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# Derivative complex, BGG correspondence, and numerical inequalities for compact Kähler manifolds

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## 1 Introduction

Given an irregular compact Kähler manifold $X$, one can form the *derivative complex* of $X$, which governs the deformation theory of the groups $H^i(X,\alpha)$ as $\alpha$ varies over $\mathrm{Pic}^0(X)$. Together with its variants, it plays a central role in a body of work involving generic vanishing theorems. The purpose of this paper is to present two new applications of this complex. First, we show that it fits neatly into the setting of the so-called Bernstein-Gel'fand-Gel'fand (BGG) correspondence between modules over an exterior algebra and linear complexes over a symmetric algebra (cf. [3, 9, 11, 12]). What comes out here is that some natural cohomology modules associated to $X$ have a surprisingly simple algebraic structure, and that conversely one can read off from these modules some basic geometric invariants associated to the Albanese mapping of $X$. Secondly, under an additional hypothesis, the derivative complex determines a vector bundle on the projectivized tangent space to $\mathrm{Pic}^0(X)$ at

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the origin. We show that this bundle, which encodes the infinitesimal behavior of the Hilbert scheme of paracanonical divisors along the canonical linear series, can be used to generate inequalities among numerical invariants of $X$.

Turning to details, we start by introducing the main homological players in our story. Let $X$ be a compact Kähler manifold of dimension $d$, with $H^1(X, \mathcal{O}_X) \neq 0$, and let $\mathbf{P} = \mathbf{P}_{\mathrm{sub}}(H^1(X, \mathcal{O}_X))$ be the projective space of one-dimensional subspaces of $H^1(X, \mathcal{O}_X)$. Thus a point in $\mathbf{P}$ is given by a non-zero vector $v \in H^1(X, \mathcal{O}_X)$, defined up to scalars. Cup product with $v$ determines a complex $\underline{\mathbf{L}}_X$ of vector bundles on $\mathbf{P}$:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-d) \otimes H^0(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{\mathbf{P}}(-d+1) \otimes H^1(X, \mathcal{O}_X) \longrightarrow \cdots \\ \longrightarrow \mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \mathcal{O}_X) \to 0. \end{aligned} \tag{1.1}$$

Letting $S = \mathrm{Sym}(H^1(X, \mathcal{O}_X)^\vee)$ be the symmetric algebra on the vector space $H^1(X, \mathcal{O}_X)^\vee$, taking global sections in $\underline{\mathbf{L}}_X$ gives rise to a linear complex $\mathbf{L}_X$ of graded $S$-modules in homological degrees 0 to $d$:

$$\begin{aligned} 0 \longrightarrow S \otimes_{\mathbf{C}} H^0(X, \mathcal{O}_X) \longrightarrow S \otimes_{\mathbf{C}} H^1(X, \mathcal{O}_X) \longrightarrow \cdots \\ \longrightarrow S \otimes_{\mathbf{C}} H^d(X, \mathcal{O}_X) \longrightarrow 0. \end{aligned} \tag{1.2}$$

These two complexes are avatars of the *derivative complex* introduced in [16] and studied for instance in [8, 18, 28, 30], which computes locally the push-forward to $\mathrm{Pic}^0(X)$ of the Poincaré line bundle on $X \times \mathrm{Pic}^0(X)$. We shall also be interested in the coherent sheaf $\mathcal{F} = \mathcal{F}_X$ on $\mathbf{P}$ arising as the cokernel of the right-most map in the complex $\underline{\mathbf{L}}_X$, so that one has an exact sequence:

$$\mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \mathcal{O}_X) \to \mathcal{F} \longrightarrow 0. \tag{1.3}$$

For reasons that will become apparent shortly, we call $\underline{\mathbf{L}}_X$ and $\mathbf{L}_X$ the BGG-*complexes* of $X$, and $\mathcal{F}_X$ its BGG-*sheaf.*

We shall be concerned with the exactness properties of $\underline{\mathbf{L}}_X$ and $\mathbf{L}_X$. Let

$$\mathrm{alb}_X : X \longrightarrow \mathrm{Alb}(X)$$

be the Albanese mapping of $X$, and let

$$k = k(X) = \dim X - \dim \mathrm{alb}_X(X)$$

be the dimension of the general fiber of $\mathrm{alb}_X$. We say that $X$ carries an *irregular fibration* if it admits a surjective morphism $X \longrightarrow Y$ with connected positive dimensional fibres onto a normal analytic variety $Y$ with the property that (any smooth model of) $Y$ has maximal Albanese dimension. These are the higher-dimensional analogues of irrational pencils in the case of surfaces. The behavior of $\mathbf{L}_X$ and $\underline{\mathbf{L}}_X$ is summarized in the following technical statement, which pulls together results from the papers cited above.

**Theorem A**

(i) *The complexes* $\mathbf{L}_X$ *and* $\underline{\mathbf{L}}_X$ *are exact in the first* $d-k$ *terms from the left, but* $\mathbf{L}_X$ *has non-trivial homology at the next term to the right.*
(ii) *Assume that* $X$ *does not carry any irregular fibrations. Then the* BGG *sheaf* $\mathcal{F}$ *is a vector bundle on* $\mathbf{P}$ *with* $\mathrm{rk}(\mathcal{F}) = \chi(\omega_X)$, *and* $\underline{\mathbf{L}}_X$ *is a resolution of* $\mathcal{F}$.

The essential goal of the present paper is to show that these exactness properties have some interesting consequences for the algebra and geometry of $X$.

The first applications concern the cohomology modules

$$P_X = \bigoplus_{i=0}^{d} H^i(X, \mathcal{O}_X), \qquad Q_X = \bigoplus_{i=0}^{d} H^i(X, \omega_X)$$

of the structure sheaf and the canonical bundle of $X$. Via cup product, we may view these as graded modules over the exterior algebra

$$E =_{\text{def}} \Lambda^* H^1(X, \mathcal{O}_X)$$

on $H^1(X, \mathcal{O}_X)$.[1] There has been a considerable amount of recent work in the commutative algebra community aimed at extending to modules over an exterior algebra aspects of the classical theory of graded modules over a polynomial ring. In the present context, it is natural to ask whether one can say anything in general about the algebraic properties of the modules $P_X$ and $Q_X$ canonically associated to a Kähler manifold $X$: for instance, in what degrees do generators and relations live?

An elementary example might be helpful here. Consider an Abelian variety $A$ of dimension $d+1$, and let $X \subseteq A$ be a smooth hypersurface of very large degree. Then by the Lefschetz theorem one has

$$\begin{aligned} H^i(X, \mathcal{O}_X) &= H^i(A, \mathcal{O}_A) = \Lambda^i H^1(X, \mathcal{O}_X) \quad \text{for } i < d, \\ H^d(X, \mathcal{O}_X) &\supsetneqq H^d(A, \mathcal{O}_A) = \Lambda^d H^1(X, \mathcal{O}_X). \end{aligned}$$

Thus $P_X$ has generators as an $E$-module in two degrees: $1 \in H^0(X, \mathcal{O}_X)$, and many new generators in $H^d(X, \mathcal{O}_X)$. If one takes the viewpoint that the simplest $E$-modules are those whose generators appear in a single degree, this means that $P_X$ is rather complicated. On the other hand, the situation with the

[1] Following the degree conventions of [12] and [11], we take $E$ to be generated in degree $-1$, and then declare that the summand $H^i(X, \omega_X)$ of $Q_X$ has degree $-i$, while $H^i(X, \mathcal{O}_X)$ has degree $d-i$ in $P_X$.

dual module $Q_X$ is quite different. Here $H^0(X, \omega_X)$ is big, and the maps

$$H^0(X, \omega_X) \otimes \Lambda^i H^1(X, \mathcal{O}_X) \longrightarrow H^i(X, \omega_X)$$

are surjective, i.e. $Q_X$ is generated in degree 0. This suggests that $Q_X$ behaves more predictably as an $E$-module than does $P_X$. Our first main result asserts that quite generally the module $Q_X$ has simple homological properties.

Specifically, given a graded $E$-module $M = \bigoplus_{i=0}^{-d} M_i$ generated in degrees $\leq 0$, one says that $E$ is $m$-regular if the generators of $M$ appear in degrees $0, -1, \ldots, -m$, the relations among these generators are in degrees $-1, \ldots, -(m+1)$, and more generally the $p^{\text{th}}$ module of syzygies of $M$ has all its generators in degrees $\geq -(p+m)$. This is the analogue for modules over the exterior algebra of the familiar notion of Castelnuovo-Mumford regularity for graded modules over a polynomial ring. As in the classical case, one should see regularity as being a measure of algebraic complexity, with small regularity corresponding to low complexity. For example, in the example above (an appropriate shift of) the module $P_X$ has worst-possible regularity $= d$.

The following result asserts that the regularity of $Q_X$ is computed by the Albanese fiber-dimension of $X$.

**Theorem B** *As above, let $X$ be a compact Kähler manifold, and let $k = \dim X - \dim \mathrm{alb}_X(X)$. Then*

$$\mathrm{reg}(Q_X) = k,$$

*i.e. $Q_X$ is $k$-regular, but not $(k-1)$-regular as an $E$-module. In particular, $X$ has maximal Albanese dimension (i.e. $k = 0$) if and only if $Q_X$ is generated in degree 0 and has a linear free resolution.*

According to the general BGG-correspondence, which we quickly review in Sect. 2, the regularity of a graded module over the exterior algebra $E$ is governed by the exactness properties of a linear complex of modules over a symmetric algebra. Our basic observation is that for the module $Q_X$, this complex is precisely $\mathbf{L}_X$. Then Theorem B becomes an immediate consequence of statement (i) of Theorem A. Note that it follows from Theorem B that the Albanese dimension of $X$ is determined by purely algebraic data encoded in $Q_X$; it would be interesting to know whether this has any applications.

