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A Subadditivity Property of Multiplier Ideals

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Dedicated to William Fulton on the occasion of his sixtieth birthday

Introduction

The purpose of this note is to establish a "subadditivity" theorem for multiplier ideals. As an application, we give a new proof of a theorem of Fujita concerning the volume of a big line bundle.

Let X be a smooth complex quasi-projective variety, and let D be an effective $\mathbb{Q}$-divisor on X. One can associate to D its *multiplier ideal* sheaf

$$\mathcal{J}(D) = \mathcal{J}(X, D) \subseteq \mathcal{O}_X,$$

whose zeros are supported on the locus at which the pair (X, D) fails to have log-terminal singularities. It is useful to think of $\mathcal{J}(D)$ as reflecting in a somewhat subtle way the singularities of D: the "worse" the singularities, the smaller the ideal. These ideals and their variants have come to play an increasingly important role in higher-dimensional geometry, largely because of their strong vanishing properties. Among the papers in which they figure prominently, we might mention [2; 4; 8; 13; 14; 19; 30; 33; 34]; see [6] for a survey.

We establish the following "subadditivity" property of these ideals.

Theorem. *Given any two effective $\mathbb{Q}$-divisors D_1 and D_2 on X, one has the relation*

$$\mathcal{J}(D_1 + D_2) \subseteq \mathcal{J}(D_1) \cdot \mathcal{J}(D_2).$$

This theorem admits several variants. In the local setting, one can associate a multiplier ideal $\mathcal{J}(\mathfrak{a})$ to any ideal $\mathfrak{a} \subseteq \mathcal{O}_X$, which in effect measures the singularities of the divisor of a general element of $\mathfrak{a}$. Then the statement becomes

$$\mathcal{J}(\mathfrak{a} \cdot \mathfrak{b}) \subseteq \mathcal{J}(\mathfrak{a}) \cdot \mathcal{J}(\mathfrak{b}).$$

On the other hand, suppose that X is a smooth projective variety and that L is a big line bundle on X. Then one can define an "asymptotic multiplier ideal" $\mathcal{J}(\|L\|) \subseteq \mathcal{O}_X$ that reflects the asymptotic behavior of the base loci of the linear series $|kL|$ for large k. In this setting, the theorem shows that

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$$\mathcal{J}(\|mL\|) \subseteq \mathcal{J}(\|L\|)^m.$$

Finally, there is an analytic analog (which in fact implies the other statements): one can attach a multiplier ideal to any plurisubharmonic function on X and then

$$\mathcal{J}(\phi + \psi) \subseteq \mathcal{J}(\phi) \cdot \mathcal{J}(\psi)$$

for any two such functions ϕ and ψ. The theorem was suggested by a somewhat weaker statement established in [7]. Esnault and Viehweg had proven some related statements by similar arguments (see [16]).

We apply the subadditivity relation to give a new proof of a theorem of Fujita [17]. Consider a smooth projective variety X of dimension n and a big line bundle L on X. The *volume* of L is defined to be the positive real number

$$v(L) = \limsup_{k\to\infty} \frac{n!}{k^n} h^0(X, \mathcal{O}(kL)).$$

If L is ample then $v(L) = \int_X c_1(L)^n$, and in general (as we shall see) it measures asymptotically the top self-intersection of the "moving part" of $|kL|$ (Proposition 3.6). Fujita has established the following.

Theorem [17]. *Given any $\varepsilon > 0$, there exists a birational modification*

$$\mu \colon X' = X'_\varepsilon \longrightarrow X$$

and a decomposition $\mu^* L \equiv E_\varepsilon + A_\varepsilon$, where $E = E_\varepsilon$ is an effective $\mathbb{Q}$-divisor and $A = A_\varepsilon$ an ample $\mathbb{Q}$-divisor, such that $(A^n) > v(L) - \varepsilon$.

This would be clear if L admitted a Zariski decomposition, so one thinks of the statement as a numerical analog of such a decomposition. Fujita's proof of this theorem is quite short but rather tricky. We give a new proof using multiplier ideals, which (to the present authors at least) seems perhaps more transparent. An outline of this approach to Fujita's theorem appears also in [7]. We hope that these ideas may find other applications in the future.

The paper is divided into three sections. In the first, we review (largely without proof) the theory of multiplier ideals from the algebro-geometric point of view, and we discuss the connections between asymptotic algebraic constructions and their analytic counterparts. The subadditivity theorem is established in Section 2, via an elementary argument using a "diagonal" trick as in [8]. The application to Fujita's theorem appears in Section 3, where we deduce as a corollary a geometric description of the volume of a big line bundle.

We thank E. Mouroukos for valuable discussions. We are especially delighted to have the opportunity to dedicate this paper to William Fulton on the occasion of his sixtieth birthday. His many contributions have done much to shape contemporary algebraic geometry. The third author in particular—having been first a student and being now a colleague of Bill's—has learned a great deal from Fulton over the years.

0. Notation and Conventions

(0.1) We work throughout with nonsingular algebraic varieties defined over the complex numbers $\mathbb{C}$.

(0.2) In general, we do not distinguish between line bundles and (linear equivalence classes of) integral divisors. In particular, given a line bundle L, we write $\mathcal{O}_X(L)$ for the corresponding invertible sheaf on X, and we use additive notation for the tensor product of line bundles. When X is a smooth variety, K_X denotes as usual the canonical divisor (class) on X.

(0.3) We write $\equiv$ for linear equivalence of $\mathbb{Q}$-divisors: two such divisors D_1, D_2 are linear equivalent if and only if there is a nonzero integer m such that $mD_1 \equiv mD_2$ in the usual sense.

1. Multiplier Ideals

In this section we review the construction and basic properties of multiplier ideals from an algebro-geometric perspective. For the most part we do not give proofs; most can be found in [10; 11; 16, Chap. 7; 19]; a detailed exposition will appear in the forthcoming book [24]. The algebraic theory closely parallels the analytic one, for which the reader may consult [5]. We also discuss in some detail the relationship between the algebraically defined asymptotic multiplier ideals $\mathcal{J}(\|L\|)$ associated to a complete linear series and their analytic counterparts.

Let X be a smooth complex quasi-projective variety, and let D be an effective $\mathbb{Q}$-divisor on X. Recall that a *log resolution* of (X, D) is a proper birational mapping

$$\mu\colon X' \to X$$

from a smooth variety X' to X having the property that $\mu^*D + \operatorname{Exc}(\mu)$ has simple normal crossing support, where $\operatorname{Exc}(\mu)$ is the sum of the exceptional divisors of μ.

Definition 1.1. The *multiplier ideal* of D is defined to be

$$\mathcal{J}(D) = \mathcal{J}(X, D) = \mu_*\mathcal{O}_{X'}(K_{X'/X} - [\mu^*D]). \tag{1}$$

Here $K_{X'/X}$ denotes the relative canonical divisor $K_{X'} - \mu^*K_X$ and, as usual, $[F]$ is the integer part or round-down of a $\mathbb{Q}$-divisor F. That $\mathcal{J}(D)$ is indeed an ideal sheaf follows from the observation that $\mathcal{J}(D) \subseteq \mu_*\mathcal{O}_{X'}(K_{X'/X}) = \mathcal{O}_X$. An important point is that this definition is independent of the choice of resolution. This can be verified directly, but it also follows from the fact that $\mathcal{J}(D)$ has an analytic interpretation.

