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Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension

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Introduction

In recent years, the equations defining a projective variety $X \subset \mathbf{P}^r$ and the syzygies among them have attracted considerable attention. Classical results giving conditions to guarantee that X is projectively normal or cut out by quadrics have emerged as the first cases of more general statements about higher syzygies. Specifically, theorems in this direction have been established for curves [G1], finite sets [GL], and abelian varieties [K]. Mukai observed that one can view the known results as dealing with embeddings defined by line bundles of the type $K_X + P$, where P is an explicit multiple of a suitably positive bundle. He suggested that in this form, analogous statements should hold on an arbitrary smooth projective variety X. Our first purpose here is to show that this is indeed the case, at least when P is a multiple of a very ample line bundle.

To give precise statements, we require some notation and definitions. Let X be a smooth complex projective variety of dimension n, and let L be a very ample line bundle on X, defining an embedding $X \subset \mathbf{P} = \mathbf{P}H^0(L)$. Denote by $S = \operatorname{Sym}^\bullet H^0(L)$ the homogeneous coordinate ring of the projective space $\mathbf{P}$, and

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consider the graded S-module $R = R(L) = \oplus H^0(X, L^d)$. Let $E_\bullet$ be a minimal graded free resolution of R:

$$\begin{array}{ccccccccc} \cdots \to & \oplus S(-a_{2,j}) & \to & \oplus S(-a_{1,j}) & \to & \oplus S(-a_{0,j}) & \to R \to 0\ . \\ & \| & & \| & & \| & \\ & E_2 & & E_1 & & E_0 & \end{array}$$

We have indicated here the fact that R has a canonical generator in degree zero. Observe that all $a_{0,j} \geqq 2$ since we are dealing with a linearly normal embedding, and that all $a_{i,j} \geqq i+1$ when $i \geqq 1$ thanks to the fact that X doesn't lie on any hyperplanes.

We shall be concerned with situations in which the first few modules of syzygies of $R(L)$ are as simple as possible:

Definition. The line bundle L satisfies *Property* (N_p) if

$$E_0 = S \quad \text{when } p \geqq 0$$

and

$$E_i = \oplus S(-i-1) \quad \text{(i.e. all } a_{i,j} = i+1) \quad \text{for } 1 \leqslant i \leqslant p\ .$$

Note that if $E_0 = S$, then $E_\bullet$ determines a resolution of the homogeneous ideal $I = I_{X/\mathbf{P}}$ of X in $\mathbf{P}$. Thus the definition may be summarized very concretely as follows:

L satisfies (N_0) $\Leftrightarrow$ X embeds in $\mathbf{P}H^0(L)$ as a projectively normal variety;
L satisfies (N_1) $\Leftrightarrow$ (N_0) holds for L, and the homogeneous ideal I of X is generated by quadrics;
L satisfies (N_2) $\Leftrightarrow$ (N_0) and (N_1) hold for L, and the module of syzygies among quadratic generators $Q_i \in I$ is spanned by relations of the form $\sum L_i \cdot Q_i = 0$, where the L_i are *linear* polynomials;

and so on. Properties (N_0) and (N_1) are what Mumford [M2] calls respectively *normal generation* and *normal presentation.*

We refer the reader for instance to [L2, Sect. 1] for a survey of some of the many classical statements on the projective normality and defining equations of finite sets, curves and abelian varieties. Concerning higher syzygies, the first geometric result was due to Green [G1], who proved that if X is a smooth curve of genus g, and if $\deg(L) \geqq 2g+1+p$, then L satisfies (N_p). This was recovered in [GL2] as a consequence of an analogous statement for finite sets. The second author conjectured in [L2] that if L is an ample line bundle on an abelian variety, then $L^{\otimes(p+3)}$ satisfies (N_p), and Kempf proved a slightly weaker result in [K]. On an arbitrary variety, Green [G2] showed that any "sufficiently positive" bundle L satisfies (N_p), but his theorem does not include any explicit conditions on L to guarantee (N_p). In fact, for a long time it was not even clear (at least to the present authors) what form an explicit result might take. Mukai then remarked that one could view the theorem for curves as asserting that if D is an ample bundle on a curve X, then (N_p) holds for $K_X + (p+3)D$. And of course put like this, one sees what shape a general statement should have.

Our first main result is the following

Theorem 1 *Let X be a smooth complex projective variety of dimension n, and let A be a very ample line bundle on X. Then (N_p) holds for the bundle $K_X + (n + 1 + p)A$. More generally, if B is any numerically effective line bundle on X, then*

$$K_X + (n + 1 + p)A + B \quad \textit{satisfies } (N_p) .$$

In other words, (N_p) holds for L as soon as L is "at least as positive as" $K_X + (n + 1 + p)A$. The theorem implies for instance that under the embedding defined by $K_X + (n + 2)A$, the homogeneous ideal of X is generated by quadrics. We remark that at least for $p = 0$ and 1, Butler [B] has constructed examples of n-folds X carrying an ample (but not very ample!) bundle D such that (N_p) fails for $K_X + (n + 1 + p)D$.

When $p = 0$, the theorem asserts the projective normality of the embedding defined by $K_X + (n + 1)A + B$ (see (0.4)). A quick proof of this fact using the Kodaira vanishing theorem was given previously with Bertram [BEL]. In fact, the argument in [BEL] gives the stronger statement that $K_X + nA + B$ is normally generated provided that it is very ample. These results on normal generation have been obtained independently by Andreatta, Ballico, and Sommese [ABS, ABS]. Starting with the projective normality of $K_X + nA + B$, one can show by the methods of the present paper that *if $p \geqq 1$, then with one exception (N_p) holds already for $K_X + (n + p)A + B$* (see §3). However in the interests of unity we focus here mainly on the theorem as stated. In a somewhat different direction, Butler [B] has established that if X is a ruled variety over a smooth curve, then $K_X + (2n + 1)D$ is normally generated provided merely that D is an ample line bundle, and he has obtained analogous statements for higher syzygies. Butler's work on scrolls was important in guiding our thinking on these questions.

