Teichmüller space of Fibonacci maps.

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§**1. Introduction.** According to Sullivan, a space $\mathcal{E}$ of unimodal maps with the same combinatorics (modulo smooth conjugacy) should be treated as an infinitely-dimensional Teichmüller space. This is a basic idea in Sullivan's approach to the Renormalization Conjecture [S1], [S2]. One of its principle ingredients is to supply $\mathcal{E}$ with the Teichmüller metric. To have such a metric one has to know, first of all, that all maps of $\mathcal{E}$ are quasi-symmetrically conjugate. This was proved in [Ji] and [JS] for some classes of non-renormalizable maps (when the critical point is not too recurrent). Here we consider a space of non-renormalizable unimodal maps with in a sense fastest possible recurrence of the critical point (called Fibonacci). Our goal is to supply this space with the Teichmüller metric.

Let f be a unimodal map with critical point c. A Fibonacci unimodal map f can be defined by saying that the closest returns of the critical point occur at the Fibonacci moments. This combinatorial type was suggested by Hofbauer and Keller [HK] as an extremal among non-renormalizable types (see [LM] for more detailed history). Its combinatorial, geometric and measure-theoretical properties were studied in [LM] under the assumptions that f is *quasi-quadratic*, i.e., it is C^2-smooth and has the quadratic-like critical point (see also [KN]). We will assume this regularity throughout the paper.

A principle object of our combinatorial considerations is a nested sequence of intervals $I^0 \supset I^1 \supset \ldots$ obtained subsequently by pulling back along the critical orbit. Our proof is based upon the geometric result of [LM] which says that the scaling factors $\mu_n = |I^n|/|I^{n-1}|$ characterizing the geometry of the Fibonacci map decay exponentially. It follows that appropriately defined renormalizations $R^n f$ are becoming purely quadratic near the critical point. This reduces the renormalization process to the iterates of quadratic maps.

The next idea is to consider a quasi-conformal continuation of f to the complex plane which is asymptotically conformal on the real line. Then we consider complex generalized renormalizations, and prove that the renormalized maps are becoming purely quadratic in the complex plane as well. Hence the geometric patterns of renormalized maps are subsequently obtained by the Thurston pull-back transformation (up to an exponentially small error) in an appropriate Teichmüller space. It follows that these patterns converge (after rescaling) to the corresponding pattern of the quadratic map $p : z \mapsto z^2 - 1$. In particular, the shape of the complex puzzle-pieces converge to the Julia set of p, see Figure 1 (this is perhaps the most unexpected result of our analysis).

To each renormalization we then associate a pair of pants Q_n by removing from the critical puzzle-piece of level n two puzzle-pieces of the next level. Using a same type of argument as above, we show that the pairs of pants Q^n and $\tilde{Q}^n$ stay on bounded distance. This yields the quasi-conformal equivalence of the critical sets of f and $\tilde{f}$.

To complete the construction of the quasi-symmetric conjugacy, we apply a Sullivan-like pull-back argument. However, this is not quite straightforward since there is no di-

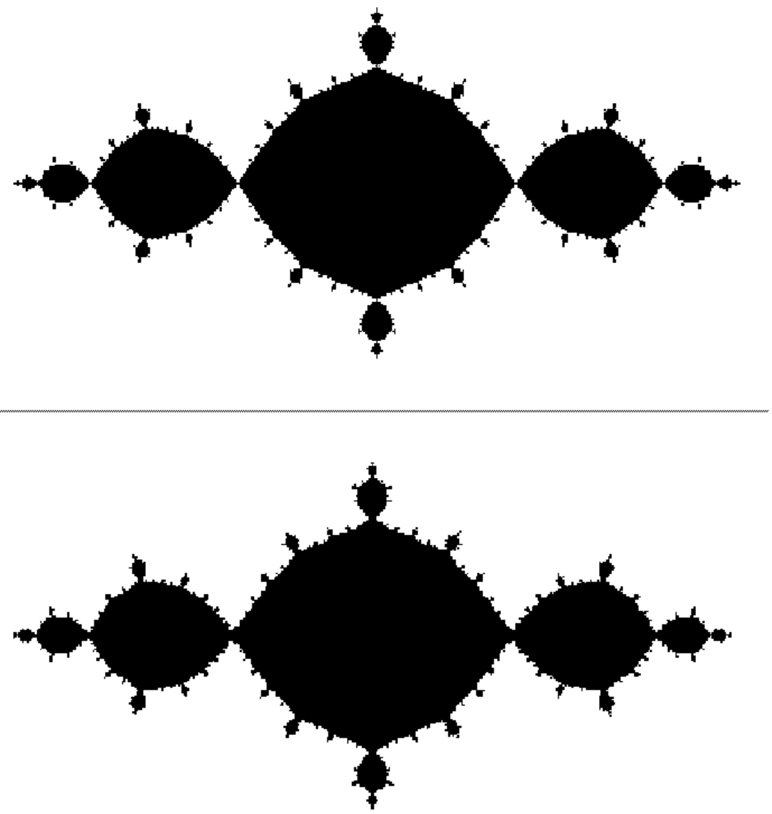

Figure 1. A Fibonacci puzzle-piece (below) versus the Julia set of $z \mapsto z^2 - 1$.
(made by S. Sutherland and B. Yarrington)

latation control away from the real line.