Using the same notation as in [7], we take a plurisubharmonic function ϕ and denote by $\mathcal{J}(\phi)$ the sheaf of germs of holomorphic functions f on X such that $\int |f|^2 e^{-2\phi}\, dV$ converges on a neighborhood of the given point. By a well-known result of Nadel [30], $\mathcal{J}(\phi)$ is always a coherent analytic sheaf, whatever the singularities of ϕ might be. In fact, this follows from Hörmander's L^2 estimates [1; 18; 20] for the $\bar\partial$ operator, combined with some elementary arguments of local algebra (Artin–Rees lemma). We need here a slightly more precise statement that can be inferred directly from the proof given in [30] (see also [4]).

Proposition 1.2. *Let ϕ be a plurisubharmonic function on a complex manifold X, and let $U \subseteq X$ be a relatively compact Stein open subset* (*with a basis of Stein* *neighborhoods of $\bar{U}$). Then the restriction $\mathcal{J}(\phi)\big|_U$ is generated as an $\mathcal{O}_U$-module by a Hilbert basis $(f_k)_{k\in\mathbb{N}}$ of the Hilbert space $\mathcal{H}^2(U,\phi,dV)$ of holomorphic functions f on U such that*

$$\int_U |f|^2 e^{-2\phi}\,dV < +\infty$$

(*with respect to any Kähler volume form dV on a neighborhood of $\bar{U}$*).

Returning to the case of an effective $\mathbb{Q}$-divisor $D = \sum a_i D_i$, let g_i be a local defining equation for D_i. Then, if ϕ denotes the plurisubharmonic function $\phi = \sum a_i \log|g_i|$, one has

$$\mathcal{J}(D) = \mathcal{J}(\phi),$$

and in particular $\mathcal{J}(D)$ is intrinsically defined. The stated equality is established in [5, (5.9)]; the essential point is that the algebro-geometric multiplier ideals satisfy the same transformation rule under birational modifications as do their analytic counterparts, so that one is reduced to the case where D has normal crossing support.

We mention two variants. First, suppose we are given an ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$. By a log resolution of $\mathfrak{a}$ we understand a mapping $\mu\colon X' \to X$ as before with the property that $\mathfrak{a}\cdot\mathcal{O}_{X'} = \mathcal{O}_{X'}(-E)$, where $E + \operatorname{Exc}(\mu)$ has simple normal crossing support. Given a rational number $c > 0$, we take such a resolution and then define

$$\mathcal{J}(c\cdot\mathfrak{a}) = \mu_*\mathcal{O}_{X'}(K_{X'/X} - [cE]);$$

again, this is independent of the choice of resolution. (More generally, given ideals $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_X$ and rational numbers $c, d > 0$, one can define $\mathcal{J}((c\cdot\mathfrak{a})\cdot(d\cdot\mathfrak{b}))$ by taking a common log resolution $\mu\colon X' \to X$ of $\mathfrak{a}$ and $\mathfrak{b}$, with $\mu^{-1}\mathfrak{a} = \mathcal{O}_{X'}(-E_1)$ and $\mu^{-1}\mathfrak{b} = \mathcal{O}_{X'}(-E_2)$, and setting $\mathcal{J}((c\cdot\mathfrak{a})\cdot(d\cdot\mathfrak{b})) = \mu_*\mathcal{O}_{X'}(K_{X'/X} - [cE_1 + dE_2])$.) If $m \in \mathbb{Z}$ is a positive integer then $\mathcal{J}(m\cdot\mathfrak{a}) = \mathcal{J}(\mathfrak{a}^m)$, and in this case these multiplier ideals were defined and studied in a more general setting by Lipman [26] (who called them "adjoint ideals"). They admit the following geometric interpretation. Working locally, assume that X is affine, view $\mathfrak{a}$ as an ideal in its coordinate ring, and take $k > c$ general $\mathbb{C}$-linear combinations of a set of generators $g_1, \dots, g_p \in \mathfrak{a}$, yielding divisors $A_1, \dots, A_k \subset X$. If $D = \frac{c}{k}(A_1 + \dots + A_k)$, then

$$\mathcal{J}(c\cdot\mathfrak{a}) = \mathcal{J}(D). \tag{2}$$

In the analytic setting, where X is an open subset of $\mathbb{C}^n$, one has $\mathcal{J}(c\cdot\mathfrak{a}) = \mathcal{J}(c\cdot\phi)$, where $\phi = \log(|g_1| + \dots + |g_p|)$.

The second variant involves linear series. Suppose that L is a line bundle on X and that $V \subset H^0(X, L)$ is a finite-dimensional vector space of sections of L, giving rise to a linear series $|V|$ of divisors on X. We now require of our log resolution $\mu\colon X' \to X$ that

$$\mu^*|V| = |W| + E,$$

where $|W|$ is a free linear series on X' and where $E + \operatorname{Exc}(\mu)$ has simple normal crossing support. In other words, we ask that the fixed locus of $\mu^*|V|$ be a divisor E with simple normal crossing support (which, in addition, meets $\mathrm{Exc}(\mu)$ nicely). Given such a log resolution and a rational number $c > 0$, we define

$$\mathcal{J}(c \cdot |V|) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - [cE]),$$

where this is once again independent of the choice of μ. If $\mathfrak{b} = \mathfrak{b}(|V|) \subseteq \mathcal{O}_X$ is the base ideal of $|V|$ then evidently $\mathcal{J}(c \cdot |V|) = \mathcal{J}(c \cdot \mathfrak{b})$, and in particular the analog of equation (2) holds for these ideals.

We now outline the main properties of these ideals that we shall require. The first is a local statement comparing a multiplier ideal with its restriction to a hyperplane. Specifically, consider an effective $\mathbb{Q}$-divisor D on a quasi-projective complex manifold X, and a smooth effective divisor $H \subset X$ that does not appear in the support of D. Then one can form two ideals on H. In the first place, the restriction $D\big|_H$ is an effective $\mathbb{Q}$-divisor on H and so one can form its multiplier ideal $\mathcal{J}\big(H, D\big|_H\big) \subseteq \mathcal{O}_H$. On the other hand, one can take the multiplier ideal $\mathcal{J}(X, D)$ of D on X and restrict it to H to obtain an ideal

$$\mathcal{J}(X, D) \cdot \mathcal{O}_H \subseteq \mathcal{O}_H.$$

A very basic fact—due in the algebro-geometric setting to Esnault–Viehweg [16]—is that one can compare these sheaves as follows.

Restriction Theorem. *In the setting just described, there is an inclusion*

$$\mathcal{J}\big(H, D\big|_H\big) \subseteq \mathcal{J}(X, D) \cdot \mathcal{O}_H.$$

One may think of this as asserting that "multiplier ideals can only get worse" upon restricting a divisor to a hyperplane. For the proof, see [16, (7.5)] or [10, (2.1)]. The essential point is that the line bundle $\mathcal{O}_{X'}(K_{X'/X} - [\mu^* D])$ appearing in equation (1) has vanishing higher direct images under μ. The same result holds true in the analytic case; namely,

$$\mathcal{J}\big(H, \phi\big|_H\big) \subseteq \mathcal{J}(X, \phi) \cdot \mathcal{O}_H$$

for every plurisubharmonic function ϕ on X (if $\phi\big|_H$ happens to be identically equal to $-\infty$ on some component of H, one agrees that $\mathcal{J}\big(H, \phi\big|_H\big)$ is identically zero on that component). In that case, the proof is completely different; it is, in fact, a direct qualitative consequence of the (deep) Ohsawa–Takegoshi L^2 extension theorem [31; 32].

