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Contact loci in arc spaces

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Abstract

We give a geometric description of the loci in the arc space defined by order of contact with a given subscheme, using the resolution of singularities. This induces an identification of the valuations defined by cylinders in the arc space with divisorial valuations. In particular, we recover the description of invariants coming from the resolution of singularities in terms of arcs and jets.

Introduction

The purpose of this paper is to study the loci of arcs on a smooth variety defined by order of contact with a fixed subscheme. Specifically, we establish a Nash-type correspondence showing that the irreducible components of these loci arise from (intersections of) exceptional divisors in a resolution of singularities. We show also that these loci account for all the valuations determined by irreducible cylinders in the arc space. Along the way, we recover in an elementary fashion (without using motivic integration) results of the third author from [Mus01] and [Mus02] relating singularities to arc spaces. Moreover, we extend these results to give a jet-theoretic interpretation of multiplier ideals.

Let X be a smooth complex variety of dimension d. Given $m \geqslant 0$ we denote by

$$X_m = \mathrm{Hom}(\mathrm{Spec}\, \mathbb{C}[t]/(t^{m+1}), X)$$

the space of mth order arcs on X. Thus X_m is a smooth variety of dimension $d(m+1)$, and the truncation morphism $\tau_{m+1,m} : X_{m+1} \longrightarrow X_m$ realizes each of these spaces as a $\mathbb{C}^d$-bundle over the previous one. The inverse limit X_∞ of the X_m parametrizes all formal arcs on X, and one writes $\psi_m : X_\infty \longrightarrow X_m$ for the natural map. These spaces have attracted a great deal of interest in recent years thanks to their central role in the theory of motivic integration (see, for example, [Bat98, DL99, DL98]). In his papers [Mus01] and [Mus02] the third author used this machine to give arc-theoretic interpretations of some of the basic invariants of higher-dimensional geometry.

Consider now a non-zero ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$ defining a subscheme $Y \subseteq X$. Given a finite or infinite arc γ on X, the order of vanishing of $\mathfrak{a}$ (or the order of contact of the corresponding scheme Y) along γ is defined in the natural way. Specifically, pulling $\mathfrak{a}$ back via γ yields an ideal (t^e) in $\mathbb{C}[t]/(t^{m+1})$ or $\mathbb{C}[[t]]$, and one sets $\mathrm{ord}_\gamma(\mathfrak{a}) = \mathrm{ord}_\gamma(Y) = e$. (Take $\mathrm{ord}_\gamma(\mathfrak{a}) = m+1$ when $\mathfrak{a}$ pulls back to the zero ideal in $\mathbb{C}[t]/(t^{m+1})$ and $\mathrm{ord}_\gamma(\mathfrak{a}) = \infty$ when it pulls back to the zero ideal in $\mathbb{C}[[t]]$.)

For a fixed integer $p \geqslant 0$, we define the *contact loci*

$$\mathrm{Cont}^p(Y) = \mathrm{Cont}^p(\mathfrak{a}) = \{\gamma \in X_\infty \mid \mathrm{ord}_\gamma(\mathfrak{a}) = p\}.$$

These are locally closed cylinders, i.e. they arise as the common pull-back of the locally closed sets

$$\mathrm{Cont}^p(Y)_m = \mathrm{Cont}^p(\mathfrak{a})_m =_{\mathrm{def}} \{\gamma \in X_m \mid \mathrm{ord}_\gamma(\mathfrak{a}) = p\} \tag{1}$$

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defined for any $m \geqslant p$. By an irreducible component W of $\mathrm{Cont}^p(\mathfrak{a})$ we mean the inverse image of a component W_m of $\mathrm{Cont}^p(\mathfrak{a})_m$ (these being in bijection for all $m \geqslant p$ via the truncation maps). The codimension of W in X_∞ is the common codimension of any of the W_m in X_m.

Our first result establishes a Nash-type correspondence showing that the irreducible components of $\mathrm{Cont}^p(\mathfrak{a})$ arise from exceptional divisors appearing in an embedded resolution of Y. Specifically, fix a log resolution $\mu : X' \longrightarrow X$ of $\mathfrak{a}$. Thus X' is a smooth variety which carries a simple normal crossing divisor $E = \sum_{i=1}^t E_i$, μ is a projective birational map, and

$$\mathfrak{a}' = \mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}\bigg(-\sum_{i=1}^t r_i E_i\bigg). \tag{2}$$

For simplicity, we assume here that all r_i are positive (we can do this if μ is an isomorphism over $X \setminus Y$), and leave the general case for the main body of the paper. We write

$$K_{X'/X} = \sum_{i=1}^t k_i E_i, \tag{3}$$

where $K_{X'/X} = K_{X'} - \mu^* K_X$ is the relative canonical divisor defined by the vanishing of $\det d\mu$. After perhaps some further blowings-up, we may (and do) assume that any non-empty intersection of the E_i is connected and hence irreducible. Observe that μ gives rise in the natural way to morphisms $\mu_m : X'_m \longrightarrow X_m$ and $\mu_\infty : X'_\infty \longrightarrow X_\infty$.

Fix next a multi-index $\nu = (\nu_i) \in \mathbb{N}^t$. We consider the 'multi-contact' locus

$$\mathrm{Cont}^\nu(E) = \{\gamma' \in X'_\infty \mid \mathrm{ord}_{\gamma'}(E_i) = \nu_i \text{ for } 1 \leqslant i \leqslant t\}.$$

These are again locally closed cylinders arising from the corresponding subsets $\mathrm{Cont}^\nu(E)_m$ of X'_m. The philosophy is that these loci can be understood very concretely thanks to the fact that E is a simple normal crossing divisor: for example, $\mathrm{Cont}^\nu(E)_m$ (if non-empty) is a smooth irreducible variety of codimension $\sum \nu_i$ in X'_m.

Our first main result describes the contact loci of Y in terms of the multi-contact loci associated to E. In particular, we see that any irreducible component of the contact locus $\mathrm{Cont}^p(\mathfrak{a})$ associated to the ideal sheaf $\mathfrak{a}$ on X arises as the image of a unique such multi-contact locus on X'. This is stated in the following theorem.

Theorem A. *For every positive integer p, we have a decomposition as a finite disjoint union*

$$\mathrm{Cont}^p(\mathfrak{a}) = \bigsqcup_\nu \mu_\infty(\mathrm{Cont}^\nu(E)),$$

over those $\nu \in \mathbb{N}^t$ such that $\sum_i \nu_i r_i = p$. Each $\mu_\infty(\mathrm{Cont}^\nu(E))$ is a constructible cylinder of codimension $\sum_i \nu_i(k_i+1)$. In particular, for every irreducible component W of $\mathrm{Cont}^p(\mathfrak{a})$ there is a unique multi-index ν as above such that $\mathrm{Cont}^\nu(E)$ dominates W.

A related result appears in [DL02, Theorem 2.4]. Theorem A yields a quick proof of results of [Mus02] relating the dimensions of the arc spaces $Y_\ell \subseteq X_\ell$ of Y to the singularities of the pair (X, Y). Keeping notation as above, recall that the log-canonical threshold of $\mathfrak{a}$, or of the pair (X, Y), is the rational number

$$\mathrm{lct}(\mathfrak{a}) = \mathrm{lct}(X, Y) =_{\mathrm{def}} \min_i \left\{\frac{k_i+1}{r_i}\right\}.$$

This is an important invariant of the singularities of the functions $f \in \mathfrak{a}$, with smaller values of $\mathrm{lct}(\mathfrak{a})$ reflecting nastier singularities (see [DK01, ELSV03, Kol97]). Now consider an irreducible component V_ℓ of Y_ℓ. Then the inverse image of V_ℓ in X_∞ is contained in the closure of an irreducible component W of the contact locus $\mathrm{Cont}^p(Y)$ for some $p \geqslant \ell + 1$. Write $\nu = (\nu_i)$ for the multi-index describing the cylinder $\mathrm{Cont}^{\nu}(E)$ dominating W. If $c = \mathrm{lct}(X, Y)$ then $k_i + 1 \geqslant cr_i$ for each of the divisors E_i, and so one finds from Theorem A:

$$\begin{aligned} \mathrm{codim}(V_\ell, X_\ell) &\geqslant \mathrm{codim}(\mu_\infty(\mathrm{Cont}^{\nu}(E)), X_\infty) \\ &= \sum \nu_i(k_i + 1) \\ &\geqslant c \cdot \sum \nu_i r_i \\ &= c \cdot p \\ &\geqslant c(\ell + 1). \end{aligned} \tag{4}$$

The reverse inequality being elementary for suitable values of ℓ, we deduce a corollary.

