A Sampling of Vector Bundle Techniques in the Study of Linear Series

by

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Introduction

The study of linear series on curves, surfaces and other algebraic varieties has always been central to algebraic geometry. Broadly speaking, the goal is to understand how a variety X can map to projective space. To begin with, one can ask for the dimensions and numerical invariants of linear series on X; this is the question addressed by the Riemann-Roch theorem, and when X is a curve it is the subject of Brill-Noether theory. Or again, it is natural to study the equations defining a given projective embedding of X, typical results here being the theorems of Noether and Enriques-Babbage-Petri concerning canonical curves. And when X has dimension two or more, it is already important simply to determine whether or not a particular linear system is base-point free or very ample: Lefschetz's theorems on abelian varieties, and Bombieri's results on pluri-canonical mappings of surfaces, are cases in point.

Interest in classical questions of this sort continues to the present day, but in recent years work in this area has started to take a new turn. Namely, vector bundles of ranks two and higher have emerged as important tools in the study of linear series. Geometrical data are encoded into a vector bundle living on the variety under study, and the desired conclusions are obtained by analyzing the cohomology or sub-bundles or moduli of the bundle so constructed. At the moment these methods constitute a point of view more than a systematic theory, but they have already proved quite powerful. The purpose of these notes is to survey some of these vector bundle techniques, and the results to which they lead.

The present exposition focuses on four groups of theorems. In §1 we discuss some work of Green [Gr] concerning the syzygies of curves of large degree. The idea here is that classical results on the equations defining algebraic curves can be generalized in a very natural way to higher syzygies. Thus let L be a very ample line bundle on a curve X, defining an embedding

$$X \subset \mathbb{P} = \mathbb{P}(H^0(L)).$$

It is well known that if $\deg(L) \geq 2g+2$, then the homogeneous ideal $I_{X/\mathbb{P}}$ of X is generated by quadrics. Green shows that if $\deg(L) \geq 2g+3$, then the module of syzygies among quadratic generators $Q_i \in I_{X/\mathbb{P}}$ is spanned by relations of the form

$$\Sigma\, L_i \cdot Q_i = 0$$

where the L_i are linear polynomials; if $\deg(L) \geq 2g+4$, then the relations among the L_i again have linear coefficients; and so on. The syzygies of a curve X are governed cohomologically by a vector bundle M_L on X canonically associated to the line bundle L. Under the hypotheses of the theorem one can control the required cohomology groups by constructing filtrations of M_L. These bundles carry a great deal of information about the pair (X, L), and at least conjecturally there is a very close connection between the geometry and the syzygies of X.

Section 2 is devoted to a discussion of some results from [GL1] which extend and unify a number of theorems concerning the normal generation of line bundles on a curve X. Specifically, we show that if L is a very ample line bundle on X, with

$$\deg(L) \geq 2g+1-2\cdot h^1(L) - \mathrm{Cliff}(X),$$

then L defines a projectively normal embedding of X. The <u>Clifford index</u> $\mathrm{Cliff}(X)$ appearing here is an integer which roughly speaking measures how general X is in the sense of moduli. In the relevant range of degrees, the failure of projective normality translates into the existence of rank two vector bundles E on X, with $\det E = \omega_X$, having an unusually large number of sections. By studying sub-bundles of E, one finds that the presence of such bundles has implications for the intrinsic geometry of X.

In §3 we explain results of [L1], the theme being that curves generating the Picard group of a K3 surface X with $\mathrm{Pic}(X) = \mathbb{Z}$ behave generically from the point of view of Brill-Noether theory. The idea here is very simple: given a curve $C \subset X$, and a line bundle A on C, under mild hypotheses one can associate to the pair (C, A) a vector bundle $F_{C,A}$ on X. Furthermore, this bundle has only trivial endomorphisms provided that the linear series $|C|$ does not contain any reducible curves. This leads to a connection between the Brill-Noether theory of the curves on X and the geometry of the moduli spaces of simple bundles on X. In particular, we will see that the Gieseker-Petri theorem on special divisors follows from a result of Mukai to the effect that these moduli spaces are always smooth. We attempt to emphasize the geometrical ideas behind the results, and we have contented ourselves at crucial points in §3 with plausibility arguments, referring to [L1] for careful proofs.

Finally, §4 is devoted to an exposition of some of Reider's work [R], in which rank two bundles lead to very quick proofs of some of the basic theorems concerning linear series on surfaces. Given a line bundle L on a surface X, one wishes to understand when $K_X \otimes L$ is base-point free or very ample. The failure of either of these properties translates via Grothendieck duality and a well-known construction of Serre's into the existence of a certain rank two vector bundle on X. Under suitable hypotheses on L, Bogomolov's theorem shows that these bundles are unstable, and Reider's results are obtained by analyzing geometrically what this instability means. We have also reproduced in §4 Miyaoka's simple proof [Myk] of Bogomolov's theorem (although we use a somewhat stronger form of this theorem than that which we prove.)

As the title of these notes suggests, there are several other results that might well have been surveyed here, but which had to be omitted due to limitations of space and time. For instance, vector bundles have been applied in several ways to prove Castelnuovo-type bounds on the regularity of incomplete linear series (c.f. [GP1], [GLP], [P], [L2]). Or again, Mukai [Muk3] has obtained some very beautiful and precise results concerning the structure of K3 surfaces and Fano varieties by studying the maps to Grassmannians determined by suitable bundles. In spite of these omissions, we hope that these notes will convey something of the flavor and scope of vector bundle techniques.

At the risk of occasionally sacrificing brevity, I have tried to keep the exposition fairly elementary and self contained, and the experts won't find anything in the way of new results here. At the same time, there are a number of open problems -- some well known -- related to this circle of ideas. These occur throughout the manuscript, but most notably in §1.5 and §2.4. The notes at the end of each section summarize work related to the material discussed.

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§0. Notation and Conventions.

(0.1). We work exclusively with varieties defined over the complex numbers $\mathbb{C}$.

(0.2). If V is a complex vector space, we denote by $\mathbb{P}(V)$ the projective space of one-dimensional quotients of V, so that $H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1)) = V$. We follow the analogous convention for the projective bundle $\mathbb{P}(E)$ associated to a vector bundle E.

(0.3). If F is a coherent sheaf on a projective variety X, we write $H^i(F)$ in place of $H^i(X, F)$ if no confusion seems likely to result. As usual, $h^i(F) = \dim H^i(F)$. If V is a complex vector space, $V \otimes_{\mathbb{C}} \mathcal{O}_X$ is the trivial vector bundle on X with fibres V; we sometimes abbreviate this simply by $V \otimes \mathcal{O}_X$.

§1. Syzygies of Curves of Large Degree.

In this section we will discuss a theorem from [Gr] concerning the syzygies of a curve embedded in projective space by a very ample linear system of large degree. Our focus will be on the geometric background of the results, and the role that vector bundles play in their proof. We refer to Mark Green's lectures at this conference for a general discussion of syzygies and other applications of Koszul cohomology.

§1.1. Background. Let X be a smooth irreducible projective curve of genus g, and let L be an ample line bundle on X, generated by its global sections. Thus L determines a morphism

$$\phi_L : X \longrightarrow \mathbb{P}(H^0(L)) = \mathbb{P}^r,$$

where $r = r(L) = h^0(L) - 1$. It is an elementary consequence of Riemann-Roch that L is very ample, i.e. that ϕ_L is an embedding, as soon as $\deg(L) \geq 2g+1$. It is then natural to study the equations defining X in $\mathbb{P}(H^0(L))$.

There are two classical results in this direction. First, a theorem of Castelnuovo [Cast], Mattuck [Mat] and Mumford [Mfd3] states that if $\deg(L) \geq 2g+1$, then the natural maps

$$\rho_m : \mathrm{Sym}^m H^0(L) \longrightarrow H^0(L^{\otimes m})$$

are surjective for all $m \geq 0$. In other words, ϕ_L embeds X as a projectively normal variety. The importance of this condition is that it reduces the problem of computing the number of hypersurfaces of degree m passing through X $[= \dim(\ker \rho_m)]$ to Riemann-Roch. The second theorem -- due to Fujita [Fjt] and St. Donat [St.-D2], strengthening earlier work of Mumford [Mfd3] -- deals with line bundles of slightly larger degree. It asserts that if $\deg(L) \geq 2g + 2$, then the homogeneous ideal $I_{X/\mathbb{P}^r}$ is generated by quadrics.

These results hint that as the degree of L grows, the equations defining the embedding $X \subset \mathbb{P}(H^0(L))$ display increasingly regular algebraic behavior. But in spite of the classical nature of the question, it is only quite recently that a precise meaning has been given to this philosophy. The basic insight is that one should focus on the whole chain of syzygies associated to a line bundle.

§1.2. Syzygies of Curves. With X and L as above, denote by S the symmetric algebra $\mathrm{Sym}^{\cdot} H^0(L)$ on $H^0(L)$. Thus S is the homogeneous coordinate ring of the projective space $\mathbb{P}(H^0(L))$. Consider now the graded ring

$$R = R(L) = \bigoplus_m H^0(X, L^m)$$

associated to L. Then R is in the natural way a finitely generated module over S, and so has a minimal graded free resolution $E_{\cdot} = E_{\cdot}(L)$:

(1.2.1)
$$0 \longrightarrow E_{r-1} \longrightarrow E_{r-2} \longrightarrow \ldots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow R \longrightarrow 0$$

where as above $r = r(L) = h^0(L) - 1$. Here each E_i is a direct sum of twists of S:

$$E_i = \bigoplus S(-a_{i,j}) ,$$

and hence in particular the maps in (1.2.1) are given by matrices of homogeneous forms. Minimality in the present context means that none of the entries in these matrices are non-zero constants. In principle, (1.2.1) is constructed by first using generators of R to define $E_0 \longrightarrow R$, then picking generators of $\ker(E_0 \longrightarrow R)$ to construct $E_1 \longrightarrow E_0$, and so on. Note that the resolution stops after $r-1 = \mathrm{codim}(X, \mathbb{P}^r)$ steps; this reflects the happy fact that $R(L)$ is automatically a Cohen-Macaulay module when X is a smooth curve.

Example 1.2.2. Let $C \subset \mathbb{P}^3$ denote the twisted cubic curve, i.e. the image of the embedding $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$ defined by the line bundle $L = \mathcal{O}_{\mathbb{P}^1}(3)$ on $\mathbb{P}^1$. If $T_0, \ldots, T_3$ denote homogeneous coordinates on $\mathbb{P}^3$, then one has the classical description of C as the locus where the 2×3 matrix

(1.2.3)
$$\begin{bmatrix} T_0 & T_1 & T_2 \\ T_1 & T_2 & T_3 \end{bmatrix}$$

has rank 1. Thus C is cut out by the three quadrics

$$\Delta_{01} = T_0T_2 - (T_1)^2 \qquad \Delta_{02} = T_0T_3 - T_1T_2 \qquad \text{and} \qquad \Delta_{12} = T_1T_3 - (T_2)^2 .$$

In fact, these generate the homogeneous ideal $I = I_{C/\mathbb{P}^3}$ of C, and furthermore $R = S/I$ thanks to the fact that $C \subset \mathbb{P}^3$ is projectively normal. Now by repeating either row of (1.2.3) and expanding the resulting determinant, one finds two relations among the $\Delta_{i,j}$:

$$T_0\Delta_{12} - T_1\Delta_{02} + T_2\Delta_{01} = 0 \qquad \text{and} \qquad T_1\Delta_{12} - T_2\Delta_{02} + T_3\Delta_{01} = 0 .$$

One can show moreover that any other relation among the $\Delta_{i,j}$ is a consequence of these. Hence the minimal resolution of $R = R(L)$ has the form

$$0 \longrightarrow S(-3)^2 \xrightarrow{\begin{bmatrix} T_0 & T_1 \\ -T_1 & -T_2 \\ T_2 & T_3 \end{bmatrix}} S(-2)^3 \xrightarrow{[\Delta_{12}, \Delta_{02}, \Delta_{01}]} S \longrightarrow R \longrightarrow 0 .$$

We remark that this is a simple special case of the Eagon-Northcott resolution of the locus defined by the vanishing of the maximal minors of a matrix (or of a vector bundle map). See [Szpr] or [GP3] for details.

<u>Example 1.2.4.</u> Let $E \subset \mathbb{P}^3$ be an elliptic curve embedded in $\mathbb{P}^3$ by a line bundle L of degree 4. Then E is the complete intersection of two quadrics Q_1 and Q_2. As above E is projectively normal, and now $R = R(L)$ has a <u>Koszul resolution</u>:

$$0 \longrightarrow S(-4) \xrightarrow{\begin{bmatrix} Q_2 \\ -Q_1 \end{bmatrix}} S(-2)^2 \xrightarrow{[Q_1\ ,\ Q_2]} S \longrightarrow R \longrightarrow 0\,.$$

<u>Remark</u>. It is very exceptional to be able to construct the whole resolution (1.2.1) explicitly, let alone to be able to do so by hand!

The way to generalize the classical results stated in §1.1 is to ask when the first few terms of the resolution (1.2.1) are as simple as possible. Specifically, one makes the following

<u>Definition 1.2.5.</u> Fix an integer $p \geq 0$. We say that the line bundle L on X satisfies **Property (N_p)** if

$$E_0(L) = S \quad \text{and} \quad E_i(L) = \oplus\, S(-i-1) \text{ for all } 1 \leq i \leq p\,.$$

(The second requirement means that if one writes $E_i = \oplus S(-a_{i,j})$, then all $a_{i,j} = i + 1$.)

Note that if $E_0 = S$, then $E_\cdot$ determines a resolution of the homogeneous ideal $I = I_{X/\mathbb{P}^r}$ of X in $\mathbb{P}(H^0(L))$. Thus (1.2.5) may be summarized very concretely as follows:

L satisfies (N_0)	$\Leftrightarrow$	ϕ_L embeds X as a projectively normal curve;
L satisfies (N_1)	$\Leftrightarrow$	(N_0) holds for L, and the homogeneous ideal I of X is generated by quadrics;
L satisfies (N_2)	$\Leftrightarrow$	(N_0) and (N_1) hold for X, and the module of syzygies among quadratic generators $Q_i \in I$ is spanned by relations of the form $$\Sigma\, L_i Q_i = 0,$$ where the L_i are <u>linear</u> polynomials;

and so on. Properties (N_0) and (N_1) are what Mumford [Mfd3] calls respectively <u>normal generation</u> and <u>normal presentation</u>.

<u>Example 1.2.6.</u> The twisted cubic $C \subset \mathbb{P}^3$ of (1.2.2) satisfies (N_2). The elliptic quartic $E \subset \mathbb{P}^3$ satisfies (N_1) but not (N_2).

The result we are <u>aiming</u> for, which extends and clarifies the theorems summarized in §1.1, is due to M. Green:

<u>**Theorem 1.2.7.**</u> ([Gr]) <u>Let</u> L <u>be a line bundle on</u> X <u>of degree</u> $2g + 1 + p$, <u>defining an embedding</u> $X \subset \mathbb{P}(H^0(L)) = \mathbb{P}^{g+p}$. <u>Then</u> L <u>satisfies property</u> (N_p).

