Technische Universität München
— Fakultät für Mathematik —

Diplomarbeit:

Parameter Rays for the Exponential Family

Markus Förster

Matrikelnr. 1962336

Aufgabensteller & Betreuer: Professor Dierk Schleicher
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Bremen, 31. Januar 2003

Markus Förster

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Abstract

This diploma thesis contributes to the understanding of dynamical systems generated by the iteration of entire functions. After the intense and successful investigation of polynomial dynamics, the question arises if and how these results can be carried over to entire transcendental functions. This thesis deals – in analogy to the discussion of the *Mandelbrot set* – with the *parameter space* of the exponential family $\{E_\lambda(z) := \lambda \cdot \exp(z) \ : \ \lambda \in \mathbb{C}^*\}$.

According to the external rays for the Mandelbrot set, we construct *parameter rays* for the exponential family. These are curves of *escaping* parameters, i.e. parameters for which the singular orbit $\{0, \lambda, \lambda e^\lambda, \dots\}$ escapes under iteration in absolute values to infinity. The parameter rays are the essential step towards a classification of the escaping parameters, and they provide structure on the exponential parameter plane. This diploma thesis builds on results by Dierk Schleicher, who constructed parameter rays for the case of bounded combinatorics.

Deutsche Zusammenfassung

Die vorliegende Diplomarbeit leistet einen Beitrag für das Verständnis von durch Iteration ganzer Funktionen erzeugten dynamischen Systemen. Nachdem seit den achtziger Jahren die Dynamik von Polynomen intensiv und mit Erfolg erforscht wurde, lenken wir nun das Augenmerk darauf, ob und wie man diese Ideen auf ganze transzendente Funktionen übertragen kann. In dieser Arbeit geht es – analog zur Diskussion der Mandelbrotmenge – um den *Parameterraum* der Exponentialfamilie $\{E_\lambda(z) := \lambda \cdot \exp(z) \ : \ \lambda \in \mathbb{C}^*\}$.

Entsprechend den externen Strahlen für die Mandelbrotmenge werden hier *Parameterstrahlen* für die Exponentialfamilie konstruiert. Das sind Kurven von *entkommenden* Parametern, also Parameter für die der singuläre Orbit $\{0, \lambda, \lambda e^\lambda, \dots\}$ unter Iteration betragsmäßig nach ∞ entkommt. Die Parameterstrahlen stellen einen wesentlichen Schritt zur Klassifikation der entkommenden Parameter dar und verleihen dem Exponentialparameterraum Struktur. Die Diplomarbeit baut auf Ergebnissen von Dierk Schleicher auf, der Parameterstrahlen für den Fall beschränkter Kombinatorik konstruiert hat.

1 The Setting

1.1 Introduction

We consider discrete dynamical systems generated by the iteration of a holomorphic function $f : \mathbb{C} \to \mathbb{C}$. This means that every $z \in \mathbb{C}$ represents a state of a dynamical system, which changes to the state $f(z)$ in the next step. The sequence

$$(z, f(z), f(f(z)), \dots, f^{\circ n}(z), \dots)$$

is called the *orbit* of z. Every $z \in \mathbb{C}$ determines thus a certain kind of dynamical behavior, such as being an equilibrium (the orbit is eventually fixed or periodic) or being *chaotic*. In this paper we are interested in the set

$$\{z \in \mathbb{C} : \lim_{n\to\infty} |f^{\circ n}(z)| = \infty\} \tag{1}$$

of so-called *escaping points*. The holomorphic functions discussed are the *exponential functions*

$$E_\lambda(z) := \lambda \exp(z) \ , \quad \lambda \in \mathbb{C} \setminus \{0\} \ .$$

So instead of only one function we investigate the behavior of a one-parameter-family of holomorphic functions depending analytically on the complex parameter. Having only one singular value, the asymptotic value 0, the exponential family can be considered as a prototypical family for transcendental entire maps of finite type — analogous to the unicritical standard polynomial families $\{z^d + c : \ c \in \mathbb{C}\}$ $(d \geq 2)$ having only the singular value c.

We distinguish between the *dynamic plane*, which describes the dependence of the dynamical system on its initial value z for a fixed parameter λ, and the *parameter plane*, the space of possible parameters $\lambda \in \mathbb{C}$. The major goal is to understand and to classify the various possibilities of the dynamics, which involves an investigation of both the parameter and the dynamic planes.

This thesis helps to endow the parameter space with structure, by marking parameters which yield a certain type of dynamics. In our case, this type is determined by the behavior of the singular orbit $(z_n)_{n\geq 1} := (0, \lambda, \lambda e^\lambda, \dots)$. We will mark those parameters for which the singular orbit *escapes* (see (1)).

The goal of understanding and classifying the dynamics has been worked on quite successfully for polynomial dynamics, beginning with the fundamental Orsay notes [DH] by A. Douady and J. Hubbard and continued by many others. It thus stands to reason that these results might be useful hints for the investigation of the exponential family.

A dynamical system generated by a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ can be basically described in terms of its *Julia set* $J(f)$, the locus of *non-normality* in the sense of Montel. Roughly speaking, that means that the Julia set consists of those initial points for which the dynamical system is unstable, i.e. responding sensitively to perturbations of the initial value. The (chaotic) dynamics on the Julia set is crucial for the understanding of the dynamical system, for the dynamics on the *Fatou set* $F(f) := \mathbb{C} \setminus J(f)$ is comparably easy to understand. For polynomials p of degree $d \geq 2$, there is an easy description of the Julia set, which is connected to the question of *escaping*: the Julia set of polynomials is the boundary of the *filled-in Julia set* $K(p) := \{z \in \mathbb{C} : \ z \text{ not escaping}\}$.

It turns out for polynomials $p(z) = z^d + c$ that the set of escaping points carries a natural structure which helps to understand the dynamics on the Julia set. In fact, either $J(p)$ is a Cantor set, which is the easiest case, or $K(p)$ is connected and simply connected. Hence there is a biholomorphic Riemann map $\varphi : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K(p)$ which transfers the structure of the rays $(\vartheta, t) \mapsto te^{2\pi i \vartheta} : \ \mathbb{S}^1 \times (1, \infty) \to \mathbb{C}$ to the set $\mathbb{C} \setminus K(p)$ of escaping points. The escaping points can thus be considered as a union of so-called *dynamic* (sometimes: *external*) *rays* $R_\vartheta(t) = \varphi(te^{2\pi i \vartheta})$ or as the set of pairs of *angles* and *potentials*

$$(\vartheta, t) \in \mathbb{S}^1 \times (1, \infty) \ ,$$

and the dynamics acts on these pairs (if φ has been chosen appropriately) via

$$(\vartheta, t) \mapsto (d\vartheta, t^d) \ .$$

Now if $J(p)$ is locally connected, which is true in many cases, then the Carathéodory Theorem asserts that every ray *lands* on $\partial K(p) = J(p)$, i.e. that the limits $\lim_{t \searrow 1} R_\vartheta(t) \in J(p)$ exist. This gives us a surjective continuous map $\mathbb{S}^1 \to J(p)$, allowing us to model $(J(p), p)$ as the unit circle $(\mathbb{S}^1, \vartheta \mapsto d\vartheta)$ modulo the equivalence relation given by angles having a common landing point. The equivalence relation is respected by the dynamics; thus one just have to understand the equivalence relation in order to understand the chaotic dynamics on $J(p)$. In other words, the key to the understanding of polynomial dynamics is to understand which rays land together. Symbolic dynamics is a very useful tool in this context, see for example [BS].

Let us come back to our original question, whether it is possible to modify these ideas so as to be applied to a class of non-polynomials such as the exponential family. Of course, any positive answer is a great progress for the understanding of transcendental dynamics. By no means one could have expected before that it was possible to generalize the beautiful polynomial results to any class of transcendental functions whatsoever. However, the research work during the last years gave a positive answer, beginning with Bob Devaney (for example [DK], [DGH]) and his coauthors, who mainly handled exponential maps for real parameters $\lambda < 1/e$ and for arbitrary parameters with bounded combinatorics, and Dierk Schleicher, who was the first one to get results for arbitrary parameters and combinatorics (for example [SZ], [S1]).

First of all, there are parameters λ for which $J(E_\lambda) = \mathbb{C}$ (for example if 0 is escaping under E_λ), so we cannot expect the concept of Julia sets to establish a sufficient structure on the dynamic plane. Instead we could for example describe attracting components (components which are attracted by a periodic orbit) or escaping points. The latter has been done in the paper [SZ] by D. Schleicher and J. Zimmer, from which the main results will be repeated in Chapter 2.1 and will be used throughout the whole paper.

As a matter of fact, the escaping points of every exponential function E_λ are organized in differentiable rays — again called *dynamic rays* — just as in the polynomial case. (But a difference is that the union of the rays does not have Hausdorff dimension 2 anymore.) Moreover, they have a unique encoding as pairs

$$(\underline{s}, t) \in \mathbb{Z}^{\mathbb{N}} \times (T, \infty) \ , \qquad \text{with some } T \geq 0 \ ,$$

where $\underline{s}$ is a sequence of integers coming from a partition of the plane fitting to E_λ (see Figure 2), and t arises from a suitable parametrization of the curve. The dynamic system acts on these pairs via

$$((s_1, s_2, s_3, \ldots), t) \mapsto ((s_2, s_3, \ldots), e^t - 1) \ .$$

The idea is to use $(\mathbb{R}^+, e^t - 1)$ as a model for escaping dynamics and to construct the rays by conjugation similar to the polynomial case (but the conjugation is constructed in the other order). The conjugation φ discussed above for the polynomials $p(z) = z^d + c$ is usually constructed by first pushing forward a given point n times by the dynamics (i.e. iterating) until we are close to ∞ (where $p \approx z^d$), and then taking n times appropriate d-th roots. Instead, we first iterate along the positive real axis, starting at some given potential t, and then choose the backward images depending on the sequence $\underline{s}$. This works only if t is big enough so that boundaries of the partition are not passed anymore. In this case, the image of some end of the real axis under the limit function is a ray. These *ray tails* can be extended down uniquely to a minimal potential $t_{\underline{s}}$ independent from the parameter λ, except for some special cases (see Theorem 2.2). The sequence $\underline{s}$ is called the *external address* and t the *potential*.

In [SZ], it has been shown that every escaping point is either on such a dynamic ray or a landing point of one, and there is a precise description about which pairs of external addresses and potentials refer to escaping points. The dynamic rays are the central tool used in this thesis. In addition to the classification of escaping points they give an interesting dimension paradox first mentioned by B. Karpińska [K], stating that the union of all rays has Hausdorff dimension 1, whereas the escaping landing points together have Hausdorff dimension 2. Günter Rottenfußer [R] has completed the same program for the cosine family $\{E_{a,b}(z) := ae^z + be^{-z} : \ a, b \in \mathbb{C}^*\}$ with the difference that the escaping set even has positive two-dimensional Lebesgue measure (using a result by C. McMullen), whereas M. Lyubich showed that for the exponential functions the Lebesgue measure equals 0.

Beyond the dynamics of particular functions we are interested in the classification of and bifurcation between different types of behavior, considering a family of functions parameterized by a complex parameter. A discussion of the parameter plane defined by such a family also helps to obtain results which can be carried back to particular dynamics: the comprehension of the parameter plane of a holomorphic family yields a deep understanding of the dynamics (and vice versa).

Let us briefly resume the results from the polynomial case. This leads us to the well known *Mandelbrot set* $\mathcal{M}$, which emerges from the discussion of the *quadratic family* $\{p_c(z) = z^2 + c : \ c \in \mathbb{C}\}$. Two possible ways to define a structure on the parameter plane are to decide whether for a given parameter c the critical orbit escapes ($|p_c^{\circ n}(0)| \to \infty$) or whether the associated Julia set is disconnected. For polynomials these two conditions are in fact equivalent, and they define the Mandelbrot set

$$\mathcal{M} := \{z \in \mathbb{C} : J(p_c) \text{ is connected}\}$$

(resp. the *Multibrot sets* for degrees $d > 2$). Parameters for which the critical point 0 escapes are called *escaping parameters*. They have the structure of rays R_ϑ, called *parameter rays*, inherited by the dynamic rays. The parameter rays are an important tool for the topological and dynamical understanding of the Mandelbrot set in a similar spirit as dynamic rays are for the Julia sets. If the famous conjecture MLC ($\mathcal{M}$ is locally connected) is true, then we get similar as for the polynomial Julia sets discussed above a nice topological model, called the *pinched disk model*. This describes $\mathcal{M}$ as the unit circle $\mathbb{S}^1$ modulo angles for which the parameter rays R_ϑ land together. Hence the parameter rays constitute an essential step towards a description of the parameter plane.

After the marvellous and highly useful investigation of polynomial parameter spaces we are interested in the generalization of this theory. How can the ideas be taken advantage from for transcendental dynamics, or at least for its easiest case, the exponential dynamics? This thesis furnishes an essential first step for the project of carrying over ideas and results from polynomial parameter space to the exponential parameter space.

Again, there are several ways to define a structure on the parameter space. One is to describe the attracting components — components of parameters for which there is an attractive orbit, which necessarily attracts the singular orbit.) D. Schleicher and L. Rempe are currently working on this task.

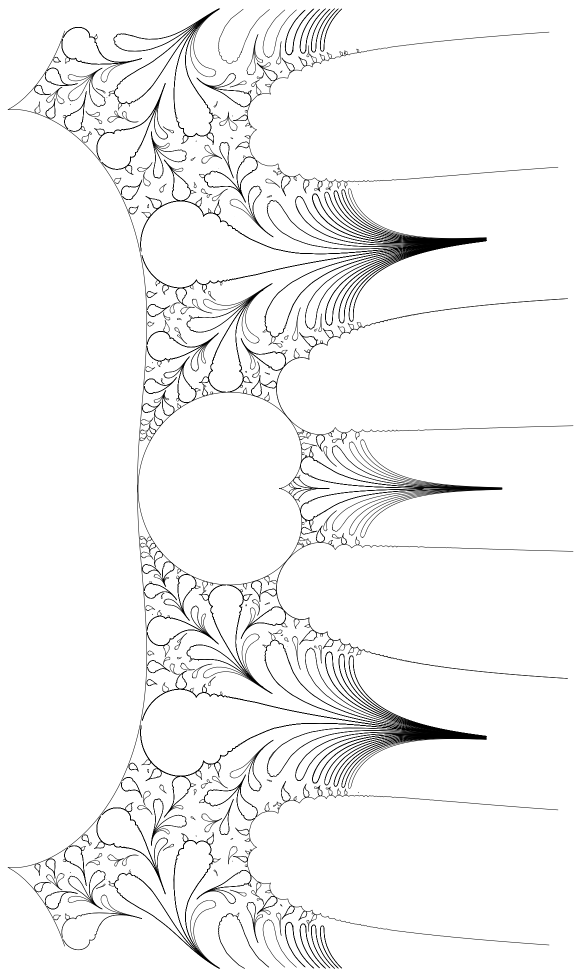

Figure 1: The exponential λ-parameter plane with some parameter rays.

Instead, we try to give a description of the *escaping parameters*, the parameters for which the singular value 0 is escaping. For every such escaping parameter the Julia set coincides with $\mathbb{C}$. Remember that for the polynomial families discussed above, these parameters form just the complement of the Mandelbrot (Multibrot) set. For the exponential family, the escaping points will be organized similarly to the escaping points within the dynamic planes. They form disjoint rays, called *parameter rays*. To be more precise, this thesis will construct the parameter rays consisting of escaping points. It is work in progress to show that every escaping parameter really is contained in one of the constructed rays or is a landing point of one.

For the construction of the parameter rays, we define a parameter κ to be on the *parameter ray* for external address $\underline{s}$ at potential t if the singular orbit is escaping under E_κ such that the singular value 0 is on the dynamic ray for $\underline{s}$ at potential t. Since for a given external address $\underline{s}$ we can locally find a uniform potential beyond which the dynamic rays behave well (ray tails), we can begin with a parameter ray tail. This is in fact a differentiable curve and can be extended down uniquely to the minimal potential $t_{\underline{s}}$, which does not depend on the parameter κ.

Theorem 3.9 will show that for every possible combinatorics $\underline{s}$ there is a parameter ray $G_{\underline{s}} : (t_{\underline{s}}, \infty) \to \mathbb{C}$ defined on the maximal interval of potentials such that there is a 1-to-1 correspondence between the parameters κ satisfying the formal definition given above and $G_{\underline{s}}(t)$. Hence we describe all escaping parameters for which 0 is contained in some dynamic ray, rather than being landing point of one. So it only remains to handle the latter case. The next step in this direction will be to classify landing points of parameter rays, distinguishing between *fast* and *slow* external addresses as done in [SZ].

In 2000, Günter Rottenfußer has written an intelligent program drawing exponential parameter rays. Older, pointwise drawn pictures of the exponential parameter space were not very accurate since the exponentially fast growing orbits led quickly to overflows in the computer calculations. Günter's picture (see Figure 1) reveals the true structure of the parameter plane. Actually it shows boundaries of attracting components, which is conjectured to form the closure of the parameter rays. This picture gives us an insight into the parameter plane. We see hyperbolic components such as the left half-plane and the main cardioid. But we get also a picture of how the parameter rays look like, being bundled in vertical distances of 2π. In fact, the parameter rays satisfy the asymptotics $G_{\underline{s}}(t) = t + 2\pi i s_1 + O(e^{-t})$ with $s_1 \in \mathbb{Z}$.

I also want to mention the research work of Lasse Rempe in Kiel, who has been working on exponential dynamics over the last years and answered a number of questions in both the dynamic and the parameter plane. A new idea by him is to slightly change the conjugation between the dynamic rays and the model dynamics on $\mathbb{R}_0^+$ by taking into account the imaginary parts. (Before and also in this paper, the conjugation uses only the real parts of the points on the ray.) In particular, this yields a faster convergence of the approximated conjugations and thus simplifies a number of proofs.