Our second line of application for Theorem A is as a mechanism for generating inequalities on numerical invariants of $X$. The search for relations among the Hodge numbers of an irregular variety has a long history, going back at least as far as the classical theorem of Castelnuovo and de Franchis giving a lower bound on the holomorphic Euler characteristic of surfaces without irrational pencils. Assuming that $X$ does not have any irregular fibrations, statement (ii) of Theorem A implies that the BGG-sheaf $\mathcal{F}$ is a vector bundle on the projective space $\mathbf{P}$, whose invariants are determined by $\underline{\mathbf{L}}_X$. Geometric facts about vector bundles on projective spaces then give rise to inequalities for $X$. Specifically, consider the formal power series:

$$\gamma(X;t) =_{\mathrm{def}} \prod_{j=1}^{d}(1-jt)^{(-1)^j h^{d,j}} \in \mathbf{Z}[[t]], \tag{1.4}$$

where $h^{i,j} = h^{i,j}(X)$. Write $q = h^1(X, \mathcal{O}_X)$ for the irregularity of $X$ (so that $q = h^{d,d-1}$), and for $1 \leq i \leq q-1$ denote by

$$\gamma_i = \gamma_i(X) \in \mathbf{Z}$$

the coefficient of $t^i$ in $\gamma(X;t)$. Thus $\gamma_i$ is a polynomial in the $h^{d,j}$. We prove:

**Theorem C** *Assume that $X$ does not carry any irregular fibrations* (*so that in particular $X$ itself has maximal Albanese dimension*). *Then*

(i) *Any Schur polynomial of weight $\leq q-1$ in the $\gamma_i$ is non-negative. In particular*

$$\gamma_i(X) \geq 0$$

*for every $1 \leq i \leq q-1$.*

(ii) *If $i$ is any index with $\chi(\omega_X) < i < q$, then $\gamma_i(X) = 0$.*

(iii) *One has $\chi(\omega_X) \geq q - d$.*

Part (i) expresses in particular the fact that the Chern classes of $\mathcal{F}$ are non-negative. For example, when $i = 1$ this yields (under the assumption of the theorem) the inequality

$$h^{d,1} - 2h^{d,2} + 3h^{d,3} - \cdots + (-1)^{d+1} \cdot d \cdot h^{d,d} \geq 0. \tag{*}$$

This includes some classically known statements (for instance if $\dim X = 3$, (*) reduces to the Castelnuovo–de Franchis-type inequality $h^{0,2} = h^{3,1} \geq 2q-3$), but the positivity of higher $\gamma_i$ and part (iii) produce new stronger results. In fact, for threefolds satisfying the hypotheses of the theorem, an inequality kindly provided by a referee and the inequality in (iii) together imply that asymptotically

$$h^{0,2} \succeq 4q \quad \text{and} \quad h^{0,3} \succeq 4q$$

while in the case of fourfolds the same plus the inequality $\gamma_2 \geq 0$ give asymptotically

$$h^{0,2} \succeq 4q, \qquad h^{0,3} \succeq 5q + \sqrt{2q}, \qquad h^{0,4} \succeq 3q + \sqrt{2q}.$$

(See Corollary 4.5 for more details.) When $X$ is a surface without irrational pencils, related methods applied to $\Omega^1_X$ yield a new inequality for $h^{1,1}$ as well. Assertion (iii) is (a slightly special case of) the main result of [30], for which we provide a simple proof. The idea is that when the rank $\mathrm{rk}(\mathcal{F}) = \chi(\omega_X)$ of $\mathcal{F}$ is small compared to $q-1 = \dim \mathbf{P}$, it is hard for such a bundle to exist, giving rise to lower bounds on $\chi(\omega_X)$. The method used here allows us to further analyze possible borderline cases when the Euler characteristic is small, and conjecture stronger inequalities. All of the inequalities above can fail when $X$ does carry irregular fibrations.

Given the role it plays in Sect. 3, it is natural to ask what is the geometric meaning of the BGG sheaf $\mathcal{F}_X$. Our last result, which could be considered as an appendix to [15] Sect. 4 and [16], shows that in fact it has a very natural interpretation. Recall that in classical terminology, a *paracanonical divisor* on $X$ is an effective divisor algebraically equivalent to a canonical divisor. The set of all such is parametrized by the Hilbert scheme (or Douady space) $\mathrm{Div}^{\{\omega\}}(X)$, which admits an Abel-Jacobi mapping

$$u : \mathrm{Div}^{\{\omega\}}(X) \longrightarrow \mathrm{Pic}^{\{\omega\}}(X)$$

to the corresponding component of the Picard torus of $X$. The projective space $|\omega_X|$ parametrizing all canonical divisors sits as a subvariety of $\mathrm{Div}^{\{\omega\}}(X)$: it is the fibre of $u$ over the point $[\omega_X] \in \mathrm{Pic}^{\{\omega\}}(X)$. On the other hand, the projectivization $\mathbf{P}(\mathcal{F}) = \mathrm{Proj}_{\mathbf{P}}(\mathrm{Sym}(\mathcal{F}))$ sits naturally in $\mathbf{P}^{q-1} \times \mathbf{P}(H^d(X, \mathcal{O}_X))$, giving rise to a morphism

$$\mathbf{P}(\mathcal{F}) \longrightarrow \mathbf{P}\big(H^d(X, \mathcal{O}_X)\big) = |\omega_X|. \tag{1.5}$$

**Theorem D** *With the notation just introduced,* $\mathbf{P}(\mathcal{F})$ *is identified via the morphism* (1.5) *with the projectivized normal cone to* $|\omega_X|$ *inside* $\mathrm{Div}^{\{\omega\}}(X)$.

Note that we do not assume here that $X$ carries no irregular fibrations. When this additional hypothesis does hold, Theorem D implies the amusing fact that whether or not the projective space $|\omega_X|$ is an irreducible component of $\mathrm{Div}^{\{\omega\}}(X)$ depends in most cases only on the Hodge numbers $h^{d,j}(X)$ (Proposition 5.5).

We conclude this Introduction with a few remarks about related work. The sheafified complex $\underline{\mathbf{L}}_X$ came up in passing in [10] and [19], but it has not up to now been exploited in a systematic fashion. Going back to Theorem B, we note that a statement of a similar type was established in [13] for the singular cohomology of the complement of an affine complex hyperplane arrangement: in fact this cohomology always has a linear resolution over the exterior algebra on its first cohomology (though it is not generated in degree zero). However whereas the result of [13] is of combinatorial genesis and does not involve using the BGG correspondence, as explained above Theorem B is ultimately based on translating Hodge-theoretic information via BGG. We note also that Catanese suggested in [5] that it might be interesting to study the BGG correspondence for holomorphic cohomology algebras. In another direction, Lombardi [23] has extended the approach of the present paper to deal with Hodge groups $H^q(X, \Omega^p_X)$ with $p, q \neq 0, d$: see Remark 4.7 for a brief description of some of his results.

Concerning the organization of the paper, in Sect. 2 we prove the main technical result Theorem A. The connection with BGG is discussed in Sect. 3. Applications to numerical inequalities occupy Sect. 4, where we also give a number of examples and variants. Finally, Theorem D appears in Sect. 5.

## 2 Proof of Theorem A

This section is devoted to the proof of the main technical result, giving the exactness of the BGG complexes. The argument proceeds in the form of three propositions. We keep notation as in the Introduction: $X$ is a compact Kähler manifold of dimension $d$, $\mathrm{alb}_X : X \longrightarrow \mathrm{Alb}(X)$ is the Albanese mapping, and

$$k = k(X) = \dim X - \dim \mathrm{alb}_X(X)$$

is the dimension of the generic fibre of $\mathrm{alb}_X$. As before, $\mathbf{P} = \mathbf{P}_{\mathrm{sub}}(H^1(X, \mathcal{O}_X))$ is the projective space of lines in $H^1(X, \mathcal{O}_X)$, and $\mathbf{L}_X$ and $\underline{\mathbf{L}}_X$ are the complexes appearing in (1.1) and (1.2).

**Proposition 2.1** *The complexes $\mathbf{L}_X$ and $\underline{\mathbf{L}}_X$ are exact in the first $d-k$ terms from the left.*

*Proof* It is sufficient to prove the exactness for $\mathbf{L}_X$, as this implies the corresponding statement for its sheafified sibling. The plan for this is to relate $\mathbf{L}_X$ to the *derivative complex* introduced and studied in [16].

Write $V = H^1(X, \mathcal{O}_X)$ and $W = V^*$, and let $\mathbf{A} = \mathrm{Spec}(\mathrm{Sym}(W))$ be the affine space corresponding to $V$, viewed as an algebraic variety. Thus a point in $\mathbf{A}$ is the same as a vector in $V$. Then there is a natural complex $\mathcal{K}^\bullet$ of trivial algebraic vector bundles on $\mathbf{A}$:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^0(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^1(X, \mathcal{O}_X) \longrightarrow \cdots \\ \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^d(X, \mathcal{O}_X) \longrightarrow 0, \end{aligned}$$

with maps given at each point of $\mathbf{A}$ by wedging with the corresponding element of $V = H^1(X, \mathcal{O}_X)$. Recalling that $\Gamma(\mathbf{A}, \mathcal{O}_{\mathbf{A}}) = S$, one sees that $\mathbf{L}_X = \Gamma(\mathbf{A}, \mathcal{K}^\bullet)$ is the complex obtained by taking global sections in $\mathcal{K}^\bullet$. As

$\mathbf{A}$ is affine, to prove the stated exactness properties of $\mathbf{L}_X$, it is equivalent to establish the analogous exactness for the complex $\mathcal{K}^\bullet$, i.e. we need to show the vanishings $\mathcal{H}^i(\mathcal{K}^\bullet) = 0$ of the cohomology sheaves of this complex in the range $i < d - k$. For this it is in turn equivalent to prove the vanishing

$$\mathcal{H}^i\big(\mathcal{K}^\bullet\big)_0 = 0 \tag{*}$$

of the stalks at the origin of these homology sheaves in the same range $i < d - k$. Indeed, (*) implies that $\mathcal{H}^i(\mathcal{K}^\bullet) = 0$ in a neighborhood of the origin. But the differential of $\mathcal{K}^\bullet$ scales linearly in radial directions through the origin, so we deduce the corresponding vanishing on all of $\mathbf{A}$.

Now let $\mathbf{V}$ be the vector space $V$, considered as a complex manifold, so that $\mathbf{V} = \mathbf{C}^q$, where $q = h^1(X, \mathcal{O}_X)$ is the irregularity of $X$. Then on $\mathbf{V}$ we can form as above a complex $(\mathcal{K}^\bullet)^{an}$

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbf{V}} \otimes H^0(X, \mathcal{O}_X) &\longrightarrow \mathcal{O}_{\mathbf{V}} \otimes H^1(X, \mathcal{O}_X) \longrightarrow \cdots \\ \longrightarrow \mathcal{O}_{\mathbf{V}} \otimes H^d(X, \mathcal{O}_X) &\longrightarrow 0, \end{aligned}$$

of coherent analytic sheaves, which is just the complex of analytic sheaves determined by the algebraic complex $\mathcal{K}^\bullet$. This analytic complex was studied in [16], where it was called the *derivative complex* $D^\bullet_{\mathcal{O}_X}$ of $\mathcal{O}_X$. Since passing from a coherent algebraic to a coherent analytic sheaf is an exact functor (cf. [33, 3.10]), one has $\mathcal{H}^i((\mathcal{K}^\bullet)^{an}) = \mathcal{H}^i((\mathcal{K}^\bullet))^{an}$. So it is equivalent for (*) to prove:

$$\mathcal{H}^i\big((\mathcal{K}^\bullet)^{an}\big)_0 = 0 \quad \text{for } i < d - k. \tag{**}$$

But this will follow immediately from a body of results surrounding generic vanishing theorems.

