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Generic Properties of Lagrangian Systems and Conservative Diffeomorphisms

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ABSTRACT

Generic Properties of Lagrangian Systems and Conservative Diffeomorphisms

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In this dissertation we study some generic properties of Lagrangian Systems and symplectic diffeomorphisms.

In Chapter 3 we give a negative answer to a conjecture proposed by Mañé, we give an example of a C^1 open set U of Lagrangians on the n-torus such that for any Lagrangian in U there exist a cohomology class $c \in H^1(\mathbb{T}^n, \mathbb{R})$ which has at least n different ergodic c-minimal measures. The minimizing measures are used in the construction of orbits connecting different regions in the phase space (Arnold diffusion), so understanding their structure should be helpful.

We also consider symplectic diffeomorphisms on compact manifolds. We prove in Chapter 4 that if a symplectic diffeomorphism is not partially hyperbolic then with an arbitrarily small C^1 perturbation we can create a totally elliptic periodic point inside any given open set on the manifold. As a consequence, a C^1 generic

symplectic diffeomorphism is either partially hyperbolic or it has dense elliptic periodic points.

From this theorems we obtain some interesting corollaries:

(1) Any C^1 robustly transitive symplectic diffeomorphism must be partially hyperbolic.

(2) Any stably ergodic symplectic diffeomorphism must be partially hyperbolic.

The second corollary is a converse of the Pugh-Shub conjecture on stable ergodicity for the symplectic case. Here a map f is stably ergodic if there exist a C^1 open neighborhood U of f such that every C^2 map in U is ergodic.

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Contents

CHAPTER 1

Introduction

One of the areas of interest in Dynamical Systems is concerned with the dynamics of Hamiltonian and Lagrangian Systems. There is a natural duality between them given by the Legendre transform, they actually represent the same problem seen from two different points of view. These types of systems are modeling many problems from different areas like mechanics, physics, chemistry and they have been extensively studied.

In this dissertation we are particularly interested in the robust properties of these systems, or the generic properties. A property is called generic if it holds for a residual subset of systems in some given topology.

We consider smooth Lagrangians $L : TM \times S^1 \to \mathbb{R}$ with the following properties

-convexity: $\frac{\partial^2 L}{\partial v^2}$ is positive definite;

-completeness: the corresponding Euler-Lagrange flow is defined on all $\mathbb{R}$;

-superlinearity: $\lim_{\|v\|\to\infty} \frac{L(x,v,t)}{\|v\|} = \infty$ uniformly with respect to $x \in M, t \in S^1$.

Here M is a connected compact Riemannian manifold, (x, v, t) are local coordinates on $TM \times S^1$ and in these coordinates the Euler-Lagrange flow is given by the equation

$$\frac{d}{dt}\frac{\partial L}{\partial v}(x,\dot{x},t)=\frac{\partial L}{\partial x}(x,\dot{x},t).$$

One can extend the Aubry-Mather theory for twist maps of the annulus in this setting and define the minimizing measures. Roughly speaking they are invariant probability measures which minimize the action of L among the ones with a fixed homology class (see [**Ma3**], [**Mt**]). They are also invariant measures minimizing the action of $L-c$ for some given closed 1-form c. The minimizing measures are a useful tool in constructing orbits connecting different regions in the phase space (the Arnold diffusion).

We can denote $\mathcal{M}_\rho$ the set of minimising measures corresponding to the homology class $\rho\in H_1(M,\mathbb{R})$ and $\mathcal{M}^c$ the set of minimising measures corresponding to the cohomology class $c\in H^1(M,\mathbb{R})$. Mañé asked how many (ergodic) measures are in each $\mathcal{M}_\rho,\mathcal{M}^c$ for C^∞ generic Lagrangians. He proved that once we fix a homology class ρ (or a cohomology class c), then for a generic Lagrangian there is a unique measure in $\mathcal{M}_\rho$ (or $\mathcal{M}^c$).

There are many possible ways to extend this results, and he proposed several of them in [**Ma3**]. One naturally asks what happens for *all* the homology or cohomology classes for generic Lagrangians. We give an example of an open set of Lagrangians on the n-torus such that for every one of them we can find a cohomology class c (depending on the Lagrangian) with at least n different ergodic measures in $\mathcal{M}^c$.

Theorem 1. *There exist a C^1 open set U of autonomous Lagrangians on the n-torus such that for each Lagrangian L in U there exist a cohomology class $c \in H^1(\mathbb{T}^n, \mathbb{R})$ such that L has at least n different ergodic c-minimal measures.*

This suggests that the best one can hope for is to prove that for a generic Lagrangian, for *every* cohomology class $c \in H^1(M, \mathbb{R})$, there are at most as many different ergodic minimising measures in $\mathcal{M}^c$ as the dimension of the first cohomology group of M. It is also conjectured that for a generic Lagrangian, for *every* homology class $\rho \in H_1(M, \mathbb{R})$, there is a unique measure in $\mathcal{M}_\rho$ (not necessarily ergodic).

We also consider M to be a compact connected Riemannian even-dimensional manifold together with a symplectic form (a non-degenerate closed 2-form). A diffeomorphism on M is called symplectic if it preserves the symplectic form. Examples of symplectic diffeomorphism are area preserving diffeomorphisms, time 1 maps and Poincaré return maps of Hamiltonian flows. Symplectic maps belong to the larger class of volume preserving diffeomorphisms, i. e. the diffeomorphisms which preserve a volume form on the manifold.

A diffeomorphism f on M has a dominated splitting if there exists an invariant splitting of the tangent bundle $TM = A \oplus B$ and an integer $l > 0$ such that

$$\|Df^l(a)\| \geq 2\|Df^l(b)\|, \forall a \in A_x, b \in B_x, x \in M.$$

In this case we say that A dominates B (or A l-dominates B).

In the case of symplectic diffeomorphisms the existence of a dominated splitting is equivalent to partial hyperbolicity in the following sense: there is an invariant splitting of the tangent bundle $TM = A \oplus B \oplus C$, with at least two of them non-trivial, such that

-A is uniformly expanding

-C is uniformly contracting

-A dominates B and B dominates C.

The partially hyperbolic diffeomorphisms are supposed to have good statistical properties. It is conjectured (Pugh, Shub) that among the volume preserving partially hyperbolic diffeomorphisms the stable ergodic ones form a C^1 open C^2 dense subset. We give a converse of this in the symplectic case, we prove that a stable ergodic map must be partially hyperbolic.

The existence of elliptic periodic points and the existence of a dominated splitting are mutually exclusive. We prove that C^1 generically these are the only two possibilities for symplectic maps.

Theorem 2. *There exist an open dense subset $\mathcal{U}$ of the set of C^1 symplectic diffeomorphisms of the compact manifold M such that for any $f \in \mathcal{U}$ exactly one of the following is true:*

-f is partially hyperbolic

-f has at least an elliptic periodic point.

There exist a residual $\mathcal{R}$ in the set of C^1 symplectic diffeomorphisms of M such that for any $f \in \mathcal{R}$ exactly one of the following is true:

-f is partially hyperbolic

-the elliptic periodic points of f are dense in M.

This is a consequence of a stronger result which we prove and which also led us to some interesting corollaries:

Theorem 3. *Suppose that f is a symplectic diffeomorphisms on M which is not partially hyperbolic and U is a fixed open set in M. Then there is an arbitrarily small C^1 perturbation g of f such that g has an elliptic periodic point in U.*

Corollary 1. *Any C^1-robustly transitive symplectic diffeomorphism must be partially hyperbolic.*

Corollary 2. *If a symplectic diffeomorphism f has a neighborhood U in the C^1 topology such that any C^∞ map in U is ergodic (this is a form of stable ergodicity) then f must be partially hyperbolic.*

Corollary 3. *The set of partially hyperbolic symplectic diffeomorphisms is equal to the interior of its closure in the C^1 topology.*

The proof uses C^1 perturbation results like Hayashi's connecting lemma and Franks lemma. Some of these results also hold for volume preserving diffeomorphisms, with partial hyperbolicity replaced by the existence of a dominated splitting.

These results enrich the known picture of C^1 generic symplectic (or volume preserving) diffeomorphisms. We know that generically they are transitive, they have dense (hyperbolic) periodic points, and every hyperbolic periodic point has transverse homoclinic points dense on the manifold. Newhouse proved that in the symplectic case C^1 generically the diffeomorphisms either have dense quasi-elliptic periodic points or they are Anosov (see [**Ne2**]). He used this to prove that a structurally stable symplectic map must be uniformly hyperbolic. Our result can be viewed as a generalization of this, the elliptic periodic points being an obstruction to robust transitivity and stable ergodicity in the same way the quasi-elliptic periodic points are for structural stability.

Similar results for general and volume preserving diffeomorphisms are obtained in [**BoDiPu**]. These results are adapted for the symplectic case in [**HoTa**]. Also in [**BcVi1**], [**BcVi2**] there are results about the Lyapunov exponents of almost all the points (instead of the type of the periodic points) for generic symplectic and volume preserving maps.

CHAPTER 2

Background

2.1. Lagrangian and Hamiltonian Systems

A *Lagrangian* is a smooth map $L : TM \times S^1 \to \mathbb{R}$, M being a compact Riemannian manifold, with the following properties:

-the Hessian derivative on the fibers is positive definite (convexity);

-L has superlinear growth on the fibers:

$$\lim_{\|v\| \to \infty} \frac{L(x, v, t)}{\|v\|} = \infty$$

uniformly with respect to $x \in M, t \in S^1$.

-the corresponding Euler-Lagrange flow on TM is complete.

The Euler-Lagrange flow is given by the following differential equation:

$$\frac{d}{dt}\frac{\partial L}{\partial v}(x, \dot{x}, t) = \frac{\partial L}{\partial x}(x, \dot{x}, t).$$

If the Lagrangian is time-independent then we call it *autonomous*. In this case the Euler-Lagrange flow is always complete. The example we keep in mind is the mechanical Lagrangian, given by the kinetic energy minus the potential energy

$$L(x,v) = \frac{1}{2} < A(x)v, v > -U(x)$$

where $A(x)$ is positive definite.

The solution curves of the Euler-Lagrange equations can be found using a fixed endpoints variational problem. For an absolutely continuous curve $\gamma : [a,b] \to M$ we can define the action functional:

$$A(\gamma) = \int_a^b L(\gamma(t), \gamma'(t), t)dt.$$

The critical curves with fixed endpoints - $\gamma(a) = x, \gamma(b) = y; x, y \in M$ fixed - are solutions of the Euler-Lagrange equations.

Applying the Legendre transform we can get a Hamiltonian system on the cotangent bundle of M.

Given an even dimensional manifold N with a symplectic form ω (non-degenerate closed 2-form) one can define canonically a map $J : T^*N \to TN$ using the formula

$$\omega(J(c), v) = c(v), \forall x \in N, c \in T_x^*N, v \in T_xN.$$

Then a *Hamiltonian* is a smooth map $H : N \times \mathbb{R} \to \mathbb{R}$ and the corresponding Hamiltonian system is given by the following equation:

$$\dot{x} = JdH(x,t)$$

where the derivative of H is taken with respect to x. If H is time-independent then the system is called again *autonomous*.

In the classical mechanical case the manifold N will be T^*M and the Hamiltonian will be the sum of the kinetic energy and the potential energy. This is the total energy and it is a constant of motion (a first integral).

2.2. Symplectic Diffeomorphisms

In what follows we will consider M to be a $2n$-dimensional compact connected Riemannian manifold and ω a symplectic form on M, i. e. a non-degenerate closed 2-form. Taking n times the wedge product of ω with itself we obtain a volume form on M. A C^r diffeomorphism f of M, $r \geq 1$, is called *symplectic* if it preserves the symplectic form, $f^*\omega = \omega$. The set of C^r symplectic diffeomorphisms of M will be denoted $\mathbf{Diff}^r_\omega(M)$ and we consider it having the uniform C^r topology. In the 2-dimensional case this is the same with the set of area preserving C^r diffeomorphisms. In higher dimensions this is just a subset of the set of C^r volume preserving diffeomorphisms (the volume form corresponding to ω). Examples of symplectic diffeomorphisms are Poincaré return maps and time 1 maps of Hamiltonian flows.

A point $p \in M$ is a *periodic point* of period k for f if $f^k(p) = p$. If all the eigenvalues of $Df^k(p)$ have the norm different than 1 then we will say that p is a *hyperbolic periodic point*. In this case we have well defined stable and unstable manifolds,

$$W^s(p) = \{x \in M, \lim_{l\to\infty} d(f^l(x), f^l(p)) = 0\} \text{ and}$$

$$W^u(p) = \{x \in M, \lim_{l \to \infty} d(f^{-l}(x), f^{-l}(p)) = 0\},$$

where d is the Riemannian metric on the manifold. In the case of symplectic maps the eigenvalues come in pairs, λ is an eigenvalue of $Df^k(p)$ if and only if λ^{-1} is an eigenvalue, so the dimension of both the stable and unstable manifolds will be n in our case. A point $q \in W^s(p) \cap W^u(p) \setminus \{p\}$ is called a *homoclinic* point of p.

A splitting of the tangent bundle $TM = \mathbf{A} \oplus \mathbf{B}$ is called *invariant* if $\mathbf{A}$ and $\mathbf{B}$ are invariant under Df. An invariant splitting $TM = \mathbf{A} \oplus \mathbf{B}$ is called *dominated* if there is an $l > 0$ such that for any $x \in M$ and any two unit vectors $u \in \mathbf{A}_x, v \in \mathbf{B}_x$ we have $\|Df^l(u)\| \geq 2\|Df^l(v)\|$. We will also say that $\mathbf{A}$ *dominates* $\mathbf{B}$. If we want to emphasize the importance of l we say that $\mathbf{A}$ *l-dominates* $\mathbf{B}$, or the splitting is *l-dominated*.