Some words about the structure of this paper: Chapter 1 briefly introduces the setting, while Chapter 2 contains old and new results about dynamic rays needed in Chapter 3, including derivatives and winding numbers of dynamic rays. The construction of the parameter rays will be done in Chapter 3.

Acknowledgements

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1.2 Basic Definitions

We are investigating the family

$$\{E_\lambda :\ z \mapsto \lambda \exp(z)\ \mid \lambda \in \mathbb{C}^*\}\ ,$$

where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Often we switch to the parametrization $E_\kappa(z) = \exp(z + \kappa)$, which can be obtained by choosing a branch of the logarithm and setting $\kappa := \log(\lambda)$. Usually we take the standard branch of the logarithm, so that $|\mathrm{Im}(\kappa)| \leq \pi$. Depending on the situation, one parametrization is sometimes more convenient than the other one and vice versa.

Throughout the following discussion (Chapters 1 and 2), we will always fix a parameter $\lambda = e^\kappa \in \mathbb{C}^*$ and investigate the dynamical system generated by the iteration of the function $E_\lambda : \mathbb{C} \to \mathbb{C}$. The goal is an understanding and a classification of the behavior depending on the initial value. In the third chapter we are going to discuss how the global behavior of the dynamics of E_λ depends on the parameter λ. This global behavior is determined by the singular orbit for our purposes. It will turn out that both discussions are surprisingly similar.

Let us now collect some notation and conventions. As usual, the iteration of a function is abbreviated by $f^{\circ(n+1)}(z) := f \circ f^{\circ n}(z) = f \circ \cdots \circ f(z)$, with $f^{\circ 0} := \mathrm{id}$. By f^{-1} we mean the relation mapping a point or a set to its preimage, which is usually not a single point.

Definition 1.1 (Basic Notation)
Let $\mathcal{S}$ be the space of integer sequences $\underline{s} = (s_1, s_2, \dots)$ with $s_1, s_2, \dots \in \mathbb{Z}$. Define the shift map *$\sigma : \mathcal{S} \to \mathcal{S}$ by $\sigma(s_1, s_2, s_3, \dots) := (s_2, s_3, \dots)$. Furthermore, let*

$$F : \mathbb{R}_0^+ \to \mathbb{R}_0^+\ , \quad F(t) := e^t - 1\ .$$

The function F will be used as a parametrization function. The reason for preferring this one over the obvious candidate exp is that F can be pulled back arbitrarily often, as opposed to the Logarithm, which runs out of its domain after finitely many steps.

Our goal is to understand the set of *escaping points* of E_λ. (Later in Chapter 3 we are going to investigate also *escaping parameters*). What is an escaping point?

Definition 1.2 (Escaping Points)
A point $z \in \mathbb{C}$ with orbit $(z_k)_{k\geq 1} := (E_\lambda^{\circ(k-1)}(z))_k$ is called an escaping point *if $|z_k| \to \infty$ as $k \to \infty$.*

Exponential functions have several characteristic properties. One of them is that the factor of expansion, i.e. the absolute value of the derivative, depends only on the real part. Two close points having large positive real parts can be mapped far apart from each other, whereas the whole left half plane is mapped into the unit disc. This is expressed by the formula $|E_\lambda'(z)| = |E_\lambda(z)| = \exp(\mathrm{Re}(z))$. On the other hand, the imaginary parts determine the angle: $\arg(E_\lambda(z)) = 2\pi \mathrm{Im}(z)$. For points with large real parts, this yields a very sensitive dependence of E_λ on the imaginary parts.

So how easy it for a point z to escape to ∞? First of all we record the following Lemma.

Lemma 1.3 (Characterization of Escaping Points)
Consider a point $z \in \mathbb{C}$ with its orbit $(z_k)_{k\geq 1} := (E_\lambda^{\circ(k-1)}(z))_k$. The point z is escaping if and only if $\mathrm{Re}(z_k) \to +\infty$ as $k \to \infty$.

Proof. This follows directly from $|z_{k+1}| = |\lambda| \exp(\mathrm{Re}(z_k))$. $\square$

We observe that we cannot expect a thick set of escaping points, since nearby points with different imaginary part may be mapped far to the left and thus are close to 0 after two iterations. For example, consider some $y \in \mathbb{R}$ and $R > 0$ and let us find points in the box $\{z \in \mathbb{C} : \mathrm{Re}(z) > R,\ \mathrm{Im}(z) \in [y-\pi, y+\pi]\}$ which stay in the right half plane $\mathbb{H}_R := \{z \in \mathbb{C} : \mathrm{Re}(z) > R\}$. Then more than half of the points have to be removed after just one iteration, since the imaginary parts provide an angle which throws them out of $\mathbb{H}_R$. The same will happen in the next step and so on. One could guess from this idea that the escaping points meet in horizontal rays of distance $2\pi i$. This intuitive thought has been confirmed by the work of B. Devaney, D. Schleicher, and others.

The above discussion of the dynamical properties of the exponential functions gave rise to the following construction. First of all, we need some structure of dynamical meaning on the plane, so as to obtain *itineraries* and thus symbolic dynamics. These are sequences of symbols assigned to each $z \in \mathbb{C}$ telling which parts of a partition are visited along the orbit $\{z, E_\kappa(z), E_\kappa^{\circ 2}(z), \dots\}$. In our case they are called *external addresses*, and they are the key to get a grip on the escaping points.

Since E_κ is $2\pi i$-periodic it makes sense to have a partition of width $2\pi i$ such that each region of the partition is the image of a branch of the logarithm. On the slit plane $\mathbb{C}' := \mathbb{C} \setminus \mathbb{R}_0^-$ we can define a biholomorphic branch $\log : \mathbb{C}' \to \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \pi\}$ of the logarithm, which we will refer to as the standard branch, denoted by Log. Therefore, we consider for E_κ the partition having $E_\kappa^{-1}(\mathbb{R}^-)$ as boundary. We code the regions by integers: The one containing $-\kappa$ will be called

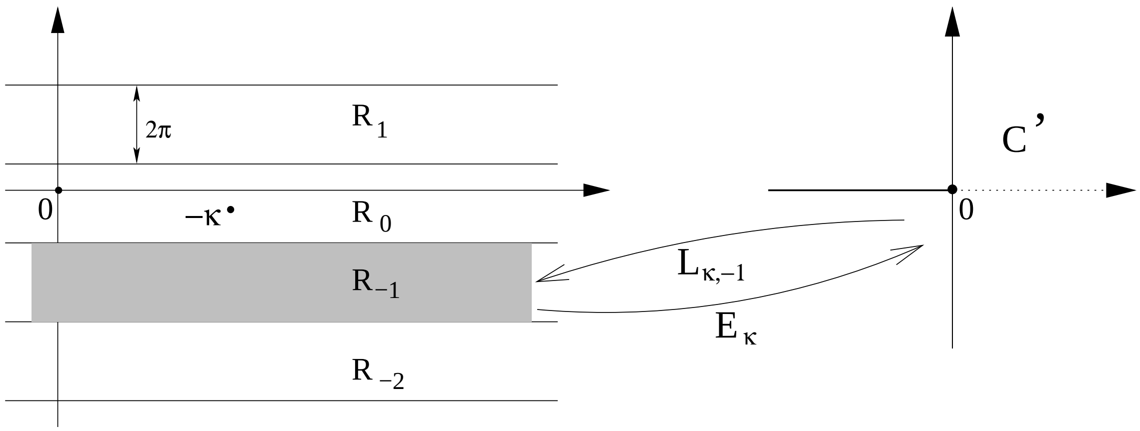

Figure 2: The (static) partition and $L_{\kappa,j}$.

R_0. (Since $|\mathrm{Im}(\kappa)| \le \pi$, the singular value 0 is contained in the closure of R_0.) In general, for every $j \in \mathbb{Z}$ define

$$R_j := \{z \in \mathbb{C} : -\mathrm{Im}(\kappa) - \pi + 2\pi j < \mathrm{Im}(z) < -\mathrm{Im}(\kappa) + \pi + 2\pi j\} \ .$$

(See Figure 2.)

Hence, every restriction $E_\kappa|_{R_j}$ has an inverse function, called $L_{\kappa,j} : \mathbb{C}' \to R_j$, defined by

$$L_{\kappa,j}(z) = \mathrm{Log} z - \kappa + 2\pi j i \ .$$

Definition 1.4 (External Addresses)
Let $z = z_1 \in \mathbb{C}$ be a number such that $z_{n+1} := E_\kappa^{\circ n}(z) \notin \mathbb{R}^-$ for all $n \ge 1$. Then define the external address *$\underline{s}(z) = (s_1, s_2, \dots) \in \mathcal{S}$ of z to be the sequence of labels such that $z_n \in R_{s_n}$ for all $n \ge 1$.*

Remark. If $(z_n)_{n\ge1}$ is the orbit from Definition 1.4 and $(s_1, s_2, \dots) = \underline{s}(z_1)$, then

$$\begin{aligned} 2\pi|s_n| &\le |\mathrm{Im}(z_n + \kappa)| + \pi \qquad \text{and} && (2)\\ |\mathrm{Im}(z_n)| &\le 2\pi|s_n| + |\mathrm{Im}(\kappa)| + \pi \ . && (3) \end{aligned}$$

This follows from the triangle inequality, applied on $|\mathrm{Im}(z_n + \kappa) - 2\pi s_n| \le \pi$.

Definition 1.5 (Exponential Boundedness)
A sequence $\underline{s} \in \mathcal{S}$ is said to be exponentially bounded *if there are constants $A \ge 1$, $x > 0$ such that $|s_k| \le AF^{\circ(k-1)}(x)$ for all $k \ge 1$. The constants will be called* growth parameters.

Remark. Note that we could drop the constant A, for if $\underline{s} \in \mathcal{S}$ is exponentially bounded then there is an $x > 0$ such that $|s_k| \leq F^{\circ(k-1)}(x)$ for all $k \geq 1$. But using A has the big advantage that x can be chosen small so as to contain more information about the sequence, i.e. providing much more useful estimates. For example, if $\underline{s}$ is constant, then $x > 0$ can be chosen arbitrarily small.

Lemma 1.6 (External Addresses are Exponentially Bounded)
Consider an arbitrary number $z = z_1 \in \mathbb{C}$ and its orbit $(z_k)_{k\geq 1} = (E_\lambda^{\circ(k-1)}(z))$. Let $\delta \geq \log|\lambda| + 5$. Then $|z_k| \leq F^{\circ(k-1)}(|z| + \delta)$ for all $k \geq 1$.

Now assume in addition that (z_k) has a well-defined external address $\underline{s}(z) =: (s_1, s_2, \dots)$ and let $\delta' \geq \max\{\delta, 2\pi\}$. Then for all $k \geq 1$

$$|s_k| \leq \frac{1}{2\pi} F^{\circ(k-1)}(|z| + \delta') \ .$$

Thus all possible external addresses are exponentially bounded.

Proof. We will prove by induction that for all $k \geq 1$

$$|z_k| + \delta \leq F^{\circ(k-1)}(|z| + \delta) \ .$$

The induction seed for $k = 1$ is immediate. For the induction step we can estimate

$$\begin{aligned} |z_{k+1}| + \delta &= |\lambda| \exp(\mathrm{Re}(z_k)) + \delta \leq |\lambda| \exp|z_k| + \delta \leq \\ &\overset{(*)}{\leq} \exp(|z_k| + \delta) - (\delta + 1)\exp|z_k| + \delta \leq \exp(|z_k| + \delta) - 1 = \\ &= F(|z_k| + \delta) \leq F(F^{\circ(k-1)}(|z| + \delta)) = F^{\circ(k)}(|z| + \delta) \ , \end{aligned}$$

where (*) follows from $|\lambda| \leq e^\delta - (\delta + 1)$, which is true for every $\delta \geq \log|\lambda| + 5$.

Now if the orbit $(z_k)_{k\geq 1}$ does not contain negative real numbers and $\delta' \geq \max\{\delta, 2\pi\}$, then we have by Formula (2)

$$2\pi|s_k| \leq |\mathrm{Im}(z_k)| + |\mathrm{Im}(\kappa)| + \pi \leq |z_k| + 2\pi \leq |z_k| + \delta' \leq F^{\circ(k-1)}(|z_1| + \delta') \ .$$

□

2 Dynamic Rays

2.1 Definition of Dynamic Rays

This section 2.1 summarizes results from [SZ], which we are going to use later on. In that paper it has been shown that the converse to Lemma 1.6 is true in the sense that for every exponentially bounded sequence $\underline{s} \in \mathcal{S}$ there are points with sufficiently large real part having the external address $\underline{s}$. It turns out that escaping points sharing an external address $\underline{s}$ are organized in a differentiable curve $g_{\kappa,\underline{s}}$. These *dynamic rays* can be maximally extended so as to contain all escaping points: In fact every escaping point is either on a unique dynamic ray at a unique position, or it is a limit of a dynamic ray. This provides a nice classification and a useful combinatorial structure on the set of escaping points, revealing interesting paradoxes concerning the Hausdorff dimension of the escaping sets and positively answering A. Eremenko's question, whether every escaping point can be connected within the escaping set to ∞.

The construction of these rays has some similarity to the construction of the Böttcher coordinates for polynomial dynamics. There, the task is to find a conjugation $\varphi : \mathbb{C} \setminus K \to \mathbb{C} \setminus \overline{\mathbb{D}}$ from the polynomial dynamics outside the filled-in Julia set (i.e. on the immediate basin of ∞) to the model dynamics $z \mapsto z^d$ outside the unit disc. This is usually done by pushing forward the dynamics to a neighborhood of ∞ (where the model is very precise) by iteration and pulling back by taking appropriate branches of the d-th root (the inverse function of the model map). In our case instead, we first push forward along the model dynamics $(\mathbb{R}_0^+, F(t) = e^t - 1)$ and then pull back using an appropriate branch $L_{\kappa,j}(z) = \text{Log}z - \kappa + 2\pi j i$ of the Logarithm.

Given an exponentially bounded sequence $\underline{s} = (s_1, s_2, s_3, \dots) \in \mathcal{S}$, we define for every $n \in \mathbb{N}$ the functions

$$g^n_{\kappa,\underline{s}}(t) := L_{\kappa,s_1} \circ \cdots \circ L_{\kappa,s_n}(F^{\circ n}(t)) \ . \tag{4}$$

If t is sufficiently large such that the logarithms can be applied, these functions are well-defined and they converge uniformly. The limit will be the end of the dynamic ray $g_{\kappa,\underline{s}}$, called *ray tail*. These are curves consisting of escaping points having external address $\underline{s}$. We will refer to the variable t as the *potential* according to the polynomial case. By construction of the g^n, the dynamical system acts on the ray tail by shifting the external address and applying the function $F(t) = e^t - 1$ on the potential: $E_\lambda(g_{\kappa,\underline{s}}(t)) = g_{\kappa,\sigma\underline{s}}(F(t))$. The idea behind is that the potentials model the real part respectively the escaping rate of the escaping points, and the external address is a coding of the imaginary part such that there is a 1-to-1 relationship between external addresses and rays.

After defining these rays for large potentials we can pull back so as to extend the ray to its maximal possible interval of potentials. This gave rise to the following definition of the *minimal potential* $t_{\underline{s}}$, which is clearly a lower bound for possible potentials, and which is independent of the parameter κ.

Definition 2.1 (Minimal Potential)
Let $\underline{s} \in \mathcal{S}$ be an external address. Define the minimal potential *by*

$$t_{\underline{s}} := \inf \left\{ t > 0 : \limsup_{n\to\infty} \frac{|s_n|}{F^{\circ(n-1)}(t)} = 0 \right\} .$$

Remark. A sequence $\underline{s} \in \mathcal{S}$ is exponentially bounded if and only if $t_{\underline{s}} < \infty$. The minimal potential of a shifted sequence is $t_{\sigma(\underline{s})} = F(t_{\underline{s}})$. Moreover, if $\underline{s} \in \mathcal{S}$ has the minimal potential $t_{\underline{s}}$, then for every $\varepsilon > 0$ there is an $A \geq 1$ such that $|s_k| \leq AF^{\circ(k-1)}(t_{\underline{s}} + \varepsilon)$, i.e. the growth parameter x can be chosen arbitrarily close to the minimal potential.

Unfortunately, it is not always possible to define the dynamic rays on their entire natural domain $(t_{\underline{s}}, \infty)$, since the singular value may be passed during the process of pull-backs. The fact that this happens only if 0 escapes does not help us, since we will deal only with exactly those *escaping* parameters.

The following Theorem and Lemma summarize the results we will need concerning dynamic rays.