Specifically, write $\mathrm{Pic}^0(X) = V/\Lambda$, let $\mathcal{P}$ be a normalized Poincaré line bundle on $X \times \mathrm{Pic}^0(X)$, and write

$$p_1 : X \times \mathrm{Pic}^0(X) \longrightarrow X, \qquad p_2 : X \times \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^0(X)$$

for the two projections. The main result of [16], Theorem 3.2, says that via the exponential map $\exp : V \to \mathrm{Pic}^0(X)$ we have the identification of the analytic stalks at the origin

$$\mathcal{H}^i\big((\mathcal{K}^\bullet)^{an}\big)_0 \cong (R^i p_{2*}\mathcal{P})_0. \tag{***}$$

On the other hand, by [30] Theorem C, we have $R^i p_{2*}\mathcal{P} = 0$ for $i < d - k$, where $p_2$ is the projection onto the second factor.[2] In view of (***), this gives (**), and we are done. $\square$

[2]This was posed as a problem in [16], and first answered in the smooth projective case by Hacon [18] and Pareschi [28]. In [30] it was simply shown that the result is equivalent to the Generic Vanishing Theorem of [15], hence it holds also in the compact Kähler case.

The next point is the non-exactness of $\mathbf{L}_X$ beyond the range specified in the previous proposition.

**Proposition 2.2** *The complex* $\mathbf{L}_X$ *is not exact at the term* $S \otimes_{\mathbf{C}} H^{d-k}(X, \mathcal{O}_X)$.

*Remark 2.3* The analogous statement for the sheafified complex $\underline{\mathbf{L}}_X$ can fail. For example, if $X = A \times \mathbf{P}^k$, with $A$ an Abelian variety of dimension $d-k$, then $\underline{\mathbf{L}}_X$—which in this case is just a Koszul complex on $\mathbf{P}^{d-k-1}$—is everywhere exact.

*Proof of Proposition 2.2*[3] As the cohomology groups involved in the construction of $\mathbf{L}_X$ are birationally invariant, one can assume that there is a surjective morphism $f : X \to Y$, with $Y$ a compact Kähler manifold of dimension $d-k$, such that $f^* : H^1(Y, \mathcal{O}_Y) \xrightarrow{\simeq} H^1(X, \mathcal{O}_X)$ and that there is a map $Y \to \mathrm{Alb}(X)$, generically finite onto its image. Noting that a surjection between compact Kähler manifolds $f$ induces injective maps

$$f^* : H^i(Y, \mathcal{O}_Y) \hookrightarrow H^i(X, \mathcal{O}_X) \quad \text{for all } i,$$

we obtain an inclusion of complexes $\mathbf{L}_Y \hookrightarrow \mathbf{L}_X$. Now the rightmost term in $\mathbf{L}_Y$ is $S \otimes_{\mathbf{C}} H^{d-k}(Y, \mathcal{O}_Y)$. By Serre duality this is non-zero, since $H^0(Y, \omega_Y) \neq 0$ by virtue of $Y$ being of maximal Albanese dimension. If $0 \neq \alpha \in H^{d-k}(Y, \mathcal{O}_Y)$ then obviously $d_Y(1 \otimes \alpha) = 0$, hence also $d_X(1 \otimes f^*\alpha) = 0$, where $d_Y$ and $d_X$ are the differentials of $\mathbf{L}_Y$ and $\mathbf{L}_X$ respectively. But $f^*\alpha \neq 0$, and $1 \otimes f^*\alpha$ cannot be in the image of $d_X$, since it has degree 0 with respect to the $S$-grading. □

*Remark 2.4* (Converse to the Generic Vanishing theorem [15]) Arguing as in the proof of Proposition 2.1, the conclusion of Proposition 2.2 is equivalent to the fact that $(R^{d-k} p_{2*}\mathcal{P})_0 \neq 0$. According to [30] Theorem 2.2, this is in turn equivalent to the fact that around the origin, the cohomological support loci of $\omega_X$ (see Proposition 2.5 below) satisfy

$$\mathrm{codim}_0 V^i(\omega_X) \geq i - k \quad \text{for all } i > 0.$$

Therefore this last condition becomes equivalent to $\dim a(X) \leq k$, which is a converse to the main result of [15] (in a strong sense, as it has the interesting consequence that the behavior of the $V^i(\omega_X)$ is dictated by their behavior around the origin).

[3] This proof was indicated to us by one of the referees; it simplifies considerably our original argument. One can also deduce the statement directly from a theorem of Kollár by passing through the BGG correspondence: see Remark 3.5 in the next section.

Finally, we record a criterion to guarantee that the BGG sheaf $\mathcal{F}$ on the projective space $\mathbf{P}$ is locally free. Recall that an *irregular fibration* of $X$ is a surjective morphism $f : X \longrightarrow Y$ with connected fibres from $X$ onto a normal variety $Y$ with $0 < \dim Y < \dim X$ having the property that a smooth model of $Y$ has maximal Albanese dimension.

**Proposition 2.5**

(i) *If $X$ has maximal Albanese dimension, then $\underline{\mathbf{L}}_X$ is a resolution of $\mathcal{F}$.*
(ii) *Suppose that $0 \in \mathrm{Pic}^0(X)$ is an isolated point of the cohomological support loci*

$$V^i(\omega_X) =_{\mathrm{def}} \left\{\alpha \in \mathrm{Pic}^0(X) \mid H^i(X, \omega_X \otimes \alpha) \neq 0\right\}$$

*for every $i > 0$. Then $\mathcal{F}$ is a vector bundle on $\mathbf{P}$, with $\mathrm{rk}(\mathcal{F}) = \chi(\omega_X)$.*
(iii) *The hypothesis of (ii) holds in particular if $X$ does not carry any irregular fibrations.*

*Proof* The first statement is the case $k = 0$ of Proposition 2.1, and (iii) follows from [16], Theorem 0.1. In general, $V^i(\omega_X)$ contains the support of the direct image $R^i p_{2*}\mathcal{P}$. Hence if the $V^i(\omega_X)$ are finite for $i > 0$, then the corresponding direct images are supported at only finitely many points, and this implies that the vector bundle maps appearing in $\underline{\mathbf{L}}_X$ are everywhere of constant rank. (Compare [10] or [19], Proposition 2.11.) Thus $\mathcal{F}$ is locally free, and its rank is computed from $\underline{\mathbf{L}}_X$. □

*Remark 2.6* We note for later reference that the main result of [16] asserts more generally that if $X$ doesn't admit any irregular fibrations, then in fact $V^i(\omega_X)$ is finite for every $i > 0$.

## 3 BGG and the canonical cohomology module

In this section we apply main technical result Theorem A to study the regularity of the canonical module $Q_X$. We also discuss a variant involving a twisted BGG complex.

We start by briefly recalling from [12] and [11] some basic facts concerning the BGG correspondence. Let $V$ be a $q$-dimensional complex vector space,[4] and let $E = \bigoplus_{i=0}^{d} \bigwedge^i V$ be the exterior algebra over $V$. Denote by $W = V^\vee$ be the dual vector space, and by $S = \mathrm{Sym}(W)$ the symmetric alge-

[4]The BGG correspondence works over any field, but in the interests of unity we stick throughout to $\mathbf{C}$.

bra over $W$. Elements of $W$ are taken to have degree 1, while those in $V$ have degree $-1$.

Consider now a finitely generated graded module $P = \bigoplus_{i=0}^{d} P_i$ over $E$. The *dual* over $E$ of the module $P$ is defined to be the $E$-module

$$Q = \widehat{P} = \bigoplus_{j=0}^{d} P_{-j}^{\vee}$$

(so that positive degrees are switched to negative ones and vice versa). The basic idea of the BGG correspondence is that the properties of $Q$ as an $E$-module are controlled by a linear complex of $S$-modules constructed from $P$. Specifically, one considers the complex $\mathbf{L}(P)$ given by

$$\cdots \longrightarrow S \otimes_{\mathbf{C}} P_{j+1} \longrightarrow S \otimes_{\mathbf{C}} P_j \longrightarrow S \otimes_{\mathbf{C}} P_{j-1} \longrightarrow \cdots$$

with differential induced by

$$s \otimes p \mapsto \sum_i x_i s \otimes e_i p,$$

where $x_i \in W$ and $e_i \in V$ are dual bases. We refer to [12] or [11] for a dictionary linking $\mathbf{L}(P)$ and $Q$.

It is natural to consider a notion of regularity for $E$-modules analogous to the theory of Castelnuovo-Mumford regularity for finitely generated $S$-modules. We limit ourselves here to modules concentrated in non-positive degrees.

**Definition 3.1** (Regularity) A finitely generated graded $E$-module $Q$ with no component of positive degree is called *$m$-regular* if it is generated in degrees 0 up to $-m$, and if its minimal free resolution has at most $m+1$ linear strands. Equivalently, $Q$ is $m$-regular if and only if

$$\mathrm{Tor}_i^E(Q, \mathbf{C})_{-i-j} = 0$$

for all $i \geq 0$ and all $j \geq m+1$.

As an immediate application of the results of Eisenbud-Fløystad-Schreyer, one has the following addendum to [12] Corollary 2.5 (cf. also [11] Theorems 7.7, 7.8), suggested to us by F.-O. Schreyer.

**Proposition 3.2** *Let $P$ be a finitely generated graded module over $E$ with no component of negative degree, say $P = \bigoplus_{i=0}^{d} P_i$. Then $Q = \widehat{P}$ is $m$-regular* *if and only if* $\mathbf{L}(P)$ *is exact at the first* $d-m$ *steps from the left*, *i.e. if and only if the sequence*

$$0 \longrightarrow S \otimes_{\mathbf{C}} P_d \longrightarrow S \otimes_{\mathbf{C}} P_{d-1} \longrightarrow \cdots \longrightarrow S \otimes_{\mathbf{C}} P_m$$

*of* $S$*-modules is exact.*

We propose to apply this machine to the canonical cohomology module. As before, let $X$ be a compact Kähler manifold of dimension $d$, and $\mathrm{alb}_X : X \longrightarrow \mathrm{Alb}(X)$ its Albanese map. Set

$$V = H^1(X, \mathcal{O}_X), \qquad E = \Lambda^* V, \qquad W = V^\vee, \qquad S = \mathrm{Sym}(W).$$

We are interested in the graded $E$-modules

$$P_X = \bigoplus_{i=0}^{d} H^i(X, \mathcal{O}_X), \qquad Q_X = \bigoplus_{i=0}^{d} H^i(X, \omega_X),$$

the $E$-module structure arising from wedge product with elements of $H^1(X, \mathcal{O}_X)$. These become dual modules (thanks to Serre duality) provided that we assign $H^i(X, \mathcal{O}_X)$ degree $d-i$, and $H^i(X, \omega_X)$ degree $-i$.

