ON THE CONNECTEDNESS OF DEGENERACY LOCI AND SPECIAL DIVISORS

BY

W. FULTON(1) and R. LAZARSFELD

Brown University
Providence, U.S.A.

Harvard University
Cambridge, U.S.A.

Introduction

Let C be a smooth complex projective curve of genus g, and let J be the Jacobian of C. Upon choosing a base-point in C, J may be identified with the set of linear equivalence classes of divisors of degree d on C. Denote by W_d^r the algebraic subvariety of J parametrizing divisors which move in a linear system of dimension at least r. A fundamental theorem of Kempf [9] and Kleiman and Laksov [11, 12] asserts that these loci are non-empty when their expected dimension

$$\varrho = g-(r+1)(g-d+r)$$

is non-negative. We complement this existence theorem with two results on the global structure of W_d^r when $\varrho>0$. First of all, for an arbitrary curve C, we prove

THEOREM I. *If $\varrho>0$, then W_d^r is connected.*

When C is generic (in the sense of moduli), deep results about the local geometry of W_d^r have been obtained by Griffiths and Harris [5] and by Gieseker [4]. Combining these with Theorem I, we deduce the

COROLLARY. *For a generic curve C, W_d^r is irreducible when $\varrho>0$.*

By a standard construction, W_d^r may be realized as the locus where a certain homomorphism of vector bundles on J drops rank. Theorem I then becomes a simple consequence of a general result—of independent interest—on the connectivity of such degeneracy loci.

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Specifically, let X be an irreducible complex projective variety of dimension n, and let

$$\sigma: E \to F$$

be a homomorphism of vector bundles of ranks e and f on X. Put

$$D_k(\sigma) = \{x \in X \mid \operatorname{rank} \sigma(x) \leq k\}.$$

If non-empty, the degeneracy locus $D_k(\sigma)$ has codimension $\leq (e-k)(f-k)$ in X. Under a suitable positivity hypothesis on the bundles in question, we prove that $D_k(\sigma)$ is connected as soon as its expected dimension is positive:

THEOREM II. *Assume that the vector bundle* $E^* \otimes F = \operatorname{Hom}(E, F)$ *is ample. Then:*

(a) $D_k(\sigma)$ *is non-empty when* $n \geq (e-k)(f-k)$,

and

(b) $D_k(\sigma)$ *is connected when* $n > (e-k)(f-k)$.

It was suggested in [3, § 10] that such a result should hold. Note that it is not assumed that $D_k(\sigma)$ actually has the expected dimension.

The present paper is divided into two parts. § 1 is devoted to the proof of Theorem II, and to a simple consequence concerning the singularities of finite mappings to projective space. The application to loci of special divisors occupies § 2. Our construction of W_d^r follows the well-known approach of Grothendieck, Mattuck, Schwartzenberger, Kempf, Kleiman and Laksov. Since only an elementary part of their work is needed, we have included details for the convenience of the reader. We remark that statement (a) of Theorem II gives rise to a simple proof of the Kempf–Kleiman–Laksov existence theorem for special divisors, bypassing the Chern class computations of the original proofs. On the other hand, those calculations lead to a formula for the cohomology class of W_d^r, which is important in enumerative questions. We recommend Chapter 3 of the forthcoming book [1], whose notation we follow, for an account of results along these lines.

Finally, a word on the proof of Theorem II may prove helpful. The strategy is to reduce the problem to proving the vanishing of certain singular cohomology groups, and then to draw on Lefschetz-type theorems to establish these vanishings. Consider for example the special case when E is a trivial line bundle and $k=0$, so that $D_0(\sigma) = Z(\sigma)$, the zero-locus of the section σ of the ample vector bundle F. If F is a line bundle, then $X - Z(\sigma)$ is affine; hence $H^i(X - Z(\sigma)) = 0$ for $i \geq \dim X + 1$, and this easily leads to a proof that $Z(\sigma)$ is connected if $\dim X \geq 2$. When $f = \operatorname{rk}(F) > 1$, σ determines a section σ^* of the line bundle $\mathcal{O}(1)$ on $\mathbf{P}(F^*)$. Since $\mathbf{P}(F^*) - Z(\sigma^*)$ fibres over $X - Z(\sigma)$ with fibres $\mathbf{C}^{f-1}$, and since $\mathcal{O}(1)$ is ample on $\mathbf{P}(F^*)$ by the ampleness of F, one deduces that $H^i(X - Z(\sigma)) = 0$ for $i \geqslant \dim X + f$; the connectedness of $Z(\sigma)$ follows when $\dim X \geqslant f+1$. This argument was used by Sommese [15], and an elaboration of this construction plays an important role in the proof of Theorem II.