A map f is called *partially hyperbolic* if there is an invariant splitting of the tangent bundle of M, $TM = \mathbf{A} \oplus \mathbf{B} \oplus \mathbf{C}$, with at least two of them nontrivial, such that

(i)$\mathbf{A}$ is uniformly expanding: there exist $\alpha > 1$ and $C > 0$ such that

$$\|Df^k(u)\| \geq C\alpha^k\|u\|, \forall u \in \mathbf{A}, k \in \mathbb{N},$$

(ii)$\mathbf{C}$ is uniformly contracting: there exist $\beta > 1$ and $D > 0$ such that

$$\|Df^k(v)\| \leq D\beta^{-k}\|v\|, \forall v \in \mathbf{C}, k \in \mathbb{N},$$

(iii)$\mathbf{A}$ dominates $\mathbf{B}$ and $\mathbf{B}$ dominates $\mathbf{C}$.

One can prove that a dominated splitting is continuous, so the angle between the two subbundles is bounded away from 0, and a small C^1-perturbation of a map with a dominated splitting also has a dominated splitting. So the set of partially hyperbolic maps is open in $\mathbf{Diff}^1_\omega(M)$. Also we make the remark that the property of dominance is independent of the Riemannian structure on the manifold, but two different structures can have different constants of dominance l. If $\mathbf{B}$ is trivial then f is called *uniformly hyperbolic* or *Anosov*. This definitions can be extended to an invariant subset N of M.

A periodic point p of f of period k is called *elliptic* if all the eigenvalues of $Df^k(p)$ are simple, non-real and of norm 1. Obviously the existence of an elliptic periodic point is an obstruction for partial hyperbolicity. We will prove here that the converse is also true generically, i.e. if a C^1-generic symplectic diffeomorphism is not partially hyperbolic then it has an elliptic periodic point (actually it has a dense set of elliptic periodic points). Here we say that a property is C^r-*generic* if it is true for a residual subset of $\mathbf{Diff}^r_\omega(M)$.

CHAPTER 3

Minimal Measures for Lagrangian Flows

3.1. Preliminaries

The holonomic measures for (autonomous) Lagrangians are defined as follows: Let C^l be the space of real-valued continuous functions on TM with at most linear growth on the fibers with the topology given by the norm

$$\|f\|_l = \sup_{(x,v)\in TM} \frac{|f(x,v)|}{1+\|v\|}.$$

We consider the set of probability measures on TM with the corresponding weak* topology from the dual of this space. If we restrict to the set of probability measures supported on closed absolutely continuous curves on the manifold and we take the closure of this in the topology mentioned above we get the *holonomic measures* $\bar{C}$. This is a convex metrizable noncompact set (see [**Ma3**]). Any probability measure μ invariant under the Lagrangian flow with $\int_{TM} \|v\| d\mu$ finite is also a holonomic measure - one can just use the ergodic theorem and the fact that $\bar{C}$ is convex and is the closure of measures supported on closed curves.

The *L-action* of a probability measure is defined as

$$A_L(\mu) = \int_{TM} L d\mu.$$

The holonomic measures with minimal action are called *minimal measures*. If we replace L by $L - c$ where c is a closed 1-form on the manifold M we get the *c-minimal measures*. They always exist and are invariant under the flow (see [**Ma3**] or [**Mt**]).

The integral of an exact 1-form on a closed absolutely continuous curve vanishes. But every 1-form is in C^l and the holonomic measures are in the closure of the probability measures supported on closed curves, so by continuity the integral of any exact 1-form on M vanishes for any holonomic measure. This shows that the $(L - c)$-action and the c-minimal measures depend only on the cohomology class of c.

One can define the *α-function*:

$$\alpha : H^1(M, \mathbb{R}) \to \mathbb{R} \ , \ \alpha(c) = -\min_{\mu \in \bar{C}} A_{L-c}(\mu).$$

This is finite, convex and has superlinear growth. Here c represents both a closed 1-form and its cohomology class. Then a holonomic measure μ is c-minimal if

$$A_{L-c}(\mu) = -\alpha(c).$$

The convex dual of this is the *β-function* which can also be defined as follows: For each holonomic measure μ the integral of any exact 1-form is 0, so one can define a linear functional on the cohomology group

$$h_\mu : H^1(M,\mathbb{R}) \to \mathbb{R}\ ,\ h_\mu(c) = \int_{TM} c\ d\mu.$$

Because of the duality between the homology and the cohomology we obtain the *rotation vector* of the holonomic measure μ, denoted $\rho(\mu) \in H_1(M,\mathbb{R})$, corresponding to the linear functional h_μ. For example if μ is the probability measure supported on the closed curve $\gamma : [a,b] \to M$ then $\rho(\mu)$ is the integer homology of γ divided by $(b-a)$. The function $\rho : \bar{C} \to H_1(M,\mathbb{R})$ is continuous. Then the β-function is

$$\beta : H_1(M,\mathbb{R}) \to \mathbb{R}\ ,\ \beta(r) = \min_{\mu\in\bar{C},\rho(\mu)=r} A_L(\mu).$$

This is again finite, convex and has superlinear growth.

The set of holonomic measures with the action in a bounded subset of $\mathbb{R}$ is compact (in the topology given above). Any minimal measure is a limit of probability measures uniformly distributed along closed curves on M such that their action is also converging to the minimal action (see again [**Ma3**] and [**Mt**]).

The set of holonomic measures minimising the action for $L-c$ is denoted by $\mathcal{M}^c$ and the set of holonomic measures with rotation number r minimising the action is denoted by $\mathcal{M}_r$. An open question is regarding the number of different ergodic measures in $\mathcal{M}^c$ and $\mathcal{M}_r$ for generic Lagrangians. Mañé proved that for a fixed homology class r (or cohomology class c) for a generic Lagrangian, $\mathcal{M}_r$ (or $\mathcal{M}^c$) has only one element. We give an example with an open set of Lagrangians such that for each one of them there exists some cohomology class c such that $\mathcal{M}^c$ has at least n different ergodic measures, where n is the dimension of the first cohomology group. This shows that Mañé's result can't be extended for generic Lagrangians for *all* cohomology classes. The corresponding question about homology is still open:

Q1. Is it true that for a generic Lagrangian for every homology class r the set $\mathcal{M}_r$ has only one element?

One could also ask how many different ergodic minimal measures are in every $\mathcal{M}^c$ for all the cohomology classes c for a generic Lagrangian:

Q2. Is it true that for a generic Lagrangian for every cohomology class c the set $\mathcal{M}^c$ contains at most n different ergodic measures ($n = dim H^1(M, \mathbb{R})$)?

The method we use in our construction is the following: we choose n mutually disjoint subsets of the manifold M carrying the same $(n-1)$-dimensional homology. We make the action small on these regions and large outside them such that the c-minimal measures are supported on these subsets for c in the corresponding $(n-1)$-dimensional subspace of $H^1(M, \mathbb{R})$. Now if we look at the graphs of the α functions restricted to each one of these regions we get n hypersurfaces (with dimension $n-1$) in an n-dimensional space, which will intersect in one point if we choose the right Lagrangian. The intersection point corresponds to n c-minimal measures with disjoint supports for the same cohomology class c and taking their ergodic decomposition we get at least n different ergodic c-minimal measures.

3.2. Proof of Theorem 1

Let $L : T\mathbb{T}^n \to \mathbb{R}$ be a Lagrangian on the $n-$torus of the form

$$L(x,v) = \sum_{i=1}^{n} a_i(x)v_i^2 + U(x)\ ;\ a_i, U : \mathbb{T}^n \to \mathbb{R}\ ;\ v = \sum_{i=1}^{n} v_i \frac{\delta}{\delta x_i} \in T_x\mathbb{T}^n.$$

Let A_i be the strips on the torus with the $n-$th coordinate x_n between $\frac{6i-1}{6n}$ and $\frac{6i+1}{6n}$, B_i the strips with x_n between $\frac{6i+2}{6n}$ and $\frac{6i+4}{6n}$ and C_i^+, C_i^- the strips between A_i and B_i respectively B_{i-1} and A_i.

On A_0 let

$$U(x) = -\delta < 0,$$

$$a_i(x) = 2,\ i = 1, ..n.$$

On $A_j,\ 1 \leq j \leq n-1$ let

$$U(x) = 0,$$

$$a_j(x) = 1,$$

$$a_i(x) = 2,\ i = 1, ..n,\ i \neq j.$$

Let $U(x) = a_i(x) = C > 0$ large, $i = 1, ..n$ on B_j and also let their partial derivatives with respect to the nth coordinate be negative on C_j^- and positive on C_j^+.

We divided the torus in n strips with low action (A_i), n strips with high action (B_i), and the transition regions (C_i^-, C_i^+).

The projection along the n-th coordinate $p : \mathbb{T}^n \to \mathbb{T}^{n-1}$ corresponds to the cohomology map $p^* : H^1(\mathbb{T}^{n-1}, \mathbb{R}) \to H^1(\mathbb{T}^n, \mathbb{R})$. Let $H = Im(p^*)$ (the range of p^*). So we restrict our attention to the cohomology classes corresponding to the first $n-1$ coordinates.

For any L' a Lagrangian C^1 close to L one can define the α function on each strip $A'_i = \mathbb{T}^{n-1} \times [\frac{6i-1}{6n} - \epsilon_0, \frac{61+1}{6n} + \epsilon_0]$ for $\epsilon_0 > 0$ small as follows: $\alpha^i_{L'}(c) = -\inf_\mu \int (L' - c) d\mu$ where the infimum is taken over the set of holonomic probability measures μ supported in TA'_i. We used again the same notation (c) for both a closed 1-form and its cohomology class in H. Another way to define $\alpha^i_{L'}$ is to construct L'_i agreeing with L' on $B_{i-1} \cup C^-_i \cup A_i \cup C^+_i \cup B_i$ and large on the rest of $\mathbb{T}^n$. For any $c \in H$ and any curve in $\mathbb{T}^n$ there exist a curve in A'_i defined on the same time interval and with the same projection on $\mathbb{T}^{n-1}$ with smaller $(L'_i - c)$-action because of the construction of L, L' and L'_i. The new curve can be the same as the initial one inside A'_i, have the n-th coordinate constant when the initial ! curve is in the transition regions around A'_i and just have the speed along the n-th coordinate less than the speed of the initial curve for the rest of it. This implies that for any $c \in H$ there is a c-minimal measure for L'_i supported in A'_i (see the proof of lemma 1). So $\alpha^i_{L'}$ is the α function of L'_i restricted to H.

For L given above and $c = (\lambda_1, \lambda_2, ..\lambda_{n-1}) \in H$ we can compute

$$\alpha^0_L(c) = -\min_\mu \int \left(-\delta + \sum_{i=1}^{n} 2v_i^2 - \sum_{i=1}^{n-1} \lambda_i v_i \right) d\mu_{(x,v)} =$$

$$= \delta - \min_{\mu} \int \left[\sum_{i=1}^{n-1} \left(2 \left(v_i - \frac{\lambda_i}{4} \right)^2 - \frac{\lambda_i^2}{8} \right) + 2v_n^2 \right] d\mu_{(x,v)} = \delta + \sum_{i=1}^{n-1} \frac{\lambda_i^2}{8}$$

where minimum is attained for any invariant probability measure μ supported on the $\sum_{i=1}^{n-1} \frac{\lambda_i}{4} \frac{\delta}{\delta x_i}$ section of TA_0;

$$\alpha_L^j(c) = - \min_{\mu} \int \left(\sum_{i=1}^{n} 2v_i^2 - v_j^2 - \sum_{i=1}^{n-1} \lambda_i v_i \right) d\mu_{(x,v)} =$$

$$= - \min_{\mu} \int \left[\sum_{i=1, i\neq j}^{n-1} \left(2 \left(v_i - \frac{\lambda_i}{4} \right)^2 - \frac{\lambda_i^2}{8} \right) + \left(v_j - \frac{\lambda_j}{2} \right)^2 - \frac{\lambda_j^2}{4} + 2v_n^2 \right] d\mu_{(x,v)} =$$

$$= \sum_{i=1}^{n-1} \frac{\lambda_i^2}{8} + \frac{\lambda_j^2}{8}$$

where minimum is attained for any invariant probability measure μ supported on the $\sum_{i=1}^{n-1} \frac{\lambda_i}{4} \frac{\delta}{\delta x_i} + \frac{\lambda_j}{4} \frac{\delta}{\delta x_j}$ section of TA_j.

We chose the coefficients a_i such that the shapes of the α functions for the strips are paraboloids 'squeezed' in one direction (different for different i's) for $i > 0$ and not squeezed but translated upward for $i = 0$. Thus it is easier to prove that all the n graphs intersect. Now the next theorem will give the proof of Theorem 1.

Theorem 4. *For L' positive definite Lagrangian C^1 close to L there exists a cohomology class with at least n different corresponding ergodic minimal measures.*

Proof. If $|L - L'| \leq \epsilon$ then clearly for any $c \in H$ we have $|\alpha_L^i(c) - \alpha_{L'}^i(c)| \leq \epsilon$ for all $i = 0, ..n-1$.

Lemma 1. *For any $c \in H$ and L' C^1 close to L we have*

$$\alpha_{L'}(c) = \max_{0 \le i \le n-1} \alpha^i_{L'}(c).$$

Proof. The set of holonomic probability measures in $T\mathbb{T}^n$ contains the ones supported in TA'_i so

$$\alpha_{L'}(c) \ge \alpha^i_{L'}(c), i = 0, ..n-1 \Rightarrow \alpha_{L'}(c) \ge \max_{0 \le i \le n-1} \alpha^i_{L'}(c).$$

On the other hand we can prove that there exist a holonomic measure μ supported on TA'_i for some i such that $A_{L'-c}(\mu) = \int (L'-c) d\mu = -\alpha_{L'}(c)$. For that it is enough to show that for any closed curve in $\mathbb{T}^n$ there is a closed curve in an A'_i with smaller average action. If the average action of the initial curves converges to $-\alpha_{L'}(c)$ the same will hold for the new ones (the action can't get smaller) and a limit measure (eventually for a subsequence, so the curves are inside the same A'_i) will verify the requirement. This will imply that

$$\alpha_{L'}(c) \le \alpha^i_{L'}(c) \le \max_{0 \le i \le n-1} \alpha^i_{L'}(c).$$

If the curve is in $B_{i-1} \cup C_i^- \cup A_i \cup C_i^+ \cup B_i$ (around one single region of low values for the Lagrangian) we can find such a curve in A'_i in the same way we did when we defined $\alpha^i_{L'}$.

