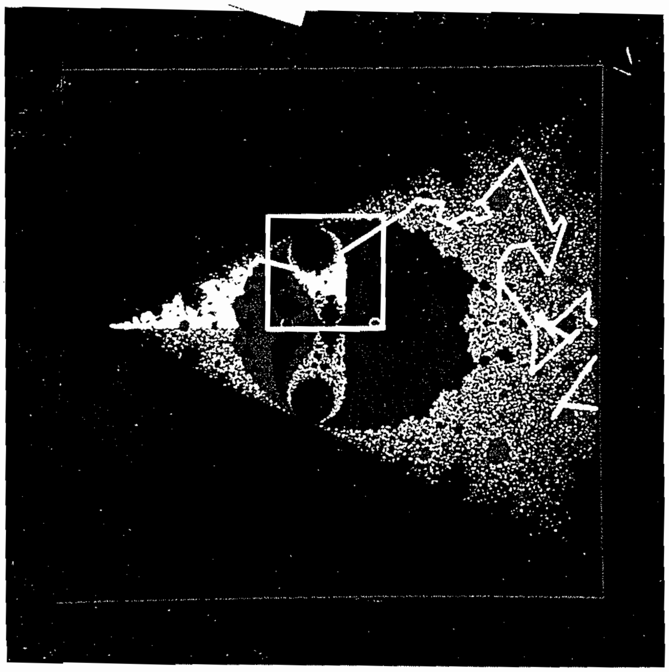

Figure 1.103. e-plane with half of boundary of M'.

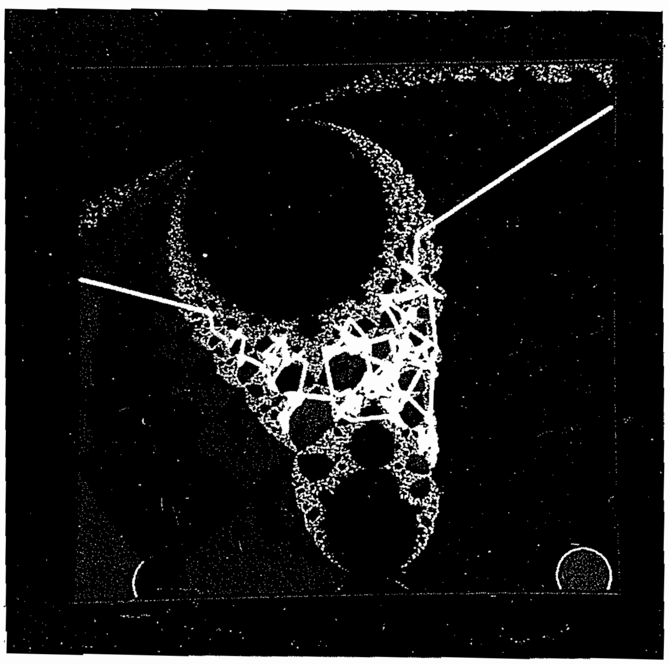

Figure 1.104. Blow up of figure 1.103.

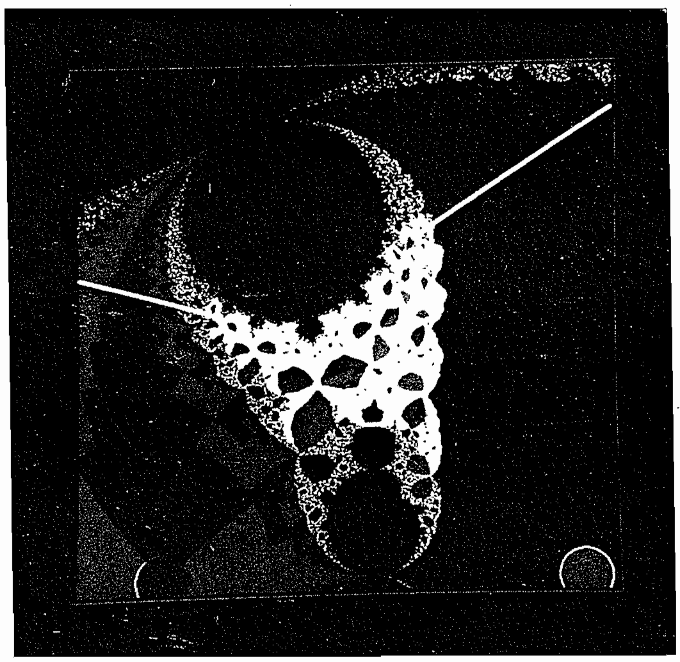

Figure 1.105. Refined boundary of M'.

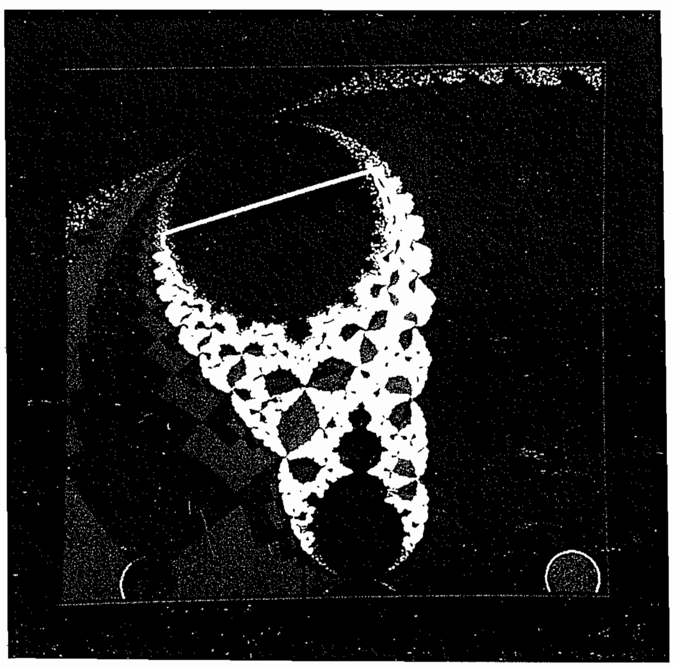

Figure 1.106. Refined boundary of M''.

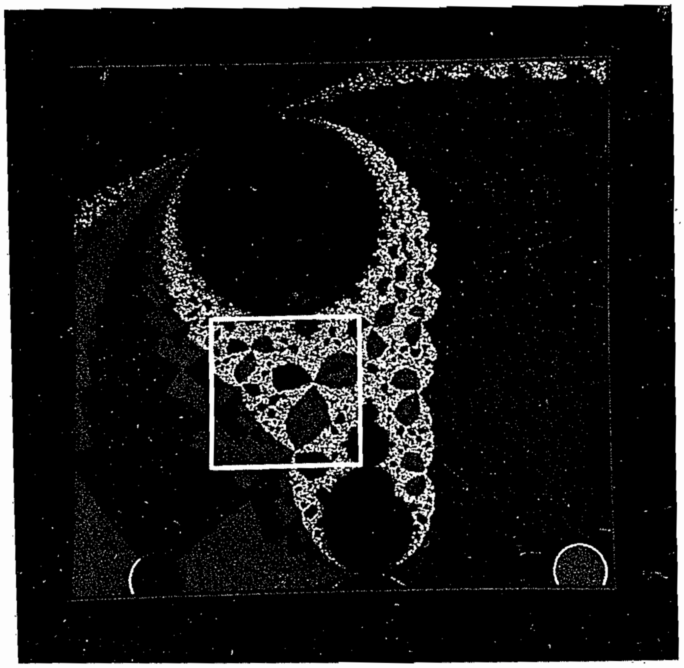

Figure 1.107. e-plane.

are in the limb of M'' which contains the regions labeled c and e in figure 1.71.

These approximations to the boundary of M' show another interesting aspect of matings among the f_e. Figure 1.110 shows the same portion of the e-plane as does figure 1.66 and figure 1.111 is outlined in figure 1.109. In figure 1.110 is a somewhat refined approximation of half of the boundary of M'. Notice how it dips below the real axis (compare with figure 1.102). If we think of M' as drawn in figure 1.63, then M' has an upper half and a lower half. It is not quite obvious from figure 1.111, but the black region in figure 1.111 marked with a white ex is in both the upper and lower half of M'. So the map from M' to the e-plane in not injective even on the interior of M'.

Another point to be made about the black regions in the e-plane is that they

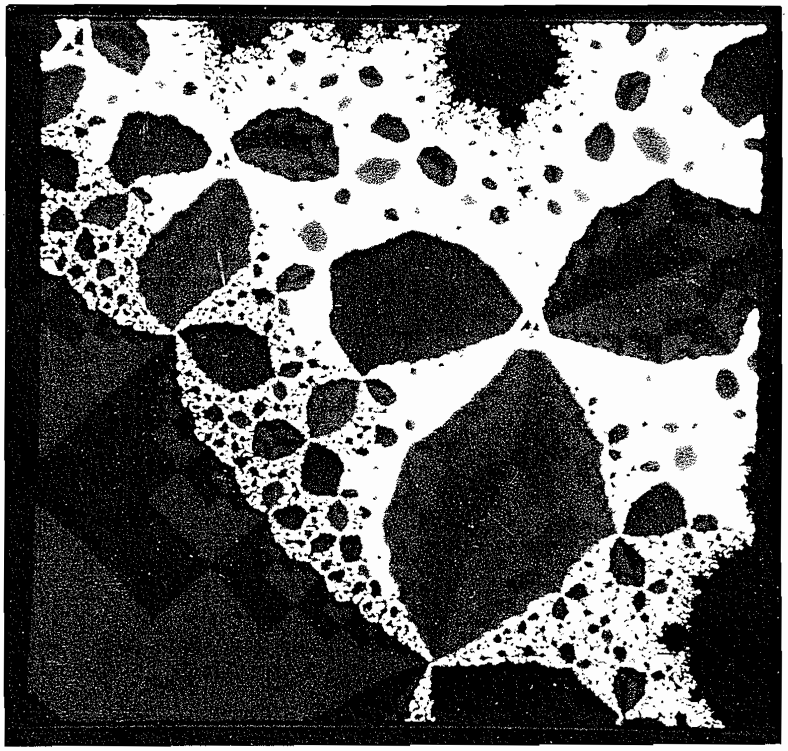

Figure 1.108. Blow up of boundary of M'.

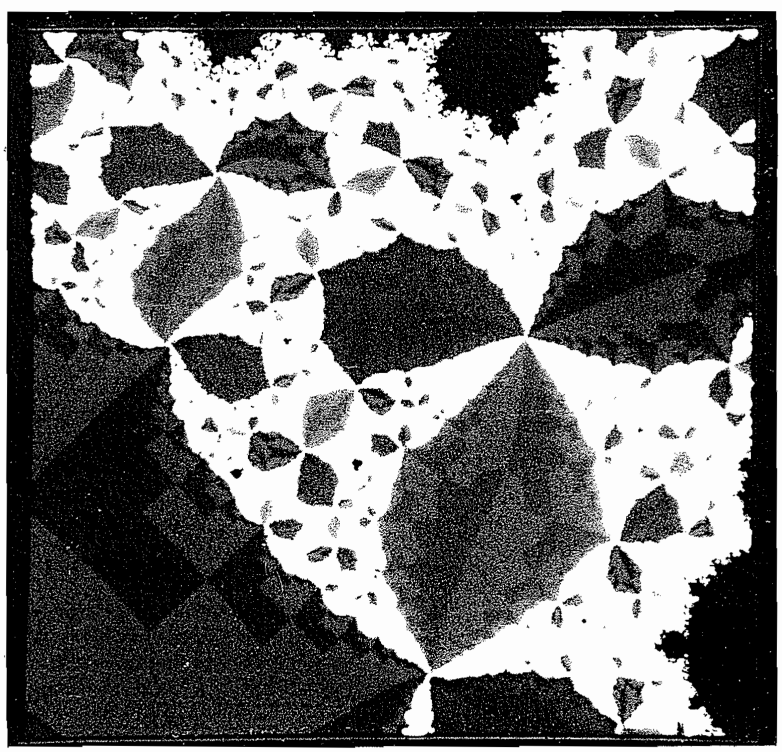

Figure 1.109. Blow up of boundary of M''.

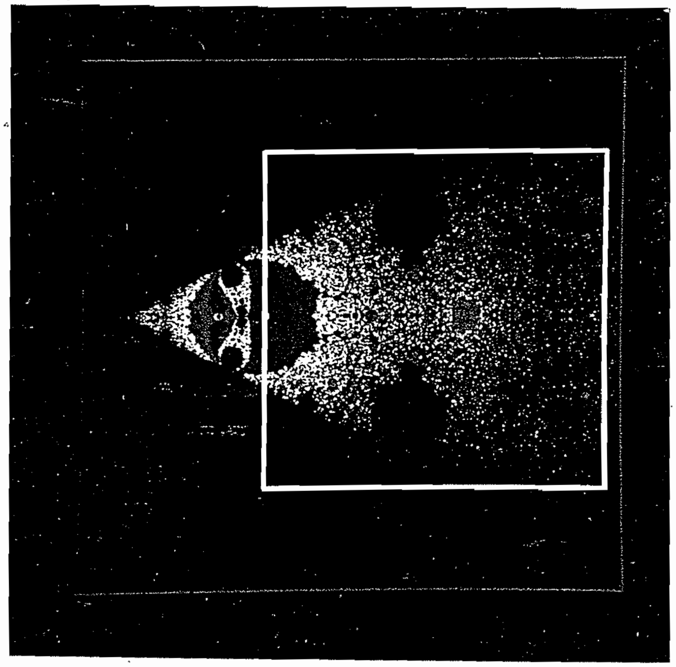

Figure 1.110. e-plane.

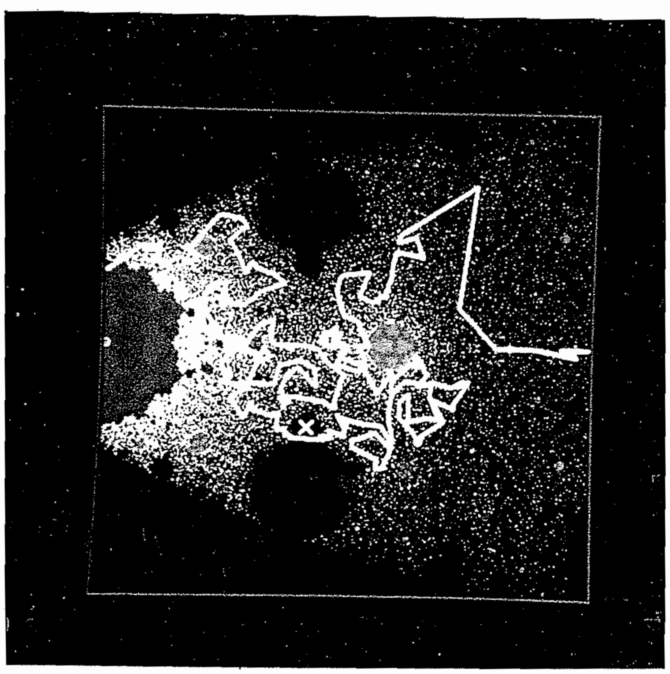

Figure 1.111. e-plane with boundary of M'.

are not all part of one or more of M', M'', and $\bar{M}''$. For $e \approx 4.3114$, the critical point $(e+1)/2e$ is periodic of period four, but f_e is not a mating. We know this because a theorem of Thurston implies that if two rational functions are conjugate to the same mating, they are conjugate to each other by a Möbius transformation. It is easy to see that none of the f_e are conjugate to each other by a Möbius transformation and we can find elsewhere in the e-plane all possible matings of $f_{c'_0}$, $f_{c''_0}$, and $f_{\bar{c}''_0}$ with f_c for which the critical point 0 is periodic of period four under f_c. It might be possible to interpret these non-matings as captures at periodic points on the boundary of $K_{c'_0}$ (theorem 8.4.1 gives a restriction on at what points).

Finally, we consider captures by $f_{c'_0}$, $f_{c''_0}$, and $f_{\bar{c}''_0}$. The sad fact is that in general we do not know how to define captures at y_1 by f_c for y_1 which are in the interior of the Hubbard tree of f_c. We prove in theorem 8.4.1 that if we gut K_c (i.e. remove the Hubbard tree and all the components of the interior of K_c through which it passes) in addition to appropriately mutilating it, then what is left does sew into the appropriate mutilated Mandelbrot set of matings according to the rule we mentioned earlier.

$K_{c''_0}$ and $K_{\bar{c}''_0}$ gutted and appropriately mutilated consists of only two pieces (see figure 1.2), so the picture we see of the gutted mutilated $K_{c''_0}$ in figures 1.68, 1.69, 1.108, and 1.109 is quite understandable. On the other hand, $K_{c'_0}$ gutted and appropriately mutilated (see figure 1.61) is a terribly disconnected set. There is one part which is not affected by the gutting, and that shows up in the tame boundary we see in figure 1.102. But the captures by $f_{c'_0}$ to the right of the large blue region are very confusing. Contemplate how they could sew themselves into M' according to our rule in light of figure 1.111.

§1.10. *Introduction for the specialist.*

This work has two main goals. One is to define captures (section 8.1) and show that some captures are also matings (theorem 8.4.1). The other is to show how a rational function of degree two can be interpreted as a mating in two different ways (theorem 11.1.1 and complement 11.1.3), a phenomenon we called shared mating. Together, theorem 8.4.1, theorem 11.1.1, and complement 11.1.3 can go a long way towards explaining some parts of the parameter space pictures presented earlier in this introduction.

There are also two lesser goals. One is to show that some captures do not exist (proposition 8.2.1). The other is to present an algorithm for determining in some cases all four participants in a shared mating (complement 11.1.2 and its proof).

All proofs in this work fall along a line of deduction to one of the results mentioned in the two preceding paragraphs.

Central to our definition of capture and proof of theorem 8.4.1 is the notion that some branched covers of S^2 to S^2 can be defined uniquely up to Thurston topological equivalence by their action on certain graphs. Making this notion precise is the purpose of embedding graphs, introduced and studied in chapters 3 and 4 (and in particular, in theorem 4.4.1). We believe embedding graphs can be of use to others.

Following Thurston and Levy, we use the following definition of mating. Polynomials naturally act on **C** with a line adjoined at infinity. Appropriately sewing two such lines together gives a branched cover from S^2 to S^2. A rational function Thurston topologically equivalent to such a branched cover is called a mating of the two polynomials. To prove the results about shared matings, we need a more complete picture of mating. Theorem 6.1.1 shows how a mating can actually be thought of as a sewing together of two filled in Julia sets.

Our results about shared matings also require a mating criterion due to Thurston (chapter 7) and the notion that one can identify a polynomial by its action on a kind of abstract Hubbard tree which we call a quadratic tree (chapter 5). Quadratic trees also provide a good example of the use of embedding graphs.

Finally, in order to specify the algorithm to determine all four participants in a shared mating, we needed an algorithm due to Douady and Hubbard for calculating the identification of S^1 induced by the Carathéodory loop (chapter 9) and the notion that for stars (i.e. direct bifurcations off the central cardioid of the Mandelbrot set) one can specify the external angle of a periodic or pre-periodic point in the Julia set by specifying an address of that point with respect to the internal structure of the filled in Julia set (chapter 10).

Chapter 2. Background and Notation.

§2.1. *General notation.*

We denote the Riemann sphere by $\hat{\mathbf{C}}$ if it is the domain of a polynomial and by $\mathbf{P}^1$ otherwise. We let D_r be the open unit disc in $\mathbf{C}$ of radius r centered at 0 and we let $D := D_1$.

Set $\mathbf{T} := \mathbf{R}/\mathbf{Z}$ and define $\mathrm{Exp} : \mathbf{T} \to \partial D$ by

$$\mathrm{Exp}(t) := e^{2\pi i t}.$$

Given $t \in \mathbf{Q}/\mathbf{Z}$, the *dynamic denominator* of t is the smallest q of the form $2^m(2^n - 1)$ such that $t = p/q$ for some p. Note that if the dynamic denominator of t is $2^m(2^n-1)$, then $2^{m+n}t = 2^m t$.

Let X be an oriented surface and let p be a point in X. Let $R = \{R_0, R_1, \ldots, R_{k-1}\}$ be a set of non-intersecting line segments in X each having p as an endpoint. By definition, σ is the *clockwise-around-p permutation of* R if σ is as follows. Choose orientation preserving co-ordinates on a neighborhood of p so that the segments R_i are straight. Using those co-ordinates, $\sigma(R_i)$ should be the next segment encountered after R_i when going around p clockwise.

§**2.2.** *Rational functions.*

An excellent introduction to the dynamics of rational functions is given in [B]. Except for slight changes in notation, we reproduce here almost word for word those results from [B] we shall need.

Definition. A family of functions $\mathcal{F}$ is normal if every sequence of functions in $\mathcal{F}$ has a sub-sequence which converges uniformly on compact subsets.

Let f be a rational function of degree greater than one.

Definition. A point $z \in \mathbf{P}^1$ is an element of the *Fatou set* F_f of f if there exists a neighborhood U of z in $\mathbf{P}^1$ such that the family of iterates $\left\{ (f^{\circ n})\Big|_U \right\}$ is a normal family. The *Julia set* J_f is the complement of the Fatou set.

Clearly J_f is closed.

Definition. The *eigenvalue* of a periodic orbit of period n is by definition

$$\lambda := (f^{\circ n})'(z_0)$$

for some z_0 in the orbit. By the chain rule, this definition is independent of the choice of z_0. A periodic orbit is

attracting if $0 < |\lambda| < 1$,

super-attracting if $\lambda = 0$,

repelling if $|\lambda| > 1$,

neutral if $|\lambda| = 1$.

Proposition 2.2.1. *If a periodic orbit is attracting or super-attracting, then it is contained in F. If it is repelling, then it is contained in J.*

Definition. A point z is *eventually periodic* if, for some n, $f^{\circ n}(z)$ is a periodic point. The point z is *preperiodic* if it is eventually periodic but not periodic.

We use the notation $(f^{\circ n})^{(k)}(z_0)$ to represent the kth derivative of $f^{\circ n}$.

Theorem 2.2.2. *Let z_0 be a point in a super-attracting periodic orbit. Suppose $k \geq 2$, $(f^{\circ n})^{(k)} \neq 0$, and*

$$(f^{\circ n})'(z_0) = (f^{\circ n})^{(2)}(z_0) = \cdots = (f^{\circ n})^{(k-1)}(z_0).$$

Then there exists a neighborhood U of z_0 and an analytic homeomorphism ϕ : $U \to D_r$ (for some r) such that $\phi(z_0) = 0$ and the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{f^{\circ n}} & U \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ D_r & \xrightarrow{z \mapsto z^k} & D_r \end{array}$$

Furthermore, such a ϕ is unique up to post-composition with multiplication by a $(k-1)$st root of unit.

Theorem. *(Sullivan) Every component of the Fatou set is eventually periodic.*

Definition. Let U be a periodic component of the Fatou set of period n and let

$$g := f^{\circ n}.$$

1) U is an *attracting domain* if U contains a point p of an attracting periodic cycle and all points of U are attracted to p under iteration of g.

2) U is a *super-attracting domain* if U contains a point p of a super-attracting periodic cycle and all points of U are attracted to p under iteration of g.

3) U is a *parabolic domain* if there exists a periodic point p in ∂U whose period divides n and all points of U are attracted to p under iteration of g.

4) U is a *Siegel disk* if U is simply connected and $g\big|_U$ is analytically conjugate to

$$(z \mapsto e^{i\theta} z).$$

5) U is a *Herman ring* of U is conformally equivalent to an annulus and $g\big|_U$ is analytically conjugate to a rigid rotation of the annulus.

Siegel disks and Herman rings are often referred to as *rotation domains.*

Theorem 2.2.3. *(Sullivan) Every periodic component of the Fatou set is either attracting, super attracting, parabolic, a Siegel disk, or a Herman ring. Furthermore, there are finitely many such domains. In the parabolic case,* $g'(p) = 1$*. The attracting and parabolic domains both contain infinite forward orbits of critical points, and the boundaries of rotation domains are contained in the closure of the forward orbit of the critical points.*

Corollary 2.2.4. *Suppose every critical point of* f *is either periodic, attracted to a periodic cycle or pre-periodic. Then every point in the Fatou set will be attracted to an attracting or super-attracting periodic cycle.*

The following is not stated in [B], but it is a standard result.

Theorem 2.2.5. *Suppose every critical point of* f *is either periodic, attracted* *to an attractive periodic cycle, or pre-periodic. Let $W \subset \mathbf{P}^1$ be a closed set containing the forward orbits of those critical points of f which are either periodic or attracted to an attractive periodic cycle. Then there an open set $U \supset \mathbf{P}^1 - W$, a number $\rho < 1$ and a metric μ on U such that $f^{-1}(U) \subset U$ and $f\big|_U$ is locally expanding with respect to μ by a factor of at least $1/\rho$.*

§2.3. *Quadratic polynomials.*

Most of definitions and results in this section come from [DH1] and [DH2].

All quadratic polynomials are conjugate by an affine map to one of the form

$$f_c(z) = z^2 + c$$

for some $c \in \mathbf{C}$.

Definition. The *filled in Julia set of f_c* is by definition,

$$K_c := \{z \mid f_c^{\circ n}(z) \not\to \infty \text{ as } n \to \infty\}.$$

K_c is closed and the boundary of K_c is $J_c := J_{f_c}$.

Theorem. *K_c is connected if and only if $0 \in K_c$ and K_c is a Cantor set if and only if $0 \notin K_c$.*

Definition. The Mandelbrot set is by definition

$$M := \{c \in \mathbf{C} \mid K_c \text{ is connected }\}$$

$$= \{c \in \mathbf{C} \mid 0 \in K_c\}.$$

Theorem. *For c in M, there is a unique analytic map*

$$\hat{\psi}_c : \hat{\mathbf{C}} - \bar{D} \to \hat{\mathbf{C}} - K_c$$

such that

$$f_c(\hat{\psi}_c(z)) = \hat{\psi}_c(z^2) \tag{2.1}$$

for all $z \in \hat{\mathbf{C}} - \bar{D}$.

Theorem 2.3.1. *Suppose c is in M and 0 is periodic, attracted to an attractive periodic cycle, or pre-periodic under* f_c. *Then* $\hat{\psi}_c$ *extends continuously to*

$$\hat{\psi}_c : \hat{\mathbf{C}} - D \to \hat{\mathbf{C}} - \mathring{K}_c$$

such that (2.1) is also satisfied for $z \in \partial D$.

Definition. In $\hat{\psi}_c$ extends as in the previous theorem, we define the *Carathéodory loop*

$$\gamma_c : \mathbf{T} \to J_c$$

of K_c (or of f_c) by

$$\gamma_c(t) := \hat{\psi}_c(\mathrm{Exp}(t)).$$

Note that γ_c is onto J_c.

Definition.

$$\mathcal{R}(K_c, t) := \left\{ \hat{\psi}_c(r \cdot \mathrm{Exp}(t)) \mid r \in [1, \infty[\right\}.$$

Notation. Let $\mathcal{D}_0$ be the set of c for which 0 is periodic under f_c. Let $\mathcal{D}_2$ be the set of c for which 0 is pre-periodic under f_c. (We will define $\mathcal{D}_1$ later).

Proposition. *If $c \in \mathcal{D}_0$, then $\overset{\circ}{K}_c \neq \emptyset$ and if $c \in \mathcal{D}_2$, then $\overset{\circ}{K}_c = \emptyset$.*

Proposition. *The components of $\overset{\circ}{K}_c$ are finite or countable.*

Proposition 2.3.2. *Let c be in $\mathcal{D}_0$. Let 0 be periodic and let U_0 be the component of $\overset{\circ}{K}_c$ containing 0. For $i = 1, 2, 3, \ldots$ let U_i be the rest of the components of $\overset{\circ}{K}_c$. For $i = 0, 1, 2, \ldots$ let i' be such that $f(U_i) = U_{i'}$. Then there is a unique set of homeomorphisms*

$$\check{\psi}_i : \bar{D} \to \bar{U}_i$$

for $i = 0, 1, 2, \ldots$ satisfying the following.

1) For $i = 0$,

$$f_c(\check{\psi}_i(z)) = \check{\psi}_{i'}(z^2)$$

for all $z \in \bar{D}$.

2) For $i = 1, 2, \ldots$,

$$\left(\check{\psi}_{i'}\right)^{-1} \circ f_c \circ \check{\psi}_i = \text{ id } .$$

3) $\check{\psi}_i$ is analytic on U_i.

Notation. By definition, the point in U_i of internal angle t and radius r is $\check{\psi}(r \cdot \operatorname{Exp}(t))$. We let

$$\mathcal{R}(U_i, t) := \left\{\check{\psi}_i(r \cdot \operatorname{Exp}(t)) \mid r \in [0,1]\right\}.$$

Proposition 2.3.3. *Let $f := f_c$, U, and μ be as in theorem 2.2.5. Let V be such that $\bar{V} \subset \overset{\circ}{U}$. Then any set of the form*

$$\mathcal{R}(K_c, t) \cap V \qquad \text{or} \qquad \mathcal{R}(U_i, t) \cap V$$

is of finite length with respect to μ.

Definition. For $c \in \mathcal{D}_0 \cup \mathcal{D}_2$ we say an arc γ in K_c is *regulated* if for all i, $\gamma \cap \bar{U}_i$ is contained in two rays of the form $\mathcal{R}(U_i, t)$.

Proposition. *For $c \in \mathcal{D}_0 \cup \mathcal{D}_2$ and for any distinct points x and y in K_c, there is a unique regulated arc from x to y which we denote by*

$$[x, y]_{K_c}.$$

Definition. A set $X \subset K_c$ is *regularly connected* if for all x and y in X,

$$[x, y]_{K_c} \subset X.$$

The *regulated envelope* $[A]$ of a set $A \subset K_c$ is the intersection of all regulatedly connected sets containing A.

Proposition 2.3.4. *Let $x_1, \ldots, x_n$ be points in K_c. The regulated envelope $[\{x_1, \ldots, x_n\}]$ of $\{x_1, \ldots, x_n\}$ is a finite topological tree.*

Remark 2.3.5. All extremities of $[\{x_1, \ldots, x_n\}]$ are in $\{x_1, \ldots, x_n\}$.

Definition. For $c \in \mathcal{D}_0 \cup \mathcal{D}_2$, the *Hubbard tree X_{H_c} of f_c* is the regulated envelope of the set

$$\{f_c^{\circ n}(0) \mid n = 0, 1, 2, \ldots\}.$$

Properties.

1) $f(X_{H_c}) \subset X_{H_c}$.

2) $X_{H_c} - \{0\}$ has at most two components and f_c is injective on the closure of each component.

3) $f_c(0)$ is an extremity of X_{H_c}.

Notation. The *external angles* of a point z in J_c are the elements of $\gamma_c^{-1}(\{z\})$.

Proposition 2.3.6. *If $z \in J_c$ has more than one external angle, some forward image of z lies on X_{H_c}.*

Claim 2.3.7. *Let q be in J_c and let $[x,y]_{K_c}$ be the regulated arc joining x to y. If q has only one external angle, then q is not in the interior of $[x,y]_{K_c}$.*

§2.4. *The Mandelbrot set.*

We defined M in the last section. One can easily see that M is closed.

Definition. A component of $\overset{\circ}{M}$ is called *hyperbolic* if it contains a point of $\mathcal{D}_0$.

Proposition. *If M_i is a hyperbolic component of $\overset{\circ}{M}$, then for all c in M_i, f_c has an attractive or super-attractive periodic cycle other than ∞. The map ϕ_i which maps a c in M_i to the eigenvalue of that cycle is an analytic isomorphism of M_i with D. That map extends to a homeomorphism from $\bar{M}_i$ to $\bar{D}$.*

Definition. With M_i and ϕ_i as above, the point in $\bar{M}_i$ at *internal angle* t *and radius* r is by definition

$$(\phi_i)^{-1}(r \cdot \mathrm{Exp}(t)).$$

The point of M_i at internal angle 0 and radius 1 is called the *root* of M_i. No point is the root of more than one component of $\overset{\circ}{M}$. By definition, $\mathcal{D}_1$ is the set of roots of hyperbolic components of $\overset{\circ}{M}$. The *center* of M_i is by definition the point in $\mathcal{D}_0 \cap M_i$.