According to Proposition 3.2, the regularity of $Q_X$ is governed by the exactness of the complex $\mathbf{L}(P_X)$ associated to $P_X$. Thus Theorem B from the Introduction follows at once from statement (i) of Theorem A in view of the following:

**Lemma 3.3** *The complex* $\mathbf{L}(P_X)$ *coincides with the complex* $\mathbf{L}_X$ *appearing in* (1.2).

*Proof* It follows easily from the definitions that the differentials of both complexes are given on the graded piece corresponding to any $p \geq 0$ and $i \geq 0$ by

$$S^{p-1}W \otimes H^{i-1}(X, \mathcal{O}_X) \longrightarrow S^p W \otimes H^i(X, \mathcal{O}_X)$$

induced by the cup-product map $V \otimes H^{i-1}(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_X)$ and the natural map $S^{p-1}W \to S^p W \otimes V$. $\square$

*Remark 3.4* (Exterior Betti numbers) The exterior graded Betti numbers of $Q_X$ are computed as the dimensions of the vector spaces $\mathrm{Tor}_i^E(Q, \mathbf{C})_{-i-j}$. When $X$ is of maximal Albanese dimension and $q(X) > \dim X$, Theorem B implies that these vanish for $i \geq 0$ and $j \geq 1$, and the $i$-th Betti number in the linear resolution of $Q_X$ is

$$b_i = \dim_{\mathbf{C}} \mathrm{Tor}_i^E(Q, \mathbf{C})_{-i} = h^0(\mathbf{P}, \mathcal{F}(i))$$

where $\mathcal{F}$ is the BGG sheaf defined in (1.3) in the Introduction. (The last equality follows from general machinery, cf. [11] Theorem 7.8.) On the other hand $\mathcal{F}$ is 0-regular in the sense of Castelnuovo-Mumford by virtue of having a linear resolution, so the higher cohomology of its nonnegative twists vanishes. Hence $b_i = \chi(\mathbf{P}, \mathcal{F}(i))$, i.e. the exterior Betti numbers are computed by the Hilbert polynomial of $\mathcal{F}$.

*Remark 3.5* (Alternative proof of Proposition 2.2) One can use the BGG correspondence to deduce the non-exactness statement of Proposition 2.2 from a theorem of Kollár and its extensions. In fact, in view of Proposition 3.2, it is equivalent to prove that $Q_X$ is not $(k-1)$-regular. To this end, observe that the main result of [21] (extended in [32] and [34] to the Kähler setting) gives the splitting $\mathbf{R}a_*\omega_X \cong \bigoplus_{j=0}^k R^j a_*\omega_X[-j]$ in the derived category of $A$. This implies that $Q_X$ can be expressed as a direct sum

$$Q_X = \bigoplus_{j=0}^{k} Q^j[j], \quad \text{with } Q^j = H^*(A, R^j a_*\omega_X).$$

Moreover this is a decomposition of $E$-modules: $E$ acts on $H^*(A, R^j a_*\omega_X)$ via cup product through the identification $H^1(X, \mathcal{O}_X) = H^1(A, \mathcal{O}_A)$, and we again consider $H^i(A, R^j a_*\omega_X)$ to live in degree $-i$. We claim next that $Q^k \neq 0$. In fact, each of the $R^j a_*\omega_X$ is supported on the $(d-k)$-dimensional Albanese image of $X$, and hence has vanishing cohomology in degrees $> d-k$. Therefore $H^d(X, \omega_X) = H^{d-k}(A, R^k a_*\omega_X)$, which shows that $Q^k \neq 0$. On the other hand, $Q^k[k]$ is concentrated in degrees $\leq -k$, and therefore $Q_X$ must have generators in degrees $\leq -k$.

*Remark 3.6* Keeping the notation of the previous Remark, the authors and C. Schnell have shown that each of the modules $Q^j$ just introduced is 0-regular. Thus the minimal $E$-resolution of $Q_X$ splits into the direct sum of the (shifted)-linear resolutions of the modules $Q^j[j]$. Details will appear in a forthcoming note.

Finally, we discuss briefly a variant involving twisted modules. Fix an element $\alpha \in \mathrm{Pic}^0(X)$, and set

$$P_\alpha = \bigoplus H^i(X, \alpha), \qquad Q_\alpha = \bigoplus H^j(X, \omega_X \otimes \alpha^{-1}).$$

With the analogous grading conventions as above, these are again dual modules over the exterior algebra $E$. Letting

$$t = t(\alpha) = \max\{i \mid H^i(X, \omega_X \otimes \alpha^{-1}) \neq 0\},$$

the BGG complexes $\mathbf{L}(P_\alpha)$ and $\underline{\mathbf{L}}(P_\alpha)$ for $P_\alpha$ take the form

$$\begin{aligned} 0 \to S \otimes H^{d-t}(X,\alpha) &\to S \otimes H^{d-t+1}(X,\alpha) \to \cdots \\ &\to S \otimes H^{d-1}(X,\alpha) \to S \otimes H^{d}(X,\alpha) \to 0 \end{aligned}$$

and

$$\begin{aligned} 0 \to \mathcal{O}_{\mathbf{P}}(-t) \otimes H^{d-t}(X,\alpha) &\to \mathcal{O}_{\mathbf{P}}(-t+1) \otimes H^{d-t+1}(X,\alpha) \to \cdots \\ &\to \mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}(X,\alpha) \to \mathcal{O}_{\mathbf{P}} \otimes H^{d}(X,\alpha) \to 0. \end{aligned}$$

Writing as before $k = k(X)$ for the generic fibre dimension of the Albanese map, it follows as above from [18, 28, 30] that $\mathbf{L}(P_\alpha)$ is exact at the first $d-t-k$ steps from the left. Hence:

**Variant 3.7** *The $E$-module $Q_\alpha$ is $k$-regular.* □

We will return to the sheafified complex $\underline{\mathbf{L}}(P_\alpha)$ later.

*Remark 3.8* (Holomorphic forms) One can also extend aspects of the present discussion to $E$-modules associated to other bundles of holomorphic forms. This is worked out by Lombardi [23], who gives some interesting applications: see Remark 4.7.

## 4 Inequalities for numerical invariants

In this section, we use the BGG-sheaf $\mathcal{F}$ to study numerical invariants of a compact Kähler manifold. The exposition proceeds in three parts. We begin by establishing Theorem C from the Introduction. In the remaining two subsections we discuss examples, applications and variants.

*Inequalities from the BGG bundle* As before, let $X$ be a compact Kähler manifold of dimension $d$, and write

$$p_g = h^0(X,\omega_X), \qquad \chi = \chi(X,\omega_X), \qquad q = h^1(X,\mathcal{O}_X), \qquad n = q-1.$$

Denote by $\mathbf{P} = \mathbf{P}_{\text{sub}}(H^1(X,\mathcal{O}_X))$, so that $\mathbf{P}$ is a projective space of dimension $n = q-1$, and by $\mathcal{F} = \mathcal{F}_X$ the BGG-sheaf on $\mathbf{P}$ introduced in (1.3). Once one knows that $\mathcal{F}$ is locally free, more or less elementary arguments with vector bundles on projective space yield inequalities for numerical invariants. As in the Introduction, for $1 \le i \le q-1$ define $\gamma_i = \gamma_i(X)$ to be the coefficient of $t^i$ in the formal power series

$$\gamma(X;t) =_{\text{def}} \prod_{j=1}^{d} (1-jt)^{(-1)^j h^{d,j}} \in \mathbf{Z}[[t]],$$

with $h^{i,j} = h^{i,j}(X)$. The following statement recapitulates Theorem C from the Introduction under a slightly weaker hypothesis (cf. Proposition 2.5).

**Theorem 4.1** *Assume that* $0 \in \mathrm{Pic}^0(X)$ *is an isolated point of* $V^i(\omega_X)$ *for every* $i > 0$. *Then*

(i) *Any Schur polynomial of weight* $\leq q-1$ *in the* $\gamma_i$ *is non-negative. In particular*

$$\gamma_i(X) \geq 0$$

*for every* $1 \leq i \leq q-1$.

(ii) *If* $i$ *is any index with* $\chi(\omega_X) < i < q$, *then* $\gamma_i(X) = 0$.

(iii) *One has* $\chi(\omega_X) \geq q-d$.

*Proof* Thanks to Proposition 2.5, the hypothesis guarantees that $\mathcal{F}$ is locally free, and has a linear resolution:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-d) \otimes H^0(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{\mathbf{P}}(-d+1) \otimes H^1(X, \mathcal{O}_X) \longrightarrow \cdots \\ \longrightarrow \mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \mathcal{O}_X) \longrightarrow \mathcal{F} \longrightarrow 0. \end{aligned} \tag{4.1}$$

Identifying as usual cohomology classes on $\mathbf{P}^n$ with integers, $\gamma(X;t)$ is then just the Chern polynomial of $\mathcal{F}$. On the other hand, as $\mathcal{F}$ is globally generated, the Chern classes $c_i(\mathcal{F})$—as well as the Schur polynomials in these—and represented by effective cycles. Thus

$$\gamma_i(X) = \deg c_i(\mathcal{F}) \geq 0.$$

The second statement follows from the fact that $c_i(\mathcal{F}) = 0$ for $i > \mathrm{rank}(\mathcal{F})$.

Turning to (iii), we may assume that $q > d$ since in any event $\chi \geq 0$ by generic vanishing. If $q - d = 1$, then the issue is to show that $\chi = \mathrm{rank}(\mathcal{F}) \geq 1$, or equivalently that $\mathcal{F} \neq 0$. But this is clear, since there are no non-trivial exact complexes of length $n$ on $\mathbf{P}^n$ whose terms are sums of line bundles of the same degree. So we may suppose finally that $q - 1 = n > d$. The quickest argument is note that chasing through (4.1) implies that $\mathcal{F}$ and its twists have vanishing cohomology in degrees $0 < j < n-d-1$. But if $\chi \leq n-d$ this means by a result of Evans–Griffith, [14] Theorem 2.4, that $\mathcal{F}$ is a direct sum of line bundles, which as before is impossible. (See [22], Example 7.3.10, for a quick proof of this splitting criterion due to Ein, based on Castelnuovo-Mumford regularity and vanishing theorems for vector bundles.)