As a immediate consequence, one obtains an analogous statement for restrictions to submanifolds of higher codimension.

Corollary 1.3. *Let $Y \subset X$ be a smooth subvariety that is not contained in the support of D. Then*

$$\mathcal{J}\big(Y, D\big|_Y\big) \subseteq \mathcal{J}(X, D) \cdot \mathcal{O}_Y,$$

where $D\big|_Y$ denotes the restriction of D to Y.

Of course, the analogous statement is still true in the analytic case, as well as for the multiplier ideals associated to linear series or ideal sheaves.

The most important global property of multiplier ideals is the following.

Nadel Vanishing Theorem. *Let X be a smooth complex projective variety, D an effective $\mathbb{Q}$-divisor, and L a line bundle on X. Assume that $L - D$ is big and nef. Then*

$$H^i\big(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(D)\big) = 0 \quad \textit{for } i > 0.$$

This follows quickly from the Kawamata–Viehweg vanishing theorem applied on a log resolution $\mu\colon X' \to X$ of (X, D). Similarly, if $V \subset H^0(X, B)$ is a linear series on X, with B a line bundle such that $L - c \cdot B$ is big and nef, then

$$H^i\big(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(c \cdot |V|)\big) = 0 \quad \text{for } i > 0.$$

Under the same hypotheses, if $\mathfrak{a} \subseteq \mathcal{O}_X$ is an ideal sheaf such that $B \otimes \mathfrak{a}$ is globally generated, then $H^i\big(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(c \cdot \mathfrak{a})\big) = 0$ when $i > 0$.

Nadel vanishing yields a simple criterion for a multiplier ideal sheaf to be globally generated. The essential point is the following elementary lemma of Mumford, which forms the basis of the theory of Castelnuovo–Mumford regularity. (We beg the reader's indulgence for the fact that we prefer to state the lemma using multiplicative notation for tensor products of line bundles, rather than working additively as we do elsewhere in the paper.)

Lemma 1.4 [29, Lecture 14]. *Let X be a projective variety, B a very ample line bundle on X, and $\mathcal{F}$ any coherent sheaf on X satisfying the vanishing*

$$H^i(X, \mathcal{F} \otimes B^{\otimes(k-i)}) = 0 \quad \textit{for } i > 0 \textit{ and } k \geq 0.$$

Then $\mathcal{F}$ is globally generated.

Although this lemma is quite standard, it seems not to be as well known as one might expect in connection with vanishing theorems (Remark 1.6). Thus we feel it is worthwhile to write out the argument.

Proof. Evaluation of sections determines a surjective map $e\colon H^0(B) \otimes_{\mathbb{C}} \mathcal{O}_X \to B$ of vector bundles on X. The corresponding Koszul complex takes the form

$$\cdots \longrightarrow \Lambda^3 H^0(B) \otimes B^{\otimes -2} \longrightarrow \Lambda^2 H^0(B) \otimes B^{\otimes -1} \longrightarrow H^0(B) \otimes \mathcal{O}_X \longrightarrow B \longrightarrow 0. \qquad (*)$$

Tensoring through by $\mathcal{F}$ and then applying the hypothesis with $k = 0$ as one chases through the resulting complex, one sees first of all that the multiplication map

$$H^0(B) \otimes H^0(\mathcal{F}) \longrightarrow H^0(\mathcal{F} \otimes B)$$

is surjective. Next tensor $(*)$ by $\mathcal{F} \otimes B$ and apply the vanishing hypothesis with $k = 1$; it follows that $H^0(B) \otimes H^0(\mathcal{F} \otimes B)$ maps onto $H^0(\mathcal{F} \otimes B^{\otimes 2})$ and hence that $H^0(B^{\otimes 2}) \otimes H^0(\mathcal{F}) \longrightarrow H^0(\mathcal{F} \otimes B^{\otimes 2})$ is also onto. Continuing, one finds that

$$H^0(X, \mathcal{F}) \otimes H^0(X, B^{\otimes m}) \longrightarrow H^0(X, \mathcal{F} \otimes B^{\otimes m}) \qquad (**)$$

is surjective for all $m \geq 0$. But since B is very ample, $\mathcal{F} \otimes B^{\otimes m}$ is globally generated for $m \gg 0$. It then follows from the surjectivity of $(**)$ that $\mathcal{F}$ itself must already be generated by its global sections. (A similar argument shows that the case $k = 0$ of the vanishing hypothesis actually implies the cases $k \geq 1$, but for present purposes we don't need this.) $\square$

COROLLARY 1.5. *In the setting of the Nadel vanishing theorem, let B be a very ample line bundle on X. Then*

$$\mathcal{O}_X(K_X + L + mB) \otimes \mathcal{J}(D)$$

is globally generated for all $m \geq \dim X$.

Proof. In fact, thanks to Nadel vanishing, the hypothesis of Mumford's lemma applies to $\mathcal{F} = \mathcal{O}_X(K_X + L + mB) \otimes \mathcal{J}(D)$ as soon as $m \geq \dim X$. $\square$

REMARK 1.6. This corollary was used by Siu in the course of his spectacular proof of the deformation invariance of plurigenera [34], where the statement was established by analytic methods. Analogous applications of Lemma 1.4 in the context of vanishing theorems have appeared implicitly or explicitly in a number of papers over the years (e.g. [12; 16; 21; 37], to name a few).

We next turn to the construction of the asymptotic multiplier ideal associated to a big linear series. In the algebro-geometric setting, the theory is due to the second author [9] and Kawamata [19]. Suppose that X is a smooth complex projective variety and that L is a big line bundle on X. Then $H^0(X, \mathcal{O}_X(kL)) \neq 0$ for $k \gg 0$ and therefore, given any rational $c > 0$, the multiplier ideal $\mathcal{J}\big(\frac{c}{k}|kL|\big)$ is defined for large k. One checks easily that

$$\mathcal{J}\big(\tfrac{c}{k} \cdot |kL|\big) \subseteq \mathcal{J}\big(\tfrac{c}{pk} \cdot |pkL|\big) \tag{$\star$}$$

for every integer $p > 0$. We assert that then the family of ideals $\big\{\mathcal{J}\big(\frac{c}{k} \cdot |kL|\big)\big\}$ ($k \gg 0$) has a *unique* maximal element. In fact, the existence of at least one maximal member follows from the ascending chain condition on ideals. On the other hand, if $\mathcal{J}\big(\frac{c}{k} \cdot |kL|\big)$ and $\mathcal{J}\big(\frac{c}{\ell} \cdot |\ell L|\big)$ are each maximal, then it follows by ($\star$) that they must both coincide with $\mathcal{J}\big(\frac{c}{k\ell} \cdot |(k\ell)L|\big)$.

DEFINITION 1.7. The *asymptotic multiplier ideal sheaf* associated to c and $|L|$,

$$\mathcal{J}(c \cdot \|L\|) = \mathcal{J}(X, c \cdot \|L\|),$$

is defined to be the unique maximal member of the family of ideals $\big\{\mathcal{J}\big(\frac{c}{k} \cdot |kL|\big)\big\}$ (k large).

