Distortion results and invariant Cantor sets of unimodal maps

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Abstract. A distortion theory is developed for $S-$unimodal maps. It will be used to get some geometric understanding of invariant Cantor sets. In particular attracting Cantor sets turn out to have Lebesgue measure zero. Furthermore the ergodic behavior of $S-$unimodal maps is classified according to a distortion property, called the Markov-property.

1. Introduction

The work presented here originated in the question whether or not attracting Cantor sets of unimodal maps have Lebesgue measure zero. This question led to a general $S-$unimodal distortion theory. As applications of this theory we got uniform proofs of the basic known ergodic properties: ergodicity, conservativity, existence of attractors. But also an answer to the original question.

Theorem A. *Cantor attractors of $S-$unimodal maps have Lebesgue measure zero.*

The strategy for studying invariant Cantor sets is constructing open, arbitrarily fine and nested covers of them. These covers are constructed in such a way that an invariance property appears: except for the component containing the critical point every component is mapped monotonically onto its image which is also a component of the cover. Finally all components are transported to the central one, that is the one containing the critical point. The main question to be answered is whether this transport has good distortion properties.

In fact the covers of the Cantor sets are part of covers of the almost the whole interval, having the same invariance property. $S-$unimodal maps having arbitrarily fine covers with uniform good distortion properties are said to have the Weak-Markov-property.

THEOREM B. *Every $S-$unimodal map not having periodic attractors has the Weak-Markov-Property.*

The tools developed for proving Theorem B are used to proof Theorem A. The ergodicity of $S-$unimodal maps is a direct consequence of the Weak-Markov-Property. For understanding stronger ergodic properties we need a stronger distortion property.
Using the Weak-Markov-Property we see that smaller and smaller intervals are transported with uniform bounded distortion to smaller and smaller central intervals. For getting stronger ergodic properties we need to find almost everywhere smaller and smaller intervals transported with uniform bounded distortion to a fixed big interval. Maps having this distortion property are said to have the Markov-Property.
Using this Markov-Property all basic ergodic properties can be proved in a uniform way: conservativity, existence of attractors, etc.
The main application of the Markov-Property is an ergodic classification of $S-$unimodal maps. In particular it can be used to classify the maps having an attracting Cantor set in the sense of Milnor ([Mi]).

THEOREM C. *An $S-$unimodal map not having a periodic attractor has the Markov-property if and only if it doesn't have a Cantor attractor.*

An appendix is added in which the basic notions of $S-$unimodal dynamics are defined.
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2. Covers and induced maps

The analytical properties of the construction presented here are based on the following fundamental Lemma. Its proof can be found in different places, for example [MMS].

LEMMA 2.1 (KOEBE-LEMMA). *Let M, T be intervals in $[0,1]$ with $M \subset T$. The components of $T \backslash M$ are denoted by L and R. For every $\epsilon > 0$ there exist $\delta > 0$ and $K > 0$*

such that the following holds. Let $f : [0,1] \to [0,1]$ *be a map with negative Schwarzian derivative. If* $f^n|T$

is monotone and

$$|f^n(L)| \geq \epsilon |f^n(M)| \text{ and } |f^n(R)| \geq \epsilon |f^n(M)|$$

then

1) $\frac{|Df^n(x)|}{|Df^n(y)|} \leq K$ *for* $x, y \in M$ *(Koebe-Lemma);*

2) T *contains a* $\delta-$*scaled neighborhood of* M *(Macroscopic-Koebe-Lemma).*

Corollary 2.2. *For every* $\rho > 0$ *there exists* $\delta > 0$ *with the following property. Let* $\{M_i | i \geq 1\}$ *be a pairwise disjoint collection of subintervals of the interval* T*. For* $i \geq 1$ *let* L_i *and* R_i *be such that*

1) L_i *is a component of* $T - \cup_{j \geq 1} M_j$ *next to* M_i*;*

2) R_i *is a component of* $T - M_i$ *next to* M_i*;*

3) $L_i \cap R_i = \emptyset$*;*

4) $|L_i| \geq \rho |M_i|$ *and* $R_i| \geq \rho |M_i|$.

If $g : T' \to T$ *is monotone, onto and has* $Sg(x) < 0$ *for* $x \in T'$ *then for* $i \geq 1$

$$|L_i'| \geq \delta |M_i'|$$

where L_i' *and* M_i' *are the preimages under* g *of respectively* L_i *and* M_i.

The Koebe-Lemma assures bounded distortion if there are big extensions of monotonicity on both sides. The next step is to develop topological instruments for studying the maximal extensions.

In this section we will fix an $S-$unimodal map $f : [0,1] \to [0,1]$ with critical point c and we assume that f does not have periodic attractors.

For $x \in [0,1]$ denote the interval $(x, \tau(x))$ by V_x (the involution τ is defined in the appendix). Furthermore define

$$\mathcal{N} = \{x \in [0,c) | V_x \cap orb(x) = \emptyset\}.$$

The points in $\mathcal{N}$ are called *nice*. Observe that every periodic orbit contains nice points. Hence $\mathcal{N}$ is not empty. Moreover since the critical point is accumulated by periodic orbits $\mathcal{N}$ also accumulates on c. Clearly $\mathcal{N}$ is closed. Fix $x \in \mathcal{N}$.

LEMMA 2.3. *For $i = 1, 2$ let $T_i \subset [0,1]$ be two different intervals such that $f^{n_i} : T_i \to V_x$ is monotone and onto for some $n_1 \leq n_2$. If $T_1 \cap T_2 \neq \emptyset$ then $T_2 \subset T_1$ and $n_1 < n_2$.*

PROOF: Suppose that $\partial T_1 \cap T_2 \neq \emptyset$. Then $n_1 \neq n_2$. And $x \in f^{n_1}(T_2)$ which implies $f^{n_2 - n_1}(x) \in V_x$. Contradiction. ∎

The set of points who visit V_x is called $C_x \subset [0,1]$. Observe that it can be described as being the union of all intervals T such that $f^n : T \to V_x$ is monotone and onto for some $n \geq 0$. Furthermore let $\Lambda_x = [0,1] - C_x$ and $D_x = f^{-1}(C_x) \cap V_x$.

LEMMA 2.4. *Let $I \subset C_x$ be a component. Then there exists $n \geq 0$ such that $f^n : I \to V_x$ is monotone and onto. Furthermore $\{I, f(I), \ldots, f^n(I) = V_x\}$ is a pairwise disjoint collection. In particular there exists only one such $n \geq 0$.*

PROOF: Lemma 2.3 implies easily that for every component I of C_x there exists $n \geq 0$ such that $f^n : I \to V_x$ is monotone and onto. Observe that this number $n \geq 0$ is defined uniquely: $c \in f^n(I)$ which implies that $f^{n+s}|I$ is not monotone for $s > 0$.

To proof the disjointness of the orbit it suffices to proof that $f^j(I) \cap V_x = \emptyset$ for all $j < n$. Suppose $f^j(I) \cap V_x \neq \emptyset$ with $j < n$. By Lemma 2.3 we get $f^j(I) \subset V_x$.

Let $j < n$ be minimal such that $f^j(I) \subset V_x$. Let $H \subset [0,1]$ be the maximal interval containing I with $f^j|H$ is monotone and $f^j(H) \subset V_x$. Suppose that a component L of $H - I$ is mapped in V_x: $\overline{f^n(L)} \subset V_x$. Then by maximality there exists $s < j$ such that $c \in \partial f^s(L)$. The minimality of j implies $f^s(I) \cap V_x = \emptyset$. Hence $x \in f^s(L)$. Because $\overline{f^j(L)} \subset V_x$ this implies $orb(x) \cap V_x \neq \emptyset$. Contradiction.