If this is not the case (the curve moves from a region of low values of the Lagrangian to another) then we can divide the curve in segments such that each one of them is in an $B_{i-1} \cup C_i^- \cup A_i \cup C_i^+ \cup B_i$ for some i.

We can also require that each such segment crosses completely one B_i for some i (it moves from C_i^+ to C_{i+1}^- or backward). We construct again a curve in A_i' with the same projection on $\mathbb{T}^{n-1}$ and with the n-th coordinate constant where the initial segment is outside A_i. Because of this the difference of actions of the two segments will be at least the minimal action needed to cross B_i for $c = 0$. To estimate this we use only the n-th coordinate and the potential for L and we find that it is greater than

$$\int_a^b C\gamma'(t)^2 + Cdt = C(b-a) + \frac{C}{b-a}\int_a^b \gamma'(t)^2 dt \int_a^b 1^2 dt \geq$$

$$\geq C(b-a) + \frac{C}{b-a}\left[\int_a^b \gamma'(t)dt\right]^2 = C(b-a) + \frac{C}{(b-a)9n^2} \geq \frac{2C}{3n}$$

where $\gamma : [a,b] \longrightarrow [\frac{6i+2}{6n}, \frac{6i+4}{6n}]$ is the projection on the n-th coordinate of the segment crossing B_i.

To make the estimation for L' suppose $|L' - L| < \frac{1}{2}$ and $C \geq 20n^2$.

If $b - a \leq 1$ then the action needed to cross B_i for L' can be at most with $\frac{1}{2}$ smaller so it is at least

$$-\frac{1}{2} + \frac{2C}{3n} \geq -\frac{1}{2} + \frac{n}{3} + 13n > 13n.$$

because $n \geq 2$ (we want at least 2 minimal measures).

If $b - a > 1$ then we use the fact that $L' \geq L - \frac{1}{2} \geq C - \frac{1}{2}$ so the action needed is at least

$$(b-a)\left(L-\frac{1}{2}\right) \geq C-\frac{1}{2} \geq 20n^2-1 > 13n.$$

So in both cases the action needed to cross B_i is greater than $13n$.

On the other hand for any $c \in H$ and $a, b \in A'_i$ there is a geodesic $\xi : [0,1] \to A'_i$ from a to b with length at most $2\sqrt{n}$ and such that $c(\xi') \geq 0$. The $(L-c)$-action of this is

$$\int_0^1 (L-c)(\xi(t), \xi'(t))dt = \int_0^1 U(\xi(t))dt + \int_0^1 \sum_{j=1}^n a_j(\xi'(t))_j^2 dt -$$

$$-\int_0^1 c(\xi'(t))dt \leq \frac{1}{2} + \int_0^1 3\|\xi'(t)\|^2 dt \leq \frac{1}{2} + 12n$$

where ϵ_0 (who gives A'_i) is small enough so $U < \frac{1}{2}$ and $a_j < 3, j = 1, ..n$ on A'_i. For $|L-L'| < \frac{1}{2}$ we get that the $(L'-c)$-action of such a curve is at most $13n$.

In conclusion, if we take $C > 20n^2$ then we can complete the segment from A'_i obtained by 'projecting' the initial segment around A_i, to a closed curve inside A'_i (by adding a geodesic as above to connect the endpoints) with less action that the initial segment and defined on an interval larger with one unit. Doing that for each segment we obtain a finite (the initial curve is absolute continuous) number of closed curves inside $\cup_{i=0}^{n-1} A_i$ with smaller total action. By dividing to a larger time interval the average action will be again smaller than the initial one. But the total average action is a convex combination of the average actions of the closed curves so at least one of them will have an action smaller (or equal) than the initial curve Q.E.D. $\square$

Remark. Actually L' has to be C^1 close enough to L so that the Lagrangian is increasing as we move away from A_i', which is true for L. This allows us to construct the curves in A_i' with smaller action. Although C^0 closeness may be enough, the C^1 condition makes the proof easier.

Lemma 2. *There exist $c \in H$ such that*

$$\alpha_{L'}^0(c) = \alpha_{L'}^1(c) = \cdots = \alpha_{L'}^{n-1}(c) = \alpha_{L'}(c)$$

if $|L - L'| < \frac{\delta}{6}$.

Proof. Let $T : \mathbb{R}^{n-1} \cong H \to \mathbb{R}^{n-1} \cong H$,

$$T(\lambda_1, \ldots, \lambda_{n-1}) = \left(\lambda_1\big(1 + \frac{\alpha_{L'}^0(c) - \alpha_{L'}^1(c)}{2\delta}\big), \ldots, \lambda_{n-1}\big(1 + \frac{\alpha_{L'}^0(c) - \alpha_{L'}^{n-1}(c)}{2\delta}\big)\right)$$

for $c = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \in \mathbb{R}^{n-1} \cong H$.

We will prove that T maps $[\sqrt{\delta}, 4\sqrt{\delta}]^{n-1}$ to itself, this implying the existence of a fixed point and consequently the claim.

First observe that $|L - L'| < \frac{\delta}{6} \Rightarrow |\alpha_L^i(c) - \alpha_{L'}^i(c)| < \frac{\delta}{6}$ for any $0 \le i \le n-1 \Rightarrow$

$$\alpha_L^0(c) - \alpha_L^i(c) - \frac{\delta}{3} < \alpha_{L'}^0(c) - \alpha_{L'}^i(c) < \alpha_L^0(c) - \alpha_L^i(c) + \frac{\delta}{3} \Rightarrow$$

$$\Rightarrow \frac{2\delta}{3} - \frac{\lambda_i^2}{8} < \alpha_{L'}^0(c) - \alpha_{L'}^i(c) < \frac{4\delta}{3} - \frac{\lambda_i^2}{8} \Rightarrow$$

$$\Rightarrow \frac{4}{3} - \frac{\lambda_i^2}{16\delta} < 1 + \frac{\alpha_{L'}^0(c) - \alpha_{L'}^i(c)}{2\delta} < \frac{5}{3} - \frac{\lambda_i^2}{16\delta}.$$

For $\lambda_i \in [\sqrt{\delta}, 4\sqrt{\delta}]$ we have λ_i, $\frac{4}{3} - \frac{\lambda_i^2}{16\delta}$ and $\frac{5}{3} - \frac{\lambda_i^2}{16\delta}$ positive so

$$\lambda_i \left(\frac{4}{3} - \frac{\lambda_i^2}{16\delta} \right) < \lambda_i \left(1 + \frac{\alpha_{L'}^0(c) - \alpha_{L'}^i(c)}{2\delta} \right) < \lambda_i \left(\frac{5}{3} - \frac{\lambda_i^2}{16\delta} \right).$$

The functions $f(x) = x\left(\frac{4}{3} - \frac{x^2}{16\delta}\right)$ and $g(x) = x\left(\frac{5}{3} - \frac{x^2}{16\delta}\right)$ map the interval $[\sqrt{\delta}, 4\sqrt{\delta}]$ to itself, so if $\lambda_i \in [\sqrt{\delta}, 4\sqrt{\delta}]$ then $T(c)_i \in [\sqrt{\delta}, 4\sqrt{\delta}]$ for any $0 \leq i \leq n-1$ which proves that indeed T maps $[\sqrt{\delta}, 4\sqrt{\delta}]^{n-1}$ to itself. This concludes the proof of the second lemma. $\square$

Now suppose that for L' close to L we have $c' \in H$ such that

$$\alpha_{L'}^0(c') = \alpha_{L'}^1(c') = \cdots = \alpha_{L'}^{n-1}(c') = \alpha_{L'}(c').$$

Then for any $0 \leq i \leq n-1$ there exist an ergodic invariant probability measure μ_i supported on TA_i' such that $A_{L'-c'}(\mu_i) = \alpha_{L'}^i(c') = \alpha_{L'}(c')$. Because the supports of these measures are disjoint they must be different. They are also c-minimal because they minimize the $(L' - c')$-action.

So for any L' a Lagrangian C^1 close to L we can find a cohomology class c which has at least n ergodic c-minimal measures with disjoint supports. $\square$

Remark. C^1 close means that one can change not only the potential U, but the entire Lagrangian, as long as the positive definiteness is preserved.

Remark. We consider here the autonomous case and the cohomology class with n ergodic minimal measure we obtain is in an $(n-1)$-dimensional subspace of the cohomology group. So maybe this result can be improved, at least for the non-autonomous case.

Remark. The same construction could work for a manifold M of the type $S^1 \times N$, creating a cohomology class c with n c-minimal measures where n is the dimension of $H^1(M)$. In the general case it is not always possible to pick the n disjoint regions of low action with the same $(n-1)$-dimensional homology. One could take some non-disjoint regions and make sure that there are no minimal measures supported on their intersection. Or one could get different results, for example for a surface of genus 2 we can take two disjoint regions with the same 2-dimensional homology and thus get a curve of cohomology classes such that each one of them has at least two different ergodic minimising measures.

CHAPTER 4

Generic Symplectic Diffeomorphisms

4.1. Preliminaries

Hyperbolicity and ellipticity are responsible for completely different dynamics for symplectic diffeomorphisms. An hyperbolic map is on one hand chaotic - it is transitive, it has sensitive dependence on the initial conditions - but on the other hand it has good statistical properties (it is ergodic if it is smooth enough - $C^{1+\alpha}$) and it has a Markov partition, and thus is similar to a subshift of finite type. The Anosov maps are structurally stable and stably ergodic (again, if they are smooth enough). It is also conjectured that among the partially hyperbolic diffeomorphisms the stable ergodic ones form an open dense set in the C^2 topology (see [**PgSh**]). Here we give a partial answer to the converse of this for symplectic diffeomorphisms, we show that a stronger form of stable ergodicity implies partial hyperbolicity. The dynamics in a neighborhood of an elliptic periodic point are different. We know from KAM theory that if the map is smooth enough then clos! e to the elliptic point we have many invariant tori (possible a positive measure set) on which the map is conjugated to a strongly irrational rotation. Also the elliptic point is accumulated by other quasi-elliptic and hyperbolic periodic points, as well as homoclinic points (see [**Ze1**]).

If for a periodic point p of period k the tangent map $Df^k(p)$ has exactly $2l$ simple non-real eigenvalues of norm 1 and the other ones have norm different from 1, then we say that p is an *l-elliptic periodic point*. Sometimes it is also called *quasi-elliptic*. S. Newhouse proved that a C^1-generic symplectic diffeomorphism is either Anosov or it has dense 1-elliptic periodic points (see [**Ne2**]). He concluded from this that a symplectic diffeomorphism is structurally stable if and only if it is uniformly hyperbolic. In dimension 2 the 1-elliptic periodic points are actually elliptic. M.-C. Arnaud proved (see [**Ar1**], [**Ar2**]) that a C^1 generic symplectic diffeomorphism in dimension 4 is either hyperbolic, or partially hyperbolic, or it has an elliptic periodic point (in our paper we consider hyperbolicity as a particular case of partial hyperbolicity for simplicity). Our results generalize this for any dimension.

We continue by mentioning some related results. C. Bonatti, L. J. Diaz and E. R. Pujals obtained similar results for the case of general and volume preserving diffeomorphisms (see [**BoDiPu**]). V. Horita and A. Tahzibi extended their methods for the symplectic case ([**HoTa**]). For general diffeomorphisms the elliptic points are substituted by sinks and sources and the partial hyperbolicity by the existence of a dominated splitting. Also the result is restricted to homoclinic classes of hyperbolic periodic points, because transitivity is needed. For the case of volume preserving maps the elliptic points are not stable under perturbations, an arbitrarily small one can make them hyperbolic, so they get a weaker result without the genericity: if a map doesn't have a dominated splitting, with an arbitrarily small perturbation one can create periodic points with all the eigenvalues equal to 1. We can get our stronger result for symplectic diffeomorphisms because of three reaso! ns: the elliptic points are stable under perturbations if there are no multiple eigenvalues (which is a generic property) - this is not true for volume preserving maps, but is true for sinks and sources for general diffeomorphisms; generically the symplectic maps are transitive - this is true in the volume preserving case but not in the general one; and only in the symplectic case the existence of a dominated splitting implies partial hyperbolicity (see the Appendix).

Other related results, obtained by J. Bochi and M. Viana (some of them announced before by R. Mañé), take into consideration the Lyapunov exponents of almost all the points instead of looking at the types of the periodic points (see [**BcVi1**], [**BcVi2**], [**Ma2**]). They prove that for a generic volume preserving diffeomorphism for almost all the points either all the Lyapunov exponents are equal to 0 or their Oseledets splitting along the orbit is dominated. Also, a generic symplectic diffeomorphism is either Anosov or almost all the points have 0 as a Lyapunov exponent (with multiplicity at least 2).

In conclusion we state some of the known C^1 generic properties of symplectic diffeomorphisms. There exist a residual subset $\mathcal{R}$ of $\mathbf{Diff}^1_\omega(M)$ such that for any $f \in \mathcal{R}$ we have:

1. The periodic points of f are dense in M.
2. Every periodic point of f is either quasi-elliptic or hyperbolic.
3. The hyperbolic periodic points of f are dense in M.

4. The stable and unstable manifolds of hyperbolic periodic points of f intersect transversally.

5. Every hyperbolic periodic point of f has homoclinic orbits.

6. The homoclinic points of f are dense in M.

7. The homoclinic points of a hyperbolic periodic point of f are dense in both the stable and unstable manifolds of the periodic point.

8. The map f is transitive.

Property 1 is a direct consequence of the C^1 closing lemma of Pugh and Robinson (see [**PgRo**]). Properties 2 and 4 were proved by Robinson (see [**Ro**]). Property 3 is a direct consequence of property 1, because we cane make a periodic point hyperbolic with a small perturbation if the period is large enough. An alternate proof can use properties 1 and 2 and the fact that a quasi-elliptic periodic point is generically accumulated by hyperbolic ones from KAM theory (see [**Ne2**]). Property 5 was proved by Takens (see [**Ta**]). It is also a consequence of Hayashi's connecting lemma and the fact that C^r generically the stable (unstable) manifolds accumulate on themselves, so also on the unstable (stable) manifolds - this proof gives also property 7 (see [**WeXi**], [**Xi**]). Property 6 is a consequence of properties 3 and 5. Property 8 is another application of Hayashi's connecting lemma and it was proved by Bonatti and Crovisier (see [**BoCr**]).