So for instance one recovers the well known facts that a rational normal curve $C \subset \mathbb{P}^r$ satisfies (N_{r-1}), while an elliptic normal curve $E \subset \mathbb{P}^r$ satisfies (N_{r-2}). Examples show that this result is in general optimal. (See Remark 1.4.2 below).

To prove the theorem, we attach to L a vector bundle M_L on X. The syzygies of L are computed cohomologically in terms of this bundle, and the theorems are reduced to proving the vanishings of certain cohomology groups.

§1.3. The Vector Bundle M_L.

As before, let X be a smooth projective curve of genus g, and let L be an ample line bundle on X, generated by its global sections. Then there is a surjective evaluation map of vector bundles

$$e_L : H^0(L) \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow L\,,$$

and we put

$$M_L = \ker e_L\,.$$

Thus M_L is a vector bundle on X of rank $r = r(L)$, and by construction one has the basic exact sequence

(1.3.1) $$0 \longrightarrow M_L \longrightarrow H^0(L) \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow L \longrightarrow 0\,.$$

Note that (1.3.1) is the pull-back via $\Phi = \Phi_L$ of a twist of the Euler sequence on $\mathbb{P}(H^0(L))$; in particular, $M_L = \Phi^*(\Omega^1_{\mathbb{P}^r}(1))$.

Example 1.3.2. If $X = \mathbb{P}^1$ and $L = \mathcal{O}_{\mathbb{P}^1}(k)$, then M_L is isomorphic to the direct sum of k copies of $\mathcal{O}_{\mathbb{P}^1}(-1)$. By contrast, if $g(X) \geq 2$, and $\deg(L) \geq 2g+1$, then M_L is stable.

The basic point for us is that M_L governs the syzygies of X in $\mathbb{P}(H^0(L))$:

Proposition 1.3.3. Assume that L is non-special, i.e. that $H^1(L) = 0$. Then L satisfies property (N_p) if and only if

$$H^1(X, \Lambda^{p+1}M_L \otimes L^k) = 0 \qquad \text{for all } k \geq 1.$$

Remark. The Proposition is particularly transparent when $p = 0$. In fact to prove it in this case, just twist (1.3.1) by L^k and take cohomology: one finds that the map $H^0(L) \otimes H^0(L^k) \longrightarrow H^0(L^{k+1})$ is surjective if and only if $H^1(M_L \otimes L^k) = 0$. This argument is closely related to the classical "base-point free pencil trick" (c.f. [ACGH]).

Proof of (1.3.3). The case $p = 0$ having just been treated, we assume henceforth that $p \geq 1$.

Step (i). Referring to (1.2.1), we claim first that L satisfies (N_p) if and only if

(*) $\qquad E_p(L)$ has no generators of degrees $\geq p + 2$.

In fact, let n_i [resp. N_i] denote the minimum [resp. the maximum] of the degrees of the generators of $E_i(L)$. Then the sequences $\{n_i\}$ and $\{N_i\}$ are both strictly increasing in i; this is clear in the first case and follows in the second from the fact that $(E.)^*$ gives (after twisting) a minimal resolution of the module $\oplus H^0(\omega_X \otimes L^n)$. On the other hand, since Φ_L does not map X into any hyperplane in $\mathbb{P}(H^0(L))$, one has $N_1 \geq n_1 \geq 2$, and hence $N_i \geq n_i \geq i+1$. But (*) means that $N_p = n_p = p+1$, and this now forces $N_i = n_i = i+1$ for all $1 \leq i \leq p$.

Step (ii). Keeping the notation of §1.2, let $\mathbb{C}$ denote the residue field of S at the irrelevant maximal ideal $\mathfrak{m} = (T_0, \ldots, T_r)$, and consider the graded modules $\mathrm{Tor}_i(R(L), \mathbb{C})$. These are finite dimensional vector spaces which may be computed from a minimal resolution of $R(L)$. Since all the maps in (1.2.1) have entries in $\mathfrak{m}$, it is evident that

$$\dim \mathrm{Tor}_i(R, \mathbb{C})_k \;=\; \text{number of generators of } E_i(L) \text{ of degree } k.$$

Hence L satisfies (N_p) if and only if $\mathrm{Tor}_p(R, \mathbb{C})_{p+k} = 0$ for $k \geq 2$.

On the other hand, one may just as well compute these Tor's by resolving $\mathbb{C}$. Explicitly, let $V = S_1 = H^0(L)$. Tensoring the Koszul resolution

$$0 \to \Lambda^{r+1}V \otimes S(-r-1) \to \ldots \to \Lambda^2 V \otimes S(-2) \to V \otimes S(-1) \to S \to \mathbb{C} \to 0$$

by $R(L)$ and taking graded pieces, one finds that $\mathrm{Tor}_p(R, \mathbb{C})_{p+k}$ is isomorphic to the homology at the middle term of the Koszul-type complex

(1.3.4)
$$\Lambda^{p+1}H^0(L) \otimes H^0(L^{k-1}) \longrightarrow \Lambda^p H^0(L) \otimes H^0(L^k) \longrightarrow \Lambda^{p-1}H^0(L) \otimes H^0(L^{k+1})$$

So we are reduced to proving the exactness of (1.3.4) for $k \geq 2$.

Step (iii). It remains only to interpret (1.3.4) in terms of M_L. Consider to this end the diagram (1.3.5) on the following page, in which the vertical and horizontal exact sequences are obtained by taking exterior powers in (1.3.1) and twisting by suitable powers of L. The complex (1.3.4) arises by taking global sections in the indicated diagonal sequence of homomorphisms. Observe that $H^1(L^m) = 0$ for all $m \geq 1$ thanks to the fact that L is non-special. Chasing through (1.3.5), one then finds that for $k \geq 2$, (1.3.4) is exact if and only if $H^1(\Lambda^{p+1}M_L \otimes L^{k-1}) = 0$. ∎

Remarks. (1). The crux of this argument is that one can compute syzygies as the cohomology of the Koszul complex (1.3.4). We again refer to [Gr] and to Green's lectures at this conference for a systematic development of this point of view.

$$
\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & \Lambda^{p+1}M_L\otimes L^{k-1} \arrow[d] & & & \\
 & \Lambda^{p+1}H^0(L)\otimes L^{k-1} \arrow[d] \arrow[dr] & & 0 \arrow[d] & \\
0 \arrow[r] & \Lambda^{p}M_L\otimes L^{k} \arrow[r] \arrow[d] & \Lambda^{p}H^0(L)\otimes L^{k} \arrow[r] \arrow[dr] & \Lambda^{p-1}M_L\otimes L^{k+1} \arrow[r] \arrow[d] & 0 \\
 & 0 & & \Lambda^{p-1}H^0(L)\otimes L^{k+1} &
\end{tikzcd}
$$

Diagram (1.3.5)

---

(2). A variant of the proposition remains valid if $H^1(L) \neq 0$. Specifically, the proof above shows that L satisfies property (N_p) if and only if the natural maps

$$\Lambda^{p+1}H^0(L)\otimes H^0(L^{k-1}) \longrightarrow H^0(\Lambda^p M_L\otimes L^k)$$

are surjective.

§1.4. **Proof of Green's Theorem.** To gain control over the cohomology groups appearing in Proposition 1.3.3, the idea is to produce a filtration of M_L with line bundle quotients. In general, such filtrations are constructed in terms of a flag of secant planes to $X \subset \mathbb{P}(H^0(L))$ (c.f. [GL2, §2]). However here we only need the simplest of these filtrations. As above, L is an ample line bundle on X which is generated by its global sections.

__Lemma 1.4.1.__ __Let__ $x_1,\ldots,x_n \in X$ __be distinct points such that:__

$L(-\Sigma x_i)$ __is generated by its global sections,__

__and__

$$h^1(L(-\Sigma x_i)) = h^1(L).$$

__Then one has an exact sequence__

$$0 \longrightarrow M_{L(-\Sigma x_i)} \longrightarrow M_L \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_X(-x_i) \longrightarrow 0$$

__of vector bundles on__ X.

__Proof.__ Put $D = \Sigma x_i$, and let $W_D = H^0(L)/H^0(L(-D))$, so that $\mathbb{P}(W_D) \subset \mathbb{P}(H^0(L))$ is the plane spanned by D. Then the inclusion $D \subset \mathbb{P}(W_D)$ determines in the natural way a sheaf homomorphism

$$u_D : W_D \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow L\otimes\mathcal{O}_D .$$

Note that for each $x \in D$, the composition of u_D with the canonical quotient $L\otimes\mathcal{O}_D \longrightarrow L\otimes\mathcal{O}_{\{x\}}$ yields a surjective map

$$W_D\otimes\mathcal{O}_X \longrightarrow L\otimes\mathcal{O}_{\{x\}} .$$

This homomorphism is characterized by the fact that it yields on global sections the one-dimensional quotient $W_D \longrightarrow H^0(L\otimes\mathcal{O}_{\{x\}})$ corresponding to the point $x \in D \subset \mathbb{P}(W_D)$. Now the hypotheses on $L(-D)$ imply that D consists of n linearly independent points in $\mathbb{P}(W_D) = \mathbb{P}^{n-1}$. It follows that W_D is canonically the direct sum of the one-dimensional vector spaces $W_i = H^0(L)/H^0(L(-x_i))$, and that u_D decomposes as the direct sum of the natural maps $v_i : W_i \longrightarrow L\otimes\mathcal{O}_{\{x\}}$. In particular,

$$\ker u_D \simeq \oplus \ker v_i \simeq \bigoplus_{i=1}^{n} \mathcal{O}_X(-x_i) .$$

On the other hand, recalling that $L(-D)$ is generated by its global sections, one has the exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^0(L(-D)) \otimes \mathcal{O}_X \arrow[r] \arrow[d, "e_{L(-D)}"] & H^0(L) \otimes \mathcal{O}_X \arrow[r] \arrow[d, "e_L"] & W_D \otimes \mathcal{O}_X \arrow[r] \arrow[d, "u_D"] & 0 \\
0 \arrow[r] & L(-D) \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & L \otimes \mathcal{O}_D \arrow[r] \arrow[d] & 0 . \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The desired exact sequence then follows by taking kernels of the vertical maps. ∎

Green's theorem is now an easy consequence of Proposition 1.3.3:

<u>Proof of Theorem 1.2.7.</u> We show that $H^1(X, \Lambda^{p+1}M_L \otimes L) = 0$, the vanishing of $H^1(\Lambda^{p+1}M_L \otimes L^k)$ for $k \geq 2$ being similar but simpler. To this end, note that the hypotheses of Lemma 1.4.1 are satisfied by $r - 1 = g + p$ <u>general</u> points $x_1, \ldots, x_{r-1} \in X$. Put $D = x_1 + \ldots + x_{r-1}$. Then $h^0(L(-D)) = 2$, so $M_{L(-D)} = L^*(D)$ and one has the exact sequence

$$0 \longrightarrow L^*(D) \longrightarrow M_L \longrightarrow \Sigma \longrightarrow 0,$$

where $\Sigma = \oplus\, \mathcal{O}_X(-x_i)$. Taking $(p+1)^{st}$ exterior powers, and twisting by L, one gets:

$$0 \longrightarrow \Lambda^p\Sigma \otimes \mathcal{O}_X(D) \longrightarrow \Lambda^{p+1}M_L \otimes L \longrightarrow \Lambda^{p+1}\Sigma \otimes L \longrightarrow 0 .$$

The term on the right is a direct sum of line bundles of degrees $(2g+1+p)-(p+1) = 2g$, and hence $H^1(\Lambda^{p+1}\Sigma \otimes L) = 0$. As for the term on the left, it is a direct sum of line bundles of the form

$$\mathcal{O}_X(x_{i_1} + \ldots + x_{i_{r-1-p}}) .$$

But

$$r - 1 - p = (g + p) - p = g,$$

and a general effective line bundle of degree $\geq g$ is non-special. Hence by choosing the x_i sufficiently generally, we can assume that $H^1(\Lambda^p\Sigma \otimes \mathcal{O}_X(D)) = 0$ and the theorem follows. ∎

<u>Remark 1.4.2.</u> Theorem (1.2.7) is in general optimal. In fact, a more sophisticated argument with the bundle M_L yields a classification of the borderline examples:

<u>Theorem</u> ([GL4]) <u>Assume that</u> $g(X) = g \geq 2$, <u>and let</u> L <u>be a line bundle of degree</u> $2g + p$ <u>on</u> X, <u>defining an embedding</u> $X \subset \mathbb{P}^{g+p}$. <u>Then</u> (N_p) <u>fails for</u> L <u>if and only if either:</u>

X <u>is hyperelliptic;</u>

<u>or</u>

ϕ_L <u>embeds</u> X <u>with a</u> $(p+2)$-<u>secant</u> p-<u>plane.</u>

When $p = 0$, the assertion in (b) should be interpreted to mean that ϕ_L is not, in fact, an embedding. The theorem implies for example that if $X \subset \mathbb{P}^{g+1}$ is a non-hyperelliptic curve of degree $2g+1$, then the homogeneous ideal of X is generated by quadrics unless X has a tri-secant line. This theorem gives a first indication of the fact that the syzygies of a curve are closely connected to its geometry. We will return to this theme in §2.4 below.

§1.5. A Conjecture for Abelian Varieties. Classical work on the equations defining algebraic curves has traditionally been complemented by analogous results for abelian varieties. Thus let X be an abelian variety of dimension g, and let L be an ample line bundle on X. A classical theorem of Lefschetz asserts that $L^{\otimes k}$ is very ample for $k \geq 3$, defining an embedding $X \subset \mathbb{P}(H^0(L^{\otimes k})) = \mathbb{P}^{n(k)}$. Koizumi [Koiz] and Sekiguchi [Sek] -- generalizing earlier work of Mumford [Mfd1, Mfd3] -- have shown that $L^{\otimes k}$ is normally generated for $k \geq 3$. When $k \geq 4$, Mumford [Mfd3] proves that $X \subset \mathbb{P}^{n(k)}$ is scheme-theoretically cut out by quadrics.

By analogy with Green's theorem, one expects such results to extend to higher syzygies. In particular, the obvious statement to hope for is:

Conjecture 1.5.1. Let L be an ample line bundle on an abelian variety X. If $k \geq p+3$, then $L^{\otimes k}$ satisfies property (N_p).

Note that if X is an elliptic curve, then the conjecture is just a restatement of (1.2.7). Kempf [Kmf2] has lately made very substantial progress on this problem: he shows that if $p \geq 1$, then $L^{\otimes k}$ satisfies (N_p) provided that $k \geq 2p+2$. In particular, if $k \geq 4$ then I_X is generated by quadrics.

§1.6. Notes.

(1.6.1) The relationship between Koszul cohomology and syzygies implicit in the proof of (1.3.3) is classical. The connection with the vector bundle M_L is then immediate, and well-known to those who have thought about these questions. However, Green's original proof of (1.2.7) does not make explicit use of this bundle.

(1.6.2). In the spirit of [St.D2], one can deduce Theorem 1.2.7 from a statement on the syzygies of finite subsets of projective space. Specifically, it is shown in [GL4] that if $S \subset \mathbb{P}^r$ consists of $2r+1-p$ points in linear general position, then X satisfies property (N_p) (suitably defined). This immediately implies (1.2.7) by taking a generic hyperplane section of the embedding $X \subset \mathbb{P}^{g+1+p}$. Conversely, it turns out that this result on finite sets can be given a proof via vector bundles very much in the spirit of the argument in §1.4.