It is well known that results on syzygies may be interpreted in terms of the vanishing of certain Koszul cohomology groups (cf. [G1, G2]). For $X = \mathbf{P}^n$, Green [G2, G4] proved a general vanishing theorem for such groups, which he, Voisin and others (c.f. [G5]) have used to make interesting infinitesimal computations in Hodge theory. In this spirit, we may view Theorem 1 as a special case of the following, which generalizes Green's statement to arbitrary varieties:

Theorem 2 *Let A be a very ample line bundle, and let B and C be numerically effective line bundles on a smooth complex projective n-fold X. Put*

$$L_d = K_X + dA + B \quad \textit{and} \quad N_f = K_X + fA + C .$$

Let $W \subset H^0(X, L_d)$ be a base-point free subspace of codimension c, and consider the Koszul-type complex

$$\Lambda^{p+1} W \otimes H^0(N_f) \to \Lambda^p W \otimes H^0(L_d \otimes N_f) \to \Lambda^{p-1} W \otimes H^0(L_d^2 \otimes N_f) .$$

If $d \geqq n + 1$ and $f \geqq (n + 1) + p + c$, then this complex is exact (in the middle).

Green's result is the case $X = \mathbf{P}^n$ and $A = \mathcal{O}_{\mathbf{P}^n}(1)$.

We hope that Theorem 2 may open the door to finding explicit formulations of theorems hitherto known precisely for $\mathbf{P}^n$ but only asymptotically in general. In this direction we prove in §3 the following, which may be seen as making precise some results of [CGGH] and [G3] concerning "sufficiently positive" divisors.

Proposition 3 *Let X be a smooth complex projective threefold, and let A be a very ample and B a nef line bundle on X.*

(1) *If $Y \in |3K_X + 16A + B|$ is a sufficiently general smooth divisor, then* $\mathrm{Pic}(Y) = \mathrm{Pic}(X)$.

(2) *If $Y \in |K_X + 8A + B|$ is any smooth divisor, then the infinitesimal Torelli theorem holds for Y, i.e. the derivative of the period mapping is injective at Y.*

Note that when $X = \mathbf{P}^3$, (1) is just the classical Noether–Lefschetz theorem and (2) is the elementary fact that infinitesimal Torelli holds for surfaces of degrees $\geqq 4$.

The proofs of the theorems combine the theory of Castelnuovo–Mumford regularity with some vanishing theorems for bundles. It is known that the syzygies of L and the complexes appearing in Theorem 2 are governed by the cohomology of a vector bundle M_L associated to $L = L_d$; roughly speaking, one has to verify that $H^1(X, \Lambda^{p+1} M_L \otimes N_f) = 0$. When $X = \mathbf{P}^n$, the required vanishing follows from the multiplicative behavior [L1, (2.7)] of Castelnuovo–Mumford regularity. In general, however, there does not seem to be a good theory of regularity on an arbitrary variety. But in the case at hand, when $L = K_X + dA + B$ with $d \geqq n + 1$, we show in effect that the best features of the case $X = \mathbf{P}^n$ persist. Specifically, we use considerations of regularity to build non-exact "resolutions" first of M_L and then of its $p + 1$-fold tensor product $T^{p+1} M_L$. The strategy is then to use the existence of these "resolutions" to read off the required vanishings. A similar tact was taken in [GLP, §1], but here we are forced to deal with the homology of the complexes so constructed. This is achieved by means of the following vanishing theorem of Le Potier–Sommese type, which combines and unifies a number of statements appearing for instance in [LeP], [S] and [SS, Chap. 5]:

(*) *Let $E_1, \ldots, E_r$ be vector bundles on X, of ranks $e_1, \ldots, e_r$, and let A be an ample line bundle on X. Assume that each E_i is generated by its global sections, and fix integers $a_1, \ldots, a_r \geqq 1$. Then*

$$H^k(X, K_X \otimes \Lambda^{a_1} E_1 \otimes \cdots \otimes \Lambda^{a_r} E_r \otimes A) = 0 \quad \text{for } k > (e_1 - a_1) + \cdots + (e_r - a_r)\,.$$

In the application, each E_i is a twist of the normal bundle N of X in $\mathbf{P}H^0(A)$. We hope that some of these techniques may find other applications in the future.

The paper is organized as follows. In §1 we review the cohomological interpretation of Property (N_p) and the Koszul complexes appearing in Theorem 2, and we prove the vanishing theorem (*). §2 is devoted to the proofs of the main results. We present in §3 some variants and applications of these results. Finally, we discuss in §4 some open problems.

0 Notation and conventions

(0.1) We work throughout over the complex numbers.

(0.2) If X is a variety and F is a coherent sheaf on X, we write $H^i(F)$ for the cohomology group $H^i(X, F)$ if no confusion seems likely. If $C_\bullet$ is a complex of sheaves on X, we denote by $\mathscr{H}_i(C_\bullet)$ the ith homology sheaf of $C_\bullet$. K_X is the canonical bundle of a smooth variety X. We will usually write L^k for the k-fold tensor power $L^{\otimes k}$ of a line bundle L. However in discussing adjunction-type bundles of the form $K_X + dA$ it seems to be traditional to use additive notation, and at the risk of notational inconsistency we have decided to respect this tradition.

(0.3) Let X be an irreducible projective variety. Recall that a line bundle B on X is *numerically effective* or *nef* if $c_1(B)\cdot\Gamma \geqq 0$ for every irreducible curve $\Gamma \subset X$. It follows from Kleiman's criterion for ampleness that if A is ample and B is nef, then $A \otimes B$ is again ample.

(0.4) The definition of Property (N_p) makes perfectly good sense as soon as L is generated by its global sections. Thus for example (N_0) means in this context that the natural maps $S^m H^0(L) \to H^0(L^m)$ are surjective for all $m \geqq 0$. When $p = 0$ and $(X, A, B) = (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), \mathcal{O}_{\mathbf{P}^n})$ Theorem 1 should be understood in this broader sense, since in this instance the bundle appearing in the statement is trivial. However as explained in the proof of Proposition 2.4, in all other cases $K_X + (n + 1 + p)A + B$ is ample.

1 Preliminaries

Cohomological interpretations

For the convenience of the reader, we begin by reviewing a cohomological criterion for Property (N_p) to hold, and for the exactness of certain Koszul complexes. Details may be found for instance in [GL2], [L2], or [G5].

