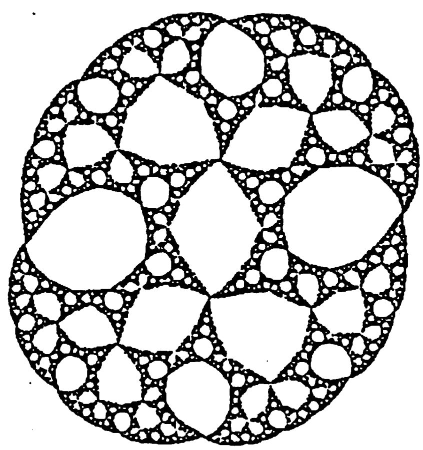
Shadowing Property for Nondegenerate Zero Entropy Piecewise Monotone Maps

by Liang Chen

April 1990



SUNY StonyBrook
Institute for Mathematical Sciences
Preprint #1990/9

SHADOWING PROPERTY FOR NONDEGENERATE ZERO ENTROPY PIECEWISE MONOTONE MAPS

Liang Chen
Institute for Mathematical Sciences
SUNY at Stony Brook

ABSTRACT

Let f be a continuous piecewise monotone map of the interval. If any two periodic orbits of f have different itineraries with respect to the partition of the turning points of f, then f is referred to as 'nondegenerate'. In this paper we prove that a nondegenerate zero entropy continuous piecewise monotone map f has the Shadowing Property if and only if 1) f does not have neutral periodic points; 2) for each turning point c of f, either the ω-limit set $\omega(c, f)$ of c contains no periodic repellors or every periodic repellor in $\omega(c, f)$ is a turning point of f in the orbit of c. As an application of this result, the Shadowing Property for the Feigenbaum map is proven.

CONTENTS

§0. Introduction.

In recent years there has been a growth of interest in piecewise monotone maps of the interval. One reason for this is that we have a decomposition theory for their nonwandering sets (cf. Jonker-Rand [JoR1], [JoR2]; Hofbauer [Hof1], [Hof2] and Nitecki [Ni]). This decomposition can be put into a quasi-filtration frame which is determined by unstable manifolds of the periodic points (cf. Nitecki [Ni]); In the 1970's, L. Block and M. Misiurewisz discovered the structure of zero entropy maps of the interval. Misiurewicz showed that any topologically transitive invariant set of such a map can be described as a single periodic orbit of period a power of 2 or a minimal set with the adding machine behavior (cf. [Mi]). Now, people naturally expect some nice dynamical properties for zero entropy piecewise monotone maps. In this paper, we will study the Shadowing Property for the family $\mathcal{N}_0$ of nondegenerate zero entropy continuous piecewise monotone maps (cf. (1.1)).

The study of the Shadowing Property has a long history (cf. [Ano], [Bow]), but for interval maps it is rather new. The recent research in this direction is mainly focused on the positive entropy maps (cf. [HYG], [CKY], [Ch], [Kan], [BoG]) and work for zero entropy is still seldom to be found in the literature.

In the light of the decomposition theory for piecewise monotone maps, we connect the Shadowing Property for $\mathcal{N}_0$ to a finitely generated filtration. From the structure of topologically invariant sets of zero entropy maps, we will find necessary conditions (4.1.1) and (4.1.3) for the Shadowing Property for the family $\mathcal{N}_0$. Based on the existence of a quasi-filtration for piecewise monotone maps (cf. [Ni]), we will construct finitely generated filtrations for nondegenerate piecewise monotone maps which satisfy (4.1.1) and then use a shadowing test for filtrations (3.4.1) which will be discussed early to prove (4.1.1) and (4.1.3) are also sufficient for the shadowing (cf.(1.2)).

§1. Definitions and statement of results.

Suppose $f : I \to I$ is a map of I to itself. Denote the set of periodic points of f by $Per(f)$, the set of the fixed points of f by $F(f)$, the nonwandering set of f by $\Omega(f)$ and the topological entropy by $h(f)$. f is called *piecewise monotone* if f has only finitely many local extrema in I. Such local extrema are called *turning points* of f and the set of turning points of f is denoted by $C(f)$. We say two periodic points of f are *m-equivalent* if they have the same itinerary with respect to the partition by turning points of f. This means there exists a compact subinterval of I containing these two periodic points so that f^k is monotone on this subinterval for every $k > 0$.

(1.1) **Definition.** We say a piecewise monotone map f is *nondegenerate* if every m-equivalence class in $Per(f)$ is trivial, in other words, any two periodic orbits have different itineraries with respect to the partition by turning points of f.

The family of nondegenerate continuous piecewise monotone maps of the interval is denoted by $\mathcal{N}$ and the family of the elements of $\mathcal{N}$ with zero entropy is denoted by $\mathcal{N}_0$.

If A is a subset of I and $\delta > 0$ is given, a sequence $\{x_j\}$ of points in A is called a *δ-pseudo orbit in A* for f provided that

$$|f(x_j) - x_{j+1}| \leq \delta \quad \textit{for every } j \geq 0.$$

In particular, we call a finite δ-pseudo orbit $\{x_0, \ldots, x_N\}$ a *(δ, N)-chain.*

Given $\varepsilon > 0$, a δ-pseudo orbit $\{x_j\}$ is said to be *ε-shadowed* by a point $y \in I$, if

$$|f^j(y) - x_j| \leq \varepsilon \quad \textit{for every } j,$$

where f^j is the *j-th* iterate of f.

f is said to have *the Shadowing Property on a subset A of I* if for any $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit in A can be ε-shadowed by a point in A.

The main result in this paper is

(1.2) **Theorem.** *$f \in \mathcal{N}_0$ has the Shadowing Property if and only if*

I. f does not have any neutral periodic point;

II. for each turning point $c \in C(f)$, either $\omega(c, f)$ contains no periodic repellors or every periodic repellor $p \in \omega(c, f)$ satisfies

$$p \in orb(c, f) \cap C(f).$$

§2. Preliminaries.

Let I be a compact interval and A a subset of I. Denote by $|A|$ the diameter of A, $Int\ A$ the interior of A and $B(c,r) = (c-r, c+r) \cap I$ the open interval centered at c with radius r. $\langle a, b\rangle$ will be used for a closed interval with endpoints a and b, irrespective of the order of a and b. If $\mathcal{S} = \{S_\ell \mid \ell \in \mathcal{L}\}$ is a collection of subsets of I, write

$$\|\mathcal{S}\| = \sup\{\ |S_\ell| \mid \ell \in \mathcal{L}\}.$$

Suppose $f : I \to I$ is a map of I to itself. Given $\varepsilon > 0$, $x \in I$ is *ε-linked* to $y \in I$ by f if there exist an integer $M > 0$ and $z \in B(x, \varepsilon)$ such that $f^M(z) = y$ and

$$|f^j(x) - f^j(z)| < \varepsilon \qquad \textit{for every } j = 0, 1, \ldots, M.$$

$x \in I$ is *linked* to $y \in I$ by f if for any $\varepsilon > 0$, x is ε-linked to y by f.

Define

$$D_k(x, \varepsilon) = f^k\Big[\bigcap_{j=0}^{k} f^{-j}[B(f^j(x), \varepsilon)]\Big].$$

Then, x is ε-linked to y by f if and only if $y \in D_M(x, \varepsilon)$ for some $M \geq 1$ (cf. [Ch]).

If f is a continuous piecewise monotone map, for $x \notin C(f)$, we define

$$\tau(x) = \begin{cases} +1 & \text{if } f \text{ is increasing at } x; \\ -1 & \text{if } f \text{ is decreasing at } x, \end{cases}$$

and define a *signature system* for each turning point of f as follows:

Suppose $c \in C(f)$, and define $\sigma_1(c)$ by

$$\sigma_1(c) = \begin{cases} -1 & \text{if } f \text{ takes a local maximum at } c; \\ +1 & \text{if } f \text{ takes a local minimum at } c. \end{cases}$$

Inductively define $\sigma_n(c)$ by

$$\sigma_{n+1}(c) = \begin{cases} \tau[f^n(c)]\sigma_n(c) & \text{if } f^n(c) \notin C(f); \\ \sigma_1(f^n(c)) & \text{if } f^n(c) \in C(f). \end{cases}$$

2.1. Periodic Attractors, periodic repellors and neutral periodic points.

Given a point $x \in I$, a *full neighborhood* (F-neighborhood) of x is an interval of the form $N(x, \varepsilon, F) = (x - \varepsilon, x + \varepsilon)$ for some $\varepsilon > 0$; the *L-neighborhood* of size ε is $N(x, \varepsilon, L) = (x - \varepsilon, x]$ and the *R-neighborhood* of size ε is $N(x, \varepsilon, R) = [x, x + \varepsilon)$. Call any one of these three types of neighborhood an *S-neighborhood* where $S = R, L, or\ F$.

Suppose $p \in Per(f)$, $f : I \to I$ continuous. We define the *S-unstable manifold* of the f orbit of p ($S = R, L,$ *or* F) by

$$U(p, f, S) := \bigcap_{\varepsilon>0} \overline{\{\bigcup_{j\geq 0} f^j[N(p, \varepsilon, S)]\}}.$$

When p is fixed under f, it is clear that $U(p, f, S)$ is a closed interval containing p—possibly equal to $\{p\}$, in which case we call it *trival* —and equals its image under f (we refer to this as *strong f-invariance*). If p is periodic under f with least period n, then each point $f^k(p)$ of the orbit of p is fixed under f^n and for given S a side at p, there exist sides $f^k(S)$ at $f^k(p)$ ($k = 1, 2, \ldots$) such that

$$f^k[U(p, f^n, S)] = U(f^k(p), f^n, f^k(S))$$

and

$$U(p, f, S) = \bigcup_{k=0}^{n-1} U(f^k(p), f^n, f^k(S)).$$

(2.1.1) **Remark.** If $f^n(p) = p$ and $S = L$ *or* R, then $U(p, f, S)$ is a finite union of closed intervals permuted by f. An endpoint of one of these intervals either belongs to a periodic orbit consisting of endpoints or else is the image under f^k ($k \leq n$) of a point interior to $U(p, \varepsilon, S)$.

Say a map f *respects side* S at a fixed point p if the image of every sufficiently small S-neighborhood of p is contained in an S-neighborhood of p; f *flips side* S at p if the image of every sufficiently small S-neighborhood of p is a T-neighborhood of p, where $T \neq S$. A map which flips both sides *exchanges sides at p*.

A piecewise monotone map respects or flips either side at p, giving three possibilities: (i) both sides are respected, (ii) one side is flipped onto the other, which is respected, or (iii) both sides flipped —that is, exchange of sides.