The next lemma presents another generic property of symplectic (or volume preserving) diffeomorphisms. The main consequence we use from it is the fact that there is an arbitrarily small C^1 perturbation of f such that the set of homoclinic points of (the continuation of) an hyperbolic periodic point x is dense in M.

Lemma 3. *There exist a residual $\mathcal{R} \in \mathbf{Diff}^1_\omega(M)$ (or in the set of volume preserving diffeomorphisms) such that for any $f \in \mathcal{R}$ and any hyperbolic periodic point x of f the set of corresponding homoclinic points is dense in M.*

Proof. It is known that for a C^1 generic symplectic (or more general volume preserving) diffeomorphism for any hyperbolic periodic point y the set of homoclinic points of y is dense in both $W^s(y)$ and $W^u(y)$ - from property 7 above. We need to prove that C^1 generically $W^s(y)$ and $W^u(y)$ are dense in M for all hyperbolic periodic points y and we are done. We will use the fact that a C^1 generic symplectic (or volume preserving) diffeomorphism is transitive (property 8) and the C^1 connection lemma.

Let f be transitive, $U \subset M$ be an open set, y a hyperbolic periodic point, $p \in W^s(y)$ and B_k the ball of radius 2^{-k} centered at p. Because f is transitive there is an iterate of U intersecting B_1: $f^{k_1}(U) \cap B_1 \neq \emptyset$. Let U_1 be an open set such that $U_1 \subset \mathrm{cl}(U_1) \subset U \cap f^{-k_1}(B_1)$. Now there is an iterate $f^{k_2}(U_1)$ of U_1 which intersects B_2, so we can choose an open set U_2 such that $U_2 \subset \mathrm{cl}(U_2) \subset U_1 \cap f^{-k_2}(B_2)$ and so on. Then $\mathrm{cl}(U_k)$ is a decreasing sequence of compact sets inside U so there is a point y_U in their intersection and its orbit accumulates on p. Because p is not periodic, we can use the connection lemma to find an arbitrarily small perturbation of f such that p is a positive iterate of y_U and the positive orbit of p is unchanged, so $y_U \in W^s(y)$. So the stable manifold of y intersects U and obviously this is also true for! small perturbations (replacing y by its continuation).

Now let us denote by $\mathcal{R}(k, U)$ the set of diffeomorphisms with the property that the stable manifold of any hyperbolic periodic point with period less than k intersects U. From what we proved before and from the fact that for an open dense set of diffeomorphisms there are finitely many periodic points of period less than k we get that $\mathcal{R}(k, U)$ is an open dense subset of $\mathbf{Diff}^1_\omega(M)$ (or the volume preserving diffeomorphisms). But there is a countable base of the topology so taking the intersection over $k \in \mathbb{N}$ and U in the countable basis we get the residual we are looking for. For $W^u(y)$ the proof is similar. $\square$

4.2. A lemma from linear algebra

We present here a result we need from linear algebra. We will first give the motivation of the result.

We consider $\mathbb{R}^{2n}$ with the canonical symplectic structure given by the 2-form $\sum_{i=1}^n dx_i \wedge dy_i$ where $x_i, y_i, i \in \{1, 2, \ldots, n\}$ are the coordinates on $\mathbb{R}^{2n}$. The norm of the vectors is the euclidean norm in $\mathbb{R}^{2n}$ and the norm of matrices is the norm of the corresponding linear operators on $\mathbb{R}^{2n}$. There exist local coordinates $\phi_j : V_j \to \mathbb{R}^{2n}, j \in J$ finite, $M = \cup_{j \in J} V_j$, such that ω has the canonical form: $\omega = (\phi_j)_*(\sum_{i=1}^n dx_i \wedge dy_i)$. These will be called *symplectic coordinates*. In these coordinates the tangent map $Df_x : T_xM \to T_{f(x)}M$ can be seen as a symplectic matrix. For each $x \in M$ we fix coordinates ϕ_j with j the smallest number such that $x \in V_j$ and write Df_x using these coordinates at x and $f(x)$. From now on when we talk about l-dominance we use the euclidean norm in these coordinates. We also use them when we talk about the di! stance between two functions, or the size of a perturbation.

Now the question we ask is the following: suppose we have two unit vectors $u, v \in T_xM$ and $\mathbb{R}u$ doesn't l-dominate $\mathbb{R}v$, $\|Df_x^l(u)\| \leq 2\|Df_x^l(v)\|$. Obviously we can perturb the tangent map along the first l iterates of x moving u in the direction of v. Now using Frank's lemma (see [**Fr**]) we can realize these purely algebraic perturbations as the tangent map of a perturbation f' of f and get that $D{f'}_x^l(u) = cDf_x^l(v)$ for some real constant c. This perturbation is supported in an arbitrarily small neighborhood of the first l iterates of x and leaves these iterates unchanged and the size of the perturbation is proportional to the size of the algebraic perturbations of the tangent map. The question is how small will this perturbation be. We prove that it depends only on l and on the upper bound of the norms of the derivatives of f. If l is arbitrarily large then the perturbation needed can be arbitrarily small.

For the 2-dimensional case or the non-symplectic case a proof can be found in [**Ma1**], [**Ma2**].

Lemma 4. *For any $\epsilon > 0$, $K > 0$, there exist an $l \in \mathbb{N}$ such that if $A_0, A_1, \ldots, A_l$ are (symplectic, with determinant 1) $2n$-dimensional matrices with $\|A_k\| \leq K, k \in \{0, 1, \ldots, l\}$ and $\|A_{l-1} \ldots A_1 A_0(u)\| \leq 2\|A_{l-1} \ldots A_1 A_0(v)\|$ for some unit vectors*

$u, v \in \mathbb{R}^{2n}$ then there exist (symplectic, with determinant 1) matrices $A'_0, A'_1, \ldots, A'_l$ with $\|A_k - A'_k\| < \epsilon, k \in \{0, 1, \ldots, l\}$, such that

$$A'_l A'_{l-1} \ldots A'_1 A'_0(u) = c A_l A_{l-1} \ldots A_1 A_0(v)$$

for some nonzero real number c.

Proof. We will define A'_j as compositions of A_j with symplectic matrices close to the identity, which clearly will work in all the three cases. We give the proof only for the case $n \geq 2$.

We will choose l later in the proof. Denote $A_k A_{k-1} \ldots A_0 = B_k$ for all $k \in \{0, 1, \ldots, l\}$. For every such k we will consider an orthonormal symplectic basis in $\mathbb{R}^{2n}$, $\{e_1^k, \ldots, e_n^k, f_1^k, \ldots, f_n^k\}$ such that $\frac{B_k(u)}{\|B_k(u)\|} = e_1^k$ and $\frac{B_k(v)}{\|B_k(v)\|} = a_{k1} e_1^k + a_{k2} e_2^k + b_{k1} f_1^k$ for some $a_{k1}, a_{k2}, b_{k1} \in \mathbb{R}, a_{k1}^2 + a_{k2}^2 + b_{k1}^2 = 1$. We can also make this choice such that a_{k2} and b_{k1} have the same sign. By symplectic basis we mean that $\omega(e_i^k, f_i^k) = -\omega(f_i^k, e_i^k) = 1$ and $\omega(e_i^k, e_j^k) = \omega(f_i^k, f_j^k) = \omega(e_i^k, f_j^k) = \omega(f_i^k, e_j^k) = 0$ for all $i, j \in \{1, 2, \ldots, n\}, i \neq j$.

We divide the proof in two steps. In the first one we prove that if the angle between the iterates of u and v is small at some point then we can use a small rotation moving one into another. In the second step, if the angles between the iterates of u and v are bounded away from 0, then at the first step we make a small rotation of u toward v, then we make small perturbations along the orbit contracting u and expanding v thus the new orbit of u will move toward v in long enough time, then we complete with another small rotation in the end to move this new orbit of u to a multiple of $B_l(v)$.

Step 1. If the angle between $B_j(u)$ and $B_j(v)$ is small enough for some $j \in \{0, 1, \dots, l\}$ then we can construct a perturbation moving $B_j(u)$ to the direction of $B_j(v)$ only at the j'th step: if $a_{j1} > 0$ let $A'_j = RA_j$ where R is a symplectic linear map such that

$$R(e_1^j) = a_{j1}e_1^j + a_{j2}e_2^j + b_{j1}f_1^j$$

$$R(e_2^j) = a_{j1}^{-1}e_2^j$$

$$R(f_1^j) = a_{j1}^{-1}f_1^j$$

$$R(f_2^j) = a_{j1}f_2^j - a_{j2}f_1^j$$

and R leaves the other vectors of the basis unchanged. This map moves $B_j(u)$ to a multiple of $B_j(v)$. If $a_{j1} < 0$ we can just replace $B_j(v)$ by $-B_j(v)$. Clearly $\|R - Id\|$ tends to 0 as $|a_{j1}|$ tends to 1, so we can find $\alpha \in (0, 1)$ depending on ϵ and K such that if $|a_{j1}| > \alpha$ then $\|R - Id\| < \frac{\epsilon}{K}$ so $\|A'_j - A_j\| \leq \|R - Id\|\|A_j\| < \epsilon$. Taking $A'_i = A_i$ for all $i \in \{0, 1, \dots, l\} \setminus \{j\}$ we get the desired sequence of perturbation.

Step 2. Now we can suppose that $|a_{k1}| \leq \alpha$ for any k. Then we get that $|\frac{a_{k1}a_{k2}}{a_{k2}^2 + b_{k1}^2}| \leq \frac{\alpha}{1-\alpha^2}$ and $|\frac{a_{k1}b_{k1}}{a_{k2}^2 + b_{k1}^2}| \leq \frac{\alpha}{1-\alpha^2}$ for all k. For a $\sigma > 1$ we can consider the symplectic linear map T_k such that

$$T_k(e_1^k) = \frac{1}{\sigma}e_1^k$$

$$T_k(e_2^k) = (\sigma - \frac{1}{\sigma})\frac{a_{k1}a_{k2}}{a_{k2}^2 + b_{k1}^2}e_1^k + \sigma e_2^k + (\sigma - \frac{1}{\sigma})\frac{a_{k1}b_{k1}}{a_{k2}^2 + b_{k1}^2}f_2^k$$

$$T_k(f_1^k) = (\sigma - \frac{1}{\sigma})\frac{a_{k1}b_{k1}}{a_{k2}^2 + b_{k1}^2}e_1^k + \sigma f_1^k - (\sigma - \frac{1}{\sigma})\frac{a_{k1}a_{k2}}{a_{k2}^2 + b_{k1}^2}f_2^k$$

$$T_k(f_2^k) = \frac{1}{\sigma}f_2^k$$

and all the other vectors of the basis are unchanged. This map has the property that $T_k B_k(u) = \frac{1}{\sigma}B_k(u)$ and $T_k B_k(v) = \sigma B_k(v)$. Also $\|T_k - Id\|$ tends to 0 as σ tends to 1 uniformly with respect to $\frac{a_{k1}a_{k2}}{a_{k2}^2+b_{k1}^2}$ and $\frac{a_{k1}b_{k1}}{a_{k2}^2+b_{k1}^2}$ on compact sets, so there exist a $\sigma_0 > 1$ depending on ϵ and K such that if $1 \leq \sigma \leq \sigma_0$ then $\|T_k - Id\| < \frac{\epsilon}{K}$ as long as $|\frac{a_{k1}a_{k2}}{a_{k2}^2+b_{k1}^2}| \leq \frac{\alpha}{1-\alpha^2}$ and $|\frac{a_{k1}b_{k1}}{a_{k2}^2+b_{k1}^2}| \leq \frac{\alpha}{1-\alpha^2}$. From now on we fix $\sigma = \sigma_0$. Let $A'_k = T_k A_k, k \in \{1, 2, \ldots, l-1\}$. For $\delta = 1 - \alpha$ the angle between u and $u + \delta v$ is small enough so we can find a symplectic map R such that $R(u) = u + \delta v$ and $\|R - Id\| < \frac{\epsilon}{K}$ (see the first step of the proof of the lemma). Let $A'_0 = A_0 R$.

From the construction of the perturbations we have that

$$A'_{l-1} \ldots A'_1 A_0(u) = \sigma^{-(l-1)} A_{l-1} \ldots A_1 A_0(u),$$

$$A'_{l-1} \ldots A'_1 A_0(v) = \sigma^{l-1} A_{l-1} \ldots A_1 A_0(v),$$

and consequently

$$A'_{l-1} \ldots A'_1 A'_0(u) = \sigma^{-(l-1)} A_{l-1} \ldots A_1 A_0(u) + \delta\sigma^{l-1} A_{l-1} \ldots A_1 A_0(v),$$

or $A'_{l-1}\dots A'_1A'_0(u) = \sigma^{-(l-1)}B_{l-1}(u) + \delta\sigma^{l-1}B_{l-1}(v)$. If we choose l such that $\sigma^{l-1} > \frac{2}{\delta}$ and we use the hypothesis of non-dominance, $\|B_{l-1}(u)\| \leq 2\|B_{l-1}(v)\|$, we get that the angle between $B_{l-1}(v)$ and $A'_{l-1}\dots A'_1A'_0(u)$ is small enough so there exist a symplectic map R' such that $R'A'_{l-1}\dots A'_1A'_0(u) = cB_{l-1}(v)$ for some real number c and $\|R' - Id\| < \frac{\epsilon}{K}$. Now if we let $A'_l = A_lR'$ then we get the conclusion of the lemma.

$\square$

Remark. The lemma is also true for the odd dimensional case (not for symplectic matrices). The proof is easier, the perturbations required are restricted to a two dimensional subspace.

Definition. For a fixed $K > 0$ we define a decreasing function $e : \mathbb{N} \to \mathbb{R}$ as follows: given an $l \in \mathbb{N}$, then we define $e(l)$ to be the smallest positive number such that for any sequence of $2n$-dimensional symplectic matrices $A_1, A_2, \dots A_l$ with $\|A_k\| \leq K, k \in \{1, 2, \dots, l\}$ and any two unit vectors $u, v \in \mathbb{R}^{2n}$ such that $\|A_{l-1}\dots A_2A_1(u)\| \leq 2\|A_{l-1}\dots A_2A_1(v)\|$ there exist symplectic matrices $A'_1, A'_2, \dots, A'_l$ with $\|A_k - A'_k\| \leq \frac{e(l)}{2}, k \in \{1, 2, \dots, l\}$, such that

$$A'_lA'_{l-1}\dots A'_2A'_1(u) = cA_lA_{l-1}\dots A_2A_1(v)$$

for some nonzero real number c. The lemma says that $\lim_{l\to\infty} e(l) = 0$.