In the last section we prove that two Fibonacci maps which stay on zero Teichmüller distance are smoothly conjugate. So this pseudo-metric is non-degenerate on the smooth equivalence classes.

We will use abbreviations qc and qs for "quasi-conformal" and "quasi-symmetric" respectively.

Remark 1. Since the rate at which the scaling factors decrease depends on the initial bounds of the map only, the dilatation of the conjugacy we construct also depends only on this data.

Remark 2. It is proved in [L] that, as in the Fibonacci case, the scaling factors of any non-renormalizable quasi-quadratic map decay exponentially. This allows us to generalize the above result to all combinatorial classes of quasi-quadratic maps. The exposition of this result is more technical, and it will be the subject of forthcoming notes. Note that for polynomial-like maps this result follows from the Yoccoz Theorem (see [H] for the exposition of this theorem, and [K] for an alternative proof based upon a pull-back argument).

Remark 3. In this paper we concentrate on the dynamical constructions, and don't touch the issue of the sharp regularity for which the theory can be built up. This issue is clearly important for a proper Teichmüller theory (compare [S2] and [G]), and will be discussed elsewhere.

§2. Asymptotically conformal continuation and generalized renormalization.

Real renormalization (see [LM]). Given a Fibonacci map f, there is a sequence of maps

$$g_n : I_0^n \cup I_1^n \to I_0^{n-1}, \quad n = 1, 2, ...$$

constructed in the following way. Let $I^0 \equiv I_0^0$ be a c-symmetric interval satisfying the property $f^n(\partial I^0) \cap I^0 = \emptyset$, $n = 1, 2, \dots$. Now given $I^{n-1} \equiv I_0^{n-1} \ni c$ by induction, let us consider the first return map $f_n : \cup I_j^n \to I^{n-1}$. Its domain of definition generally consists of infinitely many intervals $I_j^n \subset I^{n-1}$. However, for the Fibonacci map only two of them, $I^n \equiv I_0^n \ni c$ (the "central" one) and I_1^n intersect the critical set $\omega(c)$. Let us define g_n as the restriction of f_n to these two intervals. These maps satisfy the following properties:

(i) $g_n : I_1^n \to I_0^{n-1}$ is a diffeomorphism and $g_n(\partial I_0^n) \subset \partial I_0^{n-1}$;
(ii) $g_n I_0^n \supset I_0^n$ (*high return*);
(iii) $g_n c \in I_1^n$ and $g_n^2 c \in I_0^n$.

By rescaling I^n to some definite size T (e.g., $T = [0, 1]$), we obtain the generalized n-fold renormalization

$$R^n f : T_0^n \cup T_1^n \to T$$

of f. The asymptotic properties of the renormalized maps express the small scale information of the critical set $\omega(c)$.

Let us now introduce the principle geometric parameters, the scaling factors

$$\mu_n = \frac{|I^n|}{|I^{n-1}|} = \frac{|T^n|}{|T|}.$$

The main result of [LM] says that they decrease to 0 exponentially at the following rate:

$$\mu_n \sim a \left(\frac{1}{2}\right)^{n/3}. \qquad (1)$$

It follows by the Koebe principle that up to an exponentially small error the restriction of $R^n f$ to the central interval T_0^n is purely quadratic, while the restriction to T_1^n is linear. This all we need to know for the comprehensive study of f.

Asymptotically conformal continuation. Let us represent f as $h \circ \phi$ where $\phi(z) = (z - c)^2$ is the quadratic map, while h is a C^2-diffeomorphism of appropriate intervals. Let us continue h to a diffeomorphism of a bounded C^2 norm on the whole real line, and then consider the Ahlfors-Beurling continuation of h to the complex plane:

$$\hat{h}(x + iy) = \frac{1}{2y} \int_{x-y}^{x+y} h(t)dt + \frac{1}{y}\left(\int_x^{x+y} h(t)dt - \int_{x-y}^{x} h(t)dt\right).$$

This is clearly a C^2-map, and one can check by calculation that $\bar{\partial}\hat{h} = 0$ on the real line. Hence $\bar{\partial}\hat{h}/\partial\hat{h} = O(|y|)$ as $|y| \to 0$. This provides us with a C^2 extension of f which is asymptotically conformal on the real line in the sense that

$$\mu(z) \equiv \bar{\partial}\hat{f}/\partial\hat{f} = O(|y|) \tag{2}$$

as well. In what follows we denote the extended h and f by the same letters.