COROLLARY B [Mus02]. *One has*

$$\mathrm{lct}(X, Y) = \min_{\ell} \left\{ \frac{\mathrm{codim}(Y_\ell, X_\ell)}{\ell + 1} \right\}.$$

A closer look at this argument identifies geometrically the components of Y_ℓ having maximal possible dimension: they are the closures of the images of multi-contact loci on X' involving the divisors E_i computing $\mathrm{lct}(\mathfrak{a})$ (Corollary 3.3). We also recover the main result of [Mus01] relating irreducibility of Y_ℓ to the singularities of Y when Y is a local complete intersection.

Our second main result concerns the valuations defined by irreducible cylinders $C \subset X_\infty$. Assuming that C does not dominate X, it determines a valuation val_C on the function field $\mathbb{C}(X)$ of X by the rule:

$$\mathrm{val}_C(f) = \mathrm{ord}_\gamma(f) \quad \text{for general } \gamma \in C.$$

We prove that any such valuation comes from a contact locus. This is stated in the next theorem.

THEOREM C. *Let $C \subseteq X_\infty$ be an irreducible cylinder which does not dominate X, and denote by val_C the corresponding valuation on $\mathbb{C}(X)$. Then there exist an ideal $\mathfrak{a} \subseteq \mathcal{O}_X$, an integer $p > 0$ and an irreducible component W of $\mathrm{Cont}^p(\mathfrak{a})$ such that $\mathrm{val}_C = \mathrm{val}_W$. Moreover val_C is a divisorial valuation, i.e. it agrees up to a constant with the valuation given by the order of vanishing along a divisor in a suitable blow-up of X.*

The divisor in question is the exceptional divisor in a weighted blow-up of X' along the intersection of the relevant E_i from Theorem A.

Theorem A is a consequence of a result of Denef and Loeser [DL99] describing the fibers of the map $\mu_m : X'_m \longrightarrow X_m$. This is also one of the principal inputs to the theory of motivic integration, so the techniques used here are certainly not disjoint from the methods of the third author in [Mus01] and [Mus02]. However the present approach clarifies the geometric underpinnings of the results in question, and it bypasses the combinatorial complexities involved in manipulating motivic integrals.

In order to streamline the presentation, we collect in § 1 some basic results on cylinders in arc spaces of smooth varieties. The proofs of the above theorems are given in § 2. Section 3 is devoted to some variants and further applications: in particular, we complete the proof of Corollary B and explain how to recover some of the results of [Mus01]. We discuss also more general pairs of the form $(X, \alpha \cdot Y - \beta \cdot Z)$, and interpret the corresponding generalized log-canonical thresholds in terms of arc spaces. In particular, this yields a jet-theoretic description of the multiplier ideals. As an example, we treat the case of monomial subschemes in an affine space.

1. Cylinders in arc spaces

We collect in this section some basic facts on cylinders. Let X be a fixed smooth, connected d-dimensional complex variety. We have truncation morphisms $\psi_m : X_\infty \longrightarrow X_m$, and $\tau_{m+1,m} : X_{m+1} \longrightarrow X_m$, such that $\tau_{m+1,m}$ is locally trivial with fiber $\mathbb{C}^d$. Note that X_∞ is the set of $\mathbb{C}$-valued points of a scheme over $\mathbb{C}$, and we consider on X_∞ the restriction of the Zariski topology on this scheme. It is clear that this is equal to the inverse limit topology induced by the projections $\{\psi_m\}_m$. Since every $\tau_{m+1,m}$ is flat, hence open, it follows that ψ_m is open for every m.

Recall that a cylinder C in X_∞ is a subset of the form $\psi_m^{-1}(S)$, for some m, and some constructible subset $S \subseteq X_m$. We stress that all the points we consider in X_∞ and X_m are $\mathbb{C}$-valued points. The cylinders form an algebra of sets. It is clear that $C = \psi_m^{-1}(S)$ is open (closed, locally closed) if and only if S is. Moreover, we have $\overline{C} = \psi_m^{-1}(\overline{S})$, so $\overline{C}$ is a cylinder. Since the projection maps are locally trivial, it follows that C is irreducible if and only if S is. In particular, the decomposition of S in irreducible components induces a similar decomposition for C. For every cylinder $C = \psi_m^{-1}(S)$, we put $\operatorname{codim}(C) := \operatorname{codim}(S, X_m) = (m+1)d - \dim(S)$. If $C_1 \subseteq C_2$ are cylinders, then $\operatorname{codim}(C_1) \geqslant \operatorname{codim}(C_2)$. Moreover, if C_1 and C_2 are closed and irreducible, then we have equality of codimensions if and only if $C_1 = C_2$.

We say that a subset T of X_∞ is *thin* if there is a proper closed subscheme Z of X such that $T \subseteq Z_\infty$.

Proposition 1.1. *If C is a non-empty cylinder, then C is not thin.*

Proof. Suppose that Z is a proper closed subset such that $C \subseteq Z_\infty$. In particular, we have $\operatorname{codim}(C) \geqslant \operatorname{codim}(\psi_m^{-1}(Z_m))$ for every m. On the other hand, it can be shown that $\lim_{m\to\infty} \operatorname{codim}(\psi_m^{-1}(Z_m)) = \infty$ (see, for example, [Mus01, Lemma 3.7]). This gives a contradiction. ☐

Lemma 1.2. *If $C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$ are non-empty cylinders in X_∞, then $\bigcap_m C_m \neq \emptyset$.*

Proof. We give a proof following [Bat98]. Note first that a similar assertion holds for a non-increasing sequence of constructible subsets of a given variety, as we work over an uncountable field. Moreover, it follows from definition and Chevalley's theorem that, for every cylinder $C \subseteq X_\infty$, the image $\psi_p(C)$ is constructible for all p.

Consider the constructible subsets of X

$$\psi_0(C_1) \supseteq \psi_0(C_2) \supseteq \psi_0(C_3) \supseteq \cdots .$$

The above remark shows that we can find $x_0 \in \bigcap_m \psi_0(C_m)$. Replace now C_m by $C'_m := C_m \cap \psi_0^{-1}(x_0)$, which again form a non-increasing sequence of non-empty cylinders. Consider

$$\psi_1(C'_1) \supseteq \psi_1(C'_2) \supseteq \psi_1(C'_3) \supseteq \cdots ,$$

so that we can find $x_1 \in \bigcap_m \psi_m(C'_m)$. We replace now C'_m by $C''_m := C_m \cap \psi_1^{-1}(x_1)$ and continue this way, to get a sequence $\{x_p\}_p$, such that $\tau_{p+1,p}(x_{p+1}) = x_p$ for every p, and such that $x_p \in \psi_p(C_m)$ for every p and m. This defines $x = (x_p)_p \in X_\infty$, and since each C_m is a cylinder, we see that $x \in C_m$ for every m. ☐

We consider now a proper, birational morphism $\mu : X' \longrightarrow X$ between smooth varieties, with $\mu_m : X'_m \longrightarrow X_m$ the induced map on arc spaces. The next lemma shows that μ_m is surjective for all $m \in \mathbb{N}$. In fact, we will show later the stronger fact that μ_∞ is surjective.

Lemma 1.3. *If μ is as above, then $\mu_m : X'_m \longrightarrow X_m$ is surjective for every $m \in \mathbb{N}$.*

Proof. Let $Z \subset X$ be a proper closed subset such that μ is an isomorphism over $X \setminus Z$. It follows from the valuative criterion for properness that $X_\infty \setminus Z_\infty$ is contained in the image of μ_∞. On the other hand, for every $u \in X_m$, we have $\psi_m^{-1}(u) \not\subseteq Z_\infty$ by Proposition 1.1, so $u \in \operatorname{Im}(\mu_m)$. ☐

We give now a criterion for the image of a cylinder by μ_∞ to be a cylinder. If μ is as above, we denote by ψ_m and ψ'_m the projections corresponding to X and X', respectively.