Theorem. *There is a unique analytic isomorphism*

$$\Psi : \hat{\mathbf{C}} - \bar{D} \to \hat{\mathbf{C}} - M$$

such that Ψ *is tangent to the identity at* ∞. *Hence,* M *is connected.*

Theorem. *If* t *is in* $\mathbf{Q}/\mathbf{Z}$, *then the curve*

$$r \mapsto \Psi(r \cdot \operatorname{Exp}(t))$$

converges to a point $\gamma_M(t)$ *in* M *as* $r \to 1$. *If* t *has even dynamic denominator, then* $\gamma_M(t) \in \mathcal{D}_2$. *If* t *has odd dynamic denominator, then* $\gamma_M(t) \in \mathcal{D}_0$.

So this defines

$$\gamma_M : \mathbf{Q}/\mathbf{Z} \to \partial M.$$

Notation. By definition, the *external ray of* M *at angle* t is the set

$$\mathcal{R}(M,t) := \{\Psi(r \cdot \operatorname{Exp}(t)) \mid r \in]1, \infty[\}.$$

If $t \in \mathbf{Q}/\mathbf{Z}$, we say that $\mathcal{R}(M,t)$ *corresponds* to $\gamma_M(t)$. If $\gamma_M(t) \in \mathcal{D}_1$ and c is the center of the component of $\overset{\circ}{M}$ of which $\gamma_M(t)$ is the root, then we say that $\mathcal{R}(M,t)$ *corresponds* to c.

Proposition 2.4.1. *Let* $t \in \mathbf{Q}/\mathbf{Z} - \{0\}$ *have dynamic denominator* $2^n - 1$ *and let* $c \in \mathcal{D}_0$ *correspond to* $\mathcal{R}(M,t)$. *Let* z *be the root (i.e. the point at internal* *angle* 0 *and radius* 1*) of the component of* $\overset{\circ}{K}_c$ *containing* $f_c(0)$. *Then*

$$\#\gamma_M^{-1}(\gamma_M(t)) = 2 \qquad \text{and} \qquad \gamma_M^{-1}(\gamma_M(t)) = \gamma_c^{-1}(z).$$

Also $f_c^{\circ n}(0) = 0$.

Proposition. *Let* $t \in \mathbf{Q}/\mathbf{Z}$ *have even dynamic denominator. Then*

$$\gamma_M^{-1}(\gamma_M(t)) = \gamma_c^{-1}(z).$$

Proposition and Definition. The component of $\overset{\circ}{M}$ containing 0 is denoted by M_0 and is called the *central cardioid of* M. M_0 is hyperbolic. For every $t \in \mathbf{Q}/\mathbf{Z} - \{0\}$, let c_t be the point at M_0-internal angle t and radius 1. $M - \{c_t\}$ has two components, one containing 0 and the other which we denote by M_t and call the *limb of* M *attached to the central cardioid at internal angle* t. We call c_t the *root* of M_t.

$$M = \left(\bigcup_{t \in \mathbf{Q}/\mathbf{Z}} M_t \right) \cup \partial M_0.$$

Proposition and Definition. The root of a limb is the root of a hyperbolic component of $\overset{\circ}{M}$. The center of that component is called the *star* of the limb. The external rays of M corresponding to the root are said to *correspond* to the limb and the star of the limb.

Definition. The fixed points of f_c are

$$(1 \pm \sqrt{1-4c})/2.$$

If $c \in [1/4, \infty]$, they are complex conjugates. We can choose a branch of $\sqrt{1-4c}$ for c in $\mathbf{C} - [1/4, \infty]$ so that $\sqrt{1} = 1$. For those c we let

$$\beta_c := (1 + \sqrt{1-4c})/2 \qquad \text{and} \qquad \alpha_c := (1 - \sqrt{1-4c})/2.$$

Proposition. β_c is in J_c and is repulsive. For $c \in M_0$, α_c is in $\mathring{K}_c$ and is attractive. For $c \in \partial M_0$, α_c is in J_c and is neutral. For $c \in \mathbf{C} - \left(\bar{M}_0 \cup [1/4, \infty]\right)$, α_c is in J_c and is repulsive.

Proposition 2.4.2. Let t be in $\mathbf{Q}/\mathbf{Z} - \{0\}$ and let θ and θ' be the angles of the external rays of M corresponding to M_t. Let the dynamic denominator of t be $2^n - 1$. Then for all $c \in M_t$ with Carathéodory loop γ_c,

$$\begin{aligned}\gamma_c^{-1}(\alpha_c) &= \left\{2^0\theta, 2^1\theta, 2^2\theta, \ldots, 2^{n-1}\theta\right\} \\ &= \left\{2^0\theta', 2^1\theta', 2^2\theta', \ldots, 2^{n-1}\theta'\right\}\end{aligned}$$

and

$$\gamma_c^{-1}(\beta_c) = \{0\}\,.$$

Proposition 2.4.3. *Let c be a star and let $[x, y]_{K_c}$ be a regulated arc in K_c which intersects α_c. Then there exists a neighborhood U of α_c in $[x, y]_{K_c}$ such that*

$$U \cap J_c = \{\alpha_c\}\,.$$

Proposition 2.4.4. Let c be a star and let θ be the angle of an external ray of M corresponding to c. The dynamic denominator of θ can be written $2^n - 1$. Let

$$\Theta := \left\{2^0\theta, 2^1\theta, 2^2\theta, \ldots, 2^{n-1}\theta\right\}.$$

The Hubbard tree X_{H_c} of c is homeomorphic to

$$S_\theta := \{r \cdot \mathrm{Exp}(t) \mid r \in [0,1], t \in \Theta\}$$

and $f : X_{H_c} \to X_{H_c}$ is conjugate to

$$(z \mapsto z^2) : S_\theta \to S_\theta.$$

Proposition. Let c be a star and let X_{H_c} be the Hubbard tree of c. Then

$$X_{H_c} \cap J_c = \{\alpha_c\}\,.$$

Claim. *Points of the form $\gamma_M(p/(2^m))$ are dense in ∂M.*

Proposition. If f has a Carathéodory loop γ_c, then points of the form $\gamma_c(p/(2^m))$ are dense in ∂K_c.

Proposition. Points of the form $\gamma_M(p/2^m)$ have only one corresponding external ray of M.

Proposition 2.4.5. Points of the form $\gamma_c(p/2^m)$ have only one external angle.

Caution. In order to minimize the number of sub-sub-sub scripts required, in later chapters we abuse notation with statements such as "Let f be in $\mathcal{D}_0$". Of course, by that we mean, "Let $f = f_c$ for some c in $\mathcal{D}_0$."

§2.5. *Thurston's topological characterization of rational functions.*

We shall make heavy use of an algorithm and theorem due to Thurston. We reproduce here the definitions and statements. For a more complete discussion the reader can consult [Th1], [Th2], or [DH3]. Except for slight changes in notation, this section is taken almost word for word from [DH3].

Let $f : S^2 \to S^2$ be an orientation preserving branched covering map. We will call

$$\Omega_f := \{x \mid \deg_x f > 1\}$$

the *critical set* of f, and

$$P_f = \bigcup_{n>0} f^{\circ n}(\Omega_f)$$

the *post-critical set.* The mapping f will be called *critically finite* if P_f is a finite set.

Clearly there exists a smallest function

$$\nu_f : P_f \to \mathbf{N} \cup \{\infty\}$$

such that $\nu_f(x)$ is a multiple of $\nu_f(y) \cdot \deg_y f$ for each $y \in f^{-1}(x)$. We will say that the *orbifold* $O_f := (S^2, \nu_f)$ *of* f is *hyperbolic* if its Euler characteristic

$$\chi(O_f) = 2 - \sum_{x \in P_f} (1 - (1/\nu_f(x)))$$

satisfies $\chi(O_f) < 0$.

Two branched covers $f, g : S^2 \to S^2$ are *topologically equivalent* (or Thurston topologically equivalent) if and only if there exist homeomorphisms

$$\theta, \theta' : (S^2, P_f) \to (S^2, P_g)$$

such that the diagram

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\theta'} & (S^2, P_g) \\ \downarrow f & & \downarrow g \\ (S^2, P_f) & \xrightarrow{\theta} & (S^2, P_g) \end{array}$$

commutes, and θ is isotopic to θ' (rel P_f).

If γ is a simple closed curve on $S^2 - P_f$, then the set $f^{-1}(\gamma)$ is a union of disjoint simple closed curves. If γ moves continuously, then so does each component of $f^{-1}(\gamma)$.

We will need to consider systems

$$\Gamma = \{\gamma_1, \ldots, \gamma_n\}$$

of simple, closed, disjoint, non-homotopic, non-peripheral curves on $S^2 - P_f$ (γ is non-peripheral if each component $S^2 - \gamma$ contains at least 2 points of P_f). Such a system will be called a *multicurve* on $S^2 - P_f$.

A multicurve will be called *f-stable* if for any $\gamma \in \Gamma$, all the non-peripheral components of $f^{-1}(\gamma)$ are homotopic in $S^2 - P_f$ to elements of Γ.

To each f-stable multicurve Γ we can associate the *Thurston linear transformation*

$$f_\Gamma : \mathbf{R}^\Gamma \to \mathbf{R}^\Gamma$$

as follows. Let $\gamma_{i,j,\alpha}$ be the components of $f^{-1}(\gamma_j)$ homotopic to γ_i in $S^2 - P_f$. Define

$$f_\Gamma(\gamma_j) := \sum_{i,\alpha} (1/d_{i,j,\alpha})\gamma_i,$$

where

$$d_{i,j,\alpha} = \deg\left(f\Big|_{\gamma_{i,j,\alpha}} : \gamma_{i,j,\alpha} \to \gamma_j\right).$$

The Thurston transformation commutes with iteration. That is

$$(f^{\circ n})_\Gamma = (f_\Gamma)^{\circ n}.$$

Since f_Γ has a matrix with non-negative entries, there exists a largest eigenvalue $\lambda(\Gamma, f) \in \mathbf{R}_+$; the corresponding eigenvector has non-negative entries.

Thurston's criterion is the following.

Theorem 2.5.1. *A critically finite branched map $f : S^2 \to S^2$ with hyperbolic orbifold is topologically equivalent to a rational function if and only if for any f-stable multicurve Γ, we have $\lambda(\Gamma, f) < 1$. In that case the rational function is unique up to conjugation by an automorphism of $\mathbf{P}^1$.*

Douady and Hubbard completely describe the branched mappings with non-hyperbolic orbifolds in section 9 of [DH3].

To prove theorem 2.5.1, the basic construction is a mapping σ_f from an appropriate Teichmüller space to itself. The mapping σ_f will be of interest in its own right.

Definition. The Teichmüller space $\mathcal{T}_f$ is the Teichmüller space modeled on (S^2, P_f).

Remark. The space $\mathcal{T}_f$ can be constructed as the space of diffeomorphisms

$$phi : (S^2, P_f) \to \mathbf{P}^1,$$

with ϕ_1 and ϕ_2 identified if and only if there exists a Möbius transformation $h : \mathbf{P}^1 \to \mathbf{P}^1$ such that the diagram

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\phi_1} & \mathbf{P}^1 \\ \Big\downarrow \mathrm{id} & & \Big\downarrow h \\ (S^2, P_f) & \xrightarrow{\phi_2} & \mathbf{P}^1 \end{array}$$

commutes on P_f, and commutes up to isotopy (rel P_f)

Proposition. *There is an analytic map $\sigma_f : \mathcal{T}_f \to \mathcal{T}_f$ such that if $\tau \in \mathcal{T}_f$ is represented by $\phi : (S^2, P_f) \to \mathbf{P}^1$, then $\tau' := \sigma_f(\tau)$ can be represented by $\phi' : (S^2, P_f) \to \mathbf{P}^1$ with*

$$\phi \circ f \circ (\phi')^{-1} : \mathbf{P}^1 \to \mathbf{P}^1$$

analytic.

Proposition. *The mapping f is topologically equivalent to a rational function if and only if σ_f has a fixed point.*

Definition. The *Thurston's method* for f starting at τ is the sequence $\left\{\sigma_f^{\circ n}(\tau)\right\}$.

The idea of the proof of the "if" part of theorem 2.5.1 is to show that any Thurston's method for f converges to the unique fixed point of σ_f. In some cases, Thurston's method can be run on a computer. In order to do so, however, representatives of the $\sigma_f^{\circ n}(\tau)$ must be chosen.

Definition. Given a representative ϕ of some $\tau \in \mathcal{T}_f$ and $q = (q_0, q_1, q_2)$ an ordered triple of distinct points in S^2, the *Thurston's method for f normalized at* *q starting at* ϕ is the unique sequence of diffeomorphisms

$$\{\phi_n : S^2 \to \mathbf{P}^1\}_{n=0}^{\infty}$$

satisfying the following four conditions.

1) ϕ_n is a representative of $\sigma_f^{\circ n}(\tau)$.

2) $\phi_0 = \phi$.

3) $f_n := \phi_n \circ f \circ \phi_{n+1}^{-1}$ is analytic.

4) $\phi_n(q_0) = \infty$, $\phi_n(q_1) = 0$, and $\phi_n(q_2) = 1$.

We say that the normalized Thurston's method *converges* if the ϕ_n converge (not necessarily to an injective map). Note that in that case the f_n converge to a rational function called the *output*.

The following claim follows from the proof of theorem 2.5.1.

Claim. *If f has hyperbolic orbifold and is topologically equivalent to a rational function h, then any normalized Thurston's method will output a rational function conjugate to h by a Möbius transformation.*

We end this section with a useful lemma about the leading eigenvalue of matrices with non-negative entries.

Lemma 2.5.2. *Let $A = (a_{ij})$ and $A' = (a'_{ij})$ be two square matrices of the same size with $a_{ij} \geq a'_{ij} \geq 0$. Then the leading eigenvalue λ of A is greater than or equal to the leading eigenvalue λ' of A'.*

Proof 2.5.2. Let x_n (resp. x'_n) be the largest entry in A^n (resp. in $(A')^n$). By considering Jordan canonical form, one can see that

$$x_n = k\lambda^n + o(\lambda^n) \qquad \text{and} \qquad x'_n = k'(\lambda')^n + o((\lambda')^n)$$

for some non-zero constants k and k'. Since $x_n \geq x'_n$, for all n, we are done.

End 2.5.2.

§**2.6.** *A parameterization of rational functions of degree two.*

Given any rational function g of degree two, we can let h be a Möbius transformation taking one critical point of g to ∞, the other critical point to 0, and one fixed point of g to 1. It is easy to verify that $h \circ g \circ h^{-1}$ can be written in the form

$$g_{a,b}(z) := \frac{az^2 + 1 - a}{bz^2 + 1 - b}.$$

Of course g has two critical points and k fixed points for some $k \in \{1, 2, 3\}$, so there will be $2k$ pairs (a, b) such that g is conjugate by a Möbius transformation to $g_{a,b}$.

We will let

$$R_{m,n} := \left\{(a, b) \in \mathbf{C}^2 \mid g_{a,b}^{\circ(m+n)}(\infty) = g_{a,b}^{\circ m}(\infty)\right\}.$$

$R_{m,n}$ is a one complex dimensional algebraic curve in $\mathbf{C}^2$.

We let $MR_{m,n}$ be the set of $(a, b) \in R_{m,n}$ such that $g_{a,b}^{\circ j}(0)$ does not approach the cycle containing forward images of ∞ as j tends to infinity.

We let $MR_{m,n}^k$ be the pairs $(a, b) \in R_{m,n}$ such that $g_{a,b}$ has an attractive periodic cycle of period k not containing ∞. Since that cycle must attract 0 (proposition 2.2.1 and theorem 2.2.3), $MR^k_{m,n} \subset MR_{m,n}$. For every component U of $MR^k_{m,n}$, the map $\phi : U \to D$ which maps (a, b) to the eigenvalue of the attractive periodic cycle is an isomorphism. The map ϕ extends continuously to the boundary of U. We call $\phi^{-1}(1)$ the *root* of U and $\phi^{-1}(0)$ the *center*.

§**2.7.** *A topological lemma.*

Proposition 8.1.2.1. *Let A be a closed annulus, let $f : A \to A$ be a homeomorphism, and let $\tilde{A}$ be the universal covering space of A. If there is a lift $\tilde{f} : \tilde{A} \to \tilde{A}$ of f which is the identity on the boundary of $\tilde{A}$, then f is homotopic (rel ∂A) through homeomorphisms to the identity.*

Chapter 3. Embedding Graphs.

§3.1. *Definitions.*

Definition. An *embedding graph* (or *e-graph*) G consists of the following:

1) A topological space X_G which is a finite topological graph (i.e. a finite disjoint union of closed intervals modulo some indentification of endpoints).

2) A finite subset $V_G \subset X_G$ called the *vertices* containing all points of X_G which do not have neighborhoods homeomorphic to an open interval. The connected components of $X_G - V_G$ are called *edges* and the set of edges is denoted by E_G. We require that each edge has two distinct vertices in its closure. Given a vertex v, the *edges incident upon* v are by definition members of the set

$$E_G^v := \{e \in E_G \mid v \text{ is in the closure of } e\}.$$

3) For each vertex v, a cyclic permutation

$$\sigma_G^v : E_G^v \to E_G^v$$

which is non-trivial unless $\#E_G^v = 1$.

Definition. Given an e-graph G, we define the topological space $\ddot{X}_G$ (called X_G *cut*) and an associated quotient map $\pi_G : \ddot{X}_G \to X_G$ as follows (see figure 3.1. for an example). Let e be an edge of G. For each vertex v in $\bar{e}$, let $\text{away}(e, v)$ be the orientation of $\bar{e}$ such that v is encountered first while traversing $\bar{e}$ consistently with $\text{away}(e, v)$. Also, for each orientation ω of $\bar{e}$, let $\text{tip}(e, \omega)$ be the vertex encountered last while traversing $\bar{e}$ consistently with ω. Finally, for each orientation ω of $\bar{e}$, let $\bar{e}_\omega$ be a copy of $\bar{e}$, and for each point $x \in \bar{e}$, let x_e^ω be the corresponding point in $\bar{e}_\omega$. $\ddot{X}_G$ is by definition the disjoint union of the $\bar{e}_\omega$ modulo all identifications of the form

$$v_e^\omega \sim v_{\sigma_v(e)}^{\text{away}(\sigma_v(e),v)} \quad \text{for} \quad v = \text{tip}(e, \omega).$$

We give $\ddot{X}_G$ the quotient topology. We let

$$\pi_G(x_e^\omega) := x.$$

It is easy to see that π_G is closed and hence a quotient map.

The following proposition is clear.

Proposition 3.1.1. *The connected components of $\ddot{X}_G$ defined above are each homeomorphic to S^1.*

In order to make definition 3.2.1 below, we need the following proposition.

Proposition 3.1.2. *Given a finite topological graph X embedded in an oriented surface Y, for each point x in X and neighborhood U of x in Y, there is a neighborhood V of x in Y and an orientation preserving homeomorphism $\phi : V \to D$ such that*

$$\begin{aligned} \phi(V \cap X) &= \{r \cdot \text{Exp}(t/N) \in D \mid t = 0, 1, \ldots, N-1\} \\ &= \text{the standard } N\text{-fold star}, \end{aligned}$$

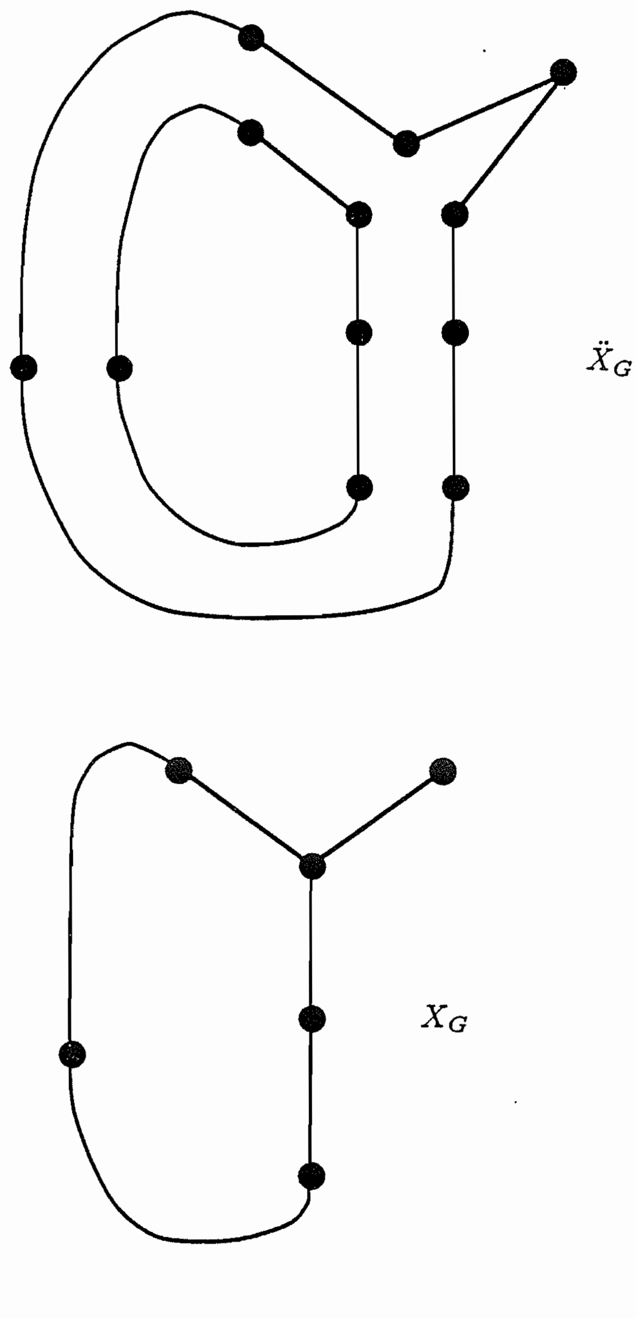

Figure 3.1. Cut of an embedding graph.

$\phi(x) = 0$, *and* $V \subset U$.

Proof 3.1.2.

Claim 3.1.2.1. *We may choose* V *such that* $V \cap X$ *is a topological star with* x *at the center,* V *is simply connected, and* ∂V *is a Jordan curve.*

Proof 3.1.2.1. Choose X' a neighborhood of x in X such that X' is a topological star with x at the center and X' is connected. Choose X'' a neighborhood of x in X such that X'' is a topological star with x at the center, X'' is connected, and $\bar{X}'' \subset X'$. We can fatten $\bar{X}''$ sufficiently little to form the desired V.

End 3.1.2.1.

Let N be the number of components of $X \cap V - \{x\}$, and let those components be labeled $X_0, X_1, \ldots, X_{N-1}$ so that $X_{n+1} = \sigma(X_n)$ where σ is the clockwise-around-x permutation of the X_i. Let V_n be the component of $V - X$ having X_n and X_{n+1} in its closure. Let

$$D_n := \left\{ r \cdot \mathrm{Exp}(t) \mid \frac{n}{N} < t < \frac{n+1}{N} \text{ and } 0 < r < 1 \right\}$$

and let

$$R_n := \left\{ r \cdot \mathrm{Exp}(\frac{n}{N}) \mid 0 < r < 1 \right\}.$$

Since the V_i are simply connected and their boundaries Jordan curves, there are orientation preserving homeomorphisms $\psi_n : \bar{D} \to \bar{V}_n$ with $\psi_n(0) = x$,

$$\psi_n(\mathrm{Exp}(n/N)) = X_n \cap \partial V$$

and

$$\psi_n(\mathrm{Exp}((n+1)/N)) = X_{n+1} \cap \partial V.$$

So ψ_n and ψ_{n+1} restricted to R_{n+1} are orientation preserving homeomorphisms onto X_{n+1}. So we may let $\gamma_n : [0,1] \to [0,1]$ be such that

$$\psi_n\left(\gamma_n(r) \cdot \mathrm{Exp}(\frac{n+1}{N})\right) = \psi_{n+1}\left(r \cdot \mathrm{Exp}(\frac{n+1}{N})\right).$$

Let

$$t_n(s) := (1-s)\left(\frac{n}{N}\right) + s\left(\frac{n+1}{N}\right),$$

and let

$$\hat{\psi}_n\left(r \cdot \mathrm{Exp}(t_n(s)\right) := \psi_n\left(((1-s)r + s\gamma_n(r)) \cdot \mathrm{Exp}(t_n(s)\right).$$

Then $\hat{\psi}_n = \hat{\psi}_{n+1}$ on $\bar{R}_{n+1}$ and $\psi := \cup_n \hat{\psi}_n$ is continuous. $\phi := \psi^{-1}$ is the map we seek.

End 3.1.2.

§**3.2.** *Assorted properties.*

Definition. Given an e-graph G and an oriented surface Y, an embedding

$$\iota : X_G \to Y$$

is an *e-graph embedding* if the cyclic permutations are those induced by Y (i.e. $\sigma_G^v(e) = e'$ implies that $\sigma(\iota(e)) = \iota(e')$ where σ is the clockwise-around-$\iota(v)$ permutation of $\iota(E_G^v)$).

Notation. If $\iota : X_G \to X$ is an e-graph embedding, by ι^{-1} we mean

$$\iota^{-1} : \iota(X_G) \to X_G.$$

Definition 3.2.1.

Let G be an e-graph, let $\ddot{X}_G$ be X_G cut, and let $\pi_G : \ddot{X}_G \to X_G$ be the associated quotient map. Given an oriented surface Y and an e-graph embedding $\iota : X_G \to Y$, we define the topological space $\ddot{Y}$ (called Y *cut along* $\iota(X_G)$), a closed map $\pi : \ddot{Y} \to Y$ (called the *associated quotient map*) and an embedding $\ddot{\iota} : \ddot{X}_G \to \ddot{Y}$ (called the *associated embedding*) so that the diagram

$$\begin{array}{ccc} \ddot{X}_G & \xrightarrow{\ddot{\iota}} & \ddot{Y} \\ \Big\downarrow{\scriptstyle \pi_G} & & \Big\downarrow{\scriptstyle \pi} \\ X_G & \xrightarrow{\iota} & Y \end{array}$$

commutes.

The points of $\ddot{Y}$ are as follows. For each point y in $Y - \iota(X_G)$, there is a unique point π^{-1} in $\ddot{Y}$. For each point x_e^ω in $\ddot{X}_G$, there is a unique point $\ddot{\iota}(x_e^\omega)$ in $\ddot{Y}$.

For each point y in $Y - \iota(X_G)$ a basis of neighborhoods of $\pi^{-1}(y)$ is as follows. Choose a neighborhood U of y such that $U \cap \iota(X_G) = \emptyset$. A basis of neighborhoods of $\pi^{-1}(y)$ is then the sets of the form $\pi^{-1}(W \cap U)$ where W is a neighborhood of y.

For each point x_e^ω in $\ddot{X}_G$ such that x is not a vertex of G, a basis of neighborhoods of $\ddot{\iota}(x_e^\omega)$ is defined as follows. For each neighborhood U of $\iota(x)$, by proposition 3.1.2 there is a neighborhood V of $\iota(x)$ and an orientation preserving homeomorphism $\phi : V \to D$ such that $\phi(\iota(x)) = 0$,

$$\phi(\iota(X_G) \cap V) = \phi(e \cap V) = \mathbf{R} \cap D,$$

and $\gamma(s) := (1-s)(1) + s(-1)$ traverses $\mathbf{R} \cap D$ consistently with $\phi(\iota(\omega))$. Let

$$W := \phi^{-1}(\{z \in D \mid \mathrm{Im}(z) < 0\})$$

and let

$$L := \left\{ z_e^\omega \in \ddot{X}_G \mid \iota(z) \in V \right\}.$$

Finally, let

$$Z := \pi^{-1}(W) \cup \ddot{\iota}(L).$$

The set of all such Z is a basis of neighborhoods of $\ddot{\iota}(x_e^\omega)$.

For each point v_e^ω in $\ddot{X}_G$ such that $v = \mathrm{tip}(e, \omega)$, a basis of neighborhoods of $\ddot{\iota}(v_e^\omega)$ is defined as follows. For each neighborhood U of $\iota(v)$, by proposition 3.1.2 there is a neighborhood V of $\iota(v)$ and an orientation preserving homeomorphism $\phi : V \to D$ such that $\phi(\iota(v)) = 0$, $\phi(e \cap V) = \mathbf{R}^+ \cap D$, and

$$\phi(\iota(X_G) \cap V) = \{r \cdot \mathrm{Exp}(t/N) \mid t = 0, 1, \ldots, N-1\}.$$

Let

$$W := \phi^{-1}\{r \cdot \mathrm{Exp}(t/N) \mid N-1 < t < N\}$$

and let

$$L := \left\{ z_e^\omega \in \ddot{X}_G \mid \iota(z) \in V \right\} \cup \left\{ z_{\sigma_v(e)}^{\mathrm{away}(\sigma_v(e), v)} \in \ddot{X}_G \mid \iota(z) \in V \right\}.$$

Finally, let

$$Z := \pi^{-1}(W) \cup \ddot{\iota}(L).$$

The set of all such Z is a basis of neighborhoods of $\ddot{\iota}(v_e^\omega)$.

It is easy to see that π is closed and that $\ddot{\iota}$ is an embedding.

End 3.2.1.

Remark. It is clear from the definition that $\ddot{Y}$ is a surface with boundary, and $\ddot{\iota}$ maps $\ddot{X}_G$ homeomorphically onto $\partial\ddot{Y}$.

Definition. An e-graph G is *connected* if X_G is connected.

Proposition 3.2.2. *Let G, $\ddot{X}_G$, π_G, Y, ι, $\ddot{Y}$, π, and $\ddot{\iota}$ be as in the definition of Y cut along $\iota(X_G)$. If G is connected and Y is homeomorphic to S^2, then the connected components of $\ddot{Y}$ are homeomorphic to $\bar{D}$.*

Proof 3.2.2. Each component of $Y - \iota(X_G)$ is of genus 0 and orientable since it is a subspace of S^2. Since X_G is connected, each component of $S^2 - \iota(X_G)$ has one boundary component. So each component of $\ddot{Y}$ is an orientable surface of genus 0 with boundary homeomorphic to S^1. We are done by the classification of orientable surfaces with boundary. **End 3.2.2.**

Proposition 3.2.3. *Let G be a connected e-graph, and let*

$$\kappa_0 : X_G \to S^2 \quad \text{and} \quad \kappa_1 : X_G \to S^2$$

be e-graph embeddings. Then there exists a homeomorphism $\phi : S^2 \to S^2$ such that

$$\phi \circ \kappa_0 = \kappa_1.$$

Proof 3.2.3.

Let $\ddot{X}_G$ be X_G cut, and let $\pi_G : \ddot{X}_G \to X_G$ be the associated quotient map. For $j = 0, 1$ let $\ddot{S}^2_j$ be S^2 cut along $\kappa_j(X_G)$, let $\pi_j : \ddot{S}^2_j \to S^2$ be the associated quotient map, and let $\ddot{\kappa}_j : \ddot{X}_G \to \ddot{S}^2_j$ be the associated embedding. So we have the commutative diagram in figure 3.2.

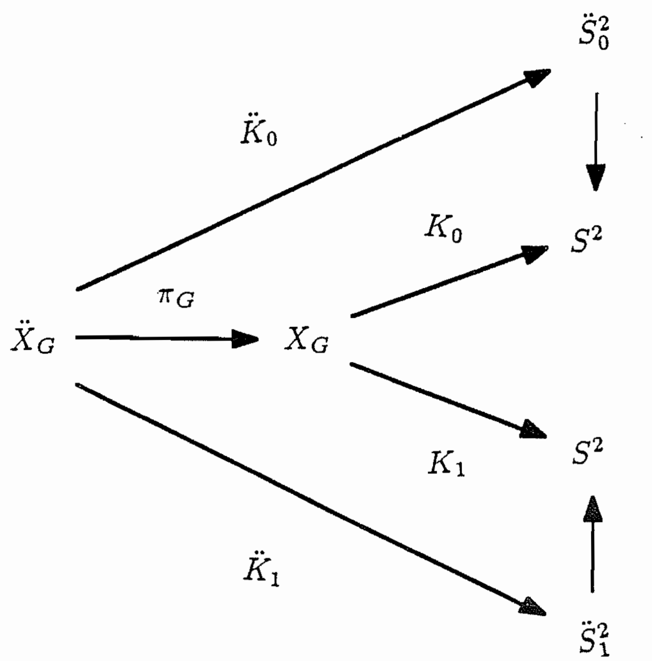

Figure 3.2. Commutative diagram.