Unless f exchanges sides, one can always pick $\tilde{S}$ for any S such that $\tilde{f} = f$ respects S and $U(p, \tilde{f}, \tilde{S}) = U(p, f, S)$. If f exchanges sides, then $\tilde{f} = f^2$ respects both sides: we pick $\tilde{S} = R$ and notice that

$$U(p, f, \tilde{S}) = U(p, f, S) = U(p, \tilde{f}, R) \cup U(p, \tilde{f}, L).$$

Write the set of pairs (p, S), $p \in Per(f)$, $S = R$ *or* L as

$$\Sigma := Per(f) \times \{R, L\}.$$

Assuming $(p,S) \in \Sigma$, $f(p) = p$. With notation $\tilde{f}, \tilde{S}$ as above, (p,S) is an *S-periodic attractor*, if $U(p,\tilde{f},\tilde{S})$ is trivial; $(p,S) \in \Sigma$ is an *S-periodic repellor*, if $U(p,\tilde{f},\tilde{S})$ is non-trivial; when p is not fixed, we call $(p,S) \in \Sigma$ an S-periodic attractor or S-periodic repellor according to which applies to a power of f fixing p.

(2.1.2) **Lemma.** ((2.10) of [Ni]) *Assuming $f \in \mathcal{N}$, $f(p) = p$ and f respects S at p, there exists an S-neighborhood V of p such that*

1) if (p,S) is an S-periodic attractor,

(a) $f(\overline{V}) \subset V$

(b) $\bigcap_n f^n(V) = \{p\}$

(c) V contains either an endpoint of I or a turning point of f

2) if (p,S) is a periodic repellor,

(a) $f(V) \supset \overline{V}$

(b) for any $N(p,\delta,S) \subset V$, $U(p,f,S) = \overline{\bigcup_n f^n[N(p,\delta,S)]}$

(c) V contains a turning point of f.

If $f \in \mathcal{N}$, then for any $p \in Per(f)$ with least period n, p is an isolated fixed point under f^n. Thus, p under f^n should be one of the following:

1) a 2-side periodic attractor;

2) a 2-side periodic repellor;

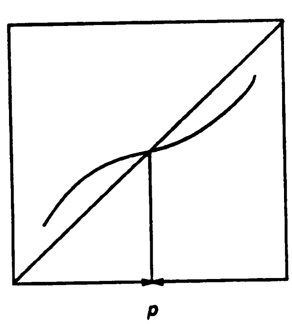
3) a 1-side periodic attractor on one side and a 1-side periodic repellor on the other;

4) a 1-side periodic attractor on the respected side and f^n flips the other side;

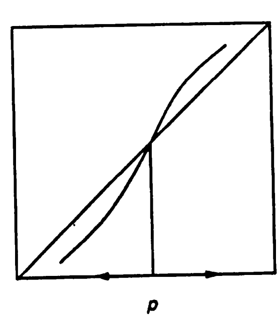
5) a 1-side periodic repellor on the respected side and f^n flips the other side.

It is easy to see that 4) gives a 2-side periodic attractor, but 5) is certainly not a 2-side periodic repellor.

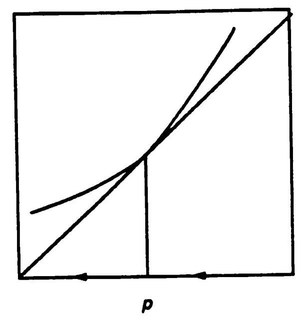
(2.1.3) **Definition.** Call p an *periodic attractor* in Case 1) or 4); a *normal periodic repellor* in 2); a *critical periodic repellor* in 5); and a *neutral periodic point* in 3) for f.



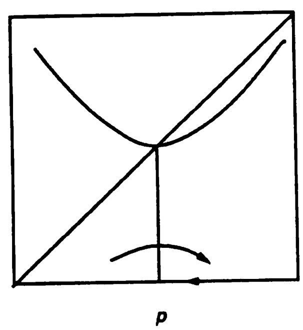
1) Periodic Attractor



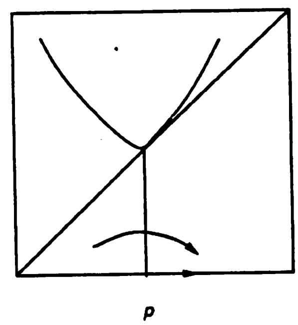
2) Normal periodic repellor



3) neutral periodic point



4) Periodic Attractor



5) Critical periodic repellor

(2.1.4) **Remark.** For $f \in \mathcal{N}$,

1) a neutral periodic point p with least period n can be characterized as $p \notin C(f)$ where the map $f^n(x) - x$ takes a local extremum;

2) the number of periodic attractors is less than or equal to the number of turning points of f. (cf. (4.1) of [Ni])

2.2. Constructing a quasi-filtration.

Suppose $f \in \mathcal{N}$. We call two pairs $(p_i, S_i) \in \Sigma$ *h-equivalent* if $U(p_1, f, S_1) = U(p_2, f, S_2)$.

Let $\mathcal{R}$ be the set of h-equivalence classes of periodic repellors in Σ, and if $(p, S) \in \alpha \in \mathcal{R}$, define the set *accessible* from α as the subset of I:

$$A(\alpha) := U(p, f, S).$$

Similar to (4.3)-(4.5) in [Ni], we define a *filtration ordering* to be a numbering of the elements of $\mathcal{R}$, say $\alpha_1, \alpha_2, \ldots$ so that

$$A(\alpha_i) \supset A(\alpha_j) \text{ implies } i \leq j$$

and a nested sequence of sets M_i, $i = 1, 2, \ldots$ by

$$M_i := \bigcup_{j \geq i} A(\alpha_j). \tag{2.2.1}$$

(2.2.2) **Remark.** The properties of M_i are summarized as follows (cf. [Ni]):

1) Each M_i is a finite union of closed intervals, strongly invariant under f (that is, $f(M_i) = M_i$). In other words, $\{M_i\}$ is a finitely generated quasi-filtration (cf. (4.5b) of [Ni])
2) If $\mathcal{R}$ is infinite, then (cf. (4.6) of [Ni])
   (i)
   $$M_\infty = \bigcap_{i=0}^{\infty} M_i$$
   is a nonempty closed invariant set
   (ii) $M_\infty \cap Per(f) = \emptyset$
   (iii) M_∞ contains the orbit of a turning point of f;
3) For $i < \infty$, if $p \in Per(f) \cap [M_i \setminus M_{i+1}]$ is a periodic repellor, then $(p, S) \in \alpha_i$ for $S = R$ or L. (cf (4.7) of [Ni])

(2.2.3) **Decomposition Theorem.** (cf. [JoR1], [Jor2], [Hof1], [Hof2] and [Ni])

Suppose $f : I \to I$ is continuous, piecewise monotone. Then the nonwandering set $\Omega(f)$ has a finite or countable decomposition into closed invariant sets

$$\Omega(f) = \tilde{\Omega}_1 \cup \ldots \cup \tilde{\Omega}_N, \qquad N \leq \infty.$$

The pairwise intersections in this decomposition are finite. Let $\Omega_i = \tilde{\Omega}_i \cap \overline{per(f)}$, $F_i = \tilde{\Omega}_i \setminus \Omega_i$.

1. For each $i < \infty$

a) F_i is finite

b) Ω_i has a finite decomposition

$$\Omega_i = \Omega_{i1} \cup \ldots \cup \Omega_{in}$$

into disjoint closed sets permuted cyclically by f. Each Ω_{ij} is either

(i) the set of fixedpoints of f^{2n} on an interval where f^n is monotone; or

(ii) a set with a dense f^{2n} orbit. In this case, either $f^n|\Omega_{ij}$ is topologically mixing or $f^{2n}|\Omega_{ij}^k$ (k=1,2) is topologically mixing, where $\Omega_{ij} = \Omega_{ij}^1 \cup \Omega_{ij}^2$ are sets meeting at a unique point and interchanged under f.

2. If there are infinitely many Ω_i, then $\Omega_\infty \neq \emptyset$ has a finite decomposition into closed invariant sets

$$\Omega_\infty = \Omega_{\infty 1} \cup \ldots \cup \Omega_{\infty k}$$

each having a generalized adding machine as a factor, and containing the ω-limit set of some turning point of f.

2.3. h-equivalence classes for zero entropy maps.

We say a periodic point p of f has a *homoclinic point* if some point interior to one of the sets $U(p, f, S)$, $S = R$ or L hits $orb(p, f)$ under f.

The following is an immediate result of (2.4) and (5.7) in [Ni] (cf. [Bl] and [BGMY])

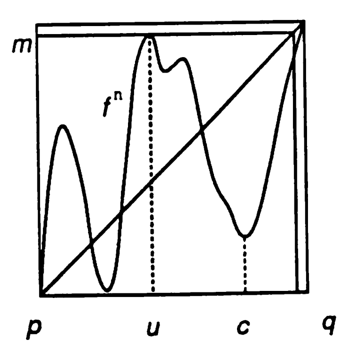
(2.3.1) **Lemma.** *Some periodic point of f has a homoclinic point if and only if $h(f) > 0$.*

As a consequence, we have

(2.3.2) **Proposition.** *Suppose $f \in \mathcal{N}$. If $p_1, p_2 \in Per(f)$ are not in the same periodic orbit and (p_1, S_1) is h-equivalent to (p_2, S_1) for some sides S_1, S_2, then either p_1 or p_2 has a homoclinic point.*

In particular, if f has zero entropy, then every h-equivalence class of Σ consists of a unique periodic orbit.

Proof. Without loss of generality we assume that p and q are fixed under some power f^n of f and $U(p, f^n, S) = U(q, f^n, T) = U$ for some sides S, T. Unless p or q is interior to U, we can assume that $U = [p, q]$. Since f is nondegenerate, there exists a turning point of f^n in (p, q). Let c be the turning point of f^n adjacent to q in (p, q), we have $f^n(z) \leq z$ for every $z \in [c, q]$. Since $c \in (p, q)$, we have $m = \max_{p<x<c} f^n(x) \geq c$ and $[p, q] = U \subset [p, m]$.



Hence, there exists $u \in (p, c)$ such that $f^n(u) = m = q$. Thus, q has a homoclinic point u. ∎

By (2.3.2), (2.2.2), (2.1.4) and a result of [Mi] (cf. Theorem 1.10 of [Ni]), we have

(2.3.3) **Proposition.** *Suppose $f \in \mathcal{N}_0$ and $\{M_i\}$ is the quasi-filtration defined as in (2.2.1). Then for each $i < \infty$, $M_i \setminus M_{i+1}$ contains only periodic orbits as its topologically transitive invariant sets. Moreover, $(I \setminus M_i) \cap \Omega(f)$ is finite.*

(2.3.4) **Proposition.** *Suppose $f \in \mathcal{N}$, $c \in C(f) \cap A(\alpha)$ is nonperiodic, p is a periodic repellor, and c is linked to p. If $p \in \alpha$, then p has a homoclinic point.*

In particular, if f has zero entropy and some nonperiodic $c \in C(f) \cap A(\alpha)$ is linked to a periodic periodic repellor p, then $p \notin \alpha$.