Complex pull-back. Given an interval $I \subset \mathbf{R}$ and $\theta \in (0, \pi/2)$, let $D_\theta(I)$ denote the domain bounded by the union of two $\mathbf{R}$-symmetric arcs of the circles which touch the real line at angle θ. In particular, $D_{\pi/2}(I) \equiv D(I)$ is the Euclidean disk with diameter I. Observe that I is a hyperbolic geodesic in the domain $\mathbf{C}\setminus(\mathbf{R}\setminus I)$ and $D_\theta(I)$ is its hyperbolic neighborhoods of radius depending only on θ.

We say than an interval $\tilde{I}$ is obtained from the I by α-scaling if these intervals are cocentric and $|\tilde{I}| = (1+\alpha)|I|$.

Lemma 1. *Let $\alpha < 1$, n be sufficiently big. Let us consider the α-scaled interval $\tilde{I}^n \supset I^n$. Let $\Delta = D(\tilde{I}^n)$, and Δ' be the pull-back of Δ by $g_{n+1}|I^{n+1}$. Then $\Delta' \subset D(\tilde{I}^{n+1})$ where $\tilde{I}^{n+1}$ is obtained from I^{n+1} by β-scaling with $\beta = \alpha + O(\mu_n)$.*

Proof. Let us skip the index n in the notations of objects of level n, while mark the objects of level $n+1$ with prime. Set $g|I' = f^p$, and let us consider the pull back $I, I_{-1}, ..., I_{-p} \equiv I'$ of I along the orbit $\{f^k c\}_{k=0}^p$. Then

$$\sum_{k=0}^{p} |I_{-k}| = O(\mu). \tag{3}$$

Since the map $f^k : I_{-k} \to I$ has the Koebe space covering I^{n-1}, the pull-back $\tilde{I}_{-k}$ of $\tilde{I}$ along the same orbit also has the total length $O(\mu)$.

Let us now take the disk Δ and pull it back along the same orbit. We obtain a sequence of pieces Δ_{-k} based upon the intervals $\tilde{I}_{-k}$. Assume by induction that $\Delta_{-l} \subset D_{\theta(k)}(\tilde{I}_{-l})$, $l = 0, ..., k < p$, with

$$\theta_l = \alpha + O(\sum_{j=0}^{l-1} |\tilde{I}_{-j}|). \tag{4}$$

Represent f as $h \circ \phi$ and carry out the next pull back in two steps: first by the diffeomorphism h and then by the quadratic map ϕ. Let $h^{-1}\tilde{I}_{-k} = L_{-k}$. If we rescale the intervals $\tilde{I}_{-k}$ and L_{-k} to the unit size, the C^1-distance from the rescaled map $H^{-1} : [0,1] \to [0,1]$ to id is $O(|I_{-k}|)$. It follows that

$$h^{-1}\Delta_{-k} \subset D_{\theta(k+1)}(L_{-k}) \tag{5}$$

with $\theta(k+1)$ as in (4).

Consider now two cases. Let first $k < p-1$. Then $\phi : L_{-k} \to \tilde{I}_{-(k+1)}$ is a diffeomorphism and by the Schwarz lemma (see the above hyperbolic interpretation of the $D_\theta(I)$)

$$\Delta_{-(k+1)} \subset D_{\theta(k+1)}(\tilde{I}_{-(k+1)}).$$

Let us now carry out the last pull-back corresponding to $k = p-1$. Then $\phi|I_{-(k+1)} = \phi|I'$ is the quadratic folding map into $L \equiv L_{-(p-1)}$. Moreover, what is important is that

$\phi I'$ covers at most half (up to an error of order $O(\mu)$) of the interval L (It follows from the high return property of g and the estimate of its non-linearity). Hence we can find an interval $K \supset L$ centered at the critical value gc such that

$$D_{\theta(p-1)}(L) \subset D(K)$$

and

$$|K| = 2|\phi I'|(1 + O(\mu)).$$

Two last equations together with (4) yield the required. □

Let us now take the Euclidean disk $\Delta = D(I^m)$ and pull it back by the maps g_n continued to the complex plane. Denote the corresponding domains by Δ_0^n and Δ_1^n, $n > m$.

Corollary 2. *If m is sufficiently big then the diam Δ_j^n is commensurable with the diam I_j^n.*

Proof. Applying the previous lemma $n-m$ times, we see that diam Δ_j^n is $|I_j^n|(1 + O(\sum_{k=m}^n \mu_k)$. Since μ_k decay exponentially, we are done. □

§3. Thurston's transformation and the shape of the complex puzzle-pieces. Let us consider the quadratic map $p : z \mapsto z^2 - 1$ and mark on $\mathbf{C}$ a set A of three points $-1, 0,$ and $a = (1 + \sqrt{5})/2$. The first two form a cycle, while the last one is fixed. Taking a conformal structure ν on the thrice punctured plane $S = \mathbf{C} \setminus A$, we can pull it back by p. This induces a "Thurston's transformation" L of the Teichmüller space T_S of thrice punctured planes into itself (compare [MT] or [DH]). The main property of L is that it strictly contracts the Teichmüller metric, and hence all trajectories $L^n\tau$ exponentially converge to the single fixed point $\tau_0 \in T_S$ represented by the standard conformal structure.