We will define $\tilde{\phi} : \ddot{S}^2_0 \to \ddot{S}^2_1$ so that $\pi_1 \circ \tilde{\phi}$ factors through π_0. For each component $\ddot{U}_0$ of $\ddot{S}^2_0$, let $\tilde{\phi}$ be $\ddot{\kappa}_1 \circ \ddot{\kappa}_0^{-1}$ on $\partial \ddot{U}_0$. Since $\ddot{\kappa}_1$ is injective, $\tilde{\phi}(\partial \ddot{U}_0)$ is the boundary component of some component $\ddot{U}_1$ of $\ddot{S}^2_1$. By proposition 6.4.1, $\ddot{U}_0$ and $\ddot{U}_1$ are both homeomorphic to $\bar{D}$, so we can extend $\tilde{\phi}$ on $\partial \ddot{U}_0$ to a homeomorphism $\tilde{\phi} : \ddot{U}_0 \to \ddot{U}_1$. Factor $\pi_1 \circ \tilde{\phi}$ through π_0 to get the desired ϕ.

End 3.2.3.

Theorem 3.2.4. *Suppose G is a connected e-graph and $\iota_t : X_G \to S^2$ for $t \in [0,1]$ is a homotopy through e-graph embeddings. Then there exists a homotopy through homeomorphims $\phi_t : S^2 \to S^2$ for $t \in [0,1]$ such that ϕ_0 is the identity, and $\phi_1 \circ \iota_0 = \iota_1$. Furthermore, if ι_t is independent of t for some subset $X'_G \subset X_G$, then ϕ_t can be chosen to be the identity on $\iota_0(X'_G)$, and ϕ_t can be chosen to be the identity on any components of $S^2 - \iota_0(X_G)$ which have boundary contained in $\iota_0(X'_G)$.*

Proof 3.2.4.

Let $\ddot{X}_G$ be X_G cut and let $\pi_G : \ddot{X}_G \to X_G$ be the associated quotient map. For each t in $[0,1]$, let $\ddot{S}^2_t$ be S^2 cut along $\iota_t(X_G)$, let $\pi_t : \ddot{S}^2_t \to S^2$ be the associated quotient map, and let $\ddot{\iota} : \ddot{X}_G \to \ddot{S}^2_t$ be the associated embedding. For each component $\ddot{Z}$ of $\ddot{X}_G$ let $\ddot{U}_t(\ddot{Z})$ be the component of $\ddot{S}^2_t$ having $\ddot{\iota}_t(\ddot{Z})$ as boundary.

Claim 3.2.4.1. *For every $\hat{t}$ in $[0,1]$, there is an interval $T_{\hat{t}}$ with $\hat{t} \in T_{\hat{t}}$ and homeomorphisms $(\phi_{\hat{t}})_t : \ddot{S}^2_0 \to \ddot{S}^2_t$ for $t \in T_{\hat{t}}$ such that*

1) $(\phi_{\hat{t}})_t \circ \ddot{\iota}_0 = \ddot{\iota}_t$,

2) $\phi_t \circ (\phi_{\hat{t}})_t$ is a homotopy, and

3) if $\hat{t} = 0$, then $(\phi_{\hat{t}})_0$ is the identity.

Proof 3.2.4.1.

For each component $\ddot{Z}$ of $\ddot{X}_G$, we will define $\phi_{\hat{t}}$ separately on $\ddot{U}_0(\ddot{Z})$. Let $U_t := \pi_t(\ddot{U}_t(\ddot{Z}))$ for all t in $[0,1]$. Choose a point y in the interior of U_0 and a tangent vector ξ to S^2 at y. Choose a point z in the interior of $U_{\hat{t}}$ and a tangent vector ζ to S^2 at z. If $\hat{t} = 0$, let $z = y$ and $\zeta = \xi$. Let I be an interval about $\hat{t}$ small enough so that z is in the interior of U_t for all t in I.

By proposition 3.2.2, $\ddot{U}_t(\ddot{Z})$ is homeomorphic to $\bar{D}$, so for t in I there is a unique conformal isomorphism

$$\psi_t : \bar{D} \to \text{interior}(\ddot{U}_t(\ddot{Z}))$$

such that

$$(\pi_t \circ \psi_t)(0) = z \qquad \text{and} \qquad (\pi_t \circ \psi_t)'(0) \in \mathbf{R}^+\zeta.$$

By the Carathéodory Extension Theorem ([Du] p. 12 or [DH1] or [G]) the ψ_t can be continuously extended to a homeomorphism

$$\psi_t : \bar{D} \to \ddot{U}_t(\ddot{Z}).$$

By the Carathéodory Convergence Theorem ([Du] p. 76) $\pi_t \circ \psi_t\big|_D$ converge uniformly in t on compact subsets of D. The bounds on the modulus of continuity at a boundary point of D in the proof of that theorem depend only on a bound on an area and on the modulus of local connectivity of $\partial\ddot{U}_t(\ddot{Z})$. In our case, we can make these bounds independently of t, so $\pi_t \circ \psi_t$ is a homotopy. Similarly, we can let

$$\alpha : \bar{D} \to \ddot{U}_0(\ddot{Z})$$

be the unique homeomorphism which is analytic on D with

$$(\pi_0 \circ \alpha)(0) = y \qquad \text{and} \qquad (\pi_0 \circ \alpha)'(0) \in \mathbf{R}^+\xi.$$

Let $\gamma_t : \ddot{Z} \to \ddot{Z}$ be given by

$$\gamma_t := \ddot{\imath}_t^{-1} \circ \psi_t \circ \alpha^{-1} \circ \ddot{\imath}_0$$

(see the commutative diagram in figure 3.3).

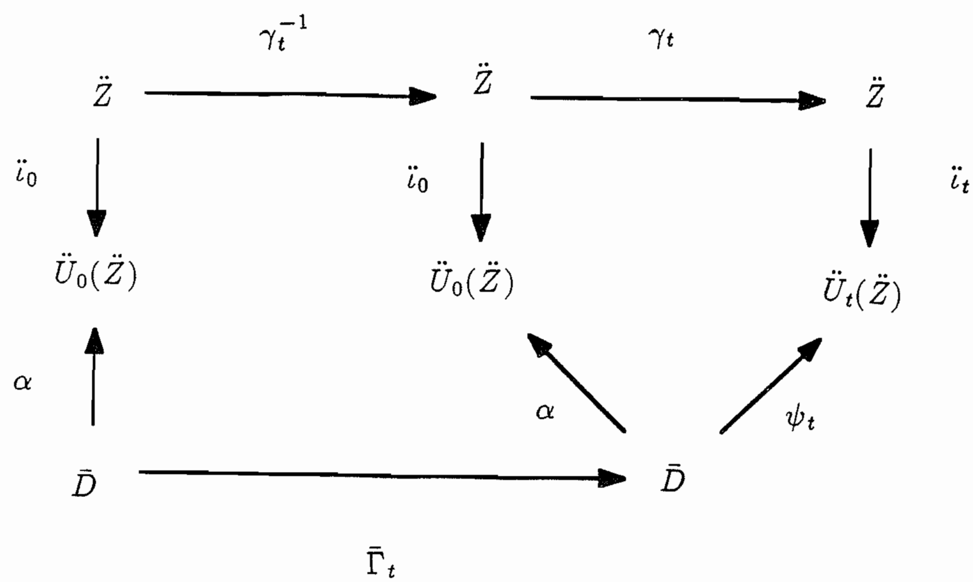

Figure 3.3. Commutative diagram.

Let $\Gamma_t : \partial D \to \partial D$ be given by

$$\Gamma_t := \alpha^{-1} \circ \ddot{\imath}_0 \circ \gamma_t^{-1} \circ \ddot{\imath}_0^{-1} \circ \alpha.$$

Γ_t is a homeomorphism, and we can extend it to a homeomorphism $\bar{\Gamma}_t : \bar{D} \to \bar{D}$ by radial projection, i.e. by letting

$$\bar{\Gamma}_t(r \cdot \mathrm{Exp}(t)) := r \cdot \Gamma_t(\mathrm{Exp}(t)).$$

We can now define $(\phi_{\hat{t}})_t$ on $\ddot{U}_0(\ddot{Z})$ by

$$(\phi_{\hat{t}})_t := \psi_t \circ \bar{\Gamma}_t \circ \alpha^{-1}.$$

Now let $T_{\hat{t}}$ be the intersection over all components of $\ddot{X}_G$ of the corresponding I, and let $(\phi_{\hat{t}})_t)$ be the union over all components of $\ddot{X}_G$ of the corresponding $(\phi_{\hat{t}})_t$.

End 3.2.4.1.

Because $[0,1]$ is compact, we can choose $\hat{t}_0 = 0, \hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n$ such that

$$\bigcup_{i=0}^{n} T_{\hat{t}_i} = [0,1] \qquad \text{and} \qquad T_{\hat{t}_i} \cap T_{\hat{t}_{i+1}} \neq \emptyset \quad \text{for} \quad i = 0,1,2,\ldots,n-1.$$

For $i = 0,1,2,\ldots,n-1$ let s_i be in $T_{\hat{t}_i} \cap T_{\hat{t}_{i+1}}$. Since each component of $\ddot{S}^2_{s_i}$ is homeomorphic to $\bar{D}$, and

$$(\phi_{\hat{t}_i})_{s_i} = (\phi_{\hat{t}_{i+1}})_{s_i} \quad \text{on} \quad \partial\ddot{S}^2_0,$$

we can let $\eta_i : [0,1] \times \ddot{S}^2_0 \to S^2_{s_i}$ be a homotopy (rel $\partial\ddot{S}^2_0$) through homeomorphisms between $(\phi_{\hat{t}_i})_{s_i}$ and $(\phi_{\hat{t}_{i+1}})_{s_i}$. The homotopy we seek is

$$\phi_{\hat{t}_0} * \eta_0 * \phi_{\hat{t}_1} * \eta_1 * \cdots * \eta_{n-1} * \phi_{\hat{t}_n},$$

where $a * b$ denotes "first do the homotopy a, then do the homotopy b."

End 3.2.4.

Supplement 3.2.5. *Suppose G is an e-graph and $\iota_t : X_G \to S^2$ for $t \in [0,1]$ is a homotopy through e-graph embeddings. Let X'_G be the subset of X_G upon which ι_t is independent of time. If each component of $S^2 - \iota_0(X_G)$ has at most two* *boundary components and at least one is contained in* $\iota_0(X'_G)$, *then there exists a homotopy through homeomorphisms* $\phi_t : S^2 \to S^2$ *for* $t \in [0,1]$ *such that*

1) ϕ_0 *is the identity,*

2) $\phi_t \circ \iota_0 = \iota_t$ *for* $t \in [0,1]$,

3) ϕ_t *is the identity on* $\iota_0(X'_G)$, *and*

4) ϕ_t *is the identity on any components of* $S^2 - \iota_0(X_G)$ *which have boundary contained in* X'_G.

Proof 3.2.5.

We only mention the differences between the proof of this supplement and the proof of theorem 3.2.4.

Because one boundary component of every component of $S^2 - \iota_0(X_G)$ is contained in $\iota_0(X'_G)$, we can choose the points y and z from the proof of claim 3.2.4.1 so that $y = z$ is in U_t for all $t \in [0,1]$. This allows us to get $\phi_t \circ \iota_0 = \iota_t$ for $t \in [0,1]$ instead of only for $t = 1$.

Because some of the components of $S^2 - \iota_0(X_G)$ are annuli instead of discs, the normalization for the corresponding Carathéodory Convergence argument is different. Let

$$A_t := \{z \in \mathbf{C} \mid 1/R_t < |z| < R_t\}$$

be such that the modulus of A_t is the same as that of U_t. Choose a

$$\zeta \in \bigcap_{t\in[0,1]} A_t$$

and map A_t to U_t conformally so that ζ maps to $y = z$.

End 3.2.5.

Chapter 4. Embedding Graph Dynamics.

§4.1. *Almost e-graph maps and an associated cut map.*

Definition. Given an e-graph G, a continuous map $f : X_G \to X_G$ is an *almost e-graph map* if $f(V_G) \subset V_G$ and f is injective on each edge of G.

Definition. Given e-graphs G and H, we say H is a *sub-e-graph* of G if

1) $X_H \subset X_G$,

2) $V_H \subset V_G$, and

3) $\sigma_H^v = \sigma_G^v$ restricted to E_H^v.

Clearly, given any closed subspace Y of X_G with $\partial Y \subset V_G$, there is a sub-e-graph H of G with $X_H = Y$. We call such an H, the *sub-e-graph of G defined by Y.*

To follow the next definition, the reader is encouraged to consult the example presented in figure 4.1.

Definition. Let G and H be e-graphs. Let $\ddot{Z}$ be a component of $\ddot{X}_G$, and let $G(\ddot{Z})$ be the e-graph defined by $\pi_G(\ddot{Z})$. Given an almost e-graph map $f : X_{G(\ddot{Z})} \to X_H$, we define $\ddot{f} : \ddot{Z} \to \ddot{X}_H$ as follows. For each edge e in E_G and orientation ω of e such that $\bar{e}_\omega$ is in $\ddot{Z}$, let $e_1, e_2, \ldots, e_n$ and $\omega_1, \omega_2, \ldots, \omega_n$ be such that e_j is in E_G, ω_j is an orientation of e_j, and as x traverses $\bar{e}$ consistently with ω, $f(x)$ traverses

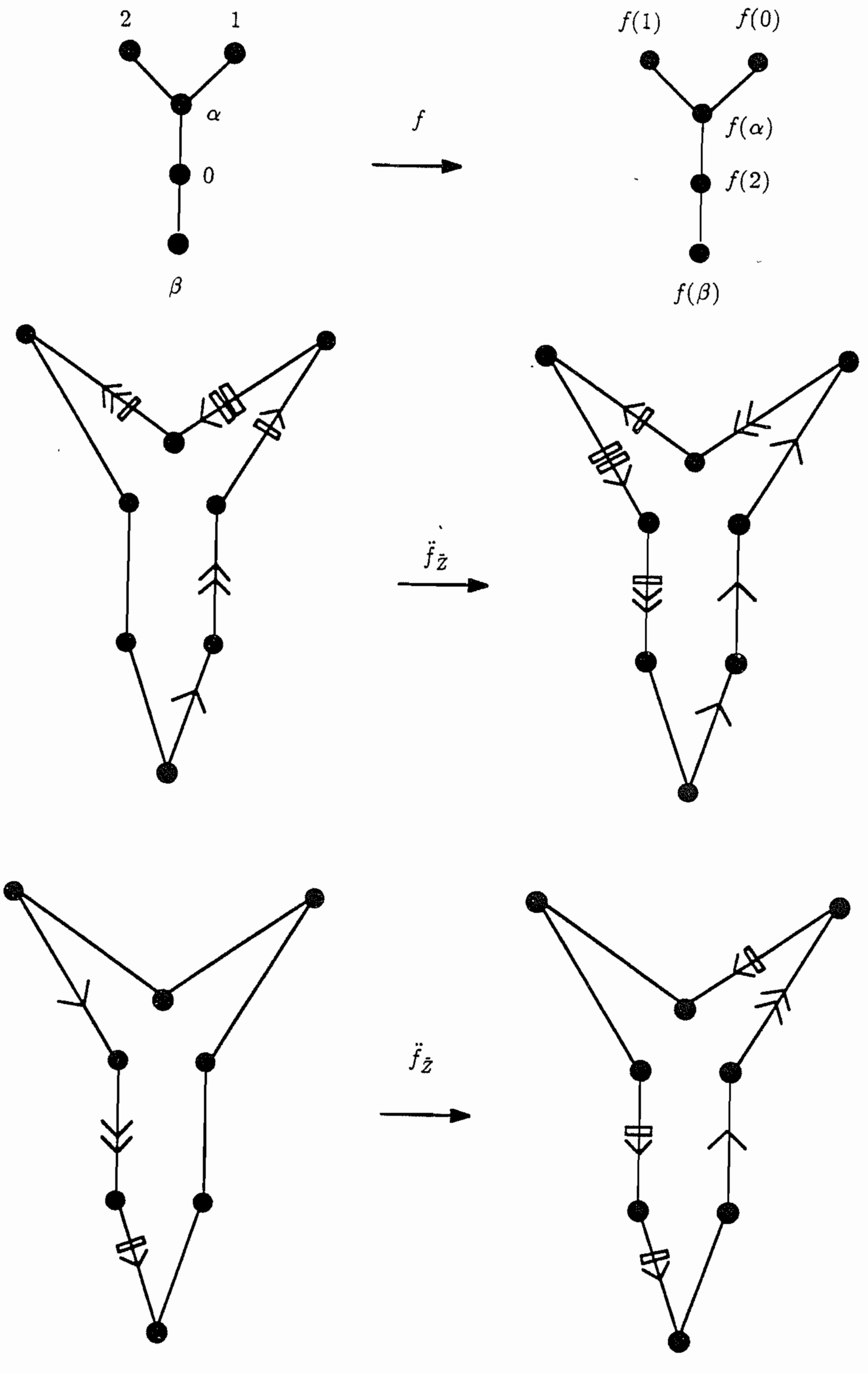

Figure 4.1. Lift of an almost e-graph map to the cut.

first $\bar{e}_1$ consistently with ω_1, then $\bar{e}_2$ consistently with ω_2, ..., then finally $\bar{e}_n$ consistently with ω_n. For each x in $e \cup \{\text{tip}(e, \omega)\}$, let j be such that $f(x)$ is in $e_j \cup \{\text{tip}(e_j, \omega_j)\}$, We set

$$\ddot{f}(x_e^\omega) := (f(x))_{e_j}^{\omega_j}.$$

§**4.2.** *The extension criterion.*

Definition. Let G and H be connected e-graphs, and let $f : X_G \to X_H$ be an almost e-graph map. For each component $\ddot{Z}$ of $\ddot{X}_G$, let $G(\ddot{Z})$ be the e-graph defined by $\pi_G(\ddot{Z})$, let $f_{\ddot{Z}}$ be f restricted to $\pi_G(\ddot{Z})$, and let $H(\ddot{Z})$ be the e-graph defined by $f_{\ddot{Z}}(\pi_G(\ddot{Z}))$. We say that f *satisfies the extension criterion* if for every component $\ddot{Z}$ of $\ddot{X}_G$, we have that $\ddot{f}_{\ddot{Z}} : \ddot{Z} \to \ddot{X}_{H(\ddot{Z})}$ is continuous and injective.

Remark. Note that in the example given in figures 4.1 through 4.4, $\ddot{f}_{\ddot{Z}}$ is neither continuous nor injective. See the example in figure 4.2 for an example of an f which does satisfy the extension criterion.

§**4.3.** *Germs of almost e-graph maps and edge dynamics.*

Definition. Given an almost e-graph map $f : X_G \to X_H$ and a vertex v of G, let U be a neighborhood of v such that for all $e \in E_G^v$, $f(e \cap U)$ is contained in some edge which we shall call $f^v(e)$. $f^v(e)$ does not depend on the choice of U, so this allows us to define

$$f^v : E_G^v \to E_H^{f(v)}.$$

f^v is called the *germ of* f *at* v.

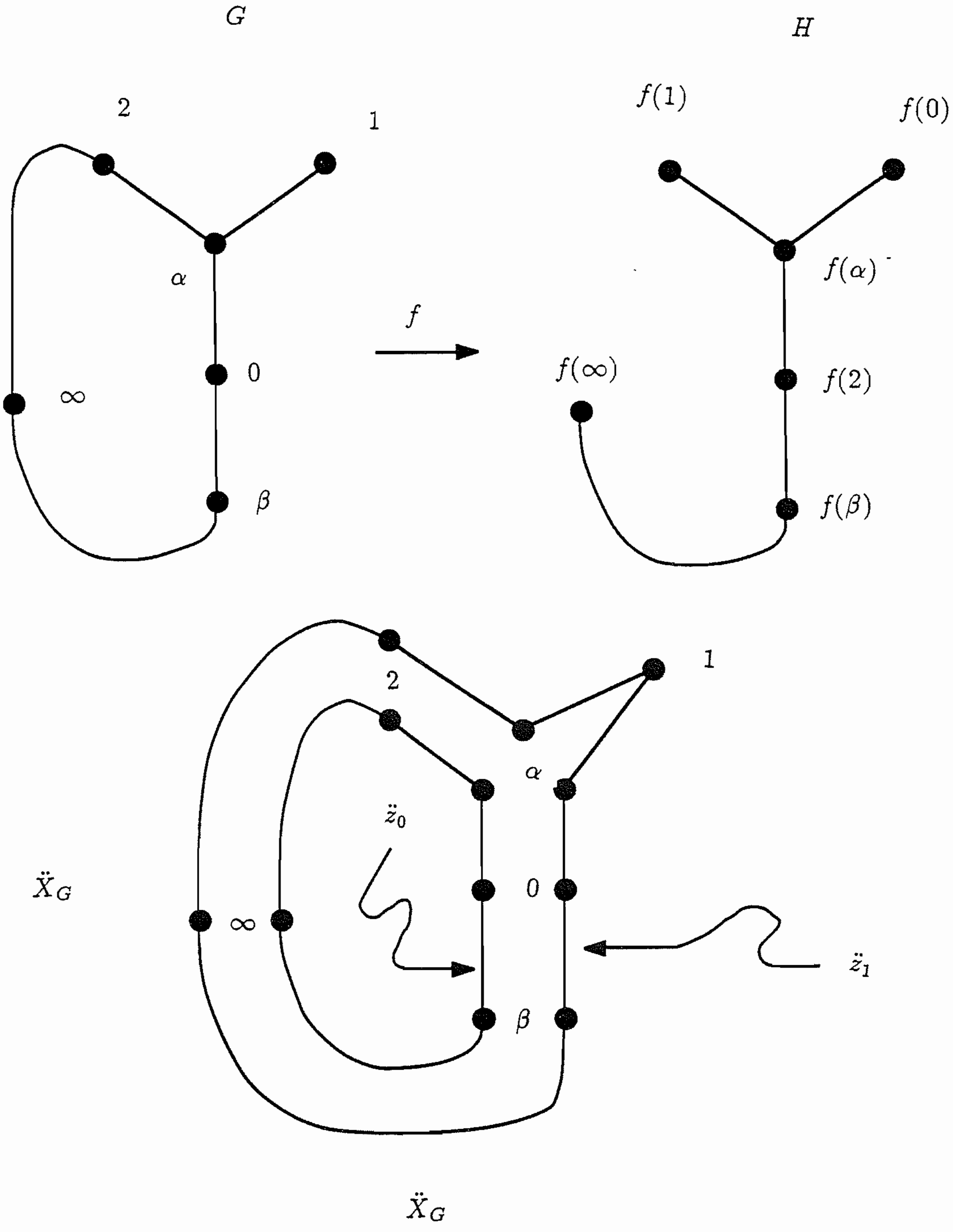

Figure 4.2. Example which satisfies extension criterion, part (a).

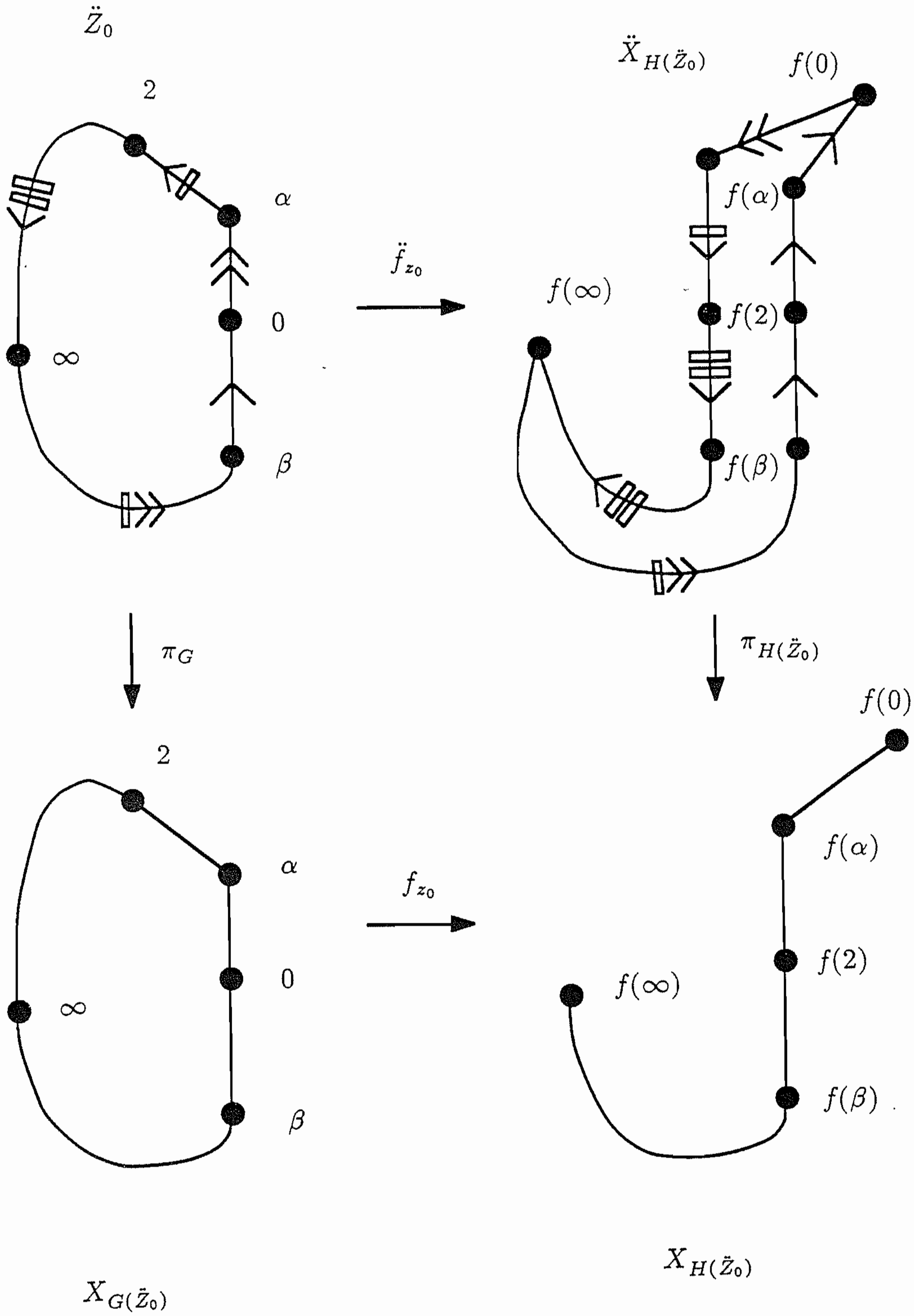

Figure 4.3. Example which satisfies extension criterion, part (b).

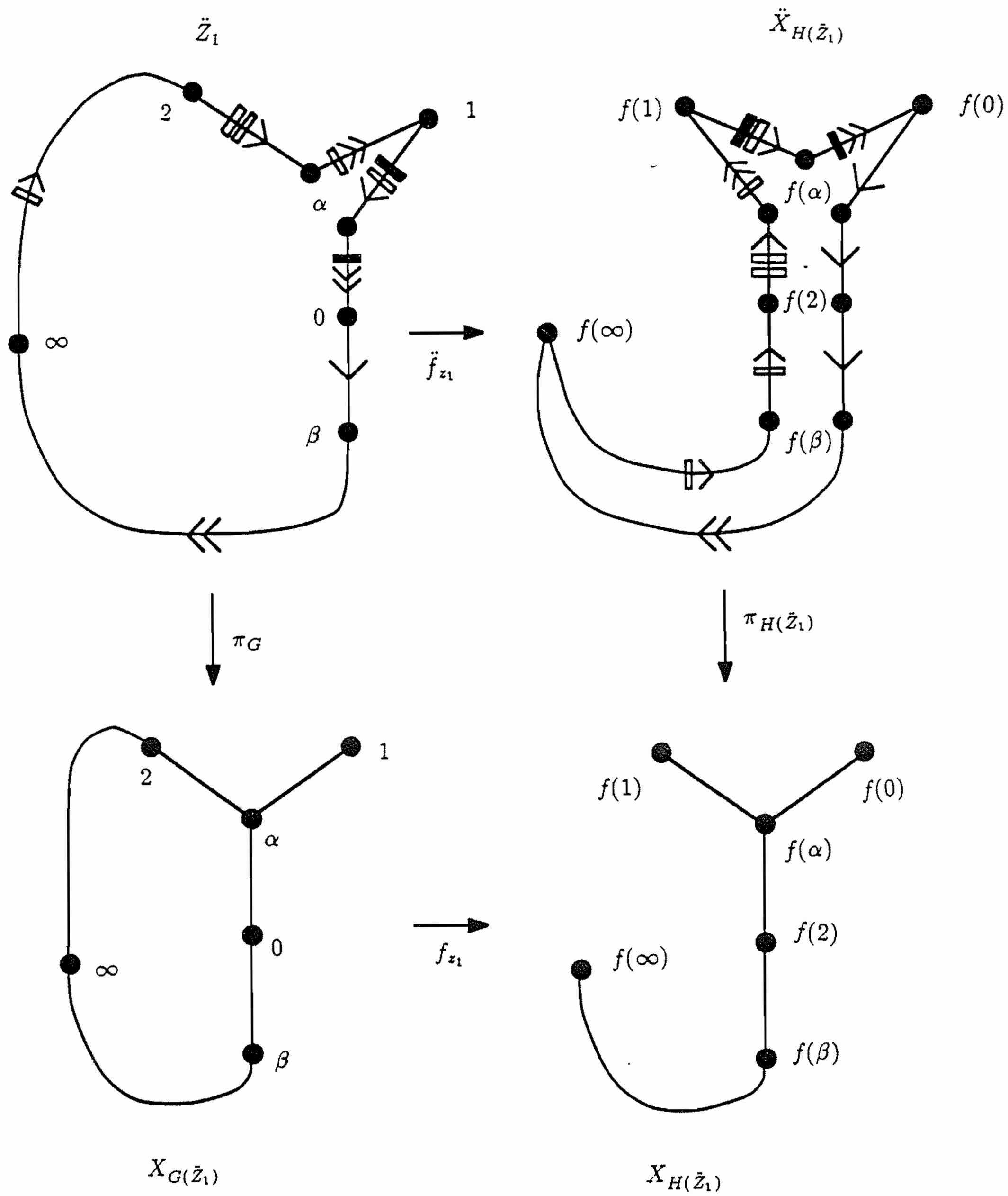

Figure 4.4. Example which satisfies extension criterion, part (c).

Definition. If $f : X_G \to X_H$ is an almost e-graph map, the set of *critical points of* f is by definition

$$\Omega_f := \{v \in V_G \mid f^v \text{ is not injective }\}.$$

If $G = H$, the *post-critical set of* f is defined as

$$P_f := \{f^{\circ n}(\Omega_f) \mid n \geq 1\}.$$

Note that $P_f \subset V_G$ and is therefore finite.

Definition. Let G and H be e-graphs, $f : X_G \to X_H$ an almost e-graph map, and ω an orientation of each edge of G and H. For each edge e of G, if $\lambda :]0,1[\to e$ is an orientation preserving parameterization of e, then $f \circ \lambda$ traverses a sequence of edges of H (possibly together with some vertices, but we are not interested in them). That sequence together with a specification for each edge traversed of whether or not the edge was traversed consistently with its orientation is by definition the *edge dynamics of* f *with respect to* ω *on* e. This data for all edges e constitutes the *edge dynamics of* f *with respect to* ω. Note that if we know ω and the edge dynamics of f with respect to ω, then we know the edge dynamics of f with respect to any known orientation of the edges of G and H. So we can, therefore, refer to this data as simply the *edge dynamics of* f.