In the same way for a given symplectic manifold M with fixed symplectic charts and for a fixed $K > 0$ we define a decreasing function $E : \mathbb{N} \to R$ as follows: for any $l \in \mathbb{N}$ we define $E(l)$ to be the smallest positive number such that for any $f \in \mathbf{Diff}^1_\omega(M)$ with $\|Df_x\| \leq K, \forall x \in M$ in the given charts and any two unit vectors $u, v \in T_xM$ for some $x \in M$ such that $\|Df^l_x(u)\| \leq 2\|Df^l_x(v)\|$ there exist a perturbation f' of f of size $\frac{E(l)}{2}$ supported on an arbitrarily small neighborhood of $\{x, f(x), \dots f^l(x)\}$ such that ${f'}^k(x) = f^k(x), 1 \leq k \leq l$ and $D{f'}^l_x(u) = cDf^l_x(v)$ for some real number c. Because of the Frank's lemma mentioned above we also have that $\lim_{l\to\infty} E(l) = 0$.

4.3. Lyapunov filtrations for the invariant manifolds of hyperbolic periodic points

For a measure preserving diffeomorphism almost all the points of the manifold have well defined Lyapunov exponents and a corresponding Oseledets splitting that give the exponential rate of expansion of the vectors in the tangent bundle under the tangent map. More specific, if $f \in \mathbf{Diff}^1(M)$ preserves a measure μ then for μ-almost every point $x \in M$ there exist real numbers $\hat{\lambda}_1(x) < \hat{\lambda}_2(x) < \cdots < \hat{\lambda}_{k(x)}(x)$ and an invariant splitting $T_xM = E^1(x) \oplus E^2(x) \oplus \cdots \oplus E^{k(x)}(x)$ with $\dim E^1(x) + \dim E^2(x) + \cdots + \dim E^{k(x)}(x) = \dim M$ such that

$$\lim_{|l|\to\infty} \frac{\log \|Df^l_x(u)\|}{l} = \hat{\lambda}_i(x), \forall u \in E^i(x), i \in \{1, 2, \dots k(x)\},$$

and for any two disjoint subsets I and J of $\{1, 2, \dots, k(x)\}$ we have

$$\lim_{|l|\to\infty} \frac{1}{l} \log \angle(\oplus_{i\in I} E^i_{f^l(x)}, \oplus_{i\in J} E^i_{f^l(x)}) = 0.$$

Such a point x is called *Lyapunov regular*, the real numbers $\hat{\lambda}_i, i \in \{1, 2, \ldots, k(x)\}$ are called the *Lyapunov exponents* of x and $E_x^i, i \in \{1, 2, \ldots, k(x)\}$ are called *Lyapunov subspaces*. The splitting it is also called the *Oseledets splitting*. The dimension of each E_x^i, $m_i(x)$, is called the *multiplicity* of $\hat{\lambda}_i(x)$.

The Lyapunov regular points $x \in M$ also have invariant forward and backward *Lyapunov filtrations* : $\{0\} = \mathbf{B}_x^0 \subset \mathbf{B}_x^1 \subset \mathbf{B}_x^2 \subset \cdots \subset \mathbf{B}_x^{k(x)} = T_xM$ and $\{0\} = \mathbf{A}_x^{k(x)+1} \subset \mathbf{A}_x^{k(x)} \subset \mathbf{A}_x^{k(x)-1} \subset \cdots \subset \mathbf{A}_x^1 = T_xM$ such that for every $i \in \{1, 2, \ldots, k(x)\}$ we have:

$$\lim_{l \to \infty} \frac{\log \|Df_x^l(v)\|}{l} = \hat{\lambda}_i(x), \forall v \in \mathbf{B}_x^i \setminus \mathbf{B}_x^{i-1},$$

$$\lim_{l \to \infty} \frac{\log \|Df_x^{-l}(u)\|}{l} = \hat{\lambda}_i(x), \forall u \in \mathbf{A}_x^i \setminus \mathbf{A}_x^{i+1}.$$

In this case $\mathbf{B}_x^i = \oplus_{j=1}^i E_x^j$ and $\mathbf{A}_x^i = \oplus_{j=i}^{k(x)} E_x^j$. Forward and/or backward Lyapunov filtrations may exist for other points on the manifold which are not Lyapunov regular.

Every periodic point is Lyapunov regular. For example, if x is a periodic point of period p then the Lyapunov exponents are of the form $\lambda = \frac{\log |\gamma|}{p}$ where γ is an eigenvalue of Df_x^p and the Lyapunov subspaces of the Oseledets splitting are given by the direct sum of the corresponding (generalized) eigenspaces. If we have different eigenvalues of the same norm (as is the case of complex eigenvalues) then the number of exponents is smaller then the number of eigenvalues.

We consider $\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_{\dim M}(x)$ be the Lyapunov exponents $\hat{\lambda}_i(x)$ of a regular point x repeated with multiplicity $m_i(x)$. Also define $\Lambda_i(x) = \lambda_i(x) + \lambda_{i+1}(x) + \cdots + \lambda_{\dim M}(x)$ and the corresponding $\hat{\Lambda}_i(x) = m_i(x)\hat{\lambda}_i(x) + m_{i+1}(x)\hat{\lambda}_{i+1}(x) + \cdots + m_{k(x)}(x)\hat{\lambda}_{k(x)}(x)$. $\Lambda_i(x)$ (or $\hat{\Lambda}_i(x)$) is actually the maximal exponential growth of the $n-i+1$-dimensional (or $m_i(x)+m_{i+1}(x)+\cdots+m_k(x)$-dimensional) volume under Df.

In the symplectic case the Lyapunov exponents come in pairs. If λ is an exponent with multiplicity m then $-\lambda$ is also an exponent with the same multiplicity m. If the dimension of the manifold M is $2n$, then when we count the eigenvalue with their multiplicity we will denote them $\lambda_{-n} \leq \lambda_{-n+1} \leq \cdots \leq \lambda_{-1} \leq 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and we have $\lambda_{-i} = -\lambda_i, \forall i \in \{1, 2, \ldots, n\}$.

The following lemma proves that the points on the stable manifold of a hyperbolic periodic point have a forward Lyapunov filtration and the points on the unstable manifold have a backward Lyapunov filtration, with the same exponents as the ones of the periodic point. As a consequence the homoclinic points will have both forward and backward Lyapunov filtrations.

Lemma 5. *Given an hyperbolic periodic point x for the diffeomorphism f of the compact manifold M with Lyapunov exponents $\hat{\lambda}_1 < \hat{\lambda}_2 < \cdots < \hat{\lambda}_k$ and corresponding Lyapunov subspaces $E^1, E^2, \ldots E^k$ then for any point p on the unstable manifold of x we have a filtration of the tangent space $\{0\} = \mathbf{A}_p^{k+1} \subset \mathbf{A}_p^k \subset \mathbf{A}_p^{k-1} \subset \cdots \subset \mathbf{A}_p^1 = T_pM$ (all the inclusions here are strict) which is invariant under Df*

and satisfy the following:

$$\lim_{l\to\infty} \mathbf{A}^j_{f^{-l}(p)} = E^k \oplus E^{k-1} \oplus \cdots \oplus E^j = \mathbf{A}^j_x,$$

$$\lim_{l\to\infty} \frac{-\log \|Df_p^{-l}(u)\|}{l} = \hat{\lambda}_j, \forall u \in \mathbf{A}^j_p \setminus \mathbf{A}^{j+1}_p,$$

and for any $j \in \{1, 2, \ldots, k\}$ *and any basis* $\{u_1, u_2, \ldots u_{d(j)}\}$ *of* $\mathbf{A}^j_p$ *we have*

$$\lim_{l\to\infty} \frac{-\log \| \bigwedge^{d(j)} Df^{-l}(u_1 \wedge u_2 \wedge \cdots \wedge u_{d(j)})\|}{l} = \hat{\Lambda}_j$$

(the $d(j)$ *dimensional volume of* $\mathbf{A}^j_p$ *decreases under backward iterates with an exponential rate of* $\hat{\Lambda}_j$*).*

Similarly for any point q *on the stable manifold of* x *we have a filtration* $\{0\} = \mathbf{B}^0_q \subset \mathbf{B}^1_q \subset \mathbf{B}^2_q \subset \cdots \subset \mathbf{B}^k_q = T_pM$ *again invariant and satisfying*

$$\lim_{l\to\infty} \mathbf{B}^j_{f^l(p)} = E^1 \oplus E^2 \oplus \cdots \oplus E^j = \mathbf{B}^j_x,$$

$$\lim_{l\to\infty} \frac{\log \|Df_p^{l}(v)\|}{l} = \hat{\lambda}_j, \forall v \in \mathbf{B}^j_p \setminus \mathbf{B}^{j-1}_p,$$

and for any $j \in \{1, 2, \ldots, k\}$ *and any basis* $\{v_1, v_2, \ldots v_{d'(j)}\}$ *of* $\mathbf{B}^j_p$ *we have*

$$\lim_{l\to\infty} \frac{\log \| \bigwedge^{d'(j)} Df^{l}(v_1 \wedge v_2 \wedge \cdots \wedge v_{d'(j)})\|}{l} = \hat{\Lambda}_k - \hat{\Lambda}_{j+1}.$$

Proof. First we observe that it is enough to define this filtrations on small local stable and unstable manifolds of x and extend them later by invariance. So we suppose p and q are on $W^u_\epsilon(x)$ respectively $W^s_\epsilon(x)$ for some small ϵ and inside a chart around x. If the map is linear or C^1 conjugated to a linear map in a neighborhood of x the proof is trivial. In general we denote $\mathbf{A}_x^j = E^k \oplus E^{k-1} \oplus \cdots \oplus E^j \subset T_xM$ and for any y close to x we denote $\mathbf{A}_x^j(y)$ the translation of $\mathbf{A}_x^j$ to T_yM using the given chart. We define $\mathbf{A}_p^j = \lim_{l\to\infty} Df^l_{f^{-l}(p)}\mathbf{A}_x^j(f^{-l}(p))$. Now one can prove that this is well defined and verifies the required properties using some cone fields in a neighborhood of x.

So let us define the cones

$$C_x^{j,\epsilon} = \mathbb{R}\{a + \epsilon' b, a \in \mathbf{A}_x^j, b \in \mathbf{B}_x^j, \|a\| = \|b\| = 1, \epsilon' < \epsilon\} =$$

$$= \{a + b, a \in \mathbf{A}_x^j, b \in \mathbf{B}_x^j, \|b\| < \epsilon\|a\|\}.$$

All this cones are positively strictly invariant, i.e. $\mathrm{cl}(Df(C_x^{j,\epsilon})) \subset C_x^{j,\epsilon}$. For $\epsilon = 0$, $\delta > 0$ fixed, by replacing eventually f by a power of f we have that any vector in $C_x^{j,0}$ expands under Df by a factor between $e^{\hat{\lambda}_j-\delta}$ and $e^{\hat{\lambda}_n+\delta}$ and the $d(j)$ dimensional volume of the $d(j)$ dimensional subspaces in $C_x^{j,0}$ (which in this case is only $C_x^{j,0}$) expands under $\bigwedge^{d(j)} Df$ by a factor between $e^{\hat{\Lambda}_j-\delta}$ and $e^{\hat{\Lambda}_j+\delta}$. Then we can find $\epsilon > 0$ such that:

$$\hat{\lambda}_j - \delta < \log\frac{\|Df(u)\|}{\|u\|} < \hat{\lambda}_n + \delta, \forall u \in C_x^{j,\epsilon},$$

$$\hat{\Lambda}_j - \delta < \log\frac{\|\bigwedge^{d(j)} Df(u_1 \wedge u_2 \wedge \cdots \wedge u_{d(j)})\|}{\|u_1 \wedge u_2 \wedge \cdots \wedge u_{d(j)}\|} < \hat{\Lambda}_j + \delta,$$

$\forall u_1, u_2, \ldots, u_{d(j)}$ linearly independent, $\operatorname{span}\{u_1, u_2, \ldots u_{d(j)}\} \subset C_x^{j,\epsilon}$.

Because f is C^1, we can construct a strictly invariant continuous cone field $C_y^{j,\epsilon}$ for y on a small neighborhood of x extending $C_x^{j,\epsilon}$ and having the properties mentioned above. For l large enough $\mathbf{A}_x^j(f^{-l}(p)) \subset C_{f^{-l}(p)}^{j,\epsilon}$ and because the cones are contracting in a neighborhood of x we get that $\mathbf{A}_p^j$ is well defined and obviously $\lim_{l\to\infty} \mathbf{A}_{f^{-l}(p)}^j = \mathbf{A}_x^j$. Also $Df^{-l}\mathbf{A}_p^j = \mathbf{A}_{f^{-l}(p)}^j \subset C_{f^{-l}(p)}^{j,\epsilon}$ for all large enough l so

$$\hat{\lambda}_j - \delta \leq \lim_{l\to\infty} \frac{\log \|Df^{-l}(u)\|}{-l} \leq \hat{\lambda}_n + \delta, \forall u \in \mathbf{A}_p^j,$$

$$\hat{\Lambda}_j - \delta \leq \lim_{l\to\infty} \frac{\log \| \bigwedge^{d(j)} Df^{-l}(u_1 \wedge u_2 \wedge \cdots \wedge u_{d(j)})}{-l} \leq \hat{\Lambda}_j + \delta$$

for any basis $u_1, u_2, \ldots u_{d(j)}$ of $\mathbf{A}_p^j$.