(1.6.3). The bundle M_Ω corresponding to the canonical line bundle $\Omega = \omega_X$ on a curve X has been considered by several authors. (Equivalently, M_Ω is the conormal bundle to X in its Jacobian.) The motivation here is a beautiful conjecture of Green concerning the syzygies of canonical curves (see §2.6 below.) In [GL2], methods along the lines of §1.4 are used to give a very simple proof of Petri's theorem on canonical curves. These techniques have been extended by Voisin [Vsn], who studies the syzygies among the quadrics defining a canonical curve.

Paranjape and Ramanan [PR] study the stability of M_Ω; in particular, they show that this bundle is stable for non-hyperelliptic X. As they observe, this leads to a very interesting question (which had occurred independently to Green and the author). Namely, assume that X is non-hyperelliptic, and fix a line bundle A on X

such that $A^{g-1} = \Omega$. Then $M_\Omega \otimes A$ is a stable bundle with trivial determinant, and hence corresponds to a unitary representation

$$\rho_A : \pi_1(X) \longrightarrow SU(g-1).$$

(The dependence on A is minor, since replacing A by another $(g-1)^{st}$ root of the canonical bundle changes ρ_A by a character of order $g-1$.) Thus there exists a canonical projective representation of $\pi_1(X)$. The problem is to construct this representation explicitly, for example by transcendental methods.

(1.6.4). If $V \subset H^0(L)$ is a base-point free linear series on a curve (or higher dimensional variety) X, then one can define a vector bundle M by the exact sequence

$$0 \longrightarrow M \longrightarrow V \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow L \longrightarrow 0.$$

At least for curves this bundle is again connected to the equations defining X, but for incomplete series the relationship is not as direct as in §1.3. (c.f. [GLP, §1]). When $X = \mathbb{P}^1$, the geometrical information encoded in these bundles is studied by Ascenzi [Asc].

§2. Projective Normality of Algebraic Curves.

In the previous section, we saw how classical results on the equations defining algebraic curves generalize to statements about higher syzygies. Here we discuss a theorem from [GL1] which extends some of these classical statements in a different direction: we stick to projective normality, but deal with embeddings defined by line bundles L of lower degree. Interestingly, now the geometry of the pair (X, L) plays a really fundamental role.

§2.1. Background and Examples. Throughout this section, X denotes a smooth irreducible curve of genus $g \geq 2$, and L is an ample line bundle on X defining a morphism $\phi_L : X \longrightarrow \mathbb{P}(H^0(L)) = \mathbb{P}^r$. Recall that L is normally generated if the natural maps $\mathrm{Sym}^m H^0(L) \longrightarrow H^0(L^{\otimes m})$ are surjective for all $m \geq 0$. Equivalently, L is normally generated if and only if L is very ample, and

$H^1(\mathbf{P}^r, \mathcal{J}_{X/\mathbf{P}^r}(m)) = 0$ for every $m \geq 0$, where $\mathcal{J}_{X/\mathbf{P}^r}$ is the ideal sheaf of X in $\mathbf{P}(H^0(L))$. As indicated in §1.1, projectively normal embeddings are usually the easiest to deal with from the point of view of postulation. So it is of interest to have criteria which guarantee that a given line bundle be normally generated.

There are a number of classical and more recent results along these lines. Probably the most famous is Noether's theorem (c.f. [Mfd3]) that the canonical bundle ω_X is normally generated if X is non-hyperelliptic. The theorem of Castelnuovo [Cast], Mattuck [Mat], and Mumford [Mfd3] recalled in §1 asserts that L is normally generated when $\deg(L) \geq 2g + 1$. Lange and Martens [LM] have have extended this result by showing that a general line bundle of degree $2g$ on a non-hyperelliptic curve is normally generated. In a similar vein, Arbarello, Cornalba, Griffiths and Harris [ACGH] observed that on a <u>general</u> curve X, a general line bundle of degree $\geq \lfloor (3g+4)/2 \rfloor$ defines a projectively normal embedding.

The diversity of these results leads one to ask whether one can obtain a unified and strengthened theorem by taking into account in some quantitative way the intrinsic geometry of X. In order to get a picture of what type of statement to expect, it is instructive to see how examples of very ample non-special bundles L which fail to be normally generated can arise as soon as $\deg(L) \leq 2g$.

<u>Example 2.1.1</u>. Let X be a hyperelliptic curve, and let L be any very ample line bundle of degree $2g$, so that L defines an embedding $X \subset \mathbf{P}^g$. Then X lies on a rational ruled surface $S \subset \mathbf{P}^g$ swept out by the hyperelliptic pencil. But S in turn is on enough quadrics to force $H^1(\mathcal{J}_{X/\mathbf{P}^g}(2)) \neq 0$. Note that the theorem quoted at the end of §1.4 implies these are the only examples of very ample but not normally generated line bundles of degree $2g$.

<u>Example 2.1.2</u>. For $\deg(L) = 2g - 1$, the possibilities are more varied:

(a). If X is hyperelliptic, then just as above L is never normally generated.

(b). If X is trigonal, then there exist line bundles L of degree $2g - 1$ with the property that Φ_L embeds X with a 4-secant line. But for reasons of Castelnuovo-Mumford regularity (c.f. [Mfd2]), a projectively normal non-special curve is always

cut out by quadrics and cubics. Hence such a line bundle cannot be normally generated.

(c). Finally, suppose that $X \subset \mathbf{P}^2$ is a smooth quintic. Pick 4 points $x_1, \ldots, x_4 \in X$, with no three collinear, and consider the linear system cut on X by cubics through the x_i. It defines an embedding $X \subset \mathbf{P}^5$ of degree eleven in which X lies on a Del Pezzo surface. As in Example 2.1.1, this puts X on too many quadrics to be projectively normal. We remark that these curves have a one dimensional family of six-secant conics in $\mathbf{P}^5$ (arising from the pencil of conics in $\mathbf{P}^2$ through the x_i).

Again it turns out that these are the only examples that occur.

The pattern that emerges from these examples is that as the degree of L becomes smaller, very ample line bundles which fail to be normally generated can exist on more and more curves. To give a more precise quantitative statement, the crucial invariant is the Clifford index of X.

§2.2. The Clifford Index and Projective Normality. Given a line bundle A on a curve X, recall that the <u>Clifford index</u> of A is the integer

$$\mathrm{Cliff}(A) = \deg(A) - 2 \cdot r(A),$$

where $r(A) = h^0(A) - 1$. The Clifford index of X itself is taken to be

$$\mathrm{Cliff}(X) = \min \{ \mathrm{Cliff}(A) \mid h^0(A) \geq 2, h^1(A) \geq 2 \}.$$

We say that a line bundle A <u>contributes to</u> the Clifford index of X if it satisfies the inequalities in the definition of $\mathrm{Cliff}(X)$; it <u>computes</u> the Clifford index of X if in addition $\mathrm{Cliff}(A) = \mathrm{Cliff}(X)$.

The Clifford index gives a rough measure, from the point of view of special linear series, of how general X is in the sense of moduli. Thus Clifford's theorem states that $\mathrm{Cliff}(X) \geq 0$ with equality if and only if X is hyperelliptic. Similarly, one can prove that $\mathrm{Cliff}(X) = 1$ if and only if X is either trigonal or a smooth plane quintic. At the other extreme, it follows from Brill-Noether theory that if X is a *general* curve of genus g, then $\mathrm{Cliff}(X) = [(g-1)/2]$, and in any event $\mathrm{Cliff}(X) \leq [(g-1)/2]$.

Remarks. (1). It follows from Riemann-Roch that for any line bundle A on X, $\mathrm{Cliff}(A) = \mathrm{Cliff}(\omega_X \otimes A^*)$. Hence by Serre duality, one may alternatively define

$$\mathrm{Cliff}(X) = \min\{\,\mathrm{Cliff}(A) \mid h^0(A) \geq 2,\ \deg(A) \leq g-1\,\}.$$

(2). The previous remark shows that the definition of $\mathrm{Cliff}(X)$ presupposes the existence of a pencil of degree $\leq g-1$ on X, and hence implicitly assumes that $g = g(X) \geq 4$. All of the results that follow remain valid if one takes $\mathrm{Cliff}(X) = 0$ for X of genus 2 or hyperelliptic of genus 3, and $\mathrm{Cliff}(X) = 1$ for X non- hyperelliptic of genus 3.

(3). G. Martens [M1] has shown that if X is a curve of Clifford index e and genus $g > (e+3)(e+2)/2$ then X carries a pencil of degree $e+2$, so that in this case the Clifford index depends only on the "gonality" of the curve. But when g is relatively small compared to the Clifford index, the picture is not currently as well understood. For instance, Ballico [Bal] has only quite recently shown that there even exist curves of arbitrary Clifford index $0 \leq e \leq [(g-1)/2]$ for a given genus g. However, if one defines the Clifford dimension of X to be the least dimension of a linear series computing the Clifford index of X, then recent work of Eisenbud, Lange, Martens and Schreyer [ELMS] suggests that it might be possible to classify all curves of given Clifford dimension. Specifically, they conjecture that besides curves of Clifford dimension 1 and smooth plane curves (which have Clifford dimension 2), there is for each $r \geq 3$ exactly one numerical family of curves of Clifford dimension r.

The following result, which applies when $h^1(L) \leq 1$, generalizes the various theorems stated in §2.1 above:

Theorem 2.2.1. ([GL1], Thm. 1) *Let* L *be a very ample line bundle on* X, *with*

$$\deg(L) \geq 2g+1-2\cdot h^1(L) - \mathrm{Cliff}(X) \tag{2.2.2}$$

(and hence $h^1(L) \leq 1$). *Then* L *is normally generated.*

Example 2.2.3. To see how the numerology works, take $L = \omega_X$, and assume that X is non-hyperelliptic. Then $\mathrm{Cliff}(X) \geq 1$, and so

$$\deg(\omega_X) = 2g-2 \geq 2g+1-2\cdot(1) - \mathrm{Cliff}(X).$$

Hence the theorem implies Noether's result. Similarly, if X is a general curve of genus g, then $\mathrm{Cliff}(X) = [(g-1)/2]$, and (2.2.1) yields the result of Arbarello *et al.*

Example 2.2.4. It follows from the theorem that if X carries a very ample line bundle of degree $2g-1$ which fails to be normally generated, then $\mathrm{Cliff}(X) \leq 1$, i.e. X is hyperelliptic, trigonal, or a smooth plane quintic. Conversely, Example 2.1.2 shows that every curve of Clifford index ≤ 1 does in fact carry such a bundle.

Remark 2.2.5. Note that (2.2.2) is equivalent to the inequality

$$\mathrm{Cliff}(L) < \mathrm{Cliff}(X). \tag{2.2.6}$$

In fact, if N is a line bundle of degree $2g+1-e$, then $\mathrm{Cliff}(N) = e-1-2\cdot h^1(N)$. It follows that (2.2.2) can only hold if $h^1(L) \leq 1$. For by definition the inequality (2.2.6) is only possible if L does not contribute to the Clifford index of X; but in the situation of Theorem 2.2.1 one has $h^0(L) \geq 2$ since L is very ample, and hence (2.2.2) implies that $h^1(L) \leq 1$.

§2.3. Proof of Theorem 2.2.1 Via Rank Two Vector Bundles. We present here the approach to Theorem 2.2.1 which appears in [GL1]. The idea is quite simple: assuming that L fails to be normally generated, we must show in view of (2.2.6) that $\mathrm{Cliff}(X) \leq \mathrm{Cliff}(L)$. Now the failure of L to to be normally generated translates into the existence of a rank two vector bundle on X having a large number of sections. The desired inequality then pops out from the existence of a line sub-bundle of suitably large degree. (Compare Reider's arguments in §4).

We start with two elementary lemmas:

Lemma 2.3.1. Let A *be any line bundle on* X. *Then*

$$h^0(A) + h^0(\omega_X \otimes A^*) = g(X) + 1 - \mathrm{Cliff}(A).$$

Proof. In fact, suppose that $h^0(A) = r+1$. Then $h^1(A) = g - d + r$ by Riemann-Roch, and the assertion follows from Serre duality. ∎

Lemma 2.3.2. *Let* L *be a very ample line bundle on* X, *defining an embedding* $X \subset \mathbb{P}(H^0(L)) = \mathbb{P}^r$, *and let* D *be a non-zero effective divisor on* X *which spans a plane* $\Lambda_D \subset \mathbb{P}^r$ *with* $\dim \Lambda_D = n$. *Then*

$$\mathrm{Cliff}(L(-D)) \le \mathrm{Cliff}(L) \iff \deg(D) \ge 2n+2.$$

Furthermore, if these conditions hold, and if $n \ge 1$, *then* $h^1(L(-D)) \ge 2$.

Proof. Since D spans an n-plane in $\mathbb{P}(H^0(L))$ one has $r(L(-D)) = r(L) - n - 1$, and the first assertion follows from the definition of the Clifford index. The second is clear from the first. ∎

Proof of Theorem 2.2.1. Suppose to the contrary that L satisfies (2.2.2) -- and hence in particular $\deg(L) \ge 2g-1-[(g-1)/2]$ -- but that L fails to be normally generated. Note to begin with that the multiplication maps

$$\mu_m : H^0(L) \otimes H^0(L^m) \longrightarrow H^0(L^{m+1})$$

are surjective for all $m \ge 3$. This may be verified by using the techniques of §1 to show that $H^1(M_L \otimes L^m) = 0$ for $m \ge 2$ in the degree range at hand (or see [Gr], Thm. 4.e.1). Hence μ_1 -- which we call simply μ -- cannot be surjective. Equivalently, its transpose

$$\mu^* : H^0(L^2)^* \longrightarrow H^0(L)^* \otimes H^0(L)^*$$

has a non-zero kernel.

On the other hand, via Serre duality $H^0(L^2)^* = \mathrm{Ext}^1(L, \omega_X \otimes L^*)$ classifies extensions of L by $\omega_X \otimes L^*$. Furthermore, μ^* is identified with the natural map

$$\mathrm{Ext}^1(L, \omega_X \otimes L^*) \longrightarrow \mathrm{Hom}(H^0(L), H^1(\omega_X \otimes L^*))$$

which takes an extension to the connecting homomorphism it determines. But by assumption μ^* fails to be injective, and hence there exists a non-split extension

$$0 \longrightarrow \omega_X \otimes L^* \longrightarrow E \longrightarrow L \longrightarrow 0 \tag{2.3.3}$$

which is exact on global sections. Observe that $\det E = \omega_X$, and that

$$h^0(E) = g + 1 - \mathrm{Cliff}(L) \tag{2.3.4}$$

thanks to (2.3.1).