§4.4. *Existence and Uniqueness of Corresponding Branched Covers.*

Theorem 4.4.1. *Suppose* G *is a connected e-graph. (Existence) Suppose* $f : X_G \to X_G$ *is an almost e-graph map which satisfies the extension criterion. If*

$\iota : X_G \to S^2$ is an e-graph embedding, then there is a post-critically finite branched cover $g : S^2 \to S^2$ which is an extension of $\iota \circ f \circ \iota^{-1}$ and has $P_g = \iota(P_f)$. (Uniqueness) Suppose for $j = 0, 1$ we have that $\iota_j : X_G \to S^2$ and $\kappa_j : X_G \to S^2$ are e-graph embeddings, g_j is a branched cover with $\Omega_{g_j} \subset \kappa_j(V_G)$, and $f_j := \iota_j^{-1} \circ g_j \circ \kappa_j$ is an almost e-graph map. If f_0 and f_1 have the same edge dynamics and ι_j is homotopic to κ_j through e-graph embeddings (rel P_{f_0}), then g_0 is topologically equivalent to g_1.

Proof 4.4.1.

Let $\ddot{X}_G$ be X_G cut, and let $\pi_G : \ddot{X}_G \to X_G$ be the associated quotient map. Let $\ddot{S}^2$ be S^2 cut along $\iota(X_G)$, let $\pi : \ddot{S}^2 \to S^2$ be the associated quotient map, and let $\ddot{\iota} : \ddot{X}_G \to \ddot{S}^2$ be the associated embedding. So we have the following commutative diagram.

$$\begin{array}{ccc} \ddot{X}_G & \xrightarrow{\ddot{\iota}} & \ddot{S}^2 \\ \big\downarrow{\scriptstyle \pi_G} & & \big\downarrow{\scriptstyle \pi} \\ X_G & \xrightarrow{\iota} & S^2 \end{array}$$

We will define $g : \ddot{S}^2 \to S^2$ in such a way that g is well defined on the quotient $\pi(\ddot{S}^2) = S^2$. The absorption of the following definitions might be facilitated by consulting the commutative diagram in figure 4.5.

For each component $\ddot{U}$ of $\ddot{S}^2$ we make the following definitions. Let $\ddot{Z}$ be the component of $\ddot{X}_G$ such that $\partial \ddot{U} = \ddot{\iota}(\ddot{Z})$. Let $G(\ddot{Z})$, $f_{\ddot{Z}}$, $H(\ddot{Z})$, and $\ddot{f}_{\ddot{Z}}$ be as in the definition of the extension criterion. Let $\ddot{\iota}_{H(\ddot{Z})} : X_{H(\ddot{Z})} \to S^2$ be the restriction of ι to $X_{H(\ddot{Z})}$. Let $\ddot{X}_{H(\ddot{Z})}$ be $X_{H(\ddot{Z})}$ cut and let $\pi_{H(\ddot{Z})} : \ddot{X}_{H(\ddot{Z})} \to X_{H(\ddot{Z})}$

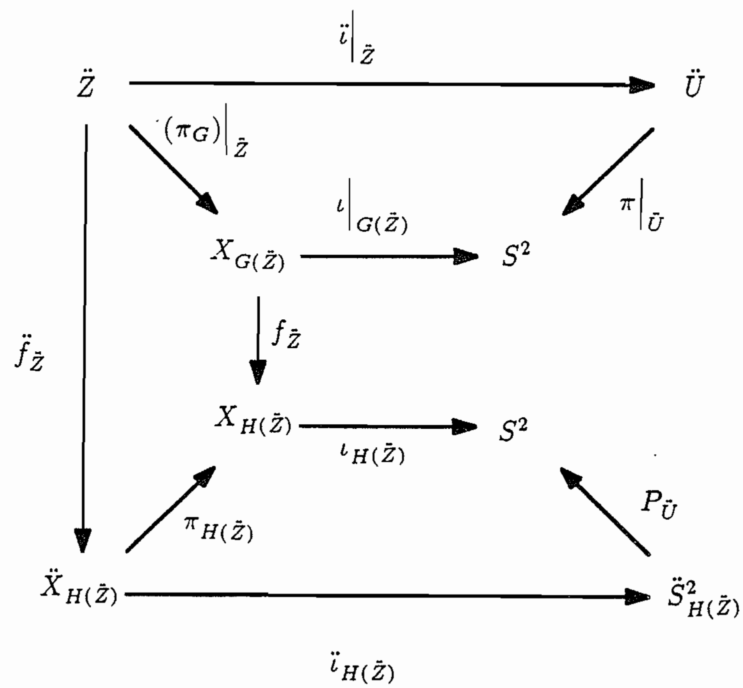

Figure 4.5. Commutative diagram.

be the associated quotient map. Let $\ddot{S}^2_{H(\ddot{Z})}$ be S^2 cut along $\iota_{H(\ddot{Z})}(X_{H(\ddot{Z})})$, let $p_{\ddot{U}} : \ddot{S}^2_{H(\ddot{Z})} \to S^2$ be the associated quotient map, and let $\ddot{\iota}_{H(\ddot{Z})} : \ddot{X}_{H(\ddot{Z})} \to \ddot{S}^2_{H(\ddot{Z})}$ be the associated embedding.

Claim 4.4.1.1. *$\ddot{f}_{\ddot{Z}}$ maps $\ddot{Z}$ homeomorphically onto a component of $\ddot{X}_{H(\ddot{Z})}$.*

Proof 4.4.1.1. By hypothesis, $\ddot{f}_{\ddot{Z}}$ is continuous and injective. Since $\ddot{X}_{H(\ddot{Z})}$ is Hausdorff, $\ddot{f}_{\ddot{Z}}$ maps $\ddot{Z}$ homeomorphically onto $\ddot{f}_{\ddot{Z}}(\ddot{Z})$. So $\ddot{f}_{\ddot{Z}}(\ddot{Z})$ is homeomorphic to S^1 and is a subset of a component of $\ddot{X}_{H(\ddot{Z})}$ which is itself homeomorphic to S^1. So $\ddot{f}_{\ddot{Z}}(\ddot{Z})$ equals that component. **End 4.4.1.1.**

By claim 4.4.1.1 we can let $\ddot{W}$ be the component of $\ddot{S}^2_{H(\ddot{Z})}$ with $\partial\ddot{W} =$

$\ddot{\imath}_{H(\ddot{Z})}(\ddot{f}_{\ddot{Z}}(\ddot{Z}))$. By proposition 3.2.2, $\ddot{W}$ and $\ddot{U}$ are homeomorphic to $\bar{D}$. Also,

$$\ddot{\imath}_{H(\ddot{Z})} \circ \ddot{f}_{\ddot{Z}} \circ (\ddot{\imath}\Big|_{\ddot{Z}})^{-1}$$

is a homeomorphism mapping $\partial\ddot{U}$ to $\partial\ddot{W}$, so we may extend it to a homeomorphism $g_{\ddot{U}} : \ddot{U} \to \ddot{W}$. The union of the

$$p_{\ddot{U}} \circ g_{\ddot{U}} : \ddot{U} \to S^2$$

factors through π giving $g : S^2 \to S^2$.

Clearly g is surjective, a local homeomorphism at exactly $S^2 - \iota(\Omega_f)$ and of finite degree. So g is a branched cover and $P_g = \iota(P_f)$ [ref A. and R. Douady].

This ends the proof of existence. The proof of uniqueness rests upon the following lemma, proved for me by A. Douady.

Lemma 4.4.1.2. *Let $V \subset S^2$ and suppose there exist continuous maps $\phi : \bar{D} \to \bar{U}$ and $\psi : \bar{D} \to \bar{U}$ such that $\phi : D \to U$ and $\psi : D \to U$ are homeomorphisms and ϕ and ψ are not constant on any arc of ∂D. Let*

$$h_2 := \left(\psi\Big|_D\right)^{-1} \circ \left(\phi\Big|_D\right).$$

So $h_2 : D \to D$ is such that $\psi \circ h_2 = \phi$. Suppose there exists an orientation preserving homeomorphism $h_1 : \partial D \to \partial D$ such that $\psi \circ h_1 = \phi$. Let $h := h_1 \cup h_2$. Then h is continuous.

Proof 4.4.1.2.

Claim 4.4.1.2.1. *h_2 extends to a continuous map $\tilde{h} = h_2 \cup \tilde{h}_1$.*

Proof 4.4.1.2.1.

Lemma 4.4.1.2.1.1. *(Carathéodory) Suppose $\zeta : \bar{D} \to S^2$ is continuous inducing a homeomorphism $\zeta : D \to U$, and that ζ is not constant on any arc of ∂D. Suppose B_n is a decreasing sequence of connected sets in D with* $\mathrm{diam}(\zeta(B_n)) \to 0$. *Then* $\mathrm{diam}(B_n) \to 0$.

Proof 4.4.1.2.1.1. Without loss of generality, replace all B_n by their closure. So by compactness,

$$\mathrm{diam}(\cap B_n) = \lim \mathrm{diam}(B_n).$$

$\cap B_n$ is connected because it is a decreasing intersection of connected compact sets. $\zeta(\cap B_n) \subset \cap\zeta(B_n)$, and since the $\zeta(B_n)$ are compact,

$$\mathrm{diam} \cap \zeta(B_n) = \lim \mathrm{diam}\zeta(B_n) = 0,$$

so $\zeta(\cap B_n)$ contains only one point. If $\cap B_n \bigcap D \neq \emptyset$, we are done since ζ restricted to D is a homeomorphism onto U. Otherwise, we are done since ζ is not constant on any arc of ∂D. **End 4.4.1.2.1.1.**

We now define $\tilde{h}_1$. Let t be a point in ∂D. Let

$$A_\epsilon := \{x \in D \mid |x - t| < \epsilon\}.$$

Because ϕ is continuous,

$$\mathrm{diam}(\phi(A_\epsilon)) \to 0 \quad \text{as} \quad \epsilon \to 0.$$

$\phi(A_\epsilon) = \psi(h_2(A_\epsilon))$, so

$$\mathrm{diam}(\psi(h_2(A_\epsilon)) \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Also, $h_2(A_\epsilon)$ is connected. So

$$\operatorname{diam}(h_2(A_\epsilon)) \to 0 \quad \text{as} \quad \epsilon \to 0$$

by lemma 4.4.1.2.1.1, and we can let

$$\tilde{h}_1(t) := \bigcap_{\epsilon \to 0} \overline{h_2(A_\epsilon)}.$$

It is straightforward to check that $\tilde{h} := h_2 \cup \tilde{h}_1$ is continuous.

End 4.4.1.2.1.

We now must show that $h_1 = \tilde{h}$. Let $\gamma := \tilde{h}_1 \circ h_1^{-1}$. We will show that γ is the identity. $\tilde{h}_1$ is a homeomorphism since by reversing the roles of ϕ and ψ we can define $\tilde{h}_1^{-1}$. Also, $\tilde{h}_1$ is orientation preserving since h_2 is. So γ is an orientation preserving homeomorphism.

For all $x \in \partial D$, $\phi(\gamma(x)) = \phi(x)$. Suppose for some t_0,

$$\gamma(\operatorname{Exp}(t_0)) \neq \operatorname{Exp}(t_0).$$

Since γ is an orientation preserving homeomorphism, there is an $\epsilon > 0$ such that $t - t_0 < \epsilon$ implies $\gamma(\operatorname{Exp}(t))$ is in the component of $\partial D - \{\operatorname{Exp}(t_0), \gamma(\operatorname{Exp}(t_0))\}$ not containing t. For all $t \in (t_0, t_0 + \epsilon)$, since ϕ is injective on D and $\phi(\gamma(\operatorname{Exp}(t))) = \phi(\operatorname{Exp}(t))$, we get that $\phi(\operatorname{Exp}(t)) = \phi(\operatorname{Exp}(t_0))$ (see figure 4.6). But this contradicts the hypothesis that ϕ is not constant on any arc of ∂D.

End 4.4.1.2.

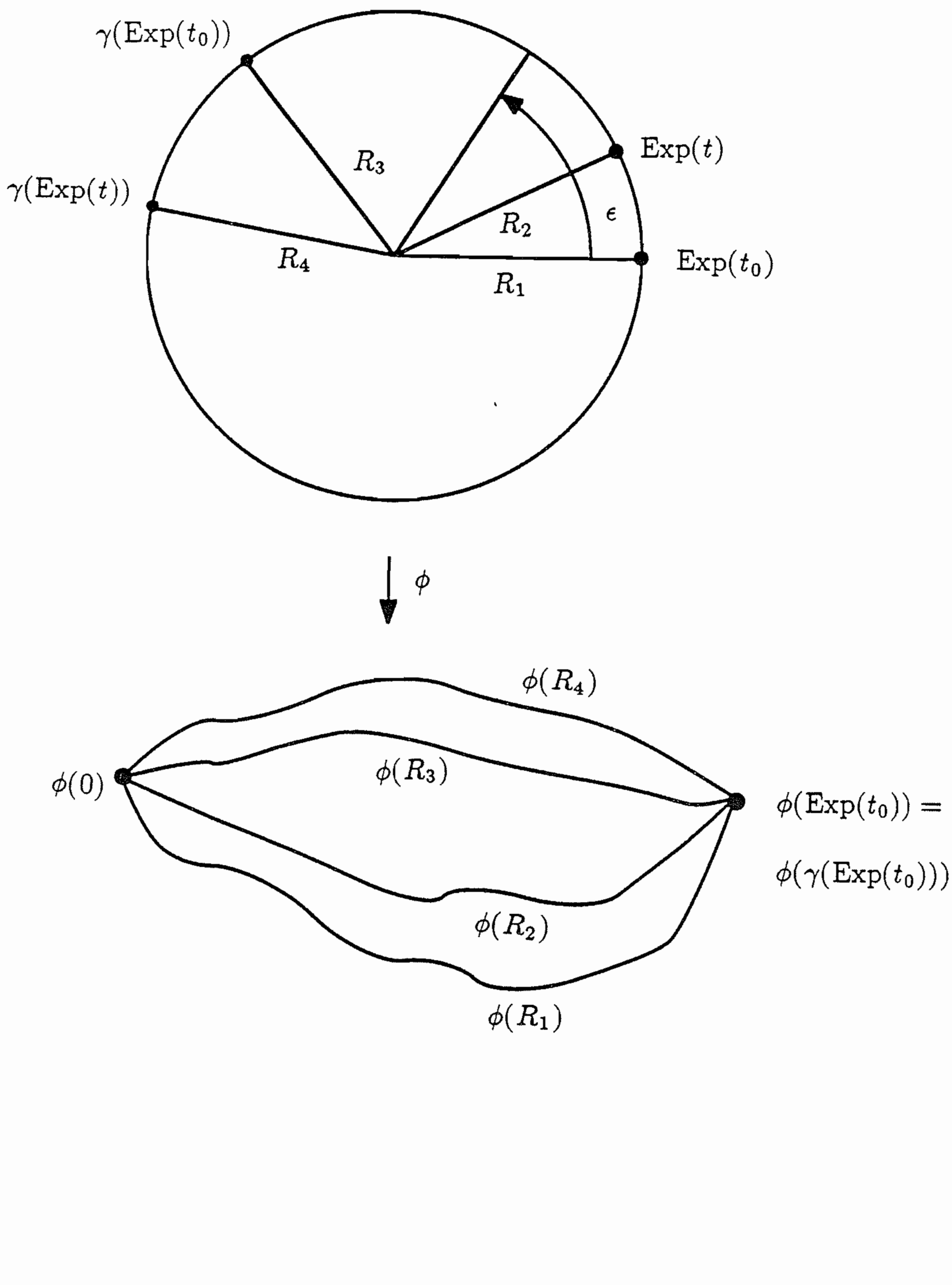

Figure 4.6. Use of hypothesis of not constant on any arc.

Before proving uniqueness, we make two simplifications. First, by theorem 3.2.4, there exist $\psi_j : S^2 \to S^2$ such that $\psi_j \circ \kappa_j = \iota_j$ and ψ_j is homotopic (rel $\kappa_j(P_{f_j})$) to the identity through homeomorphisms. So we may assume $\iota_j = \kappa_j$. Second, by proposition 3.2.3 there exists a homeomorphism $\phi : S^2 \to S^2$ such that $\phi \circ \kappa_0 = \kappa_1$. So we may assume $\kappa_0 = \kappa_1 =: \kappa$.

We will construct a homeomorphism $\theta : S^2 \to S^2$ such that $g_0 = g_1 \circ \theta$ and θ is isotopic to the identity (rel P_{g_0}). For each component U of $S^2 - \kappa(X_G)$, g_j restricted to U is a homeomorphism onto its image because $\Omega_{g_j} \cap U = \emptyset$ and U is homeomorphic to D by proposition 3.2.2. So we can let $\theta := g_1^{-1} \circ g_0$ on U. Since f_j is an almost e-graph map, for each edge e in E_G, g_j restricted to $\kappa(e)$ is a homeomorphism onto its image, so we can let $\theta := g_1^{-1} \circ g_0$ on $\kappa(e)$. We let θ be the identity on $\kappa(V_G)$. It is easy to check that θ is a homeomorphism.

Since the edge dynamics of f_0 equals that of f_1, κ and $\theta \circ \kappa$ are isotopic (rel V_G). By theorem 3.2.4 we can extend that to an isotopy from θ to some θ' which is the identity on $\kappa(X_G)$.

By proposition 3.2.2, for each component U of $S^2 - \kappa(X_G)$ there exists a continuous orientation preserving $\phi : \bar{D} \to \bar{U}$ such that ϕ restricted to D is a homeomorphism onto U. By lemma 4.4.1.2 there is a continuous map $\tilde{\theta} : \bar{D} \to \bar{D}$ such that $\phi \circ \tilde{\theta} = \theta' \circ \phi$, and $\tilde{\theta}$ is the identity on ∂D. By Alexander Shrinking we can isotope $\tilde{\theta}$ to the identity, and carry that isotopy through ϕ to an isotopy between θ' and the identity.

End 4.4.1.

§**4.5.** *e-graph maps.*

Definition. Given a cyclic permutation σ of a finite set E and some subset E_0 of E, then the *restriction of* σ *to* E_0 is defined by $e \mapsto \sigma^{\circ n}(e)$ where n is the smallest integer greater than 0 with $\sigma^{\circ n}(e) \in E_0$.

Definition. Suppose $f : X_G \to X_H$ is an almost e-graph map. For every vertex v of G, let f^v be the germ of f at v. We say that f *respects the cyclic permutations* if for every vertex v of G, and for every subset E of E_G^v upon which f^v is injective, we have

$$\sigma_0 \circ f^v = f^v \circ \sigma_1,$$

where σ_0 is $\sigma_H^{f(v)}$ restricted to $f^v(E)$ and σ_1 is σ_G^v restricted to E.

Definition. An almost e-graph map is an *e-graph map* if it respects the cyclic permutations.

Definition. Let G be an e-graph and v a vertex of G. A map $\phi : E_G^v \to \mathbf{T}$ is said to *respect the cyclic permutations* if for every e in E_G^v,

$$]\phi(e), \phi(\sigma_G^v(e))[\cap \phi(E_G^v) = \emptyset.$$

Definition. Let $f : X_G \to X_H$ be an e-graph map. Let v be a vertex of G and let f^v be the germ of f at v. f^v is said to be *quadratic* if there are maps $\phi_0 : E_G^v \to \mathbf{T}$ and $\phi_1 : E_H^{f(v)} \to \mathbf{T}$ which respect the cyclic permutations and are such that

$$\phi(f^v(e)) = 2 \cdot \phi_0(e)$$

for all e in E_G^v.

Proposition 4.5.1. *Suppose the following.*

1) $f : X_G \to X_H$ *is an e-graph map.*

2) X_G *is a loop* L *with trees attached.*

3) $f(X_G)$ *is a tree.*

4) *There exist distinct vertices* v_0 *and* v_1 *in* L *such that* f^{v_0} *and* f^{v_1} *are quadratic and* f *is injective on each component of* $L - \{v_0, v_1\}$.

Then f *satisfies the extension criterion.*

Proof 4.5.1.

Let L_0 and L_1 be the two components of $L - \{v_0, v_1\}$. Because $f(X_G)$ is a tree and because f is injective on L_0 and on L_1, we have that $f(L)$ is the unique line segment S joining $f(v_0)$ to $f(v_1)$ in $f(X_G)$.

Let $\ddot{X}_G$ be X_G cut, and let $\pi_G : \ddot{X}_G \to X_G$ be the associated quotient map. For each component $\ddot{Z}$ of $\ddot{X}_G$, we have that $\pi_G(\ddot{Z})$ is L with trees attached on only on side (see figure 4.7). We call those trees the subtrees of $\pi_G(\ddot{Z})$.

For $v = v_0, v_1$, we have that f^v maps the two edges in $E_G^v \cap L$ to the single edge in $E_H^{f(v)} \cap S$. So, since f^v is quadratic and $f(X_G)$ is a tree, f is injective on those subtrees of $\pi_G(\ddot{Z})$ attached to L at v.

So, since f is injective on each component of $L - \{v_0, v_1\}$ and since f respects the cyclic permutations, we have that f merely collapses L to S leaving the reset of $\pi_G(\ddot{Z})$ unaltered (see figure 4.8).

End 4.5.1.

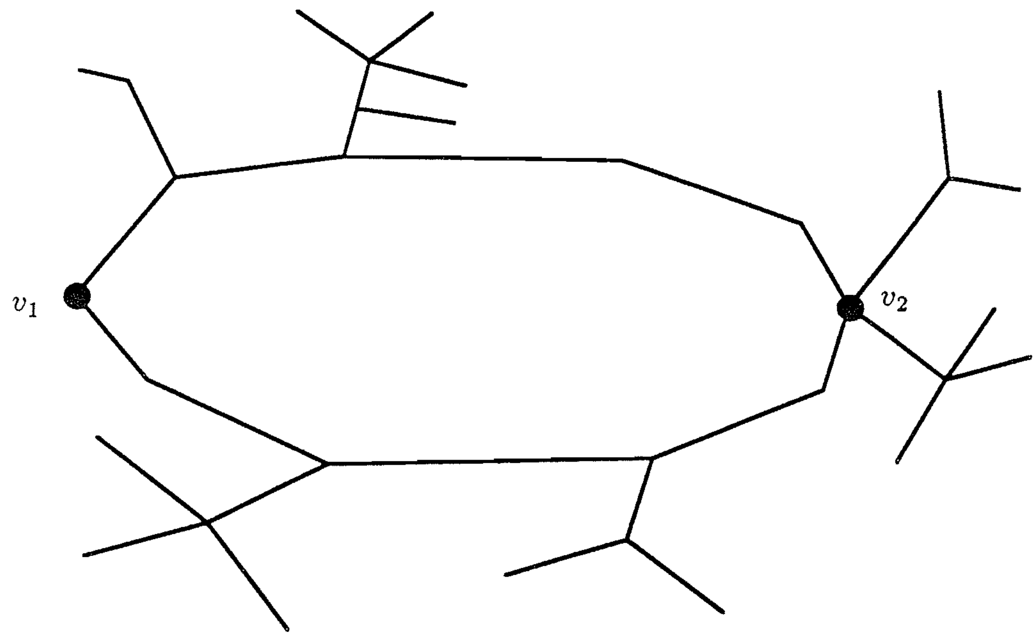

Figure 4.7. Loop with trees attached on only one side.

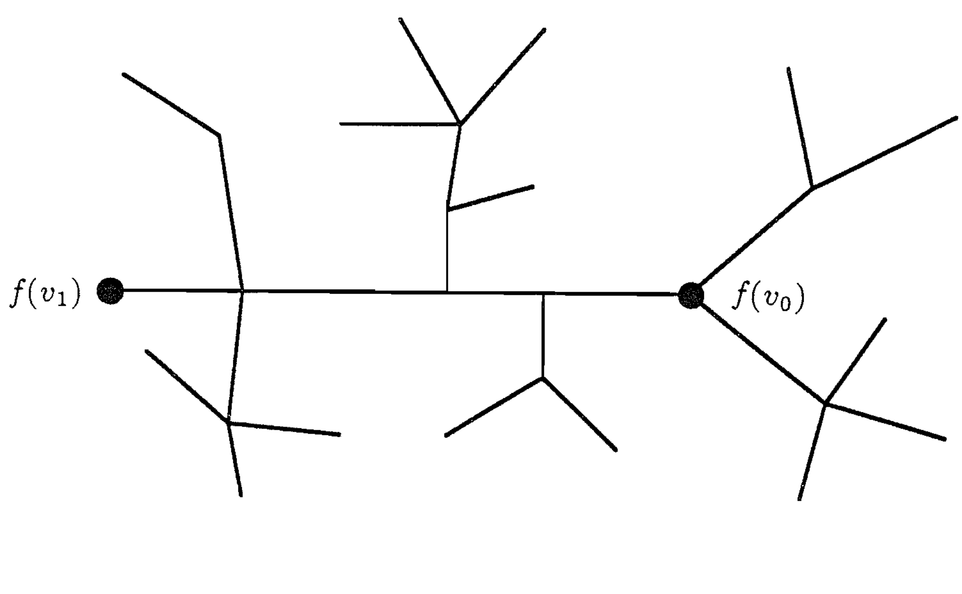

Figure 4.8. Collapse of loop with trees attached on only one side.

Chapter 5. Quadratic Trees.

§5.1. *Introduction and Definition*

Quadratic trees are a kind of abstract Hubbard tree. Every Hubbard tree defines a quadratic tree, but not conversely.

Definition. A triple (H, f, x_0) is a *Quadratic tree* if it satisfies the following:

1) H is a connected e-graph.
2) X_H is a tree (i.e. has no closed loops).
3) $x_0 \in V_H$.
4) x_0 has at most two incident edges.
5) $f : X_H \to X_H$ is an e-graph map.
6) f is injective on the closure of each component of $X_H - \{x_0\}$.
7) The set of vertices of H with only one incident edge is contained in

$$\{f^{\circ n}(x_0) \mid n \geq 0\}.$$

§5.2. *Existence and Uniqueness of Corresponding Branched Covers.*

Theorem 5.2.1. *Let (H, f, x_0) be a quadratic tree. (Existence) There is an e-graph embedding $\iota : X_H \to S^2$ and a branched cover $g : S^2 \to S^2$ of degree two* *which is an extension of $\iota \circ f \circ \iota^{-1}$ with one critical point at $\iota(x_0)$ and the other critical point fixed in $S^2 - \iota(X_H)$. (Uniqueness) Suppose for $j = 0, 1$ we have that $\iota_j : X_H \to S^2$ and $\kappa_j : X_H \to S^2$ are e-graph embeddings, $g_j : S^2 \to S^2$ is a branched cover of degree two with one critical point at $\kappa_j(x_0)$ and the other fixed in $S^2 - (\iota_j(X_H) \cup \kappa_j(X_H))$, and $f_j := \iota_j^{-1} \circ g_j \circ \kappa_j$ is an almost e-graph map. If f_0 and f_1 have the same edge dynamics and ι_j is homotopic to κ_j through e-graph embeddings (rel P_{f_0}), then g_0 is topologically equivalent to g_1.*

Proof 5.2.1.

Notation. Given two points y_0 and y_1 in a topological tree X, denote by $[y_0, y_1]_X$ the intersection of all connected subsets of X containing $\{y_0, y_1\}$. Note that if $y_0 \neq y_1$, then $[y_0, y_1]_X$ is homeomorphic to $[0, 1]$.

Notation. Let

$$x_i := f^{\circ i}(x_0).$$

Notation. Let U be the component of

$$X_H - \{x_0\}$$

not containing x_1 if it exists and the empty set otherwise.

Lemma 5.2.1.1. *f is surjective.*

Proof 5.2.1.1. $f(X_H)$ is connected and contains all the vertices of H with one incident edge. **End 5.2.1.1.**

Lemma 5.2.1.2. *There is a point $b \in U \cup \{x_0\}$ and a point $b' \in f^{-1}(\{b\})$ such that*

1) $x_0 \in [b, b']_{X_H}$ *and*

2) $f^{-1}([f(b), b]_{X_H} - \{f(b)\}) \cap U = \emptyset$.

Proof 5.2.1.2.

If $U = \emptyset$, let $b := x_0$ and let b' be the unique inverse image of b. Otherwise, define B_i inductively by

$$B_0 := [x_0, x_1]_{X_H}$$

and

$$B_{i+1} := f^{-1}(B_i) \cap (U \cup \{x_0\}).$$

Note that

$$B_i \cap B_{i+1} = f^{-i}(\{x_0\}) \cap (U \cup \{x_0\}). \tag{5.1}$$

So $B := \bigcup B_i$ is homeomorphic to an interval and so is $\bar{B}$, the closure of B in X_H. Let b be the endpoint of $\bar{B}$ not equal to x_1.

Case. *There exists $i > 0$ with $B_i = \emptyset$.*

Then $f(b) \neq b$ but by construction

$$f^{-1}([f(b), b]_{X_H} - \{f(b)\}) \cap U = \emptyset.$$

So b has no inverse image in U. So by lemma 5.2.1.1, we can let b' be the inverse image of b in $X_H - U$ and 1) is satisfied.

Case. *There does not exist i with $B_i = \emptyset$.*

Then by (5.1), b is the limit of a sequence of inverse images and is therefore fixed. Thus 2) in the statement of lemma 5.2.1.2 is satisfied. By 7) of the definition of a quadratic tree, we can let j be smallest such that x_j is a forward image of x_0 which is not in the component of $X_H - \{b\}$ containing x_1. Then

$$b \in f([x_0, x_{j-1}]_{X_H})$$

and

$$[x_0, x_{j-1}]_{X_H} \cap U = \emptyset.$$

So if we let b' be the inverse image of b in $[x_0, x_{j-1}]_{X_H}$, then b' is in $X_H - U$ and 1) is satisfied.

End 5.2.1.2.

We now construct an embedding graph G and an embedding graph map

$$f_G : X_G \to X_G$$

such that H is a sub embedding graph of G and f_G is an extension of f. To form X_G, append to X_H a new vertex which we shall call y_0 and two edges e and e'. Let e join b to y_0 and e' join b' to y_0. Let

$$f_G : X_G \to X_G$$

fix y_0, map e injectively onto the interior of $[f(b), y_0]_{(X_G - e')}$, and map e' injectively onto e. If necessary, make b and b' into vertices so that f_G respects cyclic permutations.

Claim 5.2.1.3. *f_G satisfies the extension criterion.*

Proof 5.2.1.3.

Let W and W' be the components of

$$X_G - \{x_0, y_0\}$$

containing e and e' respectively. By 2) of lemma 5.2.1.2, f_G is injective on $\bar{W}$ and on $\bar{W}'$.

X_G is a loop L containing $e \cup e' \cup \{y_0\}$ with trees attached. By 1) of lemma 5.2.1.2, L contains $\{x_0\}$. The edge e' has no inverse image, so $f_G(X_G)$ is a tree. f_G is injective on $L \cap \bar{W}$ and on $L \cap \bar{W}'$. Clearly, f^{x_0} and f^{y_0} are quadratic. So by proposition 4.5.1 we are done.

End 5.2.1.3.

Since X_H is a tree, there is an e-graph embedding

$$\iota : X_H \to S^2,$$

and $S^2 - \iota(X_G)$ is homeomorphic to D. So we can extend ι to $e \cup e' \cup \{y_0\}$. So we get the existence of the branched cover g in the statement of this theorem by claim 5.2.1.3 and theorem 4.4.1.

(Uniqueness) By theorem 4.4.1, we only need extend ι_0, ι_1, κ_0, and κ_1 to X_G so that for $j = 0, 1$ we have

1) $\Omega_{g_j} \subset \kappa_j(V_G)$,

2) the extended f_j are almost e-graph maps with the same edge dynamics, and

3) ι_j is homotopic to κ_j through e-graph embeddings (rel P_{f_0}).

For $j = 0, 1$, we will only describe the images under ι_j and κ_j of e and e'. Actually specifying ι_j and κ_j is trivial but a nuisance.