Proof. Assume that p is fixed under a power f^n of f. Since α contains exactly one periodic repellor, we have $A(\alpha) = U(p, f, S)$, thus $\langle c, f^k(p)\rangle \subset U(f^k(p), f^n, f^k(S))$ for some $1 \le k \le n$. Since c is linked to p by f and $c \in C(f)$, by definition either there is $z \in Int\ \langle c, f^k(p)\rangle$ so that $f^m(z) = p$ for some $m \ge 0$ or $p \in orb(c, f)$. When $p \in orb(c, f)$, we distingush two cases: if $c \in Int\ U(f^k(p), f^n, f^k(S))$, let $z = c$; if not, by (2.1.1) there exists $u \in Int\ \langle c, f^k(p)\rangle$ such that $f^n(u) = c$, let $z = u$. Thus, p has a homoclinic point z. ∎

§3. Shadowing Property for filtrations.

A *filtration $\mathcal{F}$ of I adapted to f* is a nested sequence of closed subsets $I = F_0 \supset F_1 \supset \ldots \supset F_k \supset \ldots$ with $f(F_k) \subset Int\ F_k$ for every k. A *quasi-attractor* generated by a filtration $\mathcal{F}$ is the intersection

$$\Lambda = \bigcap_{F \in \mathcal{F}} F.$$

Definition. A filtration $\mathcal{F}$ is *finitely generated* if every element of $\mathcal{F}$ has finitely many connected components.

3.1. Sieves and shadowing.

(3.1.1) **Lemma.** *Let K be a closed subinterval and J an open subinterval of I. If for some integer $n > 0$, $f^n(K) \subset J$, then there exists a sequence $\mathcal{M} : M_0, \ldots, M_n$ of closed subintervals of I satisfying*

(i) $f[M_\alpha] \subset Int\ M_{\alpha-1}$ for $\alpha = 1, \ldots, n$;

(ii) $M_0 = K$ and $M_n = \overline{J}$

Proof. Let $M_n = \overline{J}$, then $f^{n-1}(K)$ is a closed subinterval contained in $f^{-1}(J)$. Inductively choose a closed subinterval M_α for each $1 \le \alpha \le n-1$ such that

$$f^\alpha(K) \subset Int\ M_\alpha \subset M_\alpha \subset f^{-1}(Int\ M_{\alpha+1}).$$

and let $M_0 = K$. ∎

A sequence of closed subintervals of I satisfying (i) and (ii) of (3.1.1) is called a *sieve* for the pair (K, J). For a sieve $\mathcal{M}$ as in (3.1.1), let $\delta = \min_{0 \le \alpha \le n-1}\{dist[I - Int\ M_{\alpha+1}, f(M_\alpha)]\} > 0$, then every δ-pseudo orbit $\{x_j\}$ with $x_0 \in K$ has $x_n \in J$.

(3.1.2) **Remark.** Given $\varepsilon > 0$, by the uniformly continuity of f, we have the following:

1) If a closed subinterval K and an open subinterval J of I satisfy

(i) $|J| \le \varepsilon$;

(ii) for some $n > 0$, $|f^j(K)| < \varepsilon$ when $0 \le j \le n$ and $f^n(K) \subset U$,

then there exists a sieve $\mathcal{M} : M_0, \ldots, M_n$ for the pair (K, J) so that $\|\mathcal{M}\| < \varepsilon$;

2) Given $\varepsilon > 0$ and $N > 0$, there are $\delta > 0$ and $\sigma > 0$ such that every (δ, N)-chain $\{x_j\}_{j=0}^N$ is ε-shadowed by any point of $B(x_0, \sigma)$;

3) If $\mathcal{S} = \{S_\ell\}_{\ell \in \mathcal{L}}$ is a finite family of closed subintervals of I satisfying that $\|\mathcal{S}\| < \varepsilon$ and

$$f(S_\ell) \subset Int\ S_{\phi(\ell)}, \quad for\ \ell \in \mathcal{L}$$

where $\phi : \mathcal{L} \to \mathcal{L}$, then there is $\delta > 0$ such that every δ-pseudo orbit starting at S_ℓ is ε-shadowed by every point of S_ℓ.

(3.1.3) **Proposition.** *Suppose $\mathcal{F}$ is a finitely generated filtration of I adapted to a continuous map $f : I \to I$. If f has the Shadowing Property on I, then for every $F \in \mathcal{F}$, f has the Shadowing Property on $I \setminus F$.*

Proof. Given $F \in \mathcal{F}$, let $\eta = dist\ [I \setminus Int\ F, f(F)] > 0$, then every η-pseudo orbit $\{y_j\}$ with $y_0 \in F$ must have $y_j \in F$ for all j. For any $\varepsilon \in (0, \eta)$ choose $\delta \in (0, \eta)$ small enough that every δ-pseudo orbit in I is ε-shadowed. If $\{x_j\}$ is a δ-pseudo orbit in $I \setminus F$, then either $x_j \in I \setminus F$ for all j or $\{x_j\}$ is a (δ, n)-chain for some $n \ge 1$. Unless $\{x_j\}$ consists of a single point x_0, we can assume $x_1 \in I \setminus F$ in both cases. If y is a point ε-shadowing $\{x_j\}$, then $y \in I \setminus F$; otherwise we have $|x_1 - f(y)| \ge dist[I \setminus Int\ F, f(F)] = \eta > \varepsilon$. ∎

3.2. Totally disconnected quasi-attractors.

Suppose $\mathcal{F} = \{F_k\}$ is a finitely generated filtration of I adapted to f. An easy fact is that the quasi-attractor $\Lambda = \bigcap_k F_k$ is totally disconnected if and only if

$$\lim_{k \to \infty} \|F_k\| = 0,$$

where for every $F \in \mathcal{F}$

$$\|F\| := \max\{|U| \textit{ for all components } U \textit{ of } F\}.$$

(3.2.1) **Proposition.** *Suppose $\mathcal{F} = \{F_k\}$ is a finitely generated filtration of I adapted to a continuous map $f : I \to I$ and $\Lambda = \bigcap_k F_k$ is totally disconnected. If for every $F \in \mathcal{F}$ f has the Shadowing Property on $I \setminus F$, then f has the Shadowing Property on I.*

Proof. Given $\varepsilon > 0$, select $F \in \mathcal{F}$ so that $\|F\| \leq \varepsilon$. Let $S_1, \ldots, S_n$ be the components of F and $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a map such that $f(S_j) \subset Int\ S_{\pi(j)}$ for every j. Thus

$$\gamma = \min_{1 \leq j \leq n} \{dist[S_j, F \setminus f^{-1}(Int\ S_{\pi(j)})]\} > 0.$$

If $x \in I$ satisfies $dist(x, S_j) \leq \gamma$ for some j, then $f(x) \in Int\ S_{\pi(j)}$.

By (3.1.2), there is $\sigma \in (0, \min\{\varepsilon, \gamma\})$ such that every σ-orbit starting at a component of F can be ε-shadowed by every point of this component. By the uniform continuity of f, there is $\mu \in (0, \varepsilon)$ such that whenever $d(x, y) \leq \mu$ then $|f(x) - f(y)| \leq \sigma/2$ and by hypothesis there is $\delta \in (0, \min\{\gamma, \sigma/2\})$ such that every δ-pseudo orbit in $I \setminus F$ can be μ-shadowed. Suppose $\{x_j\}$ is δ-pseudo orbit in I. If $\{x_j\} \subset F$ or $\{x_j\} \subset I \setminus F$, we are done. Otherwise, there is $m > 0$ such that $x_j \in F$ if and only if $j \geq m$. If $x_m \in S_\ell$ for some ℓ and y μ-shadows $\{x_1, \ldots, x_{m-1}\}$ in $I \setminus F$, then $|f^m(y) - x_m| \leq \sigma < \gamma$ implies that $f^{m+1}(y) \in S_{\pi(\ell)}$. Hence, $f^m(y)$ ε-shadows $\{x_m, x_{m+1}, \ldots\}$. ∎

(3.2.2) **Example.** A subset $V \subset I$ is *trapping* for a map $f : I \to I$ provided $f(V) \subset Int\ V$. A subset $\Lambda \subset I$ is called *a periodic attractor for f in I* if there is a closed trapping subinterval W of I such that

$$\Lambda = \bigcap_{k=0}^{\infty} f^k(W).$$

We call W a *basin* of Λ.

If $\Lambda = \{a\}$ is a single point periodic attractor, we construct a filtration on W for Λ: choose $\sigma \in (0,1)$ and $k_0 \geq 0$ such that

$$f^{k_0}(W) \subset B(a,\sigma) \subset \overline{B(a,\sigma)} \subset Int\ W.$$

Let $\mathcal{M}_0 : M_0^0 \supset M_1^0 \supset \ldots \supset M_{k_0}^0$ be a sieve for the pair $(W, B(a,\sigma))$ satisfying $M_0^0 = W$ and $M_{k_0}^0 = \overline{B(a,\sigma)}$. Notice that there exists an increasing sequence $\{k_\ell\}$ satisfying

$$f^{k_\ell - k_{\ell-1}}[\overline{B(a,\sigma/2^{\ell-1})}] \subset B(a,\sigma/2^\ell).$$

By (3.1.1), define a sieve $\mathcal{M}_\ell : M_0^\ell \supset \ldots \supset M_{k_\ell - k_{\ell-1}}^\ell$ satisfying $M_0^\ell = \overline{B(a,\sigma/2^{\ell-1})}$ and $M_{k_\ell - k_{\ell-1}} = \overline{B(a,\sigma/2^\ell)}$.

Let $k_{-1} = 0$ and define $\mathcal{F} : W = F_0 \supset F_1 \ldots \supset F_k \supset \ldots$ by

$$F_k = M_{k-k_{\ell-1}}^\ell \quad for\ k_{\ell-1} \leq k \leq k_\ell.$$

By the choice of k_ℓ, we have $f^{k_\ell}(W) \subset Int\ F_{k_\ell} = B(a,\sigma/2^\ell)$. Hence, $\bigcap_k F_k = \{a\}$. By (3.1.2) for each i, f has the Shadowing Property on $W \setminus F_i$. It follows from (3.2.1) that f has the Shadowing Property on the basin W.

In general, if we suppose Λ is a periodic orbit with least period n and W is a basin of Λ consisting of n disjoint closed subintervals so that each of them contains exactly one point of Λ, then f has the Shadowing Property on W. In particular, if f is a continuous piecewise monotone map of the interval, then, for every integer $n \geq 0$, f has the Shadowing Property on $f^{-n}(W)$.

(3.2.3) **Remark.** (3.1.3) - (3.2.2) also hold for any general compact metric spaces.

3.3. Non-cyclic quasi-attractors .

(3.3.1) **Definition.** A quasi-attractor Λ is *non-cyclic* if Λ does not contain any periodic point of f.