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§ 1. Connectedness of degeneracy loci

We start with some notation. If A is a vector bundle on a variety X, $A(x)$ denotes the fibre of A at a point $x \in X$. $\mathbf{P}(A)$ is the projective bundle whose fibre over $x \in X$ is the projective space of one-dimensional *subspaces* of $A(x)$. Finally, recall that by definition, A is *ample* if the line bundle $\mathcal{O}_{\mathbf{P}}(1)$ on $\mathbf{P}(A^*)$ is ample [7].

This section is devoted to the proof of

THEOREM 1.1. *Let X be an irreducible complex projective variety of dimension n, and let*

$$\sigma: E \to F$$

be a homomorphism of vector bundles on X of ranks e and f. Assume that the vector bundle

$$E^* \otimes F = \mathrm{Hom}(E, F)$$

is ample, and let

$$D_k(\sigma) = \{x \in X \mid \operatorname{rank} \sigma(x) \leqslant k\}.$$

Then:

(a) *$D_k(\sigma)$ is non-empty if $n \geqslant (e-k)(f-k)$,*

and

(b) *$D_k(\sigma)$ is connected when $n > (e-k)(f-k)$.*

Proof. To begin with, note that it suffices to prove the theorem for normal varieties. For if $\nu: \tilde{X} \to X$ is the normalization of X, then ν gives rise to a homomorphism $\tilde{\sigma}: \nu^*(E) \to \nu^*(F)$ of vector bundles on X, and $D_k(\tilde{\sigma})$ surjects onto $D_k(\sigma)$. Moreover $\nu^*(E^*) \otimes \nu^*(F)$ is the pull-back of an ample vector bundle under a finite morphism, and hence is ample. Thus, we may assume that X is normal. Furthermore, replacing σ if necessary by its transpose, we may suppose that $f \geqslant e$.

Put $k' = e-k$, and let

$$\pi: \mathbf{G} = G_{k'}(E) \to X$$

be the Grassmannian bundle of k'-planes in E. Denote by S the rank k' tautological subbundle of π^*E on $\mathbf{G}$. Then there is a natural homomorphism

$$\tau: S \to \pi^*F,$$

defined as the composition $S \hookrightarrow \pi^*E \xrightarrow{\pi^*\sigma} \pi^*F$. Thus τ takes a k'-plane in $E(x)$ to its image under $\sigma(x)$. Let

$$Y \subseteq \mathbf{G}$$

denote the zero-locus of τ (i.e. the zero-set of the corresponding section of $S^* \otimes \pi^*F$). Then Y surjects onto $D_k(\sigma)$, so it is enough to show that Y is non-empty and connected in the appropriate range of dimensions. To this end, we will study the cohomology of $\mathbf{G}-Y$. Specifically, letting $r = n + k(e-k) = \dim \mathbf{G}$, we will prove

PROPOSITION 1.2. *If $i \geqslant r + k'f = n + (f+k)(e-k)$, then*

$$H^i(\mathbf{G}-Y;\, \mathbf{Z}) = 0.$$

Observe that the proposition indeed implies the theorem. For if $n \geqslant (e-k)(f-k)$, then

$$r = n + k(e-k) \geqslant (e-k)f,$$

and so $H^{2r}(\mathbf{G}-Y;\, \mathbf{Z}) = 0$. In particular, $\mathbf{G}-Y$ is not compact. Therefore Y, and hence also $D_k(\sigma)$, must be non-empty. Similarly, if $n > (e-k)(f-k)$, then the proposition implies that $H_{2r-1}(\mathbf{G}-Y;\, \mathbf{Q}) = 0$. Since X—and thus $\mathbf{G}$—is normal, the following lemma yields the vanishing of $H^1(\mathbf{G}, Y;\, \mathbf{Q})$, and hence the connectivity of Y.