Notation. Denote by ∞ the fixed critical point of g_j in

$$S^2 - (\iota_j(X_H) \cup \kappa_j(X_H)).$$

Case. $f_j(b) = b$.

Let $\iota_j(e)$ and $\iota_j(e')$ be disjoint curves in

$$S^2 - \iota_j(X_H)$$

such that $\iota_j(e)$ joins $\iota_j(b)$ to ∞, $\iota_j(e')$ joins $\iota_j(b')$ to ∞, and ι_j is still an embedding graph embedding. We can let $\kappa_j(e)$ be the component of $g_j^{-1}(\iota_j(e))$ which begins at $\kappa_j(b)$ and ends at ∞. We can let $\kappa_j(e')$ be the other component of $g_j^{-1}(\iota_j(e))$. So $\kappa_j(e')$ joins $\kappa_j(b')$ to ∞.

Case. $f_j(b) \neq b$.

By 2) of lemma 5.2.1.2 there is a component E of

$$g_j^{-1}\left(\iota_j([f_j(b), b]_{X_H} - \{f_j(b)\})\right)$$

such that $\kappa_j(b) \in \bar{E}$ and

$$E \cap \kappa_j(X_H) = \emptyset.$$

E is homeomorphic to a half open interval. Let $\kappa_j(e)$ be E together with a curve joining

$$E \cap g_j^{-1}(\iota_j(b))$$

to ∞ in

$$S^2 - (\kappa_j(X_H) \cup E).$$

Then if we let $\iota_j(e)$ equal $g_j(\kappa_j(e))$, we get that $\iota_j^{-1} \circ g_j \circ \kappa_j$ maps e injectively onto

$$[f_j(b), y_0]_{(X_G - e')} - \{f_j(b)\}$$

as does f_j. Now let $\iota_j(e')$ be any curve joining $\iota_j(b')$ to ∞ in

$$S^2 - (\iota_j(X_H) \cup \iota_j(e)).$$

Let $\kappa_j(e')$ be the component of $g_j^{-1}(\iota_j(e))$ which joins $\kappa_j(b')$ to ∞.

In both cases it is clear from the local nature of g_j that ι_j and κ_j are still e-graph embeddings and $\iota_0^{-1} \circ g_0 \circ \kappa_0$ is an e-graph map with the same edge dynamics as $\iota_1^{-1} \circ g_1 \circ \kappa_1$. Also, ι_j is homotopic to κ_j through embedding graph embeddings (rel P_{f_j}) since $\iota_j\Big|_{X_H}$ is homotopic to $\kappa_j\Big|_{X_H}$ through embedding graph embeddings (rel P_{f_j}),

$$\iota_j(e \cup e') \cap \iota_j(X_H) = \emptyset,$$

and

$$\kappa_j(e \cup e') \cap \kappa_j(X_H) = \emptyset.$$

End 5.2.1.

Chapter 6. Mating.

§6.1. *Non-intimate mating*

Notation. Given a branched cover $f : S^2 \to S^2$, we will let Ω_f be the set of critical points of f, $P\Omega_f$ be the set of periodic critical points of f, and P_f be the post-critical set of f.

Notation. We let

$$\bar{\mathbf{C}} := \mathbf{C} \cup \{\infty \cdot \mathrm{Exp}(t) \mid t \in \mathbf{T}\},$$

where a basis of open neighborhoods of $\infty \cdot \mathrm{Exp}(t_0)$ are the sets of the form

$$\{r \cdot \mathrm{Exp}(t) \ \mid \ r \in]R, \infty],\ t \in]t_0 - \epsilon, t_0 + \epsilon[\}.$$

Definition. Given two critically finite quadratic polynomials f_0 and f_1, we define the *non-intimate mating* of f_0 with f_1 as follows. For $i = 0, 1$ let $K_i := K_{f_i}$ and let

$$\hat{\psi}_i : \hat{\mathbf{C}} - D \to \hat{\mathbf{C}} - \mathring{K}_i$$

be the unique continuous map such that

$$f_i(\hat{\psi}_i(z)) = \hat{\psi}_i(z^2)$$

for all $z \in \hat{\mathbf{C}} - D$ and $\hat{\psi}_i$ restricted to $\hat{\mathbf{C}} - \bar{D}$ is an analytic isomorphism onto $\hat{\mathbf{C}} - K_i$ (see section 2.3). Let

$$\psi_i : \bar{\mathbf{C}} - D \to \bar{\mathbf{C}} - \mathring{K}_i$$

be defined by $\psi_i := \hat{\psi}_i$ on $\mathbf{C} - D$, and

$$\psi_i(\infty \cdot \mathrm{Exp}(t)) := \infty \cdot \mathrm{Exp}(t).$$

Note that ψ_i is continuous. Now let

$$S^2_f := \left(\bar{\mathbf{C}} \amalg \bar{\mathbf{C}}\right) / \sim$$

where

$$\psi_0(\infty \cdot \mathrm{Exp}(t)) \sim \psi_1(\infty \cdot \mathrm{Exp}(-t)).$$

Note that S^2_f is homeomorphic to S^2. We define $f : S^2_f \to S^2_f$ by

$$f := \begin{cases} f_i & \text{on } K_i; \\ (\psi_i(r \cdot \mathrm{Exp}(t)) \mapsto \psi_i(r \cdot \mathrm{Exp}(2t))) & \text{on } \bar{\mathbf{C}} - K_i. \end{cases}$$

The branched cover $f : S^2_f \to S^2_f$ is called the non-intimate mating of f_0 with f_1.

The following definition and theorem show that if a non-intimate mating is topologically equivalent to a rational function, then that rational function is a good deal more intimate than the non-intimate mating.

Notation If $f : S^2_f \to S^2_f$ is the non-intimate mating of f_0 with f_1, we let

$$\mathcal{R}_f(t) := \{\psi_0(r \cdot \mathrm{Exp}(t) \mid r \in [1, \infty]\} \cup \{\psi_1(r \cdot \mathrm{Exp}(-t)) \mid r \in [1, \infty]\}.$$

Theorem 6.1.1. *(Analytic is very intimate) If*

$$f : S^2_f \to S^2_f$$

is a non-intimate mating and

$$g : \mathbf{P}^1 \to \mathbf{P}^1$$

is analytic and topologically equivalent to f, then there exists a continuous

$$\phi : S^2_f \to \mathbf{P}^1$$

satisfying the following:

1) *ϕ is surjective.*

2) *$\phi \circ f = g \circ \phi$.*

3) *$\phi^{-1}(\mathbf{P}^1 - J_g) = \left(\mathring{K}_0 \cup \mathring{K}_1\right)$.*

4) *ϕ is injective and analytic on $\left(\mathring{K}_0 \cup \mathring{K}_1\right)$.*

5) *For each $t \in \mathbf{T}$, ϕ is constant on $\mathcal{R}_f(t)$.*

6) *ϕ is a uniform limit of homeomorphism having properties 1) and 2).*

We will prove theorem 6.1.1 below in section 6.8.

§6.2. *Historical notes on the definition of mating.*

There are quite a few definitions of mating floating around, and we do not know all we would like about their relationship. Discovering this has been somewhat painful, so we present what we know to save others the pain.

Definition. Given quadratic polynomials f_0 and f_1 having Carathéodory loops γ_0 and γ_1 respectively, *algebraic* f_0, f_1*-equivalence* is the equivalence relation generated by the following two equivalence relations.

1) $s \sim t$ if and only if $\gamma_0(s) = \gamma_0(t)$.

2) $s \sim t$ if and only if $\gamma_1(-s) = \gamma_1(-t)$.

Definition. Given quadratic polynomials f_0 and f_1 having Carathéodory loops, f_0, f_1*-equivalence* is the smallest equivalence relation the graph of which contains the closure of the graph of algebraic f_0, f_1-equivalence.

Douady first defined mating as follows.

Definition. Given quadratic polynomials f_0 and f_1 having Carathéodory loops γ_0 and γ_1 respectively and filled in Julia sets K_0 and K_1 respectively, let

$$\Sigma := K_0 \amalg K_1 / \sim,$$

where $\gamma_0(t) \sim \gamma_1(-t)$. f_0 and f_1 define a map $f : \Sigma \to \Sigma$. If f is conjugate to a rational function by a map analytic on $\mathring{K}_0$ and $\mathring{K}_1$, then that rational function is the *mating of* f_0 *with* f_1.

If algebraic f_0, f_1-equivalence and f_0, f_1-equivalence are not the same, then Σ is not homeomorphic to S^2. We therefore thought the following definition would be useful.

Definition. Given quadratic polynomials f_0 and f_1 having Carathéodory loops γ_0 and γ_1 respectively and filled in Julia sets K_0 and K_1 respectively, let

$$\Sigma' := K_0 \amalg K_1 / \sim,$$

where $\gamma_0(t) \sim \gamma_1(-t)$ and $\gamma_0(t) \sim \gamma_0(s)$ if t and s are f_0, f_1-equivalent. f_0 and f_1 define a map $f : \Sigma' \to \Sigma'$. If f is conjugate to a rational function by a map analytic on $\mathring{K}_0$ and $\mathring{K}_1$, then that rational function is the *intimate mating of* f_0 *with* f_1.

We had hoped that the fibers of the map ϕ of theorem 6.1.1 would be either points in $\mathring{K}_0 \cup \mathring{K}_1$ or sets of the form

$$\bigcup_{t \in P} \mathcal{R}_f(t)$$

for some f_0, f_1-equivalence class P. Since ϕ is a uniform limit of homeomorphisms, ϕ is cell-like. So by 5) of theorem 6.1.1, any set of the form

$$\bigcup_{t \in P} \mathcal{R}_f(t)$$

for some f_0, f_1-equivalence class P must be contained in a fiber of ϕ, but we were unable to find a reason why there could be at most one such set per fiber. So the rational function topologically equivalent to the non-intimate mating might be even more intimate than even the intimate mating.

§**6.3.** *Essential topological equivalence and very intimate mating.*

Unfortunately, topological equivalence to the non-intimate mating seems to be too strong a notion to encompass all the rational functions which seem in some sense to be matings.

Example. Let f_0 be the polynomial corresponding to the external ray of M of angle $1/7$ and let f_1 be the polynomial on the boundary of M at exterior angle

3/14. Let f be the non-intimate mating of f_0 with f_1. Let y_0 be the critical point of f_1 and let

$$y_i := f_1^{\circ i}(y_0).$$

The ray $\mathcal{R}(1/7)$ has one endpoint at y_3, $\mathcal{R}(2/7)$ has one endpoint at y_4, and $\mathcal{R}(4/7)$ has one endpoint at y_2. On the other hand, $\mathcal{R}(1/7)$, $\mathcal{R}(2/7)$, and $\mathcal{R}(4/7)$ all have their other endpoint at the fixed point α of f_0. Let

$$R := \mathcal{R}(1/7) \cup \mathcal{R}(2/7) \cup \mathcal{R}(4/7).$$

One component of $f^{-1}(R)$ is R. The other has empty intersection with P_f. So the boundary of a thin neighborhood of R is an f-stable multi-curve with eigenvalue 1.

For many such examples, however, we have run on a computer a normalized Thurston's method for the non-intimate mating. Every time, the normalized Thurston's method seemed to converge with output a rational function of degree two (Levy noticed this independently [L]). In addition, the output is reasonable in following sense. Let f_0 and f_1 be the two quadratic polynomials such that their non-intimate mating f is not topologically equivalent to a rational function, but a normalized Thurston's method converges outputting some rational function g of degree two. Now let h_n be a sequence of polynomials approaching f_1 in M such that the non-intimate mating of f_0 with h_n is topologically equivalent to a rational function g_n. Then the g_n always seem to approach g. This motivates the following somewhat unsatisfactory definition.

Definition. A critically finite branched cover $f : S^2 \to S^2$ of degree d is *essentially topologically equivalent* to a rational function g if some normalized Thurston's method for f converges outputting g and g is of degree d.

Remarks.

1) This definition is unsatisfactory in the sense that it should be topological, but we believe this will come in due course. We also believe theorem 6.1.1 holds for essential topological equivalence and that a proof can be found based upon showing that the slow mating pictures (such as figures 1.5, 1.7, 1.35 and 1.37) converge.

2) As mentioned in section 2.5, f is topologically equivalent to a rational function g only if σ_f has a fixed point $\tau \in \mathcal{T}_f$ with representatives ϕ and ϕ' such that

$$g = \phi \circ f \circ (\phi')^{-1}.$$

In that case, any normalized Thurston's method for f starting at ϕ will be the sequence $\{\phi_n\}$ with $\phi_0 \circ f \circ \phi_{n+1}^{-1} = g$ for all n. So f is essentially topologically equivalent to g.

3) As mentioned in section 2.5, if the orbifold of f is hyperbolic and f is topologically equivalent to some rational function g, then any Thurston's method for f converges to the unique fixed point of σ_f. So any normalized Thurston's method for f which converges will output a function which is conjugate to g by a Möbius transformation.

4) Aside from what we said in 3), we do not know to what extent the output of

a normalized Thurston's method depends on its starting point.

5) If the Thurston's methods for f starting at ϕ and normalized at q and at q' converge, then their outputs are conjugate by a Möbius transformation.

We now present our favorite notion of mating.

Definition. If the non-intimate mating of f_0 with f_1 is essentially topologically equivalent to a rational function g, then g is a *very intimate mating of* f_0 *with* f_1.

§6.4. *Who mates with whom?*

Douady and Hubbard noticed that matings as defined in [D] do not exist between polynomials from conjugate limbs of M. The following translates their proof to our context.

Proposition 6.4.1. *If f_0 and f_1 are in conjugate limbs of M, then the non-intimate mating of f_0 with f_1 is not topologically equivalent to a rational function.*

Proof 6.4.1.

Let f be the non-intimate mating of f_0 with f_1 and let θ and θ' be the exterior rays of M associated to the limb containing f_0. Then $\mathcal{R}_f(\theta)$ and $\mathcal{R}_f(\theta')$ both go from the fixed point α of f_0 to the fixed point α of f_1. (See figure 6.1.) Let

$$C := \mathcal{R}_f(\theta) \cup \mathcal{R}_f(\theta').$$

Let $\gamma_0 := C$ if neither of the fixed points α are post-critical and C slightly pulled back off α towards the critical values otherwise. Let Γ' be all the inverse images of γ_0. Since γ_0 is formed essentially from external rays, the curves in Γ' do not intersect essentially. So there are finitely many non-peripheral curves in Γ'. So we may let Γ be the f-stable multicurve generated by γ_0.

It is easy to see that there are curves $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ (where $2^n - 1$ is the dynamic denominator of θ) in Γ such that γ_i covers $\gamma_{i+1(\text{mod } n)}$ once under f. So the matrix of Γ has a diagonal square with eigenvalue 1. So by lemma 2.5.2 the matrix of Γ has eigenvalue greater than or equal to 1. So we are done by theorem 2.5.1.

End 6.4.1.

Conjecture. *If f_0 and f_1 are in conjugate limbs of M, then the very intimate mating of f_0 with f_1 does not exist.*

Douady and Hubbard also conjectured the converse.

Conjecture 6.4.2. *If f_0 and f_1 are not in conjugate limbs of M, then the very intimate mating of f_0 with f_1 exists.*

This conjecture seems very hard to prove, but Levy [L] and Tan [T] have partial results. There is rumor that Mary Rees may have proved some version of this conjecture. Strong confirmation of the conjecture comes from parameter space computer drawings such as those presented in the introduction.

§6.5. *Mutilated Mandelbrot sets in parameter space.*

Recall that M_θ is the limb of M attached to the central cardioid of M at

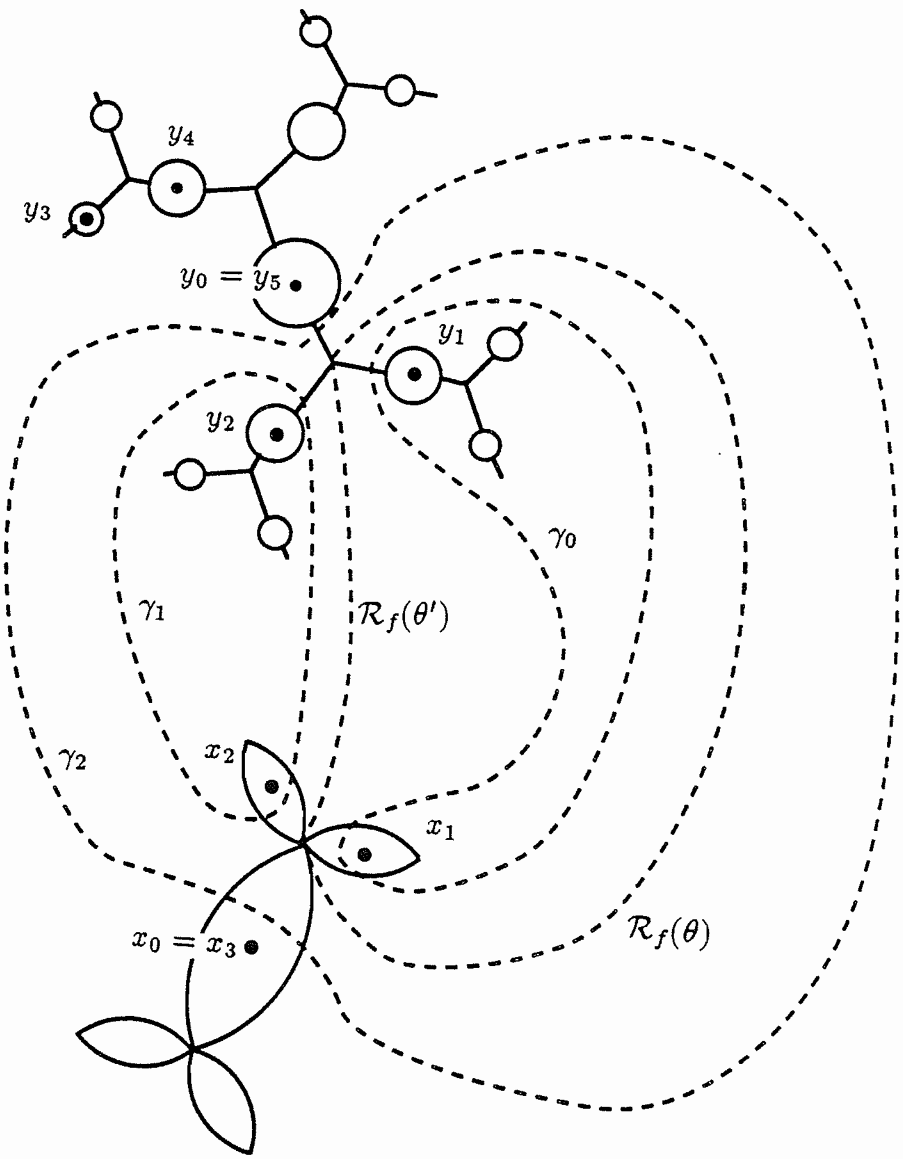

Figure 6.1. Non-intimate mating not equivalent to a rational function.

interior angle θ. Let f_0 be in $(\mathcal{D}_0 \cup \mathcal{D}_2) \cap M_\theta$ and let m and n be smallest such that

$$f_0^{\circ(m+n)}(0) = f_0^{\circ m}(0)$$

Let $MR_{m,n}$ be as in section 2.6.

Conjecture 6.5.1. *There is a continuous map*

$$\mu : (M - M_{-\theta}) \to MR_{m,n}$$

such that $\mu(f_1)$ is the very intimate mating of f_0 with f_1 for $f_1 \in \mathcal{D}_0 \cup \mathcal{D}_2$ and for those f_1 in $M - M_{-\theta}$ with an attractive periodic cycle (other than ∞), $\mu(f_1)$ has an attractive periodic cycle of the same period and with the same eigenvalue.

The evidence for this comes mainly from computer drawn parameter space pictures such as those presented in the introduction, but Douady has suggested a plan for proof of some partial results.

§**6.6.** *Please don't be too intimate.*

Before proving theorem 6.1.1, we state a conjecture we will need in the section on shared matings.

Conjecture 6.6.1. *Let $f : S_f^2 \to S_f^2$ be the non-intimate mating of f_0 with f_1 and suppose f is topologically equivalent to a rational function. Let ϕ be as in theorem 6.1.1. If a fiber of ϕ contains $\mathcal{R}_f(t_0)$ for $t_0 \in \mathbf{Q}/\mathbf{Z}$, then that fiber is exactly*

$$\bigcup_{t \in P} \mathcal{R}_f(t)$$

where P is the f_0, f_1-equivalence class of t_0.

We have never seen the kind of convolution one would expect to see in a computer drawing of the dynamical plane of a counter example. Also, Douady thinks it is true.

§6.8. *Proof of theorem 6.1.1.*

The main ideas in this proof are to be found in the work of Posdronasvili ([DH] Exposé VI).

Proposition 6.8.1. *Let $f, g : S^2 \to S^2$ be critically finite branched covers of degree two which are topologically equivalent. That is, there exist homeomorphisms*

$$\phi'_0, \phi'_1 : (S^2, P_f) \to (S^2, P_g)$$

such that

1') $\phi'_0 \circ f = g \circ \phi'_1$, and

2') ϕ'_0 is homotopic to ϕ'_1 through homeomorphisms ϕ'_t, $t \in [0,1]$, fixed on P_f.

Suppose that for every $x \in P\Omega_f$ of period n, there exists a neighborhood F'_x of the orbit of x and a neighborhood G'_x of the orbit of $\phi'_0(x) = \phi'_1(x)$ such that f restricted to F'_x and g restricted to G'_x are analytic. Then there exists homeomorphisms

$$\phi_0, \phi_1 : (S^2, P_f) \to (S^2, P_g)$$

and a neighborhood F_x of the orbit of x such that

1) $\phi_0 \circ f = g \circ \phi_1$,

2) ϕ_0 *is homotopic to* ϕ_1 *through homeomorphisms* ϕ_t, $t \in [0,1]$ *fixed on* P_f,

3) for $t \in [0,1]$, $\phi_0 = \phi_t = \phi_1$ *on* $\bigcup_{x \in P\Omega_f} F_x$, *and*

4) for $t \in [0,1]$, $\phi_0 = \phi_t = \phi_1$ *is analytic on* $\bigcup_{x \in P\Omega_f} F_x$.

Proof 6.8.1.

Let x_0 be the unique critical point of f in F'_x, and let $x_m := f^{\circ m}(x_0)$. Define y_m similarly for g. In light of theorem 2.2.2 there is a real number s with $0 < s < 1$, neighborhoods U_m of x_m, neighborhoods W_m of y_m, and analytic isomorphisms $\psi_m : D_s \to U_m$ and $\xi_m : U_m \to W_m$ such that

$$\begin{aligned}\psi_{m+1}^{-1} \circ f \circ \psi_m &= (\xi_{m+1} \circ \psi_{m+1})^{-1} \circ g \circ (\xi_{m+1} \circ \psi_{m+1}) \\ &= (z \mapsto z^2)\end{aligned} \quad \text{for} \quad m = 0 \tag{6.1}$$

and

$$\begin{aligned}\psi_{m+1}^{-1} \circ f \circ \psi_m &= (\xi_{m+1} \circ \psi_{m+1})^{-1} \circ g \circ (\xi_{m+1} \circ \psi_{m+1}) \\ &= \text{identity}\end{aligned} \quad \text{for} \quad m = 1, 2, \ldots, n-1. \tag{6.2}$$

Choose real numbers q and r such that $0 < q < r < s$ and

$$\left\{ \bigcup_{t \in [0,1]} \phi'_t(\psi_m(\partial D_r)) \right\} \cap \xi_m(\psi_m(\partial D_{q^2})) = \emptyset \quad \text{for} \quad m = 0, 1, \ldots, n-1. \tag{6.3}$$

For τ in $\mathbf{T}$, let $R_\tau : D_{q^2} \to D_{q^2}$ be given by

$$R_\tau(z) = z \cdot \text{Exp}(\tau).$$

We let Σ be the class of homeomorphisms $\sigma : S^2 \to S^2$ such that

$$\sigma = \phi'_0 \quad \text{on} \quad S^2 - \cup_m \psi_m(D_r), \tag{6.4}$$

and

$$\sigma = \xi_m \circ \left(\psi_m \circ R_{\tau_m} \circ \psi_m^{-1}\right) \quad \text{on} \quad \psi_m(D_{q^2}) \quad \text{for some} \quad \tau_m \in \mathbf{T}. \tag{6.5}$$

We now define a continuous map $T : \Sigma \to \mathbf{R}^n$ as follows. Intuitively, $T(\sigma)$ is how much one would have to unwind σ to get some fixed σ_0. Formally, let

$$A := \bar{D}_r - D_{q^2},$$

let

$$\tilde{A} := \left\{ z \in \mathbf{C} \mid \log(q^2) \leq \mathrm{Re}(z) \leq \log r \right\},$$

and let $\pi : \tilde{A} \to A$ be given by $\pi(z) = e^z$. So $\pi : \tilde{A} \to A$ is a universal covering map. By equations (6.4) and (6.5), for each σ in Σ, we can let $\lambda_m(\sigma) : A \to A$ be given by

$$\lambda_m(\sigma) := (\xi_m \circ \psi_m)^{-1} \circ \sigma \circ \psi_m.$$

By equation (6.4), $\lambda_m(\sigma)$ is the identity on ∂D_r. Let $\tilde{\lambda}_m(\sigma) : \tilde{A} \to \tilde{A}$ be the lift of $\lambda_m(\sigma)$ which is the identity on $\pi^{-1}(\partial D_r)$. We let

$$\begin{aligned} T_m(\sigma) &:= m^{\text{th}} \text{ component of } T(\sigma) \\ &:= \frac{1}{2\pi} \mathrm{Im} \left((\tilde{\lambda}_m(\sigma))(\log(q^2)) \right). \end{aligned}$$

All σ in Σ are isotopic (rel P_f) to ϕ_0'. So we can let $\tilde{\sigma}$ be the unique homeomorphism from S^2 to S^2 such that

$$\sigma \circ f = g \circ \tilde{\sigma}.$$

By equation (6.4),

$$\tilde{\sigma} = \phi_1' \quad \text{on} \quad S^2 - \cup_m \psi_m(D_r).$$

By theorem 2.2.2,

$$\tilde{\sigma} = \xi_m \circ \left(\psi_m \circ R_{\tilde{\tau}_m} \circ \psi_m^{-1}\right) \quad \text{on} \quad \psi_m(D_{q^2}) \quad \text{for some} \quad \tilde{\tau} \in \mathbf{T}.$$

By supplement 3.2.5 and equation (6.3), there is an $L(\sigma)$ in Σ such that $L(\sigma)$ is homotopic to $\tilde{\sigma}$ through homeomorphisms fixed on $\cup_m \psi_m(D_{q^2})$. If we can produce a σ such that $L(\sigma)$ is homotopic to σ through homeomorphisms fixed on $\cup_m \psi_m(D_{q^2})$, we would be done. By proposition 2.7.1 it is sufficient that $T(\sigma) = T(L(\sigma))$.

By supplement 3.2.5, T is surjective. Let σ_0 be such that $T(\sigma) = 0$. Then for all σ in Σ,

$$T(\sigma) = x \Rightarrow T(L(\sigma)) = Ax + T(L(\sigma_0)),$$

where

$$A = \begin{pmatrix} 0 & & & & \frac{1}{2} \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}.$$

Let $b := T(L(\sigma_0))$.

Since T is surjective, we need only show that $(x \mapsto Ax + b)$ has a fixed point.

Claim 6.8.1.1. *There is a norm on $\mathbf{R}^n$ with respect to which the norm of A is less than one.*

Proof 6.8.1.1. The eigenvalues of A are the n distinct roots of $\lambda^n - (-1)^n(1/2) = 0$. If we use the magnitude of the coordinates with respect to the basis of eigenvectors, then the norm of A is $(1/2)^n$. (In fact, the infimum of norms of a matrix is always the norm of the largest eigenvalue, but we do not need this general fact here.) **End 6.8.1.1.**

So there is a sufficiently large ball in $\mathbf{R}^n$ which is mapped to itself by $(x \mapsto Ax + b)$. So we are done by the Brouwer fixed point theorem.

End 6.8.1.

By the definition of topological equivalence there exist homeomorphisms

$$\phi'_t : (S^2_f, P_f) \to (\mathbf{P}^1, P_g)$$

for $t = 0, 1$ such that $\phi'_0 \circ f = g \circ \phi'_1$ and a homotopy $\Phi'_0 : S^2_f \times [0,1] \to \mathbf{P}^1$ from ϕ'_0 to ϕ'_1 through homeomorphisms fixed on P_f.

Let $P := P_f \cap \left(\overset{\circ}{K}_0 \cup \overset{\circ}{K}_1\right)$. Every point in P (resp. $\phi'_0(P) = \phi'_1(P)$) lies in the orbit of a periodic critical point of f (resp. g). Since f (resp. g) is analytic on $\overset{\circ}{K}_0 \cup \overset{\circ}{K}_1$(resp. $\mathbf{P}^1$), for every $n \geq 0$ there is a neighborhood of the orbit of every periodic critical point of period n on which $f^{\circ n}$ (resp. $g^{\circ n}$) is analytically conjugate to $z \mapsto z^2$. Let F' (resp. G') be the union of those neighborhoods of the orbits of the periodic critical points of f (resp. g). Since the distance from $\Phi'_0\left(S^2_f - F', t\right)$ to $\phi'_0(P) = \phi'_1(P)$ is continuous in t, we can choose F' small enough so that $\Phi'_0\left(S^2_f - F', [0,1]\right) \subset G$.

Proposition 6.8.1 now gives us the existence for $n = 0$ of

1) homeomorphisms $\phi_n, \phi_{n+1} : \left(S^2_f, P_f\right) \to \left(\mathbf{P}^1, P_g\right)$ such that $\phi_n \circ f = g \circ \phi_{n+1}$,
2) a homotopy $\Phi_n : S^2_f \times [0,1] \to \mathbf{P}^1$ from ϕ_n to ϕ_{n+1} through homeomorphisms fixed on P_f, and

3) a neighborhood F of the orbits of the periodic critical points of f such that

3a) for $x \in F$ and $t \in [0,1]$, $\phi_n(x) = \Phi_n(x,t) = \phi_{n+1}(x)$, and

3b) ϕ_n is analytic on $f^{-n}(F)$.

Having defined ϕ_n, ϕ_{n+1}, and Φ_n as above for $n = n_0$, we wish to do so for

$$n = n_0 + 1.$$

Indeed, since ϕ_0 is homotopic to ϕ_{n_0+1} through homeomorphisms fixing P_f,

$$(\phi_0)_* = (\phi_{n_0+1})_*$$

as isomorphisms mapping $\pi_1\left(S_f^2 - f(\Omega_f)\right)$ to $\pi_1\left(\mathbf{P}^1 - g(\Omega_g)\right)$. The existence of ϕ_1 shows that $(\phi_0)_*$ satisfies the lifting criterion, so too, therefore, does $(\phi_{n_0+1})_*$. So we can lift ϕ_{n_0+1} to ϕ_{n_0+2}. Similarly, we can lift Φ_{n_0} to Φ_{n_0+1}.

Claim 6.8.2. *For $x \in f^{-(n_0+1)}(F)$ and $t \in [0,1]$,*

$$\phi_{n_0+1}(x) = \Phi_{n_0+1}(x,t) = \phi_{n_0+2}(x).$$

Proof 6.8.2. By the induction hypothesis, for $t \in [0,1]$,

$$\phi_{n_0}(f(x)) = \Phi_{n_0}(f(x),t) = \phi_{n_0+1}(f(x)) =: y.$$

So

$$\Phi_{n_0+1}(x,t) \in g^{-1}(\{y\}).$$

But $g^{-1}(\{y\})$ is a discrete set and $\Phi_{n_0+1}(x,t)$ is a continuous function of t.

End 6.8.2.

Clearly ϕ_{n_0+1} is analytic on $f^{-(n_0+1)}(F)$.

By theorem 2.2.5, there exists a neighborhood U of J_g, a metric μ on U, and $\rho < 1$, such that

1) $g^{-1}(U) \subset U$.

2) g is locally expanding with respect to μ by a factor of at least $1/\rho$.