Taking the limit for $\delta \to 0$ we get the desired equality for the $d(j)$ dimensional volume and

$$\hat{\lambda}_j \leq \lim_{l\to\infty} \frac{\log \|Df^{-l}(u)\|}{-l} \leq \hat{\lambda}_n, \forall u \in \mathbf{A}_p^j.$$

We can see from definition that $\mathbf{A}_p^{j+1} \subset \mathbf{A}_p^j$ so for any vector $u = u_{d(j)} \in \mathbf{A}_p^j \setminus \mathbf{A}_p^{j+1}$ we can chose vectors $u_1, u_2, \ldots u_{d(j)-1}$ such that $\{u_1, u_2, \ldots u_{d(j+1)}\}$ is a basis for $\mathbf{A}_p^{j+1}$ and $\{u_1, u_2, \ldots u_{d(j)}\}$ is a basis for $\mathbf{A}_p^j$. Now using the fact that the exponential volume growth on $\mathbf{A}_p^j$ and $\mathbf{A}_p^{j+1}$ under Df^{-1} is $-\hat{\Lambda}_j$ respectively $-\hat{\Lambda}_{j+1}$ we get that the exponential volume growth under Df^{-1} on the subspace spanned by $\{u_{d(j+1)+1}, \ldots u_{d(j)}\}$ must be at least $-\hat{\Lambda}_j + \hat{\Lambda}_{j+1} = -m_j\hat{\lambda}_j$. But we know that the exponential growth of the vectors $u_{d(j+1)+1}, \ldots, u_{d(j)}$ under Df^{-1} is at most

$-\hat{\lambda}_j$ and $m_j = d(j) - d(j+1)$. From this we can conclude the required relation from the lemma.

For the subspaces $\mathbf{B}_q^j$ for q on the stable manifold of x the proof is similar. $\square$

Remark. This result can be extended for the partially hyperbolic periodic points - we can allow some zero Lyapunov exponents and consider the points on the strong stable and strong unstable manifolds.

4.4. The main perturbation result

We say that a splitting $TM = \mathbf{A} \oplus \mathbf{B}$ has *index* k if the dimension of $\mathbf{A}$ is k.

Proposition 1. *Suppose $f \in \mathbf{Diff}^1_\omega(M)$, $x \in M$ is a hyperbolic periodic point for f and $\lambda_{i+1}(f,x) - \lambda_i(f,x) > \delta > 0$. Also suppose that f does not have a l-dominated decomposition of index $n-i$. Then there is a perturbation of f of size less than $E(l)$, say g, and $y \in M$ a hyperbolic periodic point for g arbitrarily close to x such that $\Lambda_{i+1}(g,y) < \Lambda_{i+1}(f,x) - \frac{\delta}{2}$ and $\lambda_n(g,y) \leq \lambda_n(f,x)$.*

Proof. First we remark that if f doesn't have a l-dominated decomposition of index $n-i$ then the same must be true for any other function in a small C^1 neighborhood of f. Indeed, if this is not true, we can find a sequence of diffeomorphisms f_n converging to f with l-dominated splittings of index $n-i$. For a subsequence the corresponding subbundles will converge to invariant subbundles for f and by taking the limit we get that this must be also a l-dominated splitting, which is a contradiction.

The strategy is the following: first we make an arbitrarily small perturbation f_1 of f in order to create dense homoclinic points for (the continuation of) x using Lemma 3. We can choose the perturbation small enough so that the continuation of x and its new Lyapunov exponents are arbitrarily close to the old ones and there is no l-dominated splitting of index $n-i$. Let $H(x)$ be the set of homoclinic points of x which is dense in M. Then we use Lemma 5 to define the invariant subbundles $\mathbf{A}$ and $\mathbf{B}$ for the points in $H(x)$ corresponding to the Lyapunov exponents greater or equal to $\lambda_{i+1}(f_1, x)$ (using the backward iterates) respectively smaller or equal to $\lambda_i(f_1, x)$ (using the forward iterates). If $\mathbf{A}$ l-dominates $\mathbf{B}$, then we can extend these subbundles by continuity to the whole manifold M and get an l-dominated splitting, which is a contradiction.

So we can suppose that $\mathbf{A}$ doesn't l-dominate $\mathbf{B}$, which means that there exist a homoclinic point p of x and unit vectors $u \in \mathbf{A}_p$, $v \in \mathbf{B}_p$ such that $\|Df_1^l p(u)\| < 2\|Df_1^l p(v)\|$. Now we can use Lemma 4 to create a perturbation f_2 of f_1 of size less than $\frac{E(l)}{2}$ moving the vector u to a multiple of $Df_1^l p(v)$. The perturbation is supported on an arbitrarily small neighborhood of the first l iterates of p, and it doesn't change the orbit of p, so we can suppose that the orbit and the Lyapunov exponents of x are unchanged and p is still an homoclinic point of x. Also $\mathbf{A}$ is unchanged for the backward iterates of p and $\mathbf{B}$ is unchanged for the forward iterates of $f_1^l(p) = f_2^l(p)$ so we have the vector $u \in \mathbf{A}_p \cap \mathbf{B}_p$ for f_2. Without loss of generality we'll denote $f_2 = f$.

In order to finish the proof, we want to close the orbit of p with another arbitrarily small perturbation supported on a small neighborhood of x and thus get the desired hyperbolic periodic orbit with smaller Λ_{i+1}. In the end we have a perturbation of size $\frac{E(l)}{2}$ and finitely many arbitrarily small perturbations, so the total the size of the perturbation will be smaller than $E(l)$.

The next lemma shows how to close the orbit of the homoclinic point p of x.

Lemma 6. *For any p an homoclinic point for the periodic hyperbolic point x of $f \in \mathbf{Diff}^r(M)$ (or $\mathbf{Diff}^r_\omega(M)$, where ω is either a symplectic or a volume form) there exist an arbitrarily small C^r perturbation of f in $\mathbf{Diff}^r(M)$ (or $\mathbf{Diff}^r_\omega(M)$) supported in an arbitrarily small neighborhood of x such that p becomes a periodic point.*

Proof. Let V be a neighborhood of f in $\mathbf{Diff}^r(M)$ (or $\mathbf{Diff}^r_\omega(M)$). We know that there is a small neighborhood U of x where f^k is C^0-conjugated to the linear map $Df^k(x)$ on a neighborhood of the origin, where k is the period of x. By shrinking eventually the neighborhood we can suppose that the orbit of p intersected with U is inside the ϵ-stable and ϵ-unstable manifolds of x for some small $\epsilon > 0$. Let $y = f^s(p), z = f^{-t}(p), s, t > 0$ be a forward and a backward iterate of p contained in U. Using the perturbation lemma (see for example [**Ne2**]) we can find small neighborhoods $U_y \subset \hat{U}_y \subset U$ of y such that $\hat{U}_y \cap f(\hat{U}_y) = \emptyset$ and for any $y' \in U_y$ there exist $f' \in V$ with $f'(y) = f(y')$ and $f' = f$ outside $\hat{U}_y$. In the same way we can find neighborhoods $U_z \subset \hat{U}_z \subset U$ of z using f^{-1}. Now because of the conjugacy to the linear map th! ere is a point $y' \in U_y$ such that a forward iterate of y', say $f^a(y')$ is in U_z. Now let $g \in V$ be a function such that $g(y) = f(y'), g(f^{a-1}(y')) = z$ and $g = f$ outside $\hat{U}_y \cup f^{-1}(\hat{U}_z)$ as before. Then p is a periodic point for g. Obviously U can be taken arbitrarily small. $\square$

We observe that the perturbation we made consists basically of two translations,on small neighborhoods of a positive and a negative iterate of p. Also the new orbit of p spends an arbitrarily long time in the neighborhood of x because we can take as many iterates of p (positive and negative) as we want before making the required translations. So the period of p for the perturbation g_N will be $2N + k$ where k is the number of iterates away from x (which is fixed) and $2N$ is the number of iterates close to x (which can be arbitrarily large). By changing the notation we can suppose that $g_N^j(p), -N \leq j \leq N-1$ are the iterates of p close to x. Furthermore for some $s, t < N$ we have that $g_N^s(p), g_N^{s+1}(p), \ldots g_N^{2N+k-t}(p)$ is a segment of the orbit of the homoclinic point for f. We remark that p actually depends on N, but $g_N^N(p)$ and $g_N^{-N}(p)$ don't. Because $g_N^j(p), -t \leq j \leq s$ are in an arbitrarily small neighborhood and ! f is C^1, using another small perturbation we can suppose that $Dg_N(g_N^j(p)) = Df(f^{j-s}(g_N^s(p))), 0 \leq j \leq s$ and $Dg_N(g_N^{-j}(p)) = Df(f^{t-j}(g_N^{-t}(p))), 1 \leq j \leq t$ (using the coordinate chart around x). In other words, for N positive iterates of p under g_N the derivative is the same as for N iterates of the homoclinic point under f situated on a local unstable manifold of x and similarly for the negative iterates. Because of this we can define

$\mathbf{A}_{g_N^j(p)}, 0 \leq j \leq s$ as the pulled back of $\mathbf{A}_{g_N^s(p)}$ and $\mathbf{B}_{g_N^{-j}(p)}, 0 \leq j \leq t$ as the pushed forward of $\mathbf{B}_{g_N^{-t}(p)}$ under Dg_N and we have the relations:

$$-\lambda_n(f,x) \leq \lim_{N\to\infty} \frac{\log \|Dg^{-N}_{N,g_N^N(p)}(v)\|}{N} \leq -\lambda_{i+1}(f,x), \forall v \in \mathbf{A}_{g_N^N(p)},$$

$$-\lambda_n(f,x) \leq \lim_{N\to\infty} \frac{\log \|Dg^{N}_{N,g_N^{-N}(p)}(u)\|}{N} \leq \lambda_i(f,x), \forall u \in \mathbf{B}_{g_N^{-N}(p)}.$$

Also if $v_n, v_{n-1}, \ldots, v_{i+1}$ is an orthonormal basis for $\mathbf{A}_{g_N^N(p)}$ we have

$$\lim_{N\to\infty} \frac{\log \| \bigwedge^{n-i} Dg^{-N}_{N,g_N^N(p)}(v_n \wedge \cdots \wedge v_{i+1})\|}{N} = -\Lambda_{i+1}(f,x).$$

Now we will estimate $\Lambda_{i+1}(g_N, p)$. We know that

$$\Lambda_{i+1}(g_N,p) \leq \frac{\log \| \bigwedge^{n-i} Dg^{2N+k}_{N,p}\|}{2N+k}.$$

We want to choose a convenient basis $\{e_{-n}, e_{-n+1}, \ldots, e_{-1}, e_1, \ldots, e_n\}$ of T_pM such that e_j expands on the first N iterates of Dg_N with an exponential rate close to $\lambda_j(f,x)$ and consequently $e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-i}}$ expands on the first iterates of Dg_N with an exponential rate not much bigger than $\lambda_{j_1}(f,x)+\lambda_{j_2}(f,x)+\cdots+\lambda_{j_{n-i}}(f,x)$ (the reason we get an estimation only from above for the volume growth is because the angles between vectors may decrease, but that's all what we actually need). For simplicity from now on we denote $\lambda_j = \lambda_j(x,f)$.

We start by supposing that the Lyapunov subspaces at x are orthogonal (this can be done by a change of coordinates, a change of basis doesn't change the eigenvalues or the Lyapunov exponents). Then we choose an orthonormal basis $\{e'_{-n}, e'_{-n+1}, \ldots, e'_{-1}, e'_1, \ldots, e'_n\}$ of $T_{g_N^N(p)}M$ such that it agrees with the backward Lyapunov filtration for f:

$$\lim_{l\to\infty} \frac{\log \|Df^{-l}(e'_j)\|}{-l} = \lambda_j \text{ and}$$

$$\lim_{l\to\infty} \frac{\log \| \bigwedge^{n-j} Df^{-l}(e'_n \wedge e'_{n-1} \wedge \cdots \wedge e'_{j+1})\|}{-l} = \lambda_n + \cdots + \lambda_{j+1} = \Lambda_{j+1}$$

if $\lambda_{j+1} > \lambda_j$ (this can be done using Lemma 5). Also if $e'_j, \ldots e'_k$ are the vectors corresponding to a (possible multiple) exponent then the pull back under Df of the subspace generated by this vectors will converge to the corresponding Lyapunov subspace at x. Now if a Lyapunov exponent λ_j is simple then we take $e_j = \frac{Dg_N^{-N}(e'_j)}{\|Dg_N^{-N}(e'_j)\|}$. If $\lambda_{j-1} < \lambda_j = \cdots = \lambda_k < \lambda_{k+1}$ then we take $\{e_j, e_{j+1}, \ldots, e_k\}$ to be an orthonormal basis of $Dg_N^{-N}(\text{span}\{e'_j, \ldots, e'_k\})$. Then this basis of T_pM is 'almost' orthonormal, $\langle e_j, e_k\rangle$ is small for large N, and satisfies our requirements.

Now $\{e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-i}} \colon -n \le j_1 < j_2 < \cdots < j_{n-i} \le n\}$ is a basis for $\bigwedge^{n-i} T_pM$ which is again 'almost' orthonormal (for the dot product induced by the one on T_pM) so it is enough to estimate $\| \bigwedge^{n-i} Dg_N^{2N+k}(e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-i}})\|$. Suppose now that we have $\epsilon > 0$ fixed.

First case: $e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-i}} \neq e_{i+1} \wedge e_{i+2} \wedge \cdots \wedge e_n$. Then for large enough N we have

$$\log \| \bigwedge^{n-i} Dg_N^N(e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-i}})\| < N(\lambda_{j_1} + \cdots + \lambda_{j_{n-i}} + \epsilon) \le N(\Lambda_{i+1} - \delta + \epsilon)$$

and furthermore

$$\log \| \bigwedge^{n-i} Dg_N^{N+k}(e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-i}})\| < N(\Lambda_{i+1} - \delta + \epsilon) + K$$

where K is a constant which doesn't depend on N.