Let X be an irreducible complex projective variety of dimension n, and let L be an ample line bundle on X which is generated by its global sections. Then there is a canonical surjective evaluation homomorphism $e_L : H^0(X, L) \otimes_{\mathbf{C}} \mathcal{O}_X \to L$. Define

$$M_L = \ker(e_L)\,, \tag{1.1}$$

so that M_L is a vector bundle on X of rank $r(L) = h^0(X, L) - 1$. By construction M_L sits in the basic exact sequence

$$0 \to M_L \to H^0(X, L) \otimes_{\mathbf{C}} \mathcal{O}_X \to L \to 0\,. \tag{1.2}$$

Fix now a line bundle N on X and integers $p, q \geqq 0$, and set $V = H^0(X, L^V)$. Then one can form the Koszul-type complex

$$K_\bullet\colon \Lambda^{p+1} V \otimes H^0(L^{q-1} \otimes N) \to \Lambda^p V \otimes H^0(L^q \otimes N) \to \Lambda^{p-1} V \otimes H^0(L^{q+1} \otimes N)\,.$$

More generally, suppose that $W \subset V = H^0(X, L)$ is any base-point free subspace. Then one can define analogously a vector bundle M_W on X via the exact sequence

$$0 \to M_W \to W \otimes_{\mathbf{C}} \mathcal{O}_X \to L \to 0 \tag{1.3}$$

and again one can form a complex

$$K_\bullet^W\colon \Lambda^{p+1} W \otimes H^0(L^{q-1} \otimes N) \to \Lambda^p W \otimes H^0(L^q \otimes N) \to \Lambda^{p-1} W \otimes H^0(L^{q+1} \otimes N)\,.$$

Lemma 1.4 *Assume that $H^1(X, L^{q-1} \otimes N) = 0$. Then the complex $K_\bullet^W$ is exact (in the middle) if and only if*

$$H^1(X, \Lambda^{p+1} M_W \otimes L^{q-1} \otimes N) = 0 \,. \tag{1.5}$$

In particular, $K_\bullet$ is exact if and only if $H^1(X, \Lambda^{p+1} M_L \otimes L^{q-1} \otimes N) = 0$.

Proof. Taking exterior powers in (1.3) and twisting by $L^q \otimes N$ yields the exact sequence

$$(*)_{p,q}\colon\ 0 \to \Lambda^p M_W \otimes L^q \otimes N \to \Lambda^p W \otimes_{\mathbb{C}} L^q \otimes N \to \Lambda^{p-1} M_W \otimes L^{q+1} \otimes N \to 0$$

of vector bundles on X. The complex $K_\bullet^W$ may be analyzed by splicing together in the evident way several of these sequences. Specifically, let

$$d_{p,q}\colon\ \Lambda^p W \otimes H^0(L^q \otimes N) \to \Lambda^{p-1} W \otimes H^0(L^{q+1} \otimes N)\,,$$
$$d_{p+1,q-1}\colon\ \Lambda^{p+1} W \otimes H^0(L^{q-1} \otimes N) \to \Lambda^p W \otimes H^0(L^q \otimes N)\,,$$

be the maps appearing in $K_\bullet^W$. Then one finds that

$$\ker(d_{p,q}) = H^0(X, \Lambda^p M_W \otimes L^q \otimes N)\,,$$
$$\ker(d_{p+1,q-1}) = \operatorname{im}\{H^0(\Lambda^{p+1} W \otimes L^{q-1} \otimes N) \to H^0(\Lambda^p M_W \otimes L^q \otimes N)\}\,,$$

the latter map coming from $(*)_{p+1,q-1}$. The lemma then follows from the exact sequence of cohomology associated to $(*)_{p+1,q-1}$. □

As for syzygies, one has

Lemma 1.6 *Assume that L is very ample, and that $H^1(X, L^k) =$ for all $k \geqq 1$. Then L satisfies property (N_p) if and only if*

$$H^1(X, \Lambda^a M_L \otimes L^b) = 0 \quad \forall\, a \leqq p+1 \ \text{and}\ b \geqq 1\,.$$

Proof. The case $p = 0$ follows upon twisting (1.2) by powers of L and taking cohomology. So we assume $p > 0$. Then an argument with graded Tor's shows that the homology of $K_\bullet$ with $N = \mathcal{O}_X$ gives the generators in degree $p + q$ of the pth module of syzygies of the graded module $\oplus H^0(X, L^m)$. (See for instance [G3], [L2, §1.3] or [GL2, §1] for details.) Hence the assertion follows from (1.5). □

A vanishing theorem of Le Potier–Sommese type

Our next object is to record a vanishing theorem which interpolates among a number of results from [LeP], [S] and [SS, Chap. V]. The following exceedingly simple proof, which is based on an idea of Manivel [Man], was communicated to us by Demailly. In a preliminary version of this paper we gave a more elaborate argument using the approach of Schneider [Sch] and techniques of [SS].

Proposition 1.7. *Let X be a smooth complex projective variety of dimension n. Let $E_1, \ldots, E_r$ be vector bundles on X, of ranks $e_1, \ldots, e_r$, and let A be an ample line bundle on X. Assume that each E_i is generated by its global sections, and fix integers $a_1, \ldots, a_r \geqq 1$. Then*

$$H^k(X, K_X \otimes \Lambda^{a_1} E_1 \otimes \cdots \otimes \Lambda^{a_r} E_r \otimes A) = 0$$

for $k > (e_1 - a_1) + \cdots + (e_r - a_r)$.

Remark. It is enough that all the E_i be numerically effective. The conclusion of (1.7) also holds if each E_i is ample and A is numerically effective.

Proof (Demailly–Manivel). The idea is simply to reduce to Le Potier's theorem that if F is a globally generated bundle of rank f and A is an ample line bundle, then $H^k(X, K_X \otimes \Lambda^a F \otimes A) = 0$ for $k > f - a$. In fact, consider the vector bundle $F = E_1 \oplus \cdots \oplus E_r$. Then

$$\Lambda^{a_1} E_1 \otimes \cdots \otimes \Lambda^{a_r} E_r \text{ is a direct summand of } \Lambda^{(a_1 + \cdots + a_r)}(E_1 \oplus \cdots \oplus E_r) .$$

But rank $F = e_1 + \cdots + e_r$, so the result follows by applying Le Potier to F. □

2 The main theorem

Throughout this section, X denotes a smooth irreducible complex projective variety of dimension n, A is a very ample line bundle on X, and B and C are numerically effective line bundles on X. Given integers d and f, we write

$$L_d = K_X + dA + B \quad \text{and} \quad N_f = K_X + fA + C .$$

It is standard and elementary (cf. the proof of Proposition 2.4 below) that if $d \geqq n + 1$, then L_d is base-point-free [and very ample unless $d = n + 1$ and $(X, A, B) = (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), \mathcal{O}_{\mathbf{P}^n})$]. Consequently, provided that $d \geqq n + 1$, the vector bundle

$$M_d \overset{\text{def}}{=} M_{L_d}$$

of (1.1) is defined.