Let us consider the involution $\rho : T_S \to T_S$ induced by the reflection of the conformal structure about the real line. This involution commutes with L, and so the subspace T_S^* of $\mathbf{R}$-symmetric structures is L-invariant. This subspace can be identified with the set of triples on the real line up to affine transformations. We can normalize the triples, say, as follows: $\{\gamma, 0, a,\}$ $\gamma < 0$. To pull back such a triple, we should take the quadratic polynomial p_γ which fixes a and carries 0 to γ, and take the negative preimage of γ.

Let us rescale both intervals I^n and I^{n-1} to the size $T = [-a, a]$ with a as above. Let $G_n : T \to T$ be the rescaled $g_n : I^n \to I^{n-1}$ (observe that this is a non-dynamical procedure, compare [KP]). Let us select the orientation in such a way that 0 is the minimum point of G_n.

Lemma 3. *The maps G_n converge to the polynomial $p(z) = z^2 - 1$ in C^1-norm on the compact subsets of $\mathbf{C}$.*

Proof. If we pull back the Euclidean disk $\Delta = D(I^n)$, we obtain a sequence of puzzle-pieces whose diameter is commensurable with their traces on the real line (Corollary 2). By the Denjoy distortion argument,

$$Dh_n^{-1}(z) = Dh_n^{-1}(0)(1 + O(\sqrt{\mu_n})), \quad z \in \Delta,$$

so that h_n^{-1} in Δ is an exponentially small perturbation of a linear map. Rescaling, we conclude that $G_n = H_n \circ p_{\gamma(n)}$ where H_n are diffeomorpisms converging exponentially to id in C^1 on compact sets, and $p_{\gamma(n)}$ are quadratic polynomials introduced above.

Let us now consider a sequence $\tau_n \in T_S^*$ represented by triples $(G_n(0), 0, a)$. It was shown in [LM] that $|G_n(0)|/a$ stays away from 0 and 1. Hence $\tau_{n+1} = L \circ Q_n(\tau_n)$ where L is the Thurston transformation, while Q_n is exponentially close to id in the Teichmüller metric. Since L is strictly contracting, τ_n must converge to its fixed point τ_0.

We conclude that $G_n(0) \to -1$, hence $p_{\gamma(n)} \to p$ and $G_n \to p$. □

Let us consider the following topology on the space $\mathcal{K}$ of connected compact subsets K of $\mathbf{C}$. Let $\psi_K : \{z : |z| > 1\} \to \mathbf{C} \setminus K$ be the Riemann map normalized at ∞ by $\psi(z) \sim qz$ with $q > 0$. Then the topology on $\mathcal{K}$ is induced by the compact open topology on the space of univalent functions.

Let us now consider the complex pieces Δ^n based upon the intervals I^n. Here Δ^n is the g_n-pull-back of Δ^{n-1}. Rescaling of I^n to T leads to the corresponding rescaled pieces P_n.

Lemma 4. *The pieces P_n converge to the filled-in Julia set of $p(z) = z^2 - 1$.*

Proof. The piece P_n is the G_n-pull-back of P_{n-1}. By Lemma 1, $\operatorname{diam} P_n$ is bounded. Hence $G_n | P_n$ is an exponentially small perturbation of p which yields the desired. □

§**4. Qc conjugacy on the critical sets.** Let us consider the complex renormalizations of f,

$$F_n = R^n f : V_0^n \cup V_1^n \to P^n,$$

where V_i^n are the rescaled puzzle-pieces based upon the intervals T_i^n. We use the same letters for the complex extensions of different maps. In particular, let $G_n : P^n \to P^{n-1}$ is the rescaled $g_n : \Delta^n \to \Delta^{n-1}$ (see Figure 2).

Let us parametrize smoothly the boundary of the piece P^0, $\gamma : \mathbf{T} \to \partial P^0$. This parametrization can be naturally lifted to the parametrization $\gamma_1 : \mathbf{T} \to \partial P^1$, namely $G_1 \circ \gamma_1 = \gamma(z^2)$, then to the parametrization of ∂P^2 etc. We refer to these parametrizations as to the boundary markings.

Let us also consider another Fibonacci map $\tilde{f}$ whose data will be labeled by tilde. The *Teichmüller distance* between two marked puzzle-pieces is the best dilatation of qc maps between the pieces respecting the boundary marking.