The next step is to invoke a classical theorem of Segre which asserts that a rank two vector bundle F of degree d on X has a line sub-bundle $A \subset F$ of degree $\ge [(d-g+1)/2]$ (c.f. [N] or [MS] for a simple proof). Applying this to the bundle E, we arrive at an exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow \omega_X \otimes A^* \longrightarrow 0,$$

where A is a line bundle of degree $\ge [(g-1)/2]$. Taking cohomology and once again using (2.3.1), one gets

$$h^0(E) \le h^0(A) + h^0(\omega_X \otimes A^*) = g + 1 - \mathrm{Cliff}(A).$$

Hence

$$\mathrm{Cliff}(A) \le \mathrm{Cliff}(L) \tag{2.3.5}$$

by virtue of (2.3.4).

We now show that A contributes to the Clifford index of X. This will complete the proof, since then

$$\mathrm{Cliff}(X) \le \mathrm{Cliff}(A) \le \mathrm{Cliff}(L),$$

which in view of (2.2.6) contradicts the numerical hypothesis of the theorem. To begin with, note that the inequality $h^0(A) \ge 2$ is immediate. In fact the lower bound on $\deg(A)$ together with (2.2.6) and (2.3.5) imply:

$$\mathrm{Cliff}(A) \le \mathrm{Cliff}(L) < \mathrm{Cliff}(X) \le [(g-1)/2] \le \deg(A).$$

But since $\mathrm{Cliff}(A) = \deg(A) - 2\cdot r(A)$, this forces $r(A) \ge 1$. It remains only to show that $h^1(A) \ge 2$. To this end, consider the following diagram:

$$\begin{tikzcd} & & A \arrow[d, phantom, "\cap"] \arrow[dr, "\alpha"] & & \\ 0 \arrow[r] & \omega_X \otimes L^* \arrow[r] & E \arrow[r] & L \arrow[r] & 0 . \end{tikzcd}$$

We claim that the indicated map $\alpha : A \longrightarrow L$ is non-zero, and hence $A = L(-D)$ for some effective divisor D on X. In fact, recall that (2.2.2) can only hold when $\deg(L) \ge 2g-1-[(g-1)/2]$, and this implies that $\deg(\omega_X \otimes L^*) < \deg(A)$ thanks to the lower bound on $\deg(A)$. Now $D \ne 0$, or else the inclusion $A \subset E$ would split (2.3.3). Thus we are in the situation of Lemma 2.3.2, and it suffices to show that the span $\Lambda_D \subset \mathbb{P}(H^0(L))$ of D is at least a line. But $\deg(D) \ge 2\cdot\dim(\Lambda_D) + 2$, and so if $\dim(\Lambda_D) = 0$ then L is not very ample. ∎

Remark. One can deduce Theorem 2.2.1 from a more general statement to the effect that the failure of a very ample line bundle L of degree roughly $3g/2$ or greater to be normally generated is "explained" by the existence of a special secant plane. In fact, one has the following

Theorem ([GL1], Thm. 3) *Let* L *be a very ample line bundle on* X, *with*

$$\deg(L) = 2g + 1 - k.$$

Assume that $2k+1 \le g$ *if* $h^1(L) = 0$, *or that* $2k-3 \le g$ *if* $h^1(L) \ne 0$, *and consider the embedding*

$$X \subset \mathbb{P}(H^0(L)) = \mathbb{P}^r$$

defined by L. *Then* L *fails to be normally generated if and only if there exists an integer* $1 \le n \le r-2$, *and an effective divisor* D *on* X *of degree* $\ge 2n+2$, *such that*

(i). $H^1(L^2(-D)) = 0$;

and

(ii). D *spans an* n*-plane* $\Lambda_D \subset \mathbb{P}^r$ *in which* D *fails to impose independent conditions on quadrics.*

The condition in (b) is that $H^1(\mathcal{J}_{D/\Lambda}(2)) \ne 0$, where $\mathcal{J}_{D/\Lambda}$ is the ideal sheaf of D in Λ_D. When $n = 1$, so that Λ_D is a 4-secant line, this is automatic. However, when $n \ge 2$, it is not sufficient that X simply have a $(2n+2)$-secant n-plane. (Compare Example 2.1.2.(c) above.) In fact, a closer analysis shows that the divisor D constructed in the proof above has the required properties.

§2.4. Conjectures for Higher Syzygies.

It is very interesting to ask whether the results of this section extend to higher syzygies. The natural thing to hope for here is:

Conjecture 2.4.1. ([GL1], §3). *Let* L *be a very ample line bundle on* X, *with*

$$\deg(L) \ge 2g + 1 + p - 2\cdot h^1(L) - \mathrm{Cliff}(X).$$

Then L *satisfies property* (N_p) *of* §1.2 *unless* ϕ_L *embeds* X *with a* $(p+2)$*-secant* p*-plane*.

In fact, one would hope for a more precise statement along the lines of the theorem quoted at the end of the previous section. When $p = 1$, one can prove a somewhat weaker result using the techniques of §2.3. Specifically:

Proposition 2.4.2. (with M. Green) *Let* L *be a very ample line bundle with*

$$\deg(L) \ge 2g + 2 - 2\cdot h^1(L) - \mathrm{Cliff}(X),$$

defining an embedding $X \subset \mathbb{P}(H^0(L)) = \mathbb{P}^r$. Then X is scheme-theoretically cut out by quadrics unless it has a tri-secant line.

Sketch of Proof. We show that $X \subset \mathbb{P}^r$ is set-theoretically cut out by quadrics; the stronger assertion of the proposition is handled by showing that if X fails to be scheme-theoretically cut out by quadrics at $x \in X$, then $L(-x)$ fails to be normally generated. To this end, let us say that an extension

$$(*) \qquad 0 \longrightarrow \omega_X \otimes L^* \longrightarrow E \longrightarrow L \longrightarrow 0$$

has rank one if the connecting homomorphism $H^0(L) \longrightarrow H^1(\omega_X \otimes L^*) = H^0(L)^*$ it determines does. Every curve possesses rank one extensions: pick any point $x \in X$, and take (*) to be the extension given by the one-dimensional subspace of $H^0(L^2)^* = \mathrm{Ext}^1(L, \omega_X \otimes L^*)$ corresponding to $H^0(L^2(-x)) \subset H^0(L^2)$. Let us call these "insignificant" extensions. The point now is that the existence of a "significant" rank-one extension is equivalent to the failure of $X \subset \mathbb{P}(H^0(L))$ to be set-theoretically cut out by quadrics. And starting from such a rank one extension, one can argue much as in §2.3 that the inequality in the proposition cannot hold unless X has a tri-secant line. ∎

A particularly interesting case of (2.4.1) is when $L = \omega_X$. Then the conjecture asserts that the canonical bundle satisfies (N_p) if $p < \mathrm{Cliff}(X)$. On the other hand, it was proved with Green in the appendix to [Gr] that (N_p) fails for ω_X if $p \geq \mathrm{Cliff}(X)$. Hence (2.4.1) contains as a special case Green's conjecture on the syzygies of canonical curves:

Conjecture 2.4.3. ([Gr]). The Clifford index of X is equal to the least integer p for which property (N_p) fails for the canonical bundle.

So for instance the conjecture would generalize Petri's theorem [St.-D1] that the homogeneous ideal of a non-hyperelliptic canonical curve $X \subset \mathbb{P}^{g-1}$ is generated by quadrics unless X is trigonal or a smooth plane quintic, i.e. unless $\mathrm{Cliff}(X) \leq 1$. We remark that as a simple consequence of duality one can show that the least integer p for which (N_p) fails for ω_X determines completely the grading of the resolution of the canonical ring of X.

Green's conjecture has already sparked a considerable amount of work. It was verified for $g \leq 8$ by Schreyer [Sch1] who in fact worked out all possible canonical resolutions in low genera. He also observes that the conjecture fails for the generic curve of genus seven in characteristic two. The bundle M_Ω governing the syzygies of the canonical curve (c.f. §1.3) has been studied by Paranjape and Ramanan [PR]; in particular, they observe that the Clifford index is reflected in stability properties of this bundle. The most striking progress to date is due to Schreyer [Sch2] who has recently used standard basis techniques à la Petri to prove the case p = 2 of the conjecture (and even more); the same result for curves X of genus ≥ 11 is established by C. Voisin [Vsn] using some of the methods of [GL2].

Recall that if X is a general curve of genus g, then $\mathrm{Cliff}(X) = [(g-1)/2]$. Hence as a limiting case of (2.4.3) one obtains:

Conjecture 2.4.4. ([Gr]) If X has general moduli, then its canonical bundle satisfies (N_p) for $p < [(g-1)/2]$.

Drawing on some ideas of Chang and Ran, Ein [En] has proven (2.4.4) for $p \leq 3$ via an inductive degenerational argument. Interestingly enough, in order to get the induction going, Ein relies on computer-assisted calculations made by Bayer and Stillman in low genera. The idea here -- due to K. O'Grady -- is that the tangent developable surface $S \subset \mathbb{P}^g$ to the rational normal curve $C \subset \mathbb{P}^g$ of degree g should have the same syzygies as the general canonical curve. (The hyperplane sections of S are curves of degree $2g-2$ with g cusps and hence -- according to a philosophy of Eisenbud and Harris -- should behave like general canonical curves.) But the syzygies of S can be computed by machine for small g. Amazingly, it is not yet known in general that the syzygies of S are what one would expect.

Finally, we mention that another result connected to (2.4.3) is described in §3.6 below.

§2.5. Notes.

(2.5.1). Koh and Stillman [KS] have given a different -- purely algebraic -- proof of Theorem 2.2.1. Ballico and Ellia study the normal generation of general line bundles on a general curve in [BE], and at least for non-special curves, they obtain optimal results.

(2.5.2) The connection between the bundles constructed in §2.3 and projective normality is certainly not new -- it was used e.g. by Lange and Narasimhan [LN] to study rank two bundles on curves. The main novelty of the approach presented above lies in reversing the process, i.e. in using the bundles to study the algebra.

§3. Brill-Noether Theory and Vector Bundles on a K3 Surface

In this section we explain following [L1] how the analysis of certain vector bundles on a K3 surface leads to a simple proof of some of the basic results of Brill-Noether theory. At the expense of substituting plausibility arguments for careful proofs, we will attempt to emphasize more clearly than in [L1] the geometrical underpinnings of the results. In particular, we will indicate how a theorem of Mukai on the moduli spaces of vector bundles on a K3 plays a central role.

§3.1. Petri's Condition. We start with a brief review of Brill-Noether theory. Our purpose is mainly to establish notation and set the stage, and we refer to [ACGH] for a detailed account.

Throughout this section, C denotes a smooth irreducible projective curve of genus $g \geq 2$. We are interested in the varieties of special divisors on C, i.e. the loci $W^r_d(C) \subseteq \mathrm{Pic}^d(C)$ defined by

$$W^r_d(C) = \{ \text{ line bundles } A \mid \deg(A) = d \ , \ h^0(A) \geq r+1 \ \} .$$

These are determinantal subvarieties of $\mathrm{Pic}^d(C)$, i.e. locally defined by the vanishing of the minors of a suitable matrix. The postulated dimension of $W^r_d(C)$ was already known in the last century; it is given by the Brill-Noether number

$$\rho = \rho(r,d,g) = g - (r+1)\cdot(g-d+r) .$$

Elementary results about determinantal varieties apply in the case at hand to show that the actual dimension of $W^r_d(C)$ is $\geq \rho(r,d,g)$ provided that $W^r_d(C) \neq \emptyset$.

The basic facts about these varieties fall into two classes. First, one has two global results that hold for an arbitrary curve C. Namely, a theorem of Kempf [Kmf1] and Kleiman-Laksov [KL] states that $W^r_d(C)$ is non-empty as soon as $\rho(r,d,g) \geq 0$. This existence theorem is complemented by a result from [FL] that $W^r_d(C)$ is connected when $\rho(r,d,g) \geq 1$.

The second class of theorems deals with the local geometry of W^r_d on curves of general moduli. It was assumed classically that if C is a general curve of genus g, then $W^r_d(C)$ behaves like a generic determinantal variety, and it has recently been proved that this is effectively the case. Specifically, Brill and Noether asserted, and Griffiths and Harris [GH1] proved:

(3.1.1) *If C has general moduli, then* $\dim W^r_d(C) = \rho(r,d,g)$ *for every* r *and* d.

This includes the assertion that if A is any line bundle on a general curve C, then

$$\rho(A) =_{def} g(C) - h^0(A)\cdot h^1(A) \geq 0 ,$$

i.e. that $W^r_d(C) = \emptyset$ when $\rho(r,d,g) < 0$. Gieseker [Gies] then proved the fundamental fact

(3.1.2) *For C sufficiently general,* $W^r_d(C)$ *is smooth away from* $W^{r+1}_d(C)$.

We note that (3.1.2) and the connectedness theorem mentioned above imply that if C is general, then $W^r_d(C)$ is irreducible when $\rho(r,d,g) \geq 1$.

It is technically convenient to avoid working directly with the geometric statements (3.1.1) and (3.1.2), and to focus instead on the so-called Petri homomorphism

$$\mu_A : H^0(A) \otimes H^0(\omega_C \otimes A^*) \longrightarrow H^0(\omega_C)$$

defined for a given line bundle A by multiplication of sections. As explained for instance in [ACGH], the deformation theory of line bundles with sections shows that both (3.1.1) and (3.1.2) are implied by

(3.1.3) For every line bundle A on C, the homomorphism μ_A is injective.

Gieseker's theorem in [Gies] is actually that (3.1.3) holds for the general curve of genus g. This had been asserted in passing by Petri, and we will say that C satisfies Petri's condition if (3.1.3) -- and hence (3.1.1) and (3.1.2) -- hold for C.

The difficulty in establishing Petri's assertion is that one has to find a way to use the hypothesis that C is general. It would suffice for each genus g to exhibit any one curve for which the Brill-Noether-Petri package holds; but in large genera no explicit examples are known! Instead, the technique of [GH1] and [Gies] -- which goes back to classical ideas of Severi and Castelnuovo -- is to study degenerations of C. The idea is to argue that (3.1.1) and (3.1.3) cannot fail identically for one-parameter families of curves degenerating to a suitable singular limit. This line of attack involves some rather involved combinatorics.

A different approach to these questions -- one not involving any degenerations -- was suggested in [L1]. The idea is to study the Brill-Noether theory of curves on a K3 surface. As we will see, it turns out that on a "sufficiently general" K3 surface, there must exist curves for which (3.1.3) holds.

§3.2. Brill-Noether Theory on K3 Surfaces. We henceforth let X denote a K3 surface, i.e. a compact complex surface with $H^1(X, \mathcal{O}_X) = 0$ and $\omega_X = \mathcal{O}_X$. The main result for which we are aiming in this section is:

Theorem 3.2.1. ([L1]) Let $C \subset X$ be a smooth irreducible curve of genus $g \geq 2$ on the K3 surface X. Assume that every divisor in the linear series $|C|$ is reduced and irreducible. Then:

(i). For every line bundle A on C, one has $\rho(A) \geq 0$;

and

(ii). Petri's condition holds for the general member $C' \in |C|$.

Observe that the irreducibility condition on the linear series $|C|$ holds in particular when $\mathrm{Pic}(X)$ is infinite cyclic, generated by the class of C. On the other hand, it is a well-known consequence of the Hodge-theory of K3 surfaces that for any genus $g \geq 2$ one can find a 19-dimensional family of algebraic K3 surfaces X with $\mathrm{Pic}(X) = \mathbb{Z}\cdot[C]$ for $C \subset X$ a curve of genus g. Hence the theorem implies Gieseker's result that some -- and hence the general -- curve of genus g does indeed satisfy Petri's condition.