Our purpose in this section is to prove the following

Theorem 2.1 *Denote by $T^q(M_d)$ the q-fold tensor power of M_d. Then*

$$H^k(X, T^q(M_d) \otimes N_f) = 0 \quad \textit{when} \quad k > 0$$

provided that $d \geqq n + 1$ and $f \geqq n + 1 + q - k$.

Remark. Using a result from [BEL] (or [ABS]) to handle the case $q = 1$, one can prove a slightly stronger vanishing. See (3.1) below.

Note that the theorem indeed implies the geometric results for which we are aiming:

Corollary 2.2 *If $d \geqq n + 1 + p$, then the line bundle L_d satisfies Property (N_p).*

Proof. Fix $p \geqq 0$ and $d \geqq n + p + 1$. Then L_d is base-point free, and hence for every $k \geqq 1$, $L_d^{\otimes k}$ is of the form $K_X + dA + C$ for some nef line bundle C (depending on k). Hence in the first place $H^1(X, L_d^{\otimes k}) = 0$ by Kodaira. Furthermore, it follows from (2.1) that if $q \leqq p + 1$, then $H^1(X, T^q M_d \otimes L_d^{\otimes k}) = 0$ for all $k \geqq 1$. Since we are in characteristic zero, $\Lambda^q M_d$ is a direct summand of $T^q(M_d)$, and therefore the assertion follows from Lemma 1.6. □

Corollary 2.3 *Let $W \subseteq H^0(X, L_d)$ be a base-point free subspace of codimension c. If $d \geqq n+1$ and $f \geqq (n+1)+p+c$, then the complex*

$$\Lambda^{p+1} W \otimes H^0(N_f) \to \Lambda^p W \otimes H^0(L_d \otimes N_f) \to \Lambda^{p-1} W \otimes H^0(L_d^2 \otimes N_f)$$

appearing in the statement of Theorem 2 is exact.

Proof. When $W = H^0(X, L_d)$, so that $c = 0$, the statement follows from (1.4) and the theorem. In general, one argues just as in [G4] or [G5]. In brief, fix a filtration $H^0(X, L_d) = W_0 \supset W_1 \supset \cdots \supset W_c = W$ by subspaces each having codimension one in the next, and let M_i be the bundle M_{W_i} defined in (1.3). By (1.4), it suffices to prove

$$(*) \qquad H^k(X, \Lambda^p M_i \otimes L_f) = 0 \quad \text{for } k > 0 \quad \text{and } f \geqq (n+1)+p+i-k\,.$$

One argues by induction on i, the case $i = 0$ having just been treated. To this end, observe that one has exact sequences $0 \to M_{i+1} \to M_i \to \mathcal{O}_X \to 0$, and hence also

$$(**) \qquad 0 \to \Lambda^p M_{i+1} \to \Lambda^p M_i \to \Lambda^{p-1} M_{i+1} \to 0\,.$$

Twisting (**) by N_f and taking cohomology, one concludes with a downward induction on p. □

We now turn to the proof of Theorem 2.1. The following proposition, which constructs a non-exact "resolution" of M_d, plays a crucial role in all that follows.

Proposition 2.4 *Assume that $d \geqq n+1$. Then there exist finite dimensional vector spaces V_i and a complex $R_\bullet$ of vector bundles on X of the form*

$$(2.5) \qquad \begin{array}{ccccccc} \cdots \to & V_2 \otimes_{\mathbb{C}} A^{-2} & \to & V_1 \otimes_{\mathbb{C}} A^{-1} & \xrightarrow{\varepsilon} & M_d & \to 0\,. \\ & \| & & \| & & \| & \\ & R_2 & & R_1 & & R_0 & \end{array}$$

(In other words, for $i > 0$, $R_i = \oplus A^{-i}$ is a direct sum of copies of A^{-i}.) This complex has the properties:

(1) *The map ε is surjective, i.e. $\mathcal{H}_0(R_\bullet) = 0$;*
(2) *For $i \geqq 1$,*

$$\mathcal{H}_i(R_\bullet) = \Lambda^i N^* \otimes L_d\,,$$

where $N = N_{X/\mathbf{P}H^0(A)}$ denotes the normal bundle to X in $\mathbf{P}H^0(A)$, X being embedded by the complete linear series associated to A.

Remark. The construction will show that (2.5) is a bounded complex, i.e. that $R_i = 0$ for $i \gg 0$. But this is unimportant for our purposes.

Proof. We consider the embedding $X \subset \mathbf{P} = \mathbf{P}H^0(A)$ defined by A. Viewing L_d as a sheaf on $\mathbf{P}$, note to begin with that

$$(2.6) \qquad H^i(\mathbf{P}, L_d(-i)) = 0 \quad \text{for } i > 0\,.$$

In fact, since $d \geqq n+1$, $A^{\otimes(d-i)} \otimes B$ – being the tensor product of a nef and an ample line bundle – is ample when $i \leqq n$. Hence $H^i(X, L_d(-i)) = 0$ for $i > 0$ by Kodaira vanishing.

Now (2.6) means precisely that L_d is 0-regular in the sense of Castelnuovo–Mumford (cf. [M1, Chap. 14], [M2], or [EG]). This implies in the first place that L_d is generated by its global sections, as stated above.* Furthermore, L_d, like any 0-regular sheaf, admits a (finite) locally free resolution of the form

$$\cdots \to V_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbf{P}}(-2) \to V_1 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbf{P}}(-1) \to V_0 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbf{P}} \to L_d \to 0\ ,$$

where the V_i are vector spaces and $V_0 = H^0(\mathbf{P}, L_d)$. We denote by $S_\bullet$ the corresponding complex of vector bundles on $\mathbf{P}$, so that $S_i = V_i \otimes_{\mathbb{C}} \mathcal{O}_{\mathbf{P}}(-i)$.

Consider next the restriction $T_\bullet = S_\bullet \otimes \mathcal{O}_X$ of $S_\bullet$ to X. Thus $T_i = V_i \otimes_{\mathbb{C}} A^{-i}$, and we assert that

$$\mathcal{H}_i(T_\bullet) = \Lambda^i N^* \otimes L_d\ . \tag{2.7}$$

In fact, evidently $\mathcal{H}_i(T_\bullet) = \mathrm{Tor}_i^{\mathcal{O}_{\mathbf{P}}}(\mathcal{O}_X, L_d)$. On the other hand, computing this Tor via a resolution of $\mathcal{O}_X$ (or using a change of rings spectral sequence) one sees that

$$\mathrm{Tor}_i^{\mathcal{O}_{\mathbf{P}}}(\mathcal{O}_X, L_d) = \mathrm{Tor}_i^{\mathcal{O}_{\mathbf{P}}}(\mathcal{O}_X, \mathcal{O}_X) \otimes_{\mathcal{O}_X} L_d\ .$$

Hence (2.7) follows from the well-known formula $\mathrm{Tor}_i^{\mathcal{O}_{\mathbf{P}}}(\mathcal{O}_X, \mathcal{O}_X) = \Lambda^i N^*$.