Lemma 5. *The marked puzzle-pieces P^n and $\tilde{P}^n$ stay bounded Teichmüller distance apart.*

Proof. Let we have a K-qc map $H_{n-1} : P^{n-1} \to \tilde{P}^{n-1}$ of marked pieces respecting the positions of the critical points and the critical values, that is, $H_n(0) = 0$ and $H_n(\gamma_n) = \tilde{\gamma}_n$. It can be lifted to the $K(1 + O(\mu_n))$-qc map $h_n : P^n \to \tilde{P}^n$. This map respects boundary marking and 0-points but it does not respect γ-points. However, it respects these points up to exponentially small error, namely $h_n(\gamma_n$ and $\tilde{\gamma}_n$ are exponentially close.

Indeed, let $q_n \in T_1^n$ be the G_n-preimage of 0. As the length of T_n is exponentially small, the points q_n and γ_{n+1} are exponentially close. Moreover, by Lemma 4 the distance from these points to the boundary ∂P^n is bounded from below. By the Hölder continuity of qc maps we conclude that $(h_n(q_n)$ and $h_n(\gamma_n)$ are also exponentially close. As $h_n(q_n) = \tilde{q}_n$, the points $h_n(\gamma_n)$ and $\tilde{\gamma}_n$ are exponentially close as well.

As the distance from these points to the boundary $\partial \tilde{P}^n$ and from 0 is bounded from below, they are exponentially close with respect to the Poincaré metric of $\tilde{P}^n$. Hence there is a diffeomorphism $\psi : \tilde{P}^n \to \tilde{P}^n$ with exp small dilatation keeping $\partial \tilde{P}^n$ and 0 fixed, and pushing $h_n(\gamma_n)$ to $\tilde{\gamma}_n$. Then $H_n = \psi \circ h_n$ is a $(K+$ exp small)-qc map between the marked puzzle-pieces P_n and $\tilde{P}_n$ respecting the positions of the critical points and the critical values.

Proceeding in a such a way we construct uniformly qc maps between P^n and $\tilde{P}^n$ on all levels (as the exponentially small addings to dilatation sum up to a finite value). □

Let us now consider the pairs of pants $Q^n = P^n \setminus (V_0^n \cup V_1^n)$ where $V_0^n \equiv P^{n+1}$ and V_1^n with naturally marked boundary.

Lemma 6. *The pairs of pants Q^n and $\tilde{Q}^n$ stay bounded Teicmüller distance apart.*

Proof. Let us consider a K-qc homeomorphism $H_{n-1} : Q^{n-1} \to \tilde{Q}^{n-1}$ of marked pairs of pants. It follows from the previous lemma that we can extend these maps across $V_j^{n_1}$. Indeed, the previous lemma provides us with the continuation to V_0^{n-1}. Moreover, it provides us with a map $P^{n-1} \to \tilde{P}^{n-1}$ which then can be pulled back to V_1^{n-1}. Let us keep the notation H_{n-1} for this extension.

Let us now consider the pull-back $W^{n-1} \subset V_1^{n-1}$ of V_0^{n-1} by F_{n-1}. Its boundary is also naturally marked. By one more pull-back of H_{n-1} we can reconstruct it in such a way that it will respect this marking. Let us consider the annulus $A^{n-1} = P^{n-1} \setminus W^{n-1}$ with marked boundary.

The annulus $L^n = P^n \setminus V_0^n$ double covers A^{n-1} under G_n. So we can pull H^{n-1} back to a K-qc map $H^n : L^n \to \tilde{L}^n$. Moreover, this map respects the parametrization of ∂V_1^n, and hence can be restricted to the K-qc map of marked pairs of pants of level n. □

Figure 3

We are prepared to obtain the desired result of this section.

Lemma 7. *There is an* $\mathbf{R}$*-symmetric qc map which conjugate* f *and* $\tilde{f}$ *on their critical sets.*

Proof. The critical set can be represented as

$$\omega(c) = \cap_{n=1}^{\infty} \cup Q_i^n,$$

where Q_i^n are dynamically constructed disjoint pairs of pants (see Figure 3). They are obtained by univalent pull-backs of appropriate central pairs of pants. As these pull-backs have bounded dilatations, Lemma 6 implies that Q_i^n stay on bounded Teichmüller distance from $\tilde{Q}_i^n$. Gluing together all these pairs of pants, we obtain the desired result. □

§**5. Pull-back argument.** Sullivan's pull-back argument allows to construct a qc conjugacy between two polynomial-like maps as long as there is a qc conjugacy on their critical sets. In this paper we deal with asymptotically conformal maps, so that we need the dilatation control of pull-backs. Lemma 1 will provide us with such a control along the real line. However, out of the real line the dilatation can grow, so that we should stop the construction at an appropriate moment. Let us show how it works. First we need some extra analysis on the real line.

Let $f_n : \cup I_j^n \to I^{n-1}$ be the full return map to the interval I^{n-1}.