Proposition 1.4. *With the above notation, if $C' = (\psi'_m)^{-1}(S')$ is a cylinder so that S' is a union of fibers of μ_m, then $C := \mu_\infty(C')$ is a cylinder in X_∞.*

Proof. Since $S := \mu_m(S')$ is constructible, it is enough to show that $C = \psi_m^{-1}(S)$. We prove that $C \supseteq \psi_m^{-1}(S)$, the reverse inclusion being obvious. If $\gamma \in \psi_m^{-1}(S)$, consider for every $p \geqslant m$, the cylinder

$$D_p = (\psi'_p)^{-1}(\mu_p^{-1}(\psi_p(\gamma))) \subseteq X'_\infty.$$

It follows from Lemma 1.3 that $D_p \neq \emptyset$ for every $p \geqslant m$. On the other hand, it is clear that $D_p \supseteq D_{p+1}$ for $p \geqslant m$, so Lemma 1.2 shows that there is $\gamma' \in \bigcap_{p \geqslant m} D_p$. Note that $\mu_\infty(\gamma') = \gamma$, while we have $\gamma' \in C'$, as S' is a union of fibers of μ_m. □

We use this to strengthen the assertion in Lemma 1.3.

Corollary 1.5. *If $\mu : X' \longrightarrow X$ is a proper, birational morphism between smooth varieties, then μ_∞ is surjective.*

Proof. $C' = X'_\infty$ certainly satisfies the hypothesis of Proposition 1.4, so $C = \mu_\infty(X'_\infty)$ is a cylinder. On the other hand, we have seen in the proof of Lemma 1.3 that there is a proper closed subset $Z \subset X$ so that $X_\infty \setminus Z_\infty \subseteq C$. Therefore $X_\infty \setminus C$ is contained in Z_∞, so it is empty by Proposition 1.1. □

We recall now a theorem of Denef and Loeser which will play a pivotal role in our arguments. Suppose that $\mu : X' \longrightarrow X$ is a proper, birational morphism of smooth varieties. As in the Introduction, we introduce the relative canonical divisor

$$K_{X'/X} =_{\text{def}} \{\det(d\mu) = 0\},$$

an effective Cartier divisor on X', supported on the exceptional locus of μ.

Denef and Loeser Theorem [DL99]. *Given an integer $e \geqslant 0$, consider the contact locus*

$$\text{Cont}^e(K_{X'/X})_m = \{\gamma' \in X'_m \mid \text{ord}_{\gamma'}(K_{X'/X}) = e\}.$$

If $m \geqslant 2e$ then $\text{Cont}^e(K_{X'/X})_m$ is a union of fibers of $\mu_m : X'_m \longrightarrow X_m$, each of which is isomorphic to an affine space $\mathbb{A}^e$. Moreover if

$$\gamma', \gamma'' \in \text{Cont}^e(K_{X'/X})_m$$

lie in the same fiber of μ_m, then they have the same image in X'_{m-e}.

In fact, Denef and Loeser show that μ_m is a Zariski-locally trivial $\mathbb{A}^e$-bundle over the image of $\text{Cont}^e(K_{X'/X})_m$ in X_m.

Remark 1.6. The most important point for our purposes is the statement that the contact locus $\text{Cont}^e(K_{X'/X})_m \subseteq X'_m$ is a union of fibers of μ_m, and that all of these fibers are irreducible of dimension e. When $\mu : X' \longrightarrow X$ is the blowing-up of X along a smooth center this is readily checked by an explicit calculation in local coordinates. This case in turn implies the statement when μ is obtained as a composition of such blow-ups. There would be no essential loss in generality in limiting ourselves in what follows to such a composition of nice blow-ups.

Corollary 1.7. *Let $\mu : X' \longrightarrow X$ be a proper, birational morphism between smooth varieties. If $C' \subseteq X'_\infty$ is a cylinder, then the closure $\overline{\mu_\infty(C')}$ of its image is a cylinder. Moreover, if there is $e \in \mathbb{N}$ such that $C' \subseteq \text{Cont}^e(K_{X'/X})$, then $\mu_\infty(C')$ is a cylinder.*

Proof. We use the above Theorem of Denef and Loeser. Note first that, in order to prove the second assertion, we may assume that C' is irreducible. Let e be the smallest p such that $C' \cap \mathrm{Cont}^p(K_{X'/X}) \neq \emptyset$ (we have $e < \infty$ by Proposition 1.1). As $C'_\circ := C' \cap \mathrm{Cont}^e(K_{X'/X})$ is open and dense in C', it follows that $\overline{\mu_\infty(C')} = \overline{\mu_\infty(C'_\circ)}$, so that it is enough to prove the second assertion in the theorem.

Let p and $T \subseteq X'_p$ be such that $C' = (\psi'_p)^{-1}(T)$, and fix m with $m \geqslant \max\{2e, e+p\}$. If $S \subseteq X'_m$ is the inverse image of T by the canonical projection to X'_p, Proposition 1.4 shows that it is enough to check that S is a union of fibers of μ_m. This follows from Denef and Loeser's theorem: if $\delta_1 \in S$ and $\delta_2 \in X'_m$ are such that $\mu_m(\delta_1) = \mu_m(\delta_2)$, then the first assertion in the theorem implies $\delta_2 \in \mathrm{Cont}^e(K_{X'/X})_m$, and the last assertion in the theorem shows that δ_1 and δ_2 lie over the same element in X'_p, hence $\delta_2 \in S$. □

We show now that our notion of codimension for cylinders agrees with the usual one, defined in terms of the Zariski topology on X_∞.

Lemma 1.8. *If $C \subseteq X_\infty$ is an irreducible cylinder, and if $W \supseteq C$ is an irreducible closed subset of X_∞, then W is a cylinder.*

Proof. It follows from the definition of the Zariski topology that $W = \bigcap_{m \in \mathbb{N}} W^{(m)}$, where $W^{(m)} = \psi_m^{-1}(\overline{\psi_m(W)})$. Note that each $W^{(m)}$ is a closed irreducible cylinder, and we have $C \subseteq W^{(m+1)} \subseteq W^{(m)}$ for every m. Therefore $\mathrm{codim}\, W^{(m)} \leqslant \mathrm{codim}(C)$ for every m, and we deduce that $\mathrm{codim}\, W^{(m)}$ is eventually constant. This shows that there is m_0 such that $W^{(m)} = W^{(m_0)}$ for all $m \geqslant m_0$; hence $W = W^{(m_0)}$ is a cylinder. □

Corollary 1.9. *If $C \subseteq X_\infty$ is a cylinder, then* $\mathrm{codim}(C)$ *is the codimension of C, as defined using the Zariski topology on X_∞.*

Proof. We may clearly assume that C is closed and irreducible. In this case, the above lemma shows that every chain of closed irreducible subsets containing C consists of cylinders, so the assertion is obvious. □

Proposition 1.10. *Let $C \subseteq X_\infty$ be a cylinder. If we have a countable disjoint union of cylinders $\bigsqcup_{p \in \mathbb{N}} D_p \subseteq C$, whose complement in C is thin, then*

$$\mathrm{codim}(C) = \min_{p \in \mathbb{N}} \mathrm{codim}(D_p).$$

Moreover, if each D_p is irreducible (or empty), then for every irreducible component W of C there is a unique $p \in \mathbb{N}$ such that $D_p \subseteq W$, and D_p is dense in W.

Proof. Let $Z \subset X$ be a proper closed subset such that $C \subseteq Z_\infty \cup \bigcup_p D_p$. We will use the fact that $\lim_{m \to \infty} \mathrm{codim}(\psi_m^{-1}(Z_m)) = \infty$ (see [Mus01, Lemma 3.7]).

It is clear that we have $\mathrm{codim}(C) \leqslant \min_p \mathrm{codim}(D_p)$. For the reverse inequality, choose m such that $\mathrm{codim}(\psi_m^{-1}(Z_m)) > \min_p \mathrm{codim}(D_p)$. It follows from Lemma 1.2 that there is r such that

$$C \subseteq \bigcup_{p \leqslant r} D_p \cup \psi_m^{-1}(Z_m).$$

This clearly gives $\mathrm{codim}(C) \geqslant \min_p \mathrm{codim}(D_p)$.

Suppose now that every D_p is irreducible, and let W be an irreducible component of C. Choose m such that $\mathrm{codim}(\psi_m^{-1}(Z_m)) > \mathrm{codim}(W)$, and let r be such that

$$C \subseteq \bigcup_{p \leqslant r} D_p \cup \psi_m^{-1}(Z_m).$$

It follows that there is $p \leqslant r$ such that $W \subseteq \overline{D_p}$. Since D_p is irreducible, we see that we also have $D_p \subseteq W$. ☐

Remark 1.11. Under the assumption of Proposition 1.10 one knows that

$$\mu_{\mathrm{Mot}}(C) = \sum_p \mu_{\mathrm{Mot}}(D_p),$$

where μ_{Mot} is the motivic measure from [DL99]. By taking the Hodge realization of the motivic measure, the statement in the above proposition follows immediately. However, we preferred to avoid this formalism.