For a more direct argument in the case at hand that avoids Evans–Griffith, let $s \in H^0(\mathbf{P}, \mathcal{F})$ be a general section, and let $Z = \mathrm{Zeroes}(s)$. We may suppose that $Z$ is non-empty—or else we could construct a vector bundle $\mathcal{F}'$ of smaller rank having a linear resolution as in (4.1)—and smooth of dimension

$n-\chi$. Splicing together the sequence (4.1) and the Koszul complex determined by $s$, we arrive at a long exact sequence having the shape:

$$\begin{aligned}0 &\longrightarrow \mathcal{O}_{\mathbf{P}}(-d) \longrightarrow \oplus\mathcal{O}_{\mathbf{P}}(-d+1) \longrightarrow \cdots \longrightarrow \oplus\mathcal{O}_{\mathbf{P}}(-1)\\ &\longrightarrow \oplus\mathcal{O}_{\mathbf{P}} \longrightarrow \Lambda^2\mathcal{F} \longrightarrow \cdots \longrightarrow \Lambda^{\chi-1}\mathcal{F} \longrightarrow \mathcal{O}_{\mathbf{P}}(c_1) \longrightarrow \mathcal{O}_Z(c_1)\\ &\longrightarrow 0, \qquad (*)\end{aligned}$$

where $c_1 = c_1(\mathcal{F})$. Observe that $\omega_Z = \mathcal{O}_Z(c_1 - n - 1)$ by adjunction. Since $\mathcal{F}$ is globally generated, a variant of the Le Potier vanishing theorem[5] yields that

$$H^i(\mathbf{P}, \Lambda^a\mathcal{F} \otimes \omega_{\mathbf{P}}(\ell)) = 0 \quad \text{for } i > \chi - a\ ,\ \ell > 0.$$

Now twist through in (*) by $\mathcal{O}_{\mathbf{P}}(d-n-1)$. Chasing through the resulting long exact sequence, one finds that $H^{n-d-(\chi-1)}(Z, \omega_Z(d)) \neq 0$. But if $\chi \leq n-d$, this contradicts Kodaira vanishing on $Z$. □

*Remark 4.2* (Evans–Griffith Theorem) A somewhat more general form of (iii) appears in [30] and can be deduced here as well: applying Variant 4.12 and using more carefully the results of [16], one can assume only that there are no irregular fibrations $f : X \longrightarrow Y$ such that $Y$ is generically finite onto a *proper* subvariety of a complex torus (i.e. $X$ has no higher irrational pencils in Catanese's terminology [4].) The argument in [30] involved applying the Evans–Griffith syzygy theorem to the Fourier–Mukai transform of the Poincaré bundle on $X \times \mathrm{Pic}^0(X)$. The possibility mentioned in the previous proof of applying the Evans–Griffith–Ein splitting criterion to the BGG bundle $\mathcal{F}$ is related but substantialy quicker. As we have just seen the additional information that $\mathcal{F}$ admits a linear resolution allows one to bypass Evans–Griffith altogether, although as in Ein's proof we still use vanishing theorems for vector bundles.

*Hodge-number inequalities* Here we give some examples and variants of the inequalities appearing in the first assertions of Theorem 4.1. To put things in context, we start with an extended remark on what can be deduced from previous work of various authors.

[5]The statement we use is that if $\mathcal{E}$ is a nef vector bundle of rank $e$ on a smooth projective variety $V$ of dimension $n$, then

$$H^i(V, \Lambda^a\mathcal{E} \otimes \omega_V \otimes L) = 0$$

for $i > e - a$ and any ample line bundle $L$. In fact, it is equivalent to show that $H^j(V, \Lambda^a\mathcal{E}^* \otimes L^*) = 0$ for $j < n + a - e$. For this, after passing to a suitable branched covering as in [22], proof of Theorem 4.2.1, we may assume that $L = M^{\otimes a}$, in which case the statement follows from Le Potier vanishing in its usual form: see [22], Theorem 7.3.6.

*Remark 4.3* (Inequalities deduced from [4] and [6]) Catanese [4] shows that if a compact Kähler manifold $X$ admits no irregular fibrations, then the natural maps

$$\phi_k : \bigwedge^k H^1(X, \mathcal{O}_X) \longrightarrow H^k(X, \mathcal{O}_X) \tag{4.2}$$

are injective on primitive forms $\omega_1 \wedge \cdots \wedge \omega_k$, for $k \leq \dim X$. Since these correspond to the Plücker embedding of the Grassmannian $\mathbf{G}(k, V)$, one obtains the bounds $h^{0,k}(X) \geq k(q(X) - k) + 1$. This includes the classical $h^{0,2}(X) \geq 2q(X) - 3$. However, still based on Catanese's results, one of the referees points out a nice argument that provides the even stronger inequality:

$$h^{0,2}(X) \geq 4q(X) - 10 \tag{4.3}$$

provided that $\dim X \geq 3$. We thank the referee for allowing us to include the proof here.

Assume then that $X$ is a compact Kähler manifold of dimension $\geq 3$ with no irregular fibrations. By [4], for any independent 1-forms $\omega_1, \omega_2, \omega_3$ on $X$ one has $\omega_1 \wedge \omega_2 \wedge \omega_3 \neq 0$ in $H^0(X, \Omega^3_X)$. Accordingly, for any independent $\omega_1, \omega_2, \omega_3, \omega_4 \in H^0(X, \Omega^1_X)$ one has

$$\omega_1 \wedge \omega_2 \neq \omega_3 \wedge \omega_4 \in H^0(X, \Omega^2_X).$$

Writing $W = \langle \omega_1, \omega_2, \omega_3, \omega_4 \rangle \subseteq H^0(X, \Omega^1_X)$, we then have that the natural map

$$\bigwedge^2 W \to H^0(X, \Omega^2_X)$$

is injective (as the secant variety of the Grassmannian $\mathbf{G}(2, 4)$ fills up the ambient $\mathbf{P}^5$ of the Plücker embedding). Denote now by $\mathcal{E}$ the tautological sub-bundle on the Grassmannian of subspaces $\mathbf{G} = \mathbf{G}(4, H^0(X, \Omega^1_X))$. We obtain a morphism

$$\mathbf{R} := \mathbf{P}_{\mathrm{sub}}\left(\bigwedge^2 \mathcal{E}\right) \xrightarrow{\varphi} \mathbf{P}_{\mathrm{sub}}(H^0(X, \Omega^2_X))$$

which is given by sections of the line bundle $\mathcal{O}_{\mathbf{P}}(1)$. As this line bundle is big,[6] we have $\dim \mathrm{Im}(\varphi) = \dim \mathbf{R}$, and (4.3) follows.

[6] This is equivalent to the assertion that the map $\Phi : \mathbf{R} \longrightarrow \mathbf{P}_{\mathrm{sub}}(\bigwedge^2 H^0(X, \Omega^1_X))$ is generically finite over its image. But it is elementary to see that the image of $\Phi$ coincides with the secant variety of the Plucker embedding of $\mathbf{G}' = \mathbf{G}(2, H^0(X, \Omega^1_X))$, and this secant variety is well-known to have dimension $= 2 \dim \mathbf{G}' + 1 - 4 = 4q - 11 = \dim \mathbf{R}$.

Finally we note that Causin and Pirola [6] provide a more refined study in the case $k = 2$ of the homomorphism appearing in (4.2). Among other things, for $q(X) \le 2\dim X - 1$ they show that $\phi_2$ is injective, so $h^{0,2}(X) \ge \binom{q(X)}{2}$. Thus for threefolds with $q(X) = 5$ and no irregular fibrations, their result coincides with (4.3).[7]

*Example 4.4* (Theorem 4.1 in small dimensions) We unwind a few of the inequalitites in statement (i) of Theorem 4.1. We assume that $X$ has dimension $d$, and that it carries no irregular fibrations. For compactness, write $h^{0,j} = h^{0,j}(X)$ and $q = q(X)$. To begin with, the condition $\gamma_1 \ge 0$ gives a linear inequality among $q, h^{0,2}, \dots, h^{0,d-1}$. In small dimensions this becomes:

$$\begin{aligned} h^{0,2} &\ge -3 + 2q \quad \text{when } d = 3;\\ h^{0,3} &\ge 4 - 3q + 2h^{0,2} \quad \text{when } d = 4;\\ h^{0,4} &\ge -5 + 4q - 3h^{0,2} + 2h^{0,3} \quad \text{when } d = 5. \end{aligned} \tag{4.4}$$

Similarly, $\gamma_2$ is a quadratic polynomial in the same invariants, and one may solve $\gamma_2 \ge 0$ to deduce the further and stronger inequalities:

$$\begin{aligned} &h^{0,2} \ge -\frac{7}{2} + 2q + \frac{\sqrt{8q-23}}{2} \quad \text{when } d = 3;\\ &h^{0,3} \ge \frac{7}{2} - 3q + 2h^{0,2} + \frac{\sqrt{49 - 24q + 8h^{0,2}}}{2} \quad \text{when } d = 4;\\ &h^{0,4} \ge -\frac{11}{2} + 4q - 3h^{0,2} + 2h^{0,3} + \frac{\sqrt{71 + 48q - 24h^{0,2} + 8h^{0,3}}}{2}\\ &\quad \text{when } d = 5, \end{aligned} \tag{4.5}$$

where in the last case we assume that the expression under the square root is non-negative. (This is automatic when $d = 3$ since $q \ge 3$, and when $d = 4$ thanks to (4.3).) Note that equality holds in (4.4) when $X$ is an Abelian variety (in which case $\mathcal{F} = 0$). Similarly, when $X$ is a theta divisor in an Abelian variety of dimension $d + 1$, then rank $\mathcal{F} = 1$, so equality holds in each of the three instances of (4.5).

When $d = 3$ or $d = 4$, we can combine the various inequalities in play to get an asymptotic statement:

[7]The results of Catanese and Causin-Pirola actually only assume the absence of higher irrational pencils, cf. Remark 4.2.

**Corollary 4.5**

(i) *If $X$ is an irregular compact Kähler threefold with no irregular fibration, then*

$$h^{0,3}(X) \geq h^{0,2}(X) - 2,$$

*so asymptotically*

$$h^{0,2}(X) \succeq 4q(X) \quad \textit{and} \quad h^{0,3}(X) \succeq 4q(X).$$

(ii) *If $X$ is an irregular compact Kähler fourfold with no irregular fibration, then asymptotically*

$$h^{0,2}(X) \succeq 4q(X), \qquad h^{0,3}(X) \succeq 5q(X) + \sqrt{2q(X)},$$
$$h^{0,4}(X) \ \succeq \ 3q(X) + \sqrt{2q(X)}.$$

*Proof* (i) The first inequality is equivalent to the statement $\chi(\omega_X) \geq q - 3$ from statement (iii) of Theorem 4.1, and the other inequalities follow from this and (4.3).

(ii) The inequality $\chi(\omega_X) \geq q - 4$ is equivalent to

$$h^{0,4}(X) \geq (2q - 5) + \big(h^{0,3}(X) - h^{0,2}(X)\big).$$

The statement then follows by using (4.5) to bound $h^{0,3} - h^{0,2}$, and invoking the inequality $h^{0,2} \succeq 4q$ coming from (4.3). □

Similar arguments lead to a new inequality for $h^{1,1}$ on a surface. Specifically, let $X$ be a compact Kähler surface with no non-constant morphism to a curve of genus at least 2. The classical Castelnuovo-de Franchis inequality asserts that $h^{0,2}(X) \geq 2q(X) - 3$. A related result based on the Castelnuovo-de Franchis Lemma and linear algebra also bounds $h^{1,1}$ in terms of the irregularity: it is shown in [1], IV.5.4, that $h^{1,1}(X) \geq 2q(X) - 1$. The methods of the present paper yield a strengthening of this:[8]

**Proposition 4.6** *If $X$ is a compact Kähler surface without irrational pencils, then*

$$h^{1,1}(X) \geq \begin{cases} 3q(X) - 2 & \textit{if } q(X) \textit{ is even}, \\ 3q(X) - 1 & \textit{if } q(X) \textit{ is odd}. \end{cases}$$

[8] We remark that in the case of surfaces of general type without irrational pencils the inequality $h^{1,1}(X) \geq 3q(X) - 2$ could also be obtained by combining the Castelnuovo-de Franchis inequality with the deep Bogomolov-Miyaoka-Yau inequality.