One can show that there exists a positive integer k_0 such that $\mathcal{J}(c \cdot \|L\|) = \mathcal{J}\big(\frac{c}{k} \cdot |kL|\big)$ for every $k \geq k_0$. It follows easily from the definition that $\mathcal{J}(m \cdot \|L\|) = \mathcal{J}(\|mL\|)$ for every positive integer $m > 0$. (In fact, fix $m > 0$; then, for $p \gg 0$ we have $\mathcal{J}(\|mL\|) = \mathcal{J}\big(\frac{1}{p} \cdot |mpL|\big) = \mathcal{J}\big(\frac{m}{mp} \cdot |mpL|\big) = \mathcal{J}(m \cdot \|L\|)$.)

The basic facts about these asymptotic multiplier ideals are summarized in the following theorem.

THEOREM 1.8. *Let X be a nonsingular complex projective variety of dimension n, and let L be a big line bundle on X.*

(i) *The natural inclusion*

$$H^0\big(X, \mathcal{O}_X(L) \otimes \mathcal{J}(\|L\|)\big) \longrightarrow H^0\big(X, \mathcal{O}_X(L)\big)$$

is an isomorphism; that is, $\mathcal{J}(\|L\|)$ contains the base ideal $\mathfrak{b}(|L|) \subset \mathcal{O}_X$ of the linear series $|L|$.

(ii) *For any nef and big divisor P, one has the vanishing*

$$H^i\big(X, \mathcal{O}_X(K_X + L + P) \otimes \mathcal{J}(\|L\|)\big) = 0 \quad \textit{for } i > 0.$$

(iii) *If B is very ample, then $\mathcal{O}_X(K_X + L + (n+1)B) \otimes \mathcal{J}(\|L\|)$ is generated by its global sections.*

Of course, the analogous statements hold with L replaced by mL.

Proof. The first statement follows easily from the definition. For (ii) and (iii), note that $\mathcal{J}(\|L\|) = \mathcal{J}(D)$ for a suitable $\mathbb{Q}$-divisor D numerically equivalent to L. This being said, (ii) is a consequence of the Nadel vanishing theorem while (iii) follows from Corollary 1.5. □

REMARK 1.9. The definition of the asymptotic multiplier ideal $\mathcal{J}(\|L\|)$ requires only that $\kappa(X, L) \geq 0$, where $\kappa(X, L)$ is the Kodaira–Iitaka dimension of L; Theorem 1.8 remains true in this setting. When L is big—as we assumed for simplicity—the proof of Nadel vanishing shows that it suffices in statement (ii) that P be nef, and hence in (iii) one can replace the factor $(n + 1)$ by n. However we do not need these improvements here.

Finally we discuss the relation between these asymptotic multiplier ideals and their analytic counterparts. In the analytic setting, there is a concept of singular hermitian metric $h_{\min}$ with minimal singularities (see e.g. [6]), defined whenever the first Chern class $c_1(L)$ lies in the closure of the cone of effective divisors ("pseudoeffective cone"); it is therefore not even necessary that $\kappa(X, L) \geq 0$ for $h_{\min}$ to be defined, but only that L be pseudoeffective. The metric $h_{\min}$ is defined by taking any smooth hermitian metric h_∞ on L and putting $h_{\min} = h_\infty e^{-\psi_{\max}}$, where

$$\psi_{\max}(x) = \sup\{\psi(x);\ \psi \text{ usc},\ \psi \leq 0,\ i(\partial\bar\partial \log h_\infty + \psi) \geq 0\}.$$

For arbitrary sections $\sigma_1, \dots, \sigma_N \in H^0(X, kL)$, we can take

$$\psi(x) = \frac{1}{k} \log \sum_j \|\sigma_j(x)\|^2_{h_\infty} - C$$

as an admissible ψ function. We infer from this that the associated multiplier ideal sheaf $\mathcal{J}(h_{\min})$ satisfies the inclusion

$$\mathcal{J}(\|L\|) \subseteq \mathcal{J}(h_{\min}) \tag{3}$$

when $\kappa(X, L) \geq 0$. The inclusion is strict in general. In fact, let us take E to be a unitary flat vector bundle on a smooth variety C such that no nontrivial symmetric power of E or $E^\star$ has sections (such vector bundles exist already when C is a curve of genus ≥ 1), and set $U = \mathcal{O}_C \oplus E$. We take as our example $X = \mathbb{P}(U)$ and $L = \mathcal{O}_{\mathbb{P}(U)}(1)$. Then, for every $m \geq 1$, $\mathcal{O}_X(mL)$ has a unique nontrivial section that vanishes to order m along the "divisor at infinity" $H \subset \mathbb{P}(U) = X$, and hence $\mathcal{J}(\|L\|) = \mathcal{O}_X(-H)$. However, L has a smooth semipositive metric induced by the flat metric of E, so that $\mathcal{J}(h_{\min}) = \mathcal{O}_X$. It is somewhat strange (but very interesting) that the analytic setting yields "virtual sections" that do not have algebraic counterparts.

Note that, in the example just presented, the line bundle L has Iitaka dimenson 0. We conjecture that if L is big then equality should hold in (3). We will prove here a slightly weaker statement by means of an analytic analog of Theorem 1.8. If ϕ is a plurisubharmonic function then the ideal sheaves $\mathcal{J}((1+\varepsilon)\phi)$ increase as ε decreases to 0; hence there must be a maximal element, which we denote by $\mathcal{J}_+(\phi)$. This ideal always satisfies $\mathcal{J}_+(\phi) \subseteq \mathcal{J}(\phi)$. When ϕ has algebraic singularities, standard semicontinuity arguments show that $\mathcal{J}_+(\phi) = \mathcal{J}(\phi)$, but we do not know if equality always holds in the analytic case.

THEOREM 1.10. *Let X be a nonsingular complex projective variety of dimension n, and let L be a pseudoeffective line bundle on X. (Recall that the pseudoeffective cone is the closure of the cone of effective divisors on X.) Fix a singular hermitian metric h on L with nonnegative curvature current.*

(i) *For any big and nef divisor P, one has the vanishing*

$$H^i\big(X, \mathcal{O}_X(K_X + L + P) \otimes \mathcal{J}_+(h)\big) = 0 \quad \text{for } i > 0.$$

(ii) *If B is very ample, then the sheaves $\mathcal{O}_X(K_X + L + (n+1)B) \otimes \mathcal{J}(h)$ and $\mathcal{O}_X(K_X + L + (n+1)B) \otimes \mathcal{J}_+(h)$ are generated by their global sections.*

Proof. (i) is a slight variation of Nadel's vanishing theorem in its analytic form. If P is ample, the result is true with $\mathcal{J}(h)$ as well as with $\mathcal{J}_+(h)$ (the latter case being obtained by replacing h with $h^{1+\varepsilon} \otimes h_\infty^{-\varepsilon}$, where h_∞ is an arbitrary smooth metric on L; the defect of positivity of h_∞ can be compensated by the strict positivity of P). If P is big and nef, we can write $P = A + E$ with an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor E, and E can be taken arbitrarily small. We then have vanishing with $\mathcal{J}_+(h \otimes h_E)$, where h_E is the singular metric of curvature current $[E]$ on E. However, if E is so small that $\mathcal{J}(h_E^N) = \mathcal{O}_X$ $(N \gg 1)$, we do have $\mathcal{J}_+(h \otimes h_E) = \mathcal{J}_+(h)$, as follows from an elementary argument using Hölder's inequality.