2. Contact loci and valuations

This section is devoted to the proof of our main results. Keeping the notation established in the Introduction, we give first the statement and the proof of a more general version of Theorem A. Consider a smooth variety X of dimension d and a non-zero ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$ on X defining a subscheme $Y \subseteq X$. Fix a log resolution $\mu : X' \longrightarrow X$ of $\mathfrak{a}$, with $E = \sum_{i=1}^t E_i$ a simple normal crossing divisor on X' such that

$$\mathfrak{a}' = \mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}\biggl(-\sum_{i=1}^t r_i E_i\biggr), \quad K_{X'/X} = \sum_{i=1}^t k_i E_i \tag{5}$$

for some integers $r_i, k_i \geqslant 0$. Note that this time we do not assume $r_i \geqslant 1$ if $k_i \geqslant 1$. Given a multi-index $\nu = (\nu_i) \in \mathbb{N}^t$ we define the support of ν to be

$$\mathrm{supp}(\nu) = \{i \in [1,t] \mid \nu_i \neq 0\},$$

and we put

$$E_\nu = \bigcap_{i \in \mathrm{supp}(\nu)} E_i.$$

Thus E_ν is either empty or a smooth subvariety of codimension $|\mathrm{supp}(\nu)|$ in X'. Without loss of generality we will assume in addition that E_ν is connected (and hence irreducible) whenever it is non-empty. We can always arrive at this situation by a sequence of blow-ups along smooth centers. In fact, starting with an arbitrary log resolution, first blow up the d-fold intersections of the E_i; then blow up the $(d-1)$-fold intersections of their proper transforms; and so on. At the end of this process we arrive at a log resolution where the stated condition is satisfied.

Given a multi-index $\nu \in \mathbb{N}^t$ and an integer $m \geqslant \max_i \nu_i$, we consider as in the Introduction the 'multi-contact' loci

$$\mathrm{Cont}^\nu(E)_m = \{\gamma' \in X'_m \mid \mathrm{ord}_{\gamma'}(E_i) = \nu_i \text{ for } 1 \leqslant i \leqslant t\},$$

and the corresponding sets $\mathrm{Cont}^\nu(E) \subseteq X'_\infty$. Provided that $E_\nu \neq \emptyset$, a computation in local coordinates shows that $\mathrm{Cont}^\nu(E)_m$ is a smooth irreducible locally closed subset of codimension $\sum \nu_i$ in X'_m. (If E_ν is empty, then so is $\mathrm{Cont}^\nu(E)_m$.) Recall that a subset of X_∞ is called *thin* if it is contained in Z_∞ for some proper closed subscheme $Z \subset X$.

Theorem 2.1. *For every positive integer p, we have a disjoint union*

$$\bigsqcup_\nu \mu_\infty(\mathrm{Cont}^\nu(E)) \subseteq \mathrm{Cont}^p(Y),$$

where the union is over those $\nu \in \mathbb{N}^t$ such that $\sum_i \nu_i r_i = p$. For every ν as above such that $\mathrm{Cont}^\nu(E) \neq \emptyset$, its image $\mu_\infty(\mathrm{Cont}^\nu(E))$ is an irreducible cylinder of codimension $\sum_i \nu_i(k_i+1)$. Moreover, the complement in $\mathrm{Cont}^p(Y)$ of the above union is thin.

Proof. We use the theorem of Denef and Loeser. It is clear that we have a disjoint union

$$\bigsqcup_{\nu} \mathrm{Cont}^{\nu}(E) \subseteq \mu_{\infty}^{-1}(\mathrm{Cont}^{p}(Y)),$$

where ν varies over the set in the statement, and the complement is contained in the union of those $(E_i)_{\infty}$ such that $r_i = 0$. For every ν we put $e = \sum_i \nu_i k_i$, so that $\mathrm{Cont}^{\nu}(E) \subseteq \mathrm{Cont}^{e}(K_{X'/X})$. Corollary 1.7 implies that $\mu_{\infty}(\mathrm{Cont}^{\nu}(E))$ is a cylinder.

We show now that $\mu_{\infty}(\mathrm{Cont}^{\nu}(E))$ and $\mu_{\infty}(\mathrm{Cont}^{\nu'}(E))$ are disjoint if $\nu \neq \nu'$. Indeed, otherwise there are $\gamma \in \mathrm{Cont}^{\nu}(E)$ and $\gamma' \in \mathrm{Cont}^{\nu'}(E)$ with $\mu_{\infty}(\gamma) = \mu_{\infty}(\gamma')$. If e and e' correspond to ν and ν', respectively, fix $m \geqslant \max\{2e, e + \nu_i, e + \nu_i'\}_i$. The first assertion in the theorem of Denef and Loeser gives $e = e'$, and the last assertion implies that γ and γ' have the same image in X'_{m-e}, a contradiction.

Since μ_{∞} is surjective by Corollary 1.5, we get a disjoint union as in the theorem. Moreover, if ν is such that $E_{\nu} \neq \emptyset$ and if $m \gg 0$, then the Theorem of Denef and Loeser shows that the projection

$$\mathrm{Cont}^{\nu}(E)_m \longrightarrow \mu_m(\mathrm{Cont}^{\nu}(E)_m)$$

has irreducible, e-dimensional fibers (in fact, it is locally trivial with fiber $\mathbb{A}^e$). Hence

$$\dim \mu_m(\mathrm{Cont}^{\nu}(E)_m) = \dim \mathrm{Cont}^{\nu}(E)_m - e = (m+1)d - \sum_i \nu_i(k_i + 1),$$

which completes the proof of the theorem. □

Remark 2.2. Note that if μ is as in the introduction, i.e. if it is an isomorphism over $X \setminus Y$, then the disjoint union in Theorem 2.1 is finite. Moreover, it follows from the above proof that in this case the union is equal to $\mathrm{Cont}^{p}(Y)$, and we recover also the first statement of Theorem A.

Remark 2.3. As pointed out by the referee, the fact that

$$\mu_{\infty}(\mathrm{Cont}^{\nu}(E)) \cap \mu_{\infty}(\mathrm{Cont}^{\nu'}(E)) = \emptyset$$

for $\nu \neq \nu'$ can be seen also in an elementary way as follows. If $Z = \mu(E)$, then μ induces an isomorphism $X' \setminus E \longrightarrow X \setminus Z$. By the valuative criterion for properness, μ_{∞} induces a bijection $X'_{\infty} \setminus E_{\infty} \longrightarrow X_{\infty} \setminus Z_{\infty}$. As ν_i, $\nu_i' \neq \infty$ for all i, we see that $\mathrm{Cont}^{\nu}(E)$, $\mathrm{Cont}^{\nu'}(E) \subseteq X'_{\infty} \setminus E_{\infty}$, and we deduce our assertion.

In the setting of Theorem 2.1, the disjoint decomposition of $\mathrm{Cont}^{p}(Y)$ allows us to relate the irreducible components of $\mathrm{Cont}^{p}(Y)$ with the multi-contact loci $\mathrm{Cont}^{\nu}(E)$. This is a formal consequence of Proposition 1.10.

Corollary 2.4. *With the notation in Theorem 2.1, we have*

$$\mathrm{codim}\, \mathrm{Cont}^{p}(Y) = \min_{\nu} \sum_i \nu_i(k_i + 1),$$

where the minimum is over all $\nu \in \mathbb{N}^t$ such that $\sum_i \nu_i r_i = p$ and $E_{\nu} \neq \emptyset$. Moreover, for every irreducible component W of $\mathrm{Cont}^{p}(Y)$ there is a unique ν as above, such that W contains $\mu_{\infty}(\mathrm{Cont}^{\nu}(E))$ as a dense subset.

Note that this corollary allows us to describe the irreducible components of each $\mathrm{Cont}^{p}(Y)$ which have minimal codimension in terms of the numerical data of the resolution. However, while Theorem 2.1 shows that all the irreducible components are determined by the resolution, the ones of codimension larger than $\mathrm{codim}(\mathrm{Cont}^{p}(Y))$ seem to depend on more than just the numerical data.

In the next section we will see that, by considering a varying auxiliary scheme $Z \subset X$, we can describe also other components of the contact loci.