*Proof* It is well known that given any non-zero one-form $\omega \in H^0(X, \Omega^1_X)$ on such a surface $X$, the map $H^1(X, \mathcal{O}_X) \xrightarrow{\wedge\omega} H^1(X, \Omega^1_X)$ obtained by wedging with $\omega$ is injective. On the other hand, this map is naturally dual to the map $H^1(X, \Omega^1_X) \xrightarrow{\wedge\omega} H^1(X, \Omega^2_X)$, via Serre duality. Hence in the natural complex

$$0 \longrightarrow H^1(X, \mathcal{O}_X) \xrightarrow{\wedge\omega} H^1(X, \Omega^1_X) \xrightarrow{\wedge\omega} H^1(X, \Omega^2_X) \longrightarrow 0,$$

the first map is injective and the second is surjective. Globalizing, we obtain a monad of vector bundles on $\mathbf{P} := \mathbf{P}_{\mathrm{sub}}(H^0(X, \Omega^1_X))$:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-1)^q \longrightarrow \mathcal{O}_{\mathbf{P}}^{h^{1,1}(X)} \xrightarrow{\phi} \mathcal{O}_{\mathbf{P}}(1)^q \longrightarrow 0.$$

The cohomology $E$ of this monad sits in an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-1)^q \longrightarrow K \longrightarrow E \longrightarrow 0$$

where $K = \ker(\phi)$. A direct calculation shows that $\mathrm{rk}(E) = h^{1,1}(X) - 2q$ and

$$c_t(E) = \frac{1}{(1-t^2)^q} = 1 + qt^2 + \binom{q+1}{2} t^4 + \cdots,$$

with non-zero terms in all even degrees $\leq \dim \mathbf{P} = q - 1$. As $c_i(E) = 0$ for $i > \mathrm{rk}(E)$, this implies that $\mathrm{rk}(E) \geq q - 2$ if $q$ is even, and $\mathrm{rk}(E) \geq q - 1$ if $q$ is odd. □

*Remark 4.7* (Bounds for other Hodge numbers) The techniques of this paper, applied to bundles of holomorphic forms $\Omega^p_X$ as opposed to $\mathcal{O}_X$, can be used to bound other Hodge numbers for important classes of compact Kähler manifolds, where no results in the style of those of [4] in Remark 4.3 are available. This is carried out by Lombardi in [23]; for instance, on threefolds whose 1-forms have at most isolated zeros (e.g. subvarieties of Abelian varieties with ample normal bundle), there are lower bounds for *all* Hodge numbers in terms of $q(X)$. Besides those mentioned in the proof of Corollary 4.5, one has asymptotically

$$h^{1,1} \succeq 2q + \sqrt{2q} \quad \text{and} \quad h^{2,1} \succeq 3q + \sqrt{2q}.$$

*Bounds involving the Euler characteristic* In this final subsection, we make some remarks surrounding the inequality

$$\chi(X, \omega_X) \geq q(X) - \dim X \tag{4.6}$$

established in [30] and Theorem 4.1(iii) for compact Kähler manifolds with no irregular fibrations.

Note to begin with that equality holds in (4.6) when $X$ is birational to a complex torus (in which case $\chi = 0$) or to a theta divisor in a principally polarized Abelian variety (in which case $d = q - 1$ and $\chi = 1$). It was essentially established by Hacon–Pardini [19], Sect. 4, that in fact these are the only such examples with $\chi \leq 1$.

**Proposition 4.8** *Let $X$ be an irregular smooth projective complex variety with no irregular fibrations.*

(i) *If $\chi(\omega_X) = 0$, then $X$ is birational to an Abelian variety.*
(ii) *If $\chi(\omega_X) = 1 = q(X) - \dim X$, then $X$ is birational to a principal polarization in a* PPAV.

Since the statement does not appear explicitly in [19] we will briefly indicate the proof, but we stress that all the ideas are already present in that paper.

*Sketch of Proof* We again focus on the exact sequence (4.1) of bundles on $\mathbf{P} = \mathbf{P}^{q-1}$. Note that in any event $X$ has maximal Albanese dimension, and hence $q \geq d$. If $\chi = 0$, then $\mathcal{F} = 0$. In this case one reads off from (4.1) that $q = d$ and $P_1(X) = h^{0,d}(X) > 0$. On the other hand, the assumption of the theorem implies by [16] that $V^i(\omega_X)$ has only isolated points when $i > 0$, and since $\chi(\omega_X) = 0$, this implies that $V^0(\omega_X)$ also consists only of isolated points. But a result of Ein-Lazarsfeld (cf. [7], Theorem 1.7) says that a variety of maximal Albanese dimension with $V^0(\omega_X)$ zero dimensional is birational to its Albanese.

Now suppose that $\chi = 1$. Then $\mathcal{F}$ is a line bundle, and it follows that (4.1) is a twist of the standard Koszul complex, this being the unique linear complex of length $n + 1$ on $\mathbf{P}^n$ whose outer terms have rank one. In particular $h^{0,d}(X) = q$. On the other hand we have an injection $H^0(A, \Omega_A^d) \to H^0(X, \Omega_X^d)$. Indeed, since $d = q - 1$, if the pullback map were not injective we would have a $d$-wedge of independent holomorphic 1-forms on $X$ equal to 0, which by [4] Theorem 1.14 would imply the existence of an irregular fibration. Now since the two dimensions are equal, the map is in fact an isomorphism. To prove (ii), one can then use a characterization of principal polarizations due to Hacon-Pardini (cf. [19] Proposition 4.2), extending a criterion of Hacon, which says that the only other thing we need to check is $V^i(\omega_X) = \{0\}$ for all $i > 0$. But this follows from Remarks 2.6 and 4.13. $\square$

On the other hand, one expects it to be very rare to find manifolds $X$ with no irregular fibrations for which $\chi(X, \omega_X) = q(X) - \dim X \geq 2$.

**Conjecture 4.9** *If $X$ is an irregular compact Kähler manifold with no irregular fibrations and $\chi(\omega_X) \geq 2$, then*

$$\chi(\omega_X) \ > \ q(X) - \dim X$$

*when $q(X)$ is very large compared to $\chi(\omega_X)$.*

The thinking here is that if equality were to hold in (4.6), then the BGG-sheaf $\mathcal{F}$ would provide a non-split vector bundle of small rank on the projective space $\mathbf{P}$. But these should almost never exist. The fact that $\mathcal{F}$ admits a linear resolution, and the resulting relations in Theorem C (ii), should provide even more constraints.

As an example in this direction, one has the following, whose proof was shown to us by I. Coandă.

**Proposition 4.10** *Let $X$ be a compact Kähler manifold with no irregular fibrations, such that $\chi(\omega_X) = 2$ and $q(X) \geq 5$. Then $q(X) - \dim X < 2$.*

*Proof* Assume for a contradiction that $q(X) - \dim X = 2$, and consider yet again the BGG resolution (4.1) of $\mathcal{F}$. This resolution shows first of all that $\mathcal{F}$ is 0-regular in the sense of Castelnuovo-Mumford. We next claim that $H^1(\mathbf{P}, \mathcal{F}(-2)) \neq 0$, while $H^1(\mathbf{P}, \mathcal{F}(i)) = 0$ for all $i \neq -2$. Grant this for the moment. Then the $S$-module $H^1_*(\mathcal{F})$ has a non-zero summand in degrees $-2$ and higher. But [26] Theorem 1.7 asserts that a 0-regular rank 2 bundle with this property cannot exist when $n \geq 4$. As for the claim, note that (4.1) starts on the left with a twist of the Euler sequence, and thus the cokernel of the injection $\mathcal{O}_{\mathbf{P}^n}(-n+1) \to \mathcal{O}_{\mathbf{P}^n}^{n+1}(-n+2)$ is $T_{\mathbf{P}^n}(-n+1)$. One then finds that

$$H^1(\mathbf{P}^n, \mathcal{F}(i)) = H^{n-1}(\mathbf{P}^n, T_{\mathbf{P}^n}(-n+1+i)),$$

and the assertion follows from the Bott formula (cf. [27] p.8–9) and Serre duality. □

*Example 4.11* (Surfaces and the Tango bundle) The case of surfaces is particularly amusing from the present point of view. When $\dim X = 2$, (4.6) is equivalent to the classical Castelnuovo-de Franchis inequality

$$p_g(X) \geq 2q(X) - 3,$$

which holds for surfaces with no irrational pencils of genus at least 2. As soon as $q(X) \geq 4$ it has been suggested (cf. e.g. [24])—and proved in [25] for $q(X) = 5$—that there should be no such surfaces satisfying $p_g(X) =$

$2q(X) - 3$. If such a surface were to exist, its BGG bundle $\mathcal{F}$ would have a resolution:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n}(-2) \longrightarrow \mathcal{O}_{\mathbf{P}^n}(-1)^{n+1} \longrightarrow \mathcal{O}_{\mathbf{P}^n}^{2n-1} \longrightarrow \mathcal{F} \longrightarrow 0$$

where $n = q(X) - 1 \geq 3$. On the other hand, for every $n \geq 3$ there does exist a vector bundle having this shape: it is the dual of the *Tango bundle* (cf. [27] Chap. I, Sect. 4.3). It would be quite interesting to decide one way or the other whether one can in fact realize the dual Tango bundle as the BGG bundle of a surface.

Finally, we discuss a strengthening of the inequality (4.6), also appearing in [30], which involves the twisted BGG complexes introduced in Variant 3.7.

As always, let $X$ be a compact Kähler manifold, and fix any point $\alpha \in \mathrm{Pic}^0(X)$. Following [30], one defines the *generic vanishing index* of $\omega_X$ at $0 \in \mathrm{Pic}^0(X)$ to be the integer

$$\mathrm{gv}_0(X) = \min_{i>0}\{\mathrm{codim}_0 V^i(\omega_X) - i\}.$$

The basic generic vanishing theorems assert that $\mathrm{gv}_0(X) \geq 0$ when $X$ has maximal Albanese dimension, and if 0 is an isolated point of $V^i(\omega_X)$ for every $i > 0$ then

$$\mathrm{gv}_0(X) = q(X) - \dim(X).$$

The following statement, which appeared as Corollary 4.1 in [30], therefore generalizes Theorem C(iii).