In order to evaluate the last N iterates we consider an orthonormal basis $\{\bar{e}_{-n}, \bar{e}_{-n+1}, \ldots, \bar{e}_{-1}, \bar{e}_1, \ldots, \bar{e}_n\}$ of $T_{g_N^{-N}(p)}M$ which agrees with the forward Lyapunov filtration for f: $\lim_{l\to\infty} \frac{\log \|Df^l(\bar{e}_j)\|}{l} = \lambda_j$ (this can be done again using Lemma 5). Then we get that

$$\lim_{l\to\infty} \frac{\log \| \bigwedge^{n-i} Df^l(\bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_{n-i}})\|}{l} \leq \lambda_{j_1} + \cdots + \lambda_{j_{n-i}} \leq \Lambda_{i+1}.$$

So for large enough N we have $\| \bigwedge^{n-i} Dg_N^N(\bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_{n-i}})\| < \mathrm{e}^{N(\Lambda_{i+1}+\epsilon)}$ for any $-n \leq j_1 < j_2 < \cdots < j_{n-i} \leq n$. But

$$\{\bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_{n-i}} \colon -n \leq j_1 < j_2 < \cdots < j_{n-i} \leq n\}$$

is again an orthonormal basis of $\bigwedge^{n-i} T_{g_N^{-N}(p)}M$ which has dimension $\binom{n}{n-i}$ so $\| \bigwedge^{n-i} Dg_N^N|_{T_{g_N^{-N}(p)}}\| < \binom{n}{n-i}\mathrm{e}^{N(\Lambda_{i+1}+\epsilon)}$. Taking the log and combining with the inequality for the first $N+k$ iterates we get

$$\log \| \bigwedge^{n-i} Dg_N^{2N+k}(e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-i}})\| < 2N(\Lambda_{i+1} - \frac{\delta}{2} + \epsilon) + K'$$

for all large enough N where K' again doesn't depend on N. Dividing by $2N+k$ and taking N sufficiently large we get

$$\frac{\log \| \bigwedge^{n-i} Dg_N^{2N+k}(e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-i}})\|}{2N+k} < \Lambda_{i+1} - \frac{\delta}{2} + \epsilon$$

if $e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-i}} \neq e_{i+1} \wedge e_{i+2} \wedge \cdots \wedge e_n$.

Second case: the estimation for $e_{i+1} \wedge e_{i+2} \wedge \cdots \wedge e_n$. As in the first case we get for large enough N that

$$\log \| \bigwedge^{n-i} Dg_N^{N+k}(e_{i+1} \wedge e_{i+2} \wedge \cdots \wedge e_n)\| < N(\Lambda_{i+1} + \epsilon) + K.$$

We know that there is a unit vector $u \in \mathrm{span}\{e_{i+1}, e_{i+2}, \dots, e_n\} = \mathbf{A}_p^{i+1}$ such that $Dg_N^{N+k}(u) \in \mathbf{B}_{g_N^{-N}}(p) = \mathrm{span}\{\bar{e}_{-n}, \bar{e}_{-n+1}, \dots, \bar{e}_i\}$. This implies that $Dg_N^{N+k}(e_{i+1} \wedge e_{i+2} \wedge \cdots \wedge e_n)$ is inside the subspace of $\bigwedge^{n-i} T_{g_N^{-N}}(p)M$ generated by all $\bar{e}_{j_1} \wedge \bar{e}_{j_2} \wedge \cdots \wedge \bar{e}_{j_{n-i}}$ different from $\bar{e}_{i+1} \wedge \bar{e}_{i+2} \wedge \cdots \wedge \bar{e}_n$. In the same way we did for the first case we get that the norm of $\bigwedge^{n-i} Dg_N^N$ restricted to this subspace is bounded from above by $\binom{n}{n-i}\mathrm{e}^{N(\Lambda_{i+1}-\delta+\epsilon)}$ for large N. Again combining this two and taking N sufficiently large we get

$$\frac{\log \| \bigwedge^{n-i} Dg_N^{2N+k}(e_{i+1} \wedge e_{i+2} \wedge \cdots \wedge e_n)\|}{2N+k} < \Lambda_{i+1} - \frac{\delta}{2} + \epsilon.$$

To conclude, we have the inequality for all the elements of the basis, which is 'almost' orthonormal, so by taking N big enough we get the inequality for any unit vector (in $\bigwedge^{n-i} T_pM$):

$$\frac{\log \| \bigwedge^{n-i} Dg_N^{2N+k} \|}{2N+k} < \Lambda_{i+1} - \frac{\delta}{2} + 2\epsilon$$

which implies that $\Lambda_{i+1}(g_N, p) < \Lambda_{i+1}(f, x) - \frac{\delta}{2} + 2\epsilon$. Now we can do the same thing for a slightly larger δ and arbitrarily small ϵ, so eventually we get $\Lambda_{i+1}(g_N, p) < \Lambda_{i+1}(f, x) - \frac{\delta}{2}$.

The proof of the second inequality of the proposition is similar, but we don't have to take the two separate cases. It is enough to remark that the exponential rate of growth of any vector is bounded from above by λ_n around x so it's enough to take N large and we get $\lambda_n(g_N, p) < \lambda_n(f, x) + \epsilon$. But again ϵ can be chosen arbitrarily small so with another small perturbation we get $\lambda_n(g_N, p) \leq \lambda_n(f, x)$.

As a last remark because the period of p is arbitrarily large we can easily make sure that the point p is actually hyperbolic, so it doesn't have zero Lyapunov exponents. $\square$

Corollary 4. *Suppose $f \in \mathbf{Diff}^1_\omega(M)$ doesn't have a dominated decomposition of index $n-i$. Then there is an arbitrarily small C^1 perturbation g of f such that g has a periodic point x with $\lambda_i = \lambda_{i+1}$.*

Proof. We denote by $\mathrm{Per}(g)$ the set of hyperbolic periodic points of g. Suppose that the result is not true, so there exist a C^1 neighborhood V of f and a $\delta > 0$ such that for any function $g \in V$ and any $x \in \mathrm{Per}(g)$ we have $\lambda_{i+1}(g, x) - \lambda_i(g, x) > \delta$. We also let $\Lambda_{i+1}(g) = \inf_{x \in \mathrm{Per}(g)} \Lambda_{i+1}(g, x)$, $\Lambda = \liminf_{g \to f} \Lambda_{i+1}(g)$. This imply that there exist $g_n \to f$, $x_n \in \mathrm{Per}(g_n)$ such that $\Lambda_{i+1}(g_n, x_n) \to \Lambda$. We can suppose that $g_n \in V$ for all $n > 0$.

We will denote d the C^1 distance on $\mathbf{Diff}^1_\omega(M)$ using the fixed symplectic charts. Now for any $l \in \mathbb{N}$ there exist $n_l > l$ such that g_{n_l} doesn't have a l-dominated splitting of index $n-i$ (otherwise we can pass the decomposition to the limit and get one for f). We also know that $\lambda_{i+1}(g_{n_l}, x_{n_l}) - \lambda_i(g_{n_l}, x_{n_l}) > \delta$ so we can apply the proposition to find $h_l \in \mathbf{Diff}^1_\omega(M)$, $d(h_l, g_{n_l}) < E(l)$, $y_l \in \mathrm{Per}(h_l)$ such that $\Lambda_{i+1}(h_l, y_l) < \Lambda_{i+1}(g_{n_l}, x_{n_l}) - \frac{\delta}{2}$. Because $E(l) \to 0$ as $l \to \infty$ we get that $h_l \to f$. Also

$$\lim_{l\to\infty} \Lambda_{i+1}(h_l) \leq \lim_{l\to\infty} \Lambda_{i+1}(h_l, y_l) \leq \lim_{l\to\infty} \Lambda_{i+1}(g_{n_l}, x_{n_l}) - \frac{\delta}{2} = \Lambda - \frac{\delta}{2}$$

which is a contradiction. $\square$

Remark. One case in which $\lambda_i = \lambda_{i+1}$ is when the corresponding eigenvalues are complex conjugate. Periodic points of this type are used in [**BoDiPu**] to construct sinks or sources in the case of general diffeomorphisms or periodic points with all the eigenvalues of modulus 1 for volume preserving diffeomorphisms in the absence of dominance.

4.5. Proof of Theorem 3

Now we will give the proof of Theorem 3.

Proof. (Theorem 3) We will use the proposition to prove that for any open set U in M we can find an arbitrarily small perturbation of f with an elliptic periodic point in U.

So let's fix an open set U in M. Lemma 7 from the Appendix shows that for symplectic diffeomorphisms the existence of a dominated splitting is equivalent to partial hyperbolicity, so we know that there are no dominated splittings for f. We define the decreasing function $L : \mathbb{R}_+ \to \mathbb{N}$ as follows: for any $\epsilon > 0$ we let $L(\epsilon)$ to be the largest integer such that all the perturbations of f of size at most ϵ don't have a $L(\epsilon)$-dominated decomposition. Because for any $l > 0$ there are no sequences of diffeomorphisms with l-dominated splittings converging to f, there is a neighborhood V_l of f such that no function in V_l has an l-dominated splitting. This proves that $\lim_{\epsilon\to 0} L(\epsilon) = \infty$.

For any $g \in \mathbf{Diff}^1_\omega(M)$ we define $\lambda_n(g) = \inf_{x\in \mathrm{Per}(g)\cap U} \lambda_n(g,x)$ and $\lambda = \operatorname{liminf}_{g\to f} \lambda_n(g)$. Because of the C^1 closing lemma λ must be finite. If $\lambda = 0$ we are done, an arbitrarily small perturbation will make all the Lyapunov exponents of a periodic point zero and consequently we get an elliptic periodic point (making sure there are no multiple eigenvalues). So we can suppose that $\lambda > 0$.

There exist $f_k \to f$ (in the C^1 topology), $x_k \in \mathrm{Per}(f_k) \cap U$, such that $\lambda^k_n := \lambda_n(f_k, x_k) \to \lambda$. Our goal is to use the proposition several times to construct a sequence of perturbations g_k, still converging to f, and having some periodic points $y_k \in \mathrm{Per}(g_k) \cap U$ with $\lim_{k\to\infty} \lambda_n(g_k, y_k) \le (1-\alpha)\lambda$ which is a contradiction. We will choose $\alpha > 0$ later.

Suppose that $d(f, f_k) = \epsilon_k$ for $\epsilon_k > 0$ small. This means that f_k doesn't have any $L(\epsilon_k)$-dominated splitting. We denote $f_k = f_{k1}, x_k = x_{k1}$. There exist an $i_1 \in \{-1, 1, 2, \ldots, n-1\}$ such that

$$\lambda_{i_1+1}(f_{k1}, x_{k1}) - \lambda_{i_1}(f_{k1}, x_{k1}) < \frac{\lambda_n^k}{n} = \delta_1$$

Applying the proposition we can construct $f_{k2} \in \mathbf{Diff}^1_\omega(M)$, $d(f_{k2}, f_{k1}) < EL(\epsilon_k)$, with $x_{k2} \in \mathrm{Per}(f_{k2}) \cap U$ such that

$$\Lambda_{i_1+1}(f_{k2}, x_{k2}) < \Lambda_{i_1+1}(f_{k1}, x_{k1}) - \frac{\delta_1}{2},$$

$$\lambda_n(f_{k2}, x_{k2}) \leq \lambda_n^k.$$

Also

$$d(f_{k2}, f) \leq d(f_{k2}, f_{k1}) + d(f_k, f) < \epsilon_k + EL(\epsilon_k).$$

We will denote $\phi(\epsilon) = \epsilon + EL(\epsilon)$ and then we can rewrite this as $d(f_{k2}, f) < \phi(\epsilon_k)$.

If $\lambda_n(f_{k2}, x_{k2}) \leq (1-\alpha)\lambda_n^k$ then we stop and take $g_k = f_{k2}$ and $y_k = x_{k2}$. Otherwise there exist a $j_2 \in \{i_1+1, i_1+2, \ldots, n-1\}$ such that

$$\lambda_{j_2}(f_{k2}, x_{k2}) < \lambda_{j_2}(f_{k1}, x_{k1}) - \frac{\delta_1}{2n} \leq \lambda_n^k - \frac{\delta_1}{2n}.$$

So

$$\lambda_n^k \geq \lambda_n(f_{k2}, x_{k2}) > (1-\alpha)\lambda_n^k > \lambda_n^k - \frac{\delta_1}{2n} > \lambda_{j_2}(f_{k2}, x_{k2})$$

if $\alpha < \frac{1}{2n^2}$ (remember that $\delta_1 = \frac{\lambda_n^k}{n}$). This implies that there exist $i_2 \in \{j_2, j_2 + 1, \dots n-1\}$ such that

$$\lambda_{i_2+1}(f_{k2}, x_{k2}) - \lambda_{i_2}(f_{k2}, x_{k2}) > \frac{\delta_1}{2n^2} - \frac{\alpha\lambda_n^k}{n} = \frac{\lambda_n^k}{2n^3} - \frac{\alpha\lambda_n^k}{n} = \delta_2.$$

Because $d(f_{k2}, f) < \phi(\epsilon_k)$ we get that f_{k2} has no $L(\phi(\epsilon_k))$-dominated splitting. Applying again the proposition we get that there is an $f_{k3} \in \mathbf{Diff}^1_\omega(M)$, with $x_{k3} \in \mathrm{Per}(f_{k3}) \cap U$ such that

$$\Lambda_{i_2+1}(f_{k3}, x_{k3}) < \Lambda_{i_2+1}(f_{k2}, x_{k2}) - \frac{\delta_2}{2},$$

$$\lambda_n(f_{k3}, x_{k3}) \leq \lambda_n(f_{k2}, x_{k2}) \leq \lambda_n^k$$

and $d(f_{k3}, f_{k2}) < EL(\phi(\epsilon_k))$. We observe again that

$$d(f_{k3}, f) \leq d(f_{k3}, f_{k2}) + d(f_{k2}, f) < \phi(\epsilon_k) + EL(\phi(\epsilon_k)) = \phi^2(\epsilon_k).$$