Statement (ii) follows from (i), Nadel vanishing, and Lemma 1.4. Alternatively, one can argue via a straightforward adaptation of the proof given in [34], based on Skoda's L^2 estimates for ideals of holomorphic functions [35]. $\square$

THEOREM 1.11. *Let X be a projective nonsingular algebraic variety, L a big nef line bundle on X, and $h_{\min}$ its singular hermitian metric with minimal singularity. Then*

$$\mathcal{J}_+(h_{\min}) \subseteq \mathcal{J}(\|L\|) \subseteq \mathcal{J}(h_{\min}).$$

Proof. The strong version of the Ohsawa–Takegoshi L^2 extension theorem proved by Manivel [27] shows that, for every singular hermitian line bundle (L,h) with nonnegative curvature and for every smooth complete intersection subvariety $Y \subseteq X$ (actually, the hypothesis that Y is a complete intersection could probably be removed), there exists a sufficiently ample line bundle B and a surjective restriction morphism

$$H^0\big(X, \mathcal{O}_X(L+B)\otimes \mathcal{J}(h)\big) \longrightarrow H^0\big(Y, \mathcal{O}_Y(L+B)\otimes \mathcal{J}\big(h|_Y\big)\big)$$

with the following additional property: For every section on Y, there exists an extension satisfying an L^2 estimate with a constant depending only on Y (hence, independent of L). We take Y equal to a smooth 0-dimensional scheme obtained as a complete intersection of hyperplane sections of a very ample linear system $|A|$, observing that B depends only on A in that case (and hence can be taken independent of the choice of the particular 0-dimensional scheme). Fix an integer k_0 so large that $E := k_0L - B$ is effective. We apply the extension theorem to the line bundle $L' = (k-k_0)L + E$ equipped with the hermitian metric $h_{\min}^{k-k_0} \otimes h_E$ with $\operatorname{curv}(h_E) = [E]$ (and a smooth metric h_B of positive curvature on B). Then, for $k \geq k_0$ and a prescribed point $x \in X$, we select a 0-dimensional subscheme Y containing x and in this way obtain a global section σ_x of $H^0(X, kL) = H^0(X, L' + E + B)$ such that

$$\int_X \|\sigma_x(z)\|^2_{h_{\min}^{k-k_0}\otimes h_E \otimes h_B} \leq C \quad \text{while} \quad \|\sigma_x(x)\|_{h_{\min}^{k-k_0}\otimes h_E\otimes h_B} = 1.$$

From this we infer that locally $h_{\min} = e^{-2\phi}$ with $|\sigma_x(x)|^2 e^{-2(k-k_0)\phi(x) + 2\phi_E + O(1)} = 1$, hence

$$\phi(x) + \frac{1}{k-k_0}\phi_E \leq \frac{1}{k-k_0}\log|\sigma_x(x)| + C \leq \frac{1}{k-k_0}\log\sum_j |g_j(x)| + C,$$

where (g_j) is an orthonormal basis of sections of $H^0(X, kL)$. This implies that $\mathcal{J}(\|h\|)$ contains the ideal $\mathcal{J}(h_{\min}\otimes h_E^{1/(k-k_0)})$. Again, Hölder's inequality shows that this ideal contains $\mathcal{J}_+(h_{\min})$ for sufficiently large k. □

2. Subadditivity

The present section is devoted to the subadditivity theorem stated in the Introduction as well as some variants.

Let X_1 and X_2 be smooth complex quasi-projective varieties, and let D_1, D_2 be effective $\mathbb{Q}$-divisors on X_1, X_2, respectively. Fix a log resolution $\mu_i \colon X_i' \to X_i$ of the pair (X_i, D_i), $i = 1, 2$. We consider the product diagram

$$\begin{tikzcd} X_1' \arrow[d, "\mu_1"'] & X_1' \times X_2' \arrow[l, "q_1"'] \arrow[r, "q_2"] \arrow[d, "\mu_1\times\mu_2"] & X_2' \arrow[d, "\mu_2"] \\ X_1 & X_1 \times X_2 \arrow[l, "p_1"] \arrow[r, "p_2"'] & X_2, \end{tikzcd}$$

where the horizontal maps are projections.

LEMMA 2.1. *The product $\mu_1 \times \mu_2 \colon X_1' \times X_2' \to X_1 \times X_2$ is a log resolution of the pair*

$$(X_1 \times X_2,\ p_1^* D_1 + p_2^* D_2).$$

Proof. Since the exceptional set $\operatorname{Exc}(\mu_1 \times \mu_2)$ is the divisor where the derivative $d(\mu_1 \times \mu_2)$ drops rank, one sees that $\operatorname{Exc}(\mu_1 \times \mu_2) = q_1^* \operatorname{Exc}(\mu_1) + q_2^* \operatorname{Exc}(\mu_2)$. Similarly,

$$(\mu_1 \times \mu_2)^*(p_1^* D_1 + p_2^* D_2) = q_1^* \mu_1^* D_1 + q_2^* \mu_2^* D_2.$$

Therefore,

$$\begin{aligned}&\operatorname{Exc}(\mu_1 \times \mu_2) + (\mu_1 \times \mu_2)^*(p_1^* D_1 + p_2^* D_2)\\ &\qquad = q_1^*(\operatorname{Exc}(\mu_1) + \mu_1^* D_1) + q_2^*(\operatorname{Exc}(\mu_2) + \mu_2^* D_2);\end{aligned}$$

this has normal crossing support because $\operatorname{Exc}(\mu_1) + \mu_1^* D_1$ and $\operatorname{Exc}(\mu_2) + \mu_2^* D_2$ do. □

PROPOSITION 2.2. *One has*

$$\mathcal{J}(X_1 \times X_2,\ p_1^* D_1 + p_2^* D_2) = p_1^{-1} \mathcal{J}(X_1, D_1) \cdot p_2^{-1} \mathcal{J}(X_2, D_2).$$

Proof. To lighten notation we will write $D_1 \boxplus D_2$ for the exterior direct sum $p_1^* D_1 + p_2^* D_2$, so that the formula to be established is

$$\mathcal{J}(X_1 \times X_2,\ D_1 \boxplus D_2) = p_1^{-1} \mathcal{J}(X_1, D_1) \cdot p_2^{-1} \mathcal{J}(X_2, D_2).$$

The plan is to compute the multiplier ideal on the left using the log resolution $\mu_1 \times \mu_2$. Specifically,

$$\begin{aligned}&\mathcal{J}(X_1 \times X_2,\ D_1 \boxplus D_2)\\ &\qquad = (\mu_1 \times \mu_2)_* \mathcal{O}_{X_1' \times X_2'}\big(K_{X_1' \times X_2' / X_1 \times X_2} - [(\mu_1 \times \mu_2)^*(D_1 \boxplus D_2)]\big).\end{aligned}$$

To begin, observe that

$$[(\mu_1 \times \mu_2)^*(D_1 \boxplus D_2)] = [q_1^* \mu_1^* D_1] + [q_2^* \mu_2^* D_2]$$

because $q_1^* \mu_1^* D_1$ and $q_2^* \mu_2^* D_2$ have no common components. Furthermore, since q_1 and q_2 are smooth,

$$[q_1^* \mu_1^* D_1] = q_1^* [\mu_1^* D_1] \quad \text{and} \quad [q_2^* \mu_2^* D_2] = q_2^* [\mu_2^* D_2].$$