Theorem 2.2 (Dynamic Ray Tails and Dynamic Rays)
Let $\kappa \in \mathbb{C}$ be an arbitrary parameter and $\underline{s} \in \mathcal{S}$ be an exponentially bounded sequence with growth parameters A and x. Then there is a constant $t'_{|\kappa|,\underline{s}} := x + 2\log(|\kappa|+3) > t_{\underline{s}}$ and a curve $g_{\kappa,\underline{s}} : (t'_{|\kappa|,\underline{s}}, \infty) \to \mathbb{C}$, called dynamic ray tail, *with the following properties:*

1. *For every $t > t'_{|\kappa|,\underline{s}}$, the point $g_{\kappa,\underline{s}}(t)$ is escaping under E_κ and has external address $\underline{s}$.*

2. *The ray tail conjugates the dynamics to $(\mathbb{R}, F : t \mapsto e^t - 1)$: For all $t > t'_{|\kappa|,\underline{s}}$*

$$E_\kappa \circ g_{\kappa,\underline{s}}(t) = g_{\kappa,\sigma(\underline{s})} \circ F(t) .$$

3. *The map $\kappa \mapsto g_{\kappa,\underline{s}}(t)$ depends analytically on κ for fixed $t > t'_{|\kappa|,\underline{s}}$.*

4. *The ray tail satisfies the asymptotics* $\mathrm{Re}(g_{\kappa,\underline{s}}(t)) \overset{t\to\infty}{\longrightarrow} +\infty$ *along bounded imaginary parts, or more precisely,*

$$g_{\kappa,\underline{s}}(t) = t - \kappa + s_1 2\pi i + r_{\kappa,\underline{s}}(t) \ ,$$

where $|r_{\kappa,\underline{s}}(t)| < 2e^{-t}(|\kappa|+2+2\pi|s_2|+2\pi AC) = O(e^{-t})$*. (*$C := \sum_{n=2}^{\infty}\prod_{k=1}^{n-2} e^{-k} \approx 1.42$ *is a constant.)*

If $t \geq t'_{|\kappa|,\underline{s}} + \log(4A)$*, then* $|r_{\kappa,\underline{s}}(t)| < 0.82 < 1$*.*

5. *The ray tail can be extended uniquely onto a maximal interval* $(\tilde{t}_{\kappa,\underline{s}}, \infty) \subset (t_{\underline{s}}, \infty)$*. The resulting curve* $g_{\kappa,\underline{s}} : (\tilde{t}_{\kappa,\underline{s}}, \infty) \to \mathbb{C}$*, called the* dynamic ray for the external address $\underline{s}$*, consists only of escaping points. The variable* t *will be referred to as the* potential *of the escaping point* $z = g_{\kappa,\underline{s}}(t)$*.*

   If the singular orbit $(0, e^{\kappa}, e^{\kappa}e^{e^{\kappa}}, \ldots)$ *does not escape then* $\tilde{t}_{\kappa,\underline{s}} = t_{\underline{s}}$*. If the singular orbit does escape then we still have* $\tilde{t}_{\kappa,\underline{s}} = t_{\underline{s}}$*, except if the singular value* 0 *has been passed during the process of pull-backs. This happens if and only if there is an* $n \geq 1$ *and a* $t_0 > F^{\circ n}(t_{\underline{s}})$ *such that* $0 = g_{\kappa,\sigma^n(\underline{s})}(t_0)$*. In this case,* $\tilde{t}_{\underline{s}}$ *be the largest number for there is a* $k \geq 1$ *with* $F^{\circ k}(\tilde{t}_{\underline{s}}) = t_0$ *and* $0 = g_{\kappa,\sigma^k(\underline{s})}(t_0)$*.*

6. *The dynamic rays* $g_{\kappa,\underline{s}} : (\tilde{t}_{\kappa,\underline{s}}, \infty) \to \mathbb{C}$ *satisfy the above items 2 and 3. In addition, they satisfy for every* $t > \tilde{t}_{\kappa,\underline{s}}$ *the asymptotic bound*

$$E_{\kappa}^{\circ n}(g_{\kappa,\underline{s}}(t)) = F^{\circ n}(t) - \kappa + 2\pi i s_{n+1} + O\left(e^{-F^{\circ n}(t)}\right) \quad \text{for } n \to \infty \ . \qquad (5)$$

PROOF. Everything can be found in [SZ] (Proposition 3.2, Theorem 4.2, and Proposition 4.4), except for the last statement of the fourth item. Define $K := |\kappa|$ and consider some $t \geq x + 2\log(K+3) + \log(4A)$. Then we have

$$\begin{aligned}|r_{\kappa,\underline{s}}(t)| &\leq 2e^{-t}(K+2+2\pi|s_2| + 2\pi AC) \leq 2e^{-t}(K+2+2\pi A(F(x)+C)) \\ &\leq 2\frac{K+2+2\pi A(e^x+C)}{e^x 4A(K+3)^2} \leq \frac{2K+4+4\pi 2.43}{4(6K+9)} \leq \frac{1}{12} + \frac{6.57}{9} < 0.82 < 1 \quad .\end{aligned}$$

□

REMARK. The escaping points on the dynamic ray $g_{\kappa,\underline{s}}$ do not need to have the external address $\underline{s}$. (They do not have one at all if their orbit hits $\mathbb{R}^-$.)

REMARK. In fact Dierk Schleicher and Johannes Zimmer have shown in [SZ] that every escaping point z is either on a unique dynamic ray at a unique potential or is the *landing point* of one or more dynamic rays: $z = \lim_{t \searrow t_{\underline{s}}} g_{\kappa,\underline{s}}(t)$. In particular, dynamic rays cannot cross each other.

The following inequalities are taken from the proof of Proposition 3.4 of [SZ].

Lemma 2.3 (More Properties of Dynamic Rays)
Let κ be a parameter with $|\kappa| \leq K$. On the interval $(2\log(K+3), \infty)$, the curve $g_{\kappa,\underline{s}}$ is the uniform limit of the functions $g^n_{\kappa,\underline{s}} : (2\log(K+3), \infty) \to \mathbb{C}$ defined in Formula (4). They satisfy for all $t > 2\log(K+3)$ the following inequalities:

$$\begin{aligned} \operatorname{Re}\left(g^n_{\kappa,\underline{s}}(t)\right) &\geq t - (K+2)\ ; && (6)\\ \left|g^2_{\kappa,\underline{s}}(t) - g^1_{\kappa,\underline{s}}(t)\right| &< 2e^{-t}\left(K + 1 + 2\pi|s_2|\right)\ ; && (7)\\ \textit{if } n \geq 3: \qquad \left|g^n_{\kappa,\underline{s}}(t) - g^{n-1}_{\kappa,\underline{s}}(t)\right| &\leq 4\pi A e^{-t} \prod_{k=1}^{n-2} e^{-k} && (8) \end{aligned}$$

and for every $n \in \mathbb{N}$:

$$\left|\operatorname{Log}(F^{\circ(n+1)}(t)) - \kappa + 2\pi i s_{n+1} - F^{\circ n}(t)\right| \leq K + 1 + 2\pi A F^{\circ n}(x)\ . \tag{9}$$

2.2 Derivatives of Dynamic Rays

The dynamic rays constructed in [SZ] completely describe the set of escaping points for every dynamical system generated by the iteration of an exponential function E_λ. We will need some more properties of them, such as a certain degree of smoothness and estimates for low potentials. The following discussion is unfortunately a bit technical.

The differentiability of dynamic rays has already been proven in 1988 by M. Viana da Silva in [Vi], as well as for bounded combinatorics by D. Schleicher in [S1]. We will prove it once more so as to obtain good estimates on the first and second derivative at large potentials.

Theorem 2.4 (The Derivative of Dynamic Rays)
For every exponentially bounded sequence $\underline{s} \in \mathcal{S}$ and every parameter κ, the dynamic ray $g_{\kappa,\underline{s}}(t)$ is differentiable with respect to the potential t with derivative

$$g'_{\kappa,\underline{s}}(t) = \prod_{k=1}^{\infty} \frac{F^{\circ k}(t) + 1}{g_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}\ . \tag{10}$$

Moreover, for every $T > t_{\underline{s}}$ there is an $M_t \geq 0$ such that

$$\forall t \geq T : \qquad |g'_{\kappa,\underline{s}}(t) - 1| < M_T e^{-t} \ . \tag{11}$$

If A and x are growth parameters for $\underline{s}$ and $K \geq |\kappa|$, then for $t''_{K,\underline{s}} := 2x + 2\log(2K + 3 + 9\pi A) > t'_{K,\underline{s}}$ we can choose

$$M_{t''_{K,\underline{s}}} := M_K(A, x) := 6\pi A + 2K + 3 + \pi A e^x/2 \ .$$

Proof. We start with potentials $t > t''_{K,\underline{s}} > t'_{K,\underline{s}}$. Recall from Lemma 2.3 the differentiable functions $g^n_{\kappa,\underline{s}}$ defined by $g^1_{\kappa,\underline{s}}(t) = \mathrm{Log}(F(t)) - \kappa + s_1 2\pi i$ and $g^{n+1}_{\kappa,\underline{s}}(t) = L_{\kappa,s_1} \circ g^n_{\kappa,\sigma\underline{s}}(F(t))$, which are defined for all $t > t''_{K,\underline{s}} > 2\log(K+3)$. Since the functions $g^n_{\kappa,\underline{s}}$ converge uniformly to $g_{\kappa,\underline{s}}$ as $n \to \infty$, all we have to show by the Weierstraß Theorem is that $(g^n_{\kappa,\underline{s}})'_{n\in\mathbb{N}}$ converges uniformly as well. Observe that for all $t > t''_{K,\underline{s}}$, we have $F(t) > K + 2$. By Inequality (6) in Lemma 2.3 it follows for all $k, n \geq 1$, $t > t''_{K,\underline{s}}$ and $\underline{s}' \in \mathcal{S}$

$$\mathrm{Re}(g^n_{\kappa,\underline{s}'}(F^{\circ k}(t)) > F^{\circ k}(t) - (K+2) > 0 \ . \tag{12}$$

Since $F'(t) = F(t) + 1$ and $(g^1)'(t) = 1 + \frac{1}{F(t)}$, we obtain, applying the chain rule repeatedly:

$$\begin{aligned}
(g^n_{\kappa,\underline{s}})'(t) &= \frac{F'(t)}{g^{n-1}_{\kappa,\sigma\underline{s}}(F(t))} \cdot (g^{n-1}_{\kappa,\sigma\underline{s}})'(F(t)) = \cdots = \\
&= \left(\prod_{k=1}^{n-1} \frac{F'(F^{\circ(k-1)}(t))}{g^{n-k}_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}\right) \cdot (g^1_{\kappa,\sigma^{n-1}\underline{s}})'(F^{\circ(n-1)}(t)) = \\
&= \left(\prod_{k=1}^{n-1} \frac{F^{\circ k}(t) + 1}{g^{n-k}_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}\right) \cdot \left(1 + \frac{1}{F^{\circ n}(t)}\right) \ .
\end{aligned} \tag{13}$$

(Note that by (12), the denominators are non-zero.) We will show that this converges uniformly to $\prod_{k=1}^{\infty} \frac{F^{\circ k}(t)+1}{g_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}$ as $n \to \infty$.

By Formula (12), every factor of the expression in (13) is contained in the right half plane. Therefore the principal branch Log of the logarithm can be applied, which yields

$$\mathrm{Log}((g^n_{\kappa,\underline{s}})'(t)) = -\sum_{k=1}^{n-1} \mathrm{Log}\left(\frac{g^{n-k}_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}{F^{\circ k}(t) + 1}\right) + \mathrm{Log}\left(1 + \frac{1}{F^{\circ n}(t)}\right) \ . \tag{14}$$

The last summand converges uniformly as $n \to \infty$, so it is left to show that the sum in Formula (14) converges uniformly as well. For all $N, n, m \in \mathbb{N}$ with $n > 2N$ we have

$$\left|\sum_{k=1}^{n-1} \mathrm{Log}\left(\frac{g^{n-k}_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}{F^{\circ k}(t)+1}\right) - \sum_{k=1}^{n-1+m} \mathrm{Log}\left(\frac{g^{n+m-k}_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}{F^{\circ k}(t)+1}\right)\right| \le$$

$$\le \sum_{k=1}^{N} \left|\mathrm{Log}\left(g^{n-k}_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))\right) - \mathrm{Log}\left(g^{n+m-k}_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))\right)\right| +$$

$$+ \sum_{k=N+1}^{n-1} \left|\mathrm{Log}\left(g^{n-k}_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))\right) - \mathrm{Log}\left(g^{n+m-k}_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))\right)\right| +$$

$$+ \sum_{k=n}^{n-1+m} \left|\mathrm{Log}\left(\frac{g^{n+m-k}_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}{F^{\circ k}(t)+1}\right)\right| . \tag{15}$$

Fix $\varepsilon > 0$. We have to find an $N^* \in \mathbb{N}$ such that these three sums together become smaller than ε for all $t > t''_{K,\underline{s}}$, $n > 2N^*$ and $m \ge 0$. Let us start with the first sum. By Lemma (2.3) and the triangle inequality we know that the uniform convergence of $(g^n)_{n\in\mathbb{N}}$ is exponentially fast:

$$\left|g^n_{\kappa,\underline{s}}(t) - g_{\kappa,\underline{s}}(t)\right| \le \sum_{m=n+1}^{\infty}\left(4\pi A e^{-t}\prod_{l=1}^{m-2} e^{-k}\right) \le 4\pi A e^{-t} e^{-(n-1)} .$$

So for all $\delta > 0$ there is a number N_δ such that for all $n \ge N_\delta$ and $m \ge 0$ we have $\|g^n - g^{n+m}\|_\infty \le \delta/N_\delta$. Observe that since $n > 2N$, the upper indices p of the g^p all satisfy $p > N$. Thus there is an $N_1 \in \mathbb{N}$ making the first sum smaller than $\varepsilon/3$ for all $n \ge N_1$.

The existence of an $N_2 \in \mathbb{N}$ such that for all $n \ge N_2$ the second sum becomes smaller than $\varepsilon/3$ follows immediately by the exponentially fast uniform convergence of $(g^n)_{n\in\mathbb{N}}$.

Estimating the last sum is a little bit more involved. An estimate on the Taylor series using the geometric series shows for all $z \in \mathbb{C}$ with $|z| < 1$:

$$|\mathrm{Log}(1-z)| = \left|z + z^2/2 + z^3/3 + \ldots\right| < \frac{|z|}{1-|z|} \quad \text{and} \tag{16}$$

$$|1-e^z| = \left|z + z^2/2 + z^3/6 + \ldots\right| \le \frac{|z|}{1-|z|} , \tag{17}$$

and so for all $t > \log 2$

$$|\log(F(t)) - t| = |\log(1-e^{-t})| < 2e^{-t} . \tag{18}$$

Now the Formulas (7), (8), (9), and (18) give for all $t \geq t''_{K,\underline{s}}$, $p \geq 2$ and $k \geq 0$:

$$\begin{aligned}
&\left| g^p_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t)) - g^1_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t)) \right| \leq \\
&\leq \left| g^p_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t)) - g^{p-1}_{\kappa,\sigma^k s}(F^{\circ k}(t)) \right| + \cdots + \left| g^2_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t)) - g^1_{\kappa,\sigma^k s}(F^{\circ k}(t)) \right| \leq \\
&\leq 4\pi A e^{-F^{\circ k}(t)} \left(\sum_{n=2}^{p-1} \prod_{l=1}^{n-2} e^{-l} \right) + 2e^{-F^{\circ k}(t)}(K + 1 + 2\pi|s_{k+2}|) < \\
&< (2\pi AC + K + 1 + 2\pi A F^{\circ(k+1)}(x)) 2e^{-F^{\circ k}(t)} \overset{(*)}{\leq} 2\pi AC + K + 1 + 4\pi A < \\
&< 7\pi A + K + 1 \ , \qquad (19)
\end{aligned}$$

where $C = \sum_{n=2}^{\infty} \prod_{k=1}^{\infty} e^{-k} \approx 1.42$ is the same constant as in Theorem 2.2. In (*) we used that $e^{-F^{\circ k}(t)} < e^{-2} < \frac{1}{2}$ and $F^{\circ(k+1)}(x) e^{-F^{\circ k}(t)} < F^{\circ(k+1)}(x) e^{-F^{\circ k}(x)} < 1$ for all $t > t''_{K,\underline{s}}$.

Define for all $p \geq 1$ and $k \geq 1$

$$\alpha_p(k) := 1 - \frac{g^p_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}{F^{\circ k}(t) + 1} \ .$$

We can estimate this quantity independently of p, so we are going to drop the index later on. Indeed, the triangle inequality and $e^{-F^{\circ k}(t)} < \frac{1}{2}$ yield

$$\begin{aligned}
|\alpha_p(k)| &\leq \left| \frac{g^p_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t)) - g^1_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}{F^{\circ k}(t) + 1} \right| + \left| \frac{g^1_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t)) - F^{\circ k}(t) - 1}{F^{\circ k}(t) + 1} \right| \overset{(19),\,(18)}{\leq} \\
&\leq \frac{7\pi A + K + 1}{F^{\circ k}(t) + 1} + \frac{2e^{-F^{\circ k}(t)} + 1 + K + 2\pi A F^{\circ k}(x)}{F^{\circ k}(t) + 1} \leq \\
&< \frac{2K + 3 + 7\pi A + 2\pi A F^{\circ k}(x)}{F^{\circ k}(t) + 1} =: \alpha(k) \ . \qquad (20)
\end{aligned}$$

Since $t > x$, $\alpha(k)$ converges exponentially fast to 0 as $k \to \infty$. In particular, there is an N' such that for all $k \geq N'$ we have $\alpha(k) \in (0,1)$, and Formula (16) implies for every $p \geq 0$

$$|\mathrm{Log}(1 - \alpha_p(k))| < \frac{|\alpha_p(k)|}{1 - |\alpha_p(k)|} \leq \frac{\alpha(k)}{1 - \alpha(k)} \ . \qquad (21)$$

This converges to 0 exponentially fast as well, and there is an N_3 with the property that the whole third sum in Formula (15) becomes smaller than $\varepsilon/3$ for all $t > t'_{K,\underline{s}}$ if $n > N_3$.

The dynamic ray tail is thus continuously differentiable, and $g'_{\kappa,\underline{s}}$ is the uniform limit of $(g^n_{\kappa,\underline{s}})'$. Moreover, the analyticity of the pullback makes the whole dynamic ray differentiable, but we lose the above estimates.