Finally the natural map

$$e\colon T_0 = H^0(X, L_d) \otimes_{\mathbb{C}} \mathcal{O}_X \to L_d = \mathcal{H}_0(T_\bullet)$$

is just the canonical evaluation homomorphism. Hence $M_d = \ker(e)$. Setting $R_i = T_i$ for $i > 0$ and $R_0 = M_d$, the proposition follows. □

The next step is to build an analogous "resolution" of the vector bundle $T^q(M_d) \otimes N_f$. To this end, recall first the Kunneth formula:

Lemma 2.8 *Let $E_\bullet$ and $F_\bullet$ be complexes of locally free coherent sheaves on X. Assume that $E_\bullet$ and $F_\bullet$ are bounded below, and that their homology sheaves $\mathcal{H}_*(E_\bullet)$ and $\mathcal{H}_*(F_\bullet)$ are locally free. Then*

$$\mathcal{H}_i(E_\bullet \otimes F_\bullet) = \bigoplus_{p+q=i} \mathcal{H}_p(E_\bullet) \otimes \mathcal{H}_q(F_\bullet)\ .$$

Proof. This is a special case of [EGA, III.6.7.8]. □

Proposition 2.9 *Assume that $d \geqq n+1$, fix integers $q, f \geqq 1$ and as above let $N_f = K_X + fA + C$. Then there exists a complex $Q_\bullet$: $\cdots \to Q_2 \to Q_1 \to Q_0 \to 0$ of vector bundles on X with the following properties:*

(1) $Q_0 = T^q(M_d) \otimes N_f$;

(2) *For $i \geqq 1$, Q_i is a direct sum of copies of vector bundles of the form*

$$T^p(M_d) \otimes N_{f-i}$$

with $p \leqq q-1$;

(3) $\mathcal{H}_i(Q_\bullet) = 0$ *when $i \leqq q-1$;*

---

* One may argue as follows that L_d is very ample unless $d = n+1$ and $(X, A, B) = (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), \mathcal{O}_{\mathbf{P}^n})$. Since A is very ample it is enough to show that L_{d-1} is globally generated. Hence when $d > n+1$ there is nothing more to prove. So assume $d = n+1$. Then one knows by work of Sommese et al. (cf. [E]) that $K_X + nA$ is globally generated provided that $(X, A) \neq (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n})$. Hence if $B = \mathcal{O}_X$ we are done. But if $B \neq \mathcal{O}_X$, then $H^n(X, K_X \otimes B) = H^0(X, B^*)^* = 0$. This implies by Kodaira that L_n is 0-regular, and therefore globally generated.

(4) *For* $i \geqq q$,

$$\mathscr{H}_i(Q_\bullet) = \bigoplus_{\substack{a_1 + \cdots + a_q = i \\ a_1, \ldots, a_q \geqq 1}} \Lambda^{a_1} N^* \otimes \cdots \otimes \Lambda^{a_q} N^* \otimes (K_X^{q+1} \otimes A^{(qd+f)} \otimes C'),$$

where C' *is a numerically effective line bundle (depending on* i*), and* $N = N_{X/\mathbf{P}H^0(A)}$ *is the normal bundle to* X *in* $\mathbf{P}H^0(A)$.

Proof. One simply takes $Q_\bullet = T^q(R_\bullet) \otimes N_f$, where $R_\bullet$ is the complex (2.5) constructed in Proposition 2.4, and $T^q(R_\bullet)$ is its q-fold tensor product. Then assertions (1) and (2) are clear. The homology sheaves of $T^q(R_\bullet)$ are computed by repeatedly applying Lemma 2.8. But

$$\mathscr{H}_i(T^q(R_\bullet) \otimes N_f) = \mathscr{H}_i(T^q(R_\bullet)) \otimes N_f,$$

since N_f is locally free, and the proposition follows. □

The following elementary lemma allows one to deduce the vanishing of $H^k(X, Q_0)$ from a non-exact "resolution of a sheaf Q_0 as in (2.9).

Lemma 2.10 (compare [GLP, Lemma 1.6]) *Let*

$$Q_\bullet\colon \cdots \to Q_2 \to Q_1 \xrightarrow{\varepsilon} Q_0 \to 0$$

be a complex of coherent sheaves on X*, with* ε *surjective. Assume that*

(1) $H^k(X, Q_1) = H^{k+1}(X, Q_2) = \cdots = H^n(X, Q_{n-k+1}) = 0$
[i.e. $\mathrm{H}^{k+i-1}(X, Q_i) = 0$ *for all* $i \geqq 1$];
(2) $H^{k+1}(X, \mathscr{H}_1(Q_\bullet)) = H^{k+2}(X, \mathscr{H}_2(Q_\bullet)) = \cdots = H^n(X, \mathscr{H}_{n-k}(Q_\bullet)) = 0$
[i.e. $H^{k+i}(X, \mathscr{H}_i(Q_\bullet)) = 0$ *for all* $i \geqq 1$].

Then $H^k(X, Q_0) = 0$.

Proof. This is most easily verified by chopping $Q_\bullet$ into short exact sequences in the usual way, and chasing through the resulting diagram. Alternatively, one can examine the two hypercohomology spectral sequences associated to $Q_\bullet$. □

Finally, we will need to control the cohomology of twists of sheaves of the form $\Lambda^{a_1} N^* \otimes \cdots \otimes \Lambda^{a_q} N^*$. This is where the vanishing theorem proved in §1 makes its appearance.

Lemma 2.11 *Denote by* $N = N_{X/\mathbf{P}H^0(A)}$ *the normal bundle to* X *in* $\mathbf{P}H^0(A)$*, and let* C' *be any nef line bundle on* X*. Fix integers* $a_1, \ldots, a_q \geqq 1$*. Then*

$$H^k(X, \Lambda^{a_1} N^* \otimes \cdots \otimes \Lambda^{a_q} N^* \otimes (K_X^{q+1} \otimes A^r \otimes C')) = 0$$

for $k > a_1 + \cdots + a_q$ *and* $r > q(n+1) + \sum a_i$.