**Variant 4.12** *Assume that $X$ has maximal Albanese dimension. Then*

$$\chi(X, \omega_X) \geq \mathrm{gv}_0(X).$$

*Brief Sketch of Proof* The origin belongs to all the $V^i(\omega_X)$ (cf. [10], Lemma 1.8), and hence there exist a largest index $t > 0$, and an irreducible component $Z \subseteq V^t(X)$, such that $\mathrm{gv}_0(X) = \mathrm{codim}\, Z - t$. Choose a general point $\alpha \in Z$ and consider the twisted BGG complex $\underline{\mathbf{L}}(P_\alpha)$, giving a resolution of the indicated sheaf $\mathcal{F}_\alpha$:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-t) \otimes H^{d-t}(X, \alpha) \longrightarrow \mathcal{O}_{\mathbf{P}}(-t+1) \otimes H^{d-t+1}(X, \alpha) \longrightarrow \cdots \\ \longrightarrow \mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}(X, \alpha) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \alpha) \longrightarrow \mathcal{F}_\alpha \longrightarrow 0. \end{aligned} \qquad (*)$$

The sheaf $\mathcal{F}_\alpha$ is typically not locally free. But if one chooses a subspace

$$W \subseteq T_\alpha \mathrm{Pic}^0(X) = H^1(X, \mathcal{O}_X)$$

transverse to the tangent space of $Z$ at $\alpha$, and restricts (*) to the projectivization $\mathbf{P}' = \mathbf{P}_{\mathrm{sub}} W$ of $W$, then it follows as in [10], Theorem 1.2, that one gets an exact complex

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbf{P}'}(-t) \otimes H^{d-t}(X,\alpha) \longrightarrow \mathcal{O}_{\mathbf{P}'}(-t+1) \otimes H^{d-t+1}(X,\alpha) \longrightarrow \cdots \\ \longrightarrow \mathcal{O}_{\mathbf{P}'}(-1) \otimes H^{d-1}(X,\alpha) \longrightarrow \mathcal{O}_{\mathbf{P}'} \otimes H^{d}(X,\alpha) \longrightarrow \mathcal{G} \longrightarrow 0 \qquad (**) \end{aligned}$$

where $\mathcal{G}$ is a vector bundle, of rank $\chi(X,\omega_X)$. Note that

$$\dim \mathbf{P}' = \operatorname{codim} Z - 1 = \mathrm{gv}_0(\omega_X) + t - 1.$$

Now the argument proceeds much as in the proof of Theorem 4.1, using (**) in place of (4.1). □

*Remark 4.13* (Non-trivial isolated points) A similar argument gives yet another variant, which was also noted in [30].

Suppose that $\alpha \in \mathrm{Pic}^0(X)$ is a point having the property that for every $i > 0$ either $\alpha \notin V^i(\omega_X)$ or else $\alpha$ is an isolated point of $V^i(\omega_X)$. Assume furthermore $H^p(X,\alpha) \neq 0$ for some $p < d$, and let $p(\alpha)$ be the least index for which this holds. Then

$$\chi(X,\omega_X) \geq q(X) - \dim X + p(\alpha).$$

Since evidently $p(\alpha) > 0$ if $\alpha \neq 0$, this means that non-trivial isolated points improve the basic lower bound for the Euler characteristic.

## 5 The BGG sheaf and paracanonical divisors

In this section we study the geometric meaning of the BGG sheaf $\mathcal{F}$, proving Theorem D. As always, $X$ is a compact Kähler manifold of dimension $d$, but we do not exclude the possibility that it carries irregular fibrations. Keeping the notation from the Introduction, $\mathrm{Div}^{\{\omega\}}(X)$ denotes the space of divisors on $X$ lying over $\mathrm{Pic}^{\{\omega\}}(X)$, and $|\omega_X| \subseteq \mathrm{Pic}^{\{\omega\}}(X)$ is the canonical series.

The first point is to relate the Hilbert scheme of paracanonical divisors $\mathrm{Div}^{\{\omega\}}(X)$ to a suitable direct image of the Poincaré bundle on $X \times \mathrm{Pic}^0(X)$.

**Proposition 5.1** *Let $\mathcal{P}$ denote the normalized Poincaré bundle on $X \times \mathrm{Pic}^0(X)$. Then*

$$\mathrm{Div}^{\{\omega\}}(X) = \mathbf{P}\big((-1)^* R^d p_{2*}\mathcal{P}\big)$$

*as schemes over $\mathrm{Pic}^0(X)$, where $(-1) : \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^0(X)$ is the morphism given by multiplication by $-1$, and $p_1, p_2$ are the projections of $X \times \mathrm{Pic}^0(X)$ onto its factors.*

We will provide a formal proof shortly, but for a quick plausibility argument note that if $\alpha \in \mathrm{Pic}^0(X)$ is any point, then the fibre of $\mathbf{P}((-1)^* R^d p_{2*}\mathcal{P})$ over $\alpha$ is the projective space of one-dimensional quotients of $H^d(X, \alpha^{-1})$, which thanks to Serre duality is identified with the projective space parametrizing divisors in the linear series $|\omega_X \otimes \alpha|$. Granting Proposition 5.1 for the time being, we complete the

*Proof of Theorem D* It is enough to establish the stated isomorphism after pulling back by the exponential map $\exp : V = H^1(X, \mathcal{O}_X) \longrightarrow \mathrm{Pic}^0(X)$, which is étale. Then the results of [16] quoted in the proof of Proposition 2.1 imply that $\exp^*(R^d p_{2*}\mathcal{P})$ is isomorphic in a neighborhood of the origin to the cokernel of map

$$u : H^{d-1}(X, \mathcal{O}_X) \otimes \mathcal{O}_{\mathbf{V}} \longrightarrow H^d(X, \mathcal{O}_X) \otimes \mathcal{O}_{\mathbf{V}}$$

of sheaves on the affine space $\mathbf{V} = \mathbf{C}^q$ arising from the right-most terms of the BGG complex. Note that $u$ is given by a matrix of linear forms, and pulling back by $(-1)$ just multiplies the entries of this matrix by $-1$. The theorem then reduces to a general statement, appearing in the following lemma, concerning the projectivization of the cokernel of a map of trivial vector bundles on affine space defined by a matrix of linear forms. □

**Lemma 5.2** *Let $u$ be an $a \times b$ matrix of linear forms on a vector space $\mathbf{C}^q$, defining maps*

$$u : \mathcal{O}^a_{\mathbf{C}^q} \longrightarrow \mathcal{O}^b_{\mathbf{C}^q}, \qquad \overline{u} : \mathcal{O}_{\mathbf{P}^{q-1}}(-1)^a \longrightarrow \mathcal{O}^b_{\mathbf{P}^{q-1}},$$

*and set $\mathcal{A} = \mathrm{coker}(u)$, $\overline{\mathcal{A}} = \mathrm{coker}(\overline{u})$. Consider the subscheme*

$$\mathbf{P}(\mathcal{A}) \subseteq \mathbf{C}^q \times \mathbf{P}^{b-1},$$

*whose fibre $F$ over the origin $0 \in \mathbf{C}^q$ is a copy of $\mathbf{P}^{b-1}$. Then the projectivized normal cone to $F$ in $\mathbf{P}(\mathcal{A})$ is identified with $\mathbf{P}(\overline{\mathcal{A}})$ via the natural projection $\mathbf{P}(\overline{\mathcal{A}}) \longrightarrow \mathbf{P}^{b-1}$.*

*Proof* An $a \times b$ matrix $u$ of linear forms on $\mathbf{C}^q$ gives rise to an $a \times q$ matrix $\overline{w}$ of linear forms on $\mathbf{P}^{b-1}$ having the property that if $\overline{\mathcal{B}}$ is the cokernel of the resulting map

$$\overline{w} : \mathcal{O}_{\mathbf{P}^{b-1}}(-1)^a \longrightarrow \mathcal{O}^q_{\mathbf{P}^{b-1}},$$

then $\mathbf{P}(\mathcal{A}) \cong \mathbf{V}(\overline{\mathcal{B}})$ as subschemes of $\mathbf{P}^{b-1} \times \mathbf{C}^q$. (In fact, $\overline{w}$ is constructed so that $\mathbf{P}(\mathcal{A})$ and $\mathbf{V}(\overline{\mathcal{B}})$ are defined in $\mathbf{P}^{b-1} \times \mathbf{C}^q$ by the same equations.) Under this identification, the issue is to determine the projectivized normal cone to $\mathbf{V}(\overline{\mathcal{B}})$ along its zero section; equivalently, one needs to determine the exceptional divisor in the blow-up of $\mathbf{V}(\overline{\mathcal{B}})$ along this zero-section. But by virtue of [17, 8.7] this exceptional divisor is $\mathbf{P}(\overline{\mathcal{B}})$, and this in turn coincides with $\mathbf{P}(\overline{\mathcal{A}})$ as a subscheme of $\mathbf{P}^{b-1}\times\mathbf{P}^{q-1}$. □

It remains only to give the

*Proof of Proposition 5.1* As explained in [20], Theorem 3.13, there is a unique coherent sheaf $\mathcal{Q}$ on $\mathrm{Pic}^0(X)$ characterized by the property that

$$\mathcal{H}om(\mathcal{Q},\mathcal{E}) = p_{2*}(p_1^*\omega_X\otimes\mathcal{P}\otimes p_2^*\mathcal{E}) \tag{*}$$

for any sheaf $\mathcal{E}$ on $\mathrm{Pic}^0(X)$, and then $\mathrm{Div}^{\{\omega\}}(X)=\mathbf{P}(\mathcal{Q})$. So we need to establish that (*) holds with $\mathcal{Q}=(-1)^*R^dp_{2*}\mathcal{P}$. To this end, denote as usual by $\mathbf{R}\Phi_{\mathcal{P}}(\mathcal{G})=\mathbf{R}p_{2*}(p_1^*\mathcal{G}\otimes\mathcal{P})$ the Fourier-Mukai transform of a sheaf $\mathcal{G}$ on $X$. By the projection formula, one has

$$\mathbf{R}p_{2*}(p_1^*\omega_X\otimes\mathcal{P}\otimes p_2^*\mathcal{E})\cong\mathbf{R}\Phi_{\mathcal{P}}(\omega_X)\overset{\mathbf{L}}{\otimes}\mathcal{E},$$

and we claim that it suffices to prove the derived formula

$$\mathbf{R}\Phi_{\mathcal{P}}\omega_X\overset{\mathbf{L}}{\otimes}\mathcal{E}\cong\mathbf{R}\mathcal{H}om\big((-1)^*\mathbf{R}\Phi_{\mathcal{P}}\mathcal{O}_X[d],\mathcal{O}_{\mathrm{Pic}^0(X)}\big)\overset{\mathbf{L}}{\otimes}\mathcal{E}. \tag{**}$$

Indeed, suppose that (**) is known. Now

$$\mathbf{R}\mathcal{H}om\big((-1)^*\mathbf{R}\Phi_{\mathcal{P}}\mathcal{O}_X[d],\mathcal{O}_{\mathrm{Pic}^0(X)}\big)\overset{\mathbf{L}}{\otimes}\mathcal{E}\cong\mathbf{R}\mathcal{H}om((-1)^*\mathbf{R}\Phi_{\mathcal{P}}\mathcal{O}_X[d],\mathcal{E}), \tag{***}$$

so the right-hand side of (*) is computed as the $0^{\text{th}}$ cohomology sheaf of the right-hand side in (***). But there is a spectral sequence

$$E_2^{p,q} = \mathcal{E}xt^p\big((-1)^*R^{d-q}p_{2*}\mathcal{P},\mathcal{E}\big)\Rightarrow R^{p+q}\mathcal{H}om\big((-1)^*\mathbf{R}\Phi_{\mathcal{P}}\mathcal{O}_X[d],\mathcal{E}\big)$$

with $p\geq 0$ and $q\leq 0$. For degree reasons only $\mathcal{H}om((-1)^*R^dp_{2*}\mathcal{P},\mathcal{E})$ contributes to the $0^{\text{th}}$ term, so we get the required identity of sheaves.