LEMMA 1.3. *Let G be a normal projective variety of dimension r, and let $Y \subseteq G$ be a closed algebraic subset. Then there is an injection*

$$H^1(G, Y;\, \mathbf{Q}) \hookrightarrow H_{2r-1}(G-Y;\, \mathbf{Q}).$$

Proof. This follows from the exact sequence of low degree terms of the Zeeman spectral sequence (cf. [13]). Q.E.D.

Lemma 1.3 was pointed out to us by K. Vilonen. When X is smooth, one can use Lefschetz duality on $\mathbf{G}$ in place of Lemma 1.3.

The proof of Proposition 1.2 depends on a simple construction, which we now describe. Consider a homomorphism

$$h: A \to B$$

of vector bundles of ranks a and b on a variety W. Let $Y \subseteq W$ be the zero-locus of h, h being considered as a section of the vector bundle $A^*\otimes B=\mathrm{Hom}(A,B)$. Now on the projective bundle $p\colon \mathbf{P}=\mathbf{P}(\mathrm{Hom}(B,A))\to W$, there is a "tautological" map

$$p^*B\to p^*A\otimes O_{\mathbf{P}}(1).$$

Composing this with $p^*h\colon p^*A\to p^*B$, one obtains a homomorphism $h^*\colon p^*A\to p^*A\otimes O_{\mathbf{P}}(1)$. The trace of h^* then defines a section

$$\mathrm{tr}\,(h^*)\in\Gamma(\mathbf{P},O_{\mathbf{P}}(1)).$$

Denote by $Y^*\subseteq\mathbf{P}$ the zero-locus of $\mathrm{tr}\,(h^*)$. Concretely, one may think of a point in $\mathbf{P}(\mathrm{Hom}(B,A))$ as a homomorphism $\varphi\colon B(x)\to A(x)$ (defined modulo scalars), where $x=p(\varphi)\in W$, and then

$$Y^*=\{\varphi\in\mathbf{P}(\mathrm{Hom}(B,A))\,|\,\mathrm{tr}\,(\varphi\circ h(p(\varphi))=0\}.$$

The point to observe is that the projection p gives rise to a map

$$\mathbf{P}-Y^*\to W-Y$$

which is locally trivial, with fibres $\mathbf{C}^{ab-1}$. In particular,

$$H^*(W-Y)\overset{\cong}{\to}H^*(\mathbf{P}-Y^*)\tag{1.4}$$

is an isomorphism. (Compare [15].)

Proof of Proposition 1.2. We first apply the construction just described to the homomorphism $\tau\colon S\to\pi^*F$ on $\mathbf{G}$. Thus we consider the projective bundle

$$p\colon\mathbf{P}=\mathbf{P}(\mathrm{Hom}(\pi^*F,S))\to\mathbf{G},$$

and the section $\mathrm{tr}\,(\tau^*)\in\Gamma(\mathbf{P},O_{\mathbf{P}}(1))$. Let $Y^*\subseteq\mathbf{P}$ denote the zero locus of $\mathrm{tr}\,(\tau^*)$, and set

$$V^*=\mathbf{P}-Y^*.$$

In view of (1.4), the proposition is equivalent to

$$H^i(V^*;\mathbf{Z})=0\quad\text{for } i\geqslant\dim V^*+1.\tag{1.5}$$

On the other hand, consider the projective bundle $q\colon\mathbf{P}'=\mathbf{P}(\mathrm{Hom}(F,E))\to X$ over X. Then there is a natural map $g\colon\mathbf{P}\to\mathbf{P}'$, which has the following concrete description. We may think of a point in $\mathbf{P}$ as a homomorphism $\varphi\colon F(x)\to S(x)$ (mod scalars), where $x=\pi\circ p(\varphi)\in X$, and $S(x) \subseteq E(x)$ is a subspace of dimension k'. Then $g(\varphi) \in \mathbf{P}'$ is represented by the composition $F(x) \to S(x) \hookrightarrow E(x)$. Let $Z^* \subseteq \mathbf{P}'$ denote the divisor of the section $\operatorname{tr}(\sigma^*) \in \Gamma(\mathbf{P}', O_{\mathbf{P}'}(1))$ arising from $\sigma: E \to F$. Then $g^{-1}(Z^*) = Y^*$, and so g restricts to a proper morphism