Example 3.2.2. The statement of the theorem may very well fail if the linear series $|C|$ contains reducible or multiple members. In fact, suppose that $D \subset X$ is a curve of genus ≥ 2, and that $C \in |nD|$ for some integer $n \geq 2$. Then D determines a line bundle $A = \mathcal{O}_X(D)|C$ on C, and one checks that

$$\rho(A) = g(C) - h^0(A)\cdot h^1(A) < 0.$$

In other words, the Brill-Noether theorem (3.1.1) fails for A.

Example 3.2.3. One does not expect Petri's condition to hold for every curve $C' \in |C|$. For example, suppose that $X \subset \mathbb{P}^4$ is the complete intersection of a smooth quadric Q and a cubic hypersurface F. Let $L \subset \mathbb{P}^4$ be a three-plane tangent to Q at some point $q \in Q - X$, and consider the canonical curve $C = X \cap L \subset \mathbb{P}^3$. Then evidently C lies on a singular quadric in $\mathbb{P}^3$, and it is well-known that this means that Petri fails for C. On the other hand, thanks to the theorem of Noether-Lefschetz one can certainly arrange that $\mathrm{Pic}(X) = \mathbb{Z}\cdot[C]$.

The proof of Theorem 3.2.1 revolves around the study of certain vector bundles $F_{C,A}$ on X. Heuristically speaking, the role of these bundles is to encode information about the Brill-Noether theory of a given curve $C \subset X$ into a geometric object that exists globally on the K3 surface.

§3.3. The Vector Bundles $F_{C,A}$. As above C is a smooth irreducible curve on a K3 surface X. It will be convenient to define $V^r_d(C) \subset \mathrm{Pic}^d(C)$ to be the open subset of $W^r_d(C)$ consisting of line bundles A on C such that:

(i). $h^0(A) = r+1$, $\deg(A) = d$; and

(ii). Both A and $\omega_C \otimes A^*$ are generated by their global sections.

Fix now a line bundle $A \in V^r_d(C)$. We associate to the pair (C,A) a vector bundle $F_{C,A}$ on X, of rank $r+1$, as follows. Thinking of A as a sheaf on X, there is a canonical surjective evaluation map

$$e_{C,A} : H^0(A) \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow\!\!\!\!\to A$$

of $\mathcal{O}_X$-modules. Take

$$F_{C,A} =_{\text{def}} \ker e_{C,A}$$

to be its kernel. To see that $F_{C,A}$ is indeed a vector bundle, one can work locally, and suppose that $A \cong \mathcal{O}_C$. Then locally $e_{C,A}$ splits as the direct sum of the canonical map $\mathcal{O}_X \longrightarrow \mathcal{O}_C$ plus r copies of the zero map $\mathcal{O}_X \longrightarrow 0$. And since the kernel of $\mathcal{O}_X \longrightarrow \mathcal{O}_C$ is a line bundle on X, $F_{C,A}$ is locally free.

Our first task is to determine the elementary properties of these bundles. Setting $F = F_{C,A}$ one has by construction the exact sequence

(3.3.1) $$0 \longrightarrow F \longrightarrow H^0(A) \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow A \longrightarrow 0$$

of sheaves on X. Since $\omega_X = \mathcal{O}_X$, dualizing (3.3.1) gives:

(3.3.2) $$0 \longrightarrow H^0(A)^* \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow F^* \longrightarrow \omega_C \otimes A^* \longrightarrow 0 .$$

Lemma 3.3.3. One has:

(i). $c_1(F) = -[C]$, $c_2(F) = \deg(A) = d$;

(ii). F^* is generated by its global sections;

and

(iii). $H^0(F) = H^2(F^*) = 0$
$H^1(F) = H^1(F^*) = 0$
$h^0(F^*) = h^0(A) + h^1(A)$.

Proof. (i). The vector bundle map $F \longrightarrow \mathcal{O}^{r+1}$ in (3.3.1) drops rank along C, and hence $\det(F) = \mathcal{O}_X(-C)$. Let $s \in H^0(A)$ be a generic section, with divisor D, and let $W = H^0(A)/\mathbb{C}\cdot s$. Then (3.3.1) determines a vector bundle map $F \longrightarrow W \otimes_{\mathbb{C}} \mathcal{O}_X$ which drops rank exactly along the finite subscheme $D \subset X$. Hence $c_2(F) = [D]$.

(ii). This follows from (3.3.2). In fact, clearly F^* is globally generated away from C. But $\omega_C \otimes A^*$ is generated by its global sections by assumption, and every section of $\omega_C \otimes A^*$ lifts to a section of F^* thanks to the fact that $H^1(X, \mathcal{O}_X) = 0$.

(iii). It follows from the construction of $F_{C,A}$ that $H^0(F) = 0$, and that (3.3.1) is is surjective on global sections. Since $H^1(X, \mathcal{O}_X) = 0$, this latter fact implies that $H^1(F) = 0$. Hence $H^2(F^*) = H^1(F^*) = 0$ by Serre duality, and the last assertion is read off from (3.3.2). ∎

Example 3.3.4. After possibly twisting by a negative line bundle, any vector bundle F of rank $r+1$ on X is of the form $F_{C,A}$ for some curve $C \subset X$ and some line bundle A on C. In fact, suppose that $H^0(F) = H^1(F) = 0$, and that F^* is generated by its global sections. If $V \subset H^0(F^*)$ is a general subspace of dimension $r+1$, then the canonical map $ev : V \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow F^*$ will drop rank along a smooth curve C, and coker ev will be a line bundle on C, which we take to be $\omega_C \otimes A^*$.

It will turn out to be very important to be able to control the endomorphisms of $F_{C,A}$. We conclude this section with a numerical lemma in this direction:

Lemma 3.3.5. Writing $F = F_{C,A}$ as above, one has

$$\chi(F \otimes F^*) = 2\cdot h^0(F \otimes F^*) - h^1(F \otimes F^*) = 2 - 2\cdot\rho(A),$$

where $\rho(A) = g(C) - h^0(A)\cdot h^1(A)$.

Proof. The first equality follows from Serre duality. If E is a vector bundle of rank e on X, Riemann-Roch gives $\chi(E \otimes E^*) = (e-1)\cdot c_1(E)^2 - 2e\cdot c_2(E) + 2e^2$. The stated formula now follows by an explicit computation. ∎

Remark. It would be interesting to have a conceptual proof of 3.3.5.

§3.4. Irreducible Linear Systems, and the Proof of Theorem 3.2.1.(i). The presence or absence of reducible curves in the linear series $|C|$ comes into play via the following elementary but crucial:

Proposition 3.4.1. Fix a curve $C \subset X$, and a line bundle $A \in V^r_d(C)$, and set $F = F_{C,A}$. If F has non-trivial endomorphisms, i.e. if $h^0(F \otimes F^*) \geq 2$, then the linear series $|C|$ contains a reducible or multiple divisor.

Proof. Set $E = F^*$. Since $h^0(E \otimes E^*) \geq 2$, there exists by a standard argument a non-zero endomorphism $v : E \longrightarrow E$ which drops rank everywhere on X. In fact, take any endomorphism w of E, $w \neq (\text{const})\cdot 1$, and set $v = w - \lambda \cdot 1$, where λ is an eigenvalue of $w(x)$ for some $x \in X$. Then

$$\det(v) \in H^0(\det(E^*) \otimes \det(E)) = H^0(\mathcal{O}_X)$$

vanishes at x, and hence is identically zero.

Fixing such an endomorphism, let

$$N = \operatorname{im} v, \qquad M = \operatorname{coker} v,$$

and assume for simplicity that these are locally free. (The general case is only slightly more involved -- see [L1, S1].) Then one has an exact sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0,$$

and so

$$[C] = c_1(E) = c_1(N) + c_1(M)$$

in the Chow group $A_1(X) = \operatorname{Pic}(X)$. Thus it is enough to show that $c_1(N)$ and $c_1(M)$ are represented by non-zero effective curves. But N and M are quotients of E, and hence are generated by their global sections thanks to (3.3.3)(ii). This already implies that $c_1(N)$ and $c_1(M)$ are effective (or zero). Furthermore, since $H^0(E^*) = 0$, neither of these can be trivial vector bundles. But if U is a globally generated vector bundle on a projective variety, then it follows easily from Porteous' formula that $c_1(U) = 0$ if and only if U is trivial. Hence $c_1(N)$ and $c_1(M)$ must be represented by non-zero curves, and the lemma is proved. ∎

We now present two proofs of the first statement of Theorem 3.2.1. The first -- which appears in [L1] -- is very quick, but perhaps a bit unmotivated. The second, while longer, may strike the reader as more geometric. We start by recording the elementary

Lemma 3.4.2. Let A be a base-point free line bundle on a curve C, and let Δ be the divisor of base-points of $\omega_C \otimes A^*$. Then $A(\Delta)$ is also base-point free. ∎

First Proof of Theorem 3.2.1.(i). Suppose to the contrary that A is a line bundle on C with $\rho(A) < 0$. If Δ is the divisor of base-points of A, then $\rho(A(-\Delta)) < \rho(A)$. Hence replacing A by $A(-\Delta)$, we may suppose first of all that A is base-point free. Then, removing base-points from $\omega_C \otimes A^*$, it follows from (3.4.2) that we may assume that $\omega_C \otimes A^*$ is base-point free as well. Thus the vector bundle $F = F_{C,A}$ is defined. Since $\rho(A) < 0$, Lemma 3.2.5 yields

$$2 \cdot h^0(F \otimes F^*) \geq 2 - 2\cdot\rho(A) \geq 4,$$

Thus F has non-trivial endomorphisms, and it then follows from Proposition 3.4.1 that the linear series $|C|$ contains a reducible or multiple divisor. But this contradicts the hypotheses of the theorem. ∎

The second proof revolves around studying the family of all pairs (C, A) for which $F_{C,A}$ is isomorphic to a given vector bundle F. Thus let F be the vector bundle $F = F_{C_0,A_0}$ for some curve $C_0 \subset X$ and some line bundle $A_0 \in V^r_d(C_0)$. (Equivalently, as in Example 3.3.4 one could take F to be any bundle satisfying the conclusions of Lemma 3.3.3.) Put $E = F^*$. Then the exact sequence (3.3.2):

$$(*) \qquad 0 \longrightarrow H^0(A_0)^* \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow E \longrightarrow \omega_{C_0} \otimes A_0^* \longrightarrow 0$$

realizes $H^0(A_0)^*$ as an $(r+1)$-dimensional subspace of $H^0(E)$. The idea now is to vary this subspace.

We start by establishing some notation. Let

$$\mathcal{V}^r_d = \{ (C, A) \mid C \in |C_0| \text{ a smooth curve}, \ A \in V^r_d(C) \}$$

be the global family of V^r_d's over the linear series $|C_0| = \mathbb{P}^g$, and denote by

$$\pi : \mathcal{V}^r_d \longrightarrow |C_0|$$

the natural projection. Next, let

$$\mathbb{G} = \mathrm{Grass}(r+1, H^0(E))$$

be the Grassmannian of $(r+1)$-dimensional subspaces of $H^0(E)$. Given a subspace $V \subset H^0(E)$ of dimension $r+1 = \mathrm{rk}(E)$, denote by $e_V : V \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow E$ the homomorphism of vector bundles determined by evaluation of the sections in V. Since E is generated by its global sections, for general $V \subset H^0(E)$ it will be the case that e_V drops rank along a smooth curve C_V, and that $\mathrm{coker}(e_V)$ is a line bundle on C_V. Let $U \subset \mathbb{G}$ denote the Zariski-open subset of all such subspaces V. Then there is a morphism

(3.4.3) $$p : U \longrightarrow \mathcal{V}^r_d$$

defined by sending $V \subset H^0(E)$ to the pair $(C_V, \omega_{C_V} \otimes \mathrm{coker}(e_V)^*)$. The point of all this is that if A denotes the line bundle $\omega_C \otimes \mathrm{coker}(e_V)^*$ on $C = C_V$ determined by $p(V)$, then one has the familiar exact sequence

(*) $$0 \longrightarrow H^0(A)^* \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow E \longrightarrow \omega_C \otimes A^* \longrightarrow 0$$

of $\mathcal{O}_X$-modules.

Finally, we wish to study this construction infinitesimally. To this end, let

$$f : U \longrightarrow |C|$$

denote the composition $f = \pi \circ p$. Thus f maps a given subspace $V \subset H^0(E)$ to the curve $C = C_V = \{ \det(e_V) = 0 \}$, and if $A = \omega_C \otimes \mathrm{coker}(e_V)^*$, then $V = H^0(A)^*$. The

derivative of f at V now has a very simple interpretation. Namely, there is a canonical identification

$$T_V\,\mathbb{G} = T_V\,U = H^0(A) \otimes H^0(E)/H^0(A)^* = H^0(A) \otimes H^0(\omega_C \otimes A^*),$$

and one easily checks that

(3.4.4) $$df_V : T_V\,\mathbb{G} = H^0(A) \otimes H^0(\omega_C \otimes A^*) \longrightarrow T_C\,|C_0| = H^0(\omega_C)$$

is just the Petri homomorphism μ_A. Having said all this, we now give the

<u>Second Proof of Theorem 3.2.1.(i).</u> As in the first proof, we assume to the contrary that there exists a line bundle $A \in V^r_d(C)$ with $\rho(r,d,g) < 0$. The strategy is to argue geometrically that the vector bundle $E = (F_{C,A})^*$ has non-trivial endomorphisms. To this end, consider the diagram

$$\begin{tikzcd} U \arrow[rr, "p"] \arrow[dr, "f"'] & & \mathcal{V}^r_d \arrow[dl, "\pi"] \\ & |C_0| = \mathbb{P}^g & \end{tikzcd}$$

of maps introduced above. We claim that the fibres of the map p must have strictly positive dimension. In fact, pick a general point $V \in U$, giving rise to a curve $D \in |C|$ and a line bundle $B \in V^r_d(D)$ [i.e. assume that $p(V) = (D, B)$]. Set $Z = f^{-1}(f(V))$, and denote by δ the codimension in $H^0(\omega_D)$ of the image of the Petri map μ_B. Then it follows from (3.4.4) that

$$\dim Z = \dim U - g + \delta = -\rho(B) + \delta .$$

On the other hand, $\mathrm{im}(\mu_B)$ is the cotangent space at B to $W^r_d(D)$, and hence $\dim \pi^{-1}(D) \leq \delta$. But $\rho(B) = \rho(A) < 0$, and hence $\dim Z > \dim \pi^{-1}(D)$, which implies that p cannot be generically finite near V.