Notice that Λ is invariant for f. If C is a connected component of Λ, then $f(C)$ must be contained in a connected component of Λ. Hence, Λ is non-cyclic if and only if for any connected component C of Λ and any integer $n > 0$, $f^n(C) \cap C = \emptyset$. If $\{C_n\}$ is a sequence of connected components of a non-cyclic quasi-attractor with $f(C_n) \subset C_{n+1}$, then

$$\lim_{n\to\infty} |C_n| = 0. \qquad (3.3.2)$$

Suppose U, U' are two closed subintervals of I. We define

$$U \xrightarrow{f} U' \quad if \quad f(U) \subset Int\, U'.$$

A sequence

$$U_1 \xrightarrow{f} \dots \xrightarrow{f} U_n$$

is called an *f-chain*. An f-chain

$$U_1 \xrightarrow{f} \dots \xrightarrow{f} U_n \xrightarrow{f} U_1$$

is called *an f-cycle*. (n is the *length* of the f-cycle if the cycle does not contain any proper f-subcycle.)

Say an f-chain

$$V_1 \xrightarrow{f} \dots \xrightarrow{f} V_m$$

refines an f-chain

$$U_1 \xrightarrow{f} \dots \xrightarrow{f} U_n$$

if $\{U_1, \dots, U_n\} \subset \{V_1, \dots, V_m\}$ and for each s, there is t such that $V_s \subset U_t$.

For a finitely generated $\mathcal{F}$, denote by $\mathcal{U}^{(k)}$ the set of all components of F_k. Call an f-chain (f-cycle) an *f-chain (f-cycle) for* F_k if all the elements of this f-chain (f-cycle) are contained in $\mathcal{U}^{(k)}$. Call an f-chain for F_k *maximal* if it is not a proper subset of any other f-chain for F_k. Notice that every maximal f-chain for F_k contains a unique f-cycle for F_k.

A connected component C of Λ is *recurrent* if for any $k \geq 0$, C is contained in an f-cycle for F_k.

Suppose Λ is non-cyclic; for a given $\varepsilon > 0$ write

$$\mathcal{C}_\varepsilon = \{ \text{ all the components } C \text{ of } \Lambda \text{ with } |C| > \varepsilon/2 \ \}.$$

The compactness of Λ implies that $\mathcal{C}_\varepsilon$ is a finite set. Thus, there is an integer $N > 0$ such that for every $n \geq N$ we can not find $C, C' \in \mathcal{C}_\varepsilon$ with $f^n(C) \subset C'$; respectively, there is $k \geq 0$ such that every component of F_k containing no elements of $\mathcal{C}_\varepsilon$ has diameter less than ε. The main result in this section is

(3.3.3) **Lemma.** *Suppose Λ is a non-cyclic quasi-attractor generated by a finitely generated filtration $\mathcal{F}$ adapted to f. Then for each given $\varepsilon > 0$ there is $k \geq 0$ such that every maximal f-chain $\mathcal{D}$ for F_k is refined by an f-chain*

$$\mathcal{E} : E_1 \xrightarrow{f} \dots \xrightarrow{f} E_m \xrightarrow{f} \dots \xrightarrow{f} E_{m+\ell} \xrightarrow{f} E_m$$

where $E_1, \ldots, E_{m-1} \in \mathcal{U}^{(k)}$ *and*

$$\mathcal{B} : E_m \xrightarrow{f} \ldots \xrightarrow{f} E_{m+\ell} \xrightarrow{f} E_m$$

is an f*-cycle with* $\|\mathcal{B}\| < \varepsilon$.

Proof. By the uniform continuity of f, there is $\sigma \in (0, \varepsilon/2)$ such that for $j = 0, 1, \ldots, N$

$$d(f^j(x), f^j(y)) < \varepsilon/2, \quad \text{whenever } d(x, y) < \sigma.$$

If C is a recurrent element of $\mathcal{C}_\varepsilon$, then for each $k \geq 0$, C is contained in an f-cycle for F_k. Let $\{C_n\}$ be the sequence of the components of Λ satisfying $f^n(C) \subset C_n$. Then, by (3.3.2), $\lim_{n\to\infty} |C_n| = 0$. For every k, let

$$\mathcal{G}_k : U_0^{(k)} \xrightarrow{f} \ldots \xrightarrow{f} U_{\ell_k-1}^{(k)} \xrightarrow{f} U_0^{(k)}$$

be the f-cycle for F_k such that $C \subset U_0^{(k)}$. By defintion, beyond the first N indexes j, $U_j^{(k)}$ contains no elements of $\mathcal{C}_\varepsilon$. Take k big enough that $|U_j^{(k)}| \leq \varepsilon/2$ for $N \leq j \leq \ell_k - 1$ and $|U_{\ell_k-1}^{(k)}| \leq \sigma$, then

$$f^{j+1}(U_{\ell_k-1}^{(k)}) \subset Int\ U_j^{(k)} \quad \text{and} \quad |f^{j+1}(U_{\ell_k-1}^{(k)})| \leq \varepsilon/2$$

for $j = 0, \ldots, N$. By (3.1.2), we construct a sieve $\mathcal{B}' : B_{-1}, B_0, \ldots, B_N$ for the pair $(U_{\ell_k-1}^{(k)}, Int\ U_N^{(k)})$ with $\|\mathcal{B}'\| < \varepsilon/2$ satisfying $B_j \subset Int\ U_j^{(k)}$ for $0 \leq j \leq N-1$; For $j = N+1, \ldots, \ell_k - 1$ let $B_j = U_j^{(k)}$; we get a refinement of $\mathcal{G}_k$

$$\mathcal{B} : B_0 \xrightarrow{f} \ldots \xrightarrow{f} B_{\ell_k-1} = B_{-1} \xrightarrow{f} B_0.$$

with $\|\mathcal{B}\| < \varepsilon$.

If $\mathcal{D}$ is a maximal f-chain for F_k with an element containing a recurrent element of $\mathcal{C}_\varepsilon$, then let $\mathcal{B}$ be the f-cycle related to the recurrent element as above. If $E_{m-1} \in \mathcal{D} - \mathcal{B}$ is the last element in the order $\xrightarrow{f}$, then there is an element $B \in \mathcal{B}$ such that $E_{m-1} \xrightarrow{f} B$. Reindexing the elements in the set $\mathcal{D} \cup \mathcal{B}$ in the order $\xrightarrow{f}$, we get a new f-chain

$$\mathcal{E} : E_1 \xrightarrow{f} \ldots \xrightarrow{f} E_m \xrightarrow{f} \ldots \xrightarrow{f} E_{m+\ell} \xrightarrow{f} E_m$$

where $E_s \in \mathcal{D} - \mathcal{B}$ for all $s < m$, $E_s \in \mathcal{B}$ for $s \geq m$, and ℓ is the order of the cycle $\mathcal{B}$. Such an f-chain $\mathcal{E}$ refines $\mathcal{D}$; For $\mathcal{D}$ whose elements contain no recurrent elements of $\mathcal{C}_\varepsilon$, the proof is straightforward. ∎

3.4. A shadowing test for quasi-attractors.

(3.4.1) **Theorem.** *Suppose $\Lambda = \bigcap_{F\in\mathcal{F}} F$ is a quasi-attractor generated by a finitely generated filtration $\mathcal{F}$ of a compact interval I adapted to a continuous map f. If Λ is totally disconnected or non-cyclic, then f has the Shadowing Property on I if and only if for each $F \in \mathcal{F}$ f has the Shadowing Property on $I \setminus F$.*

Proof. By (3.1.3) and (3.2.1), we need prove $\Leftarrow$ for a non-cyclic Λ only.

For a given $\varepsilon > 0$, choose k big enough that (3.3.5) holds. Replace all the maximal f-chains for F_k by their refinements and let $\mathcal{R}_k$ be the set of these refinements.

Suppose

$$\mathcal{E} : E_1 \xrightarrow{f} \dots \xrightarrow{f} E_m \xrightarrow{f} \dots \xrightarrow{f} E_{m+\ell} \xrightarrow{f} E_m$$

is an f-chain in $\mathcal{R}_k$. By (3.1.1), there is $\mu_{\mathcal{E}} > 0$ such that every $\mu_{\mathcal{E}}$-pseudo orbit $\{x_j\}$ starting at E_s has $x_j \in E_{\psi^j(s)}$, where

$$\psi(s) = \begin{cases} s+1 & \text{for } s < m+\ell; \\ m & \text{for } s = m+\ell. \end{cases}$$

Let $\mu = \min\{\mu_{\mathcal{E}} |\ \mathcal{E} \in \mathcal{R}_k\}$ and $N = \#(\mathcal{U}^{(k)})$, then $m \le N$. By (3.1.2) and the definition of f-chain, there is $\sigma \in (0, \varepsilon)$ and $\tau > 0$ such that

1) Every (τ, N)-chain $\{x_0, \dots, x_N\}$ in I can be ε-shadowed by any point of $B(x_0, \sigma)$;
2) If $x \in E_s$ for some s, then $f^j[B(x,\sigma)] \subset E_{\psi^j(s)}$ for every $j = 0, 1, \dots, N$.

By the uniform continuity of f, there is $\zeta \in (0, \varepsilon)$ with $f[B(x,\zeta)] \subset B(f(x), \sigma/2)$ for every $x \in I$ and by the Shadowing Property for f on $I \setminus F_k$, there is ρ such that every ρ-pseudo orbit in $I \setminus F_k$ can be ζ-shadowed.

Let $\delta = \min\{\mu, \tau, \sigma/2, \rho\}$, it is only routine to check that every δ-pseudo orbit in I is ε-shadowed. ∎

§4. Shadowing Property for $\mathcal{N}_0$.

4.1. Necessary conditions for shadowing.

1. The 1st necessary condition for shadowing.

If f has a neutral periodic point p, it is easy to see that f fails to have the Shadowing Property in a small neighborhood of $orb(p, f)$. Thus, for $f \in \mathcal{N}_0$ to have the Shadowing Property, we must have

(4.1.1) **Condition 1.** *f does not have any neutral periodic point.*

2. The 2nd necessary condition for shadowing.

(4.1.2) **Proposition.** *Assume $f \in \mathcal{N}_0$ has the Shadowing Property. For any turning point $c \in C(f)$, if there exists a periodic repellor $p \in \omega(c, f)$, then*

i) $p \in orb(c, f)$;

ii) p is critical.

Moreover, if $c \in A(\alpha)$ for some $\alpha \in \mathcal{R}$, then $p \notin \alpha$.