Since $K_{X_1' \times X_2' / X_1 \times X_2} = q_1^*(K_{X_1'/X_1}) + q_2^*(K_{X_2'/X_2})$, it then follows that

$$\begin{aligned}&\mathcal{O}_{X_1' \times X_2'}\big(K_{X_1' \times X_2' / X_1 \times X_2} - [(\mu_1 \times \mu_2)^*(p_1^* D_1 + p_2^* D_2)]\big)\\ &\qquad = q_1^* \mathcal{O}_{X_1'}(K_{X_1'/X_1} - [\mu_1^* D_1]) \otimes q_2^* \mathcal{O}_{X_2'}(K_{X_2'/X_2} - [\mu_2^* D_2]).\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathcal{J}(X_1 \times X_2,\ D_1 \boxplus D_2)\\ &\quad = (\mu_1 \times \mu_2)_* \big(q_1^* \mathcal{O}_{X_1'}(K_{X_1'/X_1} - [\mu_1^* D_1]) \otimes q_2^* \mathcal{O}_{X_2'}(K_{X_2'/X_2} - [\mu_2^* D_2])\big)\\ &\quad = p_1^* \mu_{1*} \mathcal{O}_{X_1'}(K_{X_1'/X_1} - [\mu_1^* D_1]) \otimes p_2^* \mu_{2*} \mathcal{O}_{X_2'}(K_{X_2'/X_2} - [\mu_2^* D_2])\\ &\quad = p_1^* \mathcal{J}(X, D_1) \otimes p_2^* \mathcal{J}(X, D_2),\end{aligned}$$

thanks to the Künneth formula. But

$$p_1^*\mathcal{J}(X_1, D_1) = p_1^{-1}\mathcal{J}(X_1, D_1) \quad \text{and} \quad p_2^*\mathcal{J}(X_2, D_2) = p_2^{-1}\mathcal{J}(X_2, D_2),$$

since p_1 and p_2 are flat. Finally,

$$p_1^{-1}\mathcal{J}(X_1, D_1) \otimes p_2^{-1}\mathcal{J}(X_2, D_2) = p_1^{-1}\mathcal{J}(X_1, D_1) \cdot p_2^{-1}\mathcal{J}(X_2, D_2)$$

by virtue of the fact that $p_1^{-1}\mathcal{J}(X_1, D_1)$ is flat for p_2 (cf. [28]). This completes the proof of Proposition 2.2. □

The subadditivity property of multiplier ideals now follows immediately.

THEOREM 2.3. *Let X be a smooth complex quasi-projective variety, and let D_1 and D_2 be effective $\mathbb{Q}$-divisors on X. Then*

$$\mathcal{J}(X, D_1 + D_2) \subseteq \mathcal{J}(X, D_1) \cdot \mathcal{J}(X, D_2).$$

Proof. We apply Corollary 1.3 to the diagonal $\Delta = X \subset X \times X$. Keeping the notation of the previous proof (with $X_1 = X_2 = X$ and $\mu_1 = \mu_2 = \mu$), one has

$$\begin{aligned}\mathcal{J}(X, D_1 + D_2) &= \mathcal{J}\big(\Delta, (p_1^*D_1 + p_2^*D_2)\big|_\Delta\big) \\ &\subseteq \mathcal{J}(X \times X,\ p_1^*D_1 + p_2^*D_2) \cdot \mathcal{O}_\Delta.\end{aligned}$$

But it follows from Proposition 2.2 that

$$\mathcal{J}(X \times X,\ p_1^*D_1 + p_2^*D_2) \cdot \mathcal{O}_\Delta = \mathcal{J}(X, D_1) \cdot \mathcal{J}(X, D_2),$$

as required. □

VARIANT 2.4. *Let L be a big line bundle on a nonsingular complex projective variety X. Then, for all $m \geq 0$,*

$$\mathcal{J}(X, \|mL\|) \subseteq \mathcal{J}(X, \|L\|)^m.$$

Proof. Given m, fix $p \gg 0$ and a general divisor $D \in |mpL|$. Then

$$\mathcal{J}(\|L\|) = \mathcal{J}\big(\tfrac{1}{pm}D\big) \quad \text{and} \quad \mathcal{J}(\|mL\|) = \mathcal{J}\big(\tfrac{1}{p}D\big),$$

so the assertion follows from Theorem 2.3. □

VARIANT 2.5. *Let $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_X$ be ideals, and fix rational numbers $c, d > 0$. Then*

$$\mathcal{J}((c \cdot \mathfrak{a}) \cdot (d \cdot \mathfrak{b})) \subseteq \mathcal{J}(c \cdot \mathfrak{a}) \cdot \mathcal{J}(d \cdot \mathfrak{b}).$$

Proof. This does not follow directly from the statement of Theorem 2.3 because the divisor of a general element of $\mathfrak{a} \cdot \mathfrak{b}$ is not the sum of divisors of elements in $\mathfrak{a}$ and $\mathfrak{b}$. However, the proof Proposition 2.2 goes through to show that

$$\mathcal{J}(X \times X,\ (c \cdot p_1^{-1}\mathfrak{a}) \cdot (d \cdot p_2^{-1}\mathfrak{b})) = p_1^{-1}\mathcal{J}(X, c \cdot \mathfrak{a}) \cdot p_2^{-1}\mathcal{J}(X, d \cdot \mathfrak{b}),$$

and then, as before, one restricts to the diagonal. □

The corresponding properties of analytic multiplier ideals are proven in an analogous manner. The result is the following theorem.

THEOREM 2.6 (Analogous analytic statements).

(i) *Let X_1, X_2 be complex manifolds and let ϕ_i be a plurisubharmonic function on X_i. Then*

$$\mathcal{J}(\phi_1 \circ p_1 + \phi_2 \circ p_2) = p_1^{-1}\mathcal{J}(\phi_1) \cdot p_2^{-1}\mathcal{J}(\phi_2).$$

(ii) *Let X be a complex manifold and let ϕ, ψ be plurisubharmonic functions on X. Then*

$$\mathcal{J}(\phi + \psi) \subseteq \mathcal{J}(\phi) \cdot \mathcal{J}(\psi).$$

Proof. Only (i) requires a proof, since (ii) follows again from (i) by the restriction principle and the diagonal trick. Let us fix two relatively compact Stein open subsets, $U_1 \subset X_1$ and $U_2 \subset X_2$. Then $\mathcal{H}^2(U_1 \times U_2, \phi_1 \circ p_1 + \phi_2 \circ p_2, p_1^\star dV_1 \otimes p_2^\star dV_2)$ is the Hilbert tensor product of $\mathcal{H}^2(U_1, \phi_1, dV_1)$ and $\mathcal{H}^2(U_2, \phi_2, dV_2)$, and it admits $(f'_k \boxtimes f''_l)$ as a Hilbert basis, where (f'_k) and (f''_l) are respective Hilbert bases. Since $\mathcal{J}(\phi_1 \circ p_1 + \phi_2 \circ p_2)\big|_{U_1 \times U_2}$ is generated as an $\mathcal{O}_{U_1 \times U_2}$ module by the $(f'_k \boxtimes f''_l)$, we conclude that (i) holds true. □

3. Fujita's Theorem

Now let X be a smooth irreducible complex projective variety of dimension n, and let L be a line bundle on X. We recall the following.

DEFINITION 3.1. The *volume* of L is the real number

$$v(L) = v(X, L) = \limsup_{k \to \infty} \frac{n!}{k^n} h^0(X, \mathcal{O}(kL)).$$

(This was called the *degree* of the graded linear series $\bigoplus H^0(X, \mathcal{O}_X(kL))$ in [15], but the present terminology is more natural and seems to be becoming standard.)