It follows that there are infinitely many subspaces $V \subset H^0(E)$ such that $\mathrm{coker}\, e_V \simeq \omega_D \otimes B^*$; equivalently, $\dim \mathrm{Hom}(E, \omega_D \otimes B^*) \geq 2$. But if one fixes a "reference" sequence

$$0 \longrightarrow H^0(B)^* \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow E \longrightarrow \omega_D \otimes B^* \longrightarrow 0,$$

one sees that every homomorphism $E \longrightarrow \omega_D \otimes B^*$ lifts to an endomorphism of E. Hence $\dim \mathrm{End}(E) \geq 2$, and as in the first proof this contradicts Proposition 3.4.1. ∎

§3.5. Mukai's Theorem and the Proof of Brill-Noether-Petri. In this section, we show how the second statement of Theorem 3.2.1 follows (at least morally speaking) from a theorem of Mukai. The argument is similar to that occurring in the second proof of statement (i) of (3.2.1), except that now one has to let the bundle $F_{C,A}$ vary.

Mukai's Theorem. As above, X is a complex projective K3 surface. Recall that a vector bundle E on X is simple if E has only trivial endomorphisms, i.e. if $\mathrm{Hom}(E, E) = \mathbb{C}$. The isomorphism classes of simple bundles with given rank and Chern classes are parametrized by a quasi-projective variety $M(r, c_1, c_2)$, and Mukai's theorem is:

Theorem 3.5.1 ([Muk1, Muk2]). *The moduli space* $M(r, c_1, c_2)$ *of simple bundles on the K3 surface* X *is smooth.*

Remark. Actually, Mukai proves much more: he shows that the same statement is true for the moduli space of simple sheaves, and he also proves that these moduli spaces are symplectic, i.e. carry a nowhere-degenerate 2-form [deduced from the pairing $\mathrm{Ext}^1(E, E) \times \mathrm{Ext}^1(E, E) \longrightarrow \mathrm{Ext}^2(E, E) = \mathbb{C}$ arising from Serre duality]. Furthermore, he shows that when it has dimension two, $M(r, c_1, c_2)$ is again a K3 surface, and -- remarkably -- is able to compute the periods of this surface in terms of the periods of X.

Sketch the proof of (3.5.1) The idea is to show that the obstructions to the smoothness of $M(r, c_1, c_2)$ vanish. To this end, fix a simple vector bundle E on X. The trace gives rise to a canonical split surjection

$$(*) \qquad E^* \otimes E \longrightarrow \mathcal{O}_X,$$

and the simplicity of E implies via Serre duality that

(**) The map $H^2(X, E^* \otimes E) \longrightarrow H^2(X, \mathcal{O}_X) = \mathbb{C}$ determined by (*) is bijective.

Fix now a local artinian $\mathbb{C}$-algebra R, let $I \subset R$ be a one-dimensional square-zero ideal, set $S = R/I$, and suppose given an infinitesimal deformation $\mathcal{E}$ of E parametrized by $\mathrm{Spec}(S)$, i.e. a vector bundle $\mathcal{E}$ on $X \times \mathrm{Spec}(S)$ restricting to E on the closed fibre. The obstruction to extending $\mathcal{E}$ to a deformation over $\mathrm{Spec}(R)$ is given by an element $\alpha_{(E)} \in H^2(X, E \otimes E^*)$. On the other hand, taking determinants yields a deformation $\det \mathcal{E}$ of $\det E$ over $\mathrm{Spec}(S)$, and one has a corresponding obstruction class $\alpha_{(\det)} \in H^2(X, \mathcal{O}_X)$. Furthermore, $\alpha_{(E)}$ maps to $\alpha_{(\det)}$ under the homomorphism determined by (*). But $\mathrm{Pic}(X)$ is smooth, and hence $\alpha_{(\det)} = 0$. Therefore $\alpha_{(E)} = 0$ thanks to (**), and this proves Mukai's theorem. ∎

Proof of Theorem 3.2.1 We now give at least a "physicist's proof" of the second statement (ii) of Theorem 3.2.1. We refer the reader to [L1] for a possibly more convincing but less geometric formulation of the argument.

We suppose that the linear series $|C|$ contains no reducible curves, and we will argue that the general member $C' \in |C|$ satisfies Petri's condition. As above, let

$$\mathcal{V}^r_d = \{ (C', A') \mid C' \in |C| \text{ a smooth curve, } A' \in V^r_d(C) \},$$

and denote by $\pi = \pi^r_d : \mathcal{V}^r_d \longrightarrow |C|$ the natural map. The first point to observe is that it suffices to prove

(3.5.2) For every r and d, the variety $\mathcal{V}^r_d$ is smooth of the expected dimension $g + \rho(r,d,g)$.

Indeed, it follows from (3.5.2) by the theorem on generic smoothness that if $C' \in |C|$ is a general curve, then $V^r_d(C')$ is smooth of the expected dimension $\rho(r,d,g)$ for every r and d. This implies that the Petri homomorphism $\mu_{A'}$ is injective for every line bundle A' on C' such that both A' and $\omega_{C'} \otimes A'^*$ are generated by their global sections. A small argument using Lemma 3.4.2 then shows that C' satisfies (3.1.3). [To make the connection between smoothness of $V^r_d(C')$ and the Petri homomorphism strictly correct, one should define and deal with suitable scheme structures on $V^r_d(C')$ and $\mathcal{V}^r_d$; we will ignore this point.]

The plan is to deduce (3.5.2) from Mukai's theorem. To this end, consider pairs (E, V), where E is a simple vector bundle of rank $r+1$ on X, with $c_1(E) = [C]$, $c_2(E) = d$, and $V \subset H^0(E)$ a subspace of dimension $r+1$. Let $\mathcal{G}$ denote the family of all such pairs; then there is a natural map

$$\pi : \mathcal{G} \longrightarrow M =_{def} M(r+1, [C], d)$$

which realizes $\mathcal{G}$ as a Grassmannian bundle over M. [Absent a universal bundle over the moduli space M, the existence of $\mathcal{G}$ and the nature of the map π is somewhat suspect, but we will ignore this point too.] By Mukai's theorem $\mathcal{G}$ is smooth, and it follows from (3.3.3) and (3.3.5) that

$$\dim \mathcal{G} = 2\cdot\rho(r,d,g) + (r+1)(g-d+r) = g + \rho(r,d,g).$$

To complete the proof, we will show that one can realize $\mathcal{V}^r_d$ as an open subset of $\mathcal{G}$. To this end, let $\mathcal{U} \subseteq \mathcal{G}$ denote the open set consisting of all pairs (E, V) such that:

(i). E is generated by its global sections, and $H^1(E) = H^2(E) = 0$;

(ii). The natural map $ev : V \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow E$ determined by V drops rank on a smooth curve C_V, and $\mathrm{coker}(ev)$ is a line bundle on C_V.

Given $(E, V) \in \mathcal{U}$, let $A_V = \mathrm{coker}(e^*_V)$, so that one has exact sequences

$$(*) \qquad 0 \longrightarrow E^* \longrightarrow V^* \otimes \mathcal{O}_X \longrightarrow A_V \longrightarrow 0$$

and

$$(**) \qquad 0 \longrightarrow V \otimes \mathcal{O}_X \longrightarrow E \longrightarrow \omega_C \otimes (A_V)^* \longrightarrow 0.$$

Then $A_V \in V^r_d(C_V)$ thanks to $(*)$, and so there is a morphism

$$f : \mathcal{U} \longrightarrow \mathcal{V}^r_d \quad \text{via} \quad (E, V) \mapsto (C_V, A_V).$$

By the same token one has a natural map

$$g : \mathcal{V}^r_d \longrightarrow \mathcal{U} \quad \text{via} \quad (C, A) \mapsto (E_{C,A}, H^0(A)^*),$$

where $E_{C,A} = (F_{C,A})^*$, and $H^0(A)^* \subset H^0(E_{C,A})$ is the canonical subspace arising from 3.3.2. (Note that we are using here Proposition 3.4.1 and the irreducibility hypothesis on $|C|$ to know that the bundles $E_{C,A}$ are simple.) Then evidently $f\circ g = \mathrm{id}$. To show that $g\circ f = \mathrm{id}$, it is enough in view of $(**)$ to check that $\dim \mathrm{Hom}(E, \omega_C \otimes A^*_V) = 1$. But this follows from the simplicity of E using $(**)$ and (3.3.3)(iii). Thus $\mathcal{V}^r_d \simeq \mathcal{U}$ is an open subset of $\mathcal{G}$, as claimed, and this completes the proof. ∎

§3.6. Notes.

(3.6.1). The construction of the bundles $F_{C,A}$ makes sense for a base-point free line bundle A on a curve C on any algebraic surface X. It had been used for instance by Maruyama to produce examples of bundles. Again the main novelty here has been to apply the bundles to study the curves. Tjurin [Tjr] analyzes the bundles $F_{C,A}$ -- in a somewhat different guise -- in a more general context. He also gives a third proof of Theorem 3.2.1(i) which uses the symplectic structure on the Hilbert scheme of points on a K3.

(3.6.2). For line bundles A with $r(A) = 1$, statement (i) of Theorem 3.2.1 had been proven independently by Donagi-Morrison and Reid [Rd1] using very different techniques. Our work on these questions in part grew out of an attempt to understand and generalize their result.

(3.6.3). By further arguments using the bundles $F_{C,A}$ one can prove the following

Theorem. ([GL3]). Let $C \subset X$ be a smooth irreducible curve of genus g. Then $\mathrm{Cliff}(C) = \mathrm{Cliff}(C')$ for all smooth curves C' in the linear series $|C|$. Furthermore if $\mathrm{Cliff}(C)$ is strictly less than the generic value $[(g-1)/2]$ then there exists a line bundle L on X whose restriction to every smooth curve $C' \in |C|$ computes the Clifford index of C'.

The first statement had been conjectured by Mumford, Harris and Green, and it was observed in [Gr] that the statement would follow immediately from Green's conjecture (2.4.3) on the syzygies of canonical curves. So the theorem may be seen as added evidence for that conjecture. Using similar methods Donagi and Morrison [DM] prove a somewhat stronger result starting from a line bundle A with $r(A) = 1$.

§4. Linear Series on Surfaces: Reider's Method.

Igor Reider [R] has shown that many of the basic theorems about linear series on surfaces can be obtained as elementary consequences of the Bogomolov instability theorem. This section is devoted to an exposition of Reider's work. Miyaoka [Myk] has recently given a very simple proof of Bogomolov's theorem. His argument deserves to be better known than it is, so we reproduce it in §4.3. (However, we will use a slightly stronger form of Bogomolov's theorem than what we prove here.)

§4.1. Reider's theorem and some of its applications. Throughout this section, X denotes a smooth projective surface and L a line bundle on X. Recall that L is numerically effective (nef) if $c_1(L)\cdot C \geq 0$ for every effective curve $C \subset X$.

Reider's main technical result deals with the linear series associated to the line bundle $K_X \otimes L$, where $K = K_X$ denotes as usual the canonical bundle on X. Under mild positivity hypotheses on L, it states that the failure of this adjoint series to be base-point free or very ample is accounted for by the existence of an effective divisor $D \subset X$ with special properties:

Theorem 4.1.1. (Reider [R]) Let L be a numerically effective line bundle on X.

(i). If $c_1(L)^2 \geq 5$, and if $p \in X$ is a base point of $K_X \otimes L$, then there exists an effective divisor $D \subset X$ passing through p such that either:

(a). $c_1(L)\cdot D = 0$ and $D^2 = -1$

or

(b). $c_1(L)\cdot D = 1$ and $D^2 = 0$

(ii). If $c_1(L)^2 \geq 10$, and if $p, q \in X$ are two points (possibly infinitely near) which fail to be separated by the linear series associated to $K_X \otimes L$, then there exists an effective divisor $D \subset X$ passing through p and q such that either:

(a). $c_1(L)\cdot D = 0$ and $D^2 = -1$ or -2

or

(b). $c_1(L)\cdot D = 1$ and $D^2 = -1$ or 0

or

(c). $c_1(L)\cdot D = 2$ and $D^2 = 0$.

Example 4.1.2. The example $X = \mathbb{P}^2$ and $L = \mathcal{O}_{\mathbb{P}^2}(2)$ shows that one can't in general improve on the numerical hypotheses of statement (i). Typical examples of a divisor D as described in (i) may be obtained by taking D to be an exceptional curve of the first kind (for (a)), or as the fibre of a ruled surface (for (b)).

Reider's theorem has a surprisingly wide range of applications. To give some feel the power of (4.1.1), we sketch two of these here following [R]. The reader should consult [R] for details and numerous other corollaries.

Pluricanonical Mappings. When X is a surface of general type, one is interested in understanding the pluricanonical rational mappings

$$\Phi_m = \Phi_{|mK|} \; : X \dashrightarrow \mathbb{P}(H^0(mK)).$$

defined by multiples of the canonical bundle. Here Reider is able to give an extremely quick proof of some fundamental results of Bombieri.

Theorem 4.1.3. (Bombieri, [Bmb]). Let X be a minimal surface of general type. Then

(i). Φ_m is a morphism (i.e. the linear series $|mK|$ is base-point free) if $m \geq 4$ or if $m \geq 3$ and $K^2 \geq 2$.

(ii). Φ_m <u>is an embedding away from</u> (-2)-<u>curves if</u> $m \geq 5$, <u>or if</u> $m \geq 4$ <u>and</u> $K^2 \geq 2$, <u>or if</u> $m \geq 3$ <u>and</u> $K^2 \geq 3$.

<u>Remark 4.1.4</u>. The meaning of the statement in (b) is that Φ_m is one-to-one and unramified away from a divisor $D \subset X$ consisting of smooth rational curves with self-intersection number -2. Reider in fact indicates a proof of the more precise result that under the stated hypotheses $|mK|$ is very ample on the canonical model of X obtained by blowing down D, and he also recovers a result of Francia concerning the structure of Φ_2.

<u>Proof of (4.1.3)</u>. When X is a minimal surface of general type, the canonical bundle K_X on X is nef (c.f. [BVP, p. 73]). So we may apply Theorem 4.1.1 with $L = (m-1)K_X$, the conditions on m and K^2 being exactly what is required to guarantee that the numerical hypotheses in that theorem are fulfilled. Thus if $|mK_X|$ has a base-point, there exists an effective divisor D on X such that either $(m-1)K \cdot D = 1$ or else $(m-1)K \cdot D = 0$ and $D^2 = -1$. The first possibility is evidently excluded by the assumption that $(m-1) \geq 2$. As for the second, if $K \cdot D = 0$ then $D^2 = D \cdot (D+K) \equiv 0 \pmod 2$ by adjunction, contradicting $D^2 = -1$. This proves statement (i) of the theorem.

A similar argument shows that if mK_X fails to be very ample, so that there exists an effective divisor $D \subset X$ satisfying one of the conditions of (4.1.1)(ii), then the only possibility is that $K \cdot D = 0$ and $D^2 = -2$. But in this case a standard argument shows that each irreducible component of D is a smooth rational curve with self-intersection -2. [In fact, write $D = \Sigma D_i$. Since K is nef, $K \cdot D = 0 \Rightarrow K \cdot D_i = 0$ for all i. But then since $K^2 > 0$ the index theorem yields $(D_i)^2 = (K + D_i) \cdot D_i < 0$, which implies the stated result.] ∎

<u>Surfaces of Kodaira dimension zero</u>. Suppose now that X is a minimal surface with $\kappa(X) = 0$. Then K_X is a torsion line bundle, and so by adjunction the intersection form on X is even, i.e. $D^2 \equiv 0 \pmod 2$ for every divisor D on X. Recall that an <u>elliptic cycle</u> on X is by definition a 1-connected effective divisor $D \subset X$ with $p_a(D) = 1$ [or equivalently, since K is numerically trivial, $D^2 = 0$.]