Proof. First we show that c is linked to p. Assume not, then there are small $\varepsilon_0, \delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, the δ-pseudo orbit $\{x_j\}$ defined by

$$x_j = \begin{cases} f^j(c) & \text{for } 0 \leq j \leq k; \\ p & \text{for } j \geq k+1. \end{cases}$$

is not ε_0-shadowed.

i) If $p \notin orb(c, f)$, we show that p has a homoclinic point. Suppose f respects side S at p. Choose $\varepsilon > 0$ small enough that $N(p, \varepsilon, S) \subset U(p, f, S)$. Since c is linked to p, $p \in D_k(c, \varepsilon)$ for some index k, which we take the smallest one. If $p \notin orb(c, f)$, choose $0 < \gamma < |f^k(c) - p|$, then $p \notin D_k(c, \gamma)$ and $D_k(c, \gamma) \subset D_k(c, \varepsilon)$; this implies that $D_k(c, \gamma) \subset \langle p, f^k(c) \rangle \subset U(p, f, S)$ (cf. Fig. 4.1.2). Now $p \in D_L(c, \gamma)$ for some L; by the choice of k, $L > k$. Hence, there is a point $w \in D_k(c, \gamma) \subset \langle p, f^k(c) \rangle$ such that $f^{L-k}(w) = p$. By hypothesis, $w \neq f^k(c)$ and $w \neq p$. So $w \in Int\ U(p, f, S)$. That is, p has a homoclinic point w, contradicting the hypothesis.

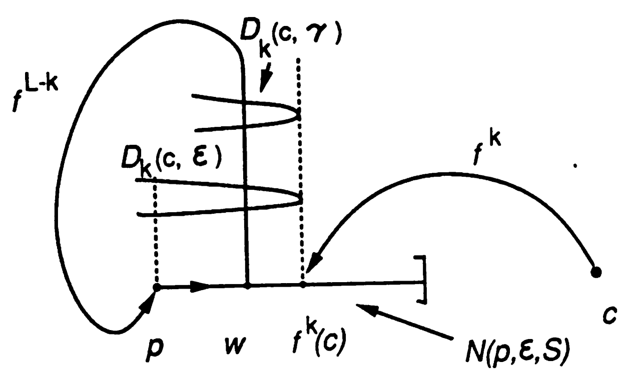


Fig. 4.1.2.

ii) Suppose that $f^k(c) = p$ for some k. If p is a normal periodic repellor, then there are $\varepsilon_0, \delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$, the δ-psedo orbit $\{x_j\}$ defined by

$$x_j = \begin{cases} f^j(c) & \text{for } 0 \leq j \leq k-1; \\ f^{j-k}[f^k(c) - \sigma_k(c)\delta/2] & \text{for } j \geq k. \end{cases}$$

is not ε_0-shadowed. ∎

From (4.1.2) we see that a second necessary condition for $f \in \mathcal{N}_0$ to have the Shadowing Property is

(4.1.3) **Condition 2.** *For each turning point $c \in C(f)$, either $\omega(c, f)$ contains no periodic repellors or every periodic repellor p in $\omega(c, f)$ satisfies*

$$p \in orb(c, f) \cap C(f).$$

For $c \in C(f)$ and some $i < \infty$, if $\omega(c, f)$ is contained in $M_i \setminus M_{i+1}$, then by (2.3.3) it is a periodic orbit. Thus, for $f \in \mathcal{N}_0$ satisfying (4.1.1), (4.1.3):

(4.1.4) *Every $c \in C(f)$ possesses one of the following properties:*

1) $\omega(c, f)$ is a periodic attractor;

2) $\omega(c, f) \subset M_\infty$;

3) the orbit of c hits a critical periodic repellor.

4.2. Existence of filtrations for $f \in \mathcal{N}$.

Assume $f \in \mathcal{N}$ satisfies (4.1.1). According to (2.2.1) and (2.2.2), there is a finitely generated quasi-filtration $\{M_i\}$ for f, that is, for each $i < \infty$, M_i possesses finitely many connected components $B_1, \ldots, B_n$. The action of f over $\{B_k\}$ defines a function $\phi_n : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $f(B_k) \subset B_{\phi_n(k)}$. Say M_i is *cyclic* if ϕ_n is a permutation over $\{1, \ldots, n\}$. In this case, $\{B_k\}$ can be renumbered so that $\phi_n(j) \equiv j + 1 \pmod{n}$. An *enlargement* N of a closed set M is a closed set such that M and N have the same number of connected components and every connected component of M is contained in exactly one connected component of N.

(4.2.1) **Remark.** 1) If M_i is strongly invariant, it is cyclic;

2) If p is in the boundary of an unstable manifold and $f^n(p) = p$ for some $n > 0$, then p is either a periodic attractor or a critical periodic repellor.

(4.2.2) **Lemma.** *For each $i < \infty$, there exists an enlargement $\hat{M}_i$ of M_i such that $f^n(\hat{M}_i) \subset Int\ \hat{M}_i$ for some $n > 0$.*

Proof. Suppose B is a connected component of M_i, by (4.2.1), there is $n > 0$ such that $f^n(B) \subset B$. Let p, q be the endpoints of B, then we have either $f^n(p) = p$ or $f^n(p) \in Int\ B$

or $f^n(p) = q$. By the continuity of f, (4.2.1) and (3.1.1), it is easy to find closed subintervals V containing p and W containing q such that $f^n(\hat{B}) \subset Int\ \hat{B}$ for $\hat{B} = B \cup V \cup W$.

Let $\{B_\ell\}$ be the set of all the connected components of M_i. Choose $\hat{B}_\ell$ as above for each B_ℓ and let $\hat{M}_i = \bigcup_\ell \hat{B}_\ell$, then

$$f^n(\hat{M}_i) \subset Int\ \hat{M}_i. \quad \blacksquare$$

(4.2.3) **Lemma.** *If $\hat{M}_i$ is as in (4.2.2), then there is a cyclic enlargement F_i for $\hat{M}_i$ such that $f(F_i) \subset Int\ F_i$ (i.e., we can take $n = 1$ in (4.2.2)).*

Proof. Suppose $\hat{B}_0, \ldots, \hat{B}_{n-1}$ are the connected components of $\hat{M}_i$. Let $A_{k,0} = \hat{B}_k$, and $A_{k,j}$ is closed subintervals inductively defined so that

$$f(A_{k,j}) \subset Int\ A_{k,j+1} \subset A_{k,j+1} \subset f^{-n+j+1}(Int\ A_{k,0}).$$

Thus, $A_{k,0}, \ldots, A_{k,n-1}$ satisfies that for each j, $f(A_{k,j}) \subset Int\ A_{k;\phi_n(j)}$, where, $\phi_n(j) \equiv j+1 \pmod{n}$.

For $\ell = 0, \ldots, n-1$, define

$$C_\ell = A_{0,\ell} \cup A_{1,\ell-1} \cup \ldots \cup A_{\ell,0} \cup A_{\ell+1,n-1} \cup \ldots \cup A_{n-1,\ell+1},$$

then

$$\begin{aligned} f(C_\ell) &\subset f(A_{0,\ell}) \cup \ldots \cup f(A_{\ell,0}) \cup f(A_{\ell+1,n-1}) \cup \ldots \cup f(A_{n-1,\ell+1}) \\ &\subset Int\ A_{0,\phi_n(\ell)} \cup \ldots \cup Int\ A_{\ell,\phi_n(0)} \cup Int\ A_{\ell+1,\phi_n(n-1)} \cup \ldots \cup Int\ A_{n-1,\phi_n(\ell+1)} \\ &\subset Int\ C_{\phi_n(\ell)}. \end{aligned}$$

Let $F_i = \bigcup_\ell C_\ell$, then $f(F_i) \subset Int\ F_i$. $\blacksquare$

If we choose the $A_{k,j}$'s carefully in the proof of (4.2.3), for $i < \infty$ we can construct an enlargement F_i of $\hat{M}_i$ so that $F_i \setminus M_i$ contains no periodic attractors, periodic repellors and turning points of f (cf. (2.3.3)). Moreover, we have the following

(4.2.4) **Theorem.** *Suppose $f \in \mathcal{N}$ does not have any neutral periodic point, then there exists a finitely generated filtration $\mathcal{F} = \{F_i\}$ adapted to f. In particular, if $\mathcal{R}$ is infinite, then the quasi-attractor*

$$F_\infty = \bigcap_{i=0}^{\infty} F_i$$

is noncyclic.

Proof. We need only to show that if $\mathcal{R}$ is infinite, then $F_\infty \cap Per(f) = \emptyset$. Notice that

$$F_\infty \setminus M_\infty \subset \bigcup_{i=0}^{\infty}(F_i \setminus M_i).$$

It follows from $Per(f) \cap M_\infty = \emptyset$ and $Per(f) \cap (F_i \setminus M_i) = \emptyset$ for all $i < \infty$ that $Per(f) \cap F_\infty = \emptyset$. ∎

(4.2.5) **Example.** Now let us consider a unimodal map given by [Ni]. This map is defined by the recursive step so as reverse the map before re-scaling, then the process yields a sequence g_i converging to a unimodal map g possessing the periods $2^k, k = 0, 1, \ldots$ and nothing else (cf. Fig. 4.2.5).

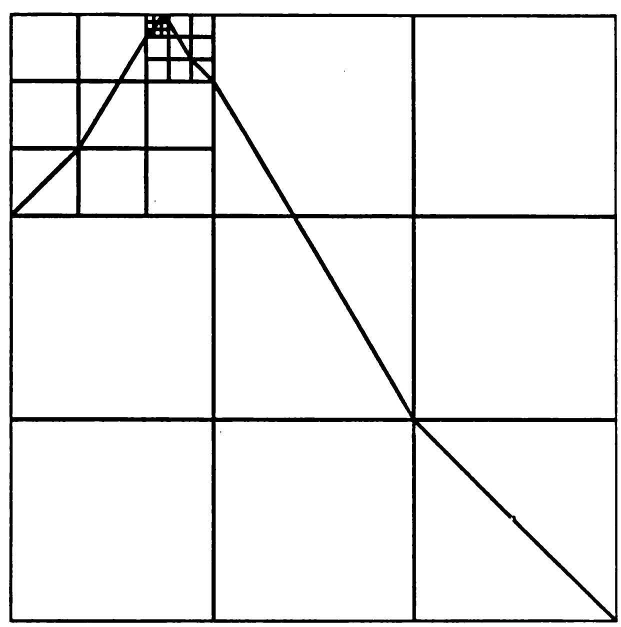
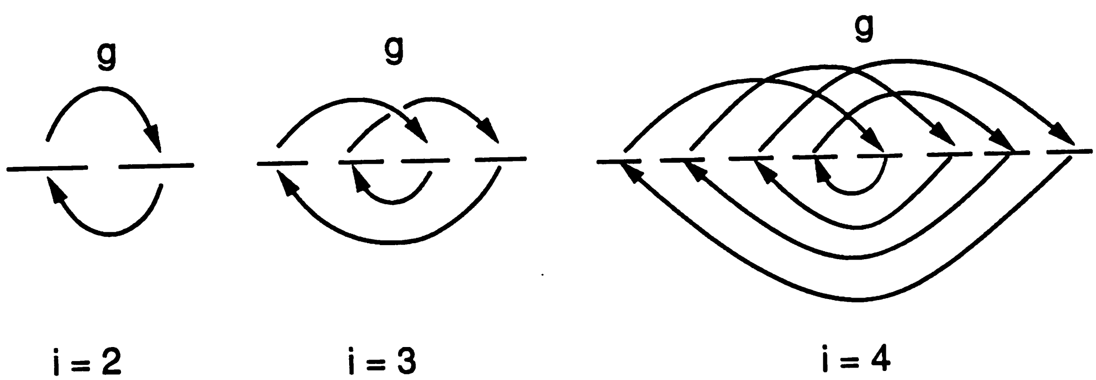


Fig. 4.2.5. The graph of g

The unstable manifolds M_i can be determined by induction: let $M_1 = [0,1]$ and M_{i+1} is obtained from M_i by deleting the middle third of each component. One endpoint of M_i is also an endpoint of M_j for all $j \geq 0$, and these points constitute the images under g of the turning point $c = 1/4$. Each component of $M_i \setminus M_{i+1}$ contains a single periodic point, which is its midpoint and has period 2^{i-1}. It is an orientation-reversing periodic repellor. The map g is linear on each component of $M_i \setminus M_{i+1}$; the slope is ± 1 except on one component where it equals $\pm 5/3$. Notice that the preimage of c under g is dense in $[0,1]$, g is nondegenerate and every periodic point of g is a periodic repellor implies that g does not have neutral periodic points.

