ON THE UNIQUENESS OF CERTAIN LINEAR SERIES ON SOME CLASSES OF CURVES

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Introduction.

This note is a report of work in progress about some problems of uniqueness for certain linear series on some curves in a projective space. The problem we started with was to show that on any smooth, non degenerate, complete intersection curve in $\mathbb{P}^3$, of degree $n > 4$ the linear series cut out by the planes is the unique simple g^3_n. Once we proved this (for the proof see § 2), we got aware that the same method of proof could be applied to obtain other uniqueness statements. The results we have achieved in this direction are exposed in § 3, and concern, for instance, projectively normal curves in $\mathbb{P}^3$, some determinantal curves in $\mathbb{P}^r$, etc. Our feeling at the present state of affairs, is that similar results should hold for several and large classes of curves in a projective space.

The key ingredient in many of the proofs of this paper turns out to be a classical method of Castelnuovo, which we briefly recall in § 1. In fact this method, and some applications of it to uniqueness questions for curves of high genus in a projective space, was the main topic of the talk given by the first named author at the CIME Conference (Acireale, June 1983). At that time most of the results exposed here were only conjectured; their proofs were achieved also thanks to the opportunity, which the Conference gave the authors, of meeting and discussing.

Notation.

We work on an algebraically closed field k of characteristic zero. If X is a k-scheme, we denote by O_X its structure sheaf. If D is any Cartier divisor on X, we denote by $O_X(D)$ the sheaf of sections of the corresponding line bundle, with $H^i(x, O_X(D))$, $h^i(X,O_X(D))$ its cohomology spaces and their dimensions over k. If X is integral and projective, $|D|$ will denote the complete linear system determined by D on X. If X is smooth, K_X will be any canonical divisor on X. If $X\subset\mathbb{P}^r$ is a variety, X will be said to be non degenerate if no hyperplane of $\mathbb{P}^r$ contains X. If $X\subset\mathbb{P}^r$ is smooth, irreducible, it is said to be projectively normal if for any $t \in \mathbb{N}$, the linear system of hypersurfaces of degree t cuts out on X a complete linear system.

1. Preliminaries: Castelnuovo's lemma.

Let C be a smooth irreducible, projective curve and let E be a divisor of degree $n > 0$ on C. Any $(r + 1)$-dimensional vector subspace V of $H^0(C,O_C(E))$ corresponds to a linear series g^r_n on C, contained in the complete series $|E|$. If D is an effective divisor of degree d on

C, we set $V(-D) = \{s \in V : (s) \geq D\}$, (s) denoting the zeroes divisor of the section $s \in V$. $V(-D)$ corresponds to a linear series having D as a fixed divisor: the linear series we get from this removing D from its divisors will be denoted by $g^r_n(-D)$.

Let D be a divisor and g^r_n a linear series on C; the non negative integer

$$c(D,\ g^r_n) = r - \dim g^r_n(-D)$$

is the so-called <u>number of conditions which</u> D <u>imposes to the linear series</u> g^r_n. Clearly it is

$$(1.1) \qquad c(D,\ g^r_n) \leq \min\{d, r+1\}$$

d being the degree of D. If g^s_m is another linear series on C, we say that g^s_m is <u>contained</u> in g^r_n, writing $g^s_m \leq g^r_n$, if there is an effective divisor D', of degree n - m, such that $g^r_n(-D')$ contains g^s_m as a linear subseries. If $g^s_m \leq g^r_n$ and D and D' have no point in common, then

$$c(D,\ g^s_m) \leq c(D,\ g^r_n)$$

If $d \leq r + 1$ and in (1.1) the equality holds, D is said to <u>impose independent conditions to</u> g^r_n. We shall assume, from now on, D formed by d distinct points. Given any integer $t \geq 0$, D is said to be in t-<u>uniform</u> <u>position</u> with respect to the g^r_n if any divisor of degree t contained in D imposes independent conditions to g^r_n. Then it is

$t \leq c(D, g^r_n)$. If $t = c(D, g^r_n)$, D is said to be in <u>uniform position</u> with respect to g^r_n.