We turn now to valuations of $\mathbb{C}(X)$ associated to cylinders in the arc space. Recall that, if C is an irreducible cylinder in X_∞, then we have defined a valuation val_C as follows. Note first that if $\gamma \in X_\infty$, and if f is a rational function on X defined in a neighborhood of $\psi_0(\gamma)$, then $\mathrm{ord}_\gamma(f)$ is well defined. If the domain of f intersects $\psi_0(C)$, then $\mathrm{val}_C(f) := \mathrm{ord}_\gamma(f)$, for general $\gamma \in C$. Note that this is a non-negative integer by Proposition 1.1. Since C is irreducible, it follows that $\mathrm{val}_C(f)$ is well defined and can be extended to a valuation of the function field of X. We will assume from now on that C does not dominate X, so val_C is non-trivial (and discrete).

It is clear that if $C_1 \subseteq C_2$ are cylinders as above, then $\mathrm{val}_{C_1}(f) \geqslant \mathrm{val}_{C_2}(f)$ for every rational function f whose domain intersects $\psi_0(C_1)$. If, moreover, C_1 is dense in C_2, then $\mathrm{val}_{C_1} = \mathrm{val}_{C_2}$. Suppose now that $\mu : X' \longrightarrow X$ is a proper, birational morphism of smooth varieties, and that $C' \subseteq X'_\infty$ is an irreducible cylinder which does not dominate X'. It follows from Corollary 1.7 that $C := \overline{\mu_\infty(C')}$ is a cylinder, and we clearly have $\mathrm{val}_{C'} = \mathrm{val}_C$.

Example 2.5. A divisorial valuation of $\mathbb{C}(X)$ is a discrete valuation associated to a prime divisor on some normal variety X' which is birational to X. If a divisorial valuation has center on X (see below), then there are infinitely many cylinders in X_∞ whose corresponding valuations agree up to a constant with the given valuation. To see this, let D be a prime divisor on X', and let val_D be the corresponding valuation. Saying that a valuation has center on X means that we may assume that we have a proper birational morphism $\mu : X' \longrightarrow X$, that X' is smooth, and that D is smooth on X'. For $p \geqslant 1$, let $C'_p = \mathrm{Cont}^p(D)$, so $\mathrm{val}_{C'_p} = p \cdot \mathrm{val}_D$. If we put $C_p = \overline{\mu_\infty(C'_p)}$, then Corollary 1.7 shows that C_p is an irreducible cylinder, and $\mathrm{val}_{C_p} = p \cdot \mathrm{val}_D$.

In the remainder of this section we show that, conversely, every valuation defined by a cylinder in X_∞ is (up to a constant multiple) a divisorial valuation. We consider first the case when the cylinder is an irreducible component of a contact locus. In this case, the assertion is a corollary of Theorem 2.1.

Corollary 2.6. *Let $Y \subset X$ be a proper closed subscheme, and let W be an irreducible component of $\mathrm{Cont}^p(Y)$ for some $p \geqslant 1$. Let $\nu \in \mathbb{N}^t$ be the multi-index given by Corollary 2.4, so that $\mu_\infty(\mathrm{Cont}^\nu(E))$ is contained and dense in W. If D is the exceptional divisor of the weighted blowing-up of (X', E) with weight ν, then $\mathrm{val}_W = q \cdot \mathrm{val}_D$, where $q = \gcd\{\nu_i \mid i \in \mathrm{supp}(\nu)\}$.*

Proof. Recall the definition of the weighted blowing-up with weight ν. Let

$$s = \mathrm{lcm}\{\nu_i \mid i \in \mathrm{supp}(\nu)\}.$$

If $T_\nu \subset X'$ is the closed subscheme defined by

$$\sum_{i \in \mathrm{supp}(\nu)} \mathcal{O}\left(-\frac{s}{\nu_i} \cdot E_i\right),$$

then the weighted blowing-up of (X', E) with weight ν is the normalized blowing-up of X' along T_ν. There is a unique prime divisor on this blowing-up which dominates T_ν; this is D.

It follows from our choice of ν that $\mathrm{val}_W = \mathrm{val}_{\mathrm{Cont}^\nu(E)}$. Let $g : X'' \longrightarrow X'$ be a proper birational map which factors through the above blowing-up and which satisfies the requirements for a log resolution. Note that we may consider D as a divisor on X''. We apply Theorem 2.1 for g.

Let C be the multi-contact locus of all arcs on X''_∞ with order q along D, and order zero along all the other divisors involved. Well-known results about weighted blow-ups show that the coefficient of D in $g^{-1}(E_i)$ is ν_i/q, and that the coefficient of D in $K_{X''/X'}$ is $-1 + \sum_i \nu_i/q$. We see that

$g_\infty(C) \subseteq \mathrm{Cont}^\nu(E)$. Moreover, both these cylinders are irreducible and have the same codimension, as

$$\mathrm{codim}\, g_\infty(C) = q \cdot \sum_i \nu_i/q = \mathrm{codim}(\mathrm{Cont}^\nu(E)).$$

This gives $\mathrm{val}_W = \mathrm{val}_C = q \cdot \mathrm{val}_D$. □

We show now that, in fact, we can describe all valuations given by cylinders using contact loci.

Theorem 2.7. *If C is an irreducible cylinder in X_∞ which does not dominate X, then there is a proper closed subscheme $Y \subset X$, a positive integer p, and an irreducible component W of $\mathrm{Cont}^p(Y)$ such that $\mathrm{val}_C = \mathrm{val}_W$. In particular, val_C is equal, up to a constant, to a divisorial valuation.*

Proof. We have to prove only the first assertion; the second one follows from this and Corollary 2.6. By replacing C with its closure, we may assume that C is closed. Moreover, it is enough to prove our assertion in the case when $X = \mathrm{Spec}(A)$ is affine. Recall that a graded sequence of ideals is a set of ideals $\mathfrak{a}_\bullet = \{\mathfrak{a}_p\}_{p \geqslant 1}$ such that $\mathfrak{a}_p \cdot \mathfrak{a}_q \subseteq \mathfrak{a}_{p+q}$ for all p and q (see [Laz04] for more on this topic). Since val_C is a valuation which is non-negative on A, if we define

$$\mathfrak{a}_p := \{f \in A \mid \mathrm{val}_C(f) \geqslant p\},$$

then $\mathfrak{a}_\bullet$ is a graded sequence of ideals. Note that, since C does not dominate X, we have $\mathfrak{a}_p \neq (0)$ for every p.

Starting with a graded sequence of ideals $\mathfrak{a}_\bullet$ as above, we get a sequence of closed cylinders as follows: for every $p \geqslant 1$, let

$$W_p = \{\gamma \in X_\infty \mid \mathrm{ord}_\gamma(f) \geqslant p \text{ for every } f \in \mathfrak{a}_p\}.$$

Since $\mathfrak{a}_\bullet$ is a graded sequence, we have $\mathfrak{a}_p^q \subseteq \mathfrak{a}_{pq}$, so that $W_{pq} \subseteq W_p$ for every p, $q \geqslant 1$. Note that in our case, it follows from definition that $C \subseteq W_p$ for all p. Moreover, since $\mathfrak{a}_p \neq (0)$, we see that W_p does not dominate X, for any p. We put $C_m := W_{m!}$ so that $C \subseteq C_{m+1} \subseteq C_m$ for every $m \geqslant 1$.

We claim that we can choose irreducible components C'_m of C_m such that $C \subseteq C'_{m+1} \subseteq C'_m$ for all m. It is clear that for every m we can choose irreducible components $C_{i,m}$ of C_i for $i \leqslant m$, so that

$$C \subseteq C_{m,m} \subseteq C_{m-1,m} \subseteq \cdots \subseteq C_{1,m}.$$

As every cylinder has finitely many irreducible components, there is an irreducible component C'_1 of C_1 such that $C'_1 = C_{1,m}$ for infinitely many m. Similarly, there is an irreducible component C'_2 of C_2, such that $C'_2 \subseteq C'_1$, and such that $C'_2 = C_{2,m}$ for infinitely many m. Continuing in this way, we deduce our claim.

Note that we have $\mathrm{codim}(C'_1) \leqslant \mathrm{codim}(C'_2) \leqslant \cdots \leqslant \mathrm{codim}(C)$. Therefore there is q such that $\mathrm{codim}(C'_m) = \mathrm{codim}(C'_q)$ for every $m \geqslant q$, hence $C'_m = C'_q$, as all C'_i are irreducible closed cylinders. Let Y be the closed subscheme defined by $\mathfrak{a}_{q!}$, and let $\tau := \min\{\mathrm{ord}_\gamma(f) \mid \gamma \in C'_q, f \in \mathfrak{a}_{q!}\}$. It is clear that $\tau \geqslant q!$ and that C'_q is the closure of an irreducible component of $\mathrm{Cont}^\tau(Y)$, so that in order to finish the proof, it is enough to show that $\mathrm{val}_C = \mathrm{val}_{C'_q}$.