Proof. We apply (1.7) to the globally generated vector bundle $N \otimes A^*$. Specifically, set $e = rkN$, and note that $\det(N \otimes A^*) = K_X \otimes A^{n+1}$. Then

$$\begin{gathered} \Lambda^{a_1} N^* \otimes \cdots \otimes \Lambda^{a_q} N^* \otimes (K_X^{q+1} \otimes A^r \otimes C') \\ \| \\ \Lambda^{a_1}(N^* \otimes A) \otimes \cdots \otimes \Lambda^{a_q}(N^* \otimes A) \otimes \{\det(N \otimes A^*)\}^q \otimes (K_X \otimes A^{r - q(n+1) - \sum a_i} \otimes C') \\ \| \\ \Lambda^{e-a_1}(N \otimes A^*) \otimes \cdots \otimes \Lambda^{e-a_q}(N \otimes A^*) \otimes (K_X \otimes A^{r - q(n+1) - \sum a_i} \otimes C'). \end{gathered}$$

But when $r > q(n+1) + \sum a_i$, the bundle $A^{r-q(n+1)-\sum a_i} \otimes C'$ is ample. Therefore the required vanishing is a consequence of Proposition 1.7. □

Now we give the

Proof of Theorem 2.1 We argue by induction on q. When $q = 0$ the theorem simply asserts the vanishings (2.6) established during the proof of Proposition 2.4. So suppose that $q > 0$, fix integers $d \geqq n+1$, and $f \geqq n+q+1-k$, and assume the theorem known for tensor powers $T^p(M_d)$ with $p \leqq q-1$.

We propose to apply Lemma 2.10 to the complex $Q_\bullet$ constructed in Proposition 2.9. To this end, we check first of all the vanishing (2.10)(1), which, in the case at hand amounts to the assertion that

$$(*) \qquad H^{k+i-1}(X, T^p(M_d) \otimes N_{f-i}) = 0 \quad \text{for all } i \geqq 1, p \leqq q-1 .$$

But since $f - i \geqq (n+q+1-k) - i \geqq n+p+1-(k+i-1)$, $(*)$ follows from the induction hypothesis.

For (2.10)(2), we are required to verify that

$$(**) \qquad H^{k+i}(X, \Lambda^{a_1} N^* \otimes \cdots \otimes \Lambda^{a_q} N^* \otimes (K_X^{q+1} \otimes A^{qd+f} \otimes C')) = 0$$

for all $i \geqq 1$ whenever $a_1 + \cdots + a_q = i$ with $a_1, \ldots, a_q \geqq 1$. But this is only non-trivial when $k + i \leqq n$, and hence we may suppose $\sum a_i \leqq n-k$. Then

$$qd + f \geqq q(n+1) + (n-k) + (q+1) > q(n+1) + \sum a_i ,$$

and therefore $(**)$ is a consequence of Lemma 2.11. This completes the proof of the Theorem. □

3 Variants and applications

Keeping notation as in §2, we start by indicating a slight strengthening of our main theorem.

Proposition 3.1 *Assume that* $(X, A, B) \neq (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), \mathcal{O}_{\mathbf{P}^n})$.

(1) *If* $p \geqq 1$, *then the line bundle* L_d *satisfies Property* (N_p) *whenever* $d \geqq n+p$.
(2) *If* $d \geqq n+1$ *and* $f \geqq n+p+c$, *then the complex appearing in the statement of Theorem 2 is exact.*

Sketch of proof. As in §2, it is enough to prove that if $q \geqq 1$, then

(3.2) $\quad H^k(X, T^q(M_d) \otimes N_f) = 0 \quad$ for $k > 0$, $d \geqq n+1$, and $f \geqq n+q-k$.

To treat the case $q = 1$, we use a result from [BEL] (or essentially also from [ABS]) to the effect that with the stated exception, the natural map

$$H^0(X, L_d) \otimes H^0(X, N_f) \to H^0(X, L_d \otimes N_f)$$

is surjective provided that $d, f \geqq n$. This implies that $H^1(X, M_d \otimes N_f) = 0$ when $d, f \geqq n$. One finds from (1.2) that $H^k(X, M_d \otimes N_f) = 0$ for $k \geqq 2$, $d \geqq n+1$, and

$f \geqq n + 1 - k$, which proves (3.2) for $q = 1$. Now so long as $d \geqq n + 1$, the "resolutions" constructed in (2.4) and (2.9) still exist. And checking the numbers in the proof of Theorem 2.1 shows that if $q \geqq 2$ the induction works just as before to yield (3.2). □

Theorem 1 ties up in an amusing way with a result of Mumford's. Consider a smooth projective subvariety $X \subset \mathbf{P}^r$ having degree d, and write H for a hyperplane divisor on X. Mumford [M1] proved that X is scheme-theoretically defined by quadrics in the embedding $X \subset \mathbf{P}^N$ defined by the linear system $|dH|$. This generalizes as follows:

Proposition 3.3 *Let $X \subset \mathbf{P}^r$ be a smooth complex projective variety of degree d, and let H be the hyperplane divisor of X. Then the line bundle $\mathcal{O}_X(kH)$ satisfies property (N_{k+1-d}).*

Proof. Assume that X has dimension n. It is enough to show that

(*) the divisor $B = (d - n - 2)H - K_X$ moves in a base point free linear system

for then the assertion follows from (2.2). But this is elementary and presumably well-known; we learned the following argument from Mumford. Specifically, (*) is clear if $r \leqq n + 1$. So assume $r \geqq n + 2$ and consider a linear projection $f: X \to \mathbf{P}^{n+1}$. If D_f is the double point divisor of f, then $D_f \equiv (d - n - 2)H - K_X$ thanks to the double point formula. But by varying the center of projection, we can choose f so that D_f misses any given point of X, and (*) follows. □

Another application of these ideas concerns Koszul rings. Recall that a graded $\mathbb{C}$-algebra $R = \bigoplus_{i \geqq 0} R_i$ with $R_0 = \mathbb{C}$ is said to be *Koszul* if $\operatorname{Tor}_i^R(k, k)$ is purely of degree i for $i \geqq 1$, where k denotes the residue field of R. This is a regularity condition on R that has been of interest to algebraists for some time, and was recently introduced into geometric questions by Kempf [K]. For the ring $R(L)$ associated to a very ample line bundle L on a variety X, the property of being Koszul is somewhat stronger than (N_1). Butler [B] has shown that if L is a line bundle of degree $\geqq 2g + 2$ on a curve of genus g, then the corresponding ring is Koszul. It is natural to ask for analogous statements for adjunction-type bundles on an arbitrary smooth variety. In this direction, Pareschi has used the techniques of the present paper to prove the following:

Theorem [P] *Let X be a smooth complex projective variety of dimension n, and let A be a very ample and B a nef line bundle on X. Put $L_d = K_X + dA + B$, and consider the graded ring $R = R(L_d)$ associated to L_d. If $d \geqq n + 2$, then R is a Koszul ring.*

Pareschi also proves a strengthening in the spirit of Proposition 3.1 above.