Before stating Castelnuovo's lemma, we recall the definition of minimal sum of some linear series $g^{r_i}_{n_i}$, $i = 1, \dots, k$, on C. Let $V^{(i)}$ be the vector sub-space of $H^o(C, O_C(E_i))$ corresponding to $g^{r_i}_{n_i}$, $i = 1, \dots, k$, and consider the linear map

$$f : \bigotimes_{i=1}^{k} V^{(i)} \to H^o(C, O_C(\sum_1^K {}_i E_i))$$

such that

$$f(\bigotimes_{i=1}^{k} s^{(i)}) = \prod_{i=1}^{k} s^{(i)}$$

The linear series corresponding to Im f is the so-called <u>minimal sum</u> of the given series $g^{r_i}_{n_i}$, and will be denoted by $\bigoplus_{i=1}^{k} g^{r_i}_{n_i}$. If $g^{r_i}_{n_i} = g^r_n$ for any $i = 1, \dots, k$, we simply set $k\, g^r_n = \bigoplus_{i=1}^{k} g^{r_i}_{n_i}$. The basic fact about minimal sums is the following:

(1.2) <u>Lemma</u> (Castelnuovo, cfr. [C]). <u>Let</u> $g^{r_i}_{n_i}$, $i = 1, \dots, k$ <u>be linear series on</u> C <u>and</u> D <u>an effective divisor of degree</u> d <u>on</u> C <u>formed by distinct points</u>. <u>If</u> D <u>is in</u> t_i-<u>uniform position with respect to</u> $g^{r_i}_{n_i}$, $i = 1, \dots, k$, <u>then</u>:

(i) <u>if</u> $t = \sum_1^k {}_i t_i - k + 1 < d$, D <u>is in</u> t-<u>uniform position with respect to</u> $\bigotimes_{i=1}^{k} g^{r_i}_{n_i}$;

(ii) if $t \geq d$, D imposes independent conditions to $\bigoplus_{i=1}^{k} g_{n_i}^{r_i}$.

We omit the proof, which can be found, for example, in [CI]. The problem, in order to apply the above lemma, is to verify, for a given divisor D, the conditions of uniform position. For this reason it is useful to have a few "uniformity criteria"; the following two will be enough for our purposes.

(1.3) Proposition (Bertini, cfr. [B]; Harris, cfr [H1]). Let g_n^r be a linear series, without base points, not composed of an involution on C, and let D be its generic divisor. If g_m^s is any linear series on C, then D is in uniform position with respect to g_m^s. In particular D is in r-uniform position with respect to g_n^r.

(1.4) Proposition (Accola, cfr [A]). Let g_n^r, g_m^s be distinct linear series, without base points on C, not composed of the same involution, with $s \geq r$. Then the generic divisor of g_n^s is in $(r + 1)$-uniform position with respect to g_n^r.

For the proofs of the above propositions we refer to the quoted references. A proof of (1.3) will also be found in [CI], where Castelnuovo's lemma is applied to study linear series on curves of the following types:

(i) curves in a projective space with high genus with respect to the degree;

(ii) in particular, smooth plane curves;

(iii) subcanonical curves in a projective space, namely curves whose canonical divisors are linearly equivalent to a multiple of a hyperplane section; in particular complete intersection curves in a projective space.

In what follows we shall focus our attention on curves of the third kind above and, taking the point of view of [CI], we shall prove a number of uniqueness theorems for some linear series on sub-canonical curves and, in particular, for complete intersections in $\mathbb{P}^3$. Later on we shall show how to extend these results to more general classes of curves.

2. Subcanonical curves.

Let $\Gamma \subset \mathbb{P}^r$, $r \geq 2$, be an irreducible, non degenerate, complete curve of degree n and let

$$p : C \to \Gamma \subset \mathbb{P}^r$$

be its normalization. The morphism p of C to $\mathbb{P}^r$ corresponds to a linear series g^r_n on C, pull-back, via p, of the linear series cut out on Γ by the hyperplanes in $\mathbb{P}^r$. We denote by H the generic divisor of this g^r_n. For each integer k, the linear series kg^r_n is pull-back, via p, of the linear series cut out Γ by the hypersurfaces of degree k in $\mathbb{P}^r$.

Let us put now

$$l(\Gamma) = \max \{t \in \mathbb{Z} : h^0(C, O_C(k_C - tH)) \neq 0\}$$

We call $l(\Gamma)$ the <u>level</u> of Γ. If K_C is linearly equivalent to $l(\Gamma)H$, Γ is said to be subcanonical of level $l(\Gamma)$.

(2.1) <u>Proposition</u>. <u>If</u> $l = l(\Gamma) \geq 1$, $n \leq r(l + 1)$ <u>and</u> g^s_m <u>is a linear series on</u> C <u>with</u> $m \leq n$, $s \geq r$, <u>then</u> $g^s_m = g^r_n$.