Lemma 8. *Let* $I^n \equiv J_0, J_{-1}, \ldots$ *be any pull-back (finite or infinite) of the interval* I^n *Then*

$$\sum |J_{-k}| = O(\mu_n).$$

Proof. Denote by $\mathcal{J}$ the union of the intervals in the pull-back. Let us first assume that the intervals J_{-k} don't intersect I^n. Let $K_0 \equiv J_0, K_1, \ldots$ be the piece of the pull-back which belongs to I^{n-1}, $\mathcal{K} = \mathcal{J} \cap I^{n-1}$ be the union of these intervals. This is actually the pull-back under the map f_n. This map is expanding with bounded distortion on I_j^n (actually very strongly expanding and almost linear on I_j^n). Hence

$$\sum |K_j| = O(\mu_n). \tag{6}$$

Let us now consider all intervals L_i obtained by pulling I^{n-1} back which are maximal in the sense that they don't belong to another pull-back interval. In other words, there is an $m = m(i)$ such that $f^m L_i = I^{n-1}$ but $f^l L_i \cap I^{n-1} = \emptyset$. These intervals are mutually disjoint (and cover almost everything).

Let now $\mathcal{K}_i = \mathcal{J} \cap L_i$. Then $f^{m(i)}$ maps $\mathcal{K}_i$ with bounded distortion (actually almost linearly) onto $\mathcal{K}$. Hence $\operatorname{dens}(\mathcal{K}_i | L_i) = O(\mu_n)$. Summing up over i we get the claim.

Assume now that there are intervals in I^n but there are no ones in I^{n+1}. Let J_{-l} be the first interval belonging to I^n. Then for the further pull-back we can repeat the same argument on level n instead of $n-1$ (taking into account that the Poincaré lengths of I_j^{n+1} in I^n are $O(\mu_n)$).

In general case let us divide the pull-back into the pieces $\mathcal{J}_l$ between the first landing at I^l and the first landing at I^{l+1}. Let us pull I^l along the corresponding piece. This pull-back does not intersect I^{l+1} either, and according to the previous considerations its total length is $O(\mu_l)$. All the more this is true for the total length of $\mathcal{J}_l$.

Hence the total length of $\mathcal{J}$ is $O(\sum_{l\geq n}\mu_l)=O(\mu_n)$. □

Let us now state the complex version of the above lemma.

Lemma 9. *Let $\Omega=D(I^n),\Omega_{-1},\ldots$ be any pull-back of the disk Ω along the real line. Then*

$$\sum \text{diam}\Omega_{-n}=O(\mu_n).$$

Proof. Let $\mathcal{W}$ denote the union of the disks in this pull-back. As in the above argument, let us decompose it into the strings $\mathcal{W}_j$ in between levels j and $j+1$. Let Ω^j be the first puzzle-piece in the j th string.

On the other hand, let Δ^j denote the pull-backs of Ω based upon the intervals I^j. Then by the Markov property of the whole family of pull-backs, $\Omega^j\subset\Delta^j$. Hence the pull-back $\mathcal{W}_j$ can be inscribed into the corresponding pull-back of $\mathcal{D}_j$ of the puzzle-piece Δ^j.

It follows from Lemma 1 that the sum of the diameters of pieces in $\mathcal{D}_j$ is commensurable with the total length of its trace on the real line. By the previous lemma, the latter is $O(\mu_n)$, and we are done. □

Let us now select a high level n and consider the complex renormalization $F_n: V_0^n\cup V_1^n\to P^n$. Let us re-denote all these objects as $F:U_0^1\cup U_1^1\to U^0$. As above, the corresponding objects for another Fibonacci map $\tilde{f}$ will be labeled with the tilde. The following statement shows that two renormalizations of sufficiently high order are qc-conjugate.

Proposition 10. *There is a qc map $U^0\to\tilde{U}^0$ which conjugate F and $\tilde{F}$ on the real line.*

Proof. By Lemma 7, there is a qc map $h_0:U^0\to\tilde{U}^0$ which conjugate F to $\tilde{F}$ on the critical sets and on the $\partial(U_0^1\cup U_1^1)$. Let us start to pull it back.

Let U_j^n denote the family of puzzle-pieces of depth n (that is, the components of $F^{-n}U^0$) *which meet the real line.* Let us assume by induction that we have already constructed a qc map $h_n:U^0\to\tilde{U}^0$ which conjugate F to $\tilde{F}$ on their critical sets and on $(U_0^1\cup U_1^1)\smallsetminus\text{int}(\cup U_j^n)$. Then construct h_{n+1} as the lift of h_n to all puzzle-pieces U_j^n.