To get Formula (11), we consider potentials $t \geq t''_{K,\underline{s}} = 2x + 2\log(K + 3 + 9\pi A)$, so in particular $t > t'_{K,\underline{s}}$ and $t > 2x + 2\log(3 + 9\pi) > 2x + 5$. Note that

$$t > 2x + 5 \quad \Longrightarrow \quad e^{t/2} - t \geq e^x \ , \tag{22}$$

for if $e^x \geq t$ then $e^x + t \leq 2e^x < e^{x+1} < e^{t/2}$, and otherwise $e^x + t < 2t$, which is less than $e^{t/2}$ for all $t \geq 5$.

For every p and k and every $t \geq t''_{K,\underline{s}}$ we have $|\alpha_p(k)| < \frac{1}{2}$, because

$$\begin{aligned} |\alpha_p(k)| \ &\leq \ \alpha(1) = \frac{2K + 3 + 7\pi A + 2\pi A F(x)}{F(t) + 1} < \frac{(2K + 3 + 9\pi A)e^x}{e^{t''_{K,\underline{s}}}} = \\ &\leq \frac{1}{\exp(x + \log(2K + 3 + 9\pi A))} < \frac{1}{2} \ . \end{aligned}$$

Therefore, using (14) and (21) we estimate

$$\left|\mathrm{Log}(g'_{\kappa,\underline{s}}(t))\right| = \left|\sum_{k=1}^{\infty} \mathrm{Log}\left(\frac{g_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}{F^{\circ k}(t) + 1}\right)\right| \leq \sum_{k=1}^{\infty} \frac{\alpha(k)}{1 - \alpha(k)} < 2\sum_{k=1}^{\infty} \alpha(k) \ .$$

Furthermore, we will show separately in Lemma 2.5 that for each $k \geq 1$,

$$\sum_{k=1}^{\infty} \alpha(k) = \frac{D}{F^{\circ k}(t) + 1} + D'\frac{F^{\circ k}(x)}{F^{\circ k}(t) + 1} < M'e^{-t}$$

for some constants D, D' and $M' := 3D + D'(e^x + 1)$ involving only A and K. Hence $\left|\mathrm{Log}(g'_{\kappa,\underline{s}}(t))\right| \leq 2M'e^{-t}$. By $e^t > (e^x + t)^2$ we have $2M'e^{-t} < 1/2$. Therefore using (17) and $M := M'/4$, we arrive at the inequality

$$|1 - g_{\kappa,\underline{s}}(t)'| \leq \frac{2M'e^{-t}}{1 - 2M'e^{-t}} < 4M'e^{-t} = Me^{-t} \ .$$

Finally, $M_K(A, x) = 6\pi A + 2K + 3 + \pi A e^x/2 > \frac{1}{4}(3(2K + 3 + 7\pi A) + 2\pi A(e^x + 1) = M'/4$ completes the proof of the explicit estimate.

The existence of the constants M_T follows immediately from this, because the continuous function $g'_{\kappa,\underline{s}}(t)$ is bounded on the compact interval $[T, t''_{K,\underline{s}}]$. $\square$

Lemma 2.5 (An Estimate on the Function F)
If $x \geq 0$ and $t \geq 5$ are real numbers such that $t \geq 2x + 5$ then

$$\sum_{k=1}^{\infty} \frac{1}{F^{\circ k}(t) + 1} < 3e^{-t} \quad \textit{and} \quad \sum_{k=1}^{\infty} \frac{F^{\circ k}(x)}{F^{\circ k}(t) + 1} < (e^x + 1)e^{-t} \ .$$

PROOF. Define $M_k(x) := \exp(-F^{\circ(k-1)}(x))$. We claim that for every $k \geq 2$

$$\frac{F^{\circ k}(x)}{F^{\circ k}(t)} \leq M_k(x)e^{-t} \,, \quad \text{and thus} \quad \sum_{k=1}^{\infty} \frac{F^{\circ k}(x)}{F^{\circ k}(t)+1} \leq \left(e^x + \sum_{k=2}^{\infty} M_k(x)\right) e^{-t} \,.$$

In order to show this, we first prove by induction that for every $k \geq 2$

$$F^{\circ(k-1)}(t/2) - t \geq F^{\circ(k-1)}(x) \quad \text{and} \quad F^{\circ(k-1)}(t) - F^{\circ(k-1)}(t/2) \geq F^{\circ(k-1)}(x) \,.$$

For $k = 2$, the first claim follows from Formula (22), since $t \geq 2x + 5$. Furthermore, $0 \leq (e^t - t)^2 \leq e^{2t} - 2e^t + t^2$ and thus $e^{2t} + 2e^t + t^2 \geq 4e^t$, so taking square roots yields $e^t + t \geq 2e^{t/2}$ or $e^t - e^{t/2} \geq e^{t/2} - t$. Hence $F(t) - F(t/2) = e^t - e^{t/2} \geq e^{t/2} - t \geq F(x)$.

Now assume that the above estimates are true for some k. Then $F^{\circ k}(t/2) - F^{\circ k}(x) \geq F^{\circ(k-1)}(t/2) - F^{\circ(k-1)}(x) \geq t$, because F is expansive on $\mathbb{R}^+$ due to $F'(s) = e^s > 1$ for all $s > 0$.

For the second claim we use that $e^a + e^b < e^{a+b}$ if $a, b \geq 0$, and so if $c = a + b$ also $e^{c-b} < e^c - e^b$. Thus it follows from $\exp\left(F^{\circ(k-1)}(t) - F^{\circ(k-1)}(x)\right) \geq \exp\left(F^{\circ(k-1)}(t/2)\right)$ that

$$\exp\left(F^{\circ(k-1)}(t)\right) - \exp\left(F^{\circ(k-1)}(x)\right) \geq \exp\left(F^{\circ(k-1)}(t/2)\right) \,,$$

which completes the claim after replacing the exponentials by the F-function.

Therefore we get for every $k \geq 2$

$$\begin{aligned} \frac{F^{\circ k}(x)}{F^{\circ k}(t)+1} &\leq \frac{F^{\circ k}(x)}{F^{\circ k}(t)} \leq \exp\left(F^{\circ(k-1)}(x) - F^{\circ(k-1)}(t) + t\right) e^{-t} \leq \\ &\leq \exp\left(t - F^{\circ(k-1)}(t/2)\right) e^{-t} \leq \exp\left(-F^{\circ(k-1)}(x)\right) e^{-t} = M_k(x)e^{-t} \,. \end{aligned}$$

We will show now that $\sum_{k=2}^{\infty} M_k(x) < 1$. The sum is monotone increasing in x, so suppose w.l.o.g. $x \geq 1$. This implies $F^{\circ(k-1)}(x) > (k-1)x$ for every $k \geq 2$ (for $F^{\circ 1}(1) = e - 1 > 1$, and thus

$$\sum_{k=2}^{\infty} M_k(x) = \sum_{k=2}^{\infty} e^{-F^{\circ(k-1)}(x)} < \sum_{k=2}^{\infty} e^{-(k-1)x} < \sum_{k=1}^{\infty} 2^{-k} = 1 \,. \tag{23}$$

So $\sum_{k=1}^{\infty} \frac{F^{\circ k}(x)}{F^{\circ k}(t)} < (e^x + 1)\, e^{-t}$, which shows the second claim. Finally, (23) yields

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{F^{\circ k}(t)+1} &= e^{-t} + \sum_{k=2}^{\infty} M_k(t) \leq e^{-t}\left(1 + \sum_{k=2}^{\infty} e^{-(k-2)t}\right) < e^{-t}\left(1 + \sum_{k=0}^{\infty} 2^{-k}\right) \\ &= 3e^{-t} \,. \end{aligned}$$

□

Proposition 2.6 (The Second Derivative of Dynamic Rays)
Every dynamic ray $g_{\underline{s}} : (t_{\underline{s}}, \infty) \to \mathbb{C}$ is twice continuously differentiable. Moreover, given growth parameters A and x for $\underline{s}$, there is a constant $C' := e^{t''_{K,\underline{s}}/2} + 3$ such that for all $t > t''_{K,\underline{s}} = 2x + 2\log(2K + 3 + 9\pi A)$

$$|g''_{\kappa,\underline{s}}(t)| < 2C'e^{-t} \qquad \text{and} \qquad \left|\frac{g''_{\kappa,\underline{s}}(t)}{g'_{\kappa,\underline{s}}(t)}\right| < C'e^{-t} \ .$$

Proof. At first let us formally differentiate equation (10) from Theorem 2.4:

$$\begin{aligned}
\frac{d}{dt}\mathrm{Log}\left(g'_{\kappa,\underline{s}}(t)\right) &= \sum_{k=1}^{\infty}\left(\frac{d}{dt}\mathrm{Log}(F^{\circ k}(t)+1) - \frac{d}{dt}\mathrm{Log}\left(g_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))\right)\right) = \\
&= \sum_{k=1}^{\infty}\left(\frac{(d/dt)F^{\circ k}(t)}{F^{\circ k}(t)+1} - \frac{\left((d/dt)F^{\circ k}(t)\right)\cdot g'_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}{g_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}\right) = \\
&= \sum_{k=1}^{\infty}\frac{(d/dt)F^{\circ k}(t)}{F^{\circ k}(t)+1}\left(1 - g'_{\kappa,\sigma^k\underline{s}}\left(F^{\circ k}(t)\right)\frac{F^{\circ k}(t)+1}{g_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}\right) = \\
&\overset{(*)}{=} \sum_{k=1}^{\infty}\frac{\prod_{m=1}^{k}(F^{\circ m}(t)+1)}{F^{\circ k}(t)+1}\left(1 - g'_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))\frac{F^{\circ k}(t)+1}{g_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}\right) \ ,
\end{aligned} \tag{24}$$

where the equality (*) follows from the chain rule:

$$\begin{aligned}
\frac{d}{dt}F^{\circ k}(t) &= \frac{d}{dt}\left(F(F^{\circ(k-1)}(t))\right) = \left(\frac{d}{dt}F^{\circ(k-1)}(t)\right)\exp\left(F^{\circ(k-1)}(t)\right) = \\
&= \left(\frac{d}{dt}F^{\circ(k-1)}(t)\right)\left(F^{\circ k}(t)+1\right) = \cdots = \prod_{m=1}^{k}(F^{\circ m}(t)+1) \ .
\end{aligned}$$

Assume $t > t''_{K,\underline{s}} \geq t'_{K,\underline{s}}$. Recall that by (20), we have for such t

$$\left|\frac{g_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t))}{F^{\circ k}(t)+1} - 1\right| = |\alpha_1(k)| \leq \alpha(k) = \frac{2K+3+7\pi A+2\pi AF^{\circ n}(x)}{F^{\circ k}(t)+1} \ .$$

Remember that if x and A are growth parameters of $\underline{s}$, then $F^{\circ k}(x)$ and A are valid growth parameters for $\sigma^k\underline{s}$. Since $t > t''_{K,\underline{s}}$ yields $F^{\circ k}(t) > t''_{K,\sigma^k\underline{s}}$ for all $k \in \mathbb{N}$, Estimation (11) in Theorem 2.4 shows

$$\left|g'_{\kappa,\sigma^k\underline{s}}(F^{\circ k}(t)) - 1\right| < M_K\left(A, F^{\circ k}(x)\right)\cdot\exp(-F^{\circ k}(t)) \ ,$$

with $M_K(A, F^{\circ k}(x)) = \frac{\pi A}{2} \exp\left(F^{\circ k}(x)\right) + 6\pi A + 2K + 3$. This allows us to estimate the k-th summand of the sum in (24), using that $|1 - wz| = |z| \cdot |z^{-1} - w| \leq |z|(|w-1| + |z^{-1} - 1|)$ for all $z, w \in \mathbb{C}^*$, as follows:

$$|k\text{-th summand}| < \left(\prod_{m=1}^{k-1} (1 + F^{\circ m}(t))\right) \cdot \left| \frac{F^{\circ k}(t) + 1}{g_{\kappa, \sigma^k \underline{s}}(F^{\circ k}(t))} \right| \cdot$$
$$\cdot \left(\frac{\pi A \exp\left(F^{\circ k}(x)/2 + 6\pi A + 2K + 3\right)}{\exp\left(F^{\circ k}(t)\right)} + \frac{2K + 3 + 7\pi A + 2\pi A F^{\circ k}(x)}{F^{\circ k}(t) + 1} \right) \tag{25}$$

The factor in the middle is irrelevant because it almost 1 on the ray tail by Theorem 2.2. Thus if we show that

$$\left(\prod_{m=1}^{k-1} (1 + F^{\circ m}(t))\right) \cdot \left(\frac{\frac{\pi A}{2} F^{\circ (k+1)}(x)}{F^{\circ (k+1)}(t)} + \frac{2\pi A F^{\circ k}(x)}{F^{\circ k}(t)} \right) \leq$$
$$\leq \left(\prod_{m=1}^{k-1} (1 + F^{\circ m}(t))\right) \cdot \frac{5\pi A}{2} \cdot \frac{F^{\circ k}(x)}{F^{\circ k}(t)} \longrightarrow 0 \tag{26}$$

exponentially fast, then we have proven that the sum in Equation (24) really exists.

To do so, let us decompose the denominator $F^{\circ k}(t)$ of (26) into its three third roots: We will show that one of them bounds $F^{\circ k}(x)$, while another one handles the factor within the parentheses. The remaining denominator of $\sqrt[3]{F^{\circ k}(t)}$ suffices to let the whole product converge to 0 exponentially fast. Clearly we have $F^{\circ (k-1)}(t) > 3F^{\circ (k-1)}(x)$ for every $k \geq 2$, since $t > 2x$. Applying the exponential function yields $\sqrt[3]{F^{\circ k}(t)} > F^{\circ k}(x)$, which shows the first claim. Similarly, every $k \geq 2$ satisfies $(k-1) \cdot F^{\circ (k-2)}(t) < F^{\circ (k-1)}(t)/3$. This implies $\sum_{1 \leq m < k} F^{\circ (m-1)}(t) < F^{\circ (k-1)}(t)/3$. Taking the exponential function gives the desired second estimate. Summarizing, we have

$$\left| \frac{d}{dt} \mathrm{Log}\left(g'_{\kappa, \underline{s}}(t)\right) \right| \leq |1\text{-st summand}| + \frac{5\pi A}{2} \sum_{k=2}^{\infty} \frac{1}{\sqrt[3]{F^{\circ k}(t)}} < \infty\,. \tag{27}$$

Hence we obtain convergence of the partial sums to $\frac{d}{dt}\mathrm{Log}(g')$. In fact, the convergence is uniform on $[t''_{K,\underline{s}}, \infty]$, since the rate of convergence is bounded below by the convergence of the lowest potential considered. Every partial sum is continuous in t, so the limit is continuous. Moreover, the partial sums are derivatives of a sequence which converges to $\mathrm{Log}(g')$. Therefore g' is also continuously differentiable, having the logarithmic derivative as claimed in Equation (24). Some end of the dynamic ray is thus twice continuously differentiable, and the analytic pullback extends this property to the whole dynamic ray.

In order to prove the estimates for potentials $t > t''_{K,\underline{s}} = 2x + 2\log(2K+3+9\pi A)$, we recall the estimate (27). For every $k \geq 2$ and $t > t''_{K,\underline{s}}$, we have $\sqrt[3]{F^{\circ k}(t)} > F^{\circ(k-1)}(t) + 1$, as already showed above. Furthermore, we can estimate the first summand, following (25):

$$|\text{1-st summand}| < O(\exp(e^{-t})) + \frac{2K + 3 + 7\pi A + 2\pi A e^x}{e^t} < e^{t''_{K,\underline{s}}/2} e^{-t} ,$$

since $e^{t''_{K,\underline{s}}/2} = e^x(2K + 3 + 9\pi A)$. Therefore, Lemma 2.5 yields

$$\begin{aligned} \left|\frac{g''(t)}{g'(t)}\right| &= \left|\frac{d}{dt}\text{Log}\left(g'_{\kappa,\underline{s}}(t)\right)\right| \leq e^{t''_{K,\underline{s}}/2-t} + \frac{5\pi A}{2}\sum_{k=2}^{\infty} \frac{1}{F^{\circ(k-1)}(t)+1} < \\ &< \left(e^{t''_{K,\underline{s}}/2} + 3\right) e^{-t} = C' e^{-t} . \end{aligned}$$

To get the first estimate, we use that $|g'_{\kappa,\underline{s}}(t) - 1| < M_K(A,x)e^{-t}$ implies $|g'(t)| \leq 1 + M_K(A,x)e^{-t} < 2$ for $t > t''_{K,\underline{s}} > \log(M_K(A,x))$. So the above estimate yields $|g''(t)| < C'|g'(t)|e^{-t} < 2C'e^{-t}$. □

2.3 Winding Numbers of Dynamic Rays

The estimates on the derivatives of dynamic rays will be used as a tool for providing an estimate on the *winding number* of dynamic rays. Since the logarithm unwraps angles to imaginary parts, this helps us controlling the dynamic rays for small potentials. We will need that to show the important Proposition 3.4.

Some ideas of the following discussion, which is in most parts taken from [S1], were originally a contribution by Niklas Beisert.

Remember the definition of winding numbers for closed curves: Consider a closed C^1-curve $\varphi : [t_0, t_1] \to \mathbb{C}$, i.e. $\varphi(t_0) = \varphi(t_1)$. Let $a \in \mathbb{C} \setminus \varphi([t_0, t_1])$ be an arbitrary point outside the graph. Then we define *the winding number of φ around a* by

$$\begin{aligned} \eta(\varphi, a) &:= \frac{1}{2\pi}\int_{t_0}^{t_1} \text{Im}\frac{\dot\varphi(t)}{\varphi(t)-a}dt = \frac{1}{2\pi}\int_{t_0}^{t_1} \text{Im}\frac{\partial}{\partial t}\left(\text{Log}(\varphi(t)-a)\right)dt = \\ &= \frac{1}{2\pi}\int_{t_0}^{t_1} d\arg(\varphi(t)-a) . \end{aligned} \tag{28}$$

It turns out in this case that η is always an integer number which counts the number of turns of φ around a. We will now carry over this concept to a certain kind of curves which are not closed, so that we will be able to apply the results on dynamic rays.