<u>Proof</u>. Of course we can reduce to the case g_m^s base points free. Let D be the generic divisor in g_m^s. If $g_n^r \neq g_m^s$, by proposition (1.4) and Castelnuovo's lemma, we would get

$$c(D, |K_C|) \geq c(D, l\, g_n^r) \geq \min \{m, l\, r + 1\}$$

By Riemann-Roch theorem, it is

$$c(D, |K_C|) = m - \dim |D| \leq m - s < m$$

and therefore we should have

$$c(D, |K_C|) \geq l\, r + 1$$

Since

$$c(D, |K_C|) \leq m - s \leq n - r$$

we should also get

$$n - r \geq c(D, K_C) \geq l\, r + 1$$

contradicting the hypotheses.

With the further assumption that Γ is projectively normal and subcanonical, and some more arithmetical conditions on n and $l(\Gamma)$ one has more information.

(2.2) <u>Theorem</u>. <u>Let</u> Γ <u>be a subcanonical</u>, <u>projectively normal curve of level</u> $l \geq 2$, <u>with</u> $r(l + 1) < n \leq r(l + 1) + l - 1$. <u>If</u> g_m^s <u>is a linear series on</u> C, <u>not composed of an involution</u>, <u>with</u> $m \leq n$, $r \leq s$, <u>then</u> $g_m^s = g_n^r$.

<u>Proof</u>. We again can assume g_m^s base points free. If $g_n^r \neq g_m^s$, the gener-

ic divisor D in g^s_m should be in $(r + 1)$-uniform position with respect to g^r_n, so that, by proposition (1.3)

$$c(D, 2g^r_n) \geq \min \{m, 2r + 1\} = 2r + 1 \tag{2.3}$$

the last equality holding, because

$$m > m - s \geq c(D, |K_C|) \geq c(D, 2g^r_n)$$

Moreover, D is in uniform position with respect to $2g^r_n$. Let $l = 2k$. Applying Castelnuovo's lemma we then get

$$n - r \geq m - s \geq c(D, |K_C|) \geq c(D, k(2g^r_n)) \geq$$

$$\geq k(c(D, 2g^r_n) - 1) + 1$$

whence, by the hypotheses

$$c(D, 2g^r_n) \leq 2r + 3 - \frac{2}{l}$$

Similarly, if l is odd, we have

$$c(D, 2g^r_n) \leq 2r + 3 - \frac{2}{l-1}$$

so that, in any case, it is

$$c(D, 2g^r_n) \leq 2r + 2 \tag{2.4}$$

Assume now that Γ has not maximal genus in $\mathbb{P}^r$. If in (2.3) the equality held, since

$$n > r(l + 1) \geq 3r \geq 2r + 2$$

the points of D would lie on a rational normal curve $\Gamma_0 \subset \mathbb{P}^r$ (see [H2]).

On the other hand, the ideal of Γ in $\mathbb{P}^r$ is generated by forms of degree $\leq 1 + 1$ (see [AS], thms (4.3), (4.7)). Since $n > r(1 + 1)$, any such a form would contain Γ_0 as well as Γ, which is impossible. Let us discuss the case in which the equality holds in (2.4). If this happened the points of D would lie either on a rational normal curve, or on an irreducible curve $\Gamma_1 \subset \mathbb{P}^r$ of degree $r + 1$ and arithmetic genus 1 (see [H3]). It is also easy to see that no point of D would be singular on Γ_1. Then it would be

$$c(D, |K_C|) = c(D, l\, g^r_n) = \begin{cases} n, \text{ if } n < l(r + 1) \\ \text{either } n \text{ or } n - 1 \text{ if } n = l(r + 1) \\ l(r + 1), \text{ if } n > l(r + 1) \end{cases}$$

But this leads to contradictions. In the first two cases, for instance, we would get

$$m - s \geq c(D, |K_C|) \geq n - 1 \geq m - 1$$

whence $r \leq s \leq 1$. In the last case it would be

$$n - r \geq m - s \geq c(D, |K_C|) \geq l(r + 1)$$

whence $n \geq r(l + 1) + 1$. Finally, if Γ has maximal genus the theorem follows by theorem (2.11) of [CI] or by the results of Accola (see [A]).