Again if $\lambda_n(f_{k3}, x_{k3}) \leq (1-\alpha)\lambda_n^k$ then we let $g_k = f_{k3}$ and $y_k = x_{k3}$ and we stop. Otherwise, again under the condition that α is sufficiently small, there will be a gap of size at least $\delta_3 = \frac{\delta_2}{2n^2} - \frac{\alpha\lambda_n^k}{n}$ between $\lambda_{i_3+1}(f_{k3}, x_{k3})$ and $\lambda_{i_3}(f_{k3}, x_{k3})$ for some $i_3 > i_2$ and we apply again the proposition to lower Λ_{i_3+1} by at least $\frac{\delta_3}{2}$ for a perturbation f_{k4} and a hyperbolic periodic point x_{k4} in U. The distance from f_{k4} to f will be less that $\phi^3(\epsilon_k)$. We can repeat this argument and in the end α can be chosen sufficiently small (depends only on n) such that after a finite number of such perturbations (at most n) we actually lower λ_n by $\alpha\lambda_n^k$. So we can find indeed $g_k \in \mathbf{Diff}^1_\omega(M)$, $d(g_k, f) < \phi^n(\epsilon_k)$ where $\epsilon_k = d(f_k, f) \to 0$, and! $y_k \in \mathrm{Per}(g_k) \cap U$ such that $\lambda_n(g_k, y_k) \leq (1-\alpha)\lambda_n^k$. Then

$$\lim_{k\to\infty} \lambda_n(g_k, y_k) \leq \lim_{k\to\infty} (1-\alpha)\lambda_n^k = (1-\alpha)\lambda.$$

We also know that $\lim_{\epsilon\to 0} L(\epsilon) = \infty, \lim_{l\to\infty} E(l) = 0$ so $\lim_{\epsilon\to 0} \phi(\epsilon) = 0$ and furthermore $\lim_{\epsilon\to 0} \phi^n(\epsilon) = 0$ which shows that g_k converges to f in $\mathbf{Diff}^1_\omega(M)$ and we are done because we reached a contradiction. □

4.6. Proof of Theorem 2 and its corollaries

Proof. (Theorem 2) It is true that the set of partially hyperbolic diffeomorphisms in $\mathbf{Diff}^1_\omega(M)$, let's denote it $\mathcal{PH}$, is open and the set of diffeomorphism with elliptic periodic points, denoted $\mathcal{E}$, is also open. Now if f is not in $\mathcal{PH}$ then we apply Theorem 3 and get that f must be in the closure of $\mathcal{E}$. This proves that $\mathcal{U} = \mathcal{PH} \cup \mathcal{E}$ is open dense. For the second part, if we denote by $\mathcal{E}^\delta$ the set of diffeomorphisms with δ-dense elliptic periodic points, applying Theorem 3 for a finite number of open subsets one can prove that $\mathcal{PH} \cup \mathcal{E}^\delta$ is open dense in $\mathbf{Diff}^1_\omega(M)$, so taking $\mathcal{R}$ their intersection for $\delta = \frac{1}{k}, k \in \mathbb{N}$ we get the result. □

A map f is called C^1 *robustly transitive* if there is a C^1 neighborhood of f such that every map in this neighborhood is transitive. An example is any the Anosov map. It is known that the existence of an elliptic periodic point is an obstruction for C^1 robust transitivity. So as a simple consequence we get Corollary 1.

A C^2 map f is called *stable ergodic* if there is a C^1 neighborhood of f such that every C^2 map in this neighborhood is ergodic. Again this is true for Anosov maps. Using the fact that $\mathbf{Diff}^2_\omega(M)$ and $\mathbf{Diff}^\infty_\omega(M)$ are dense in $\mathbf{Diff}^1_\omega(M)$ in the C^1 topology (see [**Ze2**]) we can get Corollary 2.

The proof of Corollary 3 is by contradiction.

Remark. Theorem 3 is true also for the volume preserving diffeomorphisms, with the partial hyperbolicity replaced by the existence of a dominated splitting. The proof is similar to the symplectic case.

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APPENDIX

Dominated Splittings for Symplectic Diffeomorphisms

In this appendix we prove that for a symplectic diffeomorphism the existence of a dominated splitting implies partial hyperbolicity.

Suppose we have a symplectic structure on a $2n$-dimensional vector space V, i. e. there is a non-degenerate skew symmetric bilinear functional on $V \times V$ denoted ω. Two subspaces A and B of V are called *skew orthogonal* if $\omega(a,b) = 0, \forall a \in A, b \in B$. A subspace A is called *symplectic* if ω restricted to A is non-degenerate. The *skew orthogonal complement* of a subspace A is $A^\omega = \{v \in V : \omega(a,v) = 0, \forall a \in A\}$. We have that $(A^\omega)^\omega = A$ and if the dimension of A is d then the dimension of A^ω is $2n-d$. A subspace is symplectic if and only if $A \cap A^\omega = \emptyset$. If $A = A^\omega$ then we say that A is a *Lagrangian subspace*. The Lagrangian subspaces are the maximal subspaces such that ω restricted to them is trivial (or they are a subset of their skew orthogonal complement).

Lemma 7. *If $f \in \mathbf{Diff}^1_\omega(M)$ has a dominated splitting, then it is partially hyperbolic. More precisely, say that $\mathbf{A}\oplus\mathbf{B}$ is an invariant splitting of TM,* $\dim A = i \leq n$ *and $\mathbf{A}$ l-dominates $\mathbf{B}$ for some l then there is a splitting $\mathbf{C} \oplus \mathbf{D}$ of $\mathbf{B}$ such that* $\dim \mathbf{D} = i$, $\dim \mathbf{C} = 2n-2i$, *$\mathbf{C}$ and $\mathbf{A}\oplus\mathbf{D}$ are symplectic and skew orthogonal, $\mathbf{A}$ is uniformly expanding, $\mathbf{D}$ is uniformly contracting, and $\mathbf{A}$ l'-dominates $\mathbf{C}$, $\mathbf{C}$*

l'-dominates $\mathbf{D}$ for some $l' \geq l$. In particular if $i = n$ then $\mathbf{C} = 0$ and f is hyperbolic.

Proof. We define $\mathbf{C} = \mathbf{A}^\omega \cap \mathbf{B}$, $\mathbf{D} = \mathbf{C}^\omega \cap \mathbf{B}$. We remark that $\mathbf{A}$ and $\mathbf{B}$ must be continuous because of the dominance, so $\mathbf{C}$ and $\mathbf{D}$ also must be continuous. There exist $M > 0$ such that for any $x \in M$ and any two vectors $u, v \in T_xM$ we have $|\omega(u, v)| \leq M\|u\|\|v\|$. We divide the proof in two cases.

First case: $i < n$. For any $x \in M$ there are vectors $b_1, b_2 \in \mathbf{B}_x$ with $\omega(b_1, b_2) \neq 0$. Using the continuity of $\mathbf{B}$ and the compactness of M we can find an $m > 0$ such that for any $x \in M$ there are two vectors $b_1^x, b_2^x \in T_xM$ such that $\omega(b_1^x, b_2^x) \geq m\|b_1^x\|\|b_2^x\|$. Then if we take b_1^x and b_2^x to be unit vectors and any other unit vector $a^x \in \mathbf{A}_x$ we get

$$m \leq \omega(b_1^x, b_2^x) = \omega(Df^{kl}(b_1^x), Df^{kl}(b_2^x)) \leq M\|Df^{kl}(b_1^x)\|\|Df^{kl}(b_2^x)\| \leq$$

$$\frac{M}{2^{2k}}\|Df^{kl}(a^x)\|^2 \text{ or } \|Df^{kl}(a^x)\| \geq \sqrt{\frac{2^{2k}m}{M}}, \forall x \in M, a^x \in \mathbf{A}_x.$$

Here we used the dominance hypothesis. Now if we take k large enough so that $\frac{2^{2k}m}{M} > 1$ we get that $\mathbf{A}$ must be uniformly expanding.

For any $x \in M$ and $a_1, a_2 \in \mathbf{A}_x$ we have

$$\omega(a_1, a_2) = \lim_{k\to\infty} \omega(Df^{-k}(a_1), Df^{-k}(a_2)) = 0$$

because $\mathbf{A}$ is expanding so ω restricted to $\mathbf{A}$ is trivial. Then $\mathbf{A} \subset \mathbf{A}^\omega$ and the dimension of $\mathbf{C} = \mathbf{A}^\omega \cap \mathbf{B}$ must be $2n - 2i$. Now $\mathbf{A} \subset (\mathbf{A} \oplus \mathbf{C})^\omega$ and the dimension of $(\mathbf{A} \oplus \mathbf{C})^\omega$ is i so $\mathbf{A} = (\mathbf{A} \oplus \mathbf{C})^\omega$ and consequently $\mathbf{C} \cap \mathbf{C}^\omega = \emptyset$ so $\mathbf{C}$ is symplectic. Also we know that $\mathbf{A} \subset \mathbf{C}^\omega$ and the dimension of $\mathbf{C}^\omega$ is $2i$ so the dimension of $\mathbf{D} = \mathbf{C}^\omega \cap \mathbf{B}$ must be i. From construction we have that $\mathbf{C}$ and $\mathbf{A} \oplus \mathbf{D}$ are skew orthogonal. Also the fact that $\mathbf{C}$ is symplectic implies that also $\mathbf{A} \oplus \mathbf{D}$ is symplectic and ω restricted to $\mathbf{D}$ is trivial.

What is left now to prove is that $\mathbf{D}$ is uniformly contracting and $\mathbf{C}$ dominates $\mathbf{D}$. As we remarked before $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{D}$ are continuous, M is compact and $\mathbf{C}$, $\mathbf{A} \oplus \mathbf{D}$ are symplectic, ω restricted to $\mathbf{A}$ and $\mathbf{D}$ is trivial, so there exists $m > 0$ such that for any any $x \in M, u \in \mathbf{A}_x$ (or $\mathbf{D}_x, \mathbf{C}_x$) there exist $v \in \mathbf{D}_x$ (or $\mathbf{A}_x$ respectively $\mathbf{C}_x$) such that $\omega(u, v) \geq m\|u\|\|v\|$.

Suppose we have $x \in M, d \in \mathbf{D}_x$. As we saw before we can find $a \in \mathbf{A}_x$ such that $\omega(a, d) \geq m\|a\|\|d\|$. Then

$$m\|a\|\|d\| \leq \omega(a, d) = \omega(Df^{-k}(a), Df^{-k}(b)) \leq M\|Df^{-k}(a)\|\|Df^{-k}(d)\|$$

So $\|Df^{-k}(d)\| \geq \frac{m}{M} \frac{\|a\|}{\|Df^{-k}(a)\|}\|d\|$ and because $\mathbf{A}$ is uniformly expanding if we take k large enough we get that $\mathbf{D}$ must be uniformly contracting.

We know that $\mathbf{A}$ l-dominates $\mathbf{C}$ because $\mathbf{C} \subset \mathbf{B}$. So let's take $x \in M, d \in \mathbf{D}_x, c \in \mathbf{C}_x, \|c\| = \|d\| = 1$. As before we pick $a \in \mathbf{A}_x, \|a\| = 1, \omega(a, d) \geq$

$m\|a\|\|d\| = m$. Then

$$m \leq \omega(a,d) = \omega(Df^{-kl}(a), Df^{-kl}(d)) \leq M\|Df^{-kl}(a)\|\|Df^{-kl}(d)\|.$$

There exist $c' \in \mathbf{C}_x$, $\|c'\| = 1$ such that

$$\omega(Df^{-kl}(c), Df^{-kl}(c')) \geq m\|Df^{-kl}(c)\|\|Df^{-kl}(c')\|.$$

From the fact that $\mathbf{A}$ l-dominates $\mathbf{C}$ we get that $\|Df^{-kl}(a)\| \leq 2^{-k}\|Df^{-kl}(c')\|$. Combining this two inequalities we get

$$\|Df^{-kl}(a)\| \leq \frac{2^{-k}}{m\|Df^{-kl}(c)\|}\omega(Df^{-kl}(c), Df^{-kl}(c')) =$$

$$= \frac{2^{-k}}{m\|Df^{-kl}(c)\|}\omega(c,c') \leq \frac{2^{-k}}{m\|Df^{-kl}(c)\|}M\|c\|\|c'\| = \frac{2^{-k}M}{m\|Df^{-kl}(c)\|}$$

and furthermore

$$\frac{\|Df^{-kl}(d)\|}{\|Df^{-kl}(c)\|} \geq \frac{2^k m^2}{M^2}$$

which proves that $\mathbf{C}$ also dominates $\mathbf{D}$ if we take again k large enough and we are done.

Second case: $i = n$. In this case we only have to prove that $\mathbf{A}$ is uniformly expanding and $\mathbf{B}$ is uniformly contracting. For any $x \in M$ we have that either ω restricted to $\mathbf{B}_x$ is trivial or ω restricted to $\mathbf{B}_x$ is not trivial, and as in the proof of the first case we get that $\mathbf{A}_x$ is expanded so ω restricted to $\mathbf{A}_x$ must be trivial. If we take any $x \in M$ and $a \in \mathbf{A}_x$, because ω restricted to $\mathbf{A}_x$ or to $\mathbf{B}_x$ is trivial, we can find an $b \in \mathbf{B}_x$ such that $\omega(a,b) \neq 0$ and vice-versa. These observations, together with the continuity of $\mathbf{A}$ and $\mathbf{B}$ and the compactness of M, show that there must be again an $m > 0$ such that for any $x \in M$ and for any $a \in \mathbf{A}_x$ ($b \in \mathbf{B}_x$) there exist $b \in \mathbf{B}_x$ ($a \in \mathbf{A}_x$) such that $\omega(a,b) \geq m\|a\|\|b\|$. Now let's suppose that $\mathbf{A}$ is not unifor! mly expanding, so for any large k there exist $x \in M$ and $a \in \mathbf{A}_x$ such that $\|Df^{kl}(a)\| < 2$. From the dominance condition we get that $\mathbf{B}_x$ must be contracting, $\|Df^{kl}(v)\| < 2^{1-k}, \forall v \in \mathbf{B}_x$. We know that we can find $b \in \mathbf{B}_x$ such that $\omega(a,b) \geq m\|a\|\|b\|$. Then we get

$$m\|a\|\|b\| \leq \omega(a,b) = \omega(Df^{kl}(a), Df^{kl}(b)) \leq M\|Df^{kl}(a)\|\|Df^{kl}(b)\| <$$

$$< 2^{1-k}M\|Df^{kl}(a)\|$$

and from here we find that $\|Df^{kl}(a)\| > 2^{k-1}\frac{m}{M}$. But we can take k arbitrarily large so $2^{k-1}\frac{m}{M}$ becomes larger that 2 and we get a contradiction. The proof that $\mathbf{B}$ is uniformly contracting is similar. In particular in this case we get that $\mathbf{A}$ and $\mathbf{B}$ must be Lagrangian. $\square$