$$h: V^* = \mathbf{P} - Y^* \to \mathbf{P}' - Z^* = U^*,$$

where U^* denotes the complement of Z^* in $\mathbf{P}'$. But since $\operatorname{Hom}(E, F)$ is an ample vector bundle on X, the line bundle $O_{\mathbf{P}'}(1)$ is ample on $\mathbf{P}' = \mathbf{P}(\operatorname{Hom}(E, F)^*)$. Thus U^* *is an affine variety*. The strategy now is to use the Leray spectral sequence for h to deduce (1.5) from a theorem on the vanishing of the cohomology of an affine variety.

To this end, we analyze the fibres of h. For $0 \leqslant l \leqslant k' - 1$, define a subvariety $\mathbf{P}'_l \subseteq \mathbf{P}'$ by

$$\mathbf{P}'_l = \{\varphi \in \mathbf{P}(\operatorname{Hom}(F, E)) \mid \operatorname{rank} \varphi \leqslant k' - l\},$$

and let

$$U_l^* = U^* \cap \mathbf{P}'_l.$$

Then each U_l^* is an affine variety, with $U_{l+1}^* \subseteq U_l^*$, and

$$\operatorname{codim}_{U_0^*} U_l^* = 2kl + l^2 + (f-e)l. \text{ (1)}$$

If $\varphi' \in \mathbf{P}'$ is represented by a homomorphism $\varphi': F(x) \to E(x)$ $(x = q(\varphi') \in X)$, and if $\varphi \in \mathbf{P}$ is represented by $\varphi: F(x) \to S(x)$, where $S(x)$ is a k'-plane in $E(x)$, then $\varphi \in h^{-1}(\varphi')$ if and only if φ' coincides (mod scalars) with the composition $F(x) \to S(x) \hookrightarrow E(x)$. Hence h maps V^* birationally onto U_0^*. Moreover the fibre of h over a point $\varphi' \in U_l^* - U_{l+1}^*$ is a Grassmannian $G(l, l+k)$, and so has dimension lk. Since $\operatorname{codim}_{U_0^*} U_l^* \geqslant 2lk$, the following lemma applies to the map $h: V^* \to U_0^*$ to yield (1.5).

LEMMA 1.6. *Let $f: X \to Y$ be a proper surjective morphism of irreducible varieties, with Y affine. Assume that for each $d \geqslant 0$ the set*

$$Y_d = \{y \in Y \mid \dim f^{-1}(y) \geqslant d\}$$

has codimension $\geqslant 2d$ in Y (so that in particular f is generically finite). Then

$$H^i(X, \mathbf{Z}) = 0 \quad \text{for } i \geqslant \dim X + 1.$$

Proof. Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_* \mathbf{Z}) \Rightarrow H^{p+q}(X, \mathbf{Z})$$

(1) Recall that we are assuming that $f \geqslant e$.

for f. Since f is proper, the sheaves

$$R^{2d-1}f_*\mathbf{Z},\ R^{2d}f_*\mathbf{Z}$$

are supported on Y_d. But these are constructible sheaves on the affine variety Y_d, and so their cohomology vanishes in degrees above the dimension of Y_d (cf. Artin [2, Exp. XIV, § 2, § 3], [16]):

$$H^p(Y, R^{2d-1}f_*\mathbf{Z}) = H^p(Y, R^{2d}f_*\mathbf{Z}) = 0 \quad \text{for } p \geqslant \dim Y_d+1.$$

Since $\dim Y_d+2d \leqslant \dim Y = \dim X$, one thus has $E_2^{p,q}=0$ for $p+q \geqslant \dim X+1$, and the lemma follows. Q.E.D.

This completes the proof of Theorem 1.1.

Remark 1.7. At least if X is non-singular, the same proof works in characteristic p using étale cohomology in place of singular cohomology. In the part of the argument preceding Lemma 1.3, $H^1_c(G-Y)$ is used in place of $H^1(G, Y)$, and Poincaré duality between $H^1_c(G-Y)$ and $H^{2r-1}(G-Y)$ replaces Lemma 1.3 (cf. Deligne [2, Exp. XVIII]).