So we conclude that $f^j : H \to V_x$ is monotone and onto. Furthermore H contains the component I of C_x. The definition of C_x implies that $H = I$. In particular $j = n$. Contradiction. ∎

Lemma 2.4 states the invariance property of the covers C_x discussed in the introduction.

It enables us to define the *Transfer map*

$$T_x : C_x \to V_x$$

and also the *Poincaré map*

$$R_x : D_x \to V_x$$

by $R_x = T_x \circ f$.

The next lemma shows that these induced maps are defined almost everywhere.

LEMMA 2.5. *If $x \in \mathcal{N}$ then*

1) $|C_x| = 1$ *(and* $|\Lambda_x| = 0$*);*

2) $|D_x| = |V_x|$.

Furthermore

3) Λ_x *is invariant;*

4) if $y \in \Lambda_x$ is such that $orb(y) \cap \overline{V}_x = \emptyset$ then Λ_x accumulates from both sides on y.

PROOF: Take $y \in \Lambda_x$ and assume $f(y) \notin \Lambda_x$, say $f(y) \in I \in C_x$. If $c_1 \notin I$ then $f^{-1}(I) \subset C_x$. Hence $c_1 \in I$. Because $x \in \mathcal{N}$ we get easily that $f^{-1}(I) \subset V_x$. This gives the contradiction $y \in V_x$. So Λ_x is invariant.

To prove the first two statements it suffices to proof $|\Lambda_x| = 0$. Because Λ_x is a closed 0-dimensional invariant set not containing the critical point a well-known lemma in [Mi] states that Λ_x has Lebesgue measure zero.

Take $y \in \Lambda_x$ with $orb(y) \cap \overline{V_x} = \emptyset$. Suppose that y is not accumulated from both sides by Λ_x. Hence y is a boundary point of a component of C_x. Lemma 2.4 states that the orbit of y passes through the boundary of V_x. Contradiction. ∎

As a consequence of the previous lemma the set

$$\mathcal{R} = \{x \in [0,1] | c \in \omega(x)\}$$

has Lebesgue measure 1.

The branches of the Transfer map are monotone. To apply the Koebe-Lemma we have to know how much the monotonicity can be extended.

Suppose $c_1 \in C_x$. So there exists a component $S_x \subset C_x$ with $c_1 \in S_x$. Let $\psi(x) = \partial f^{-1}(S_x) \cap [0, c)$ and define $U_x = V_{\psi(x)} = f^{-1}(S_x)$.

We get $S_x \subset (f(V_x), 1)$ because x is nice. Hence $U_x = V_{\psi(x)} \subset V_x$. Using lemma 2.4 we get $orb(f(\psi(x))) \cap V_x = \emptyset$. In particular $\psi(x) \in \mathcal{N}$.

To finish the definition of the increasing function $\psi : \mathcal{N} \to \mathcal{N}$ let $\psi(x) = x$ if $c_1 \neq C_x$.

The pair (V_x, U_x) is called a *transfer range*. If V_x contains a $\delta-$scaled neighborhood of U_x then the pair is called a $\delta-$transfer range.

PROPOSITION 2.6. *Let $x \in \mathcal{N}$ and $I \subset C_{\psi(x)}$ be a component, say $T_{\psi(x)}|I = f^n|I$. Then there exists an interval T_I containing I such that*

$$f^n : T_I \to V_x$$

is monotone and onto.

Combining the Koebe-Lemma with this proposition we get that the branches of $T_{\psi(x)} : C_{\psi(x)} \to U_x$ have uniformly bounded distortion. The bound just depends on the space in V_x around U_x.

A pair (T, I) of intervals with the property that for some $n \geq 0$, called the *transfer time*,

1) $f^n : T \to V_x$ is monotone and onto;

2) $f^n(I) = U_x$

is called *TI-pair* for (V_x, U_x). As in lemma 2.4 observe that the time $n \geq 0$ is defined uniquely. As we saw above every TI-pair for a certain transfer range has uniform bounded distortion on the middle part.

PROOF OF PROPOSITION 2.6: Let I be a component of $C_{\psi(x)}$, say $T_{\psi(x)}|I = f^n|I$, and let $H \subset [0, 1]$ be the maximal interval containing I with $f^n|H$ is monotone and $f^n(H) \subset V_x$. Suppose by contradiction that the component L of $H - I$ is mapped in V_x: $\overline{f^n(L)} \subset V_x$. Then by maximality there exists $j < n$ such that $c \in \partial f^j(L)$. We know from lemma 2.4 that $f^j(I) \cap U_x = \emptyset$. Hence $\psi(x) \in f^j(L)$. Because $\overline{f^n(L)} \subset V_x$ this implies $orb(\psi(x)) \cap V_x \neq \emptyset$. Contradiction. ∎

The intersection behavior of the TI-pairs is formulated in

LEMMA 2.7. *For $i = 1, 2$ let (T_i, I_i) be TI-pairs for (V_x, U_x) with transfer times respectively n_1 and n_2. If $T_2 \cap T_1 \neq \emptyset$ and $n_2 \geq n_1$ then*

$$T_2 \subset T_1 - I_1 \text{ or } T_2 \subset I_1$$

and $n_2 > n_1$.

PROOF: From lemma 2.3 we get $T_2 \subset T_1$ and $n_2 < n_1$. Suppose, by contradiction, $T_2 \cap \partial I_1 \neq \emptyset$. Then $\psi(x) \in f^{n_1}(T_2)$. Hence $f^{n_2 - n_1}(\psi(x)) \in V_x$. Contradiction. ∎

We finish with the definition of the last type of induced maps. Let E_x be the union of all intervals I which are part of a TI-pair (T_I, I) for (V_x, U_x) with *positive* transfer time. Again lemma 2.7 implies that the connected components of E_x are intervals which are part of a TI-pair for (V_x, U_x). This defines the *Markov* map

$$M_x : E_x \to U_x.$$

Directly after defining the first two induced maps we where able to show that they are defined almost everywhere. For Markov maps the situation is not that simple. Section 4 will deal with the question whether or not the Markov maps are defined almost everywhere. There we will show that Markov-maps are defined almost everywhere only in the case of absence of Cantor attractors.

3. The Weak-Markov-Property

In this section we are going to prove a general distortion result for $S-$unimodal maps, called the weak-Markov-property.

DEFINITION 3.1. *An $S-$unimodal map is said to satisfy the Weak-Markov-Property if there exist $K > 0$, a set $\mathcal{G}^w \subset [0, 1]$ with full Lebesgue measure and a set $\mathcal{D} \subset \mathcal{N}$ accumulating at the critical point such that for every $x \in \mathcal{G}^w$ and for every $y \in \mathcal{D}$ there exist $t \geq 0$ and an interval $I \ni x$ such that*

$$f^t : I \to V_y$$

is a diffeomorphism with distortion bounded by K.

THEOREM 3.2. *Let f be an unimodal map without periodic attractor. There exists $\delta > 0$ and $\mathcal{D} \subset \mathcal{N}$ accumulating at the critical point such that for every $x \in \mathcal{D}$ the following property holds.*
If I is a component of C_x, say $T_x|I = f^n|I$ then f^n maps the maximal interval containing I on which f^n is monotone over a $\delta-$scaled neighborhood of V_x.