Let $F_1 = [0,1]$ and inductively construct F_i for $i \geq 2$. Assume F_i has been defined which is an enlargement of M_i and contains no periodic orbits of periods less than 2^{i-1}. Since M_i contains a single repelling periodic orbit $x_0, \ldots, x_{2^{i-1}-1}$ of period 2^{i-1}, so does F_i. Delete a small open neighborhood $B_\varepsilon(i) = (x_0 - \varepsilon, x_0 + \varepsilon) \cup \ldots \cup (x_{2^{i-1}-1} - \varepsilon, x_{2^{i-1}-1} + \varepsilon)$ of this orbit from F_i, we define $F_{i+1} = F_i \setminus B_\varepsilon(i)$. When ε small enough, we must have $g(F_{i+1}) \subset Int\ F_{i+1}$ and F_{i+1} is an enlargement of M_{i+1}. Under the action of g, the components of F_i form an f-cycle of length 2^{i-1}.



Thus, $F_\infty = \bigcap_i F_i$ is an f-cycle of length 2^∞, that is, $\{F_i\}$ is a finitely generated filtration for g generating a noncyclic quasi-attractor F_∞.

4.3. Shadowing on $K_N^{(i)}$.

Assume $f \in \mathcal{N}$ satisfies (4.1.1). Let $\mathcal{F}$ be a filtration as in (4.2.4), Λ_i the set of all the periodic attractors in $F_i \setminus F_{i+1}$ and W_i a basin of Λ_i. Define

$$K^{(i)} := \bigcap_{k=0}^{\infty} f^{-k}[F_i \setminus (F_{i+1} \cup W_i)].$$

Since $f(F_{i+1} \cup W_i) \subset Int\ (F_{i+1} \cup W_i)$, $K^{(i)}$ is closed and invariant under f.

(4.3.1) **Proposition.** *If $f \in \mathcal{N}_0$, then, for $i < \infty$,* *1) the nonwandering set $\Omega_i = \Omega(f) \cap K^{(i)}$ of f in $K^{(i)}$ consists of a single periodic repellor orbit.*

2) if $(p, S) \in \alpha_i$, then $p \in \Omega_i$ and $K^{(i)} = orb(p, f)$.

Proof. 1) is an immediate consequence of (2.3.2) and (2.3.3); 2) is implied by an easy fact: if there exists $x \in U(p, S, f) \setminus orb(p, f)$ with $\omega(x, f) = orb(p, f)$, then p has a homoclinic point (cf. Fig. 4.3.1).

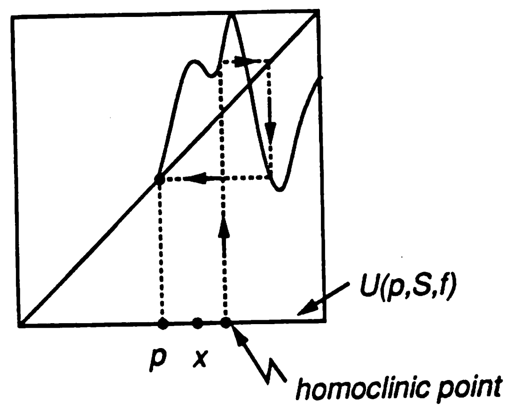


Fig. 4.3.1.

Now, assume $f \in \mathcal{N}_0$ satisfies (4.1.1) and (4.1.3). From (2.3.3) and (2.3.4) it follows that for each $i < \infty$ every nonperiodic $c \in C(f) \cap F_i$ satisfies

$$\omega(c, f) \subset F_{i+1} \cup \Lambda_i.$$

Thus, there is n such that every nonperiodic point in $C(f) \cap F_i$ is contained in

$$U_n = \bigcup_{j=0}^{n} f^{-j}(F_{i+1} \cup W_i),$$

where W_i is a carefully chosen basin of Λ_i so that the set $\mathcal{V}_n^{(i)}$ of the connected components of $F_i - U_n$ is finite. If some turning point in an element of $\mathcal{V}_n^{(i)}$ is periodic, then by (4.1.4) and the choice of U_n it is a critical periodic repellor.

Let $\Im_0^{(i)}$ be the set consisting of either those $C \in \mathcal{V}_n^{(i)}$ which contain no critical periodic repellors or the respected side connected components of $C \setminus \{p\}$ when $C \in \mathcal{V}_n^{(i)}$ contains a critical periodic repellor p. $\mathcal{V}_n^{(i)}$ is finite, so is $\Im_0^{(i)}$. Represent $\mathcal{V}_n^{(i)}$ by $\Im_0^{(i)} = \{I_1, \ldots, I_m\}$. Let

$$\begin{aligned}\Im_n^{(i)} &:= \bigwedge_{k=0}^{n-1} f^{-k}(\Im_0^{(i)}) \\ &= \{\, I_{a_0 \ldots a_{n-1}} \mid I_{a_0 \ldots a_{n-1}} = I_{a_0} \cap f^{-1}(I_{a_1}) \cap \ldots \cap f^{-(n-1)}(I_{a_{n-1}}) \neq \emptyset\}\end{aligned}$$

and

$$\Gamma_n^{(i)} := \{\underline{\alpha} \in \prod_{\ell=0}^{n-1} E_\ell |\ I_{\underline{\alpha}} \in \Im_n^{(i)}\}$$

where $E_\ell = \{1, \ldots, m\}$. It is easy to check $\Im_n^{(i)}$ has the following properties:

1) for each $\underline{\alpha} \in S_n^{(i)}$, $f|I_{\underline{\alpha}}$ is a homeomorphism from $I_{\underline{\alpha}}$ onto its image;

2) for any two $I_{\underline{\alpha}}, I_{\underline{\beta}} \in \Im_n^{(i)}$, either $I_{\underline{\beta}} \cap f(I_{\underline{\alpha}}) = \emptyset$ or $I_{\underline{\beta}} \subset f(I_{\underline{\alpha}})$.

Define $\sigma_n : \Gamma_n^{(i)} \to \Gamma_{n-1}^{(i)}$ by $\sigma_n(a_0 \ldots a_{n-1}) = a_1 \ldots a_{n-1}$, then for all $\underline{\alpha} \in \Gamma_n^{(i)}$, $f(I_{\underline{\alpha}}) = I_{\sigma_n(\underline{\alpha})}$.

Let

$$K_n^{(i)} := \bigcup \{I_{\underline{\alpha}} |\ \underline{\alpha} \in \Gamma_n^{(i)}\}$$

and call it *the n^{th}-approximation of $K^{(i)}$*. The main result in this section is

(4.3.2) **Proposition.** *For given integer $n, L \geq 1$, there is $\eta > 0$ such that if $\{x_0, \ldots, x_{L-1}\}$ is a (η, L)-chain in $K_n^{(i)}$ and $\underline{\alpha}_0, \ldots, \underline{\alpha}_{L-1}$ are the elements of $\Gamma_n^{(i)}$ with $x_j \in I_{\underline{\alpha}_j}$, then there exists $\underline{\beta} \in \Gamma_{L+n-1}^{(i)}$ such that for all j, $0 \leq j \leq L-1$, $f^j(I_{\underline{\beta}}) \subset I_{\underline{\alpha}_j}$.*

Proof. Define $h_n : K_n^{(i)} \to \Gamma_n^{(i)}$ by $h_n(x) = \underline{\alpha}$ if $x \in I_{\underline{\alpha}}$, $\underline{\alpha} \in \Gamma_n^{(i)}$. Then we have a commutative diagram as follows:

$$\begin{array}{ccc} K_{n+1}^{(i)} & \xrightarrow{\imath} & K_n^{(i)} \\ {\scriptstyle h_{n+1}} \downarrow & & \downarrow {\scriptstyle h_n} \\ \Gamma_{n+1}^{(i)} & \underset{\jmath_n}{\longrightarrow} & \Gamma_n^{(i)} \end{array}.$$

where $\imath$ is the inclusion and $\jmath_n$ is defined as $\jmath_n(a_0 \ldots a_n) = a_0 \ldots a_{n-1}$. Under the relative topology of I in $K_n^{(i)}$ and the discrete topology in $\Gamma_n^{(i)}$, $h_n : K_n^{(i)} \to \Gamma_n^{(i)}$ is continuous. Similarly, we have a commutative diagram:

$$\begin{array}{ccc} K_{n+1}^{(i)} & \xrightarrow{f} & K_n^{(i)} \\ {\scriptstyle h_{n+1}} \downarrow & & \downarrow {\scriptstyle h_n} \\ \Gamma_{n+1}^{(i)} & \underset{\sigma_n}{\longrightarrow} & \Gamma_n^{(i)} \end{array}$$

Suppose $\{x_0, \ldots, x_{L-1}\} \subset K_n^{(i)}$ and $\underline{\alpha}_j = h_n(x_j)$, where

$$\underline{\alpha}_j = a_0^j a_1^j \ldots a_{n-1}^j \qquad for\ j = 0, \ldots, L-1.$$

By the continuity of h_n, choose η small enough that if $\{x_0, \ldots, x_{L-1}\}$ is an (η, L)-chain in $K_n^{(i)}$, then

$$a_s^r = a_{s'}^{r'}, \quad \text{if} \quad r + s = r' + s'$$

for $1 \leq r+s \leq L+n-3$. Write

$$b_{r+s} = a_s^r \qquad for\ 0 \leq r+s \leq L+n-2,$$

then

$$\underline{\alpha}_j = b_j b_{j+1} \dots b_{j+n-1} \qquad for\ j = 0, \dots, L-1.$$

If let $\underline{\beta} = b_0 b_1 \dots b_{L+n-2} \in \Gamma_{L+n-1}^{(i)}$, then for all j, $0 \leq j \leq L-1$, $f^j(I_{\underline{\beta}}) \subset I_{\underline{\alpha}_j}$. ∎

Applying (4.3.2) to $L = 1$, we have

(4.3.3) **Corollary.** *Given $n \geq 0$, there is $\delta > 0$ such that if $x, y \in K_n^{(i)}$ with $|f(x) - y| < \delta$ and $x \in I_{\underline{\alpha}}, y \in I_{\underline{\alpha}'}$ for some $\underline{\alpha}, \underline{\alpha}' \in \Gamma_n^{(i)}$, then*

$$I_{\underline{\alpha}'} \subset f(I_{\underline{\alpha}}).$$

In particular, any δ-pseudo orbit $\{x_j\}$ in $K_n^{(i)}$ has

$$I_{\underline{\alpha}_j} \subset f(I_{\underline{\alpha}_{j-1}}),$$

where $\underline{\alpha}_j \in \Gamma_n^{(i)}$ satisfies $x_j \in I_{\underline{\alpha}_j}$.