<u>Theorem 4.1.5.</u> (Beauville). <u>Let</u> L <u>be a numerically effective line bundle on</u> X.

(i). <u>Assume that</u> $c_1(L)^2 \geq 6$. <u>If</u> $K_X \otimes L$ <u>fails to be base-point free, then</u> X <u>contains an elliptic cycle</u> D.

(ii). <u>If</u> $c_1(L)^2 \geq 10$, <u>and if the mapping</u> $\Phi_{K \otimes L}$ <u>fails to be an embedding outside of</u> (-2)-<u>curves, then</u> X <u>contains an elliptic cycle</u> D <u>with</u> $c_1(L) \cdot D = 1$ or 2.

<u>Remark.</u> Reider shows that the converses of these statements also hold.

<u>Sketch of proof of (4.1.5)</u>. We content ourselves with outlining an argument for (i). Assume that $K \otimes L$ has a base-point. Bearing in mind that the intersection form on X is even, Reider's theorem (4.1.1) yields an effective divisor D such that

$$D \cdot c_1(L) = 1 \quad \text{and} \quad D^2 = 0.$$

To complete the proof, it remains only to show that D is 1-connected. In fact, suppose to the contrary that $D = D_1 + D_2$ for some non-zero effective divisors D_i with $D_1 \cdot D_2 \leq 0$. Then

$$(*). \qquad 0 = D^2 = (D_1)^2 + 2D_1 \cdot D_2 + (D_2)^2 \leq (D_1)^2 + (D_2)^2 .$$

Now since L is nef and $c_1(L) \cdot D = 1$, we can assume that

$$c_1(L) \cdot D_1 = 1 \quad \text{and} \quad c_1(L) \cdot D_2 = 0 .$$

The index theorem then implies that $(D_2)^2 < 0$, and so $(D_1)^2 > 0$ thanks to (*). Now let $d = c_1(L)^2 \geq 6$. Then $(c_1(L) - dD_1) \cdot c_1(L) = 0$ and so $(c_1(L) - dD_1)^2 \leq 0$ by the index theorem again. Recalling that $c_1(L) \cdot D_1 = 1$ it follows that $d(D_1)^2 \leq 1$, which is impossible. ∎

§4.2. Stable bundles and Bogomolov's Theorem. In this section we review some of the basic facts required concerning stable and unstable vector bundles.

We start by recalling the definitions in the one dimensional case. Thus let C be a smooth irreducible projective curve over $\mathbb{C}$, and let E be a vector bundle on C of rank e. The *slope* of E is the rational number

$$\mu(E) = \deg(E) / \mathrm{rk}(E).$$

One says that E is *stable* [resp. *semistable*] if for every non-zero coherent subsheaf $F \subset E$ with $\mathrm{rk}(F) < \mathrm{rk}(E)$ one has the inequality

$$\mu(F) < \mu(E) \quad [\text{ resp. } \mu(F) \le \mu(E)].$$

E is *unstable* if it is neither stable nor semistable. (The concept of stability -- which was introduced by Mumford -- arises when one tries to construct moduli spaces for vector bundles; it is only by restricting to semistable bundles that one gets a reasonable theory. It turns out that stability also has a natural differential geometric interpretation. The reader may consult for instance [Nev] or [Kob] for a fuller discussion.) Among the basic properties of stability, we shall need the following:

(4.2.1) (i). Stability and semi-stability are invariant under twisting by line bundles and under replacing E by its dual E^*.

(ii). If E is semistable, and $\deg(E) < 0$, then $H^0(E) = 0$.

(iii). If E is semistable then so is its symmetric power $S^n(E)$ for every $n \ge 1$.

Facts (i) and (ii) are immediate; for (iii), see [Myk].

Now suppose that E is a vector bundle on an projective algebraic surface X. We fix an ample divisor H on X, and define the H-*slope* of E to be

$$\mu_H(E) = c_1(E) \cdot H / \mathrm{rk}(E).$$

(The slope of an arbitrary torsion-free subsheaf is defined analogously.) One says that E is H-*stable* [resp. H-*semistable*] if

$$\mu_H(F) < \mu_H(E) \quad [\text{resp. } \mu_H(F) \le \mu_H(E)]$$

for every torsion-free subsheaf $F \subset E$ with $\mathrm{rk}(F) < \mathrm{rk}(E)$; E is H-*unstable* if it is neither stable nor semistable. The connection with stability on curves is given by a result of Mumford, Mehta and Ramanathan:

(4.2.2) Let E be an H-semistable bundle on X. Then for all sufficiently large integers $m >> 0$, and for all sufficiently general smooth divisors $D \in | mH |$, the restriction $E|D$ of E to the curve D is a semistable bundle on D.

See [MR] for details.

The basic result for which we are aiming is a theorem of Bogomolov which gives a numerical criterion for instability:

Theorem 4.2.3. ([Bogomolov]). *Let* E *be a bundle of rank* e *on a surface* X. *If*

$$c_1(E)^2 > \frac{2e}{(e-1)} c_2(E)$$

then E *is* H-*unstable with respect to any ample divisor* H *on* X.

Remark 4.2.4. For future reference, let us spell out explicitly what this says when $\mathrm{rk}(E) = 2$. Let H be any ample divisor on X, and suppose that

$$c_1(E)^2 > 4 \cdot c_2(E).$$

Then there exists a rank one torsion-free subsheaf $A \subset E$ with

$$(*) \qquad 2 \cdot c_1(A) \cdot H < c_1(E) \cdot H.$$

Possibly replacing A by its double dual A^{**}, we may suppose first of all that A is a line bundle. Then if necessary replacing A by $A(\Delta)$ for some effective divisor Δ, we may suppose that the bundle homomorphism $A \longrightarrow E$ determined by inclusion of sheaves vanishes only on a finite subscheme $Z \subset X$. Putting $B = L \otimes A^*$ we finally obtain an exact sequence

$$(4.2.5) \qquad 0 \longrightarrow A \longrightarrow E \longrightarrow B \otimes \mathcal{I}_{Z/X} \longrightarrow 0,$$

where $\mathcal{I}_{Z/X}$ is the ideal sheaf of Z in X. Note that then (*) is equivalent to the inequality

$$(4.2.6) \qquad (c_1(A) - c_1(B)) \cdot H > 0.$$

In fact, more is true: Bogomolov proves that one can find A and B so that (4.2.6) holds simultaneously for every ample H, and that

$$(4.2.7) \qquad (c_1(A) - c_1(B))^2 > 0$$

(see [R]). In other words, the divisor class $c_1(A) - c_1(B)$ is in the positive cone of the Neron Severi group $NS(X)_{\mathbb{R}}$, ~~i.e. it is a limit of ample divisors~~. It is in this stronger form that we will use Bogomolov's theorem. See [Rd2] for a nice discussion and a proof.

§4.3. Miyaoka's proof of Bogomolov's theorem. The idea -- which goes back to Bogomolov -- is to study for $n \geq 1$ the auxiliary bundle

$$F_n = S^{ne}(E) \otimes (\det E^*)^{\otimes n}.$$

Lemma 4.3.1. (i). $\det F_n = \mathcal{O}_X$

(ii). $\chi(X, F_n) = \dfrac{(ne)^{e+1}}{(e+1)!} \left\{ \dfrac{e-1}{2e} \cdot c_1(E)^2 - c_2(E) \right\} + P(n),$

where $P(n)$ is a polynomial in n of degree $\leq e$.

Proof. (i). To avoid a calculation, one can for instance argue that it suffices to prove this under the assumption that $\det E = L^{\otimes e}$ for some line bundle L on X. But in this case the assertion is clear, since then $S^{ne}(E) \otimes (\det E^*)^{\otimes n} = S^{ne}(E \otimes L^*)$, and $E \otimes L^*$ has trivial determinant.

(ii). Consider the line bundle

$$A = \mathcal{O}_{\mathbb{P}(E)}(e) \otimes \pi^* \det(E)^*$$

on the projectivization $\pi : \mathbb{P}(E) \longrightarrow X$ of E, and let

$$\xi = c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) \quad \text{and} \quad c_i = \pi^*(c_i(E)).$$

Then

$$\chi(X, F_n) = \chi(\mathbb{P}(E), A^{\otimes n}),$$

while by Riemann-Roch

$$\chi(\mathbb{P}(E), A^{\otimes n}) = \frac{n^{e+1}}{(e+1)!} (e \cdot \xi - c_1)^{e+1} + O(n^e).$$

But using the relation $\xi^e - c_1 \cdot \xi^{e-1} + c_2 \cdot \xi^{e-2} = 0$, one finds that

$$\left(\xi - \frac{c_1}{e}\right)^{e+1} = \frac{e-1}{2e} c_1^2 - c_2,$$

and the result follows. ∎

Proof of Theorem 4.2.3. (Miyaoka). Fix an ample divisor H on X, and let E be a bundle of rank e. Note first off that it suffices to prove:

(4.3.2). If E is H-semistable then one has the bounds

$$h^0(X, F_n) \leq Q(n) \quad \text{and} \quad h^2(X, F_n) \leq R(n),$$

where $Q(n)$ and $R(n)$ are polynomials of degree $\leq e-1$ in n.

In fact, if the inequality in the statement of Bogomolov's theorem holds, then it follows from the previous lemma that $\chi(X, F_n)$ grows like n^{e+1}. But this is inconsistent with (4.3.2), and hence E cannot be semi-stable. Next, observe that (4.3.2) is in turn a consequence of

(4.3.3). If E is H-semistable, then for any line bundle L on X one has

$$h^0(X, F_n \otimes L) \leq Q_L(n),$$

where $Q_L(n)$ is a polynomial of degree $\leq e-1$ in n depending on L.

Indeed, this obviously implies the first bound in (4.3.2), while by Serre duality

$$h^2(X, F_n) = h^0(X, F_n^* \otimes \omega_X).$$

But since we are in characteristic zero,

$$(F_n)^* = S^{ne}(E^*) \otimes (\det E)^{\otimes n},$$

and so the required bound on H^2 follows by applying (4.3.3) to the H-semistable bundle E^*.

So it remains only to prove (4.3.3). To this end we start by proving the analogous statement for bundles on a curve:

(4.3.4) Let C be a smooth irreducible projective curve, let E be a semistable bundle of rank e on C, and let $F_n = S^{ne}(E) \otimes (\det E^*)^{\otimes n}$. Then for any line bundle L on C one has $h^0(C, F_n \otimes L) \leq Q_L(n)$, for some polynomial $Q_L(n)$ of degree $\leq e-1$ in n.

In fact, choose an effective divisor $D \subset C$ of degree $d > \deg(L)$, and consider the exact sequence:

$$0 \longrightarrow (F_n \otimes L)(-D) \longrightarrow F_n \otimes L \longrightarrow (F_n \otimes L) \mid D \longrightarrow 0 .$$

Then $(F_n \otimes L)(-D)$ is a semistable bundle of negative degree (since $\deg F_n = 0$), and hence has no global sections. Thus

$$h^0(C, F_n \otimes L) < h^0(D, (F_n \otimes L) \otimes \mathcal{O}_D) = d \binom{ne+e-1}{e-1},$$

as claimed.

Returning to the situation of (4.3.3), we invoke the theorem (4.2.2) of Mumford, Mehta and Ramanathan to conclude that for all sufficiently large integers m and all sufficiently general curves $C \in |mH|$, the restriction $E|C$ of E to C is semistable. Then $F_n|C$ is also semistable thanks to (4.2.1)(iii), and by taking $m >> 0$ we may assume that $L(-mH)$ is a negative line bundle, and hence that

$$c_1((F_n \otimes L)(-mH)) \cdot mH < 0$$

for all n. It then follows from (4.2.1)(ii) that $H^0(C, (F_n \otimes L)(-C) \otimes \mathcal{O}_C) = 0$. But this holds for the general element $C \in |mH|$, and therefore $H^0(X, (F_n \otimes L)(-C)) = 0$. Hence

$$h^0(X, F_n \otimes L) \leq h^0(C, (F_n \otimes L)|C),$$

and the required bound on $h^0(X, F_n \otimes L)$ now follows from (4.3.4). ∎

§4.4. **Proof of Reider's Theorem.** We will focus on statement (i) of (4.1.1). The first step in Reider's argument is to use an old construction of Serre's to pass from the hypotheses of the theorem to the existence of a suitable rank two vector bundle E on X. (See for instance [OSS] or [GH2] for a general description of Serre's construction.) Specifically, suppose that $p \in X$ is a base-point of $K_X \otimes L$. Then p determines in the natural way a non-zero cohomology class $\alpha_p \in H^1(X, \mathcal{J}_p \otimes K_X \otimes L)$ mapping to zero in $H^1(X, L)$, where $\mathcal{J}_p$ denotes the ideal sheaf of p. On the other hand, Grothendieck duality yields an isomorphism

$$H^1(X, \mathcal{J}_p \otimes K_X \otimes L)^* = \mathrm{Ext}^1(\mathcal{J}_p \otimes L, \mathcal{O}_X),$$

and it follows that the evident map $\eta : \mathrm{Ext}^1(L, \mathcal{O}_X) \longrightarrow \mathrm{Ext}^1(\mathcal{J}_p \otimes L, \mathcal{O}_X)$ is not surjective. Choose an element $e \in \mathrm{Ext}^1(\mathcal{J}_p \otimes L, \mathcal{O}_X)$ not lying in the image of η. Then e defines a non-split extension

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\ s\ } E \longrightarrow \mathcal{J}_p \otimes L \longrightarrow 0, \tag{4.4.1}$$

and the hypothesis on e implies that the sheaf E appearing in (4.4.1) is locally free. In short, we have produced a rank two vector bundle E on X, with $\det E = L$, having a section $s \in H^0(X, E)$ vanishing precisely on p.

Note that

$$c_1(E) = c_1(L) \quad \text{and} \quad c_2(E) = 1.$$

But then the hypothesis $c_1(L)^2 \geq 5$ means that $c_1(E)^2 - 4 \cdot c_2(E) > 0$, and hence E is unstable with respect to any ample divisor H thanks to Bogomolov's theorem (4.2.3). Using the version of that theorem stated in (4.2.4), we conclude the existence of a diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & A \arrow[d] \arrow[dr, "t'"] & & \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[dr, "t"'] & E \arrow[r] \arrow[d] & \mathcal{J}_p \otimes L \arrow[r] & 0, \\
 & & B \otimes \mathcal{J}_Z \arrow[d] & & \\
 & & 0 & &
\end{tikzcd} \tag{4.4.2}$$

where $Z \subset X$ is a finite set, $\mathcal{J}_Z$ its ideal sheaf, and where A and B are line bundles such that

$$A \otimes B = L,$$

and $c_1(A) - c_1(B)$ is in the positive cone of $NS(X)_{\mathbb{R}}$, i.e.

$$(c_1(A) - c_1(B)) \cdot H > 0 \quad \text{for every ample } H,$$

and

$$(c_1(A) - c_1(B))^2 > 0.$$

We claim that the maps t (and hence also t') appearing in the diagram (4.4.2) are non-zero. For otherwise there would exist a non-zero homomorphism $L \longrightarrow B$, but $c_1(B) - c_1(L) = -c_1(A)$ cannot be represented by an effective divisor since $2c_1(A) \cdot H > c_1(L) \cdot H \geq 0$ for any ample H. Let $D \subset X$ be the effective divisor defined by the vanishing of t [or equivalently of t']. Then D passes through p thanks to the fact that the section $s \in H^0(E)$ in (4.1.1) vanishes at p, and one has

$$B = \mathcal{O}_X(D) \quad \text{and} \quad A = L(-D).$$

The point now is to show that D satisfies the numerical conditions in the statement of the theorem.