Remark 1.8. The locus $D_k(\sigma)$ has a natural scheme structure, given locally by the vanishing of the $(k+1)\times(k+1)$ minors of σ. If X is non-singular, and $D_k(\sigma)$ has the expected codimension $(e-k)(f-k)$, then $D_k(\sigma)$ is a Cohen-Macaulay scheme [10]. If, in addition, $D_k(\sigma)$ is non-singular in codimension one—for example, if $D_{k-1}(\sigma)$ is the singular locus of $D_k(\sigma)$—then the connectivity of $D_k(\sigma)$ is equivalent to the *irreducibility* of $D_k(\sigma)$. Indeed, if S is the singular locus of $D_k(\sigma)$, and $D_k(\sigma)$ is connected, and if the local ring of $D_k(\sigma)$ has depth at least two at every point of S, then a theorem of Hartshorne's [6] asserts that $D_k(\sigma)-S$ is connected.

Remark 1.9. One expects a connectedness theorem such as Theorem 1.1 to extend to a Lefschetz-type result on the vanishing of higher relative homology or homotopy groups (cf. [3, § 9]). If X is non-singular, Proposition 1.2 and duality give (with notation as before)

$$H_i(G, Y; \mathbf{Z}) = 0 \quad \text{for } i \leqslant n-(e-k)(f-k). \tag{*}$$

The corresponding groups $H^i(X, D_k(\sigma))$ need not vanish, however. For example, the Segre variety $\mathbf{P}^n\times\mathbf{P}^1$ in $X=\mathbf{P}^{2n+1}$ is the degeneracy locus $D_1(\sigma)$, where σ is a $2\times(n+1)$ matrix of linear forms, but $H_3(X, D_1(\sigma)) \neq 0$, even for large n. Nonetheless there is a partial result, namely that the canonical map

$$H_i(D_k(\sigma); \mathbf{Z}) \to H_i(X; \mathbf{Z})$$

is *surjective* for $i \leqslant n-(e-k)(f-k)$. This follows from (*) and the observation that in the diagram

$$\begin{array}{ccc} H_i(Y;\mathbf{Z}) & \longrightarrow & H_i(G;\mathbf{Z}) \\ \downarrow & & \downarrow \\ H_i(D_k(\sigma);\mathbf{Z}) & \longrightarrow & H_i(X;\mathbf{Z}), \end{array}$$

the right-hand vertical map is always surjective.

We close this section with a simple application of Theorem 1.1. Given a morphism $f\colon X^n \to Y^m$ of non-singular varieties, with $m \geqslant n$, let df_x denote the induced map of tangent spaces at $x \in X$. Set

$$S_i(f) = \{x \in X \mid \operatorname{rank} df_x \leqslant n-i\}.$$

Theorem 1.1 may be applied to the vector bundle homomorphism

$$df\colon T_X \to f^*T_Y.$$

If $\Omega_X^1 \otimes f^*T_Y$ is ample, then $S_i(f)$ will be non-empty if $i(m-n+i) \leqslant n$, and connected if $i(m-n+i) < n$. These hypotheses are satisfied, for example, if $Y = \mathbf{P}^m$, if Ω_X^1 is generated by its global sections, and if $f\colon X \to \mathbf{P}^m$ is *any* finite morphism. The following result applies to more general varieties X.

PROPOSITION 1.10. *Let L be a very ample line bundle on a smooth n-dimensional variety X. Let $f\colon X \to \mathbf{P}^m$ be the morphism defined by a base-point free linear system in $|L^{\otimes k}|$, for some $k \geqslant 2$. Then $S_i(f)$ is non-empty if $i(m-n+i) \leqslant n$, and connected if $i(m-n+i) < n$.*

Proof. Let $X \hookrightarrow \mathbf{P}^N$ be the embedding defined by L, so that $f\colon X \to \mathbf{P}^m$ is given by homogeneous forms $F_0, \ldots, F_m$ of degree k in the coordinates $X_0, \ldots, X_N$ of $\mathbf{P}^N$. Consider the Euler sequence

$$0 \to O_{\mathbf{P}^N} \to O_{\mathbf{P}^N}(1)^{\oplus(N+1)} \to T_{\mathbf{P}^N} \to 0.$$