The Koebe-lemma implies directly the main consequence of this Theorem:

THEOREM 3.3 (THE WEAK-MARKOV-PROPERTY). *Every $S-$unimodal map without periodic attractor satisfies the Weak-Markov-Property.*

During the proof we will see that the number $\delta > 0$ only depends on the power of the critical point and that the set $\mathcal{D}$ has a topological definition.
During the proof of Theorem 3.3 we will see that in general the monotone extensions of the branches $f^n : I \to U_x$ with $I \subset C_x$ will have images much bigger than the transfer range. So Theorem 3.3 does not learn us something about the geometry of transfer ranges. However in our study of the Lebesgue measure of the critical limit sets we need a better understanding of the geometry of transfer ranges. That is why we need the following stronger form of Theorem 3.3.

THEOREM 3.4. *Let f be an only finitely renormalizable unimodal map having a non-periodic recurrent critical point. Then there exists a sequence $\{(U_n, V_n)\}_{n\geq 0}$ of $\delta-$transfer ranges with $|V_n| \to 0$.*

In this section we are going to prove Theorem 3.2 and 3.4. Fix an $S-$unimodal map f whose critical point c is recurrent. Furthermore we assume f not to have periodic attractors.

PROPOSITION 3.5. *There exist $\delta, \rho > 0$ such that for $x \in \mathcal{N}$ the following holds.*

1) *Assume $c \in R_x(U_x)$. Let $I \subset C_{\psi(x)}$ be a component, say $T_{\psi(x)}|I = f^k|I$, and T the maximal interval containing I for which f^k is monotone. Then $f^k(T)$ contains a $\delta-$scaled neighborhood of U_x.*
2) *Assume $c \notin R_x(U_x)$ and $|V_x| \leq (1 + \rho)|U_x|$. If $T_x|S_x = f^n|S_x$ and if T is the maximal interval containing S_x such that $f^n|T$ is monotone then $f^n(T)$ contains a $\delta-$scaled*

neighborhood of $[R_x(U_x), c]$.

PROOF: Let $T_x|S_x = f^n|S_x$ and $M = f(U_x)$. We are going to study the orbit $\{M, f(M), ..., f^n(M)\}$. M is contained in S_x. Hence by lemma 2.4 this orbit consists of pairwise disjoint intervals. Choose $m \in \{0, 1, 2, ..., n\}$ such that $|f^m(M)| \leq |Df|_0^2 |f^j(M)|$ for all $j \leq n$ and $f^m(M)$ has neighbors on both sides. This means that both components of $[0, 1] - f^m(M)$ contains intervals of the form $f^i(M)$ with $i \leq n$. Observe that by taking m' in such a way that $f^{m'}(M)$ is the smallest one we can take $m \in \{m', m'+1, m'+2\}$ having the described property. Let $f^l(M)$ and $f^r(M)$ be the direct neighbors of $f^m(M)$. Let H be the maximal interval containing M for which $f^m|H$ is monotone and $f^m(H) \subset [f^l(M), f^r(M)]$. We claim

$$f^m(H) = [f^l(M), f^r(M)].$$

Fix a component L of $H - M$ and assume that $\overline{f^m(L)} \subset [f^l(M), f^r(M)]$. by maximality there exists $j \leq n$ with $c \in \partial f^j(L)$. Because $f^j(M) \cap U_x = \emptyset$ we see that $f^{m-j-1}(M) \subset \overline{f^m(L)} \subset [f^l(M), f^r(M)]$. But $m - j - 1 \leq m - 1$. Hence $[f^l(M), f^r(M)]$ contains at least four intervals of the form $f^i(M)$, $i \leq n$. This contradiction finish the proof of the claim.

From the definition of m, l, r and the Macroscopic-Koebe-lemma we get a universal constant $\delta_1 > 0$ (only depending on $|Df|_0$) such that H contains a δ_1-scaled neighborhood of M. Because the critical point of f is non-flat there exists a universal constant $\delta_2 > 0$ such that

$$H' = f^{-1}(H) \text{ contains a } \delta_2 - \text{scaled neighborhood of } U_x.$$

PROOF OF STATEMENT 1: Assume that $c \in f^n(M)$. Take a component I of $D_{\psi(x)}$, say with transfer time $k \geq 0$. Denote by T the maximal interval containing I on which f^k is monotone and $f^k(T) \subset H'$. We claim $f^k(T) = H'$ which proves statement 1.

To prove this, fix a component L of $T - I$ and suppose that $\overline{f^k(L)} \subset H'$. From the maximality of T we get $j < k$ with $c \in \partial f^j(L)$. But $f^j(I) \cap U_x = \emptyset$. Therefore $f^k(L)$ contains $f^{k-j-1}(M)$ in its closure which is contained in H'. Because f^{m+1} is monotone on the component of $H' - \{c\}$ which contains $f^k(L)$, the iterate $f^{m+1+k-j-1}$ maps the closure of M monotonically into the interior of $[f^l(M), f^r(M)]$. In particular, because $c \in f^n(M)$, we have $k+m-j \le n$. Again this implies that $[f^l(M), f^r(M)]$ contains at least four intervals of the form $f^i(M), i \le n$. Contradiction.

PROOF OF STATEMENT 2: Let T be the maximal interval containing S_x on which f^n is monotone. The components of $T - M$ are denoted by L, R, say $c \in f^n(R)$. Because $f^n(S_x) = V_x$ and $c \notin f^n(M)$ the interval $f^n(R)$ contains a component of $V_x - \{c\}$. So using the non-flatness of the critical point we get a constant $\delta_3 > 0$ such that

$$|f^n(R)| \ge \delta_3 |[f^n(M), c]|.$$

So we only have to study $f^n(L)$. We claim

$$L' \subset f^n(L)$$

where L' is the component of $H' - U_x$ which lies on the same side of c as $f^n(L)$. Once this claim is proved statement 2 follows by taking $\rho > 0$ sufficiently small.

Indeed, suppose by contradiction $\overline{f^n(L)} \subset L'$. Again the maximality of T assures the existence of $j < n$ with $c \in \partial f^j(L)$. Observe $f^j(M) \cap V_x = \emptyset$. So we get $f^{n-j-1}(M) \subset \overline{f^n(L)} \subset L'$. Consider the interval $[f^j(M), c]$ (if $f^j(M)$ and $f^{n-j-1}(M)$ are on the same side of c. Otherwise consider $\tau([f^j(M), c])$.

The map $f^{n-j} : [f^j(M), c] \to [f^n(M), f^{n-j-1}(M)]$ is monotone and onto. Because f doesn't have a periodic attractor we get

$$f^j(M) \subset (f^{n-j-1}(M), f^n(M)).$$

Because $f^{n-j-1}(M) \subset L'$ and hence $f^{n-j-1+m+1}(M) \subset [f^l(M), f^r(M)]$ we get $n-j-1+m+1 > n$. Hence

$$j < m.$$

Furthermore $[f^{n-j-1}(M), c] \subset [L', c]$ and $f^{m+1}|[L', c]$ is monotone. Because $j < m$ the map $f^{j+1} : [f^{n-j-1}(M), c] \to [f^j(M), f^n(M)]$ is monotone and onto. Furthermore we have $[f^j(M), f^n(M)] \subset [f^{n-j-1}(M), c]$. Hence f has a periodic attractor. Contradiction. ∎

Shortly speaking Proposition 3.5 states that the central branch of the Poincaré map R_x has a quadratic shape.