Since the puzzle-pieces U_j^n shrink to points, the sequence h_n has the continuous pointwise limit h which conjugate F and $\tilde{F}$ on the real line. Moreover, by (2) and Lemma 9, the h_n has uniformly bounded dilatations. Hence h is qc. □

Let us re-denote I^n by $J\equiv J^0$, and let $\Delta=D(J)$. Let us now consider the full first return map f_1 to Δ. Its domain intersects the real line by the union of intervals $J_j^1\equiv I_j^{n+1}$. Let Δ_j^1 be the pull-back of Δ intersecting the real line by I_j^{n+1}, $\mathcal{D}^1=\cup\Delta_j^1$ (see Figure 4).

The goal of the next three lemmas is to construct a qc map $h:\Delta\to\tilde{\Delta}$ which conjugate $f_1|\partial\mathcal{D}$ to $\tilde{f}_1|\partial\tilde{\mathcal{D}}$ (as well as $f_1|\omega(c)$ to $\tilde{f}_1|\omega(\tilde{c})$). This will be the starting data for the pull-back argument. The problem is that the boundary $\partial\mathcal{D}$ is not piecewise-smooth.

Given a set U, denote by U^+ the intersection of U with the upper half-plane.

Lemma 11. *The topological discs* Δ^1_j *are pairwise disjoint. The set* $W = (\Delta \smallsetminus \mathcal{D})^+$ *is a quasi-disk.*

Proof. The map $f_n : \Delta^1_j \to \Delta$ has exponentially small non-linearity. Hence Δ^1_j is a minor distorted round disk. On the other hand, the intervals J^1_i and J^1_j are exponentially small as compared with the gap G_{ij} in between. It follows that the disks Δ^1_i and Δ^1_j are disjoint.

Let $\Gamma = \partial W$. It follows from the previous discussion that this curve is rectifiable. Take two close points $z, \zeta \in \Gamma$. Let δ be the shortest path connecting z and ζ in $\Gamma \cup \mathbf{R}$ (it is "typically" the union of an interval of the real line and two almost circle arcs), and γ be the shortest arc in Γ connecting z and ζ. Then the length of δ is commensurable with both the length of γ and the dist (z, ζ). □

Figure 4

For the further discussion it is convenient to make a more special choice of the interval J (compare [GJ], [Y], [JS]). Namely, let α be the fixed point of f with negative multiplier $\sigma \equiv f'(\alpha)$. Let $\mathcal{Y}^{(0)}$ be the partition of T by α into two intervals. Pulling this partition back, we obtain partitions $\mathcal{Y}^{(n)}$ by n-fold preimages of α. Let us call the elements of this partition *the puzzle-pieces of depth* n. The element containing c is called *critical.* We select $J = [\beta, \beta']$ as the critical puzzle-piece of sufficiently high depth N.

Set $\tau = \log|\tilde{\sigma}|/\log|\sigma|$.

Let us now start with a qc $\mathbf{R}$-symmetric map $H : \Delta \to \tilde{\Delta}$ which carries the critical set of f_1 to the critical set of $\tilde{f}_1$ and such that

$$|H(z) - \tilde{\beta}| \asymp |z - \beta|^\tau. \tag{7}$$

Moreover, let H commutes with the symmetry around c induced by f and $\tilde{f}$.

Pull H back to a map $h : \mathcal{D} \to \tilde{\mathcal{D}}$. Since the union $\cup J^1_j$ is dense in J, this map can be continued to a homeomorphism $h : J \to \tilde{J}$. Let also $h|\partial\Delta = H$. This defines h on the topological semi-circle $S = \partial\Delta^+$. Since S and $\tilde{S}$ are piecewise smooth curves, we can naturally define the notion of a quasi-symmetric map between them.

Lemma 12. *The map* $h : S \to \tilde{S}$ *is quasi-symmetric.*

Proof. Let us consider a continuation $H : T \to \tilde{T}$ of $H : J \to \tilde{J}$ which carries the puzzle-pieces of depth N to the corresponding puzzle-pieces, and has the asymptotics (7) near the boundary points of these puzzle-pieces.

Let K be the expanding Cantor set of points which never land at J. Each component L of $T \setminus K$ (a "gap") is a monotone pull-back of J with bounded distortion. So we can pull the map H back to qs maps on all gaps L. These maps clearly glue together to a homeomorphism $\phi : T \to \tilde{T}$ which respect the dynamics on the Cantor sets K and $\tilde{K}$. Moreover, if we rescale the corresponding gaps L and $\tilde{L}$ to the unit size then the rescaled ϕ near the boundary points will have asymptotics (7) uniformly in L.

Furthermore, it easily follows from the bounded distortion properties of expanding dynamics that $\phi|K$ can be extended to a qs conjugacy ψ in a neighborhood of K. This conjugacy must have the same asymtotics (7) on the rescaled gaps (since the conjugacy near the fixed points have such asymptotics). It follows that ϕ and ψ are commensurable on the gaps, and hence ϕ is qs on the whole interval.