Define E to be the kernel of the composition $L^{\oplus(N+1)} \to T_{\mathbf{P}^N} \mid X \to N_{X|\mathbf{P}^N}$, and let $F = (L^{\otimes k})^{\oplus(m+1)}$. The Jacobian matrix $(\partial F_i/\partial X_j)$ determines a morphism $J\colon E \to F$ so that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & O_X & \longrightarrow & E & \longrightarrow & T_X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cdot k} & & \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle df} & & \\ 0 & \longrightarrow & O_X & \longrightarrow & F & \longrightarrow & f^*T_{\mathbf{P}^m} & \longrightarrow & 0 \end{array}$$

commutes. Then $S_i(f) = D_{n+1-i}(J)$, and $E^* \otimes F$ is ample, since it is a quotient of a direct sum of copies of the ample line bundle $L^* \otimes L^{\otimes k} = L^{\otimes(k-1)}$. Q.E.D.

§ 2. Application to special divisors

Let C be a non-singular complex projective curve of genus g, let $J=\operatorname{Pic}^0(C)$ be the Jacobian of C, and fix once and for all a base-point $P_0 \in C$. Given $x \in J$, denote by L_x the corresponding line bundle of degree zero on C, and set

$$W_d^r = \{x \in J \mid h^0(C, L_x \otimes O_C(dP_0)) \geq r+1\}.$$

Thus W_d^r parametrizes classes of divisors of degree d which move in a linear system of (projective) dimension at least r. By a well-known construction which we review below, if $m \gg 0$ is a sufficiently large integer, then evaluation at t distinct points $P_1, ..., P_t$ yields a map

$$H^0(C, L_x \otimes O_C(mP_0)) \to \bigoplus_{i=1}^{t} \mathbf{C}(P_i)$$

which globalizes to a homomorphism of vector bundles

$$\sigma_t \colon E_m \to F_t \tag{2.1}$$

on J, of ranks $m+1-g$ and t respectively. Noting that

$$\ker \sigma(x) = H^0(C, L_x \otimes O_C(mP_0 - \sum P_i)),$$

and taking $t=m-d$, one sees that $W_d^r \cong D_k(\sigma_{m-d})$, where $k=m-g-r$. In particular, the expected dimension of W_d^r is given by the Brill-Noether number

$$\varrho = g-(r+1)(g-d+r).$$

For our purposes, the basic fact is

LEMMA 2.2. *For any $m \geq 2g$ and $t \geq 1$, $E_m^* \otimes F_t$ is an ample vector bundle on J.*

Grant the lemma for the moment. Then Theorem 1.1 (a) yields the result of Kempf [9] and Kleiman-Laksov [11, 12] that W_d^r is non-empty when $\varrho \geq 0$, while Theorem 1.1 (b) implies

THEOREM 2.3. *W_d^r is connected if $\varrho > 0$.*

Furthermore, by Remark 1.9, *the homomorphisms*

$$H_i(W_d^r; \mathbf{Z}) \to H_i(J; \mathbf{Z})$$

induced by inclusion are surjective for $i \leq \varrho$.

On special curves, the loci W_d^r may well be reducible even when the Brill-Noether number ϱ is positive. For example if C is trigonal and non-hyperelliptic, of genus 5, then W_4^1 has two irreducible (but intersecting!) one-dimensional components. In fact, if D is a trigonal divisor, then W_4^1 is swept out by the two families

$$\{|D+P|\}_{P\in C},\ \{|K-D-P|\}_{P\in C}.$$

On a *generic* curve of genus g, however, this cannot happen. Indeed, Griffiths and Harris [5] have shown that for a general curve, W_d^r has pure dimension ϱ, and Gieseker [4] has very recently proved that, for general C, W_d^r is singular only along W_d^{r+1}. Hence from Remark 1.8 one deduces

COROLLARY 2.4. *If C is a generic curve, then W_d^r is irreducible when $\varrho > 0$.*