Corollary 3.6. *Let f be an unimodal not having periodic attractors. There exist $\rho > 0$ and $K < \infty$ with the following properties. Let $x \in \mathcal{N}$ and suppose $T_x|S_x = f^n$. If $|V_x| \leq (1+\rho)|U_x|$ then*

1) For all $y_1, y_2 \in f(U_x)$

$$\frac{1}{K} \leq \frac{|Df^n(y_1)|}{|Df^n(y_2)|} \leq K;$$

2) For all $y \in U_x$

$$|Df^{n+1}(y)| \leq K.$$

proof: The first statement follows directly by applying the Koebe-lemma and the previous Proposition.

To prove the second statement we define a point $m \in \overline{U_x}$. Let $m = \psi(x)$ if $|Df^{n+1}(\psi(x))| \leq \frac{3}{1-\rho}$. If $|Df^{n+1}(\psi(x))| > \frac{3}{1-\rho}$ let m be the closest point in U_x to $\psi(x)$ such that $|Df^{n+1}(m)| = \frac{3}{1-\rho}$ (because $Df^{n+1}(c) = 0$ such an m exists).

First we are going to show

$$\frac{|[m,c]|}{|[\psi(x),c]|} \geq \frac{1}{3}.$$

We may assume $m \neq \psi(x)$. Because $f^{n+1}(U_x) \subset V_x$ we have

$$|V_x| \geq |f^{n+1}([\psi(x),m])| \geq \frac{3}{1-\rho}|[\psi(x),m]|.$$

Hence

$$\frac{|[m,c]|}{|[\psi(x),c]|} = \frac{|[\psi(x),c]| - |[\psi(x),m]|}{|[\psi(x),c]|} \geq$$

$$\geq 1 - 2\frac{1-\rho}{3}\frac{|V_x|}{|U_x|} \geq 1 - 2\frac{1-\rho}{3}\frac{1}{1-\rho} = \frac{1}{3}.$$

Because c is non-flat we get for all $y \in U_x$

$$|Df^{n+1}(y)| = \frac{|Df^{n+1}(y)|}{|Df^{n+1}(m)|}.|Df^{n+1}(m)| \leq \frac{|Df(y)|}{|Df(m)|}K\frac{3}{1-\rho} \leq$$

$$\leq B\frac{|x-c|^\alpha}{|m-c|^\alpha} \leq B\frac{|\psi(x)-c|^\alpha}{|m-c|^\alpha} \leq B3^\alpha = A$$

where α is the order of the critical point and B a positive constant depending on ρ, K and the behavior of f around the critical point. ∎

LEMMA 3.7. *There exists $\delta > 0$ such that for all $x \in \mathcal{N}$ with*

1) $c \notin R_x(U_x)$;

2) $R_x(c) \in V_x - U_x$

V_x *contains a* $\delta-$*scaled neighborhood of* U_x.

PROOF: Let $\delta > 0$ be given by proposition 3.5 and denote the part of the $\frac{\delta}{2}-$scaled neighborhood of $[R_x(U_x), c]$ which lies on the same side of c as $R_x(U_x)$ by H'. The pair of intervals $P \subset Q$ is such that

1) $f(U_x) \subset P$ and $P \subset S_x$;

2) $f^n(P) = [R_x(U_x), c]$;

3) $f^n : Q \to [H', c)$ is monotone and onto (where $R_x|U_x = f^{n+1}|U_x$);

Proposition 3.5 implies that such intervals exist. Let $|V_x| = (1+\rho)|U_x|$ and assume that ρ is small. Furthermore the Koebe-Lemma tells us that $f^n|Q$ has bounded distortion. We have to show that ρ is not too small. Observe that $|R_x(U_x)| = O(\rho)$. This follows from assumption 1) and 2).

Consider $U = f^{-1}(Q)$ as a neighborhood of U_x. Using the bounded distortion of $f^n|Q$ and the fact that the critical point of f is of order α it is easy to show that U is a $O(\rho^{-\frac{1}{\alpha}})-$scaled neighborhood of U_x. This bound implies that for ρ small $H' \subset U$.

Suppose that ρ is sufficiently small to assure that U will contain H'. Now let H be the component of $U - \{c\}$ containing $R_x(U_x)$. Then

$$f^{n+1} : H \to H' \subset H$$

is monotone. Hence f has a periodic attractor. Contradiction: ρ is away from zero. ∎

LEMMA 3.8. *There exist $\rho, \delta > 0$ such that for all $x \in \mathcal{N}$ with*

1) $|V_x| \leq (1+\rho)|U_x|$;

2) there exists $p \in U_x$ *with* $R_x(p) = p$ *and* $V_p \subset R_x(V_p)$

V_p *contains a* $\delta-$*scaled neighborhood of* U_p.

PROOF: Observe that this periodic point is nice: $p \in \mathcal{N}$. Let $M \subset V_p$ be the maximal interval with $R_x(M) \cap V_p = \emptyset$. Easily we get $U_p \subset M$. Let ρ be given by corollary 3.6.

Then we get a universal bound on $DR_x|U_x$: $|DR_x(y)| \leq A$ for all $y \in V_p$. This implies that both components of $V_p - M$ are bigger than $\frac{1}{A}|V_p|$. Hence

$$|U_p| \leq |M| \leq (1 - \frac{2}{A})|V_p|.$$

Let $\delta = \frac{1}{A}$. ∎

PROOF OF THEOREM 3.2: If there exists a neighborhood V of c such that $\overline{V} \cap \overline{orb\{c\}} = \emptyset$ then let $\mathcal{D} = \mathcal{N} \cap V$. The theorem follows from the fact that the boundary of the images of the maximal intervals of monotonicity consists of critical values, points in the orbit of c. Hence we may assume that the critical point is recurrent.

First we are going to define the sequence of closest approach to c. Let $c_n = f^n(c)$ and define

$$q(1) = 1;$$
$$q(n+1) = \min\{t \in \mathbf{N} | c_t \in V_{c_{q(n)}}\}.$$

Because $c \in \omega(c)$ and c is not periodic the sequence $\{q(n)\}_{n\geq 0}$ is well defined. Since c is an accumulation point of $\mathcal{N}$ there are infinitely many $n \geq 1$ for which $(V_{c_{q(n-1)}} - V_{c_{q(n)}}) \cap \mathcal{N} \neq \emptyset$. For those $n \geq 1$ define

$$x(n) = \sup\{(V_{c_{q(n-1)}} - V_{c_{q(n)}}) \cap \mathcal{N} \cap [0, c)\}.$$

Observe that the points $c_{q(n)}$ are not in $\mathcal{N}$: the supremum is in fact a maximum. Furthermore $x(n) \in (V_{c_{q(n-1)}} - V_{c_{q(n)}} \cap \mathcal{N})$. Hence for $i < q(n)$ $f^i(c) \notin V_{x(n)}$ and $f^{q(n)}(c) \in V_{c_{q(n)}} \subset V_{x(n)}$. In particular $R_{x(n)}|U_{x(n)} = f^{q(n)}$.

Let $\mathcal{D} = \{\psi(x(n))\}$. We distinguish two cases.