Observe now that $h : J \to \tilde{J}$ is the pull back of ϕ by the almost quadratic maps $f|J$ and $\tilde{f}|\tilde{J}$. Hence $h|J$ is qs and has asymptotics (7) near the boundary. Since it has the same asymptotics on the opposite side of β, β' on the arc $S \setminus J$, it is qs on S. □

Lemma 13. *The map* $h : \partial W \to \partial \tilde{W}$ *allows a qc extension to* $W \to \tilde{W}$.

Proof. Let E be the exterior component of $\mathbf{C} \setminus S$. By the previous lemma, there is a qc extension of h from S to $h_0 : E \to \tilde{E}$ (which change the original values of h below the real line).

We can now glue $h : \mathcal{D}^+ \to \tilde{\mathcal{D}}^+$ with h_0 to a qc map $h_* : \mathbf{C} \setminus W \to \tilde{\mathbf{C}} \setminus W$ (since they agree on the real line). Since W is a quasi-disk (by Lemma 11), h_* can be reflected to the interior of W, and this is a desired extension. □

Corollary 14. *There is an* $\mathbf{R}$*-symmetric qc map* $h : \Delta \to \tilde{\Delta}$ *which conjugates* f_1 *to* $\tilde{f}_1$ *on the critical sets and on the boundary of* $\mathcal{D}$.

Proof. Lemma 13 gives us a desired qc extension of the original h from $\mathcal{D} \cup \partial\Delta$ to Δ. □

Now we are ready to prove the main result.

Theorem I. *Any two Fibonacci quasi-quadratic maps are qc conjugate.*

Proof. Starting with the qc map h given by Corollary 14, we can go through the pull-back argument in the same way as in Proposition 10. This provides us with a qs conjugacy between the return maps f_1 and $\tilde{f}_1$. Then we can spread it around the whole interval T as in the proof of Lemma 12. □

§**6. Teichmüller metric.** Let K_h denote the dilatation of a qc map h. Given two Fibonacci maps f and g and the qs conjugacy between them, the Teichmüller pseudo-distance $\text{dist}_T(f, g)$ is defined as the infimum of $\log K_h$ for all qc extensions of h.

Theorem II. *If* dist $_T(f,g)=0$ *then* f *and* g *are smoothly conjugate.*

Proof. Our first step is the same as Sullivan's [S1]: If dist $_T(f,g)=0$ then the multipliers of the corresponding periodic orbits of the maps are equal. However, as we don't have yet a proper thermodynamical formalism for unimodal maps, we will proceed by a concrete geometric analysis.

The next observation is that the parameter a in (1) must be the same for f and g. Indeed, it can be explicitly expressed via the multipliers of the fixed points of the return maps $g_n: I^n \to I^{n-1}$ (since the g_n are asymptotically quadratic). By [LM] this already yields the smoothness of the conjugacy on the critical sets.

Let us now take a point $x \in I^n \setminus I^{n-1}$ and push it forward by iterates of g_n till the first moment it lands in I^n (if any), then apply the iterates of g_{n+1} till the first moment it lands in I^{n+1}, etc. This provides us with a nested sequence of intervals around x whose lengths can be expressed (up to a bounded error) through the scaling factors and the multipliers of appropriate periodic points (by shadowing). This implies that h is Lipschitz continuous. Moreover, when we approach the critical point, then the errors in the above argument exponentially decrease. Hence h is smooth at the critical point.

Given now any pair of intervals $I \supset J$, let us show that

$$\left|\frac{|hJ|}{|J|} : \frac{|hI|}{|I|} - 1\right| = O(|I|). \tag{8}$$

This is enough to prove locally at any point a. By the previous considerations, this is true at the critical point. Since the critical set $\omega(c)$ is minimal, this is also true for any $a \in \omega(c)$.

Let now $a \notin \omega(c)$, and I be a tiny interval around I. Remark that almost all ponts $x \in I$ eventually return back to I. Let us take the pull-back of I corresponding to this return. This provides us with the covering of almost all of I by intervals L_k. The distortion of the return map g is $O(|I|)$ on the all $L_k's$. Let σ_k be the multiplier of the g-fixed point in L_k. Then we conclude that

$$\left|\frac{|I|}{|L_k|} : \sigma_k - 1\right| = O(|I|), \tag{9}$$

and the analogous estimate holds for the second map. Since the corresponding multipliers of these maps are equal, we obtain (8) with $J = L_k$. Repeating now this procedure for returns of higher order, we obtain an arbitrarily fine covering of almost the whole of I by intervals for which (8) hold. This implies (8) for any $J \subset I$.

Let $\epsilon_n = 1/2^n$, and let us consider the sequence of functions

$$\rho_n(x) = \frac{h(x+\epsilon_n) - h(x-\epsilon_n)}{2\epsilon_n}.$$

According to (8) and Lipschitz continuity

$$|\rho_n(x) - \rho_{n+1}(x)| = O(\epsilon_n) \tag{10}$$

uniformly in x. Hence the ρ_n uniformly converge to the derivative of h.

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