3) $\phi_n\left(S_f^2 - F\right) \subset U$ for all $n \geq 0$.

Claim 6.8.3. *The ϕ_n converge uniformly.*

Proof 6.8.3.

Since $\phi_n = \phi_{n+1}$ on $F \cup P_f$, we only have to show that the ϕ_n converge uniformly on $S_f^2 - (F \cup P_f)$.

Let $\lambda_{n,x}$ be the path defined by

$$\lambda_{n,x}(t) := \Phi_n(x,t).$$

So $\lambda_{n,x}$ starts at $\phi_n(x)$ and ends at $\phi_{n+1}(x)$. Note that by the definition of Φ_n,

$$\lambda_{n,f(x)} = g \circ \lambda_{n+1,x}.$$

Also note that if $x \notin P_f$, then

$$\lambda_{n,x}([0,1]) \cap P_g = \emptyset$$

since

$$\Phi_n\left(P_f,[0,1]\right) = P_g.$$

For $x \notin F \cup P_f$, let $\Lambda_{n,x}$ be the set of differentiable paths in $U - P_g$ which are path homotopic to $\lambda_{n,x}$ in $\mathbf{P}^1 - P_g$. Since $\Lambda_{n,x} \neq \emptyset$, we may let

$$e_n(x) := \inf \{l_\mu(\lambda) \mid \lambda \in \Lambda_{n,x}\}.$$

Claim 6.8.3.1. $e_{n+1}(x) \leq \rho \cdot e_n(f(x))$.

Proof 6.8.3.1.

For $\epsilon > 0$, let $\lambda \in \Lambda_{n,f(x)}$ be such that

$$l_\mu(\lambda) \leq e_n(f(x)) + \epsilon.$$

We may lift the path homotopy between $\lambda_{n,f(x)}$ and λ to a path homotopy between $\lambda_{n+1,x}$ and some other path, say $\tilde{\lambda}$. Since g is analytic, $\tilde{\lambda}$ is differentiable. Since $g^{-1}(U) \subset U$, the image of $\tilde{\lambda}$ is contained in U. Since $g(P_g) \subset P_g$, the path homotopy between $\tilde{\lambda}$ and $\lambda_{n+1,x}$ lies in $\mathbf{P}^1 - P_g$. So $\tilde{\lambda} \in \Lambda_{n+1,x}$.

Since $g \circ \tilde{\lambda} = \lambda$,

$$l_\mu(\tilde{\lambda}) \leq \rho \cdot l_\mu(\lambda).$$

So

$$e_{n+1}(x) \leq l_\mu(\tilde{\lambda}) \leq \rho \cdot l_\mu(\lambda) \leq \rho \cdot (e_n(f(x)) + \epsilon).$$

Now let $\epsilon \to 0$.

End 6.8.3.1.

Since $\phi_n = \phi_{n+1}$ on P_f, if $x_0 \in P_f$, then $e_n(x) \to 0$ as $x \to x_0$. So we may continuously extend e_n to equal zero on P_f, and claim 6.8.3.1 still holds. So

$$\sup \left\{e_n(x) \mid x \in S^2_f - F\right\} \leq \rho^n \cdot \sup \left\{e_0(x) \mid x \in S^2_f - F\right\},$$

with the supremum on the right hand side being finite since e_0 is continuous and $S_f^2 - F$ is compact. Since the usual metric on the sphere is at most a constant times μ, we are done.

End 6.8.3.

Let ϕ be the limit of the ϕ_n. ϕ is surjective since it is the uniform limit of surjective maps on a compact space. It is also clear that $\phi \circ f = g \circ \phi$.

Claim 6.8.4. *For all* $n \geq 0$,

$$\phi^{-1}(\phi_n(f^{-n}(F))) = f^{-n}(F)$$

and

$$\phi = \phi_n \text{ on } f^{-n}(F).$$

Proof 6.8.4. Since $\phi_{n+m} = \phi_n$ on $f^{-n}(F)$ for $m \geq 0$,

$$\phi = \phi_n = \phi_{n+m} \tag{6.6}$$

on $f^{-n}(F)$. Since the ϕ_{n+m} are injective,

$$\phi_{n+m}^{-1}(\phi_n(f^{-n}(F))) = f^{-n}(F) \tag{6.7}$$

for $m \geq 0$. Since $\phi_n(f^{-1}(F))$ is open, equations (6.7) and (6.6) imply that

$$\phi^{-1}(\phi_n(f^{-n}(F))) = f^{-n}(F).$$

End 6.8.4.

Claim. $\phi^{-1}(\mathbf{P}^1 - J_g) = \overset{\circ}{K}_0 \cup \overset{\circ}{K}_1$.

Proof. By corollary 2.2.4,

$$\bigcup_{n=0}^{\infty} g^{-n}(\phi_0(F)) = \mathbf{P}^1 - J_g.$$

Since $g^{-n}(\phi_0(F)) = \phi_n(f^{-n}(F))$,

$$\bigcup_{n=0}^{\infty} \phi_n(f^{-n}(F)) = \mathbf{P}^1 - J_g.$$

So by claim 6.8.4,

$$\phi^{-1}(\mathbf{P}^1 - J_g) = \bigcup_{n=0}^{\infty} f^{-n}(F).$$

Again by corollary 2.2.4,

$$\bigcup_{n=0}^{\infty} f^{-n}(F) = \overset{\circ}{K}_0 \cup \overset{\circ}{K}_1 . \tag{6.8}$$

End.

Claim. *ϕ is injective on $\overset{\circ}{K}_0 \cup \overset{\circ}{K}_1$.*

Proof. Let $x, y \in \overset{\circ}{K}_0 \cup \overset{\circ}{K}_1$ with $\phi(x) = \phi(y)$. By equation (6.8), we can choose n large enough so that x and y are in $f^{-n}(F)$. By claim 6.8.4,

$$\phi_n(x) = \phi(x) = \phi(y) = \phi_n(y).$$

Since the ϕ_n are injective, $x = y$. **End.**

ϕ is analytic on $\overset{\circ}{K}_0 \cup \overset{\circ}{K}_1$ by claim 6.8.4 since ϕ_n is analytic on $f^{-n}(F)$.

Claim 6.8.5. *For each $t \in \mathbf{T}$, ϕ is constant on $\mathcal{R}_f(t)$.*

Proof 6.8.5.

Let

$$A := \bigcup_{t\in\mathbf{T}} \mathcal{R}_f(t).$$

Define the continuous surjection $a : [-1, 1] \times \mathbf{T} \to A$ by

$$a(r,t) := \begin{cases} \psi_0\left(\frac{1}{r}\mathrm{Exp}(t)\right) & \text{if } r \geq 0; \\ \psi_1\left(\frac{1}{r}\mathrm{Exp}(-t)\right) & \text{if } r \leq 0. \end{cases}$$

Let the path $\lambda_{r,t}$ be given by $\lambda_{r,t}(s) := a(sr, t)$.

Let $\Lambda_{n,r,t}$ be the set of differentiable paths which are path homotopic in $\phi_n(A)$ to $\phi_n \circ \lambda_{r,t}$. Since $\Lambda_{n,r,t} \neq \emptyset$, we may let

$$e_n(r,t) := \inf\left\{l_\mu(\lambda) \mid \lambda \in \Lambda_{n,r,t}\right\}.$$

Claim 6.8.5.1. $e_{n+1}(r,t) \leq \rho \cdot e_n(r, 2t)$.

Proof 6.8.5.1.

Let $\epsilon > 0$ and let $\lambda \in \Lambda_{n,r,2t}$ be such that $l_\mu(\lambda) \leq e_n(r, 2t) + \epsilon)$. Since

$$\phi_n \circ f = g \circ \phi_{n+1}$$

and $f : A \to A$ is a covering space, $g : \phi_{n+1}(A) \to \phi_n(A)$ is a covering space and we may lift the path homotopy in $\phi_n(A)$ between $\phi_n \circ \lambda_{r,2t}$ and λ to a path homotopy in $\phi_{n+1}(A)$ between $\phi_{n+1} \circ \lambda_{r,t}$ and some other path, say $\tilde{\lambda}$. So $\tilde{\lambda}$ is in $\Lambda_{n+1,r,t}$. Since $g \circ \tilde{\lambda} = \lambda$,

$$l_\mu(\tilde{\lambda}) \leq \rho \cdot l_\mu(\lambda).$$

So

$$e_{n+1}(r,t) \leq l_\mu(\tilde{\lambda}) \leq \rho \cdot l_\mu(\lambda) \leq \rho \cdot (e_n(r,2t) + \epsilon).$$

Letting $\epsilon \to 0$ we are done.

End 6.8.5.1.

By induction,

$$e_n(r,t) \leq \rho^n \cdot e_0(r, 2^n t).$$

Since e_0 is continuous and $[-1,1] \times \mathbf{T}$ is compact,

$$\sup \{e_0(r,t) \mid r \in [-1,1], t \in \mathbf{T}\} < \infty.$$

So $e_n(r,t) \to 0$ as $n \to \infty$. That is

$$d_\mu (\phi_n(a(r,t)), \phi_n(a(0,t))) \to 0 \qquad \text{as} \qquad n \to \infty.$$

End 6.8.5.

Chapter 7. Thurston's Mating Criterion.

§7.1. *Definitions*

Definition. If $f : S^2 \to S^2$ is a branched cover of degree two and λ is a closed path in $S^2 - f(\Omega_f)$ which separates the critical values of f, then there is a unique path $\tilde{\lambda}$ in $S^2 - f^{-1}(f(\Omega_f))$ such that $f \circ \tilde{\lambda} = (s \mapsto \lambda(2s))$. We call $\tilde{\lambda}$ the *double lift of* λ *by* f.

Definition. Let $f : S^2 \to S^2$ be a critically finite branched cover of degree two and let K be a compact subset of S^2. A simple closed path λ in $S^2 - (P_f \cup K)$ is an *equator of* f *in the complement of* K if

1) λ separates the orbits of the two critical points, and
2) λ is homotopic to $\tilde{\lambda}$ in $S^2 - (P_f \cup K)$, where $\tilde{\lambda}$ is the double lift of λ by f.

The following claim follows directly from supplement 3.2.5.

Claim 7.1.1. *Let $f : S^2 \to S^2$ be a branched cover of degree two. Let K be a compact subset of S^2 having two components and containing the orbits of the two critical points of f. If λ is an equator of f in the complement of K, then there is a homeomorphism $\theta_{\lambda,K} : S^2 \to S^2$ homotopic to the identity through homeomorphisms fixed on $P_f \cup K$ such that $\theta_{\lambda,K} \circ \tilde{\lambda} = \lambda$.*

Notation. Let $f_{\lambda,K} := f \circ \theta_{\lambda,K}^{-1}$.

The following four items are obvious.

1) $f_{\lambda,K} = f$ on K.

2) $f_{\lambda,K}$ is topologically equivalent to f.

3) $f_{\lambda,K} \circ \lambda = (s \mapsto \lambda(2s))$.

4) The quotient space obtained from S^2 by mapping the image of λ to a single point is homeomorphic to two spheres joined at a point. $f_{\lambda,K}$ is well defined on the quotient and maps each topological sphere to itself as a branched map of degree two with the image of λ being a fixed critical point. Call the restriction of $f_{\lambda,K}$ to the i$^{\text{th}}$ topological sphere $f_{\lambda,K,i}$.

§**7.2.** *Thurston's mating criterion.*

The following theorem is due to Thurston.

Theorem 7.2.1. *If $f : S^2 \to S^2$ is a critically finite branched cover of degree two which is topologically equivalent to a rational function and λ is an equator of f in the complement of some compact K, then $f_{\lambda,K,i}$ is topologically equivalent to some quadratic polynomial h_i for $i = 0, 1$. Furthermore, f is topologically equivalent to the non-intimate mating of h_0 with h_1.*

Proof 7.2.1.

Suppose $f_{\lambda,K,i}$ were not topologically equivalent to a rational function. Then by theorem 2.5.1, there would be an $f_{\lambda,K,i}$-stable multicurve Γ with eigenvalue greater than or equal to one. But then Γ would also be an $f_{\lambda,K}$-stable multicurve with eigenvalue greater than or equal to one, contradicting the fact that $f_{\lambda,K}$ is topologically equivalent to f.

So $f_{\lambda,K,i}$ is topologically equivalent to some rational function of degree two. That rational function has a fixed critical point, so it is conjugate to a quadratic polynomial h_i. Let $h : S^2_h \to S^2_h$ be the non-intimate mating of h_0 with h_1. Recall the notation that

$$S^2_h := (\bar{\mathbf{C}} \amalg \bar{\mathbf{C}})/\sim$$

where

$$\psi_0(\infty \cdot \operatorname{Exp}(t)) \sim \psi_1(\infty \cdot \operatorname{Exp}(-t)).$$

Call $\{\psi_0(\infty \cdot \operatorname{Exp}(t)) \mid t \in \mathbf{T}\} \subset S^2_h$ the *equator* of S^2_h. Since $f_{\lambda,K,i}$ is topologically equivalent to h_i, there are obvious homeomorphisms

$$\phi_0, \phi_1 : (S^2_h - \text{equator}, P_h) \to (\mathbf{P}^1 - \text{image}(\lambda), P_f)$$

such that

$$\phi_0 \circ h = f_{\lambda,K} \circ \phi_1$$

and ϕ_0 is homotopic to ϕ_1 through homeomorphisms fixing P_h.

It may not be possible to extend the ϕ_i continuously to the equator of S^2_h, so we do the following. Let A be a closed annulus in $S^2_h - P_h$ containing the equator. Let

$$B := \phi_0(A) \cup \text{image}(\lambda).$$

Let $\phi_0' : S_h^2 \to S^2$ be a homeomorphism mapping A onto B and equal to ϕ_0 on $S_h^2 - \overset{\circ}{A}$. Now,

$$h : h^{-1}(A) \to A$$

and

$$f_{\lambda,K} : f_{\lambda,K}^{-1}(B) \to B$$

are covering maps of degree two, so we can let

$$\phi_1' : h^{-1}(A) \to f_{\lambda,K}^{-1}(B)$$

be a lift of ϕ_0' which agrees with ϕ_1 on ∂A. Extend ϕ_1' by setting it equal to ϕ_1 on

$$S_h^2 - h^{-1}(A).$$

ϕ_1' is a homeomorphism and is homotopic to ϕ_0' through homeomorphisms fixed on P_h by supplement 3.2.5.

End 7.2.1.

Chapter 8. Captures.

§8.1. *Definition.*

Let f_0 be in $\mathcal{D}_0 \cup \mathcal{D}_2$. Let x_0 be the critical point 0 of f_0 and let

$$x_i := f_0^{\circ i}(x_0).$$

Definition. A periodic or pre-periodic point y in K_{f_0} not equal to x_1 and not in the interior of the Hubbard tree of f_0 is called a *capture site of* f_0.

Let y_1 be a capture site of f_0. Let y_0' and y_0'' be the inverse images of y_1 under f_0, and for $i = 2, 3, \dots$ let

$$y_i := f_0^{\circ(i-1)}(y_1).$$

Let

$$X := \{x_i \mid i = 0, 1, 2, \dots\},$$

and let

$$Y := (\{y_i \mid i = 1, 2, 3, \dots\} \cup \{y_0', y_0''\}) - X.$$

X and Y are finite sets, so if we let X_G' be the regular envelope in K_{f_0} of $X \cup Y$, then X_G' is a finite topological tree (proposition 2.3.4).

Claim 8.1.1. *y_0' and y_0'' are extremities of X_G'.*

Proof 8.1.1.

For $p \in X \cup Y$, let $\eta(p)$ be the number of edges of X'_G incident upon p. From the local behavior of f_0, we get that for $p \neq x_0$,

$$\eta(p) \leq \eta(f_0(p)). \tag{8.1}$$

If one

$$y \in Y - (X \cup \{y'_0, y''_0\})$$

is an extremity of X'_G, then we are done by (8.1). So suppose all

$$y \in Y - (X \cup \{y'_0, y''_0\})$$

are not extremities of X'_G. Let X_H be the Hubbard tree of f_0. Neither y'_0 nor y''_0 is in the interior of X_H, because if so, by the local nature of f_0, y_1 would be in the interior of X_H (recall that $y_1 \neq x_1$). So suppose y'_0 is not an extremity of X'_G. Then y'_0 and y''_0 must be in the same component of $X'_G - \{x_0\}$, contradicting the fact that f_0 is injective on each component of $X'_G - \{x_0\}$.

End 8.1.1.

Now let X_G be X'_G with y'_0 and y''_0 identified to a single point we shall call y_0. We form the embedding graph G with topological space X_G as follows. Let the vertices of G be the projection of $X \cup Y$ together with any points in X_G not having a neighborhood homeomorphic to an interval. For all vertices other than y_0, let the cyclic permutation of the incident edges be that induced by the embedding of

X'_G in $\mathbf{P}^1$. Let $\sigma_G^{y_0}$ be the unique (by claim 8.1.1) non-trivial permutation of $E_G^{y_0}$. Obviously

$$f_0 : X'_G \to X'_G$$

factors through the projection to X_G giving

$$f : X_G \to X_G$$

an almost e-graph map.

Theorem 8.1.2. *(Existence) There is an e-graph embedding*

$$\iota : X_G \to S^2$$

and a branched cover

$$g : S^2 \to S^2$$

of degree two which is an extension of $\iota \circ f \circ \iota^{-1}$ with one critical point at $\iota(x_0)$ and the other at $\iota(y_0)$. (Uniqueness) Suppose for $j = 0, 1$ we have that $\iota_j : X_G \to S^2$ and $\kappa_j : X_G \to S^2$ are e-graph embeddings, $g_j : S^2 \to S^2$ is a branched cover of degree two with the critical point at $\kappa_j(x_0)$ and the other at $\kappa_j(y_0)$, and $f_j := \iota_j^{-1} \circ g_j \circ \kappa_j$ is an almost e-graph map. If f_0 and f_1 have the same edge dynamics and ι_j is homotopic to κ_j through e-graph embeddings (rel P_{f_0}), then g_0 is topologically equivalent to g_1.

Proof 8.1.2.

(Existence)

Claim 8.1.2.1. *$f : X_G \to X_G$ respects boundaries.*

Proof 8.1.2.1.

Let e' (resp. e'') be the unique (by claim 8.1.1) edge of X'_G incident upon y'_0 (resp. y''_0). By claim 8.1.1, y'_0 and y''_0 are extremities of X'_G. So the only possible inverse image under $f : X'_G \to X'_G$ of y'_0 or y''_0 is a forward image of y_1. So at least one of y'_0 or y''_0 has no inverse image under $f : X'_G \to X'_G$. So at least one of e' or e'' has no inverse image under f. So $f(X_G)$ is a tree.

f is injective on the components of $X_F - \{x_0, y_0\}$ containing e' and e'' respectively, and X_G is a loop with trees attached. Clearly f^{X_0} and f^{x_1} are quadratic. So we are done by proposition 4.5.1.

End 8.1.2.1.

Since X_G is a tree embeddable in S^2 with two extremities identified, there is an e-graph embedding $\iota : X_G \to S^2$. So we get the existence of the branched cover g by claim 8.1.2.1 and theorem 4.4.1.

(Uniqueness) Uniqueness follows immediately from theorem 4.4.1.

End 8.1.2.

Theorem 8.1.2 allows us to make the following definitions.

Definition. The branched cover g given by theorem 8.1.2 is the *topological capture at y_1 by f_0.*

Definition. If a rational function is essentially topologically equivalent to the topological capture at y_1 by f_0, we say that rational function is the *capture at y_1 by f_0.*

Definition The tree X'_G is called the *tree of the capture at* y_1 *by* f_0.

§8.2. *At where are there captures?*

Definition. Let f_0 be in $(\mathcal{D}_0 \cup \mathcal{D}_2) \cap L$ where L is some limb of M. Let θ and θ' be the angles of the external rays of M corresponding to L. Let α be the fixed point α of f_0. Then

$$\mathcal{R}(K_{f_0}, \theta) \cup \mathcal{R}(K_{f_0}, \theta') \cup \{\infty, \alpha\} =: C$$

is a simple closed curve. Let U be the connected component of $\mathbf{P}^1 - C$ containing the critical value. Then the *mutilated filled in Julia set of* f_0 is

$$MK_{f_0} := K_{f_0} - U.$$

If f_0 is $z \mapsto z^2$, then we let

$$MK_{f_0} := K_{f_0}.$$

Proposition 8.2.1. *Let* f_0 *be in* $\mathcal{D}_0 \cup \mathcal{D}_2$ *and let* y_1 *be a capture site of* f_0. *If* y_1 *is not in* MK_{f_0}, *then the topological capture at* y_1 *by* f_0 *is not topologically equivalent to a rational function.*

Proof 8.2.1.

Let x_1, X_F, ι, and f_ι be as in the definition of the topological capture at y_1 by f_0. Let $\alpha \in X_F$ correspond to the fixed point α of f_0, and suppose there are k edges of X_F incident upon α.

Claim 8.2.1.1. *There are k connected components of $X_F - \{\alpha\}$.*

Proof 8.2.1.1. Let $-\alpha$ be the inverse image of α other than α. Since $y_1 \notin MK_{f_0}$, the segment in X_F joining y_1 to the Hubbard tree intersects the Hubbard tree at some point p in $[x_1, \alpha]_{X_F - \{y_0\}}$ (recall that x_1 is an extremity of the Hubbard tree). So the segments joining y_0 to the Hubbard tree intersect the Hubbard tree in $[\alpha, -\alpha]_{X_F - \{y_0\}}$. So no two components of $X_F - \{\alpha, y_0\}$ are joined by adding y_0. **End 8.2.1.1.**

Let W_0 (resp. W_1) be the component of $X_F - \{\alpha\}$ containing $\{x_0\}$ (resp. $\{x_1\}$). f permutes the edges incident upon α and is injective on all components of $X_F - \{\alpha\}$ other than W_0, so we can let W_j be the component of $X_F - \{\alpha\}$ containing $\{x_j\}$ for $j = 2, 3, \ldots, k-1$.

Let A be a small enough neighborhood of $\iota(\alpha)$ so that $A \cap \iota(V_F) = \{\iota(\alpha)\}$ (see figure 8.1). For $j = 0, 1, 2, \ldots$, k-1, let γ_j be a simple closed curve in $\left(S^2 - \iota(K_F)\right) \cup A$ with $\gamma_j \cap \iota(K_F)$ a single point in $A \cap W_j$. γ_j is in $S^2 - P_{f_\iota}$.

Since $y_1 \notin MK_{f_0}$, $y_1 \in W_1$. Since f maps W_j injectively onto $W_{j+1 (\text{mod } k)}$ for $j = 1, 2, \ldots, k-1$, $y_j \in W_j$ and $y_j \neq x_j$ for $j = 0, 1, 2, \ldots, k-1$. So the γ_j are not peripheral, and none separate the critical values of f_ι. So $f_\iota^{-1}(\gamma_j)$ has two components each mapping to γ_j with degree one. This means that $\gamma_{j-1 (\text{mod } k)}$ is isotopic (rel P_{f_ι}) to one of the inverse images of γ_j, and no two γ_j for $j = 0, 1, 2, \ldots, k-1$ intersect.

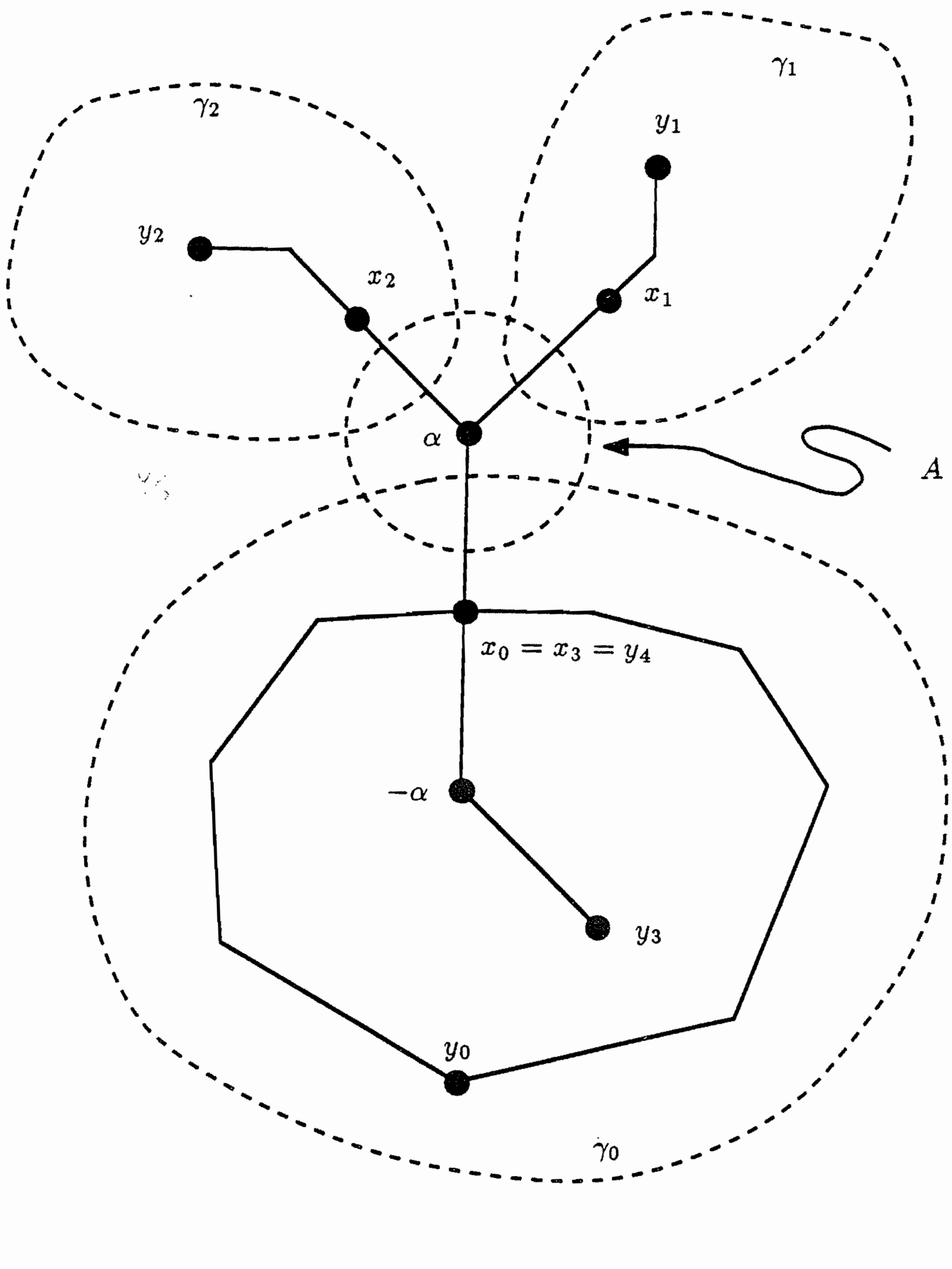

Figure 8.1. Capture not equivalent to a rational function.

Now consider all the curves in

$$\Gamma' := \bigcup_{n=0}^{\infty} f^{-n}(\gamma_j).$$

Suppose two curves $\gamma, \gamma' \in \Gamma'$ were to intersect. Let n be large enough so that $f^{\circ n}(\gamma)$ and $f^{\circ n}(\gamma')$ are both in $\{\gamma_0, \gamma_1, \ldots, \gamma_{k-1}\}$. But $f^{\circ n}(\gamma)$ and $f^{\circ n}(\gamma')$ intersect contradicting the fact that no two γ_j intersect for $j = 0, 1, 2, \ldots, k-1$. So no two curves in Γ' intersect. So if we let Γ be the non-peripheral curves in Γ', then Γ is finite. So Γ is an f-stable multicurve with a sub-block of its matrix having eigenvalue 1. So by lemma 2.5.2 and theorem 2.5.1 we are done.

End 8.2.1.

Proposition 8.2.1 suggests the following conjecture.

Conjecture. *Let f_0 be in $\mathcal{D}_0 \cup \mathcal{D}_2$ and let y_1 be a capture site of f_0. If y_1 is not in MK_{f_0}, then the capture at y_1 by f_0 does not exist.*

Conjecture. *Let f_0 be in $\mathcal{D}_0 \cup \mathcal{D}_2$ and let y_1 be a capture site of f_0. If $y_1 \in MK_{f_0}$, then the capture at y_1 by f_0 exists.*

§8.3. *Mutilated gutted filled in Julia sets in parameter space.*

Definition. For $f_0 \in \mathcal{D}_0 \cup \mathcal{D}_2$, the *guts of the filled in Julia set of f_0* is the Hubbard tree of f_0 together with the components of the interior of K_{f_0} which intersect the Hubbard tree. We denote the guts of the filled in Julia set of f_0 by GK_{f_0}.

Conjecture. *Suppose f_0 is in $\mathcal{D}_0 \cup \mathcal{D}_2$. Let m and n be smallest such that*

$$f_0^{\circ(m+n)}(0) = f_0^{\circ m}(0).$$

Then there is a continuous map

$$\mu : MK_{f_0} - GK_{f_0} \to R_{m,n}$$

satisfying the following.

1) If $y_1 \in MK_{f_0} - GK_{f_0}$ is a pre-periodic capture site of f_0, then $\mu(y_1)$ is the capture at y_1 by f_0.

2) If $y_1 \in MK_{f_0} - GK_{f_0}$ is a periodic capture site of f_0 of period k, then $\mu(y_1)$ is the root of the component of $MR^k_{m,n}$ having the capture at y_1 by f_0 as center.

3) If f_0 is in $\mathcal{D}_0$ and $y_1 \in MK_{f_0} - GK_{f_0}$ lands in the component of the interior of K_{f_0} containing 0 for the first time after k applications of f_0 at interior angle θ and radius r, then $g_{\mu(y_1)}(0)$ lands in the component of $\mathbf{P}^1 - J_{g_{\mu(y_1)}}$ containing ∞ for the first time after k applications of $g_{\mu(y_1)}$ at interior angle θ and radius r.

§**8.4.** *Matings in parameter space of mutilated Mandelbrot sets with mutilated gutted filled in Julia sets.*

So far we have conjectured that given some quadratic polynomial f_0, there will be in parameter space a mutilated Mandelbrot set of matings with f_0 and a mutilated gutted filled in Julia set of captures by f_0. The following theorem says that if they are there, then for $t \in \mathbf{Q}/\mathbf{Z}$ with dynamic denominator of the form 2^m or $2^n - 1$ they sew according to the rule $\gamma_M(t)$ sews to $\gamma_{K_{f_0}}(-t)$ as discussed in the introduction. Recall from section 2.4 that points of the form $\gamma_M(t)$ (resp. $\gamma_{K_{f_0}}(-t)$) for $t \in \mathbf{Q}/\mathbf{Z}$ with dynamic denominator of the form 2^m are dense in the boundary of M (resp. K_{f_0}).

Theorem 8.4.1. *Let f_0 be in $\mathcal{D}_0 \cup \mathcal{D}_2$. Let f_1 be either in $\mathcal{D}_0$ or on the boundary of M with corresponding external ray of diadic angle θ_1. Let γ_0 be the Carathéodory loop of f_0. If $\gamma_0(-\theta_1)$ is a capture site of f_0, then the non-intimate mating of f_0 with f_1 is topologically equivalent to the topological capture at $\gamma_0(-\theta_1)$ by f_0.*

Theorem 8.4.1 will be proved below.

Remark. We believe that theorem 8.4.1 could be proved for all $f_1 \in \mathcal{D}_0 \cup \mathcal{D}_2$ if we had a good topological definition of essential topological equivalence.

Proposition 8.4.2. *Let f_0 and f_1 be in $\mathcal{D}_0 \cup \mathcal{D}_2$, let θ_1 be the angle of an external ray of M corresponding to f_1, and let γ_0 be the Carathéodory loop of f_0. Suppose f_0 and f_1 satisfy the following.*

1) $\gamma_0(-\theta_1)$ is a capture site of f_0.

2) All points if the form $f_0^{\circ j}(\gamma_0(-\theta_1))$ are extremities of the tree of the capture at $\gamma_0(\theta_1)$ by f_0.