Finally we give an example of how Theorem 2 may be used to render explicit some infinitesimal Hodge-theoretic computations previously known only asymptotically.

Proposition 3.4 *Let X be a smooth complex projective threefold, and let A be a very ample and B a nef line bundle on X.*

(1) *If $Y \in |3K_X + 16A + B|$ is a sufficiently general smooth divisor, then* $\operatorname{Pic}(Y) = \operatorname{Pic}(X)$

(2) *If* $Y \in |K_X + 8A + B|$ *is any smooth divisor, then the infinitesimal Torelli theorem holds for* Y, *i.e. the derivative of the period mapping is injective at* Y.

Remark. When $X = \mathbf{P}^3$, statement (1) is just the usual Noether–Lefschetz theorem, while (2) is the elementary fact that infinitesimal Torelli holds for surfaces of degree $\geqq 4$. Both statements were known to hold for "sufficiently positive" divisors [CGGH, G3]. The novelty here lies in giving an *explicit* meaning to "sufficiently ample". One could deduce (1) from a global theorem of Moisezon [Mois], but the present result – being infinitesimal in nature – is arguably more elementary.

Proof. The Hodge-theoretic part of the argument is a standard application of the ideas surrounding infinitesimal variations of Hodge structures. But for the benefit of the reader, we will recall briefly the approach of [CGGH] and [G3].

For (1), let $N = N_{Y/X}$ be the normal bundle of Y in X. Then $\mathrm{Pic}^0(Y) = \mathrm{Pic}^0(X)$ thanks to the Lefschetz theorems, so it suffices to prove that $H^{1,1}_{\mathbb{Z}}(Y) = H^{1,1}_{\mathbb{Z}}(X)$ for Y sufficiently general in the appropriate linear series. Since in any event the cokernel of the map $H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is torsion free, it suffices in turn to prove that if D is any divisor on a sufficiently general Y, then $[D] \in \mathrm{im}\{H^1(X, \Omega^1_X) \to H^1(Y, \Omega^1_Y)\}$. Consider to this end the natural maps

$$\gamma : H^1(Y, \Omega^1_Y) \to H^2(Y, N^*)$$

$$\beta : H^2(Y, N^*) \to \mathrm{Hom}(H^0(Y, N), H^2(Y, \mathcal{O}_Y)) ,$$

and set $\alpha = \beta \circ \gamma$. The theory of [CGGH] identifies $\ker(\alpha)$ as those (1, 1)-classes on Y that to first order remain of type (1, 1) under all infinitesimal deformations of Y in X. So it's enough to prove that for any smooth $Y \in |3K_X + 16A + B|$,

$$\ker \alpha = \mathrm{im}\{H^1(X, \Omega^1_X) \to H^1(Y, \Omega^1_Y)\} .$$

This is turn will follow if we show

(i) β is injective;
(ii) The map $H^1(X, \Omega^1_X) \to H^1(Y, \Omega^1_X | Y)$ is surjective.

For (i) it is equivalent by duality to check the surjectivity of the multiplication

$$H^0(Y, N) \otimes H^0(Y, K_Y) \to H^0(Y, N \otimes K_Y) .$$

Set $L = \mathcal{O}_X(Y) = 3K_X + 16A + B$, so that $N = L | Y$ and $K_Y = (K_X \otimes L) | Y$. Then an evident commutative diagram shows that it's enough to prove the surjectivity of

$$(*) \qquad H^0(X, L) \otimes H^0(X, K_X \otimes L) \to H^0(X, K_X \otimes L^2) .$$

Now $L \otimes K_X = K_X + 4A + C$ where $C = 3K_X + 12A + B$ is nef. Hence the surjectivity of (*) follows from the case $p = 0$ of Theorem 2.

Turning to (ii), it suffices to prove $H^2(X, \Omega^1_X(-Y)) = 0$, or dually

$$(**) \qquad H^1(X, \Omega^2_X \otimes L) = 0$$

To this end we apply the Griffiths vanishing theorem (cf. [SS, Theorem 5.52]). Specifically, $\Omega^2_X \otimes A^3$ – being a quotient of $\Omega^2_{\mathbf{P}H^0(A)}(3)$ – is generated by its global sections, and $\det(\Omega_X^2 \otimes A^3) = 2K_X + 9A$. So for any nef line bundle B, Griffiths vanishing gives

$$H^1(X, \Omega_X^2(3A + (2K_X + 9A) + (K_X + 4A + B))) = 0 .$$

This yields (*) and with it statement (1) of the proposition.

For (2) we will assume that $(X, A, B) \neq (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1), \mathcal{O}_{\mathbf{P}^3})$ (where the statement is evident), in which case $K_X + 4A + B$ is ample. As above, put $L = \mathcal{O}_X(Y) = K_X + 8A + B$. Arguing as in [G3, §1] it is enough this time to prove

(i) $H^1(Y, \Omega_X^1 \otimes K_Y^{-1}) = 0$
(ii) $H^0(X, K_X \otimes L) \otimes H^0(X, K_X \otimes L^2) \to H^0(X, K_X^2 \otimes L^3)$ is surjective.

In fact, (i) implies that $H^1(Y, \Theta_Y) \subset H^2(Y, N^* \otimes K_Y^{-1})$, i.e. that $H^1(Y, \Theta_Y)^*$ is a quotient of $H^0(Y, N \otimes K_Y^2)$; and it follows from (ii) and the hypothesis on L that the multiplication

$$H^0(Y, K_Y) \otimes H^0(Y, N \otimes K_Y) \to H^0(Y, N \otimes K_Y^2)$$

is surjective. But the coderivative of the period maps factors through the resulting composition

$$H^0(Y, K_Y) \otimes H^0(Y, N \otimes K_Y) \to H^1(Y, \Theta_Y)^* ,$$

so the assertion of the proposition follows. Now (i) is a consequence of the exact sequence

$$0 \to \Omega_X^1 \otimes K_X^{-1} \otimes L^{-2} \to \Omega_X^1 \otimes K_X^{-1} \otimes L^{-1} \to \Omega_X^1 \otimes K_Y^{-1} \to 0 .$$

Indeed, when $L = K_X + 8A + B$, $H^1(\Omega_X^1 \otimes K_X^{-1} \otimes L^{-1}) = 0$ by Nakano vanishing, and $H^2(\Omega_X^1 \otimes K_X^{-1} \otimes L^{-2}) = 0$ by Griffiths vanishing, as above. Similarly (ii) follows from Theorem 2. □

Remark. Using Proposition 3.1, one can give a slightly stronger statement when $X \neq \mathbf{P}^n$. We leave this to the interested reader.