Definition 2.7 (Winding Numbers)
Let $\gamma : (t_0, \infty) \to \mathbb{C}$ *be a* C^1*-curve and* $a \notin \gamma(t_0, \infty)$*. If the integral* $\int_{t_0}^{\infty} |d\arg(\gamma(t) - a)|$ *exists (i.e. is finite), define the* winding number of γ around a *by*

$$\eta(\gamma, a) := \frac{1}{2\pi} \int_{t_0}^{\infty} \mathrm{Im} \frac{\dot{\gamma}(t)}{\gamma(t) - a} dt = \frac{1}{2\pi} \int_{t_0}^{\infty} d\arg(\gamma(t) - a) \ .$$

Definition 2.8 (Admissible Curves)
By an admissible curve *we mean a* C^2*-curve* $\gamma : (t_0, \infty) \to \mathbb{C}$ *with* $t_0 \geq -\infty$*, having the following properties:*

- $\lim_{t \to \infty} \mathrm{Re}(\gamma(t)) = +\infty$
- $\dot{\gamma}(t) \neq 0$ *everywhere*
- $\dot{\gamma}(t) \longrightarrow 1$ *as* $t \to \infty$
- $\mathrm{Im}(\ddot{\gamma}(t)) \longrightarrow 0$ *as* $t \to \infty$

Lemma 2.9 (Admissible Curves Have Winding Numbers)
If $\gamma : (t_0, \infty) \to \mathbb{C}$ *is an admissible curve, then*

1. *If* $a \in \mathbb{C} \setminus \overline{\gamma(t_0, \infty)}$ *and*
$$|\dot{\gamma}(t)| \not\longrightarrow \infty \quad \text{as } t \searrow t_0 \ ,$$
then the winding number $\eta(\gamma, a)$ *is defined.*
2. *For every* $t_1 > t_0$*,* $\eta(\dot{\gamma}|_{(t_1,\infty)}, 0)$ *is defined.*
3. *If* $a = \gamma(t_1)$ *for some* $t_1 > t_0$*, then* $\eta(\gamma|_{(t_1,\infty)}, a)$ *is defined.*

Proof. In all the statements we have integrands that are locally Riemann integrable. Therefore we only have to show that the integrals are finite. For this in turn it is sufficient to show that $|\frac{\dot{\varphi}}{\varphi - \tilde{a}}|$ is bounded (where φ is the respective curve and $\tilde{a} = 0$ for the second case and $\tilde{a} = a$ otherwise).

For $a \notin \overline{\gamma(t_0, \infty)}$ we can clearly bound the denominator $\gamma - a$ below by some $\varepsilon > 0$ and the numerator $\dot{\varphi}$ above, using the conditions on admissible curves and $|\dot{\gamma}(t)| \not\to \infty$ as $t \searrow t_0$.

In the second case, there is a T such that $|\dot{\gamma}(t)| > 1/2$ for all $t > T$, and on the compact interval $[t_1, T]$, the continuous function $|\dot{\gamma}(t)|$ is bounded below by some $\varepsilon > 0$. We thus have $0 \notin \overline{\dot\gamma(t_1,\infty)}$. Since, $\mathrm{Im}(\ddot\gamma(t))$ converges to 0, the same reasoning as for the first item can be applied.

In the last case we might get into trouble as $t \searrow t_1$, since the denominator tends to 0. Let $t > t_0$. By the Taylor Theorem applied on $\gamma \in C^2$ there is a $\xi \in [t_0, t]$ with

$$\gamma(t_1) = \gamma(t) + \dot\gamma(t)(t_1 - t) + \ddot\gamma(t)(\xi - t)^2/2 \ .$$

Therefore

$$\begin{aligned}\left|\mathrm{Im}\frac{\dot\gamma(t)}{\gamma(t)-\gamma(t_1)}\right| &= \left|\mathrm{Im}\frac{\gamma(t_1)-\gamma(t)-\ddot\gamma(t)(\xi-t)^2/2}{(\gamma(t)-\gamma(t_1))(t_1-t)}\right| = \\ &= \left|\mathrm{Im}\frac{\ddot\gamma(t)(\xi-t)^2/2}{(\gamma(t_1)-\gamma(t))(t_1-t)}\right| \le \left|\mathrm{Im}\frac{\ddot\gamma(t)(t_1-t)}{2(\gamma(t_1)-\gamma(t))}\right| \ .\end{aligned}$$

So $\lim_{t\searrow t_1}\left|\mathrm{Im}\frac{\dot\gamma(t)}{\gamma(t)-\gamma(t_1)}\right| \le \lim_{t\searrow t_1}\left|\mathrm{Im}\frac{\ddot\gamma(t)}{2\dot\gamma(t)}\right| = \left|\mathrm{Im}\frac{\ddot\gamma(t_1)}{2\dot\gamma(t_1)}\right|$. Up to the sign this is half the integrand for the winding number of $\dot\gamma$, which we have shown already to exist. □

Lemma 2.10 (The Winding Number of a Curve and of its Derivative)

1. *For every closed C^2-curve $\varphi : [t_0, t_1] \to \mathbb{C}$ with $\dot\varphi \neq 0$ and every $a \notin \varphi[t_1, \infty)$*

$$\eta(\varphi, a) = \eta(\dot\varphi, 0) \ .$$

2. *Let $\tilde\gamma : (t_0, \infty) \to \mathbb{C}$ be an admissible curve, and $\gamma := \tilde\gamma|_{(t_1,\infty)}$ its restriction for some $t_1 > t_0$. Then for every $a \notin \gamma[t_1, \infty)$*

$$|\eta(\gamma, a)| \le |\eta(\dot\gamma, 0)| + 2 \ .$$

Proof. Recall the definition of the winding number for a closed curve in (28), which is an integer number. If there is a homotopy H between some closed C^1-curves φ_1 and $\varphi_2 : [t_0, t_1] \to \mathbb{C}$ on $\mathbb{C} \setminus \{a\}$ then $\eta(\varphi_1, a) = \eta(\varphi_2, a)$. (A homotopy is a continuous mapping $H : [t_0, t_1] \times [0, 1] \to \mathbb{C} \setminus \{a\}$ such that $H(t, 0) = \varphi_1(t)$ and $H(t, 1) = \varphi_2(t)$.) This follows from the fact that $t \mapsto \eta(H(\bullet, t), a)$ is continuous with values in $\mathbb{Z}$. Moreover, if φ_1 and φ_2 are C^2-curves and H is continuously differentiable with $\dot H([t_0, t_1] \times [0, 1]) \subset \mathbb{C} \setminus \{0\}$, then also $\eta(\dot\varphi_1, 0) = \eta(\dot\varphi_2, 0)$.

Now consider the curve φ from the first claim and let $n := \eta(\varphi, a) \in \mathbb{Z}$. There is a C^1-homotopy with $\dot H(\bullet, t) \neq 0$ for all $t \in [0, 1]$ between φ and an n-fold loop around

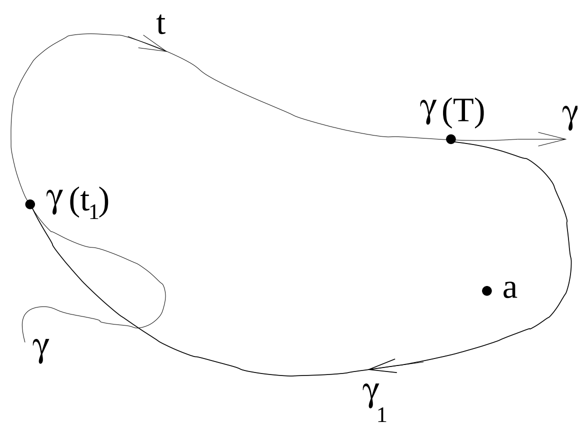

Figure 3: For the Proof of Lemma 2.10.

a. Parameterizing this loop by arc-length we get the curve $\tilde{\varphi} : [0, 2\pi n] \to \mathbb{C} \setminus \{a\}$, $\tilde{\varphi}(t) := a + e^{\mathrm{sgn}(n)it}$, having the same winding numbers. (In particular, if $n = 0$, then $\tilde{\varphi} \equiv a + 1$.) We calculate

$$\begin{aligned} \eta(\dot{\varphi}, 0) &= \eta(\dot{\tilde{\varphi}}, 0) = \int_0^{2\pi n} d \arg \dot{\tilde{\varphi}} = \int_0^{2\pi n} d \arg(\mathrm{sgn}(n) i e^{\mathrm{sgn}(n)it}) = \\ &= \int_0^{2\pi n} d \arg(\mathrm{sgn}(n) i) + d \arg(e^{\mathrm{sgn}(n)it}) = \int_0^{2\pi n} 0 + d \arg(\tilde{\varphi} - a) = \\ &= \eta(\tilde{\varphi}, a) = \eta(\varphi, a) \ . \end{aligned}$$

This shows the first claim.

If $\gamma : (t_1, \infty) \to \mathbb{C}$ is an admissible curve, then $\mathrm{Re}(\gamma)$ converges to ∞, while $\mathrm{Im}(\gamma)$ converges to some fixed number. Hence for every $\varepsilon > 0$ there is a $T > t_1$ such that

$$\begin{aligned} \left|\eta(\gamma, a) - \eta(\gamma|_{(t_1,T)}, a)\right| &< \varepsilon/2 \quad \text{and} \\ \left|\eta(\dot{\gamma}, 0) - \eta(\dot{\gamma}|_{(t_1,T)}, 0)\right| &< \varepsilon/2 \ . \end{aligned}$$

It follows that $|\eta(\gamma, a) - \eta(\dot{\gamma}, 0)| < \left|\eta(\gamma|_{(t_1,T)}, 0), a) - \eta(\dot{\gamma}|_{(t_1,T)}, 0), 0)\right| + \varepsilon$.

Now according to Figure 3, find an arc $\gamma_1 : (T - \delta, T + 1 + \delta) \to \mathbb{C} \setminus \{a\}$ with

$$\begin{aligned} \frac{d^k \gamma_1}{dt^k}(T) = \frac{d^k \gamma}{dt^k}(T) \ , \quad \frac{d^k \gamma_1}{dt^k}(T+1) = \frac{d^k \gamma}{dt^k}(T+1) \quad \text{and} \\ \dot{\gamma}_1(t) \neq 0 \qquad \text{for all } t \in [T, T+1] \text{ and } k = 0, 1, 2 \ . \end{aligned}$$

We can clearly find such an arc satisfying $|\eta(\gamma_1, a)| \leq 1$ and $|\eta(\dot{\gamma}_1, 0)| \leq 1$. By applying the first part of the Lemma on the concatenated curve $\varphi := \gamma|_{[t_1,T]} \vee \gamma_1$:

$[t_1, T+1] \to \mathbb{C} \setminus \{a\}$ we get

$$\begin{aligned} |\eta(\gamma, a) - \eta(\dot\gamma, 0)| &< \left|\eta(\gamma|_{(t_1,T)}, 0), a) - \eta(\dot\gamma|_{(t_1,T)}, 0), 0)\right| + \varepsilon \\ &\leq |\eta(\varphi, a) - \eta(\dot\varphi, 0)| + 2 + \varepsilon = 2 + \varepsilon\ . \end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrary small, the second claim is proven. $\square$

Proposition 2.11 (Winding Numbers and Pullback)
Let $\gamma_0 : (t_0, \infty) \to \mathbb{C}$ be an admissible curve such that $|\eta(\dot\gamma_0, 0)| \leq 2$. For every $n \in \mathbb{N}$ choose $a_n \in \mathbb{C} \setminus \overline{\gamma_n(t_0, \infty)}$ and define $\gamma_{n+1} := \log(\gamma_n - a_n) : (t_0, \infty) \to \mathbb{C}$. Then

$$\forall t_1 > t_0,\ \forall a \notin \overline{\gamma_n(t_0, \infty)} : \quad |\eta(\gamma_n|_{(t_1,\infty)}, b)| \leq 2^{n+2}\ .$$

Proof. Note that since $\gamma_n - a_n \neq 0$, a branch of the logarithm can be applied. The choice of branch is inessential, because we are dealing only with $d\log$.

We will show by induction that $|\eta(\dot\gamma_n, 0)| \leq 2^{n+2} - 2$ for all $n \in \mathbb{N}$. This estimate will also justify that all the winding numbers are defined. The induction seed $n = 0$ follows by the assumption $|\eta(\dot\gamma_0, 0)| \leq 2$.

For the induction step, let a, γ, $\tilde\gamma$ denote a_n, γ_n, $\gamma_{n+1} = \log(\gamma_n - a)$ respectively. We estimate

$$\begin{aligned} |\eta(\dot{\tilde\gamma}, 0)| &= \left|\eta\left(\frac{\dot\gamma}{\gamma - a}, 0\right)\right| = \frac{1}{2\pi}\left|\int d\left(\operatorname{Im}\log\left(\frac{\dot\gamma}{\gamma - a}\right)\right)\right| = \\ &= \frac{1}{2\pi}\left|\int d(\operatorname{Im}\log\dot\gamma - \operatorname{Im}\log(\gamma - a))\right| = \\ &= \frac{1}{2\pi}\left|\int \operatorname{Im}\left(\frac{\ddot\gamma}{\dot\gamma}\right) - \operatorname{Im}\left(\frac{\dot\gamma}{\gamma - a}\right) dt\right| \leq \\ &\leq \frac{1}{2\pi}\left|\int \operatorname{Im}\left(\frac{\ddot\gamma}{\dot\gamma}\right) dt\right| + \frac{1}{2\pi}\left|\int \operatorname{Im}\left(\frac{\dot\gamma}{\gamma - a}\right) dt\right| = \\ &= |\eta(\dot\gamma, 0)| + |\eta(\gamma, a)| \overset{\text{Lemma 2.10}}{\leq} 2|\eta(\dot\gamma, 0)| + 2 \leq \\ &\leq 2(2^{n+2} - 2) + 2 = 2^{n+3} - 2\ . \end{aligned}$$

So we have by Lemma 2.10 $|\eta(\gamma_n, a)| \leq |\eta(\dot\gamma_n, 0)| + 2 \leq 2^{n+2}$ for all $n \in \mathbb{N}$. $\square$

Lemma 2.12 (Winding Number of Dynamic Rays)
Consider an arbitrary parameter κ and an exponentially bounded sequence $\underline{s} \in \mathcal{S}$. Let $z = g_{\kappa,\underline{s}}(t_0)$ be any point on the dynamic ray of address $\underline{s}$ with potential $t_0 > t_{\underline{s}}$. Choose the growth parameters A and x of $\underline{s}$ such that $t_{\underline{s}} < x < t_0$. If $n \in \mathbb{N}$ is big enough such that

$$F^{\circ n}(t_0) > t_n^\star := 2F^{\circ n}(x) + 2\log(2|\kappa| + 3 + 9\pi A) ,$$

then

$$|\eta(g_{\kappa,\underline{s}}(t_0,\infty), z)| \leq 2^n + 2 .$$

Proof. Let $K := |\kappa|$. Since $t_n^\star = t''_{K,\sigma^n\underline{s}}$, we can apply Theorem 2.4 and Proposition 2.6 on $g_{\kappa,\sigma^n\underline{s}}(t)$ for potentials $t > t_n^\star$. The curve $\gamma : (0,\infty) \to \mathbb{C}$, $\gamma(t) := g_{\kappa,\sigma^n\underline{s}}(t+t_n^\star)$ is thus admissible. ($0 \notin \dot\gamma$ follows from Formula (11).) We estimate

$$\begin{aligned} |\eta(\dot\gamma, 0)| &\leq \frac{1}{2\pi}\int_0^\infty \left|\operatorname{Im}\left(\frac{\ddot\gamma}{\dot\gamma}\right)\right| dt \leq \frac{1}{2\pi}\int_0^\infty \left|\frac{g''_{\kappa,\sigma^n\underline{s}}(t+t_n^\star)}{g'_{\kappa,\sigma^n\underline{s}}(t+t_n^\star)}\right| dt \overset{(*)}{<} \\ &< \frac{1}{2\pi}\int_0^\infty \left(e^{t_n^\star/2}+3\right) e^{-t-t_n^\star} dt = \left(e^{t_n^\star/2}+3\right)\frac{e^{-t_n^\star}}{2\pi} < \frac{2}{2\pi} < 2 , \end{aligned}$$

where (*) follows from Proposition 2.6. The absolute value of the winding number of the end of $g'_{\kappa,\sigma^{n-1}\underline{s}}$ to the right of $E_\lambda^{\circ n}(z)$ is thus less than 2. If we define $\gamma_0 := \gamma$ and $\gamma_{k+1} := L_{\kappa,s_{n-k}}(\gamma_k)$ then γ_n is the initial dynamic ray starting at t_0, and applying Proposition 2.11 settles the claim. $\square$

The following Proposition generalizes Lemma II.7.2 from [S1].

Proposition 2.13 (The Behavior of the Singular Orbit)
Let λ be a parameter such that $\operatorname{Re}(\lambda) > 3$. Suppose there is an $n \geq 3$ with the property that the first n points of the singular orbit $(z_k)_{k\geq 1} := (0, \lambda, \dots) = (E_\lambda^{\circ(k-1)}(0))_{k\geq 1}$ are contained in $\{z \in \mathbb{C} : \ |\operatorname{Im}(z)| \leq |\operatorname{Re}(z)|\}$, and suppose z_n is the first orbit point with $\operatorname{Re}(z_n) < 0$. Then E_λ has an attracting periodic orbit of exact period n. The same is true for $n = 2$ if $\operatorname{Re}(\lambda) < -9$.