3) The orbit of $\gamma_0(-\theta_1)$ under f_0 has the same number of points as the orbit of the critical value of f_1.

Then the non-intimate mating of f_0 with f_1 is topologically equivalent to the *topological capture at $\gamma_0(-\theta_1)$ by f_0.*

Proof 8.4.2.

Let $g : S_g^2 \to S_g^2$ be the non-intimate mating of f_0 with f_1. Let $y_0 \in S_g^2$ be the critical point of f_1 and let $y_i := f_1^{\circ i}(y_0) = g^{\circ i}(y_0)$. Consider the tree of the capture at $\gamma_0(-\theta_1)$ by f_0 as it sits in K_0 of S_g^2. For $i = 1, 2, 3, \ldots$, extend that tree by adding on the rays $\mathcal{R}_g(-2^{(i-1)}\theta_1)$, the points y_1, and if $f_1 \in \mathcal{D}_0$, the internal rays of $\mathring{K}_1$ going from $\gamma_1(2^{(i-1)}\theta_1)$ to y_i. Also add in the rays $\mathcal{R}_g(\theta_1/2)$ and $\mathcal{R}_g((\theta_1/2)+(1/2))$, the point y_0, and if $f_1 \in \mathcal{D}_0$, the internal rays of $\mathring{K}_1$ going from $\gamma_1(\theta_1/2)$ and $\gamma_1((\theta_1/2)+(1/2))$ to y_1. Call this new graph X_g.

Conditions 2) and 3) in the statement of the proposition imply that X_g is homeomorphic to the graph X_F as defined in the definition of captures (recall that the inverse images of $\gamma_0(-\theta_1)$ are always extremities of the tree of the capture). Make X_g into an embedding graph in the obvious way, and then $g : X_g \to X_g$ has the same edge dynamics as $f : X_F \to X_F$ of the definition of captures. We are therefore done by the uniqueness part of theorem 8.1.2.

End 8.4.2.

Proof 8.4.1.

Let x_0 be the critical point of f_0 and let $x_k := f^{\circ k}(x_0)$. Let $y_1 = \gamma_0(\theta_1)$ and let $y_i := f^{\circ(i-1)}(y_1)$. Finally, let X'_G be the tree of the capture at y_1 by f_0.

Claim 8.4.1.1. *For $i = 1, 2, 3, \ldots$, the y_i have only one external angle.*

Proof 8.4.1.1.

Suppose some y_j has two or more external angles. If θ_1 is diadic, this contradicts fact 2.4.5. So suppose f_1 is in $\mathcal{D}_0$. So θ_1 is periodic under angle doubling, and so y_1 is periodic under f_0.

Let X_H be the Hubbard tree of f_0. Since y_j has at least two external angles, by fact 2.3.6, some forward image of y_j is in X_H.

Claim 8.4.1.1.1. *Neither y_j nor any forward image of y_j equals x_0.*

Proof 8.4.1.1.1. Suppose one did. If f_0 is in $\mathcal{D}_0$, then since y_1 is periodic, y_1 would be in the interior of K_{f_0} (proposition 2.2.1). If f_0 is in $\mathcal{D}_2$, then y_j, and hence y_1, would not be periodic. **End 8.4.1.1.1.**

Claim 8.4.1.1.2. *No forward image of y_j is in the interior of X_H.*

Proof 8.4.1.1.2. Suppose one were. Since by claim 8.4.1.1.1 no forward image of y_j equals x_0, all forward images after the one in X_H would be in the interior of X_H. But y_1 is periodic. So y_1 would be in the interior of X_H. But y_1 is a capture site. **End 8.4.1.1.2.**

So some forward image of y_j is an extremity of X_H. The extremities of X_H are among the x_k (remark 2.3.5). Since y_1 is periodic and the x_k are forward invariant, y_i is some x_{k_i} for $i = 1, 2, 3, \ldots$. So X'_G is the regular envelope of the x_k's and one other point (namely that inverse image of y_1 which is not in the orbit of y_1). So by claim 8.4.1.1.2, $f_0(y_j)$ is an extremity of X'_G. Since $y_j \neq x_0$, this contradicts the fact that f_0 is a local homeomorphism at points other than x_0.

End 8.4.1.1.

Part 1) of the hypothesis of proposition 8.4.2 is satisfied by hypothesis. Part 2) is satisfied by claim 8.4.1.1.

Part 3) is satisfied if θ_1 is diadic because all points with a diadic external angle have only one external angle (fact 2.4.5). If the dynamic denominator of θ_1 is of the form $2^n - 1$, the orbit of the critical value of f_1 contains n points (proposition 2.4.1). On the other hand, by claim 8.4.1.1 the orbit of y_1 has n points.

End 8.4.1.

Chapter 9. Calculating the Identifications Induced by the Carathéodory Loop.

§9.1. *Definitions and Statements.*

Let $\theta \in \mathbf{Q}/\mathbf{Z}$ and let $c \in \mathcal{D}_0 \cup \mathcal{D}_2$ correspond to θ. Let $\gamma := \gamma_{K_c}$. For $t \in \mathbf{T}$ we define an object, $\hat{\epsilon}(t)$, which can be effectively computed for $t \in \mathbf{Q}/\mathbf{Z}$, such that

$$\hat{\epsilon}(t_1) = \hat{\epsilon}(t_2) \iff \gamma(t_1) = \gamma(t_2).$$

Let $\theta_1 := \frac{\theta}{2}$ and $\theta_2 := \frac{\theta}{2} + \frac{1}{2}$.

Let q_θ be the dynamic denominator of θ.

Given a partition $\omega : \mathbf{T} \to S$, we define the associated sequence

$$\hat{\omega}(t) := \omega(2^0 t), \omega(2^1 t), \omega(2^2 t), \ldots .$$

Of course, if t is rational, then $\hat{\omega}(t)$ repeats after some point and so is actually a finite object.

Case. *$c \in \mathcal{D}_2$:*

Let $\eta : \mathbf{T} \to \{0, 1, 2\}$ be given by

$$\eta(t) := \begin{cases} 0 & \text{if } t \in \{\theta_1, \theta_2\} \ ; \\ 1 & \text{if } t \in \,]\theta_1, \theta_2[\ ; \\ 2 & \text{otherwise.} \end{cases}$$

Let

$$T_0 := \{t \in \mathbf{T} \mid \gamma(t) = 0\}\,.$$

Proposition 9.1.1.

$$T_0 = \{t \in \mathbf{Q}/\mathbf{Z} \mid \text{ dynamic denominator of } 2t = q_\theta \text{ and } \hat{\eta}(2t) = \hat{\eta}(\theta)\}\,.$$

Proposition 9.1.1 is proved below.

Let

$$\epsilon(t) := \begin{cases} 0 & \text{if } t \in T_0 \ ; \\ 1 & \text{if } t \in]\theta_1, \theta_2[- T_0 \ ; \\ 2 & \text{otherwise .} \end{cases}$$

Case. *$c \in \mathcal{D}_0$:*

Let $\eta_+ : \mathbf{T} \to \{1, 2\}$ and $\eta_- : \mathbf{T} \to \{1, 2\}$ be given by

$$\eta_+ := \begin{cases} 1 & \text{if } t \in]\theta_1, \theta_2] \ ; \\ 2 & \text{otherwise .} \end{cases}$$

$$\eta_- := \begin{cases} 1 & \text{if } t \in [\theta_1, \theta_2[\ ; \\ 2 & \text{otherwise .} \end{cases}$$

Let

$$T_+ := \{p/q_\theta \mid \widehat{\eta_+}(p/q_\theta) = \widehat{\eta_+}(\theta)\} \text{ and}$$

$$T_- := \{p/q_\theta \mid \widehat{\eta_-}(p/q_\theta) = \widehat{\eta_-}(\theta)\}\,.$$

Let θ' be the unique angle such that $\gamma_M(\theta') = \gamma_M(\theta)$ and $\theta' \neq \theta$.

Proposition 9.1.2.

$$T_+ - \{\theta\} \neq \emptyset \iff \theta' > \theta.$$

$$T_- - \{\theta\} \neq \emptyset \iff \theta' < \theta.$$

Proposition 9.1.2 is proved below.

Let

$$\epsilon := \begin{cases} \eta_+ & \text{if } \theta' > \theta \ ; \\ \eta_- & \text{if } \theta' < \theta \ . \end{cases}$$

Theorem 9.1.3. *(Douady - Hubbard)*

$$\hat{\epsilon}(t_1) = \hat{\epsilon}(t_2) \iff \gamma(t_1) = \gamma(t_2).$$

Theorem 9.1.3 is proved below.

Remark. Propositions 9.1.1 and 9.1.2 give an effective way to compute ϵ. In fact, Pierre Lavaurs has proved that there is anther way to determine if $\theta' > \theta$ [La]. His method is probably usually less computationally expensive than that given by proposition 9.1.2, but we will not discuss this any further.

§9.2. *Proofs.*

If $c \in \mathcal{D}_0$, let n be the period of 0. For $k = 1, 2, \ldots, n$ let U_k be the component of $\overset{\circ}{K}_c$ containing $P_c^{\circ k}(0)$. As stated in proposition 2.3.2, there are unique analytic isomorphisms $\psi_k : D \to U_k$ such that $\psi_{k+1} = P_c \circ \psi_k$ for $k = 1, 2, \ldots, n-1$ and $P_c(\psi_n(z)) = \psi_1(z^2)$. Let

$$\mathcal{R}_k(t) := \left\{ \psi_k \left(re^{2\pi i t} \right) \mid r \in]0, 1[\right\}.$$

As in figure 9.1 let

$$L_0 := \mathcal{R}(K_c, \theta_1) \cup \{\gamma(\theta_1)\} \cup \mathcal{R}_n(0) \cup \{0\} \cup \mathcal{R}_n(1/2) \cup \{\gamma(\theta_2)\} \cup \mathcal{R}(K_c, \theta_2).$$

If $c \in \mathcal{D}_2$, then as in figure 9.2, let

$$L_0 := \mathcal{R}(K_c, \theta_1) \cup \{0\} \cup \mathcal{R}(K_c, \theta_2).$$

Two points z_0 and z_1 are said to be on the *same side of* L_0 if $z_0, z_1 \in \mathbf{C} - L_0 \Rightarrow z_0$ and z_1 are in the same component of $\mathbf{C} - L_0$.

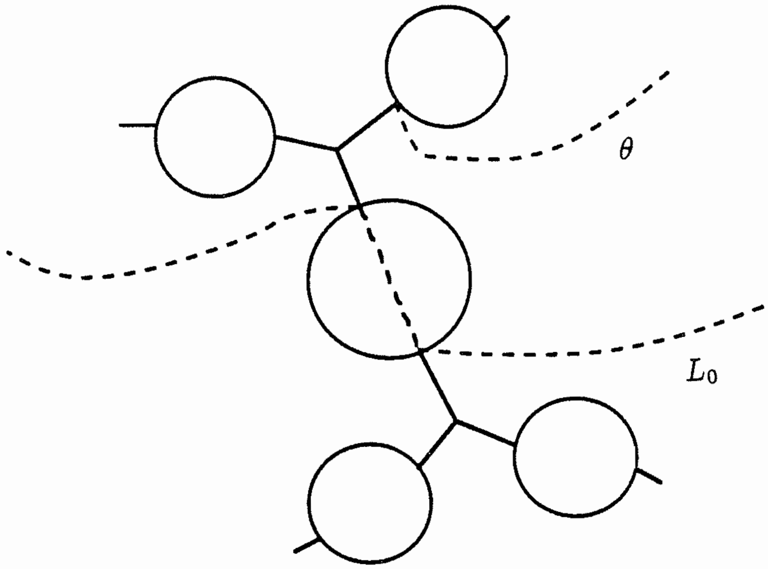

Figure 9.1. Definition of L_0 for $c \in \mathcal{D}_0$.

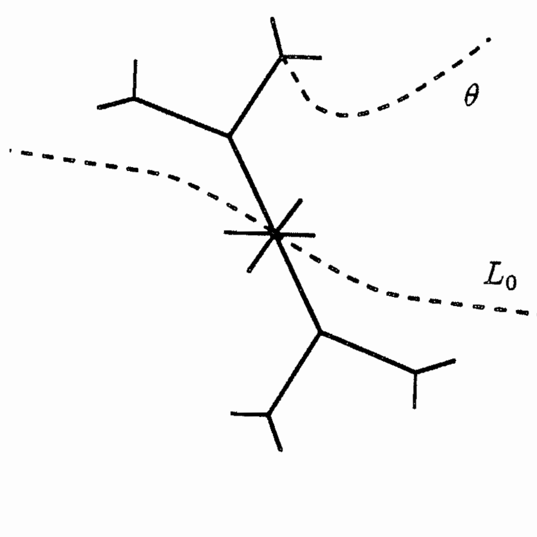

Figure 9.2. Definition of L_0 for $c \in \mathcal{D}_2$.

Theorem 9.2.1. *If a partition ω has the property that $\omega(t_1) = \omega(t_2) \Rightarrow \gamma(t_1)$ and $\gamma(t_2)$ are on the same side of L_0, then*

$$\hat{\omega}(t_1) = \hat{\omega}(t_2) \Rightarrow \gamma(t_1) = \gamma(t_2).$$

Proof 9.2.1.

As in theorem 2.2.5, we define a neighborhood U of J_c and a metric on U with respect to which P_c is expanding by a factor $\geq \rho > 1$.

Definition. A path in U, ν, joining z_0 to z_1 is called *admissible* if no lift under ${P_c}^{\circ k}$ of ν is forced to cross L_0 (i.e. for $k = 1, 2, \ldots$ and for all $\widetilde{z_0} \in {P_c}^{-k}(\{z_0\})$ there is a lift $\widetilde{\nu}$ of ν under ${P_c}^{\circ k}$ such that $\widetilde{\nu}$ begins at $\widetilde{z_0}$ and $\widetilde{\nu}$ ends at a point on the same side of L_0 as $\widetilde{z_0}$. Of course, once we have specified that $\widetilde{\nu}$ begins at $\widetilde{z_0}$, $\widetilde{\nu}$ is completely determined unless ν passes through the critical value.).

Lemma 9.2.1.1. *There exists $l_{\max}$ such that any two points in J_c can be joined by an admissible path of length less than or equal to $l_{\max}$.*

Before proving lemma 9.2.1.1, we show how it can be used to prove theorem 9.2.1.

Suppose $\hat{\omega}(t_1) = \hat{\omega}(t_2)$ but $\gamma(t_1) \neq \gamma(t_2)$. Let N be large enough so that

$$\left(\frac{1}{\rho}\right)^N l_{\max} < \inf\left\{l(v) \mid \nu \text{ joins } \gamma(t_1) \text{ to } \gamma(t_2)\right\}.$$

By claim 9.2.1.1 we can let ν_N be an admissible path of length $\leq l_{\max}$ joining ${P_c}^{\circ N}(\gamma(t_1))$ to ${P_c}^{\circ N}(\gamma(t_2))$. For $k = N, N-1, N-2, \ldots, 1$ we inductively define ν_{k-1} as follows. Having defined an admissible ν_k joining ${P_c}^{\circ k}(\gamma(t_1))$ to ${P_c}^{\circ k}(\gamma(t_2))$, we can let ν_{k-1} be the lift of ν_k beginning at ${P_c}^{\circ(k-1)}(\gamma(t_1))$ such that ν_{k-1} begins and ends on the same side of L_0. By hypothesis

$$\gamma(2^{k-1}t_1) = {P_c}^{\circ(k-1)}(\gamma(t_1))$$

and

$$\gamma(2^{k-1}t_2) = {P_c}^{\circ(k-1)}(\gamma(t_2))$$

are on the same side of L_0. So ν_{k-1} ends at ${P_c}^{\circ(k-1)}(\gamma(t_2))$. Since ν_k is admissible, so too is ν_{k-1}.

So $\nu_0 \in \{\nu \mid \nu \text{ joins } \gamma(t_1) \text{ to } \gamma(t_2)\}$, contradicting the fact that by the expansiveness of P_c,

$$l(\nu_0) \leq \left(\frac{1}{\rho}\right)^N l(\nu_n) \leq \left(\frac{1}{\rho}\right)^N l_{\max} < \inf\left\{l(v) \mid \nu \text{ joins } \gamma(t_1) \text{ to } \gamma(t_2)\right\}.$$

Proof 9.2.1.1.

Let $\gamma(t_1)$ and $\gamma(t_2)$ be two points in J_c with $t_1 < t_2$. By theorem 2.3.1we can let

$$\psi : \widehat{\mathbf{C}} - D \to \widehat{\mathbf{C}} - \overset{\circ}{K}_c$$

be the unique analytic map such that $\psi(\infty) = \infty$, $\psi'(\infty) = 1$, and $\psi(z^2) = f(\psi(z))$. Let R be such that $C := \psi(\partial D_R) \subset U$.

Consider the path from $\gamma(t_1)$ to $\gamma(t_2)$ given as follows. Follow $\mathcal{R}(K_c, t_1)$ out from J_c until hitting C. Follow C counter-clockwise until hitting $\mathcal{R}(K_c, t_2)$. Then follow $\mathcal{R}(K_c, t_2)$ into J_c. This path may fail to be admissible precisely because it crosses

$$\bigcup_{j=1}^{\infty} P_c^{\circ j}(L_0).$$

But

$$\bigcup_{j=1}^{\infty} P_c^{\circ j}(L_0)$$

is a finite union of sets of the form $\mathcal{R}(K_c, 2^j\theta)$ where $j \geq 0$. The path crosses each $\mathcal{R}(K_c, 2^j\theta)$ at most once. We remove these finitely many crossings by adding the following detours.

If $c \in \mathcal{D}_2$, stop at the point of crossing and then follow $\mathcal{R}(K_c, 2^j\theta)$ into J_c (see figure 9.3). Then follow $\mathcal{R}(K_c, 2^j\theta)$ back out to the point of crossing (see figure 9.4).

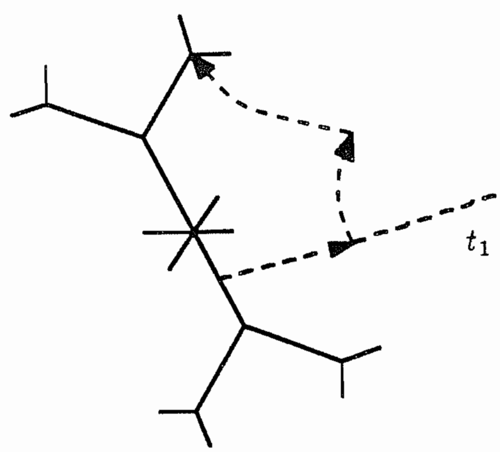

Figure 9.3. Construction of admissible path for $c \in \mathcal{D}_2$, part (a).

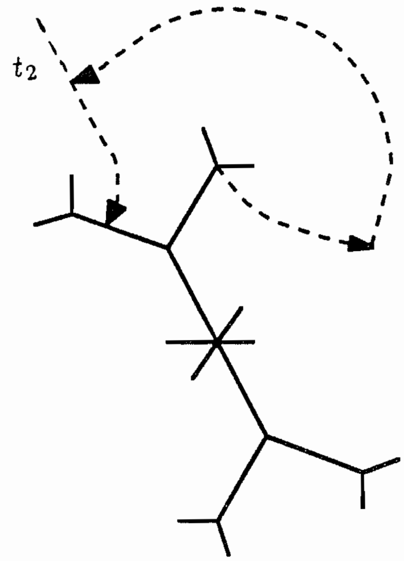

Figure 9.4. Construction of admissible path for $c \in \mathcal{D}_2$, part (b).

For $c \in \mathcal{D}_0$, as for $c \in \mathcal{D}_2$, follow $\mathcal{R}(K_c, 2^j\theta)$ into J_c. But then follow $\mathcal{R}_{j+1}(0)$ in towards $P_c^{\circ(j+1)}(0)$, stopping at some point $\psi_{j+1}(r)$ where r is such that $\psi_k(\partial D_r) \subset U$ for $k = 1, 2, \ldots, n$. Follow $\psi_{j+1}(\partial D_r)$ once around clockwise (see figure 9.5). Then follow $\mathcal{R}_{j+1}(0)$ back out to J_c and follow $\mathcal{R}(K_c, 2^j\theta)$ back out to the point of crossing (see figure 9.6).

Clearly the original path is of finite length. The detours are of finite length by proposition 2.3.3. The detours introduce no new crossings, so the new path is admissible and of finite length. We can let $l_{\max}$ be the sum of the lengths of all the possible detours plus a bound on the length of the original path.

End 9.2.1.1 and 9.2.1.

Proof 9.1.3 ($\Rightarrow$). The partition ϵ satisfies the hypothesis of theorem 9.2.1

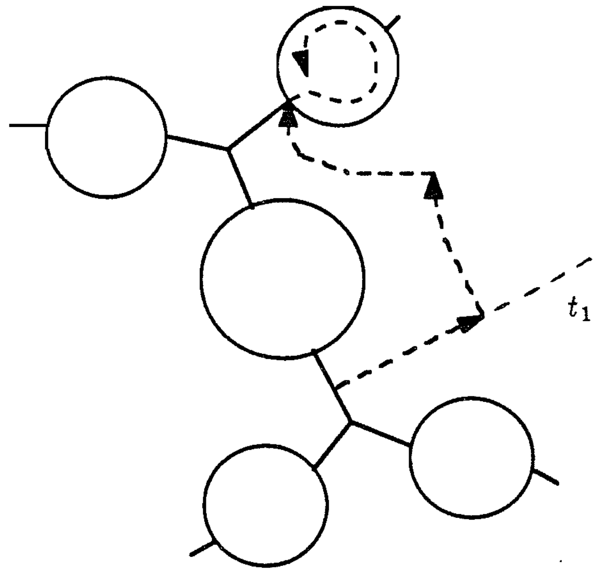

Figure 9.5. Construction of admissible path for $c \in \mathcal{D}_0$, part (a).

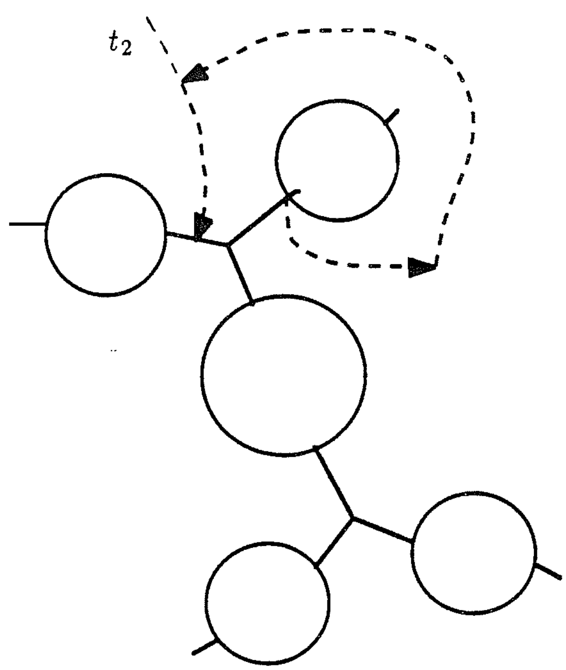

Figure 9.6. Construction of admissible path for $c \in \mathcal{D}_0$, part (b).

because external rays do not cross. **End 9.1.3** ($\Rightarrow$).

Proof 9.1.3 ($\Leftarrow$).

Obviously we are done if we can show that

$$\gamma(t_1) = \gamma(t_2) \Rightarrow \epsilon(t_1) = \epsilon(t_2).$$

Case. *$\gamma(t_j) \in \mathbf{C} - L_0$:*

If $\gamma(t_j)$ is in the same component of $\mathbf{C} - L_0$ as is $\gamma(0)$, then $\epsilon(t_j) = 2$. Otherwise $\epsilon(t_j) = 1$.

Case. *$\gamma(t_j) \in L_0$ and $c \in \mathcal{D}_2$:*

By definition $\epsilon(t_j) = 0$.

Case. *$\gamma(t_j) \in L_0$ and $c \in \mathcal{D}_0$:*

We assume $\theta' > \theta$ and $\gamma(t_j) = \gamma(\theta_2)$. The other three cases are proved in the same way.

Let $\theta_1' := \frac{\theta'}{2}$ and $\theta_2' := \frac{\theta'}{2} + \frac{1}{2}$. We have $t \in]\theta, \theta'[\Rightarrow \gamma(t) \neq \gamma(\theta)$. So

$$t \in]\theta_2, \theta_2'[\Rightarrow \gamma(t) \neq \gamma(\theta_2). \tag{9.1}$$

Now, $\gamma(\theta_2') = \gamma(\theta_1)$, so

$$t \in [\theta_2', 1] \cup [0, \theta_1] \Rightarrow \gamma(t) \neq \gamma(\theta_2). \tag{9.2}$$

So (9.1) and (9.2) give $t \in T_2 \Rightarrow \gamma(t) \neq \gamma(\theta_2)$. So t_1 and t_2 are in T_1.

End 9.1.3 ($\Leftarrow$).

Proof 9.1.1. In light of theorem 9.1.3,

$$T_0 = \{t \in \mathbf{Q}/\mathbf{Z} \mid \text{ dynamic denominator of } 2t = q_\theta \text{ and } \hat{\epsilon}(2t) = \hat{\epsilon}(\theta)\}\,.$$

The dynamic denominator of any $t \in T_0$ is $2q_\theta$, so

$$\{t \in \mathbf{Q}/\mathbf{Z} \mid \text{ dynamic denominator of } 2t = q_\theta\} \cap T_0 = \emptyset.$$

So

$$T_0 = \{t \in \mathbf{Q}/\mathbf{Z} \mid \text{ dynamic denominator of } 2t = q_\theta \text{ and } \hat{\epsilon}(2t) = \hat{\epsilon}(\theta)\}$$

$$= \{t \in \mathbf{Q}/\mathbf{Z} \mid \text{ dynamic denominator of } 2t = q_\theta \text{ and } \hat{\eta}(2t) = \hat{\eta}(\theta)\}\,.$$

End 9.1.1.

Proof 9.1.2.

Suppose $\theta' > \theta$. Then $\eta_+ = \epsilon$. So by theorem 9.1.3, $\theta' \in T_+ - \{\theta\}$. To show that $T_- - \{\theta\} = \emptyset$ we note that η_- satisfies the hypothesis of theorem 9.2.1, so

$$T_- \subset \{t \in \mathbf{T} \mid \gamma(t) = \gamma(\theta)\}\,.$$

So by theorem 9.1.3, if n is the period of θ and $\gamma(t) = \gamma(\theta)$, then $\epsilon(2^{n-1}\theta) = \epsilon(2^{n-1}t)$. But $2^{n-1}t \notin \{\theta_1, \theta_2\}$ and $2^{n-1}\theta \in \{\theta_1, \theta_2\}$, so

$$\eta_-(2^{n-1}t) = \epsilon(2^{n-1}t) = \epsilon(2^{n-1}\theta) \neq \eta_-(2^{n-1}).$$

So in fact $T_- = \{\theta\}$.

Similarly, $\theta' < \theta \Rightarrow (\theta' \in T_- - \{\theta\}$ and $T_+ = \{\theta\})$.

End 9.1.2.

Chapter 10. Stars.

§10.1. *Addresses.*

Definition. Let h be a star and let p be an inverse image of the fixed point α of h. We denote by σ_p the clockwise-around-p permutation of the internal rays of K_h incident upon p.

Claim 10.1.1. *If $q \in J_h$ has more than one external angle, then q is an inverse image of α.*

Proof 10.1.1. By fact 2.3.6, some forward image of q is in the Hubbard tree of h. So we are done by claim 2.3.7. **End 10.1.1.**

Definition 10.1.2.

Let h be the star corresponding to the exterior ray of M at angle θ. Let p be a point in J_h. We define the *address of* p as follows.

Let $[\alpha, p]_{K_h}$ be the regulated arc joining the fixed point α of h to p, and let

$$P := [\alpha, p]_{K_h} \cap J_h - \{p\}\,.$$

Claim 10.1.2.1.

$$P \subset \bigcup_{n=0}^{\infty} h^{-n}(\{\alpha\}).$$

Proof 10.1.2.1. Let q be in P. If $q = \alpha$ we are done. So by claim 2.3.7, q has at least two external angles. So we are done by claim 10.1.1. **End 10.1.2.1.**

Claim 10.1.2.2. *P is discrete.*

Proof 10.1.2.2. Let q be in P. By claim 10.1.2.1 there is a neighborhood U of q and an $n \geq 0$ such that $h^{\circ n}$ maps U homeomorphically onto a neighborhood of α. Then $h^{\circ n}$ maps $U \cap [\alpha, p]_{K_h}$ to a regulated arc which intersects α. Then we are done by proposition 2.4.3. **End 10.1.2.2.**

It follows from claim 10.1.2.2 that we can set

$$P = \begin{cases} \{p_0, p_1, p_2, \ldots\} & \text{if } P \text{ is infinite;} \\ \{p_0, p_1, p_2, \ldots, p_{N-1}\} & \text{if } \#P = N \end{cases}$$

so that p_i is closer in $[\alpha, p]_{K_h}$ to α than is p_j if and only if $i < j$. If $\#P = N < \infty$, we let $p_N := p$. Since P is discrete, for $j = 1, 2, 3, \ldots$ if P is infinite and for $j = 1, 2, 3, \ldots, N$ if $\#P = N$, there is a component W_j of $\overset{\circ}{K}_h$ such that p_{j-1} is on the boundary of W_j at some internal angle s_j. Let W_0 be the component of $\overset{\circ}{K}_h$ containing the critical point.

Let the dynamic denominator of θ be $2^n - 1$. Let $m_j \in \mathbf{Z}/n\mathbf{Z}$ be such that

$$\mathcal{R}(W_{j+1}, 0) = \sigma_{p_i}^{\circ m_j}(\mathcal{R}(W_j, s_j)).$$

If $p = \alpha$, then the address of p is empty. If p is on the boundary of some W_j, then the address of p is

$$(m_0, s_1, m_1, s_2, m_2, \ldots, s_{j-1}, m_{j-1}, s_j).$$

Otherwise, the address of p is

$$(m_0, s_1, m_1, s_2, m_2, \ldots).$$

End 10.1.2.

Claim. *The map* $p \mapsto \text{address}(p)$ *is injective.*

Proof. Let p and p' be two distinct points in J_h. As in the section on properties of regulated arcs on page 16 of [DH1], there is a $q \in K_h - \{p, p'\}$ with

$$[\alpha, p]_{K_h} \cap [\alpha, p']_{K_h} = [\alpha, q]_{K_h}.$$

So q is either in P or the center of one of the W_j's. In either case, the claim follows easily. **End.**

Claim 10.1.3. *If* p *has infinite address* $(m_0, s_1, m_1, \ldots)$, *then all the* s_j *are diadic.*

Proof 10.1.3. Since α is at interior angle 0 for every component of $\overset{\circ}{K}_h$ with α on its boundary, all inverse images if α are at diadic internal angles. **End 10.1.3.**

§**10.2.** *Address dynamics.*

Definition. Let h be the star corresponding to the external ray of M at angle θ, and let the dynamic denominator of θ be $2^n - 1$. Then $m \in \mathbf{Z}/n\mathbf{Z}$ is the *rotation of* h *at* α if

$$\mathcal{R}(h(W_0), 0) = \sigma_\alpha^{\circ m}(\mathcal{R}(W_0, 0)),$$

where W_0 is the component of $\overset{\circ}{K}_h$ containing the critical point of h.

Claim 10.2.1. *Let h be a star with rotation m at α. Let p be a point in J_h with infinite address*

$$(m_0, s_1, m_1, s_2, m_2, \ldots).$$

If $m_0 \neq 0$, then

$$\text{address}(h(p)) = (m_0 + m, s_1, m_1, s_2, m_2, \ldots).$$

If $m_0 = 0$ and $s_1 \neq 1/2$, then

$$\text{address}(h(p)) = (m, 2s_1, m_1, s_2, m_2, \ldots).$$

If $m_0 = 0$ and $s_1 = 1/2$, then

$$\text{address}(h(p)) = (m + m_1, s_2, m_2, s_3, m_3, \ldots).$$

The proof if claim 10.2.1 is trivial.