The hypothesis of projective normality on Γ in (2.2) is too strong (see [AS], remark (4.6)). Moreover also the hypothesis Γ subcanonical is too strong. It could be easily replaced by the hypothesis that the ideal of Γ is generated by forms of degree $\leq 1 + 1$.

Theorem (2.2) and proposition (2.1), already proved in [CI], readily apply to smooth complete intersection curves. One has the:

(2.5) Corollary. *Let $C \subset \mathbb{P}^3$ be a smooth complete intersection of two surfaces of degrees* h, k, *with* $h \leq k$. If $h = 2$, $k > 2$ *and if* $h = 3, 4$, *the linear series cut out on* C *by planes of* $\mathbb{P}^3$ *is the unique simple* g^3_{hk} *on* C. *A similar results holds for any smooth complete intersection of two quadrics and a hypersurface of degree* $h \geq 2$ in $\mathbb{P}^4$.

The above corollary, which has been independently proved by P. Maroscia (see [M]), suggests the problem of seeing if an analogous results holds, more generally, for any complete intersection curve of positive level. To this question we can give an affirmative answer, at least for curves in $\mathbb{P}^3$.

(2.6) Theorem. *Let $C \subset \mathbb{P}^3$ be a smooth, complete intersection of two surfaces of degrees* h,k *with* $4 < h \leq k$. *If* g^s_m *is a linear series on* C, *without base points*, *not composed of a pencil*, *with* $s \geq 2$, $m \leq hk$ *then* $g^s_m \leq |H|$.

Proof. Let D be the generic divisor in g^s_m, and let x_0 be the least integer for which the following property holds: there exists a surface G of degree x_0 in $\mathbb{P}^3$ such that

(i) $G \not\supseteq C$;

(ii) G cuts out on C a divisor containing D.

We denote by $D + D'$ the divisor cut out by G on C, and consider the linear system Σ formed by all surfaces of degree x_0 which either contain C or cut out on C a divisor containing D'. It is not difficult, using the minimality of x_0 and the projective normality of C, to see that

Σ has no fixed components. Thus, if F', F" are generic elements in Σ, the 1-dimensional subscheme Γ of $\mathbb{P}^3$, of degree x_0^2, complete intersection of F', F" can be considered. Let F now be the generic surface of degree k containing C. Since the ideal of C is generated in degree at most k, F can be chosen to be smooth, and to contain no component of Γ, so that the cycle of intersection $F\cdot\Gamma$ is well defined, its degree being x_0^2k. Now D' has degree at least $(x_0 - 1)h\,k$, and simple arguments of local algebra show that

$$x_0^2k - (x_0 - 1)h\,k = k\,(x_0^2 - hx_0 + h) \geq 0$$

Since $h > 4$, this can only happen if either $x_0 = 1$, in which case the theorem is proved, or $x_0 > h - 2$. We shall now prove that only the first case can occur. Let us put

$$h + k - 4 = a(h - 2) + b\ , \quad 0 \leq b < h - 2$$

Since C is subcanonical of level $h + k - 4$, applying proposition(1.3) and Castelnuovo's lemma, one gets

$$\begin{aligned} h\,k - 2 \geq m - s \geq c(D, |K_C|) &= c(D,\ a((h - 2)H) + b\ H) \\ &\geq a(c(D, |(h - 2)H|) - 1) + 3b + 1 \end{aligned}$$

Hence, by easy computations, we have

$$c(D, |(h - 2)H|) \leq h(h - 2)$$

and, since

$$h(h - 2) \leq \binom{h+1}{3} - 1 = \dim\ |(h - 2)H|$$

it is $x_0 \leq h - 2$ (compare [R]).

By virtue of theorem (2.6), corollary (2.5) can be extended to any value of $h > 4$. Anyhow the disappointing feature in theorem (2.6) is the hypothesis of simplicity on g_m^s. It is likely that it may be removed, but it seems that further assumptions are necessary. For example, we are able to prove the following:

(2.7) Proposition. The conclusions of theorem (2.6) still hold without assuming g_m^s simple, as soon as $h \geq 12$.

The idea of the proof is as follows. The linear series $g_m^s \oplus |H|$ is certainly simple. Thus everything amounts to prove that if $h \geq 12$ and $g_{m'}^{s'}$ is a simple linear series, without base points on C, with $s' \geq 2$, $m' \leq 2\,hk$, then $g_{m'}^{s'} \leq |2H|$. This can be done by the same reasoning of the proof of theorem (2.7).