Proof. We start the proof by claiming that if $n \geq 2$ is defined as in the statement above, then

$$e|\lambda|\exp(\operatorname{Re}(z_n)) < (|z_n|+1)^{-2} . \tag{29}$$

We estimate for all $1 \leq k < n$

$$\begin{aligned} (|z_k|+1)^2 &\leq (\sqrt{2}\operatorname{Re}(z_k)+1)^2 < 3\exp(\operatorname{Re}(z_k)) < \\ &< |\lambda|\cdot|\exp(z_k)| = |z_{k+1}| < |z_{k+1}|+1 . \end{aligned} \tag{30}$$

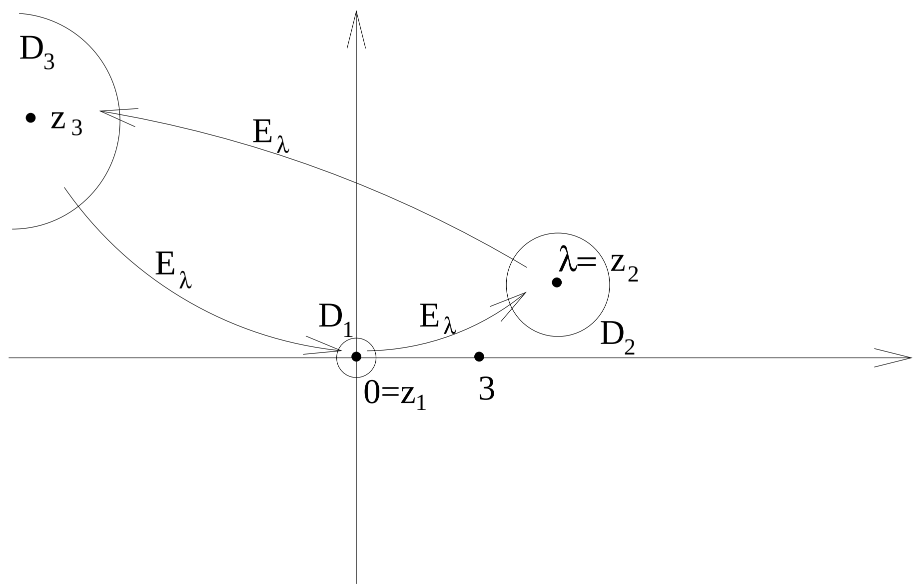

Figure 4: The setting of Proposition 2.13.

For the first inequality we used that $|z| \leq \sqrt{2}|\mathrm{Re}(z)|$ for all $z \in \{|\mathrm{Im}(z)| \leq |\mathrm{Re}(z)|\}$, and for the second inequality we used that $(\sqrt{2}x+1)^2 < 3e^x$ for every $x \geq 0$. In particular, we see that for all $1 \leq k < n$ we have $|z_k| < |z_{k+1}|$.

If $n = 2$, then $\mathrm{Re}(\lambda) < -9$. Since $\xi \mapsto e^\xi - e\sqrt{2}\xi(\sqrt{2}\xi + 1)^2$ is positive on the interval $[9, \infty]$, we thus get

$$e|\lambda|(|\lambda|+1)^2 \leq e\sqrt{2}|\mathrm{Re}(\lambda)|(\sqrt{2}|\mathrm{Re}(\lambda)|+1)^2 < \exp(|\mathrm{Re}(\lambda)|) \; .$$

This shows (29) for $n = 2$, using $|\lambda| = |z_n|$ and $|\mathrm{Re}(\lambda)| = -\mathrm{Re}(z_n)$.

Therefore assume $n \geq 3$, so that

$$|z_n| \geq |z_3| = |\lambda e^\lambda| > 3e^3 > 60 \quad \text{and thus} \quad |\mathrm{Re}(z_n)| > (\sqrt{2})^{-1} 60 > 40 \; .$$

Furthermore, we get

$$(|\mathrm{Re}(z_n)|+1)^2 < 2|\mathrm{Re}(z_n)|^2 \quad \text{and} \quad 4e(\mathrm{Re}(z_n))^3 < \exp(\mathrm{Re}(z_n)) \; ,$$

since the functions $\xi \mapsto 2\xi^2 - (\xi+1)^2$ and $\xi \mapsto 4ex^3 < e^x$ are positive on $[9, \infty]$. Putting all this together we arrive at

$$\begin{aligned} e|\lambda|(|z_n|+1)^2 &\leq 2e|\lambda||z_n|^2 \leq 2e \cdot 2|\lambda||\mathrm{Re}(z_n)|^2 < 4e|\mathrm{Re}(z_n)|^3 < \\ &< \exp(|\mathrm{Re}(z_n)|) \; . \end{aligned}$$

This shows estimate (29) for $n \geq 3$, using $|\mathrm{Re}(z_n)| = -\mathrm{Re}(z_n)$.

Define $D_n := B_1(z_n)$ and let D_{n-1} be the component of $E_\lambda^{-1}(D_n)$ containing z_{n-1}. For all $w_1 \neq w_2 \in D_{n-1}$ we deduce from the intermediate value theorem

$$\begin{aligned} \frac{|E_\lambda(w_1) - E_\lambda(w_2)|}{|w_1 - w_2|} &\in \left(\inf_{D_{n-1}} |E'_\lambda(z)|, \sup_{D_{n-1}} |E'_\lambda(z)|\right) = \left(\inf_{D_{n-1}} |E_\lambda(z)|, \sup_{D_{n-1}} |E_\lambda(z)|\right) = \\ &= \left(\inf_{D_n} |z|, \sup_{D_n} |z|\right) = \Big(|z_n| - 1, |z_n| + 1\Big) \ . \end{aligned} \tag{31}$$

This yields the inequalities

$$|w - z_{n-1}| \leq \frac{1}{|z_n| - 1} \quad \text{and} \quad |E_\lambda(w) - z_n| \leq \frac{|w - z_{n-1}|}{(|z_n| + 1)^{-1}} \ .$$

for all $w \in D_{n-1}$. It follows from the first equation that D_{n-1} does not contain 0 (so that we can define a logarithm on D_{n-1}), and from the second one that $B_{\rho_{n-1}}(z_{n-1}) \subset D_{n-1}$ for $\rho_{n-1} := (|z_n| + 1)$. (Check that $w \in B_{\rho_{n-1}}(z_{n-1})$ implies $E_\lambda(w) \in D_n$.)

By doing this step $n-1$ times we obtain open sets $D_1 \ni 0, D_2 \ni \lambda, \ldots, D_n$ with the property that every D_k contains the ball $B_{\rho_k}(z_k)$ of radius $\rho_k := \prod_{l=k+1}^{n}(|z_l| + 1)^{-1}$. (See Figure 4.) Note that in (31), we have to replace $(|z_n| - 1, |z_n| + 1)$ by $(|z_{k+1}| - 1, |z_{k+1}| + \rho_{k+1})$ in the k-th step.

In particular, we have an open set D_1 around 0 containing the disc $B_{\rho_1}(0)$ with $E_\lambda^{\circ(n-1)}(D_1) = D_n$. By induction, estimate (30) yields for all $1 \leq k < n$

$$\prod_{l=2}^{k}(|z_l| + 1) < (|z_k| + 1)^2 < |z_{k+1}| + 1 \ . \tag{32}$$

The induction seed $k = 1$ is immediate, and $\prod_{l=2}^{k+1}(|z_l|+1) < (|z_{k+1}|+1)(|z_k|+1)^2 < (|z_{k+1}|+1)^2$ shows the induction step. So we have $\rho_2 \geq (|z_n|+1)^{-2}$. But since every $w \in D_n$ satisfies $\mathrm{Re}(w) \leq \mathrm{Re}(z_n) + 1$, we get by formula (29) for every $z \in D_1$

$$\begin{aligned} |E_\lambda^{\circ n}(z)| &= \left|E_\lambda\left(E_\lambda^{\circ(n-1)}(z) \in D_n\right)\right| \leq |\lambda| \exp(\mathrm{Re}(z_n) + 1) = \\ &= e|\lambda| \exp(\mathrm{Re}(z_n)) < (|z_n| + 1)^{-2} \ . \end{aligned}$$

Therefore $E_\lambda^{\circ n}(D_1) \subset B_{\rho_1}(0) \subset D_1$, which yields an attractive cycle of exact period n. $\square$

3 Construction of Parameter Rays

3.1 Definition and Parameter Ray Tails

We want to turn our attention now to the parameter space by varying the parameter κ within the strip $\{z \in \mathbb{C} : |\mathrm{Im}(z)| \leq \pi\}$. For a better understanding of the bifurcation locus of the Exponential Family it would be nice to establish a structure like by the successful treatment of the dynamic plane in [SZ]. Surprisingly, there is indeed a lot of similarity between the discussion of the parameter space and the dynamic spaces.

We are interested in the *escaping parameters*, which are those parameters for which the singular value is an escaping point. To handle them we will construct rays again, called *parameter rays*, which distinguish the escaping parameters by the external address and the potential of the singular value 0.

The construction is closely related to the construction of dynamic rays. We first start the construction for large potentials, where it is comparably easy. Then we will extend this *parameter ray tail* on the full domain of potentials $(t_{\underline{s}}, \infty)$. Every escaping parameter for which 0 is on a ray (rather than being landing point of a ray) is contained in exactly one parameter ray. This will be proven in Theorem 3.9. We believe that actually every escaping parameter is either on a parameter ray or a landing point of some parameter ray. (Work in progress.)

Definition 3.1 (Parameter Rays)
Suppose that, for some parameter $\kappa \in \mathbb{C}$, the dynamic ray for external address $\underline{s} \in \mathcal{S}$ contains the singular value 0 at some potential $t > t_{\underline{s}}$:

$$g_{\kappa,\underline{s}}(t) = 0 \ . \tag{33}$$

Then we say, "The parameter κ is on the parameter ray *of external address $\underline{s}$ at potential t."*

Remark. At this point, the term 'parameter ray' is not yet justified as it is defined pointwise. But the notion of ray will make sense, as we will show that for every given exponentially bounded external address $\underline{s}$ and for every potential $t > t_{\underline{s}}$ there is exactly one parameter κ with $g_{\kappa,\underline{s}}(t) = 0$, and this parameter depends continuously on t.

In order to construct the parameter ray for a given external address $\underline{s}$, we begin with assigning a parameter to every sufficiently large potential. After this we show that the choice is unique and varies continuously with the potential, and that we can extend this ray tail continuously and uniquely to the full domain of potentials $t > t_{\underline{s}}$.

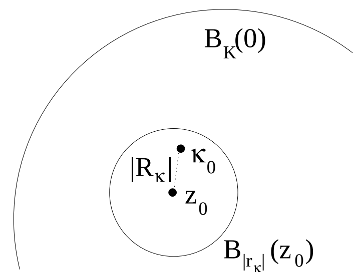

Figure 5: The setting in the Proof of Proposition 3.2.

Proposition 3.2 (Existence of Parameter Ray Tails)
Let $\underline{s} \in \mathcal{S}$ be exponentially bounded with growth parameters A and x. Then there is a constant $t'_{\underline{s}} := 18 + 2x + 2\log(4A) > t_{\underline{s}}$ and a unique map $G_{\underline{s}} : [t'_{\underline{s}}, \infty) \to \mathbb{C}$ such that for every $t \geq t'_{\underline{s}}$, the parameter $\kappa = G_{\underline{s}}(t)$ is on the parameter ray for external address $\underline{s}$ at potential t and such that $|G_{\underline{s}}(t)| < 2\pi t$ for all t. The parameter κ is a simple root of Equation (33), and the parameter ray tail carries the asymptotics

$$G_{\underline{s}}(t) = t + s_1 2\pi i + R_{\underline{s}}(t) \quad \textit{with} \quad |R_{\underline{s}}(t)| < 2e^{-t}(2\pi(t + |s_2| + AC) + 2) < 1 \ .$$

Proof. Consider an arbitrary fixed potential $t \geq t'_{\underline{s}}$ and define $K := 2\pi t$. We want to find a zero κ_0 of the map $\kappa \mapsto g_{\kappa,\underline{s}}(t)$ within the disk $B_K(0)$.

Since $t \geq t'_{\underline{s}} \geq 18 + 2\log 4 > 20$ implies $t/2 > 2\log(2\pi t + 3)$, we estimate

$$t = t/2 + t/2 > 2\log(2\pi t + 3) + x + \log(4A) = x + 2\log(K + 3) + \log(4A) \ .$$

Therefore t is on the dynamic ray tail of any $g_{\kappa,\underline{s}}$ with $|\kappa| < K$. More precisely, $t > t'_{K,\underline{s}} + \log(4A)$, and Theorem 2.2 thus provides for all $\kappa \in B_K(0)$:

$$g_{\kappa,\underline{s}}(t) = t - \kappa + 2\pi s_1 i + r_{\kappa,\underline{s}}(t) \quad \text{with } |r_{\kappa,\underline{s}}(t)| < 0.82 < 1 \ .$$

Now for given κ, define $z_0 := t + s_1 2\pi i$, so that $g_{\kappa,\underline{s}}(t) = z_0 - \kappa + r_{\kappa,\underline{s}}(t)$. Since $|r_{\kappa,\underline{s}}| < 0.82$, we have $g_{\kappa,\underline{s}}(t) \neq 0$ for $|z_0 - \kappa| \geq 0.82$. Within the disk $B_K(0)$, the only parameters κ with $g_{\kappa,\underline{s}}(t) = 0$ are thus contained in the disk $B_{0.82}(z_0)$.

Let $g : B_K(0) \to \mathbb{C}$ be defined by $g(\kappa) := g_{\kappa,\underline{s}}(t)$ and consider the curve $\tilde{\kappa}(\tau) := z_0 + e^{2\pi i \tau}$ with $\tau \in [0, 2\pi]$. Note that every κ in the range of $\tilde{\kappa}$ satisfies $|\kappa| \leq |t| + |s_1 2\pi| + 1 \leq t + \pi t + 1 < 2\pi t = K$. The curve $g \circ \tilde{\kappa}(\tau) = -e^{2\pi i \tau} + r_{\tilde{\kappa}(\tau),\underline{s}}(t)$ turns exactly once around the origin because it stays within the annulus $\{z : \ 0.18 < |z| < 1.82\}$. By Theorem 2.2, g is holomorphic in κ within any open set of parameters where $g_{\kappa,\underline{s}}(t)$ is defined. We thus obtain a holomorphic map $\tilde{g}(z) := g(z - z_0) : B_1(0) \to \mathbb{C}$ with the property that the winding number of 0 with respect to the curve $\tilde{g}(\mathbb{S}^1)$ is 1. By Rouché's Theorem, $\tilde{g}$ has exactly on root within $B_1(0)$, and there is exactly one κ_0 with $|\kappa_0| < K$ for which $g_{\kappa_0,\underline{s}}(t) = 0$, being a simple root of $\kappa \mapsto g_{\kappa,\underline{s}}(t)$.

It follows that $G_{\underline{s}}$ has the asymptotic form $t+2\pi s_1 i+R_{\underline{s}}(t)$, where $R_{\underline{s}}(t)$ satisfies the same bounds as $r_{K,\underline{s}}(t)$ in Theorem 2.2, substituting K by $2\pi t$. $\square$

Lemma 3.3 (A Bound on the Singular Orbit)
Let $s \in \mathcal{S}$ be exponentially bounded and let κ be a parameter such that $g_{\kappa,\underline{s}}(t) = 0$ for some potential $t > t_{\underline{s}}$. Choose the growth parameters x and A such that $t_{\underline{s}} < x < t$. Let $n \in \mathbb{N}$ be chosen large enough so that

$$F^{\circ n}(t) > t_n^\star := 2F^{\circ n}(x) + 2\log(2K + 3 + 9\pi A)\ .$$

Then the singular orbit $(z_k)_{k\in\mathbb{N}} = (E_\lambda^{\circ(k-1)}(0))_{k\in\mathbb{N}}$ satisfies for all $k \in \mathbb{N}$

$$|\mathrm{Im}(z_k)| \leq 2\pi(2^{n+2} + 1 + |s_k - s_1|)\ . \tag{34}$$

Proof. Using Lemma 2.12, we have $|\eta(g_{\kappa,\underline{s}}(t,\infty), 0)| \leq 2^{n+2}$. Therefore, the preimages of this ray provide a partition of the dynamic plane such that the region containing 0 is contained in $\{z \in \mathbb{C} : |\mathrm{Im}(z)| \leq (2^n + 1)2\pi\}$. Since the vertical distance of this strip to the one containing z_k is $2\pi|s_k - s_1|$ and dynamic rays cannot cross the boundary of the above partition, we get Formula (34). $\square$

Proposition 3.4 (A Bound on the Growth of Parameter Rays)
For every exponentially bounded external address $\underline{s} \in \mathcal{S}$ there is a continuous function $\xi_{\underline{s}} : (t_{\underline{s}}, \infty) \to \mathbb{R}$ such that for every κ and every $t > t_{\underline{s}}$

$$g_{\kappa,\underline{s}}(t) = 0 \qquad \Longrightarrow \qquad |\mathrm{Re}(\kappa)| \leq \xi_{\underline{s}}(t)\ .$$

Moreover, there is a constant $T_{\underline{s}} > t'_{\underline{s}}$ such that for $t > T_{\underline{s}}$, $\xi(t) := t + 1$ is a valid bound.