LOW CASE. $c \notin R_{x(n)}(U_{x(n)})$: Because f doesn't have periodic attractors and $c \notin R_{x(n)}(U_{x(n)})$ we get $\psi(x(n)) > x(n)$. Hence, by using the definition of $x(n)$ we get $\psi(x(n)) \in V_{c_{q(n)}}$ or $U_{x(n)} \subset V_{q(n)}$. So we can apply Lemma 3.7: $V_{x(n)}$ is a $\delta-$scaled neighborhood of $U_{x(n)}$. Use proposition 2.6, the fundamental property of transfer ranges, to finish the proof.

HIGH CASE. $c \in R_{x(n)}(U_{x(n)})$: The statement follows directly from proposition 3.5. ∎

PROOF OF THEOREM 3.4: The transfer ranges will be defined in terms of the points $y(n) \in \mathcal{N}$. These points are an adjustment of the points $x(n)$ from the proof of theorem 3.2 and will be defined below. Let $\delta > 0$ be the minimum of the numbers δ given by lemma 3.7 and 3.8. We distinguish two cases.

LOW CASE. $c \notin R_{x(n)}(U_{x(n)})$: The definition of $x(n)$ implies that $R_{x(n)}(c) = c_{q(n)} \notin U_{x(n)}$. Hence we can apply Lemma 3.7: $V_{x(n)}$ is a $\delta-$scaled neighborhood of $U_{x(n)}$. Let $y(n) = x(n)$.

HIGH CASE. $c \in R_{x(n)}(U_{x(n)})$: Let $\rho > 0$ be given by lemma 3.8. If $|V_{x(n)}| \geq (1+\rho)|U_{x(n)}|$ then let $y(n) = x(n)$. Assume that this metrical property doesn't hold. The map $R_{x(n)}|U_{x(n)}$ has a periodic point $p \in U_{x(n)}$. Because f is only finitely renormalizable we may assume (by taking n large enough) that $V_p \subset R_{x(n)}(V_p)$. Let $y(n) = p$ and apply lemma 3.8.

Because $c_{q(n)} \to c$ it follows that $y(n) \to c$. ∎

Let us finish this section with a simple consequence of the Weak-Markov-Property. It implies directly the ergodicity.

LEMMA 3.9. *Let f be a $S-$unimodal map without periodic attractor and $\mathcal{D} \subset [0,1]$ the set given by the Weak-Markov-Property (Theorem 3.3). If $X \subset [0,1]$ is an invariant set with positive Lebesgue measure then*

$$\lim_{\mathcal{D} \ni x \to c} \frac{|X \cap V_x|}{|V_x|} = 1.$$

PROOF: Using lemma 2.5(1) and $|X| > 0$ we get a density point $y \in X$ of X with $y \in C_x$ for $x \in \mathcal{D}$. For every $x \in \mathcal{D}$ let I_x be the component of C_x containing y. Say $f^{n_x} : I_x \to V_x$ is diffeomorphic with distortion bounded by K. Furthermore the contraction principle implies $|I_x| \to 0$ if $\mathcal{D} \ni x \to c$. Consider

$$\begin{aligned} \lim_{\mathcal{D} \ni x \to c} \frac{|X^c \cap V_x|}{|V_x|} &\leq \lim_{\mathcal{D} \ni x \to c} \frac{|T_x(X^c \cap I_x)|}{|T_x(I_x)|} \\ &= \lim_{\mathcal{D} \ni x \to c} \frac{\int_{X^c \cap I_x} |DT_x(t)| dt}{|DT_x(\beta_n)||I_x|} \end{aligned}$$

for some $\beta_n \in I_n$. Thus

$$\lim_{\mathcal{D}\ni x\to c} \frac{|X^c \cap V_x|}{|V_x|} \leq \lim_{\mathcal{D}\ni x\to c} K \frac{|X^c \cap I_x|}{|I_x|} = 0.$$

This finishes the proof. ∎

4. The Markov-Property

In this section we will study a second distortion result, called the *Markov-Property*. This property is much stronger than the Weak-Markov-Property and serves for studying more complicated ergodic properties.

DEFINITION 4.1. *An $S-$unimodal map is said to satisfy the Markov-Property if there exists a set $\mathcal{G} \subset [0,1]$ with full Lebesgue measure and $y \in \mathcal{N}$ such that for every $x \in \mathcal{G}$ there exist a sequence of TI-pairs (T_n, I_n) for the transfer range (V_y, U_y) with $x \in I_n$ and $|I_n| \to 0$.*
In particular, if $U_y \neq V_y$, the maps

$$I_n \to U_y$$

are diffeomorphisms with uniform bounded distortion.

We can reformulate the Markov-Property in terms of Markov-maps:

LEMMA 4.2. *An unimodal map f has the Markov-Property iff there exists $x \in \mathcal{N}$ such that the Markov map $M_x : E_x \to U_x$ has full domain:*

$$|E_x| = 1.$$

PROPOSITION 4.3. *Let f be an $S-$unimodal map without periodic attractor. If the limit set of the critical point is not minimal then f has the Markov-property.*

Let us prove this proposition for the $S-$unimodal map f satisfying the two conditions of the proposition. In particular f is not infinitely renormalisable.
Let us first deal with the case when f is Misiurewicz, $c \notin \omega(c)$. Take $x \in \mathcal{N}$ such that $\overline{orb(c_1)} \cap V_x = \emptyset$. Then $c_1 \notin C_x$ and $\psi(x) = x$. In this situation it is easy to show that

$$E_x = (C_x - V_x) \cup (f^{-1}(C_x) \cap V_x).$$

This implies, by using lemma 2.5, that M_x has full domain: $|E_x| = 1$.

In the sequel we will assume that the critical point c is recurrent.

LEMMA 4.4. *There exists $x \in \mathcal{N}$ and a sequence of intervals K_n, $n \geq 1$ such that for $n \geq 1$*

1) There are no renormalisations possible in V_x. In particular $\psi(x) > x$;

2) $\partial K_n \subset \Lambda_x$ and $K_n \cap V_x = \emptyset$;

3) $K_n \cap \omega(c) \neq \emptyset$;

4) $|K_n| \to 0$.

PROOF: Let $q \in \omega(c)$ be such that $c \notin \omega(q)$. This is possible because $\omega(c)$ is not minimal. Choose $x \in \mathcal{N}$ such that

$$orb(q) \cap \overline{V_x} = \emptyset.$$

and such that there are no renormalisations possible in V_x. Then $q \in \Lambda_x$ and q is accumulated from both sides by Λ_x (see lemma 2.5). Hence we can take a sequence of intervals $q \in K_n$ with $|K_n| \to 0$ and $\partial K_n \subset \Lambda_x$ accumulating at $q \in \omega(c)$. ∎

Let $x \in \mathcal{N}$ and the intervals K_n be given by lemma 4.4. Because $K_n \cap \omega(c) \neq \emptyset$ there exists a sequence $t_n \to \infty$ with $c_j \notin K_n$ for $j < t_n$ and $c_{t_n} \in K_n$. We claim that for every $n \geq 1$ there exists an interval K_n' containing c_1 such that

$$f^{t_n - 1} : K_n' \to K_n$$

is monotone and onto. Indeed let K_n' be the maximal interval containing c_1 with $f^{t_n-1}|K_n'$ is monotone and $f^{t_n-1}(K_n') \subset K_n$. Assume by contradiction that $\overline{f^{t_n-1}(L_n')} \subset K_n$ where L_n' is a component of $K_n' - \{c_1\}$. Then the maximality of K_n' implies the existence of $j < t_n - 1$ such that $c \in \partial f^j(L_n')$. So $f^{t_n-1-j}(c) \in K_n$. Contradicting the fact that c_{t_n} is the first critical value in K_n. This proves the claim.