So it remains only to prove (**), for which it suffices to establish that

$$\mathbf{R}\Phi_{\mathcal{P}}\omega_X\cong\mathbf{R}\mathcal{H}om\big((-1)^*\mathbf{R}\Phi_{\mathcal{P}}\mathcal{O}_X[d],\mathcal{O}_{\mathrm{Pic}^0(X)}\big)\cong(\mathbf{R}\Phi_{\mathcal{P}^\vee}\mathcal{O}_X)^\vee[-d].$$

But this is precisely the commutation of Grothendieck duality with integral functors (see for instance [29] Lemma 2.2[9]). □

[9]This is proved in [29] in the context of smooth projective varieties, but as indicated in [30] the same proof works on complex manifolds, due to the fact that the analogue of Grothendieck duality holds by [31].

*Remark 5.3* Note that via the BGG correspondence, one can read off the sheaf $\mathcal{F}$ just from the structure of $Q_X$ as an $E$-module. Thus Theorem D admits the picturesque corollary that $Q_X$ determines the normal cone to the canonical linear series in $\mathrm{Div}^{\{\omega\}}(X)$.

As an application, we consider the question of whether the canonical series $|\omega_X|$ is an irreducible component of the space $\mathrm{Div}^{\{\omega\}}(X)$ of paracanonical divisors: following Beauville [2], one says that $|\omega_X|$ is *exorbitant* if this happens. An immediate consequence of Theorem D is

**Corollary 5.4** *The canonical linear series* $|\omega_X|$ *is exorbitant if and only if the mapping* $\mathbf{P}(\mathcal{F}) \longrightarrow \mathbf{P}(H^d(X, \mathcal{O}_X)) = |\omega_X|$ *in* (1.5) *fails to be surjective.*

Under some additional hypotheses, the criterion in the Corollary can be checked numerically. An amusing consequence of this is that in the setting of Theorem 4.1, the exorbitance of the canonical series actually depends only on the Hodge numbers of $X$. In fact:

**Proposition 5.5** *Assume that the hypotheses of Theorem C (or, more generally, Theorem 4.1) are satisfied, and that*

$$p_g - \chi \leq q - 1. \qquad (*)$$

*Then* $|\omega_X|$ *is exorbitant if and only if the codimension* $(p_g - \chi)$ *Segre number of* $\mathcal{F}^\vee$ *vanishes, i.e. if and only if:*

$$s_{1^{\times(p_g-\chi)}}(\gamma_1, \ldots, \gamma_{q-1}) = 0,$$

*where the quantity in question indicates the Schur function associated to the partition* $(1, \ldots, 1)(p_g - \chi)$ *times.*

Observe that if $\chi > 0$ then $\mathrm{Div}^{\{\omega\}}(X)$ has a unique component of dimension $q + \chi - 1$ dominating $\mathrm{Pic}^0(X)$, so if (*) fails in this case, then necessarily $|\omega_X|$ is exorbitant.

*Proof of Proposition 5.5* According to Corollary 5.4, $|\omega_X|$ is exorbitant if and only if the natural mapping

$$\mathbf{P}(\mathcal{F}) \longrightarrow \mathbf{P}\big(H^d(X, \mathcal{O}_X)\big) = \mathbf{P}^{p_g - 1}$$

fails to be surjective. But the Segre number in question computes the degree in $\mathbf{P}^{q-1}$ of the preimage of a general point in the target, and the statement follows. □

*Example 5.6* Suppose that $X$ is a surface without irrational pencils. Then $p_g - \chi = q - 1$, and the BGG complex takes the form

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-2) \longrightarrow \mathcal{O}_{\mathbf{P}}(-1)^q \longrightarrow \mathcal{O}_{\mathbf{P}}^{p_g} \longrightarrow \mathcal{F} \longrightarrow 0.$$

In this case the Segre number appearing in Proposition 5.5 is the coefficient of $t^{q-1}$ in $(1+t)^q/(1+2t)$, and this is $=0$ if $q$ is even, and $=1$ if $q$ is odd. Thus $|\omega_X|$ is exorbitant if and only if $q$ is even, a fact observed by Beauville in [2], Sect. 4.

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## References

1. Barth, W., Hulek, K., Peters, C., Van de Ven, A.: Compact Complex Surfaces. Springer, Berlin (2004)
2. Beauville, A.: Annulation du $H^1$ et systèmes paracanoniques sur les surfaces. J. Reine Angew. Math. **388**, 149–157 (1988)
3. Bernstein, I.N., Gel'fand, I.M., Gel'fand, S.I.: Algebraic vector bundles on $P^n$ and problems of linear algebra. Funkc. Anal. Prilozh. **12**(3), 66–67 (1978)
4. Catanese, F.: Moduli and classification of irregular Kaehler manifolds (and algebraic varieties) with Albanese general type fibrations. Invent. Math. **104**, 389–407 (1991)
5. Catanese, F.: From Abel's heritage: transcendental objects in algebraic geometry and their algebraization. In: The Legacy of Niels Henrik Abel, pp. 349–394. Springer, Berlin (2004)
6. Causin, A., Pirola, G.P.: Hermitian matrices and cohomology of Kähler varieties. Manuscr. Math. **121**, 157–168 (2006)
7. Chen, J.A., Hacon, Ch.: Characterization of Abelian varieties. Invent. Math. **143**(2), 435–447 (2001)
8. Clemens, H., Hacon, Ch.: Deformations of the trivial line bundle and vanishing theorems. Am. J. Math. **124**(4), 769–815 (2002)
9. Coandă, I.: On the Bernstein-Gel'fand-Gel'fand correspondence and a result of Eisenbud, Flystad, and Schreyer. J. Math. Kyoto Univ. **43**(2), 429–439 (2003)
10. Ein, L., Lazarsfeld, R.: Singularities of theta divisors and the birational geometry of irregular varieties. J. Am. Math. Soc. **10**, 243–258 (1997)
11. Eisenbud, D.: The Geometry of Syzygies. Springer, Berlin (2005)
12. Eisenbud, D., Fløystad, G., Schreyer, F.-O.: Sheaf cohomology and free resolutions over the exterior algebra. Trans. Am. Math. Soc. **355**(11), 4397–4426 (2003)
13. Eisenbud, D., Popescu, S., Yuzvinsky, S.: Hyperplane arrangement cohomology and monomials in the exterior algebra. Trans. Am. Math. Soc. **355**(11), 4365–4383 (2003)
14. Evans, E.G., Griffith, P.: The syzygy problem. Ann. Math. **114**, 323–333 (1981)
15. Green, M., Lazarsfeld, R.: Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville. Invent. Math. **90**, 389–407 (1987)
16. Green, M., Lazarsfeld, R.: Higher obstructions to deforming cohomology groups of line bundles. J. Am. Math. Soc. **1**(4), 87–103 (1991)

17. Grothendieck, A.: Eléments de géométrie algébrique II. Publ. Res. Math. Inst. Hautes Études Sci. **8** (1961)
18. Hacon, Ch.: A derived category approach to generic vanishing. J. Reine Angew. Math. **575**, 173–187 (2004)
19. Hacon, Ch., Pardini, R.: On the birational geometry of varieties of maximal Albanese dimension. J. Reine Angew. Math. **546**, 177–199 (2002)
20. Kleiman, S.L.: The Picard scheme, Fundamental algebraic geometry. In: Math. Surveys Monogr., vol. 123, pp. 235–321. Amer. Math. Soc., Providence (2005)
21. Kollár, J.: Higher direct images of dualizing sheaves II. Ann. Math. **124**, 171–202 (1986)
22. Lazarsfeld, R.: Positivity in Algebraic Geometry I & II. Ergebnisse der Mathematik und ihrer Grenzgebiete, vols. 48, 49. Springer, Berlin (2004)
23. Lombardi, L.: in preparation
24. Mendes Lopes, M., Pardini, R.: On surfaces with $p_g = 2q - 3$. Preprint arXiv:0811.0390
25. Mendes Lopes, M., Pardini, R., Pirola, G.P.: On surfaces of general type with $q = 5$. Preprint arXiv:1003.5991
26. Mohan Kumar, N., Rao, A.P.: Buchsbaum bundles on $\mathbf{P}^n$. J. Pure Appl. Algebra **152**, 195–199 (2000)
27. Okonek, Ch., Schneider, M., Spindler, H.: Vector Bundles on Complex Projective Spaces. Birkhäuser, Boston (1980)
28. Pareschi, G.: Generic vanishing, Gaussian maps, and Fourier-Mukai transform. Preprint math.AG/0310026
29. Pareschi, G., Popa, M.: $GV$-sheaves, Fourier-Mukai transform, and generic vanishing. Preprint math/0608127, Am. J. Math. (to appear)
30. Pareschi, G., Popa, M.: Strong generic vanishing and a higher dimensional Castelnuovo-de Franchis inequality. Duke Math. J. **150**(2), 269–285 (2009)
31. Ramis, J.-P., Rouget, G., Verdier, J.-L.: Dualité relative en géométrie analitique complexe. Invent. Math. **13**, 261–283 (1971)
32. Saito, M.: Decomposition theorem for proper Kähler morphisms. Tohoku Math. J. **42**(2), 127–147 (1990)
33. Serre, J.P.: Géométrie algébrique et géométrie analytique. Ann. Inst. Fourier **6**, 1–42 (1955–1956)
34. Takegoshi, K.: Higher direct images of canonical sheaves tensorized with semi-positive vector bundles by proper Kähler morphisms. Math. Ann. **303**(3), 389–416 (1995)