4.4. Shadowing for the family $\mathcal{N}_0$.

In this section, we shall prove (1.2). Assume $f \in \mathcal{N}_0$ satisfies (4.1.1) and (4.1.3).

Suppose $I = [a, b]$ and $c_0 = a < c_1 < \dots < c_q = b$ are the turning points of $f \in \mathcal{N}_0$. Let $J_w = [c_w, c_{w+1}]$ and

$$J_{w_0 w_1 \dots w_s} = \bigcap_{j=0}^{s} f^{-j}(J_{w_j})$$

where $w, w_j \in \{0, 1, \dots, q-1\}$. By definition, we have $f(J_{w_0 \dots w_s}) \subset J_{w_1 \dots w_s}$.

Thus, for a given i we can choose n big enough that $K_n^{(i)}$ is disjoint from the orbits of the turning points of f whose ω-limit sets are contained in $F_{i+1} \cup W_i$. Then, any turning point c in $K_n^{(i)}$ is a critical periodic repellor and by definiton $Int\ K_n^{(i)} \cap orb(c, f) = \emptyset$. Furthermore,

$$Int\ K_n^{(i)} \cap \bigcup_{j=0}^{\infty} f^j[C(f)] = \emptyset. \tag{4.4.1}$$

This implies that if $K_n^{(i)} \subset J_v$, then for those $w_0, \dots, w_{s-1}$ with $J_{w_0 \dots w_{s-1} v} \neq \emptyset$

$$f^s[f^{-s}(K_n^{(i)}) \cap J_{w_0 \dots w_{s-1} v}] = K_n^{(i)}. \tag{4.4.2}$$

Moreover, we have

(4.4.3) **Lemma.** *For every $s \geq 0$, if $K_n^{(i)} \subset J_v$, then*

$$f^{-s}(K_n^{(i)}) = \bigcup_{w_0,\ldots,w_{s-1}} [f^{-s}(K_n^{(i)}) \cap J_{w_0\ldots w_{s-1}v}].$$

(4.4.4) **Lemma.** *Given $\varrho > 0$ and s, there is $\tau > 0$ so that for every $x \in f^{-s}(K_n^{(i)})$ and $y \in B(f^s(x), \tau) \cap K_n^{(i)}$, there is $z \in B(x, \varrho) \cap f^{-s}(K_n^{(i)})$ satisfying: (i) $f^s(z) = y$ and (ii) for every $j = 0, 1, \ldots, s$, $|f^j(x) - f^j(z)| < \varrho$.*

Proof. By the uniformly continuity of f, given s and ϱ, there is $\sigma \in (0, \varrho)$ such that for every $j = 0, 1, \ldots, s$, $|f^j(x) - f^j(y)| < \varrho$ whenever $|x - y| < \sigma$.

If $x \in f^{-s}(K_n^{(i)})$, then $x \in f^{-s}(K_n^{(i)}) \cap J_{w_0\ldots w_s}$ for some $w_0, \ldots, w_s$. Notice that $g_{w_0\ldots w_s} = f^s | J_{w_0\ldots w_s}$ is a homeomorphism of $J_{w_0\ldots w_s}$ onto $f^s(J_{w_0\ldots w_s})$, hence there is $\tau > 0$ such that for every $p \in f^s(J_{w_0\ldots w_s})$

$$g_{w_0\ldots w_s}^{-1}[B(p, \tau) \cap f^s(J_{w_0\ldots w_s})] \subset B(g_{w_0\ldots w_s}^{-1}(p), \sigma).$$

Then for any point $y \in B(f^s(x), \tau) \cap K_n^{(i)}$, we have $z = g_{w_0\ldots w_s}^{-1}(y) \in B(g_{w_0\ldots w_s}^{-1} \circ f^s(x), \sigma) \cap f^{-s}(K_n^{(i)}) = B(x, \sigma) \cap f^{-s}(K_n^{(i)})$. This implies that $f^s(z) = y$ and $|f^j(x) - f^j(z)| < \varrho$ for every $j = 0, 1, \ldots, s$. ∎

For a given n, write $N_n^{(i)} = f^{-n}(F_{i+1})$, $V_n^{(i)} = f^{-n}(W_i)$, then

$$F_i = K_n^{(i)} \cup V_n^{(i)} \cup N_n^{(i)}.$$

Given $\ell > 0$, by induction, for each $0 \leq i \leq \ell$ we have

$$\begin{aligned} F_i = & [K_n^{(i)} \cup f^{-n}(K_n^{(i+1)}) \cup \ldots \cup f^{-(\ell-i-1)n}(K_n^{(\ell-1)})] \\ & \cup [V_n^{(i)} \cup f^{-n}(V_n^{(i+1)}) \cup \ldots \cup f^{-(\ell-i-1)n}(V_n^{(\ell-1)})] \cup f^{-(\ell-i)n}(F_\ell). \end{aligned}$$

(4.4.5) **Remark.** i) It is easy to see that the set

$$E_n^{i,\ell} = [V_n^{(i)} \cup f^{-n}(V_n^{(i+1)}) \cup \ldots \cup f^{-(\ell-i-1)n}(V_n^{(\ell-1)})]$$

is a basin of the periodic attractors $\Lambda_i \cup \Lambda_{i+1} \cup \ldots \cup \Lambda_{\ell-1}$;

ii) $f[f^{-(\ell-i)n}(F_\ell)] \subset Int\ f^{-(\ell-i)n}(F_\ell)$.

(4.4.6) **Lemma.** *Assume $A(\alpha_j) \subset A(\alpha_i)$. For any $\varepsilon > 0$, there exist N, L, m and $\delta > 0$ such that if $\{x_k\}$ is a δ-pseudo orbit satisfying that for some $l \geq 0$, $x_k \in K_L^{(i)}$ for all $k \leq l$ and $x_k \in K_N^{(j)}$ for all $k \geq l+m$, then $\{x_k\}$ is ε-shadowed by a point in $K_L^{(i)}$.*

Proof. Choose L big enough that $\|K_L^{(i)}\| \leq \varepsilon/2$ and there is $\eta \in (0, \varepsilon)$ so that (4.3.3) holds in $K_L^{(i)}$. By assumption, there is $m > 0$ such that for $(p_j, S_j) \in \alpha_j$, $N(p_j, \varepsilon, S_j) \subset f^m(K_L^{(i)})$. Thus, there exists $\tau > 0$ so that (4.4.4) holds for $\varrho = \eta/2$ and $s = m$. Then choose N big enough that $\|K_N^{(j)}\| \leq \min\{\tau, \varepsilon/2\}$ and $K_N^{(j)} \subset f^m(K_L^{(i)})$. By the uniformly continuity of f, there is $\sigma > 0$ such that every (σ, m)-chain $\{z_0, \ldots, z_m\}$ is $\varepsilon/2$-shadowed by z_0 (cf. (3.1.2)). Let $\delta = \min\{\tau, \eta/2, \sigma\}$.

If $\{x_k\}$ is a δ-pseudo orbit satisfies that for some $l \geq 0$,

$$x_k \in K_L^{(i)} \textit{ for all } k \leq l \textit{ and } x_k \in K_N^{(j)} \textit{ for all } k \geq l+m, \tag{4.4.7}$$

then by (4.3.3), $I_{\underline{\alpha}_{k+1}}^{(i)} \subset f(I_{\underline{\alpha}_k}^{(i)})$, where $I_{\underline{\alpha}_k}^{(i)}$ is a component of $K_L^{(i)}$ with $x_k \in I_{\underline{\alpha}_k}^{(i)}$ for $k \leq l$ and $I_{\underline{\alpha}_{k'+1}}^{(j)} \subset f(I_{\underline{\alpha}_{k'}}^{(j)})$, where $I_{\underline{\alpha}_{k'}}^{(j)}$ is a component of $K_N^{(j)}$ with $x_{k'} \in I_{\underline{\alpha}_{k'}}^{(j)}$ for $k' \geq l+m$. Thus, there exist $a \in \bigcap_{0 \leq k \leq l} f^{-k}(I_{\underline{\alpha}_k}^{(i)})$, $b = x_l$ and $c \in \bigcap_{k \geq l+m} f^{-k}(I_{\underline{\alpha}_k}^{(i)})$. Then, a $\varepsilon/2$-shadows $\{x_0, \ldots, x_l\}$ since $f^k(a), x_k \in I_{\underline{\alpha}_k}^{(i)}$ for $k \leq l$ and $\|K_L^{(i)}\| \leq \varepsilon/2$; similarly, c $\varepsilon/2$-shadows $\{x_{l+m}, \ldots\}$ and b $\varepsilon/2$-shadows $\{x_l, \ldots, x_{l+m}\}$.

Now, to show that $\{x_k\}$ is ε-shadowed, we need only to show $\{y_k\}$ defined by

$$y_k = \begin{cases} f^k(a) & k \leq l-1; \\ f^{k-l}(b) & l \leq k \leq l+m-1; \\ f^{k-(l+m)}(c) & k \geq l+m \end{cases}$$

is $\varepsilon/2$-shadowed.

Since $K_N^{(j)} \subset f^m(K_L^{(i)})$ and $|f^m(b) - c| \leq \delta \leq \tau$, by (4.4.4), there exists $b' \in K_L^{(i)}$ such that $f^m(b') = c$ and $|f^k(b') - f^k(b)| \leq \eta/2 \leq \varepsilon/2$ for $k = l, \ldots, l+m$. Notice that $|f^l(a) - b'| \leq |f^l(a) - b| + |b - b'| \leq \delta + \eta/2 \leq \eta$, thus $\{a, f(a), \ldots, f^{l-1}(a), b'\}$ is an (η, l)-chain and there exists $a' \in K_L^{(i)}$ such that $f^l(a') = b'$ and $|f^k(a) - f^k(a')| \leq \|K_L^{(i)}\| \leq \varepsilon/2$ for $k \leq l$. This implies that a' $\varepsilon/2$-shadows $\{y_k\}$. ∎

We call the orbit of a' in the proof of (4.4.6) a *typical orbit* from $K_N^{(i)}$ to $K_L^{(j)}$.