The remainder of this section is devoted to the proof of Lemma 2.2. To fix notation, we start by reviewing the construction of the vector bundle homomorphism (2.1). Let $\mathcal{L}$ be the universal line bundle on $J\times C$, so that $\mathcal{L}|p^{-1}(x)=L_x$, where $p\colon J\times C\to J$ and $q\colon J\times C\to C$ are the projections. Letting $\mathcal{L}(P)$ denote the line bundle $\mathcal{L}|q^{-1}(P)$ on J, we normalize $\mathcal{L}$ (by tensoring with a line bundle from J) so that $\mathcal{L}(P_0)=O_J$. Fix $m\geqslant 2g-1$; thus $H^1(C, L_x\otimes O_C(mP_0))=0$ for all $x\in J$. Then

$$E_m \stackrel{\text{def}}{=} p_*(\mathcal{L}\otimes q^* O_C(mP_0))$$

is a vector bundle of rank $m+1-g$, and pushing forward $\mathcal{L}\otimes q^*O_C(mP_0)$ via p_* commutes with base change. Let D_t be the divisor $P_1+\ldots+P_t$ on C, and set

$$F_t = \bigoplus_{i=1}^{t} \mathcal{L}(P_i) = p_*(\mathcal{L}\otimes q^* O_{D_t}).$$

The homomorphism (2.1) arises by taking the direct images of the last two terms in the exact sequence

$$0\to \mathcal{L}\otimes q^*O_C(mP_0-D_t)\to \mathcal{L}\otimes q^*O_C(mP_0)\to \mathcal{L}\otimes q^*O_{D_t}\to 0 \tag{2.5}$$

of sheaves on $J\times C$.

Turning to the verification that

$$E_m^*\otimes F_t = \bigoplus_{i=1}^{t} E_m^*\otimes \mathcal{L}(P_i)$$

is an ample vector bundle, the first point to observe is that it suffices to prove the ampleness of E_m^*. Indeed, since a direct sum of vector bundles is ample if and only if each summand is [7], it is certainly enough to check the ampleness of $E^*\otimes\mathcal{L}(P)$ for an arbitrary point $P\in C$. Since $\mathcal{L}(P_0)=O_J$, by varying P over C one exhibits $\mathcal{L}(P)$ as a deformation of a trivial line bundle. But quite generally, if X is an irreducible projective variety, and L a line bundle on X algebraically equivalent to zero, then a vector bundle E on X is ample if and only if $E \otimes L$ is. For if $\pi: \mathbf{P}(E^*) \to X$ is the projectivization of E^*, then we may identify $\mathbf{P}(E^*)$ with $\mathbf{P}(E^* \otimes L^*)$, and the assertion becomes that $O_{\mathbf{P}}(1)$ is an ample line bundle on $\mathbf{P}(E^*)$ if and only if $O_{\mathbf{P}}(1) \otimes \pi^* L^*$ is. Observing that $c_1(O_{\mathbf{P}}(1) \otimes p^* L^*)$ is numerically equivalent to $c_1(O_{\mathbf{P}}(1))$, this follows from Nakai's criterion (cf. [8, Chapter I]).

The ampleness of E_m^* is equivalent, by definition, to the ampleness of the line bundle $O_{\mathbf{P}}(1)$ on $\mathbf{P} = \mathbf{P}(E_m)$. Letting $t=1$ and $D_i = P_0$ in (2.5) and taking direct images on J, one obtains (for $m \geqslant 2g$) an exact sequence

$$0 \to E_{m-1} \to E_m \to O_J \to 0.$$

Such a sequence gives rise to a section s of $O_{\mathbf{P}}(1)$ whose divisor is exactly the subvariety $\mathbf{P}(E_{m-1}) \subseteq \mathbf{P}(E_m)$. The crucial geometric fact is then the following (cf. [14] or [1]):

$\mathbf{P}(E_m)$ is isomorphic to C_m, the m-th symmetric product of C, and the divisor $\mathbf{P}(E_{m-1})$ corresponds to the $(m-1)$-th symmetric product C_{m-1}, embedded in C_m via the map $D \to D + P_0$. (2.6)

Granting (2.6), the ampleness of $O_{\mathbf{P}}(1)$—and hence Lemma 2.2—follows from