Since $C \subseteq C'_q$, we have $\mathrm{val}_C \geqslant \mathrm{val}_{C'_q}$ on A. Fix $f \in A$, and let us show that $m := \mathrm{val}_C(f) \leqslant \mathrm{val}_{C'_q}(f)$. By multiplying f with g such that $\mathrm{val}_C(g) > 0$, we may assume $m \geqslant 1$. Moreover, by taking a suitable power of f, we may assume that $m = p!$ for some $p \geqslant q$. By definition we have $f \in \mathfrak{a}_m$, hence

$$C'_q = C'_p \subseteq W_{p!} \subseteq \{\gamma \in X_\infty \mid \mathrm{ord}_\gamma(f) \geqslant m\}.$$

This gives $\mathrm{val}_{C'_q}(f) \geqslant m$, and completes the proof of the theorem. □

Remark 2.8. It follows from Theorem 2.7 that to each irreducible cylinder C which does not dominate X we may associate a (unique) divisor D over X such that $\mathrm{val}_C = \lambda \cdot \mathrm{val}_D$ for some $\lambda > 0$ (of course, we identify two divisors over X which give the same valuation). This map is obviously not injective, but it is surjective by Example 2.5.

Suppose now that $Y \subset X$ is a fixed proper closed subscheme. It would be interesting to understand which divisors appear from the irreducible components of contact loci of Y. One can consider this as an embedded version of Nash's problem [Nas95]. If Y is a variety, and if $\psi_0 : Y_\infty \longrightarrow Y$ is the canonical projection, then Nash described an injective map from the set of irreducible components of $\psi_0^{-1}(Y_{\mathrm{sing}})$ to the set of 'essential' exceptional divisors over Y (divisors which appear in every resolution of Y). He conjectured that this map is surjective, but a counterexample has been recently found in [IK03].

In the next section we will use Theorem 2.1 to describe certain divisors which are associated to distinguished irreducible components of the contact loci of Y, namely the divisors which compute generalized versions of the log-canonical threshold.

3. Applications to log-canonical thresholds

We discuss here some applications of Theorem 2.1. As before, X is a smooth irreducible variety of dimension d, $Y \subseteq X$ is a subscheme defined by a non-zero ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$, and $\mu : X' \longrightarrow X$ is a log resolution of (X, Y) with

$$\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}\Big(-\sum r_i E_i\Big), \quad K_{X'/X} = \sum k_i E_i.$$

3.1 Arc spaces of subschemes

We indicate how to recover some of the results of [Mus01] and [Mus02] relating singularities of the pair (X, Y) to the properties of the arc spaces $Y_\ell \subseteq X_\ell$.

Note that for every irreducible cylinder $C \subseteq X_\infty$ there is a subcylinder $C_0 \subseteq C$ which is open in C, such that $\mathrm{ord}_\gamma(\mathfrak{a})$ is constant for $\gamma \in C_0$. We denote this positive integer by $\mathrm{ord}_C(\mathfrak{a})$ or $\mathrm{ord}_C(Y)$. It is clear that, if $p = \mathrm{ord}_C(Y)$, then C is contained in the closure of an irreducible component of $\mathrm{Cont}^p(Y)$.

Suppose now that V_ℓ is an irreducible component of Y_ℓ, and let $V = \psi_\ell^{-1}(V_\ell) \subseteq X_\infty$. Note that V is an irreducible component of

$$\psi_\ell^{-1}(Y_\ell) = \mathrm{Cont}^{\geqslant(\ell+1)}(Y) =_{\mathrm{def}} \{\gamma \in X_\infty \mid \mathrm{ord}_\gamma(Y) \geqslant \ell + 1\}. \tag{6}$$

Let $p = \mathrm{ord}_V(Y)$, so $p \geqslant \ell + 1$. Note that we might have strict inequality (Example 3.1). In any case, V is the closure of an irreducible component of $\mathrm{Cont}^p(Y)$. Conversely, the closure of every irreducible component of $\mathrm{Cont}^p(Y)$ is the inverse image of an irreducible component of Y_p. Therefore our analysis of the contact loci $\mathrm{Cont}^p(Y)$ gives complete control over the arc spaces of Y.

Example 3.1. Let $X = \mathbb{A}^1$ with coordinate t, and let $Y \subset X$ be defined by $(t^e) \subseteq \mathbb{C}[t]$ for a fixed integer $e \geqslant 2$. Then $V_1 = Y_1$ is the irreducible subset of X_1 consisting of arcs centered at the origin. However, $\mathrm{ord}_V(Y) = e > 1$.

We start by completing the proof of Corollary B, namely we prove the following.

Corollary 3.2. *The log-canonical threshold of* (X, Y) *is given by*

$$\mathrm{lct}(X, Y) = \min_\ell \left\{ \frac{\mathrm{codim}(Y_\ell, X_\ell)}{\ell + 1} \right\}.$$

Proof. We saw in the Introduction that Theorem A implies the inequality

$$\operatorname{codim}(Y_\ell, X_\ell) \geqslant c(\ell+1), \tag{7}$$

where $c = \operatorname{lct}(X,Y)$, so it remains only to prove the reverse inequality for suitable ℓ. But this is immediate. In fact, it follows from the definition of $\operatorname{lct}(X,Y)$ that there exists an index i (say $i=1$) for which $k_1+1 = cr_1$. Let $\nu = (1,0,0,\ldots,0)$ be the multi-index with $\nu_1 = 1$ and $\nu_i = 0$ for $i > 1$. It follows from Theorem 2.1 that $\mu_\infty(\operatorname{Cont}^\nu(E))$ is a subcylinder of $\operatorname{Cont}^{r_1}(Y)$ of codimension k_1+1. If $\ell = r_1 - 1$, then the closure of this subcylinder can be written as $\psi_\ell^{-1}(V)$ for some closed subset $V \subseteq Y_\ell$, with $\operatorname{codim}(V, X_\ell) = c(\ell+1)$. The first part of the proof implies that V must be an irreducible component of Y_ℓ, so we are done. □

The argument just completed leads to an explicit description of the components of Y_ℓ having maximal possible dimension. Keeping notation as before, let us say that one of the divisors $E_i \subseteq X'$ *computes the log-canonical threshold* of (X,Y) if $\operatorname{lct}(X,Y) = (k_i+1)/r_i$. Note that in general there may be several divisors E_i that compute this threshold.

Corollary 3.3. *Let $V_\ell \subseteq Y_\ell$ be an irreducible component of maximal possible dimension, i.e. with*

$$\operatorname{codim}(V_\ell, X_\ell) = (\ell+1)\cdot \operatorname{lct}(X,Y),$$

and let $V = \psi_\ell^{-1}(V_\ell)$ be the corresponding subset of X_∞. Then $\operatorname{ord}_V(\mathfrak{a}) = \ell+1$ and V is dominated by a multi-contact locus $\operatorname{Cont}^\nu(E)$ where

$$\nu_i \neq 0 \Longrightarrow E_i \text{ computes the log-canonical threshold of } (X,Y).$$

Conversely the image of any such multi-contact locus $\operatorname{Cont}^\nu(E)$ determines a component of Y_ℓ of maximal possible dimension.

Proof. Write $c = \operatorname{lct}(X,Y)$. We return to the proof of Corollary B. Specifically, V is contained in the closure of some irreducible component W of $\operatorname{Cont}^p(\mathfrak{a})$, which in turn is dominated by some multi-contact locus $\operatorname{Cont}^\nu(E)$. However, $\operatorname{codim}(V_\ell, X_\ell) = c(\ell+1)$ by hypothesis, and hence equality must hold in all the inequalities appearing in Equation (4). Therefore

$$p = \sum \nu_i r_i = (\ell+1).$$

Moreover, since in any event $k_i + 1 \geqslant cr_i$ for all i, we deduce that $\nu_i(k_i+1) = c\nu_i r_i$ for each index i. In particular, if $\nu_i \neq 0$ then E_i computes the log-canonical threshold of (X,Y). We leave the converse to the reader. □

Similar arguments allow one to eliminate motivic integration from the main results of the third author in [Mus01]. For example, we have the following corollary.