To this end, we claim first that:

$$0 \leq D \cdot (c_1(L) - D) \leq 1. \tag{4.4.3}$$

In fact, let D_i be an irreducible component of D, and consider the restriction of the diagram (4.4.2) to D_i. Since the map t vanishes on D_i, one obtains an injective homomorphism of sheaves $0 \longrightarrow \mathcal{O}_{D_i} \longrightarrow L(-D) \otimes \mathcal{O}_{D_i}$. But this implies that $D_i \cdot (c_1(L) - D) \geq 0$, and it follows that $D \cdot (c_1(L) - D) \geq 0$. For the other inequality, computing from the vertical sequence in (4.4.2) yields *

$$1 = c_2(E) = D \cdot (c_1(L) - D) + \deg(Z),$$

and (4.4.3) follows.

The next point to observe is

* No! Argue as follows. Lemma L nef $L \cdot E + F$ ($E, F \ne 0$)
$\Rightarrow E \cdot F \geq 0$ ($L \cdot E \geq 0$, $L \cdot F \geq 0$, then HIT)
$L(-2D)$ in pos cone $\Rightarrow$ $n(L(-2D))$ eff $\Rightarrow$ $n(L(-D))$ eff
$n \gg 0$ nD eff

(4.4.4). $$D^2 \le 0 .$$

In fact, the inequality $c_1(L)^2 = (c_1(L(-D)) + D)^2 \ge 5$ plus the upper bound on $c_1(L(-D))\cdot D$ given in (4.4.3) implies

(*) $$D^2 + c_1(L(-D))^2 \ge 3.$$

On the other hand, $c_1(L(-2D)) = c_1(A) - c_1(B)$ is ~~belonging~~ in the positive cone of $NS(X)_{\mathbb{R}}$ -- ~~is a limit of ample divisors~~, and L is nef. Consequently

$$c_1(L)\cdot c_1(L(-2D)) \;=\; (c_1(L(-D)) + D)\cdot(c_1(L(-D)) - D) \;\ge\; 0,$$

or equivalently $c_1(L(-D))^2 \ge D^2$. Plugging this into (*) yields

(**) $$c_1(L(-D))^2 \ge 2.$$

But now apply the Hodge index theorem and (4.4.3):

$$c_1(L(-D))^2 \cdot D^2 \;\le\; (\, c_1(L(-D))\cdot D \,)^2 \;\le\; 1 .$$

The desired inequality (4.4.4) then follows immediately from (**).

Reider's theorem now follows easily. Indeed, since L is nef and D is effective we have $c_1(L)\cdot D \ge 0$. Then (4.4.3) yields $-D^2 \le D\cdot(c_1(L) - D) \le 1$, and hence

$$D^2 = 0 \quad\text{or}\quad -1$$

thanks to (4.4.4). We consider these two cases separately. Suppose first that $D^2 = 0$. Then $0 \le D\cdot c_1(L) \le 1$ by (4.4.3). But the index theorem rules out the possibility $D^2 = D\cdot(c_1(L)) = 0$, so $c_1(L)\cdot D = 1$, as claimed. Finally, suppose that $D^2 = -1$. Recalling that $c_1(L)\cdot D \ge 0$ it follows immediately from (4.4.2) that $c_1(L)\cdot D = 1$, and this completes the proof of statement (i) of Reider's theorem.

The proof of part (ii) of the theorem is similar. Assuming that $K_X \otimes L$ is generated by its global sections but fails to separate two points $p, q \in X$, one constructs a rank two vector bundle E as an extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow \mathcal{I}_{\{p,q\}} \otimes L \longrightarrow 0 .$$

One has $c_1(E) = c_1(L)$ and $c_2(E) = 2$, so E is unstable as soon as $c_1(L)^2 \ge 9$. Just as above this leads to the existence of an effective divisor $D \subset X$, and arguments as above show that D has the required properties. (The hypothesis $c_1(L)^2 \ne 9$ is used to rule out the possibility that $D^2 = 1$ and $L = \mathcal{O}_X(3D)$.) We refer to Reider's paper [R] for details. ∎

§4.5. Notes.

(4.5.1). Reider's method has its antecedents in a proof of the Ramanujan vanishing theorem given by Mumford, and reproduced in [Rd2]. Specifically, given a numerically effective line bundle L on a surface X, with $c_1(L)^2 > 0$, one wants to show that $H^1(X, L^*) = 0$. If not, there exists a non-split extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow L \longrightarrow 0,$$

and the hypothesis $c_1(L)^2 = c_1(E)^2 - 4\cdot c_2(E) > 0$ yields the instability of E. An argument with the Hodge index theorem now yields a contradiction.

(4.5.2). Sakai [Sk] has recently shown that one can prove Reider's result using a vanishing theorem of Miyaoka. In fact, he recovers (4.1.1) as a consequence of a theorem stating that if L with Iitaka-Kodaira dimension $\kappa(X,L) = 2$ (i.e. L is "big") and $c_1(L)^2 > 0$, and if $H^1(X, L^*) \ne 0$, then there exists a non-zero effective divisor D on X such that $D\cdot(c_1(L) - D) \le 0$, and $L(-2D)$ is again a big line bundle.

(4.5.3). As Mukai remarks, Reider's theorem suggests that on a surface X, the adjoint bundles $K_X \otimes L$ with L suitably positive have good properties analogous to line bundles of degree greater than $2g + \epsilon$ (ϵ a small positive integer) on a curve. This leads Mukai to raise the very interesting question of whether there are results on the syzygies of the bundles $K_X \otimes L$, analogous to to the theorem of Green for curves discussed in §1. To begin with, one can ask whether there are any useful conditions on L that imply the normal generation of $K_X \otimes L$ (possibly with a few exceptions).

References.

[ACGH] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris, Geometry of Algebraic Curves, Vol 1, Springer 1985.

[Asc] M.-G. Ascenzi, The restricted tangent bundle of a rational curve on a quadric in $\mathbb{P}^3$, Proc. AMS 98 (1986), pp. 561-566.

[Bal] E. Ballico, On the Clifford index of algebraic curves, Proc. AMS 97 (1986), pp. 217 -218.

[BE] E. Ballico and P. Ellia, On postulation of curves: embeddings by complete linear series, Arch. Math. 43 (1984), pp. 244 - 249.

[BVP] W. Barth, C. Peters and A. Van de Ven, Compact Complex Surfaces, Springer, 1984.

[Bmb] E. Bombieri, Canonical models of surfaces of general type, Publ. Math. IHES 42 (1973), pp. 171-219.

[Cast] G. Castelnuovo, Sui Multipli di uni serie lineare di gruppi di punti appartenente ad una curva algebrica, Rend. Circ. Mat. Palermo 7 (1893), pp. 99-119.

[DM] R. Donagi and D. Morrison, Linear systems on K3 sections, to appear.

[En] L. Ein, Some remarks on the syzygies of general canonical curves, J. Diff. Geom. 26 (1987), pp. 361-366.

[ELMS] D. Eisenbud, H. Lange, G. Martens and F. Schreyer, The Clifford dimension of a projective curve, I, preprint.

[Fjt] T. Fujita, Defining equations for certain types of polarized varieties, in: Complex Analysis and Algebraic Geometry (Cambridge Univ. Press, 1977), pp. 165 - 173.

[FL] W. Fulton and R. Lazarsfeld, On the connectedness of degeneracy loci and special divisors, Acta Math. 146 (1981) pp. 271 - 283.

[Gies] D. Gieseker, Stable curves and special divisors: Petri's conjecture, Invent. Math 66 (1982), pp. 251 - 275.

[Gr] M. Green, Koszul cohomology and the geometry of projective varieties, J. Diff. Geom. 19 (1984), pp. 125-171.

[GL1] M. Green and R. Lazarsfeld, On the projective normality of complete linear series on an algebraic curve, Invent. Math. 83 (1986), pp. 73-90.

[GL2] M. Green and R. Lazarsfeld, A simple proof of Petri's theorem on canonical curves, In Geometry Today, Progress in Math., Birkhauser (1986).

[GL3] M. Green and R. Lazarsfeld, Special divisors on curves on a K3 surface, Invent. Math. 89 (1987), pp. 357-370.

[GL4] M. Green and R. Lazarsfeld, Some results on the syzygies of finite sets and algebraic curves, Compositio Math. 67 (1988), pp. 301-314.

[GH1] P. Griffiths and J. Harris, The dimension of the variety of special linear systems on a general curve, Duke math. J. 47 (1980) pp. 233 - 272.

[GH2] P. Griffiths and J. Harris, Residues and zero-cycles on algebraic varieties, Ann. Math. 108 (1978), pp. 4621-505.

[GLP] L. Gruson, R. Lazarsfeld, and C. Peskine, On a theorem of Castelnuovo and the equations defining space curves, Invent. Math. 72 (1983), pp. 491-506.

[GP1] L. Gruson and C. Peskine, Postulation des courbes gauches, in Algebraic Geometry -- Open Problems, Lect. Notes in Math. 997 (1983), pp. 218 - 227.

[GP3] L. Gruson and C. Peskine, Courbes de l'espace projectif: variétés de sécantes, in Enumerative Geometry and Classical Algebraic Geometry, Progress in Math. 24 (1982), pp. 1-31.

[Kmf1] G. Kempf, Schubert methods with an application to algebraic curves, Publ. Math. Centrum, Amsterdam (1971)

[Kmf2] G. Kempf, The projective coordinate ring of abelian varieties, preprint

[KL] S. Kleiman and D. Laksov, On the existence of special divisors, Am. J. Math. 94 (1972), pp. 431 - 436.

[Kob] S. Kobyashi, Differential Geometry of Complex Vector Bundles, Princeton Univ. Press (1988).

[Koiz] S. Koizumi, Theta relations and projective normality of abelian varieties, Am. J. Math 98 (1976), pp. 865 - 889.

[KS] K. Koh and M. Stillman, to appear

[LM] H. Lange and G. Martens, Normal generation and presentation of line bundles of low degree, J. Reine Angew Math. 356 (1985), pp. 1-18.

[LN] H. Lange and M. Narasimhan, Maximal subbundles of rank two vector bundles on curves, Math. Ann. 266 (1983), pp. 55 - 72.

[L1] R. Lazarsfeld, Brill-Noether-Petri without degenerations, J. Diff. Geom. 23 (1986), pp. 299-307.

[L2] R. Lazarsfeld, A Sharp Castelnuovo bound for smooth surfaces, Duke Math. J. 55 (1987), pp. 423-428.

[M1] G. Martens, Uber den Clifford-Index algebraischer Kurven, J. Reine Angew. Math. 320, (1980) pp. 68 - 85.

[Mat] A. Mattuck, Symmetric products and Jacobians, Am. J. Math 83 (1961), pp. 189-206.

[MR] V. Mehta and A. Ramanathan, Semi-stable sheaves on projective varieties and their restrictions to curves, Math. Ann. 258 (1982), pp. 213-224.

[Myk] Y. Miyaoka, The Chern classes and Kodaira dimension of a minimal variety, in Algebraic Geometry -- Sendai 1985, Adv. Studies in Pure Math 10, pp. 449-476.

[Muk1] S. Mukai, Sympletic structure of the moduli space of sheaves on an abelian or K3 surface, Invent. Math 77 (1984), pp. 101 - 116.

[Muk2] S. Mukai, On the moduli space of bundles on K3 surfaces, I, in: Vector bundles on algebraic varieties (Oxford Univ. Press) pp. 341 - 413.

[Muk3] S. Mukai, Curves, K3 surfaces and Fano 3-folds of genus ≤ 10, to appear.

[MS] S. Mukai and F. Sakai, Maximal subbundles of vector bundles on a curve, Manuscripta Math. 52 (1985), pp. 251 - 256.

[Mfd1] D. Mumford, On the equations defining abelian varieties, Invent. Math 1 (1966), pp. 287 - 354.

[Mfd2] D. Mumford, Lectures on curves on an algebraic surface, Annals of Math. Studies, No. 59 (1966).

[Mfd3] D. Mumford, Varieties defined by quadratic equations, Corso CIME 1969, in: Questions on algebraic varieties, Rome (1970), pp. 30 - 100.

[N] M. Nagata, On selfintersection number of vector bundles of rank 2 on Riemann surface, Nagoya Math. J. 37 (1970), pp. 191 - 196.

[New] P. Newstead, Introduction to moduli problems and moduli spaces, Tata Lectures on Math 51.

[OSS] C. Okoneck, H. Spindler and M. Schneider, Vector Bundles on Complex Projective Spaces, Progress in Math. 3 (1980), Birkhauser.

[P] H. Pinkham, A Castelnuovo bound for smooth surfaces, Invent. Math. 83 (1986), pp. 321-332.

[PR] A. Paranjape and S. Ramanan, On the canonical ring of an algebraic curve, to appear.

[Rd1] M. Reid, Special linear systems on curves lying on a K3 surface, J. London Math. Soc. 13 (1976), pp. 454 - 458.

[Rd2] M. Reid, Bogomolov's theorem $c_1^2 \leq 4c_2$, in Proc. Intern. Symp. on Alg. Geom., Kyoto (1977), pp. 633 - 642.

[R] I. Reider, Vector bundles of rank 2 and linear systems on algebraic surfaces, Ann. Math. 127 (1988), pp. 309-316.

[Sak] F. Sakai, Reider-Serrano's method on normal surfaces, to appear.

[Sch1] F. Schreyer, Syzygies of canonical curves and special linear series, Math. Ann. 275 (1986), pp. 105 - 137.

[Sch2] F. Schreyer, A standard basis approach to the syzygies of canonical curves, to appear

[Sek] T. Sekiguchi, On the normal generation by a line bundle on an abelian variety, Proc. Jap. Acad. 54 (1978), pp. 185-188.

[Szpr] L. Szpiro, Travaux de Kempf, Kleiman, Laksov sur les diviseurs exceptionelles, Sem. Bourbaki, June 1972, exp. 417.

[St.-D1] B. St.-Donat, On Petri's analysis of the linear system of quadrics through a canonical curve, Math. Ann. 206 (1973), pp. 157 - 175.

[St.-D2] B. St.-Donat, Sur les equations définissant une ccourbe algébrique, C.R. Acad. Sci. Paris, Ser. A 274 (1972), pp. 324-327.

[Tjr] A. Tjurin, Cycles, curves and vector bundles on an algebraic surface, Duke Math J. 55 (1987)

[Vsn] C. Voisin, Courbes tétragonales et cohomologie de Koszul, J. Reine angew. Math. 387 (1988), pp. 111 - 121.