(4.4.8) **Proposition.** *Suppose $f \in \mathcal{N}_0$ satisfies (4.1.1) and (4.1.3), then for every $\ell > 0$, f has the Shadowing Property on $I \setminus F_\ell$.*

Proof. Given ℓ and $\varepsilon > 0$, inductively choose L_i for $i = 0, 1, \ldots, \ell - 1$ and $\delta > 0$ so that if $A(\alpha_j) \subset A(\alpha_i)$, then there is m_{ij} such that $K_{L_j}^{(j)} \subset f^{m_{ij}}(K_{L_i}^{(i)})$ and $\|K_{L_i}^{(i)}\|$ is small enough that any δ-pseudo orbit $\{x_k\}$ of form (4.4.7) is ε-shadowed by a typical orbit from $K_{L_i}^{(i)}$ to $K_{L_j}^{(j)}$. It is clear that if $A(\alpha_i) \cap A(\alpha_j) = \emptyset$ for $i < j$, then there is no typical orbits from $K_{L_i}^{(i)}$ to $K_{L_j}^{(j)}$. Moreover, by the uniformly continuity of f, δ can be chosen small enough that there is no δ-orbits of form (4.4.7) for these i and j. The ε-shadowing of δ-pseudo orbits in $I \setminus F_\ell$ is equivalent to that for those δ-pseudo orbits which are piecewise typical orbits. For any two pieces of typical orbits, we can extend one of them as in the proof of (4.4.6) to get a new typical orbit. Repeating this process for the new typical orbits, finally we can get actual orbits as follows:

1) actual orbits eventually in F_ℓ;
2) actual orbits eventually in $E_n^{i,\ell}$ for some $0 \le i \le \ell - 1$;
3) actual eventually periodic orbits in $I \setminus (F_\ell \cup \bigcup_{i=0}^{\ell-1} E_n^{i,\ell})$.

These actual orbits $\ell\varepsilon$-shadow the corresponding piecewisely typical orbits and we are done. ∎

(4.4.9) **Proof of (1.2).** Our main theorem (1.2) has been shown by (4.4.8) and (3.4.1).

4.5. Examples in the unimodal case.

A map $f : [-1,1] \to [-1,1]$ is $\mathcal{S}$*-unimodal* if

1) $f \in C^3[-1,1]$, $f(-1) = f(1)$, $f(0) = 1$;
2) $f'(x) \neq 0$ except $x = 0$ and $f''(0) < 0$;
3) The Schwartzian derivative $Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 < 0$ for all $x \neq 0$.

If f is an $\mathcal{S}$-unimodal map, $C(f) = \{-1, 0, 1\}$ and $\omega(-1, f) = \omega(0, f) = \omega(1, f)$, denoted by ω. A periodic orbit $\mathcal{O}$ of f with least period n is *stable* if $|(f^n)'(x)| \le 1$ for all $x \in \mathcal{O}$.

It is known (cf. [CE]) that an $\mathcal{S}$-unimodal map f has at most one stable periodic orbit. If this stable periodic orbit $\mathcal{O}$ exists, then $\omega = \mathcal{O}$. If f has no stable periodic orbits, then the iterated preimages of 0 under f are dense in $[-1,1]$; meanwhile, f is nondegenerate and does not have neutral periodic points (since a neutral periodic point p with least period n satisfies $|(f^n)'(p)| = 1$). In particular, if a zero-entropy $\mathcal{S}$-unimodal map has no stable periodic orbits, then, up to topological conjugacies, it is unique. The following is an example of zero-entropy $\mathcal{S}$-unimodal maps having no stable periodic orbit:

(4.5.1) **Example.** The quadratic map $f_\mu : [-1,1] \to [-1,1]$ defined by $f_\mu(x) = 1 - \mu x^2$ is $\mathcal{S}$-unimodal. Let μ_n be the value of μ for which f_{μ_n} has a stable periodic orbit of period 2^n. Feigenbaum and Coullet-Tresser made the following observations:

1) μ_n approaches some number $\mu_\infty = 1.401155...$;
2) the ratio $(\mu_n - \mu_{n+1})/(\mu_{n+1} - \mu_{n+2})$ approaches a universal number $\delta = 4.66920...$;
3) $\frac{1}{\lambda^m} f_{\mu_\infty}^{2^m}(\lambda^m x)$ approaches a universal function

$$f(x) \sim 1 - 1.52763x^2 + 0.104815x^4 - 0.0267057...x^6 + \dots$$

f satisfies the equation

$$\phi \circ \phi(\lambda x) = \lambda \phi(x)$$

where $\lambda = \phi(1)$.

Some authors refer to f_{μ_∞} as *the Feigenbaum map* (cf. Fig. 4.5.1). This map corresponds to a "stable periodic orbit of length 2^∞". In other words, the set ω is contained in an attracting Cantor set Λ which contains no periodic points. This implies that f_{μ_∞} has no stable periodic orbits. f_{μ_∞} possesses a single periodic orbit of period 2^k for each $k = 0, 1, \dots$, and nothing else. Thus, $h(f_{\mu_\infty}) = 0$, furthermore f_{μ_∞} has the Shadowing Property.

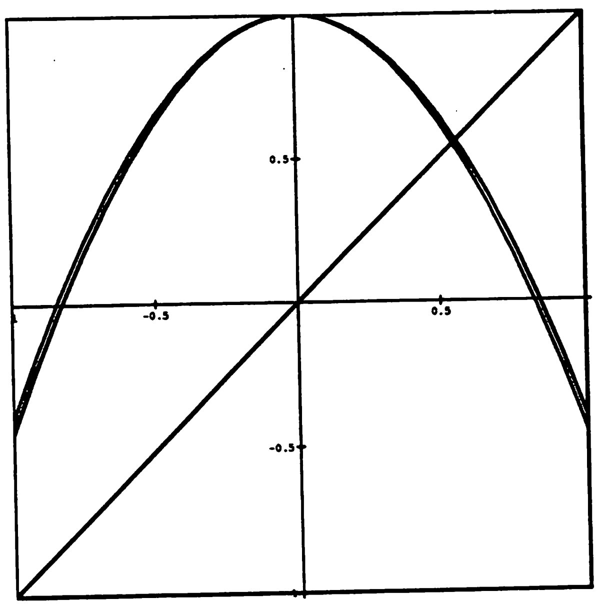


Fig. 4.5.1. The graph of f_{μ_∞}, f

(4.5.2) **Example.** The non-smooth map defined in (4.2.5) has the Shadowing Property, for the ω-limit set $\omega(c, g)$ is contained in a Cantor set which is semi-conjugate to an adding machine, so it is disjoint from $Per(g)$.

Acknowledgement. I would like to thank the Institute for Mathematical Sciences for support during the time that this paper was completed. I would also like to thank for helpful discussions with my adviser Z. Nitecki, for useful suggestions from J. Milnor, E. Coven and D. Benardet.

References

[Ano] D.V.Anosov, On a class of invariant sets for smooth dynamical systems, Proc. 5th International Conf. on Nonlinear Oscillations, vol.2 (1970: Math. Inst. Ukrainian Acad. Sci., Kiev), 39-45.

[BGMY] L.Block, J.Guckenheimer, M.Misiurewicz and L.Young, Periodic points and topological entropy of one dimensional maps, *Lect. Notes Math. 819 (Springer 1980) 18-34.*

[Bl] L.Block, Homoclinic points of mappings of the interval, *Proc. AMS 72(1978) 576-580.*

[Bow] R.Bowen, On Axiom A diffeomorphisms, CBMS Regional Conference Series in Math., No.35, 1978.

[BoG] A.Boyarsky, P.Góra, The pseudo-orbit shadowing property for Markov operators in the space of probability density functions, *preprint.* (1989)

[Ch] L.Chen, Linking and the Shadowing Property for piecewise monotone maps, 1989, *to appear.*

[CE] P.Collet, J.P.Eckman, *Iterated maps on the interval as dynamical systems*, Birkhäuser, 1980.

[CKY] E.M.Coven, I.Kan, J.A.Yorke, Pseudo-orbit shadowing in the family of tent maps, *Trans. Amer. Math. Soc., v.308 no.1(1988),227-241.*

[HYG] S.M.Hammel,J.A.Yorke,C.Grebogi, Do numerical orbit of chaotic dynamical processes represent true orbit, *J. pf Compleity* **3**, 136-145 (1987)

[Hof1] F.Hofbauer, on intrinsic ergodicity of piecewise monotonic transformations with positive entropy, (I,II), *Israel J. Math. 34* (1979) 213-237.

[Hof2] ———, The structure of piecewise monotonic transformations, *Ergodic Theory and Dynamical Systems* **1**(1981), 159-178.

[JoR1] L. Jonker and D. Rand, Une borne inférieure pour l'entropie de certaines applications de l'intervalle dans lui-même, *C. R. Acad. Sci. Paris 287(A)* (1978) 501-502.

[JoR2] ———, Bifurcations in one dimension, I. The nonwandering set, *Invent. Math.* **62**(1981), 347-365.

[Kan] I.Kan, Shadowing Property of Quadratic Maps, *preprint.* (1989)

[Mi1] M.Misiurewicz, Structure of mappings of an interval with zero entropy, *Publ. Math. IHES.*(1980)

[Mi2] ————, Invariant measures for continuous transformations of $[0,1]$ with zero topological entropy, *Lect. Notes Math. 729(1980) 144-152.*

[Ni] Z.Nitecki, Topological dynamics on the interval, *Ergodic Theory and Dynamical Systems II, Proceedings*, Birkhäuser, 1982, 1-73.

StonyBrook
Institute for Mathematical Sciences

SUNY, Stony Brook, New York 11794-3651
telephone: (516) 632-7318
email: IMS@math.sunysb.edu

AVAILABLE AND FORTHCOMING PREPRINTS:

1990/1. B. BIELEFELD (editor), Conformal Dynamics Problem List.
/2. A. M. BLOKH & M. YU. LYUBICH, Measurable Dynamics of S-Unimodal Maps of the Interval.
/3. J. J. P. VEERMAN & F. M. TANGERMAN, On Aubry Mather Sets.
/4. A. E. EREMENKO & M. YU. LYUBICH, Dynamical Properties of some Classes of Entire Functions.
/5. J. MILNOR, Dynamics in One Complex Variable: Introductory Lectures.
/6. J. MILNOR, Remarks on Iterated Cubic Maps.
/7. J. J. P. VEERMAN & F. M. TANGERMAN, Intersection Properties of Invariant Manifolds in Certain Twist Maps.
/8. J. J. P. VEERMAN & F. M. TANGERMAN, Scalings in Circle Maps (I).
/9. L. CHEN, Shadowing Property for Nondegenerate Zero Entropy Piecewise Monotone Maps.

Publication of this preprint series is made possible in part by a grant from the Paul and Gabriella Rosenbaum Foundation