Corollary 3.4 [Mus01]. *Let $Y \subseteq X$ be a reduced and irreducible locally complete intersection subvariety of codimension f. Then the arc space Y_ℓ is irreducible for all $\ell \geqslant 1$ if and only if Y has at worst rational singularities.*

Proof. Following [Mus01] let $\mu : X' \longrightarrow X$ be a log resolution of (X,Y) which dominates the blowing-up of X along Y. Write E_1 for the (reduced and irreducible) exceptional divisor created by this blow-up, so that

$$r_1 = 1, \quad k_1 = f-1,$$

and otherwise keep notation and assumptions as above. It is established in [Mus01], Theorem 2.1 and Remark 2.2 that Y has at worst rational singularities if and only if $k_i \geqslant fr_i$ for every index $i \geqslant 2$. So we are reduced to showing:

$$Y_\ell \text{ is irreducible for all } \ell \geqslant 1 \quad \Longleftrightarrow \quad k_i \geqslant fr_i \text{ for } i \geqslant 2. \tag{$*$}$$

(It is the proof of this equivalence in [Mus01] that uses motivic integration.)

Assuming that $k_i \geqslant fr_i$ for $i \geqslant 2$ we show that each Y_ℓ is irreducible. Note to begin with that Y_ℓ has one 'main component' Y_ℓ^{main}, namely the closure of the arc space $(Y_{\mathrm{reg}})_\ell$: this component is dominated by the multi-contact locus $\mathrm{Cont}^{(\ell+1,0,0,\ldots,0)}(E)$ described by the multi-index $\nu = (\ell+1,0,0,\ldots,0)$. Suppose for a contradiction that there is a further component V_ℓ of Y_ℓ. In the usual way, $V = \psi_\ell^{-1}(V_\ell)$ lies in the closure of an irreducible component W of $\mathrm{Cont}^p(Y)$ for $p \geqslant \ell+1$, which via Corollary 2.6 is dominated by a multi-contact locus $\mathrm{Cont}^\nu(E)$ for some $\nu = (\nu_i) \neq (\ell+1,0,\ldots,0)$. Since $Y \subseteq X$ is a local complete intersection of codimension f, we have in any event $\mathrm{codim}(W, X_\ell) \leqslant (\ell+1)\cdot f$. In view of the hypothesis $k_i \geqslant fr_i$ for $i \geqslant 2$ we then find the inequalities

$$\begin{aligned}(\ell+1)\cdot f &\geqslant \mathrm{codim}(W)\\ &= \sum_{i\geqslant 1}\nu_i(k_i+1)\\ &= \nu_1\cdot f + \sum_{i\geqslant 2}\nu_i(k_i+1)\\ &\geqslant f\cdot\sum_{i\geqslant 1}\nu_i r_i + \sum_{i\geqslant 2}\nu_i\\ &= fp + \sum_{i\geqslant 2}\nu_i.\end{aligned}$$

However, since $p \geqslant \ell+1$ this forces $\nu_i = 0$ for $i \geqslant 2$, a contradiction.

Conversely, suppose that $k_i < fr_i$ for some $i \geqslant 2$: say $k_2 \leqslant fr_2 - 1$. Setting $\nu = (0,1,0,\ldots,0)$ and $\ell = r_2 - 1$, the multi-contact locus $\mathrm{Cont}^\nu(E)$ maps to an irreducible set $W_\ell \subseteq Y_\ell$ with

$$\mathrm{codim}(W_\ell, X_\ell) \leqslant (\ell+1)f = \mathrm{codim}(Y_\ell^{\mathrm{main}}, X_\ell),$$

and therefore Y_ℓ cannot be irreducible. □

3.2 Generalized log-canonical thresholds

We extend now the above results to take into account also an extra scheme Z. We fix two proper closed subschemes Y, $Z \subseteq X$ defined by the ideal sheaves $\mathfrak{a}$ and $\mathfrak{b}$, respectively. We use the previous notation for $\mu : X' \longrightarrow X$, but this time we assume that μ is a log resolution for $Y \cup Z$. We write $\mathfrak{b}\cdot\mathcal{O}_{X'} = \mathcal{O}_{X'}(-\sum s_i E_i)$. Having fixed also $\beta \in \mathbb{Q}_+$, we define the log-canonical threshold $\mathrm{lct}(X, Y; \beta\cdot Z)$ to be the largest $\alpha \in \mathbb{Q}_+$ such that $(X, \alpha\cdot Y - \beta\cdot Z)$ is log canonical, i.e. $k_i + 1 \geqslant \alpha\cdot r_i - \beta\cdot s_i$ for all i. Therefore

$$\mathrm{lct}(X, Y; \beta\cdot Z) = \min_i\left\{\frac{k_i+1+\beta\cdot s_i}{r_i}\right\}.$$

It is standard to see that the definition does not depend on the particular log resolution [Kol97].

The following corollary is a generalization of Corollary 3.2 to this setting. This time we state the formula in terms of the contact loci, and leave the corresponding statement in terms of arc spaces to the reader.

Corollary 3.5. *We have*

$$\mathrm{lct}(X, Y; \beta\cdot Z) = \min_C\left\{\frac{\mathrm{codim}(C) + \beta\cdot\mathrm{ord}_C(Z)}{\mathrm{ord}_C(Y)}\right\},$$

where C runs over the irreducible cylinders in X_∞ which do not dominate X; in fact, it is enough to let C run only over the irreducible components of $\mathrm{Cont}^p(Y)$, for $p \geqslant 1$.

Proof. Let $c = \mathrm{lct}(X, Y; \beta\cdot Z)$. We show first that, if C is an irreducible cylinder in X_∞, then $\mathrm{codim}(C) \geqslant c\cdot\mathrm{ord}_C(Y) - \beta\cdot\mathrm{ord}_C(Z)$. Let $p = \mathrm{ord}_C(Y)$, and let $C_0 \subseteq C$ be an open subcylinder such that $C_0 \subseteq \mathrm{Cont}^p(Y)$. Let W be an irreducible component of $\mathrm{Cont}^p(Y)$ containing C_0, and let ν be the multi-index corresponding to W by Corollary 2.4. We have

$$\begin{aligned}\mathrm{codim}(C) &= \mathrm{codim}(C_0)\\ &\geqslant \mathrm{codim}(W)\\ &= \sum \nu_i(k_i+1)\\ &\geqslant \sum_i \nu_i(c\cdot r_i - \beta\cdot s_i)\\ &= c\cdot p - \beta\cdot \mathrm{ord}_W(Z)\\ &\geqslant c\cdot p - \beta\cdot \mathrm{ord}_C(Z),\end{aligned}$$

as required.

In order to finish the proof, it is enough to show that there is $p \geqslant 1$ and an irreducible component V of $\mathrm{Cont}^p(Y)$ such that $\mathrm{codim}(V) = c{\cdot}p - \beta{\cdot}\mathrm{ord}_V(Z)$. For this, let i be such that $c = (k_i+1+\beta{\cdot}s_i)/r_i$, and take $p = r_i$ and ν such that $\nu_i = 1$ and $\nu_j = 0$ if $j \neq i$. We may take V to be the closure of $\mu_\infty(\mathrm{Cont}^\nu(E))$ in $\mathrm{Cont}^p(Y)$. □

We have a similar generalization of Corollary 3.3. Again, we phrase the result in terms of irreducible components of contact loci of Y. We say that a divisor E_i computes $\mathrm{lct}(X, Y; \beta\cdot Z)$ if $\mathrm{lct}(X, Y; \beta\cdot Y) = (k_i + 1 + \beta\cdot s_i)/r_i$.

Corollary 3.6. *Let W be an irreducible component of $\mathrm{Cont}^p(Y)$, with $p \geqslant 1$, such that*

$$\mathrm{codim}(W) = p\cdot \mathrm{lct}(X, Y; \beta\cdot Z) - \beta\cdot \mathrm{ord}_W(Z). \tag{8}$$

Then W is dominated by a multi-contact locus $\mathrm{Cont}^\nu(E)$ where

$$\nu_i \neq 0 \quad \Longrightarrow \quad E_i \text{ computes } \mathrm{lct}(X, Y; \beta\cdot Z).$$

Conversely, the image of any such multi-contact locus determines an irreducible component of $\mathrm{Cont}^p(Y)$ as above.

The proof is similar to that of Corollary 3.3, so we omit it.

Remark 3.7. Recall that we have defined in the previous section a map from the irreducible components of $\mathrm{Cont}^p(Y)$ with $p \geqslant 1$ to the divisors over X. One can interpret the above corollary as saying that, for every Z and β, every divisor D which computes $\mathrm{lct}(X, Y; \beta\cdot Z)$ is in the image of this map. Moreover, the irreducible components which correspond to these divisors are precisely the ones satisfying (8).