Proof. Suppose κ is a parameter with $|\mathrm{Im}(\kappa)| \leq \pi$ and $g_{\kappa,\underline{s}}(t) = 0$. Choose the growth parameters A, x of $\underline{s}$ such that $t_{\underline{s}} < x < t$. Clearly, $\mathrm{Re}(\kappa)$ is bounded below by -1, since parameters κ with $\mathrm{Re}(\kappa) < -1$ ($\lambda = e^\kappa$ is contained in the main cardioid) yield attractive dynamics. Thus we only have to find an upper bound for $\mathrm{Re}(\kappa)$.

Let $n \in \mathbb{N}$ be chosen big enough so that both $n \geq 3$ and

$$F^{\circ(n-1)}(t) > t_{n-1}^\star = 2F^{\circ(n-1)}(x) + 2\log(2|\kappa| + 3 + 9\pi A)\ . \tag{35}$$

Therefore we can apply Lemma 3.3 on $z_k := E_\lambda^{\circ(k-1)}(0)$ and we get for all $1 \leq k \leq n$ the estimate $|\mathrm{Im}(z_k)| \leq 2\pi(2^{n+2} + \max_{1\leq l\leq n}\{|s_l - s_1|\} + 1) \leq 2\pi + 2^{n+3}\pi + 2\pi A(F^{\circ(n-1)}(x) + x)$.

We claim that

$$h := \max\{e^t, 2\pi A(F^{\circ(n-1)}(x) + x) + 2^{n+3}\pi + 2\log(2|\kappa| + 3 + 9\pi A)\} \tag{36}$$

is an (implicit) upper bound for $|\mathrm{Re}(\lambda)| = |\mathrm{Re}(e^\kappa)|$. Note that $h \geq t^\star_{n-1}$ and $h \geq |\mathrm{Im}(z_k)|$ for all $1 \leq k \leq n$.

Suppose $|\mathrm{Re}(\lambda)| \geq h$. If $\mathrm{Re}(\lambda) \geq h$ then we either have $(z_k)_{1\leq k\leq n} = (0, \lambda, \lambda e^\lambda, \dots) \subset \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ or there will be an attractive orbit due to Proposition 2.13, since the points $z_1, \dots, z_n$ satisfy $|\mathrm{Im}(z_k)| \leq h \leq |\mathrm{Re}(z_k)|$. But there cannot be an attractive orbit for an escaping parameter such as κ. So the singular orbit must grow along the first n orbit elements: for all $k < n$

$$|\mathrm{Re}(z_{k+1})| \geq |z_{k+1}|/\sqrt{2} = |\lambda|\exp(\mathrm{Re}(z_k))/\sqrt{2} > 4\exp(\mathrm{Re}(z_k))$$

(using $|\lambda| \geq h \geq 2\pi > 4\sqrt{2}$). So in particular we have $\mathrm{Re}(z_n) > 4\exp^{\circ(n-2)}(h)$. On the other hand, the potential t_n of z_n satisfies $t_n = F^{\circ(n-1)}(t) < \exp^{\circ(n-2)}(e^t) < \exp^{\circ(n-2)}(h)$, and since z_n is on the ray tail, Theorem 2.2 provides the estimate

$$3 \leq 3\exp^{\circ(n-2)}(h) < |\mathrm{Re}(z_n) - t_n| \leq |\kappa| + 2e^{-t_n}(|\kappa| + 2 + 2\pi|s_n| + 2\pi AC) < 1\ .$$

This is a contradiction.

If $\mathrm{Re}(\lambda) \leq -h$, we get $\mathrm{Re}(\lambda) \leq -h \leq -2^6\pi < -9$ and we still have $|\mathrm{Im}(\lambda)| < h < |\mathrm{Re}(\lambda)|$. So we can apply Proposition 2.13 again to get a contradiction.

Hence

$$\begin{aligned}
\mathrm{Re}(\kappa) &= \log(|\lambda|) \leq \log(|\mathrm{Re}(\lambda)| + |\mathrm{Im}(\lambda)|) \leq \log(2h) = \\
&= \max\{t + \log 2, \log 4 + \log(\pi A(F^{\circ(n-1)}(x) + x)) + \log(2^{n+2}\pi) + \\
&\quad + \log\log(2|\kappa| + 3 + 9\pi A)\} \leq \\
&\overset{(1)}{\leq} \max\{t + 1, \log 4 + \log(\pi A) + \log(F^{\circ(n-1)}(x) + x) + \log\pi + \\
&\quad + (n+2)\log 2 + \log\log 2 + \log\log|\kappa| + \log\log(3 + 9\pi A)\} \leq \\
&\overset{(2)}{\leq} \max\{t + 1, 5 + 2\log(\pi A) + 2\log(F^{\circ(n-1)}(x) + x) + 2(n+2) + \\
&\quad + 2\log\log(3 + 9\pi A)\} =: \max\{t + 1, \tilde{h}'(n)\} =: \tilde{h}(t, n)\ .
\end{aligned}$$

In (1) and (2), the relation $\log(a + b) \leq \log a + \log b$ (true for every $a > 1$ and $b \geq 1 + 1/(a-1)$, for example $a, b \geq 2$) has been used several times. In the step marked by (2), we applied the estimations $\log 4 + \log\pi + \log\log 2 \leq 5/2$ and

$$\log\log(|\kappa|) \leq \log\log(\mathrm{Re}(\kappa) + \pi) \overset{(3)}{\leq} \mathrm{Re}(\kappa) - \log\log(\mathrm{Re}(\kappa) + \pi) \leq \alpha - \log\log(|\kappa|)\ ,$$

where the inequality (3) follows from the positivity of $\varphi(a) := a - 2\log\log(a+\pi)$ on $[1,\infty)$, and α is any upper bound for $\mathrm{Re}(\kappa)$. (In our case α is the second argument of the maximum in the line on which (2) has been applied.)

Let us first show the existence of a continuous function $\xi_{\underline{s}} : (t_{\underline{s}}, \infty) \to \mathbb{R}$ bounding $|\mathrm{Re}(\kappa)|$ for given $\underline{s}$. The problem with $\tilde{h}(t,n)$ is that we do not know anything about n, which can be arbitrary big as soon as the potential gets closer to $t_{\underline{s}}$. But for every $t_0 > t_{\underline{s}}$ there is an $N(t_0)$ which suffices as n for every κ with potential $t \geq t_0$: We either have $\mathrm{Re}(\kappa) < t+1$, or we find a minimal $n = n(\kappa)$ satisfying (35), which is equivalent to

$$2|\kappa| + 3 + 9\pi A < \exp\left(F^{\circ(n-1)}(t)/2 - F^{\circ(n-1)}(x)\right) . \tag{37}$$

Using $|\kappa| \leq \pi + \tilde{h}'(n(\kappa))$, we can sharpen this condition on $n(\kappa)$ for all $t \geq t_0$ to

$$\begin{aligned} 2|\kappa| + 3 + 9\pi A \leq 4\log\left(F^{\circ(n(\kappa)-1)}(x) + x\right) + 4n(\kappa) + \mathrm{const}(A,x) \overset{!}{<} \\ < \exp\left(F^{\circ(n(\kappa)-1)}(t_0)/2 - F^{\circ(n(\kappa)-1)}(x)\right) . \end{aligned}$$

Obviously, there is an $N(t_0) \in \mathbb{N}$ independent of $t \geq t_0$ and κ, such that the condition "!" is true for all $n \geq N(t_0)$. Therefore the desired function $\xi_{\underline{s}}$ exists, which can be chosen to be any continuous dominant of $\tilde{h}(t, N(t))$.

It is left to show that there is a $T_{\underline{s}}$ such that every parameter κ with potential $t \geq T_{\underline{s}}$ satisfies $\mathrm{Re}(\kappa) \leq t+1$. First of all, set

$$T'_{\underline{s}} := \tilde{h}'(3) + 7 \geq t'_{\underline{s}} = 18 + 2x + \log(4A) .$$

For every parameter κ there is a minimal $n \geq 3$ such that (37) holds. We have seen that this leads to the estimate $\mathrm{Re}(\kappa) < \tilde{h}(t,n)$. As pointed out above, there is an $N := N(T'_{\underline{s}})$ such that $n \leq N$.

If κ is a parameter for which $n = 3$ suffices, then the we obviously have $\mathrm{Re}(\kappa) < \tilde{h}(t,3) = t+1$, since $\tilde{h}'(3) < t+1$.

Now assume $n > 3$ and $\mathrm{Re}(\kappa) \in [t+1, \tilde{h}'(n)]$. We will show that then Formula (37) is also true for $n-1$ instead of n, which is a contradiction. We can choose $x \in (t_{\underline{s}}, t)$ satisfying $x \geq 1$, which implies $F^{\circ(n-1)}(x) > 4F^{\circ(n-2)}(x)$ for all $n > 3$. Since $t > F(x)$ and $|\kappa| \leq \tilde{h}'(n) + \pi$, we get

$$\begin{aligned} &F^{\circ(n-2)}(t)/2 - F^{\circ(n-2)}(x) > F^{\circ(n-1)}(x)/2 - F^{\circ(n-2)}(x) > F^{\circ(n-2)}(x) \geq \\ &\geq \log F^{\circ(n-1)}(x) > \log F^{\circ(n-1)}(x) + (|\kappa| - \pi - \tilde{h}'(n))/2 = \\ &= |\kappa|/2 - \pi/2 - 5/2 - \log(\pi A) - \log x - (n+2) - \log\log(3 + 9\pi A) \geq \\ &\geq |\kappa|/2 - N + \mathrm{const}(A,x) . \end{aligned}$$

By possible enlarging $T'_{\underline{s}}$ to some $T_{\underline{s}} \geq T'_{\underline{s}}$ we may assume that for every $\alpha \geq T_{\underline{s}}$ we have $\alpha/2 - N + \mathrm{const}(A,x) \geq \log(2\alpha + 3 + 9\pi A)$. For all $t > T_{\underline{s}}$ we get $|\kappa| \geq t+1 > T_{\underline{s}}$, and thus

$$F^{\circ(n-2)}(t)/2 - F^{\circ(n-2)}(x) > \log(2|\kappa| + 3 + 9\pi A) \ .$$

Hence (37) holds for $n-1$ as well. $\square$

Corollary 3.5 (Parameter Ray Tails Are Unique)
Let $\underline{s} \in \mathcal{S}$ be an exponentially bounded sequence. Then for every $t > T_{\underline{s}}$ there is one and only one parameter κ on the parameter ray at address $\underline{s}$ and potential t, where $T_{\underline{s}} > t'_{\underline{s}}$ is the constant from Proposition 3.4. The multiplicity of κ as a root of $\kappa \mapsto g_{\kappa,\underline{s}}(t)$ is 1, and thus $(\partial/\partial\kappa) g_{\kappa,\underline{s}}(t) \neq 0$.

Proof. On the one hand, Proposition 3.2 provides existence and uniqueness under the condition that $|\kappa| < 2\pi t$. On the other hand, the bound in Proposition 3.4 gives $|\kappa| \leq |\mathrm{Re}(\kappa)| + |\mathrm{Im}(\kappa)| < t + 1 + \pi < 2\pi t$. $\square$

3.2 Parameter Rays at Their Full Length

Lemma 3.6 (The Domain of Definition of $\kappa \mapsto g_{\kappa,\underline{s}}(t)$)
Fix an exponentially bounded sequence $\underline{s} \in \mathcal{S}$.

- *For every open ball $B := B_\varepsilon(\kappa_0)$ of parameters (with $\kappa_0 \in \mathbb{C}$ and $\varepsilon > 0$ arbitrary) and every compact interval $I \subset (t_{\underline{s}}, \infty)$ of potentials there is an $N \in \mathbb{N}$ such that $g_{\kappa,\sigma^n \underline{s}}(F^{\circ n}(t))$ is defined for all $n \geq N$, $\kappa \in B$, and $t \in I$;*
- *For each potential $t_0 > t_{\underline{s}}$, the set $D_{t_0,\underline{s}} := \{\kappa \in \mathbb{C} : \ g_{\kappa,\underline{s}}(t_0) \text{ is defined}\} \subset \mathbb{C}$ is open;*
- *For every $\kappa_0 \in D_{t_0,\underline{s}}$ there are neighborhoods $I \subset \mathbb{R}$ and $\Lambda \subset \mathbb{C}$ of t_0 and κ_0 respectively such that $g_{\kappa,\underline{s}}(t)$ is defined for all $t \in I$ and $\kappa \in \Lambda$.*

Proof. Recall that if $t > t_{\underline{s}}$, the only possible reason for $g_{\kappa,\underline{s}}(t)$ not to be defined is the existence of an $n \geq 1$ such that $g_{\kappa,\sigma^n\underline{s}}(t_0) = 0$ with $t_0 \geq F^{\circ n}(t)$ (following Theorem 2.2).

For the first claim, let $K := |\kappa_0| + \varepsilon$, so that $|\kappa| < K$ for all $\kappa \in B$. Define the growth parameters x and A of $\underline{s}$ such that $t_{\underline{s}} < x < \inf I$. Now just take N big enough so that both $F^{\circ N}(t) > F^{\circ N}(x) + 2\log(K+3) + \log(4A)$ and $F^{\circ N}(t) > K + 1$

for all $t \in I$. This implies by Theorem 2.2 that for all $\kappa \in B$, $F^{\circ N}(t)$ is on the dynamic ray tail of $g_{\kappa,\sigma^N \underline{s}}$ with $|r_{\kappa,\sigma^N}(F^{\circ N}(t))| < 1$, so that

$$\begin{aligned} \operatorname{Re}\left(g_{\kappa,\sigma^N \underline{s}}(F^{\circ N}(t'))\right) &\geq F^{\circ N}(t') - K - |r_{K,\sigma^N \underline{s}}(F^{\circ N}(t'))| > \\ &> F^{\circ N}(t) - K - 1 > 0 \end{aligned} \tag{38}$$

for all $t' \geq t$. This shows the first statement.

The third claim implies the second claim, so that it is only left to show the third item. Assume that (t_0, κ_0) is a pair of a potential $t_0 > t_{\underline{s}}$ and a parameter $\kappa_0 \in \mathbb{C}$ such that $\kappa_0 \in D_{t_0,\underline{s}}$. If the statement was wrong then there would be sequences $(t_n)_{n\geq 1}$ and $(\kappa_n)_{n\geq 1}$ with $|t_n - t_0| \leq 1/n$ and $|\kappa_n - \kappa_0| \leq 1/n$ for all $n \geq 1$, such that

$$\forall n \geq 1 \;\; \exists N_n \geq 1 \;,\; \exists t'_n \geq F^{\circ N_n}(t_n) : \quad g_{\kappa_n,\sigma^{N_n}\underline{s}}(t'_n) = 0 \;.$$

By the first step above, we may assume by passing to a subsequence that all the N_n are equal to some N_0. Furthermore, the sequence $(t'_n)_{n\geq 1}$ is contained in some compact interval $[F^{\circ N_0}(t_*), t^*]$, where $t_* = \inf_n t'_n$ and t^* is some potential on the ray tail, beyond which we have good control, compare (38). So by passing to a subsequence once more we may assume that $(t'_n)_n$ converges to some $t'_0 \geq F^{\circ N_0}(t_0)$. Since the map $(t,\kappa) \mapsto g_{\kappa,\sigma^{N_0}\underline{s}}(t)$ is sequential continuous wherever it is defined, it follows from $g_{\kappa_n,\sigma^{N_0}\underline{s}}(t'_n) = 0$ for all $n \geq 1$ that

$$\lim_{n\to\infty} g_{\kappa_n,\sigma^{N_0}\underline{s}}(t'_n) = 0 \;, \quad \text{and so} \quad g_{\kappa_0,\sigma^{N_0}\underline{s}}(t'_0) = 0 \;.$$

(Note that we do not leave the set of pairs (t,κ) for which $g_{\kappa,\sigma^{N_0}\underline{s}}(t)$ is defined.) This contradicts the assumption $\kappa_0 \in D_{t_0,\underline{s}}$. $\square$

The following Lemma will be needed several times in the following discussion.

Lemma 3.7 (Discreteness of Zeros of Dynamic Rays)
Let $\underline{s} \in \mathcal{S}$ be exponentially bounded and $t > t_{\underline{s}}$ be any potential.

- *The set* $Z_{t,\underline{s}} := \{\kappa : g_{\kappa,\underline{s}}(t) \text{ is defined and } g_{\kappa,\underline{s}}(t) = 0\}$ *is discrete in* $\mathbb{C}$.
- *For every $\kappa \in Z_{t,\underline{s}}$ there are neighborhoods $\Lambda \subset \mathbb{C}$ and $I \subset \mathbb{R}$ containing κ and t respectively, such that for every $t' \in I$, the number of elements of $Z_{t',\underline{s}} \cap \Lambda$ (counting multiplicities) equals the finite multiplicity of κ as a root of the map $\kappa \mapsto g_{\kappa,\underline{s}}(t)$.*

PROOF. Assume that $Z_{t,\underline{s}}$ is not discrete. By Lemma 3.6 there is an open ball B of parameters κ containing a limit point of $Z_{t,\underline{s}}$ on which $g_{\kappa,\underline{s}}(t)$ is holomorphic. Due to the identity theorem we have $B \subset Z_{t,\underline{s}}$. In [Ye] (Theorem 3 and Corollary 4), Zhuan Ye has shown that if κ is an escaping parameter then E_κ is not *J-stable*. But on B we would have J-stability: for every $\kappa \in B$ the Julia set equals $\mathbb{C}$. (This can for example also be found in [Ye], Theorem A.) This is a contradiction.