Thus, L is big if and only if $v(L) > 0$. If L is ample, or merely nef and big, then asymptotic Riemann–Roch shows that

$$h^0(X, \mathcal{O}_X(kL)) = \frac{k^n}{n!}(L^n) + o(k^n),$$

so that in this case $v(L) = (L^n)$ is the top self-intersection number of L. If D is a $\mathbb{Q}$-divisor on X then the volume $v(D)$ is defined analogously, with the limit taken over k so that kD is an integral divisor.

Fujita's theorem asserts that "most of" the volume of L can be accounted for by the volume an ample $\mathbb{Q}$-divisor on a modification.

THEOREM 3.2 [17]. *Let L be a big line bundle on X, and fix $\varepsilon > 0$. Then there exists a birational modification*

$$\mu \colon X' \to X$$

(depending on ε) and a decomposition $\mu^*L \equiv E + A$ (also depending on ε), with E an effective $\mathbb{Q}$-divisor and A an ample $\mathbb{Q}$-divisor on S', such that

$$v(X', A) = (A^n) \geq v(X, L) - \varepsilon.$$

Conversely, given a decomposition $\mu^*L \equiv E + A$ as in Theorem 3.2, one evidently has $v(X', A) = (A^n) \leq v(X, L)$. So the essential content of Fujita's theorem is that the volume of any big line bundle can be approximated arbitrarily closely by the volume of an ample $\mathbb{Q}$-divisor (on a modification). This statement initially arose in connection with alegbro-geometric analogs of the work [4] of the first author (cf. [15; 23, Sec. 7]). A geometric reinterpretation appears in Proposition 3.6.

Remark 3.3. Suppose that L admits a *Zariski decomposition*. That is, assume there exists a birational modification $\mu\colon X' \to X$ as well as a decomposition $\mu^*L = P + N$, where P and N are $\mathbb{Q}$-divisors and with P nef, having the property that

$$H^0(X, \mathcal{O}_X(kL)) = H^0(X', \mathcal{O}_{X'}([kP]))$$

for all $k \geq 0$. Then $v(X, L) = v(X', P) = (P^n)$; that is, the volume of L is computed by the volume of a nef divisor on a modification. Although it is known that such decompositions do not exist in general [3], Fujita's theorem shows that an approximate asymptotic statement does hold.

Fujita's proof is quite short but rather tricky: it is an argument by contradiction revolving around the Hodge index theorem. The purpose of this section is to use the subadditivity property of multiplier ideals to give a new proof that may seem a bit more transparent. (To a certain extent, one can see the present argument as extending to all dimensions the proof for surfaces, due to Fernandez del Busto, appearing in [23, Sec. 7]).

We begin with two lemmas. The first, due to Kodaira, is a standard consequence of asymptotic Riemann–Roch (cf. [22, (VI.2.16)]).

Lemma 3.4 (Kodaira's Lemma). *Given a big line bundle L and any ample bundle A on X, there is a positive integer $m_0 > 0$ such that $m_0L = A + E$ for some effective divisor E.*

The second (somewhat technical) lemma shows that one can perturb L slightly without greatly affecting its volume.

Lemma 3.5. *Let G be an arbitrary line bundle. For every $\varepsilon > 0$, there exists a positive integer m and a sequence $\ell_\nu \uparrow +\infty$ such that*

$$h^0(X, \ell_\nu(mL - G)) \geq \frac{\ell_\nu^n m^n}{n!}(v(L) - \varepsilon).$$

In other words,

$$v(mL - G) \geq m^n(v(L) - \varepsilon)$$

for m sufficiently large.

Proof. Clearly, $v(mL - G) \geq v(mL - (G + E))$ for every effective divisor E. We can take E so large that $G + E$ is very ample, and we are thus reduced to the case where G itself is very ample by replacing G with $G + E$. By definition of $v(L)$, there exists a sequence $k_\nu \uparrow +\infty$ such that

$$h^0(X, \mathcal{O}_X(k_\nu L)) \geq \frac{k_\nu^n}{n!}\left(v(L) - \frac{\varepsilon}{2}\right).$$

We now fix an integer $m \gg 1$ (to be chosen precisely later) and put $\ell_\nu = [k_\nu/m]$, so that $k_\nu = \ell_\nu m + r_\nu$, $0 \leq r_\nu < m$. Then

$$\ell_\nu(mL - G) = k_\nu L - (r_\nu L + \ell_\nu G).$$

Now fix a constant $a \in \mathbf{N}$ such that $aG - rL$ is an effective divisor for each $0 \leq r < m$. Then $maG - r_\nu L$ is effective, and hence

$$h^0\big(X, \mathcal{O}_X(\ell_\nu(mL - G))\big) \geq h^0\big(X, \mathcal{O}_X(k_\nu L - (\ell_\nu + am)G)\big).$$

We select a smooth divisor D in the very ample linear system $|G|$. By looking at global sections associated with the exact sequences of sheaves

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_X(-(j+1)D) \otimes \mathcal{O}_X(k_\nu L) \longrightarrow \mathcal{O}_X(-jD) \otimes \mathcal{O}_X(k_\nu L) \\ \longrightarrow \mathcal{O}_D(k_\nu L - jD) \longrightarrow 0 \end{aligned}$$

for $0 \leq j < s$, we infer inductively that

$$\begin{aligned} h^0(X, \mathcal{O}_X(k_\nu L - sD)) &\geq h^0(X, \mathcal{O}_X(k_\nu L)) - \sum_{0 \leq j < s} h^0(D, \mathcal{O}_D(k_\nu L - jD)) \\ &\geq h^0(X, \mathcal{O}_X(k_\nu L)) - s h^0(D, \mathcal{O}_D(k_\nu L)) \\ &\geq \frac{k_\nu^n}{n!}\left(v(L) - \frac{\varepsilon}{2}\right) - sCk_\nu^{n-1}, \end{aligned}$$

where C depends only on L and G. Hence, putting $s = \ell_\nu + am$ yields

$$\begin{aligned} h^0\big(X, \mathcal{O}_X(\ell_\nu(mL - G))\big) &\geq \frac{k_\nu^n}{n!}\left(v(L) - \frac{\varepsilon}{2}\right) - C(\ell_\nu + am)k_\nu^{n-1} \\ &\geq \frac{\ell_\nu^n m^n}{n!}\left(v(L) - \frac{\varepsilon}{2}\right) - C(\ell_\nu + am)(\ell_\nu + 1)^{n-1} m^{n-1}, \end{aligned}$$

and the desired conclusion follows by taking $\ell_\nu \gg m \gg 1$. □

Proof of Fujita's Theorem. Note to begin with that it is enough to produce a big and nef divisor A satisfying the conclusion of the theorem. For by Kodaira's lemma one can write $A \equiv E + A'$, where E is an effective $\mathbb{Q}$-divisor and A' is an ample $\mathbb{Q}$-divisor. Then

$$E + A \equiv E + \delta E + (1 - \delta)A + \delta A',$$

where $A'' \stackrel{\text{def}}{=} (1 - \delta)A + \delta A'$ is ample and the top self-intersection number $((A'')^n)$ approaches (A^n) as closely as we want.