LEMMA 2.7. *For all $m \geqslant 1$, C_{m-1} is an ample divisor on C_m.*

Proof. Fix a k-dimensional subvariety $V \subseteq C_m$. By Nakai's criterion, it suffices to show that the intersection number $\zeta^k \cdot V$ is positive, where ζ denotes the numerical equivalence class of C_{m-1}. For any $P \in C$, let $C_{m-1}(P)$ be the divisor on C_m obtained by embedding C_{m-1} in C_m via the map $D \to D + P$. Then $C_{m-1} = C_{m-1}(P_0)$ is algebraically—and hence numerically—equivalent to $C_{m-1}(P)$ for any $P \in C$. Now given V as above, then $C_{m-1}(P)$ meets V in a non-empty divisor on V for almost every $P \in C$. (Indeed, this is simply the assertion that given any k-dimensional family of divisors on C, almost every $P \in C$ is contained in some but not all of the divisors in the family.) Therefore, for k generic points $P_1, \ldots, P_k \in C$, the intersection

$$C_{m-1}(P_1) \cap \ldots \cap C_{m-1}(P_k) \cap V$$

is a finite non-empty set. Since $\zeta^k \cdot V$ is the degree of the corresponding intersection cycle, the positivity of $\zeta^k \cdot V$ follows. Q.E.D.

Alternatively, one could prove the lemma by observing that if $w: C^m \to C_m$ is the map to C_m from the Cartesian product of C with itself m times, then $w^* O(C_{m-1}) = \otimes_{i=1}^m \pi_i^* O_C(P_0)$, where $\pi_i\colon C^m \to C$ is projection onto the ith factor. Thus $w^*\mathcal{O}(C_{m-1})$ is ample. But since $w\colon C^m \to C_m$ is finite and surjective, the ampleness of $\mathcal{O}(C_{m-1})$ is equivalent to that of $w^*\mathcal{O}(C_{m-1})$.

Finally, for the convenience of the reader, we sketch a proof of (2.6). Let

$$u\colon C_m \to J$$

be the Abel map, which takes a divisor D of degree m to the divisor class of $D - mP_0$. We first show that there is a morphism $v\colon C_m \to \mathbf{P}(E_m)$, compatible with projections to J, such that

$$C_m \times_{\mathbf{P}(E_m)} \mathbf{P}(E_{m-1}) = C_{m-1}.$$

To this end, let $\mathcal{D}_m \subseteq C_m \times C$ be the universal divisor of degree m: $\mathcal{D}_m = \{(D, P) \mid P \in D\}$. Bearing in mind the chosen normalization of $\mathcal{L}$, it follows from the universal property of J that

$$(u \times 1)^*(\mathcal{L} \otimes q^*(mP_0)) = \mathcal{O}(\mathcal{D}_m) \otimes p'^*\mathcal{O}_{C_m}(-C_{m-1}),$$

where $p'\colon C_m \times C \to C_m$ denotes the projection. Since taking the direct image of $\mathcal{L} \otimes q^*\mathcal{O}_C(mP_0)$ on J commutes with base-change, one then has

$$p'_*\mathcal{O}(\mathcal{D}_m) \otimes \mathcal{O}_C(-C_{m-1}) = u^*E_m.$$

Now the canonical section of $\mathcal{O}(\mathcal{D}_m)$ gives rise to a nowhere vanishing section of $p'_*\mathcal{O}(\mathcal{D}_m)$, and so one obtains an inclusion

$$\mathcal{O}(-C_{m-1}) \hookrightarrow u^*E_m$$

of vector bundles on C_m. But this is equivalent to giving a morphism $v\colon C_m \to \mathbf{P}(E_m)$ over J such that $v^{-1}(\mathbf{P}(E_{m-1})) = C_{m-1}$. Note next that v is bijective. Indeed, it suffices to check this fibre by fibre over J, where it is clear. Then since $\mathbf{P}(E_m)$ is smooth, it follows that v is an isomorphism. (In positive characteristic, one would observe in addition that since $C_m \times_{\mathbf{P}(E_m)} \mathbf{P}(E_{m-1}) = C_{m-1}$ is smooth, v must be separable.) Q.E.D.

Remark 2.8. The proof of Lemma 2.2 works for curves over an arbitrary algebraically closed field. Granting the results of [4] and [5] in positive characteristic, and making use of Remark 1.7, one deduces that Theorem 2.3 and Corollary 2.4 are also valid in arbitrary characteristic.

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