For the second claim, consider a small loop γ around κ, bounding an open neighborhood $\Lambda \subset D_{t,\underline{s}}(\kappa)$ of κ, such that κ is the only root of the map $\kappa' \mapsto g_{\kappa',\underline{s}}(t)$ within Λ. This is possible since $D_{t,\underline{s}}$ is open and $Z_{t,\underline{s}}$ is discrete. The multiplicity of κ as a zero equals $|\eta(g_{\gamma,\underline{s}}(t), 0)|$. (The winding number η has been defined in Formula (28) of section 2.3.) But by Rouché's Theorem and continuity of $(\kappa, t) \mapsto g_{\kappa,\underline{s}}(t)$, we have $|\eta(g_{\gamma,\underline{s}}(t'), 0)| = |\eta(g_{\gamma,\underline{s}}(t), 0)|$ for potentials t' sufficiently close to t. This completes the proof. $\square$

Theorem 3.8 (Continuous Local Extension of Parameter Rays)
Suppose κ_0 is on a parameter ray for some external address $\underline{s} \in \mathcal{S}$ at potential $t_0 > t_{\underline{s}}$. Then there is an open interval I containing t_0 and a map $G_{\underline{s}} : I \to \mathbb{C}$ with $G_{\underline{s}}(t_0) = \kappa_0$ such that for all $t \in I$, the parameter $G_{\underline{s}}(t)$ is on the parameter ray for external address $\underline{s}$ at potential t. The map $G_{\underline{s}}$ may be chosen to be continuous at t_0.

PROOF. At least we know by Lemma 3.6 that there are neighborhoods I and Λ of t_0 and κ_0 respectively on which $g_{\kappa,\underline{s}}(t)$ is defined for all $t \in I$ and $\kappa \in \Lambda$. By Lemma 3.7 it follows that if we choose Λ and I sufficiently small around κ_0 and t_0, then for every $t \in I$ there is a parameter $\kappa \in \Lambda$ on the parameter ray for $\underline{s}$ at potential t. This defines a map $G_{\underline{s}}(t) := \kappa$, possibly involving a choice. This map $G_{\underline{s}}$ is continuous at t_0, since we can find such an interval I as above for arbitrary small neighborhoods Λ around κ_0. $\square$

We are now ready to state and to prove the main result of this paper.

Theorem 3.9 (Parameter Rays at Their Full Length)
For every exponentially bounded sequence $\underline{s} \in \mathcal{S}$ there is a unique curve $G_{\underline{s}} : (t_{\underline{s}}, \infty) \to \mathbb{C}$ such that every parameter $\kappa = G_{\underline{s}}(t)$ is an escaping parameter with external address $\underline{s}$ and potential t. Conversely, every parameter κ for which there are $\underline{s} \in \mathcal{S}$ and $t > t_{\underline{s}}$ such that $g_{\kappa,\underline{s}}(t) = 0$ satisfies $G_{\underline{s}}(t) = \kappa$. The curve $G_{\underline{s}}$ thus realizes the notion of parameter ray from definition 3.1 uniquely.

PROOF. Corollary 3.5 provides for large potentials $t > T_{\underline{s}}$ a unique choice for $G_{\underline{s}}(t)$, which depends continuously on t. Let

$$I := \{t_* > t_{\underline{s}} : \ G_{\underline{s}} \text{ exists for all } t > t_*\} \ni T_{\underline{s}}$$

be the maximal interval of potentials onto which the parameter ray tail can be extended. The set I is non-empty, and it follows from Theorem 3.8 that I is open. We will now show that I is also closed in $(t_{\underline{s}}, \infty)$. Let $t_* := \inf I$ and suppose $t_* > t_{\underline{s}}$. Then by Proposition 3.4 and $|\mathrm{Im}(\kappa)| \leq \pi$, the set $\{G_{\underline{s}}(t) : \ t \in (t_*, t_*+1)\}$ is contained in a compact set. Thus the set L of all limits $\lim_{t \searrow t_*} G_{\underline{s}}(t)$ is a nonempty compact subset of $\mathbb{C}$.

We first have to be sure that for every $\kappa_0 \in L$ the dynamic ray $g_{\kappa_0,\underline{s}}$ is defined for the potential t_*. Otherwise there would be an $n \geq 1$ such that the potential t of 0 (i.e. $g_{\kappa_0,\underline{s}}(t) = 0$) satisfies $t \geq F^{\circ n}(t_*)$. By Formula (5) of Theorem 2.2, we know for large k that $\mathrm{Re}(E_{\kappa_0}^{\circ k}(0)) \geq F^{\circ(k+n)}(t) - \mathrm{Re}(\kappa_0) - 1$. But $\mathrm{Re}(E_\kappa^{\circ k}(0)) = F^{\circ k}(t) - \mathrm{Re}(\kappa) + O(1)$ for $\kappa = G_{\underline{s}}(t)$ and $t > t_*$, which contradicts the continuity of the map $\kappa \mapsto E_\kappa^{\circ k}(0)$.

We want to prove now that every $\kappa_0 \in L$ satisfies $g_{\kappa_0,\underline{s}}(t_*) = 0$. But if this was wrong, then by continuity of the map $(\kappa, t) \mapsto g_{\kappa,\underline{s}}(t)$ there were nearby potentials $t > t_*$ with $g_{G_{\underline{s}}(t),\underline{s}}(t) \neq 0$, which is a contradiction. It follows that I is closed in $(t_{\underline{s}}, \infty)$, so $I = (t_{\underline{s}}, \infty)$. Hence $G_{\underline{s}}(t)$ can be defined for all potentials greater than the the minimal potential.

Roots κ of the equation $g_{\kappa,\underline{s}}(t) = 0$ are isolated by Lemma 3.7, so L is a non-empty finite set. We know that for sufficiently large potentials, there is a unique $G_{\underline{s}}(t)$. And if there was a potential $t_0 > t_{\underline{s}}$ with more than one choice for $G_{\underline{s}}(t_0)$, then we could perturb all these branches for potentials t sufficiently close to t_0 and run a similar proof as above, but for increasing instead of decreasing potentials: let t^* be the finite supremum of all potentials for which there is more than one choice for $G_{\underline{s}}(t)$. Then for the maximal potential itself, $G_{\underline{s}}(t^*)$ must be unique by Theorem 3.8 (or we could further extend the different choices). Therefore, several branches of $G_{\underline{s}}$ for $t_{\underline{s}} < t < t^*$ must converge to the same point $G_{\underline{s}}(t^*)$. But this is ruled out by Lemma 3.7 again: $\kappa^* := G_{\underline{s}}(t^*)$ must be a simple root of $\kappa \mapsto g_{\kappa,\underline{s}}(t^*)$, since $G_{\underline{s}}(t)$ is unique for all $t > t^*$. This yields uniqueness for all t in a neighborhood of t^*, which is a contradiction to the choice of t^* as the supremum of potentials without uniqueness.

It follows that $G_{\underline{s}}$ is defined and unique for all potentials greater than the minimal potential, and continuity is now guaranteed by Theorem 3.8. Finally, injectivity of $G_{\underline{s}}$ and disjointness of the rays for different combinatorics follow from the fact that the singular value can be at most on one dynamic ray, having at most one potential. $\square$

3.3 Derivative of Parameter Rays

After the Lemmas concerning the domain of definition of the maps $\kappa \mapsto g_{\kappa,\underline{s}}(t)$, we are ready to discuss analogously to the results in Chapter 2.2 the derivative of dynamic rays with respect to the parameter. At first glance, the following Proposition may look immediate, since we have the asymptotic form $g_{\kappa,\underline{s}}(t) = t - \kappa + 2\pi i s_1 + O(e^{-t})$. However, there is no direct way to handle $(\partial/\partial\kappa)O(e^{-t})$.

Proposition 3.10 (The Derivative of Dynamic Rays w.r.t. the Parameter)

Let $\underline{s} \in \mathcal{S}$ be an exponentially bounded sequence, and $\kappa_0 \in \mathbb{C}$ be a parameter. Then there is a neighborhood $U \subset \mathbb{C}$ of κ_0 such that for all $t > t_{\underline{s}}$, $\kappa \mapsto (\partial/\partial\kappa)g_{\kappa,\underline{s}}(t) : U \to \mathbb{C}$ is a holomorphic function, satisfying the asymptotics

$$\lim_{t\to\infty} \frac{\partial}{\partial\kappa} g_{\kappa,\underline{s}}(t)\bigg|_{\kappa=\kappa_0} = -1 \ .$$

Furthermore, for every $t > t_{\underline{s}}$

$$\lim_{n\to\infty} \frac{\partial}{\partial\kappa} E_\kappa^{\circ n}(g_{\kappa,\underline{s}}(t))\bigg|_{\kappa=\kappa_0} = -1 \ .$$

Proof. We know by Lemma 3.6 that there is a neighborhood U of κ_0, say contained in the ball $B_K(0)$ of some radius $K > 0$, such that $g_{\kappa,\underline{s}}(t)$ is defined for all $\kappa \in U$. Recall from Lemma 2.3 that the ray ends $g_{\kappa,\underline{s}}(2\log(K+3), \infty)$ are the uniform limits of the holomorphic functions $g^n_{\kappa,\underline{s}} : (2\log(K+3), \infty) \to \mathbb{C}$ defined by

$$g^n_{\kappa,\underline{s}}(t) := L_{\kappa,s_1} \circ \cdots \circ L_{\kappa,s_n}(F^{\circ n}(t))$$

with $L_{\kappa,m}(z) := \mathrm{Log}(z) - \kappa + 2\pi i m$. As already pointed out in [SZ] (Proposition 3.4), the Weierstraß Theorem asserts that $\kappa \mapsto (\partial/\partial\kappa)g_{\kappa,\underline{s}}(t)$ is holomorphic and satisfies

$$\frac{\partial}{\partial\kappa} g_{\kappa,\underline{s}}(t) = \lim_{n\to\infty} \frac{\partial}{\partial\kappa} g^n_{\kappa,\underline{s}}(t)$$

if the limit exists. Assume $t > \max\{K+2, 2\log(K+3)\}$. We calculate $(\partial/\partial\kappa)g^n_{\kappa,\underline{s}}(t)$ for arbitrary $n > 0$ as follows. Define for all $n, m \geq 1$ the numbers $a_{n,m} \in \mathbb{C}$ and $a_m \in \mathbb{C}$ by

$$a_{n,m} = a_{n,m}(t,\underline{s}) := g^{n-m+1}_{\kappa,\sigma^m\underline{s}}(F^{\circ m}(t)) \ , \quad a_m := \lim_{n\to\infty} a_{n,m} = g_{\kappa,\sigma^m\underline{s}}(F^{\circ m}(t)) \ .$$

By Formula (6) in Lemma 2.3 there is a $T_K = \max\{K+3, 2\log(K+3)\}$ such that $\mathrm{Re}(g^n_{\kappa,\tilde{\underline{s}}}(t)) > 1$ for all $t > T_K$ and all $\tilde{\underline{s}} \in \mathcal{S}$. Therefore all the $a_{n,m}$ and a_n are non-zero if we restrict to potentials $t > T_K$. We show by induction that for all $n \in \mathbb{N}$

and $t > T_K$

$$\frac{\partial}{\partial \kappa} g^{n+1}_{\kappa,\underline{s}}(t) = -1 + \sum_{m=1}^{n} \prod_{l=1}^{m} \frac{1}{a_{n,l}} \ . \tag{39}$$

The induction seed follows from $(\partial/\partial\kappa) g^1_{\kappa,\underline{s}}(t) = (\partial/\partial\kappa)(\mathrm{Log}(F(t)) - \kappa + 2\pi i s_1) = -1$. Assume that the formula is right for some $n-1 \in \mathbb{N}$. Then

$$\begin{aligned}
\frac{\partial}{\partial \kappa} g^{n+1}_{\kappa,\underline{s}}(t) &= \frac{\partial}{\partial \kappa}\left(\mathrm{Log}\ g^n_{\kappa,\sigma\underline{s}}(F(t)) - \kappa + 2\pi i s_1\right) = -1 + \frac{(\partial/\partial\kappa) g^n_{\kappa,\sigma\underline{s}}(F(t))}{g^n_{\kappa,\sigma\underline{s}}(F(t))} = \\
&= -1 + \frac{1}{a_{n,1}(t,\underline{s})}\left(-1 + \sum_{m=1}^{n-1}\prod_{l=1}^{m} \frac{1}{a_{n-1,l}(F(t),\sigma\underline{s})}\right) = \\
&\overset{(*)}{=} -1 - \frac{1}{a_{n,1}} + \sum_{m=1}^{n-1}\prod_{l=1}^{m+1} \frac{1}{a_{n,l}(t,\underline{s})} = -1 + \sum_{m=1}^{n}\prod_{l=1}^{m} \frac{1}{a_{n,l}} \ ,
\end{aligned}$$

where (*) follows from the relationship $a_{n-1,l}(F(t),\sigma\underline{s}) = a_{n,l+1}(t,\underline{s})$.

We claim that for all $t > T_K$

$$\frac{\partial}{\partial \kappa} g_{\kappa,\underline{s}}(t) = -1 + \sum_{m=1}^{\infty} \prod_{l=1}^{m} \frac{1}{a_l} \ . \tag{40}$$

Firstly, this sum exists by the ratio test, since $|\prod_{l=1}^{m+1} \frac{1}{a_l} / \prod_{l=1}^{m} \frac{1}{a_l}| = |1/a_{m+1}| < 1/2 < 1$ for large m. It is left to show that the expression in (39) converges to the expression in (40). But

$$\begin{aligned}
&\left|\left(-1 + \sum_{m=1}^{\infty}\prod_{l=1}^{m}\frac{1}{a_l}\right) - \left(-1 + \sum_{m=1}^{n}\prod_{l=1}^{m}\frac{1}{a_{n,l}}\right)\right| \leq \\
&\leq \sum_{m=n+1}^{\infty}\prod_{l=1}^{m}\frac{1}{|a_l|} + \sum_{m=1}^{n}\left|\prod_{l=1}^{m}\frac{1}{a_l} - \prod_{l=1}^{m}\frac{1}{a_{n,l}}\right| \longrightarrow 0 + 0 = 0 \ :
\end{aligned}$$

This is clear for the first limit, and for the second one it is by

$$\sum_{m=1}^{n}\left|\prod_{l=1}^{m}\frac{1}{a_l} - \prod_{l=1}^{m}\frac{1}{a_{n,l}}\right| \leq n \cdot \left|\frac{1}{a_1} - \frac{1}{a_{n,1}}\right| \leq n \cdot \frac{|a_{n,1} - a_1|}{|a_1||a_{n,1}|}$$

enough to show that $a_{n,1} \to a_1$ exponentially fast. But this follows from Formula (8) in Lemma 2.3 and the triangle inequality:

$$\left|g^n_{\kappa,\underline{s}}(t) - g_{\kappa,\underline{s}}(t)\right| \leq \sum_{m=n+1}^{\infty}\left(4\pi A e^{-t}\prod_{l=1}^{m-2} e^{-l}\right) \leq 4\pi A e^{-t} e^{-(n-1)} \ .$$

This shows Formula (40). Since

$$\left|\sum_{m=1}^{\infty}\prod_{l=1}^{m}\frac{1}{a_l(t,\underline{s})}\right| \le \sum_{m=1}^{\infty} a_1(t,\underline{s})^{-m} = \frac{a_1(t,\underline{s})}{1-a_1(t,\underline{s})} \stackrel{t\to\infty}{\longrightarrow} 0 \ ,$$

it follows that

$$\lim_{t\to\infty} \frac{\partial}{\partial\kappa} g_{\kappa,\underline{s}}(t)\bigg|_{\kappa=\kappa_0} = \lim_{t\to\infty}\left(-1+\sum_{m=1}^{\infty}\prod_{l=1}^{m}\frac{1}{a_l(t,\underline{s})}\right) = -1 \ .$$

The second statement follows from the functional equation

$$E_\kappa^{\circ n}(g_{\kappa,\underline{s}}(t)) = g_{\kappa,\sigma^n\underline{s}}(F^{\circ n}(t))$$

and the fact that the rate of convergence of $|a_m(t,\underline{s})| \to \infty$ has been estimated above independently of $\underline{s}$. □

Corollary 3.11 (Parameter Rays are Differentiable)
For every exponentially bounded external address $\underline{s} \in \mathcal{S}$*, the parameter ray* $G_{\underline{s}} : (t_{\underline{s}}, \infty) \to \mathbb{C}^*$ *is a continuously differentiable curve with*

$$G'_{\underline{s}}(t) = -\frac{(\partial/\partial t)\ g_{\kappa,\underline{s}}(t)}{(\partial/\partial\kappa)\ g_{\kappa,\underline{s}}(t)} \ ,$$

satisfying the asymptotics $\lim_{t\to\infty}\mathrm{Re}(G_{\underline{s}}(t)) = \infty$ *and* $\lim_{t\to\infty} G'_{\underline{s}}(t) = 1$.

Proof. By the implicit function theorem, all we need to show is that $(\partial/\partial\kappa)g_{\kappa,\underline{s}}(t) \neq 0$ for all (κ, t) for which $g_{\kappa,\underline{s}}(t) = 0$. If $(\partial/\partial\kappa)g_{\kappa,\underline{s}}(t) \neq 0$ would be true for some $t > t_{\underline{s}}$, then $g_{\kappa,\underline{s}}(t)$ would have a multiple root κ_0. But this contradicts Theorem 3.9.

The first limit follows from Proposition 3.2, the second one follows from

$$\frac{\partial}{\partial t} g_{\kappa,\underline{s}}(t) \longrightarrow 1 \quad \text{and} \quad \frac{\partial}{\partial\kappa} g_{\kappa,\underline{s}}(t) \longrightarrow -1$$

by Theorem 2.2 and Lemma 3.10. □

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