Remark 3.8. Suppose that $X = \mathrm{Spec}(A)$ is affine, that Y is defined by the ideal $\mathfrak{a} \subset A$, and that Z is defined by a principal ideal (f). It follows from definition that $\lambda < \mathrm{lct}(X, Y; Z)$ if and only if $f \in \mathcal{I}(X, \lambda\cdot\mathfrak{a})$, the multiplier ideal of $\mathfrak{a}$ with coefficient λ (we refer to [Laz04] for the basics on multiplier ideals). One can therefore interpret Corollary 3.5 as giving an arc-theoretic interpretation of multiplier ideals.

We end with an example: monomial ideals in the polynomial ring. For a non-zero monomial ideal $\mathfrak{a}$ in the polynomial ring $R = \mathbb{C}[T_1, \ldots, T_d]$, the Newton polyhedron of $\mathfrak{a}$, denoted by $P_{\mathfrak{a}}$, is the convex hull in $\mathbb{R}^n$ of the set $\{\mathbf{u} \in \mathbb{N}^d \mid T^{\mathbf{u}} \in \mathfrak{a}\}$. For $\mathbf{u} = (\mathbf{u}_i) \in \mathbb{N}^d$, we use the notation $T^{\mathbf{u}} = \prod_i T_i^{\mathbf{u}_i}$.

Proposition 3.9. *Let $X = \mathbb{A}^d$, and Y, $Z \hookrightarrow X$ subschemes defined by non-zero monomial ideals $\mathfrak{a}$ and $\mathfrak{b} \subseteq R$, respectively, where $R = \mathbb{C}[T_1, \ldots, T_d]$. For every α, $\beta \in \mathbb{Q}_+$, with $\alpha \neq 0$, we have $(X, \alpha \cdot Y - \beta \cdot Z)$ log-canonical if and only if*

$$\beta \cdot P_{\mathfrak{b}} + \mathbf{e} \subseteq \alpha \cdot P_{\mathfrak{a}},$$

where $\mathbf{e} = (1, \ldots, 1)$.

Proof. For every $\mathbf{q} \in \mathbb{N}^d$, consider the multi-contact locus $C_{\mathbf{q}} \subseteq X_\infty$, consisting of those arcs with order $\mathbf{q}_i$ along the divisor defined by T_i, for all i. Every contact locus of Y is a union of such cylinders. It follows from Corollary 3.5 that $(X, \alpha \cdot Y - \beta \cdot Z)$ is log canonical if and only if

$$\beta \cdot \operatorname{ord}_{C_{\mathbf{q}}}(Z) \geqslant \alpha \cdot \operatorname{ord}_{C_{\mathbf{q}}}(Y) - \operatorname{codim}(C_{\mathbf{q}}), \tag{9}$$

for every $\mathbf{q}$.

It is easy to see that $\operatorname{codim}(C_{\mathbf{q}}) = \sum_i \mathbf{q}_i$. Moreover, it is clear that we have $\operatorname{ord}_{C_{\mathbf{q}}}(Y) = \inf\{\sum_i \mathbf{u}_i \mathbf{q}_i \mid \mathbf{u} \in P_{\mathfrak{a}}\}$, and a similar formula for $\operatorname{ord}_{C_{\mathbf{q}}}(Z)$. If we consider the dual polyhedron of $P_{\mathfrak{a}}$,

$$P_{\mathfrak{a}}^\circ := \left\{ \mathbf{p} \in \mathbb{R}^d \;\middle|\; \sum_i \mathbf{p}_i \mathbf{u}_i \geqslant 1 \text{ for all } \mathbf{u} \in P_{\mathfrak{a}} \right\},$$

then we see that $\operatorname{ord}_{C_{\mathbf{q}}}(Y) \geqslant m$ if and only if $(1/m)\mathbf{q} \in P_{\mathfrak{a}}^\circ$.

Condition (9) then says that for every $\mathbf{q} \in \mathbb{N}^d$ and every $m \in \mathbb{N}^*$, such that $(1/m)\mathbf{q} \in P_{\mathfrak{a}}^\circ$, we have

$$\beta/m \cdot \inf_{\mathbf{v} \in P_{\mathfrak{b}}} \left(\sum_i \mathbf{v}_i \mathbf{q}_i \right) + \sum_i \mathbf{q}_i/m \geqslant \alpha.$$

Since $P_{\mathfrak{a}}^\circ$ is a rational polyhedron, this is the same as saying that for every $\mathbf{q}' \in P_{\mathfrak{a}}^\circ$, we have

$$\beta \cdot \inf_{\mathbf{v} \in P_{\mathfrak{b}}} \left(\sum_i \mathbf{q}'_i \mathbf{v}_i \right) + \sum_i \mathbf{q}'_i \geqslant \alpha.$$

Since $(P_{\mathfrak{a}}^\circ)^\circ = P_{\mathfrak{a}}$ and $\alpha > 0$, the above condition is equivalent to $\beta \cdot P_{\mathfrak{b}} + \mathbf{e} \subseteq \alpha \cdot P_{\mathfrak{a}}$. ☐

Corollary 3.10 [How01]. *If $X = \mathbb{A}^d$ and $\mathfrak{a} \subseteq R = \mathbb{C}[T_1, \ldots, T_d]$ is a non-zero monomial ideal, then for every $\alpha \in \mathbb{Q}_+^*$, the multiplier ideal of $\mathfrak{a}$ with coefficient α is given by*

$$\mathcal{I}(X, \alpha \cdot \mathfrak{a}) = (T^{\mathbf{u}} \mid \mathbf{u} + \mathbf{e} \in \operatorname{Int}(\alpha \cdot P_{\mathfrak{a}})).$$

Proof. Note that multiplier ideals of monomial ideals are monomial (for example, because we can find a log resolution which is equivariant with respect to the standard $(\mathbb{C}^*)^d$-action on $\mathbb{A}^d$). If Z is the subscheme defined by $T^{\mathbf{u}}$, note that $T^{\mathbf{u}} \in \mathcal{I}(X, \alpha \cdot \mathfrak{a})$ if and only if $(X, \lambda \cdot Y - Z)$ is log canonical for some $\lambda > \alpha$. The assertion of the corollary follows now from the above proposition. ☐

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References

Bat98 V. V. Batyrev, *Stringy Hodge numbers of varieties with Gorenstein canonical singularities*, in *Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)* (World Scientific, River Edge, NJ, 1998), 1–32.

DK01 J.-P. Demailly and J. Kollár, *Semicontinuity of complex singularity exponents and Kähler–Einstein metrics on Fano orbifolds*, Ann. Sci. École Norm. Sup. **34** (2001), 525–556.

DL98 J. Denef and F. Loeser, *Motivic Igusa zeta function*, J. Algebraic Geom. **7** (1998), 505–537.

DL99 J. Denef and F. Loeser, *Germs of arcs on singular varieties and motivic integration*, Invent. Math. **135** (1999), 201–232.

DL02 J. Denef and F. Loeser, *Lefschetz numbers of iterates of the monodromy and truncated arcs*, Topology **41** (2002), 1031–1040.

ELSV03 L. Ein, R. Lazarsfeld, K. E. Smith and D. Varolin, *Jumping coefficients of multiplier ideals*, Preprint (2003), math.AG/0303002.

How01 J. Howald, *Multiplier ideals of monomial ideals*, Trans. Amer. Math. Soc. **353** (2001), 2665–2671.

IK03 S. Ishii and J. Kollár, *The Nash problem on arc families of singularities*, Duke Math. J. **120** (2003), 601–620.

Kol97 J. Kollár, *Singularities of pairs*, in *Algebraic geometry, Santa Cruz, 1995*, Proc. Sympos. Pure Math., vol. 62 (American Mathematical Society, Providence, RI, 1997), 221–286.

Laz04 R. Lazarsfeld, *Positivity in algebraic geometry*, in preparation (2004).

Mus01 M. Mustaţă, *Jet schemes of locally complete intersection canonical singularities, with an appendix by D. Eisenbud and E. Frenkel*, Invent. Math. **145** (2001), 397–424.

Mus02 M. Mustaţă, *Singularities of pairs via jet schemes*, J. Amer. Math. Soc. **15** (2002), 599–615.

Nas95 J. F. Nash, Jr., *Arc structure of singularities*, in *A celebration of John F. Nash, Jr.*, Duke Math. J. **81** (1995), 31–38.

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