We are going to use the components of $C_{\psi(x)}$, which belongs to TI-pairs, in K_n to show that the domain of the Markov map M_x is has positive upper density in c. This is done by pulling back the TI-pairs into K_n', close to c_1. And then one step more to get them close to c. However we have to be careful. The whole TI-pairs have to be pulled back. The collection $\hat{\mathcal{M}}_n$ defined below will contain the components for which this is impossible.

The statements in the lemma below follow directly from respectively proposition 2.6, the definition of C_x and lemma 2.5. We will leave the the proofs to the reader.

LEMMA 4.5. *If I is a component of $C_{\psi(x)}$ with $I \cap K_n \neq \emptyset$ then there exists an interval T containing I such that*

1) (T, I) is a TI-pair for (V_x, U_x);

2) $T \subset K_n$.

Furthermore

$$|C_{\psi(x)} \cap K_n| = |K_n|.$$

The map $f^{t_n - 1} : K_n' \to K_n$ is monotone and differentiable hence we can pull back the TI-pairs in K_n into K_n' and form the pairwise disjoint collection:

$$\mathcal{I}_n = \{I \subset K_n' | f^{t_n-1}(I) \text{ is a component of } C_{\psi(x)} \cap K_n\}.$$

From the previous lemma we get for every $I \in \mathcal{I}_n$ an interval $T_I \subset K_n'$ such that (T_I, I) is a TI-pair for (V_x, U_x). Furthermore $| \cup \mathcal{I}_n| = |K_n'|$.

Let $\hat{\mathcal{M}}_n$ consists of $I \in \mathcal{I}_n$ with

1) $I \cap [0, c_1] \neq \emptyset$;

2) $c_1 \in T_I$.

From lemma 2.7 we know that the collection $\{T_I | I \in \hat{\mathcal{M}}_n\}$ consists of nested intervals. In particular we can count $\hat{\mathcal{M}}_n = \{\hat{M}_i | i \geq 0\}$ such that $T_{\hat{M}_{i+1}} \subset T_{\hat{M}_i} - \hat{M}_i$. Denote by $\hat{L}_i$ the component of $K_n' - \cup_{i \geq 1} \hat{M}_i$ next to $\hat{M}_i$ such that $c_1 \notin [0, \hat{L}_i]$.

LEMMA 4.6. *If $I \cap \hat{L}_i \neq \emptyset$ for some $i \geq 1$ and $I \in \mathcal{I}_n$ then $T_I \subset \hat{L}_i$.*
Furthermore there exists $\hat{\rho} > 0$ such that $|\hat{L}_i| \geq \hat{\rho} |\hat{M}_i|$ for all $i \geq 0$.

PROOF: The first statement follows directly from lemma 2.7. The Koebe-Lemma gives $\hat{\rho} > 0$ such that $|T_I|$ contains a $\hat{\rho}-$scaled neighborhood of I where (T_I, I) is a TI-pair for (V_x, U_x). Again the statement follows from lemma 2.7: $\hat{L}_i$ contains a component of $T_{\hat{M}_i} - \hat{M}_i$. ∎

Let T_n be the symmetric interval $f^{-1}(K_n')$. Observe that $\cup_{i \geq 0} \hat{L}_i \subset f(T_n)$. So define

$$E_n = f^{-1}(\cup_{i \geq 0} \hat{L}_i).$$

from the previous lemma we get

LEMMA 4.7. *Every $y \in E_n$ has a TI-pair for (V_x, U_x). Furthermore this TI-pair is inside T_n.*

The next step is to show that $E_n \subset T_n$ has some universal metrical properties. Denote the components of $T_n - E_n$ by $\mathcal{M}_n = f^{-1}(\hat{\mathcal{M}}_n)$. Again $\mathcal{M}_n$ is countable, say $\mathcal{M}_n = \{M_i | i \geq 0\}$. If $c \in \cup \mathcal{M}_n$ then the corresponding component is M_0. If not we define $M_0 = \emptyset$.
For $i \geq 1$ let L_i be the component of $T_n - \cup_{i \geq 0} M_i$ next to M_i but not between M_i and c. Furthermore let R_i be the component of $T_n - M_i$ containing c. If $M_0 \neq \emptyset$ define R_0 and L_0 to be the components of $T_n - \cup_{i \geq 0} M_i$ next to M_0.

LEMMA 4.8. *There exists $\rho > 0$ such that for $i \geq 0$*

$$|L_i| \geq \rho |M_i| \text{ and } |R_i| \geq \rho |M_i|.$$

This bound is independent of $n \geq 1$.

PROOF: For $i \geq 1$ R_i contains one component of $T_n - \{c\}$. From the non-flatness of the critical and the fact that T_n is a symmetric interval we get $\rho_1 > 0$ with $|R_i| \geq \rho_1 |M_i|$.
Now $L_i \cup M_i$ is mapped monotonically onto $\hat{M}_j \cup \hat{L}_j$ for some $j \geq 1$. The space given by lemma 4.5 can be pulled back without being distorted to much because the non-flatness of the critical point. Hence we get a constant $\rho_2 > 0$, depending on the behavior of the critical point such that

$$|L_i| \geq \rho_2 |M_i|.$$

For $i = 0$ the situation is even better: L_0 and R_0 are mapped onto some $\hat{L}_j$ and M_0 into the corresponding $\hat{M}_j$. Again the non-flatness allow us to pull back space. ∎

PROOF OF PROPOSITION 4.3: Let $\mathcal{R}$ be the set of points whose orbit accumulates onto c. In section 2 we saw $|\mathcal{R}| = 1$. We are going to show that the domain of the Markov map M_x has a positive upper density in every point $y \in \mathcal{R}$. From $|\mathcal{R}| = 1$ it follows that $|E_x| = 1$. Take $y \in \mathcal{R}$ and $n \geq 1$. The first time the orbit of y hits T_n is denoted by s_n. Let H_n be the maximal interval containing y such that f^{s_n} maps H_n monotonically into T_n. Before we saw that there are no renormalisations possible inside U_x. Hence the Lemma from the appendix gives

$$f^{s_n}(H_n) = T_n.$$

Now let $A_n \subset H_n$ be the preimage of E_n. By using corollary 2.2 we get a universal constant $\rho > 0$ such that

$$|A_n| \geq \rho |H_n|.$$

The last observation to be made is $A_n \subset E_x$. The TI-pairs in E_n are part of T_n hence they can be pulled back into H_n. So every point in A_n has a TI-pair hence is in E_x. Thus the upper density of E_x in y is bigger than ρ. ∎

Let us finish this section with the fundamental distortion property which holds for Markov maps.

Fix $x \in \mathcal{N}$ and consider the Markov map $M_x : E_x \to U_x$. Let the collection $\mathcal{B}_0$ consists of the branches of M_x. That is it consists of the components of E_x. Define inductively the collections of branches of M_x^n as follows. An interval $I \subset U_x$ is in $\mathcal{B}_{n+1}$ if $M_x(I) \in \mathcal{B}_n$.

PROPOSITION 4.9. *If f satisfies the Markov-property, say $M_x : E_x \to U_x$ has full domain, then there exists $K > 0$ such that*

$$\frac{1}{K} \leq \frac{|DM_x^n(y_1)|}{|DM_x^n(y_2)|} \leq K$$

for $y_1, y_2 \in I \in \mathcal{B}_n$ and $n \geq 1$.