§**10.3.** *Eventually periodic addresses.*

Definition. An infinite address $(m_0, s_1, m_1, \ldots)$ is *eventually periodic* if there is a ν and a κ such that for $j \geq \nu$, $m_{j+\kappa} = m_j$ and $s_{j+\kappa} = s_j$. The smallest such ν is called the *onset of periodicity* and the smallest such κ is called the *period of the address.*

Caution. The period of a point and the period of the address of a point need not be the same.

Proposition 10.3.1. *Let h be the star corresponding to the external ray of M at angle θ. Let p be a point in J_h with eventually periodic address $(m_0, s_1, m_1, \ldots)$. Let κ be the period of the address of p and let ν be the onset of periodicity. Then*

1) p has exactly one external angle, θ_p,

2) θ_p is rational (i.e. p is eventually periodic), and

3) there is an algorithm to compute θ_p given θ, κ, ν, and $(m_0, s_1, m_1, \ldots, s_\nu, m_\nu)$.

Furthermore, if $\nu = 1$, then the dynamic denominator of θ_p is odd (i.e. p is periodic).

Proof 10.3.1.

It is obvious from claim 10.2.1 that p is either periodic or eventually periodic, that p is periodic of $\nu = 1$, and that no forward image of p is α. Since no forward image of p is α, by claim 10.1.1, p only has one external angle.

From θ one can compute m, the rotation of h at α, since m is such that

$$2\theta = \sigma^{\circ m}(\theta)$$

where σ is the cyclic permutation of

$$\Theta := \{\theta, 2\theta, 2^2\theta, \ldots\}$$

which carries each angle in Θ to the next smallest one. It is obvious from claim 10.2.1 that one can compute the dynamic denominator of θ_p. Now, given any j, it is possible to compute from $(m_0, s_1, s_2, \ldots, s_j, m_j)$ in which connected component of

$$\mathbf{T} - \bigcup_{i=0}^{j} (t \mapsto 2t)^{-1}(\Theta)$$

θ_p must lie. But for large enough j, there will be at most one angle with dynamic denominator equal to that of θ_p in each component.

End 10.3.1.

Chapter 11. Shared Matings.

§11.1. *Statements.*

Theorem 11.1.1. *Let f_0 and f_1 be quadratic polynomials with*

$$f_0 \in \mathcal{D}_0 \qquad \text{and} \qquad f_1 \in (\mathcal{D}_0 \cup \mathcal{D}_2) \cap L$$

where L is some limb of M. Let θ_0' and θ_0'' be the angles of external rays of M corresponding to f_0. Let θ_1 be the angle of an exterior ray of M corresponding to f_1, and let θ be the angle of one of the two external rays of M corresponding to L. Let $f : S_f^2 \to S_f^2$ be the non-intimate mating of f_0 with f_1. If

1) f is topologically equivalent to a rational function,

2) $\{-\theta_0', -\theta_0''\} \cap \{2^n\theta \mid n \geq 0\} \neq \emptyset$, and

3) $\theta \notin \{2^n\theta_1 \mid n \geq 1\}$,

then f is also topologically equivalent to the non-intimate mating of the star of L with some quadratic polynomial.

Complement 11.1.2. *Conjecture 6.6.1 implies that if θ_1 is diadic, then up to conjugacy, there is only one quadratic polynomial h_1 such that f is topologically* *equivalent to the non-intimate mating of the star of L with h_1, and there is an algorithm to find h_1.*

Complement 11.1.3. *Let θ_0 be the unique element in $\{\theta_0', \theta_0''\} \cap \{-2^n\theta \mid n \geq 0\}$. Let $2^N - 1$ be the dynamic denominator of θ_0 and θ. Let $t_0 := 2^{N-1}\theta_0$ and let $t_0' := t_0 + 1/2$. Let T be the component of $\mathbf{T} - \{t_0, t_0'\}$ containing $\{0\}$. Inductively define t_j by*

$$t_{j+1} := \{t_j/2, (t_j/2) + (1/2)\} \cap (T \cup \{t_0\}).$$

There is a k_0 such that

$$-t_{k_0} \notin \{\theta, 2\theta, 2^2\theta, \ldots, 2^{N-1}\theta\}.$$

Let k be the smallest such k_0. If $k > 1$, then conjecture 6.6.1 implies that if we fix f_0 and let f_1 vary through all diadic points in L, then the h_1 produced by the algorithm of complement 11.1.2 all lie in the same limb of M.

§11.2. *Proof of theorem 11.1.1.*

Let θ_0 be the unique element in $\{\theta_0', \theta_0''\} \cap \{-2^n\theta \mid n \geq 0\}$ (see figure 11.1). Let $x_0 \in S_f^2$ be the critical point of f_0 and let $x_i := f^{\circ i}(x_0)$. Let U_i be the connected component of the interior of K_0 containing $\{x_i\}$. Let $\mathcal{R}_i$ be the ray at angle zero in the interior of U_i. By definition, $\mathcal{R}_i$ is without endpoints. For $i = 0, 1, 2, \ldots$ let r_i be the endpoint of $\mathcal{R}_f(2^i\theta_0)$ in K_0, let s_i be the endpoint of $\mathcal{R}_f(2^i\theta_0)$ in K_1, and let

$$S_i := \mathcal{R}_f(2^i\theta_0) - \{r_i, s_i\}.$$

Let

$$e_i := \mathcal{R}_i \cup \{r_i\} \cup S_i.$$

So e_i is homeomorphic to an open interval and x_i and s_i are in its closure. Let α_1 be the fixed point α of f_1.

By fact 2.4.2 and the fact that $-\theta_0 \in \{2^n\theta \mid n \geq 0\}$, we get that $s_i = \alpha_1$.

Claim 11.2.1. *The e_i are disjoint.*

Proof 11.2.1.

Clearly the $\mathcal{R}_i$ and the S_i are disjoint. So suppose $j \neq k$ yet $r_j = r_k$.

One possibility is that $r_n = r_j$ for all n. In that case, f_0 and f_1 are in conjugate limbs of M, and that together with the fact that the non-intimate mating of f_0 with f_1 is topologically equivalent to a rational function contradicts proposition 6.4.1.

The other possibility is that there exists an l with $r_l \neq r_j$ (see figure 11.2). Let σ be the clockwise-around-α_1 permutation of the S_i. Without loss of generality, we assume that $S_l = \sigma(S_k)$. If $S_m = \sigma(S_j)$, then $r_l = r_m$ because the iterate of f which carries S_k onto S_j will carry S_l onto S_m. But then S_m would have to cross either S_j or S_k.

End 11.2.1.

Let

$$X_H := \bigcup_{i=0}^{\infty}(\{x_i\} \cup e_i \cup \{\alpha_1\}).$$

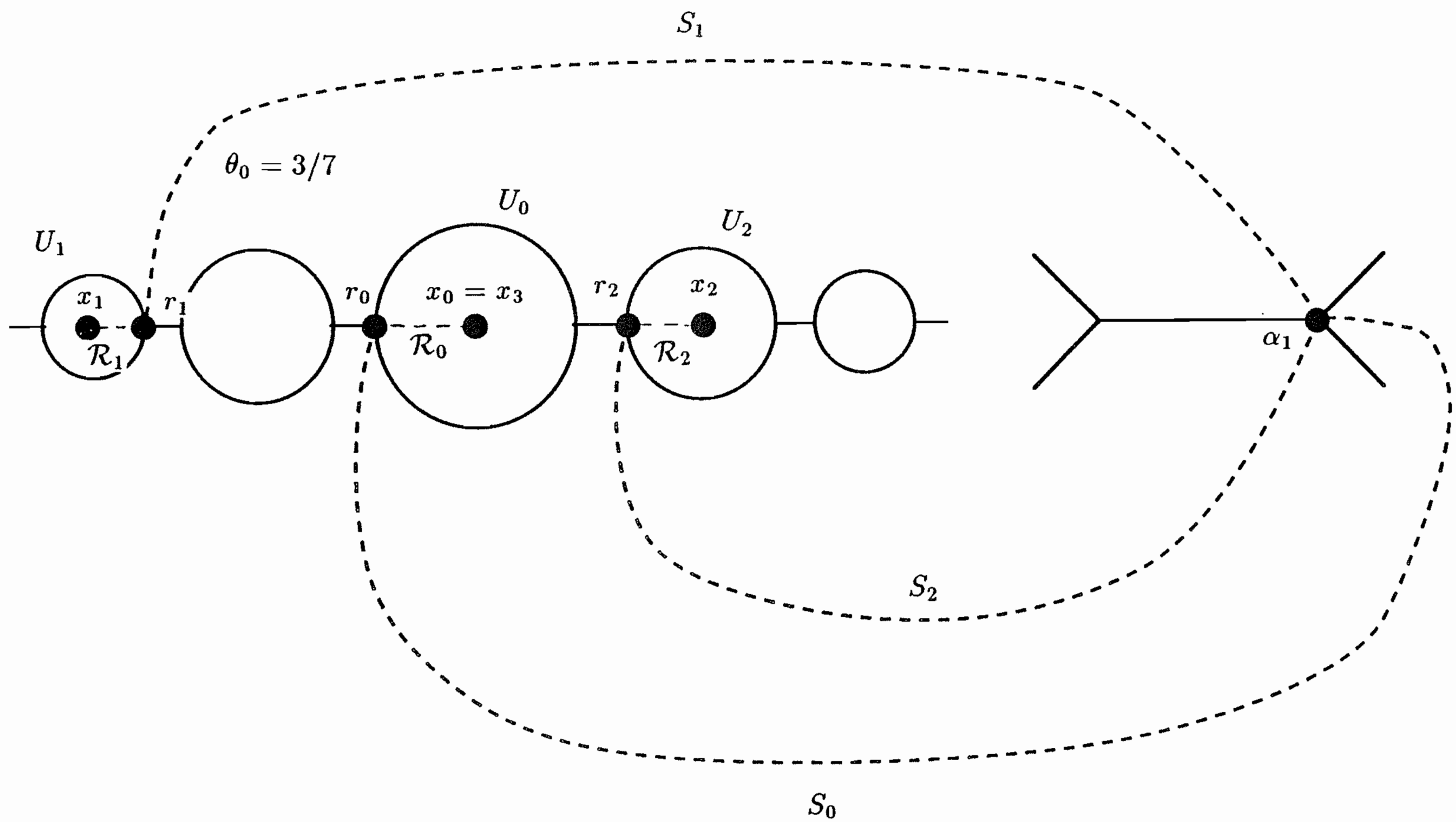

Figure 11.1. Existence of star in shared mating.

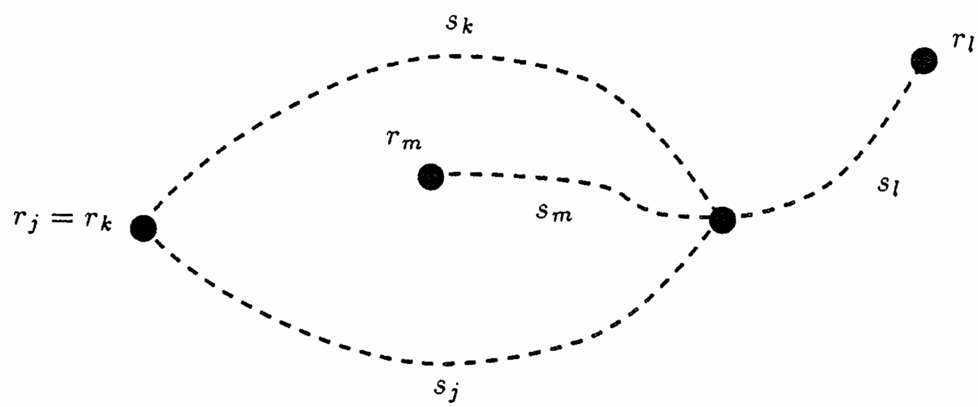

Figure 11.2. Why there does not exist an l with $r_l \neq r_j$.

X_H is a star with center at α_1, endpoints equal to the x_i, and edges in $S^2_f - P_f$. $f^{-1}(X_H)$ is X_H together with another star which we shall call $-X_H$. Let $-\alpha_1$ be the inverse image of α_1 not equal to α_1. Then

$$-X_H \cap X_H = \{x_0\},$$

the endpoints of $-X_H$ are not in P_f, the center of $-X_H$ is $-\alpha_1$, and the edges of $-X_H$ are in $S^2_f - P_f$. Since by hypothesis,

$$\theta \notin \{2^n\theta_1 \mid n \geq 1\},$$

we get that $-\alpha_1 \notin P_f$. So $-X_H$ is contractible (rel P_f) to $\{x_0\}$.

So if we let $K := X_H$ and let λ be a simple closed path parameterizing the boundary of a sufficiently slight fattening of X_H, then λ is an equator of f in the complement of K. So by theorem 7.2.1 (Thurston's mating criterion), $f_{\lambda,K,i}$ is topologically equivalent some quadratic polynomial h_i for $i = 0, 1$ and f is topologically equivalent to the non-intimate mating of h_0 with h_1.

We have only left to show that h_0 is the star of L. If we define the embedding graph H to have topological space X_H, vertices α_1 and the x_i, and edges the e_i, then (H, f, x_0) is the quadratic tree of the star of L (see fact 2.4.4). So we are done by theorem 5.2.1.

§**11.3.** *Proof of complements 11.1.2 and 11.1.3.*

Notation. Let h_0 be the star of L and suppose f is topologically equivalent to

$$h : S^2_h \to S^2_h$$

which is the non-intimate mating of h_0 with h_1.

Notation. Let $g : \mathbf{P}^1 \to \mathbf{P}^1$ be the rational function topologically equivalent to f and h. Let $\phi_f : S^2_f \to \mathbf{P}^1$ and $\phi_h : S^2_h \to \mathbf{P}^1$ be the maps given by theorem 6.1.1.

Notation. Let $K_{f,0} \subset S^2_f$ be K_{f_0}. Let $\alpha_{f,0}$ and $\beta_{f,0}$ be the fixed points α and β respectively of $K_{f,0}$. Let $\gamma_{f,0}$ be the Carathéodory loop of $K_{f,0}$. Similarly define $K_{f,1}$, $\alpha_{f,1}$, $\beta_{f,1}$, $\gamma_{f,1}$, $K_{h,0}$, $\alpha_{h,0}$, $\beta_{h,0}$, $\gamma_{h,0}$, $K_{h,1}$, $\alpha_{h,1}$, $\beta_{h,1}$, and $\gamma_{h,1}$.

Lemma 11.3.1. *Let*

$$U^1, U^2, \ldots, U^N$$

be N distinct components of $\overset{\circ}{K}_{f,1}$. For $n = 1, 2, \ldots, N$ let

$$V^n := \phi_h^{-1}(\phi_f(U^n)).$$

So

$$V^1, V^2, \ldots, V^N$$

are N distinct components of $\overset{\circ}{K}_{h,0}$. For $n = 1, 2, \ldots, N$ let $r_n \in \mathbf{T}$ be such that

$$\gamma_{f,0}(r_n) \in \bar{U}^n,$$

let $s_n \in \mathbf{T}$ be the internal angle in $\bar{U}^n$ of $\gamma_{f,0}(r_n)$, and let p_n be the point in ∂V^n of internal angle s_n in V^n. Suppose there exists a point

$$a \in \bigcup_{i=0}^{\infty} f^{-i}(\alpha_{f,1})$$

such that $\gamma_{f,1}(-r_n) = a$ for $n = 1, 2, \ldots, N$. Then $p_1 = p_2 = \ldots = p_N$ and the cyclic permutation of $\{1, 2, \ldots, N\}$ induced by the clockwise-around-a permutation of

$$\{\mathcal{R}_f(r_n) \mid n = 1, 2, \ldots, N\}$$

is the same as that induced by the clockwise-around-p_1 permutation of

$$\{\mathcal{R}(V^n, s_n) \mid n = 1, 2, \ldots, N\}.$$

Proof 11.3.1.

By theorem 6.1.1, $\phi_f(a)$ is in the closure of $\phi_f(U'), \phi_f(U^2), \ldots, \phi_f(U^N)$, which are N distinct components of $\mathbf{P}^1 - J_g$. By conjecture 6.6.1 $\phi_f(a)$ is not in the closure of any other components of $\mathbf{P}^1 - J_g$. So the N distinct components of $\overset{\circ}{K}_{h,0}$ having p_1 in their closure map by ϕ_h to $\phi_f(U^1), \phi_f(U^2), \ldots, \phi_f(U^N)$. So since ϕ_h and ϕ_f preserve internal angles, each V^n has p_1 in its boundary at internal angle s_n.

The cyclic permutation of $\{1, 2, \ldots, N\}$ induced by the clockwise-around-$\phi_f(a)$ permutation of the set of internal rays at angle s_n in $\phi_f(U^n)$ is the same as the two mentioned in the statement of the lemma.

End 11.3.1.

Let θ_0 be the unique element in $\{\theta_0', \theta_0''\} \cap \{-2^n\theta \mid n \geq 0\}$.

Claim 11.3.2. *There is an angle $\theta_\beta \in \mathbf{Q}/\mathbf{Z}$ such that*

$$K_{h,0} \cap \phi_h^{-1}(\phi_f(\beta_{f,0})) = \{\gamma_{h,0}(2^j\theta_\beta) \mid j = 0, 1, 2, \ldots\}$$

and an algorithm to compute θ_β which has only θ_0 and θ as input.

Proof 11.3.2.

Let $2^N - 1$ be the dynamic denominator of θ_0 and θ. Let $t_0 := 2^{N-1}\theta_0$ and let $t_0' := t_0 + (1/2)$. Let T be the component of $\mathbf{T} - \{t_0, t_0'\}$ containing $\{0\}$. Inductively define t_j and t_j' by

$$t_{j+1} := \{t_j/2, (t_j/2) + (1/2)\} \cap (T \cup \{t_0\})$$

and

$$t_{j+1}' := \{t_j'/2, (t_j'/2) + (1/2)\} \cap (T \cup \{t_0\})$$

(see figure 11.3).

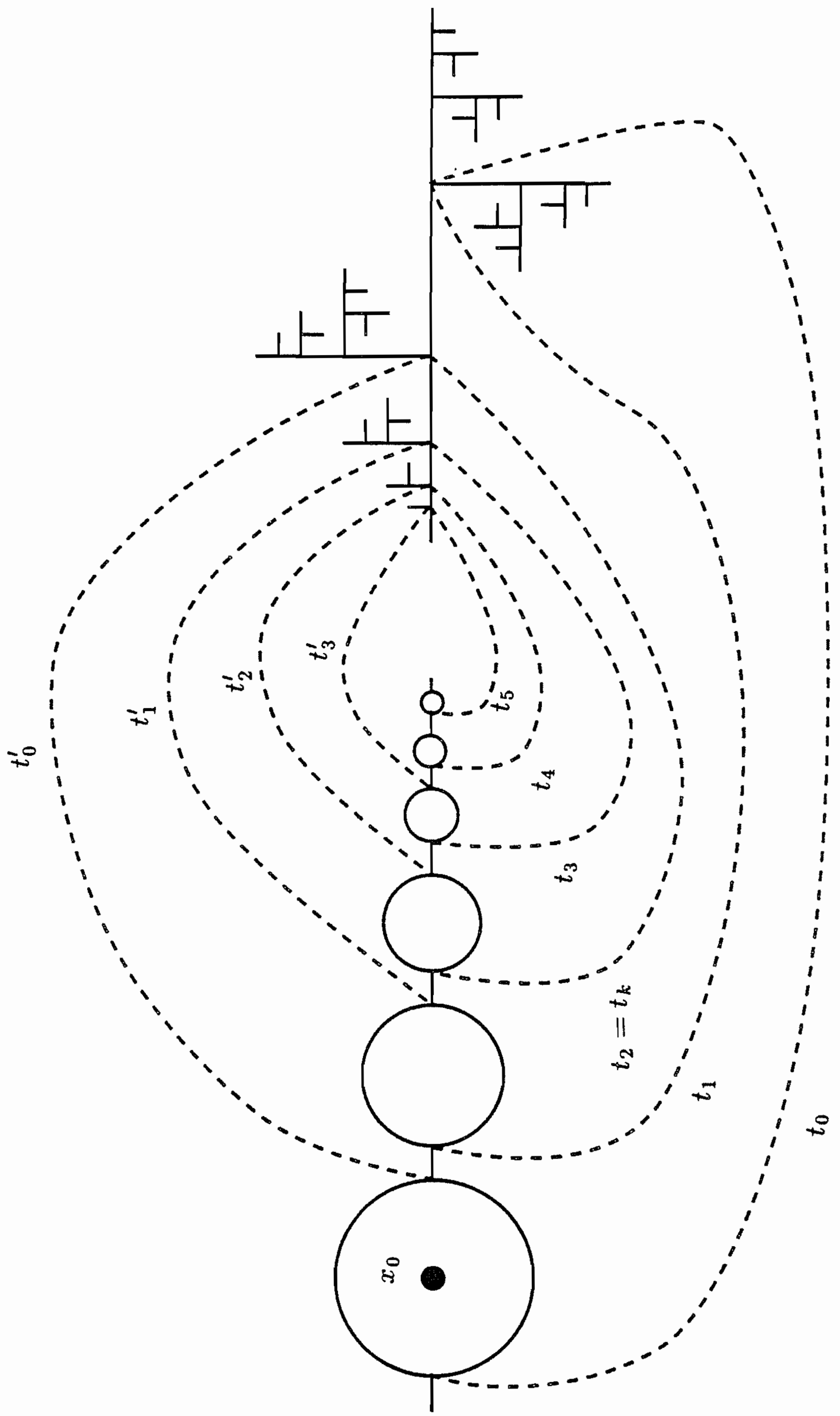

Figure 11.3. Determination of θ_β.

Claim. *There is a $k_0 < N$ such that*

$$-t_{k_0} \notin \{\theta, 2\theta, 2^2\theta, \ldots, 2^{N-1}\theta\}.$$

Proof. Suppose not. Then since $\theta_0 = 2t_0$ is an inverse image of t_0 under $t \mapsto 2t$,

$$2t_0 \in T \cup \{t_0\}.$$

This is only possible if $t_0 = 0$ or $t_0 = 1/2$. But t_0 has an odd dynamic denominator.

End.

We let k be the smallest such k_0 and note that

$$1 \leq k < N.$$

This is the k mentioned in the statement of complement 11.1.3.

Let $\alpha^j := \gamma_{f,1}(-t_j)$. So $\alpha^0 = \alpha_{f,1}$ and $f(\alpha^{j+1}) = \alpha^j$.

For $j = 0, 1, 2, \ldots, k-1$

$$\gamma_{f,1}(-t_j) = \alpha^0, \tag{11.1}$$

and for $j \geq 0$,

$$\gamma_{f,1}(-t'_j) = \gamma_{f,1}(-t_{j+k}) = \alpha^{j+k+1}. \tag{11.2}$$

Since $t_j \to 0$ as $j \to \infty$,

$$\alpha^j \to \beta_{f,1} \quad \text{as} \quad j \to \infty. \tag{11.3}$$

For each j, let ρ_j be the clockwise-around-α^j permutation of

$$\left\{\mathcal{R}_f(-t) \mid t \in \gamma_{f,1}^{-1}(\alpha^j)\right\}.$$

f preserves this clockwise ordering. That is, for $j = 1, 2, 3, \ldots$

$$\rho_j^{\circ l}(\mathcal{R}(t)) = \mathcal{R}(s) \quad \Rightarrow \quad \rho_{j-1}^{\circ l}(\mathcal{R}(2t)) = \mathcal{R}(2s). \tag{11.4}$$

So there is a unique m_1 such that

$$\mathcal{R}_f(t_{j+k}) = \rho_{j+1}^{\circ m_1}(\mathcal{R}_f(t'_j)) \tag{11.5}$$

for $j = 0, 1, 2, \ldots$. Let m be the rotation of h_0 at α (see section 10.2). Note that for $i = 0, 1, 2, \ldots, k-1$,

$$\rho_0^{\circ(-im)}(\mathcal{R}_f(t_0)) = \mathcal{R}_f(t_i) \tag{11.6}$$

(see figure 11.3).

For $i = 0, 1, 2, \ldots, k-1$ let ρ_i be the point in J_{h_0} with address

$$(-im, 1/2, m_1, 1/2, m_1, \ldots).$$

Lemma 11.3.1 together with (11.1), (11.2), (11.5), (11.6), and (11.3) give us that

$$p_i \in K_{h,0} \cap \phi_h^{-1}(\phi_f(\beta_{f,0})). \tag{11.7}$$

Claim 11.3.2.1. *$h_0(p_i) = p_{i-1 (\text{mod } k)}$ for $i = 0, 1, 2, \ldots, k-1$.*

Proof 11.3.2.1.

By (11.5),

$$\rho_1^{\circ m_1}(\mathcal{R}_f(t'_0)) = \mathcal{R}_f(t_k).$$

Also,

$$2t_0' = \theta_0 \qquad \text{and} \qquad 2t_k = t_{k-1}.$$

So by (11.4),

$$\rho_0^{\circ m_1}(\mathcal{R}_f(\theta_0)) = \mathcal{R}_f(t_{k-1}). \tag{11.8}$$

Since $2t_0 = \theta_0$, by the definition of m,

$$\rho_0^{\circ m}(\mathcal{R}_f(t_0)) = \mathcal{R}_f(\theta_0). \tag{11.9}$$

Equations (11.8) and (11.9) give that

$$\rho_0^{\circ(m+m_1)}(\mathcal{R}_f(t_0)) = \mathcal{R}_f(t_{k-1}). \tag{11.10}$$

By (11.6),

$$\rho_0^{\circ(-(k-1)m)}(\mathcal{R}_f(t_0)) = \mathcal{R}_f(t_{k-1}). \tag{11.11}$$

So by (11.10) and (11.11),

$$-(k-1)m = m + m_1 \pmod{N}.$$

Now

$$\text{address}(p_0) = (0, 1/2, m_1, 1/2, m_1, \ldots).$$

So by claim 10.2.1,

$$\begin{aligned}\text{address}(h_0(p_0)) &= (m + m_1, 1/2, m_1, 1/2, m_1, \ldots) \\ &= (-(k-1)m, 1/2, m_1, 1/2, m_1, \ldots) \\ &= \text{address}(p_{k-1}).\end{aligned}$$

For $i = 1, 2, \ldots, k-1$, by claim 10.2.1,

$$\begin{aligned}\text{address}(h_0(p_i)) &= (-im + m, 1/2, m_1, 1/2, m_1, \ldots) \\ &= \text{address}(p_{i-1}).\end{aligned}$$

End 11.3.2.1.

By 1) of Proposition 10.3.1, $\gamma_{h,0}^{-1}(p_0)$ only has one element which we shall call θ_β. By claim 11.3.2.1, the dynamic denominator of θ_β is $2^k - 1$ and

$$-2^i\theta_\beta = -\gamma_{h,0}^{-1}(p_{-i(\text{mod } k)}).$$

Claim 11.3.2.2. *There is a single point ζ in*

$$K_{h,1} \cap \phi_h^{-1}(\phi_f(\beta_{f,0}))$$

and ζ has exactly k external angles which are $-2^i\theta_\beta$ for $i = 0, 1, 2, \ldots, k-1$.

Proof 11.3.2.2.

Let

$$Z_1 := K_{h,1} \cap \phi_h^{-1}(\phi_f(\beta_{f,0})),$$

and suppose there were more than one point in Z_1. Let

$$Z_0 := K_{h,0} \cap \phi_h^{-1}(\phi_f(\beta_{f,0})).$$

Since there is more than one point in Z_1, by conjecture 6.6.1 at least one point in Z_0 would have to have more than one external angle. Since h_0 is a star, the only points in $J_{h,0}$ having more than one external angle have dynamic denominators of the form $2^l(2^N-1)$ (see claim 10.1.1). By claim 11.3.2 and (11.7), there are points in Z_0 having external with dynamic denominator $2^k - 1$. By conjecture 6.6.1, all points in Z_0 have external angles with the same dynamic denominator. So one point in Z_0 has an external angle with dynamic denominator of the form $2^l(2^N-1)$ and of the form $2^k - 1$. But $k < N$.

So there is only one point ζ in Z_1 and ζ is a fixed point of h_1. Angle doubling acts transitively on the external angles of a fixed point (Fact 2.4.2) and $-\theta_\beta$ is one of the external angles of ζ.

End 11.3.2.2.

So

$$K_{h,0} \cap \phi_h^{-1}(\phi_f(\beta_{f,0})) = \left\{\gamma_{h,0}(2^i\theta_\beta) \mid i = 0, 1, 2, \ldots\right\}.$$

By Proposition 10.3.1, there is an algorithm to compute θ_β having only θ_0 and θ as input because that is enough to calculate the address of p_0.

End 11.3.2.

If $k \neq 1$, then ζ must be the fixed point α of h_1, but the sets of angles incident upon the fixed points α of two polynomials in M are the same if and only if the two polynomials are in the same limb of M (fact 2.4.2).

We have only left to describe the algorithm to find h_1.

Let $\xi : \mathbf{T} \to \mathbf{T}$ be given by $\xi(t) = 2t$. Let

$$\xi_0 := \left(\xi\Big|_{T\cup\{t_0\}}\right)^{-1}$$

and let

$$\xi_1 := \left(\xi\Big|_{\mathbf{T}-(T\cup\{t_0\})}\right)^{-1}.$$

Let $2^{N_1} - 1$ be the dynamic denominator of θ_1. Let S be the component of

$$\xi^{-N_1}(T \cup \{t_0\})$$

containing $-\theta_1$. For $i = 0, 1, 2, \ldots, k-1$ let s'_i be the unique element of $\zeta^{-N_1}(\{t'_i\})$ in S. Since the address of $\gamma_{h,0}(s'_i)$ is finite, we can calculate it by calculating the angles incident upon all the points in

$$\bigcup_{n=0}^{N_1+k} f_1^{-n}(\alpha_{f,1}).$$

This we can do by chapter 9.

If

$$(m_{0,i}, s_{1,i}, m_{1,i}, \ldots, s_{l,i}, m_{l,i}, 1/2)$$

is the address of $\gamma_{h,0}(s'_i)$, let q_i be the point in $J_{h,0}$ with address

$$(m_{0,i}, s_{1,i}, m_{1,i}, \ldots, s_{l,i}, m_{l,i}, 1/2, m_1, 1/2, m_1, 1/2, m_1, \ldots).$$

By considering appropriate inverse images of the t_i and t'_i, one can see that

$$q_i \in K_{h,0} \cap \phi_h^{-1}(\phi_f(\gamma_{f,0}(\theta_1))) \tag{11.12}$$

and

$$h_0^{\circ N_i}(q_i) = p_i. \tag{11.13}$$

By proposition 10.3.1, $\gamma_{h,0}^{-1}(q_i)$ has only one element which we shall call $\hat{\theta}_i$.

Claim 11.3.3. *There is a single point $\hat{\zeta}$ in*

$$K_{h,1} \cap \phi_h^{-1}(\phi_f(\gamma_{f,1}(\theta_1))),$$

and $\hat{\zeta}$ has exactly k exterior angles which are $\hat{\theta}_i$ for $i = 0, 1, 2, \ldots, k-1$.

Proof 11.3.3.

Let

$$\hat{Z} := K_{h,1} \cap \phi_h^{-1}(\phi_f(\gamma_{f,1}(\theta_1))),$$

and suppose there were more than one point in $\hat{Z}$. Just as in the proof of claim 11.3.2.2, we would get that the exterior angles of points in $\hat{Z}$ would have dynamic denominators of the form $2^l(2^N - 1)$. By (11.13),

$$f^{\circ N_i}(\hat{Z}) \in \{\zeta\}. \tag{11.14}$$

So the dynamic denominator of the exterior angles of points in $\hat{Z}$ is $2^{N_1}(2^k - 1)$. But $k < N$. So there is a single point $\hat{\zeta}$ in $\hat{Z}$.

By (11.14), $\hat{\zeta}$ has exactly k exterior angles. By (11.12), they are as claimed.

End 11.3.3.

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