Fix now a very ample bundle B on X, set $G = K_X + (n+1)B$, and for $m \geq 0$ put

$$M_m = mL - G.$$

We can suppose that G is very ample, and we choose a divisor $D \in |G|$. Then multiplication by ℓD determines, for every $\ell \geq 0$, an inclusion $\mathcal{O}_X(\ell M_m) \hookrightarrow \mathcal{O}_X(\ell mL)$ of sheaves and therefore an injection

$$H^0(X, \mathcal{O}_X(\ell M_m)) \subseteq H^0(X, \mathcal{O}_X(\ell mL)).$$

Given $\varepsilon > 0$, we use Lemma 3.5 to fix $m \gg 0$ such that

$$v(M_m) \geq m^n(v(L) - \varepsilon). \tag{4}$$

We further assume that m is sufficiently large so that M_m is big.

Having fixed $m \gg 0$ satisfying (4), we will produce an ideal sheaf $\mathcal{J} = \mathcal{J}_m \subset \mathcal{O}_X$ (depending on m) such that

$$\mathcal{O}_X(mL) \otimes \mathcal{J} \text{ is globally generated;} \tag{5}$$

$$H^0(X, \mathcal{O}_X(\ell M_m)) \subseteq H^0(X, \mathcal{O}_X(\ell mL) \otimes \mathcal{J}^\ell) \text{ for all } \ell \geq 1. \tag{6}$$

Granting for the time being the existence of $\mathcal{J}$, we complete the proof. Let $\mu\colon X' \to X$ be a log resolution of $\mathcal{J}$, so that $\mu^{-1}\mathcal{J} = \mathcal{O}_{X'}(-E_m)$ for some effective divisor E_m on X'. It follows from (5) that

$$A_m \overset{\text{def}}{=} \mu^*(mL) - E_m$$

is globally generated and hence nef. Using (6), we find

$$\begin{aligned} H^0(X, \mathcal{O}_X(\ell M_m)) &\subseteq H^0(X, \mathcal{O}_X(\ell mL) \otimes \mathcal{J}^\ell) \\ &\subseteq H^0(X', \mathcal{O}_{X'}(\mu^*(\ell mL) - \ell E_m)) \\ &= H^0(X', \mathcal{O}_{X'}(\ell A_m)) \end{aligned}$$

(which shows in particular that A_m is big). This implies that

$$\begin{aligned} ((A_m)^n) &= v(X', A_m) \\ &\geq v(X, M_m) \\ &\geq m^n(v(L) - \varepsilon), \end{aligned}$$

so Fujita's theorem follows upon setting $A = \frac{1}{m}A_m$ and $E = \frac{1}{m}E_m$.

Turning to the construction of $\mathcal{J}$, set

$$\mathcal{J} = \mathcal{J}(X, \|M_m\|).$$

Since $mL = M_m + (K_X + (n+1)B)$, (5) follows from Theorem 1.8(iii) applied to M_m. As for (6), we first apply Theorem 1.8(i) to ℓM_m, together with the subadditivity property in the form of Variant 2.4, to conclude that

$$\begin{aligned} H^0(X, \mathcal{O}_X(\ell M_m)) &= H^0(X, \mathcal{O}_X(\ell M_m) \otimes \mathcal{J}(\|\ell M_m\|)) \\ &\subseteq H^0(X, \mathcal{O}_X(\ell M_m) \otimes \mathcal{J}(\|M_m\|)^\ell). \end{aligned} \tag{7}$$

Now the sheaf homomorphism

$$\mathcal{O}_X(\ell M_m) \otimes \mathcal{J}(\|M_m\|)^\ell \xrightarrow{\cdot \ell D} \mathcal{O}_X(\ell m L) \otimes \mathcal{J}(\|M_m\|)^\ell$$

evidently remains injective for all ℓ, and consequently

$$H^0(X, \mathcal{O}_X(\ell M_m) \otimes \mathcal{J}(\|M_m\|)^\ell) \subseteq H^0(X, \mathcal{O}_X(\ell m L) \otimes \mathcal{J}(\|M_m\|)^\ell). \tag{8}$$

The required inclusion (6) follows by combining (7) and (8). This completes the proof of Fujita's theorem. □

We conclude by using Fujita's theorem to establish a geometric interpretation of the volume $v(L)$. Suppose as before that X is a smooth projective variety of dimension n and that L is a big line bundle on X. Given a large integer $k \gg 0$, denote by $B_k \subseteq X$ the base locus of the linear series $|kL|$. The *moving self-intersection number* $(kL)^{[n]}$ of $|kL|$ is defined by choosing n general divisors $D_1, \ldots, D_n \in |kL|$ and putting

$$(kL)^{[n]} = \#(D_1 \cap \cdots \cap D_n \cap (X - B_k)).$$

In other words, we simply count the number of intersection points away from the base locus of n general divisors in the linear series $|kL|$. This notion arises, for example, in Matsusaka's proof of his "big theorem" (cf. [25]).

We show that the volume $v(L)$ of L measures the rate of growth with respect to k of these moving self-intersection numbers. The following result is implicit in [36] and was undoubtably known also to Fujita.

PROPOSITION 3.6. *Assume as before that L is a big line bundle on a smooth projective variety X. Then one has*

$$v(L) = \limsup_{k \to \infty} \frac{(kL)^{[n]}}{k^n}.$$

Proof. We start by interpreting $(kL)^{[n]}$ geometrically. Let $\mu_k \colon X_k \to X$ be a log resolution of $|kL|$ with $\mu_k^*|kL| = |V_k| + F_k$, where

$$P_k \overset{\text{def}}{=} \mu_k^*(kL) - F_k$$

is free and $H^0(X, \mathcal{O}_X(kL)) = V_k = H^0(X_k, \mathcal{O}_{X_k}(P_k))$, so that $B_k = \mu_k(F_k)$. Then evidently $(kL)^{[n]}$ counts the number of intersection points of n general divisors in P_k, and consequently

$$(kL)^{[n]} = ((P_k)^n).$$

We have $((P_k)^n) = v(X_k, P_k)$ for $k \gg 0$ since then P_k is big (and nef), and $v(X, kL) \geq v(X_k, P_k)$ since P_k embeds in $\mu_k^*(kL)$. Hence

$$v(X, kL) \geq (kL)^{[n]} \quad \text{for } k \gg 0.$$

On the other hand, an argument in the spirit of Lemma 3.5 shows that $v(X, kL) = k^n \cdot v(X, L)$ [15, Lemma 3.4], so we conclude that

$$v(L) \geq \frac{(kL)^{[n]}}{k^n}. \tag{†}$$

for every $k \gg 0$.

For the reverse inequality we use Fujita's theorem. Fix $\varepsilon > 0$, and consider the decomposition $\mu^* L = A + E$ on $\mu\colon X' \to X$ constructed in (3.2). Let k be any positive integer such that kA is integral and globally generated. By taking a common resolution we can assume that X_k dominates X', and hence we can write

$$\mu_k^* kL \equiv A_k + E_k$$

with A_k globally generated and

$$((A_k)^n) \geq k^n \cdot (v(X, L) - \varepsilon).$$

But then $H^0(X_k, A_k)$ gives rise to a free linear subseries of $H^0(X_k, P_k)$, and consequently

$$((A_k)^n) \leq ((P_k)^n) = (kL)^{[n]}.$$

Therefore

$$\frac{(kL)^{[n]}}{k^n} \geq v(X, L) - \varepsilon. \tag{‡}$$

But (‡) holds for any sufficiently large and divisible k, and in view of (†) the proposition follows. □

Note added in proof. The subadditivity theorem has recently been used to establish some surprising results concerning symbolic powers of radical ideals on a smooth variety. See L. Ein, R. Lazarsfeld, and K. E. Smith, "Uniform bounds and symbolic powers on smooth varieties" (to appear).

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