3. Extension of the above results. Final remarks.

Once theorem (2.6) has been proved, a natural problem is to look for other, or larger, classes of curves in $\mathbb{P}^3$, or preferably in $\mathbb{P}^r$, $r \geq 2$, for which an analogous result holds. A first extension can be made to smooth projectively normal curves in $\mathbb{P}^3$. Let C be such a curve. Then it is well known that the homogeneous ideal of C can be minimally generated by the minors of maximal order of a homogeneous matrix of forms of the type $u\times(u+1)$, which we write in the form

$$A = \begin{pmatrix} f_{11} & \cdots\cdots & f_{1,u+1} \\ & & \\ f_{u1} & \cdots\cdots & f_{u,u+1} \end{pmatrix}$$

We put $m_{ij} = \deg f_{ij}$. It is known that A can be taken such that

$$m_{u,1} \leq m_{u-1,1} \leq \cdots \leq m_{1,1} \leq \cdots \leq m_{1,u+1}$$

In this case $u = 1$ if and only if C is a complete intersection: therefore we shall assume $u \geq 2$. We have the:

(3.1) Theorem. Let $m_{11} \geq 9$ and let n be the degree of C. If g_m^s is a linear series on C, without base points, not composed of a pencil, with $s \geq 2$, $m \leq n$, then $g_m^s \leq |H|$.

Proof. We put $h = m_{1,1}$, and call k the highest degree of a minor of A. Simple computations show that

$$n > hk \tag{3.2}$$

Moreover it is

$$l(C) \geq h + k - 4 \tag{3.3}$$

(see [G], pg 36). Now the proof goes like that of theorem (2.6), the role of F being played by the generic surface of degree k through C. One has

$$x_0^2 - \frac{n}{k} x_0 + \frac{n}{k} \geq 0$$

By (3.2) and by the hypotheses, it is $n > 4k$, implying that either $x_0 = 1$, or $x_0 > \frac{n}{k} - 2$. But the second case cannot happen: in fact, we set $q = [\frac{n}{k}]$ and show that $x_0 \leq q - 2$. If

$$h + k - 4 = a(q - 2) + b, \quad 0 \leq b < q - 2$$

one can easily see that $a \geq 1$. Then applying Castelnuovo's lemma and (3.3) we get

$$n - 2 \geq a(c(D, |(q - 2)H|) - 1) + 3b + 1$$

whence

$$c(D, |(q-2)H|) \leq 2(q-1)(q-2)$$

Now it is not difficult to check that

$$\dim |(q-2)H| = \binom{q+1}{3} - 1$$

Thus $c(D, |(q-2)H)|) \leq \dim |(q-2)H|$ if $q \geq 9$, and this is certainly the case by virtue of (3.2), since $h \geq 9$. So the theorem is proved.

The hypothesis $m_{1,1} \geq 9$ in (3.1) is probably too strong. It is likely that the theorem can be substantially improved. By contrast the statement is not true for any $m_{1,1}$: there are in fact projectively normal, but not special, curves. But even for projectively normal, special curves the uniqueness can fail: a counterexample is the projection in $\mathbb{P}^3$ of a canonical curve of degree 8 in $\mathbb{P}^4$ from a generic point on it.

It seems quite difficult to extend rather precise statements like theorems (2.6) and (3.1), to curves in $\mathbb{P}^r$, $r > 3$. Anyhow, if one seeks for sort of asymptotic results, the same strategy of the proofs of these theorems appears to be very fruitful. Moreover the use of Castelnuovo's lemma can be avoided, so that the hypothesis of simplicity can be removed. As an example we state here a rather general asymptotic uniqueness theorem for determinantal curves, which can be proved in the same vein as theorems (2.6) and (3.1), but with some more technical devices:

(3.4) Theorem. Let E,F be vector bundles of ranks n, $n+r-2$ respectively on $\mathbb{P}^r$, $r \geq 2$. Then there exists an integer $c(E,F)$, such that for any $a \geq c(E,F)$ and for any homomorphism $u : E \to F(a)$ which *drops rank on a smooth irreducible curve* C *of degree* n, *the following happens*:

if g^s_m *is a linear series on* C *without base points*, *with* $m \leq n$, *then* $g^s_m \leq |H|$, H *being a hyperplane section of* C. *In particular* C *has no* g^r_m *for* $m < n$ *and* $|H|$ *is the only* g^r_n *on* C.

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