PROOF: Suppose that f also satisfies the Misiurewicz property, that is, the critical point is not recurrent. In this case we may assume that the critical orbit does not intersect V_x. Then it is easy to see that the branches of M_x^n have essentially bigger monotone extensions. Using the Koebe-Lemma the proposition can be proved.

In the other case, the critical orbit is recurrent, we also find easily monotone extensions. Indeed, using lemma 2.7 it is easy to see that every branch of M_x^n has a monotone extension to V_x. Again the Koebe-Lemma finishes the proof. ∎

COROLLARY 4.10. *If f satisfies the Markov-property then every 0-dimensional closed invariant set has Lebesgue measure zero.*

PROOF: Let $X \subset [0,1]$ be a 0-dimensional closed invariant set and let $y \in \mathcal{N}$ be such that the Markov-Property holds for the transfer range (V_y, U_y).

Suppose $|X| > 0$. Then there exists a density point x of X with $x \in X \cap \mathcal{G}$. Let $\{(T_n, I_n)\}_{n>0}$ be the TI-pairs for (V_y, U_y) given by the Markov-Property. As in the proof of lemma 3.9 we show

$$\lim_{n\to\infty} \frac{|X^c \cap U_x|}{|U_x|} \leq \lim_{n\to\infty} K \frac{|X^c \cap I_n|}{|I_n|} = 0.$$

Hence, because X is closed, $U_x \subset X$. Contradiction. ∎

5. The Ergodic Properties of Unimodal maps

In this section the distortion results from the previous sections will be used to give a measure theoretical description of $S-$unimodal dynamics. First we will use the developed technics to prove two fundamental theorems which were already proved by Blokh and Lyubich in [BL1]: the ergodicity and the Ergodic Classification Theorem. However we will include the relation with the Markov-Property.

THEOREM 5.1. *Every $S-$unimodal map which doesn't have periodic attractors is ergodic.*

The ergodicity follows directly from lemma 3.9.

We will fix in this section an $S-$unimodal map f without periodic attractor. Its critical point is denoted by c.

Let $n \geq 0$ and $x \in [0,1]$. The maximal interval containing x on which f^n is monotone is denoted by $T_n(x)$. Furthermore the components of $T_n(x) - \{x\}$ are denoted by respectively $L_n(x)$ and $R_n(x)$. Define $r_n : [0,1] \to \mathbf{R}$ by

$$r_n(x) = min\{|L_n(x)|, |R_n(x)|\}.$$

This functions turn out to be fundamental for understanding the ergodic theory of unimodal maps.

Furthermore define

$$r(x) = \limsup_{n\to\infty} r_n(x)$$

for every $x \in [0,1]$.

The ergodicity of f implies the existence of a number $r \geq 0$ such that

$$r(x) = r \text{ for a.e. } x \in [0,1].$$

PROPOSITION 5.2. *Let f be an $S-$unimodal map without periodic attractor. The map f has the Markov-Property if and only if $r > 0$.*

PROOF: The Markov-Property easily implies $r > 0$.

Suppose we have $r > 0$ for a map f not satisfying the Markov-Property. Using the Contraction Principle we get that the function $\delta : [0,1] \to [0,1]$

$$\delta(\epsilon) = \sup\{|I| | f^n \text{ maps the interval } I \text{ monotonically onto } T \text{ with } |T| < \epsilon\}$$

tends to zero as $\epsilon \to 0$.

Take $x \in \mathcal{N}$ close enough to the critical point c to get $\delta(\frac{1}{2}|V_x|) < \frac{1}{2}r$.

Because f doesn't satisfy the Markov-Property, we get $|E_x| < 1$, which implies, by using the ergodicity, that for almost every point $y \in [0,1]$ the exists $n_y \geq 0$ such that for every $n \geq n_y$ with $f^n(y) \in U_x$

$$r_n(y) < \frac{1}{2}|V_x|.$$

Now take such a point y and $n \geq n_y$. Let $s \geq 0$ be the smallest number such that $f^{n+s}(y) \in U_x$. Suppose that $r_{n+s}(y)$ is determined by the piece $R_{n+s}(y)$ of $T_{n+s}(y)$ (see definition $r_n(y)$). Now we claim that one piece, say $R_n(y)$ of $T_n(y)$ is mapped monotonically onto $R_{n+s}(y)$. In fact this follows easily from the fact that $R_{n+s}(y) \subset V_x$ and that $orb(\partial U_x) \cap V_x = \emptyset$.

Then we conclude that $r_n(y) < \delta(\frac{1}{2}|V_x|) < \frac{1}{2}r$ for all $n \geq n_y$. Hence $\limsup r_n < \frac{1}{2}r$. Contradiction. ∎

Using the number $r \geq 0$ we can give The Ergodic Classification of $S-$unimodal maps.

$$\mathcal{P} = \{f \in \mathcal{U} | f \text{ has a periodic attractor }\};$$

$$\mathcal{C} = \{f \in \mathcal{U} | r = 0\};$$

$$\mathcal{I} = \{f \in \mathcal{U} | r > 0\}.$$

Let us remember the definition given in [Mr] of an attractor (the ergodicity of our maps makes the definition a bit simpler).

DEFINITION 5.3. *A closed invariant set $A \subset [0,1]$ is called an attractor if for almost every $x \in [0,1]$*

$$\omega(x) = A.$$

THEOREM 5.4 (THE ERGODIC CLASSIFICATION THEOREM). *Every $S-$unimodal map has an attractor A. It can be of three different types:*

1) if $f \in \mathcal{P}$ then A is a periodic orbit;

2) if $f \in \mathcal{C}$ then $A = \omega(c)$ which is a minimal Cantor set;

3) if $f \in \mathcal{I}$ then A is the orbit of a periodic interval.

In particular $f \in \mathcal{I}$ if and only if f has the Markov-Property.

PROOF: Consider an $S-$unimodal map f. If $f \in \mathcal{P}$ we are finished.
If $f \in \mathcal{C}$ then f doesn't have the Markov-Property. Hence using Proposition 4.3 we get that $\omega(c)$ is a minimal Cantor set. Now $r = 0$ implies that almost all orbits accumulates on $\omega(c)$. Again using the minimality of $\omega(c)$ the attractor turns out to be $\omega(c)$.
Take $f \in \mathcal{I}$. By lemma 5.2 we get the Markov-Property for f, say $|E_x| = 1$ for some $x \in \mathcal{N}$. It easily follows from Proposition 4.9 that for almost every $x \in [0,1]$

$$U_x \subset \omega(x)$$

which implies $[c_2, c_1] \subset \omega(x)$. Because $\omega(x) \subset [c_2, c_1]$ for every $x \in [0,1]$ we get $A = [c_2, c_1] = \omega(x)$ for almost every $x \in [0,1]$. ∎

THEOREM 5.5. *If f is a $S-$unimodal map whose limit set $\omega(c)$ of the critical point is zero-dimensional then*

$$|\omega(c)| = 0.$$

In particular if $f \in \mathcal{C} \cup \mathcal{P}$ we have $|A| = 0$.

PROOF: Suppose that the critical point is not recurrent. Then there exists $x \in \mathcal{N}$ with $\omega(c) \cap V_x = \emptyset$ or $\omega(c) \subset \Lambda_x$. From lemma 2.5 we know $|\Lambda_x| = 0$ which implies the theorem.