Remark. Observe that it is the use of Griffiths vanishing in the previous proof that the accounts for the somewhat strange shape of statement (1): the point is that the argument requires that K_X appear with coefficient $\geqq 3$ in L. Paoletti has constructed examples showing that – contrary to what one might expect – there cannot exist a universal constant α such that $H^1(X, \Omega_X^2(K_X + \alpha A)) = 0$ for all very ample line bundles A on all threefolds X.

Remark. One can generalize Green's bound [G4] on the codimension of components for which the statement of (3.4)(1) fails. Keeping notation as in (3.4), let $L = 3K + dA + B$, and let $\Sigma \subset |L|$ be an irreducible component of the set of all smooth surfaces $Y \in |L|$ such that $\mathrm{Pic}(Y) \neq \mathrm{Pic}(X)$. Note that the tangent space $T_Y\Sigma$ to Σ at Y sits in $H^0(X, L)/\mathbb{C}s$, where $s \in H^0(X, L)$ is the section defining Y, and hence determines in the natural way a subspace $W_Y\Sigma \subset H^0(X, L)$. Then arguing as in [G4] and the proof of (3.4) one finds the following:

(*) *Assume that at a general point $Y \in \Sigma$ the linear system $W_Y\Sigma \subset H^0(X, L)$ is base-point free. Then* $\mathrm{codim}(\Sigma, |3K_X + dA + B|) \geqq d - 16$.

Note that when $X = \mathbf{P}^3$ the condition on $W_Y \Sigma$ is automatic. The examples of Kim [Kim] suggest that some sort of hypotheses are needed to obtain a bound in general.

4 Open problems

In conclusion, we present a number of open problems.

The reader will note that Theorem 1 does not imply Green's result on syzygies of curves, because we have always assumed that A is very ample. The most naive (and rash) hope in general would be the following

Possibility. If X is a smooth complex projective variety of dimension n and D is any ample divisor on X, then (N_p) holds for $K_X + (n + 2 + p)D$.

This does yield the result for curves, but at the moment it seems completely out of reach. For example, a celebrated conjecture of Fujita asserts that $K_X + (n + 2)D$ is very ample whenever D is ample, but in spite of the very interesting work of Demailly [D], this is unknown already when $n = 3$. So if one wants to study syzygies, it seems that for the time being one should set one's sights lower. One place to start might be

Problem 4.1 Can one extend the results of the present paper to deal with bundles of the form $K_X + dA + B$ assuming only that A is ample and generated by its global sections?

One hopes for statements having the shape of Theorems 1 and 2, although perhaps the numbers will have to be adjusted a bit.

In view of Reider's work [R] and other recent progress, it does seem realistic to ask for optimal results for surfaces. Hence we pose the following, which is essentially due to Mukai:

Conjecture 4.2 *If X is a smooth projective surface, and D is an ample divisor on X, then $K_X + (p + 4)D$ satisfies Property (N_p).*

This would already be very exciting to know when $p = 0$ – even this case seems to require new ideas.

Adjunction-type bundles $K_X + dA$ have been the focus of considerable study, notably by Fujita, and Sommese and his school (cf. [Sed]). Much of this work is concerned with classifying situations in which the bundles in question fail to be very ample. It seems to us natural to study also exceptional algebraic and geometric behavior when $K_X + dA$ is very ample. In this direction we pose:

Problem 4.3 Can one say anything about the classification of very ample line bundles A on smooth n-folds X where (N_p) fails for $K_X + (n + p - 1)A$?

In other words, we are asking about the "borderline" cases in Proposition 3.1. For curves, one has a very rich conjectural picture of the interaction of geometry with syzygies (cf. [GL1, §3]). Although probably the most one can hope for in general is a much coarser overview, still one expects that (4.3) could lead to some picture of how geometry can affect syzygies in higher dimensions.

In a related direction, in the case of curves one has a good sense – again partly conjectural [GL1, loc. cit.] – of the bahavior of the whole minimal resolution $E_\bullet$ of the graded ring $R(L)$ of a line bundle of large degree. It is natural to ask for at least a rough picture in general:

Problem 4.4 Let A be a very ample line bundle on a smooth n-fold X, and set $L_d = K_X + dA$. If $d \gg 0$, what can one say about the overall shape of the resolution of $R(L_d)$?

One might want to assume here that $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n$, so that $R(L_d)$ is Cohen–Macaulay. In general it is our impression that the varieties whose resolutions have been studied by algebraists are for the most part either close to being rational (e.g. determinantal) or of very small dimension (e.g. finite sets or curves). Hence it would not be surprising if there were interesting but accessible phenomina that have so far escaped notice.

The next question was suggested by W. Fulton. Let G be a semi-simple algebraic group, and consider an irreducible representation $V = V(\lambda)$ of G with highest weight λ. Let $X = G/P(\lambda) \subset \mathbf{P}V(\lambda)$ be the corresponding homogeneous space, so that X carries a very ample homogeneous line bundle $L(\lambda)$ with $H^0(X, L(\lambda)) = V(\lambda)$. Then we define $p(\lambda)$ to be the largest integer p for which (N_p) holds for $L(\lambda)$.

Problem 4.5 Compute $p(\lambda)$ group-theoretically.

It seems that the value of this invariant is already unknown for the various embeddings of $\mathbf{P}^n$. Kempf and his school have studied the syzygies among the Plucker quadrics.

Finally, we have given in Proposition 3.4 an illustration of how the results of the present paper can be used to find explicit formulations of Hodge-theoretic statements proved by infinitesimal techniques.

Problem 4.6 Can one give precise versions of other results (e.g. from [CGGH]*) known to hold for "sufficiently positive" divisors or complete intersections in a given variety X?*

The difficulty here seems to lie in controlling the sort of groups for which we invoked Griffiths vanishing in the proof of (3.4), the problem being that the numbers threaten to become quite messy. Perhaps with some new ideas one could circumvent these calculations, although one will have to keep in mind the examples of Paoletti mentioned above.

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Note added in proof: M.S. Ravi has made important progress in connection with Problem 4.6. (An effective version of Nori's theorem, to appear)