The second case deals with the situation when $c \in \omega(c)$ but the limit set is not minimal. In this case the map has the Markov-property. Corollary 4.10 tells that every 0-dimensional closed invariant set has Lebesgue measure zero. In particular $|\omega(c)| = 0$.

Now assume that $c \in \omega(c)$ and that $\omega(c)$ is minimal. Suppose f is infinitely renormalizable. So $f|\omega(c)$ is injective. Together with lemma 3.9 we get $|\omega(c)| = 0$. Hence we may assume that f is not renormalizable.

Let $\{(V_{x_n}, U_{x_n})\}_{n\geq 1}$ be the sequence of $\delta-$transfer ranges given by theorem 3.4. Fix $n \geq 1$. The minimality of $\omega(c)$ and lemma 2.5(3) implies that every point $x \in \omega(c)$ is contained in some component $I \subset C_{\psi(x_n)}$. So we get a finite collection of intervals T_I covering $\omega(c)$. Here the intervals T_I form together with the intervals I, $I \cap \omega(c) \neq \emptyset$, a TI-pair for (V_{x_n}, U_{x_n}). Using the fact that these intervals T_I are nested, see lemma 2.7, and form a finite collection we can find a component I_n of $C_{\psi(x_n)}$ such that

1) $I_n \cap \omega(c) \neq \emptyset$;

2) $(T_{I_n} - I_n) \cap \omega(c) = \emptyset$;

Because (T_{I_n}, I_n) is a TI-pair for the $\delta-$transfer range (V_{x_n}, U_{x_n}) we get a $\rho_1 > 0$, not depending on $n \geq 1$ such that both components of $T_{I_n} - I_n$ are bigger than $\rho_1 |I_n|$. Observe that these components do not contain points of $\omega(c)$: we found some space in the Cantor set $\omega(c)$.

Let $t_n \geq 1$ be minimal such that $c_{t_n} \in I_n$. Using 2) above we get as before an interval T_n' containing c_1 such that $f^{t_n - 1} : T_n' \to T_{I_n}$ is an onto diffeomorphism. Let $T_n = f^{-1}(T_n')$ and $M_n \subset T_n$ be the maximal interval with $f^{t_n}(M_n) \subset I_n$. Let L_n and R_n be the components of $T_n - M_n$. Using the fact that the critical point is non-flat we find $\rho > 0$ such that

1) $(T_n - M_n) \cap \omega(c) = \emptyset$;

2) $|L_n| = |R_n| \geq \rho |M_n|$.

Now we are able to prove that $\omega(c)$ does not have density points. For this let $x \in \omega(c)$. Because $c \in \omega(x)$ we can define $s_n \geq 0$ to be the smallest number with $f^{s_n}(x) \in T_n$. Then apply the Lemma from the appendix and we find for every $n \geq 1$ an interval K_n around x such that $f^{s_n} : K_n \to T_n$ is an onto diffeomorphism. Again the Contraction-principle assures that $|K_n| \to 0$. Using the Macroscopic-Koebe-Lemma and property 1) and 2) of T_n above we see that x is not a density point of $\omega(c)$. Hence $|\omega(c)| = 0$. ∎

THEOREM 5.7. *Let A be the topological attractor of $f \in \mathcal{I}$. Then $f|A$ is conservative.*

PROOF: Let $D \subset A$ with positive Lebesgue measure. The map f has the Markov-Property. Hence by Lemma 4.2 there exists $x \in \mathcal{N}$ with $|E_x| = 1$. Because $D \subset A$ there exists $C \subset U_x$ with positive Lebesgue measure such that $f^n(C) \subset D$ for some $n \geq 0$. Let $D_0 \subset D$ be the set of points whose orbits return to D. Now by using the Markov-Property we see that almost every point of D has a positive upper density for D_0. Hence $|D_0| = |D|$. ∎

All the results are stated for $S-$unimodal maps. In fact it can be shown that the orbits of the intervals considered in the proofs of section 3 satisfy the necessary disjointness conditions needed for applying the C^2-Koebe-Lemma. Hence the results in this section also hold for C^2-unimodal maps. The difficulties for proving the C^2 versions of the Theorems in this paper occur when dealing with the orbits of TI-pairs. These orbits do not satisfy some disjointness conditions needed for applying the C^2-Koebe-Lemma. However the intervals in these orbits are nested, as described by Lemma 2.7. This property makes it possible to make the necessary estimates for applying the C^2-Koebe-Lemma. All results stated in this paper will turn out to be true for C^2-unimodal maps.

Appendix: Basic notions in real 1-dimensional dynamics

Let $f : X \to X$ be a measurable map on the borel measure space (X, λ). We will give some basic definitions dealing with the ergodic theory of this map.

0) The orbit $\{x, f(x), f^2(x), ...\}$ of a point x is denoted by $orb(x)$ and the set of all limits of the $orb(x)$ is denoted by $\omega(c)$.
1) A borel set $A \subset X$ is invariant iff $f(X) \subset X$.
2) f is called ergodic iff X cannot be written as the union of two disjoint invariant sets both with positive measure.
3) f is called conservative iff for every set $D \subset X$ with $\lambda(D) > 0$ the first return map f_D on D can be defined in almost every point of D.
4) A closed invariant set D is called minimal if the orbit of every point in D is dense in D.
5) acip stands for absolutely continuous invariant probability measure. And acim stands for $\sigma-$finite absolutely continuous invariant measure.

Dealing with maps on the interval we will use the following notation:

6) ∂D is the boundary of the interval D;

7) The Lebesgue measure will be denoted by $|.|$;

8) A $\delta-$scaled neighborhood T of the interval I is an interval such that both components of $T - I$ have length $\delta|I|$.

9) Denote the maximal interval containing $x \in [0,1]$ on which f^s is monotone by $T_s(x)$.

The collection $\mathcal{U}$ consists of the $S-$unimodal maps. This are maps $f : [0,1] \to [0,1]$ having the following property:

1) f has negative Schwarzian derivative;

2) $f(0) = f(1) = 0$;

3) there is exactly one point $c \in [0,1]$ where the derivative of f vanishes. Furthermore this *critical point* is non-flat: around c the map behaves like $x \to x^\alpha$ $(\alpha > 1)$. The number $\alpha > 1$ is called the order of the critical point.

For every $S-$unimodal map f the homeomorphism τ is defined to be the order reversing map satisfying $f \circ \tau = f$. The non-flatness of the critical point implies that this map is Lipschitz. Furthermore intervals of the form $(x, \tau(x))$ are called symmetric.

An $S-$unimodal map f is called renormalisable if there exists a symmetric interval V such that the first return map to V is of the form $f^n|V$, for some $n \geq 0$, and up to scaling $S-$unimodal. $f^n|V$ is called a renormalisation of f. It is called infinitely renormalisable if there are arbitrarily small symmetric intervals on which f can be renormalised.
The limit set of the critical point of an infinitely renormalisable $S-$unimodal map is a minimal Cantor set. Furthermore the map acts like a homeomorphism on it.

Lemma. *Let $f \in \mathcal{U}$ be non-renormalisable and V a symmetric interval. If $s \geq 0$ is minimal such that $f^s(x) \in V$, $x \in [0,1]$, then $V \subset f^s(T_s(x))$.*

Contraction Principle. *Let $f \in \mathcal{U}$ without periodic attractor. Then for every $x \in [0,1]$*

$$|T_s(x)| \to 0$$

when $s \to \infty$.

The proofs of the above lemmas can be found for example in [MMS].

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