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CONTINUITY OF CONVEX HULL BOUNDARIES

LINDA KEEN AND CAROLINE SERIES

In this paper we consider families of finitely generated Kleinian groups $\{G_\mu\}$ that depend holomorphically on a parameter μ which varies in an arbitrary connected domain in $\mathbb{C}$. The groups G_μ are quasiconformally conjugate. We denote the boundary of the convex hull of the limit set of G_μ by $\partial\mathcal{C}(G_\mu)$. The quotient $\partial\mathcal{C}(G_\mu)/G_\mu$ is a union of pleated surfaces each carrying a hyperbolic structure. We fix our attention on one component S_μ and we address the problem of how it varies with μ. We prove that both the hyperbolic structure and the bending measure of the pleating lamination of S_μ are continuous functions of μ.

1. Introduction. A discrete subgroup $G \subset PSL(2,\mathbb{C})$ is both a subgroup of aut$(\hat{\mathbb{C}})$ and a group of isometries of hyperbolic 3-space, $\mathbb{H}^3$. The regular set $\Omega = \Omega(G)$ is the subset of $\hat{\mathbb{C}}$ on which the elements of G form a normal family, and the limit set $\Lambda(G)$ is its complement. An important object of study in the Thurston theory of hyperbolic 3-manifolds is the boundary in $\mathbb{H}^3$ of the convex hull of $\Lambda(G)$. This boundary carries all of the essential geometric information about G. Its connected components are examples of what Thurston calls pleated surfaces. They carry an intrinsic hyperbolic metric with respect to which they are complete hyperbolic surfaces. Denote the convex hull boundary by $\partial\mathcal{C} = \partial\mathcal{C}(G)$. Each component of $\partial\mathcal{C}$ "faces" a certain component of Ω; more precisely, each component of $\partial\mathcal{C}$ is the image of a component of Ω under the retraction map defined in Section 2. Topologically, but not conformally, the components of $\partial\mathcal{C}/G$ are equivalent to the components of Ω/G determined by this correspondence.

Suppose now that $\{G_\mu\}$ is a family of Kleinian groups depending holomorphically on a parameter μ that varies in a connected domain $D \subset \mathbb{C}$ in such a way that the groups G_μ are all quasiconformally conjugate. For the sake of readability we often drop the G and write $\Omega(\mu)$ for $\Omega(G_\mu)$ and $\partial\mathcal{C}(\mu)$ for $\partial\mathcal{C}(G_\mu)$ etc. Let $\partial\mathcal{C}^*(\mu)$ be a particular component of $\partial\mathcal{C}(\mu)$ that faces a connected component $\Omega^*(\mu)$ of $\Omega(\mu)$. For example, $\{G_\mu\}$ might be a family of quasi-Fuchsian groups and $\Omega^*(\mu)$ one of the two components of the regular set.

In this paper we address the problem of how the pleated surface $S_\mu = \partial\mathcal{C}^*(\mu)/\operatorname{Stab}(\Omega^*(\mu))$ varies with μ, where $\operatorname{Stab}(\Omega^*(\mu))$ is the stabilizer of the component Ω^* in G_μ. A pleated surface consists of a union of totally geodesic pieces and bending lines; this set of bending lines, known as the bending or pleating locus, is a geodesic lamination on the surface. This lamination supports a natural transverse measure, the bending measure, that measures the angle through which successive flat pieces are bent as one moves along the transversal.

What we prove here is that both the hyperbolic structure and the bending measure of S_μ are continuous functions of μ. Precise statements of our results are at the beginning of Section 4.

There is a natural topology on the space of all pleated surfaces homeomorphic to a given surface. This is defined carefully in [**3**, Section 5.2]. To see that our surfaces vary continuously in this topology, one needs the continuity of the hyperbolic structure on S_μ. The result then follows easily using methods similar to those given here. It is proved in [**3**, Section 5.3] that the map from pleated surfaces to laminations with the Thurston topology is continuous. This is a weaker version of our result. All the notions, convex hull boundary, pleated surface, bending measure, etc. were introduced by Thurston in [**14**]. This paper depends heavily on the detailed account of these topics in [**5**]. A more rapid introduction to pleated surfaces is also to be found in [**13**].

We were originally led to these questions in the course of our investigations of the Maskit embedding of the Teichmüller space of a punctured torus [**7**, **9**]. Our work there is the first step in a program to define new moduli (called pleating coordinates) for more general spaces of Kleinian groups. These coordinates reflect the geometry of the convex hull boundary. As is already apparent in [**7**, **9**] and [**8**], the results in this paper play a crucial role in this plan. Although our results here are certainly not unexpected, the proofs are surprisingly non-trivial and do not appear in the literature.

The outline of our paper is as follows. In Section 2 we carefully describe the general setup outlined above. In Section 3 we recall the definitions of geodesic laminations, pleated surfaces, etc. that we need. In Section 4 we prove our theorems.

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2. The general setup. In this section we establish notation and explain the general situation that we discuss in this paper. As in the introduction, $\mathbb{H}^3$ is hyperbolic three space. Its boundary is the sphere at infinity, $\hat{\mathbb{C}}$. Let $G \subset \operatorname{aut}(\hat{\mathbb{C}})$ be a finitely generated Kleinian group. For simplicity, we always assume that our groups are torsion free. Then G acts by isometries on $\mathbb{H}^3$ and by conformal automorphisms on $\hat{\mathbb{C}}$. That part of $\hat{\mathbb{C}}$ on which the elements of G form a normal family is the regular set $\Omega = \Omega(G)$ of G and the complement is the limit set $\Lambda = \Lambda(G)$. Alternatively, Λ is the closure of the set of fixed points of loxodromic elements of G.

We form the convex hull $\mathcal{C} = \mathcal{C}(G)$ in $\mathbb{H}^3$ of Λ; that is, $\mathcal{C}$ is the intersection with $\mathbb{H}^3$ of all closed hyperbolic half-spaces of $\mathbb{H}^3 \cup \hat{\mathbb{C}}$ containing Λ. Let $\partial\mathcal{C} = \partial\mathcal{C}(G)$ be the boundary of $\mathcal{C}$. Then $\partial\mathcal{C}$ carries an intrinsic hyperbolic metric (see [**5**, Theorem 1.12.1]).

Let Ω^* be a fixed component of $\Omega(G)$. The assumption that G is finitely generated implies, by the Ahlfors finiteness theorem, that $\Omega^*/\operatorname{Stab}(\Omega^*)$ is a compact Riemann surface of finite genus from which at most finitely many points have been removed. In general, the limit set Λ is not contained in a circle. If it is, the convex core is contained in a two-dimensional hyperbolic plane (see [**14**, Chap. 8]). We make the convention in this case that the boundary of the core is "two sided". More precisely, if C is a circle on $\hat{\mathbb{C}}$ and if $\Lambda = C$, then $\partial\mathcal{C}$ is the two sided hyperbolic plane spanned by C and, if the group G is Fuchsian, each side is a separate component. If $\Lambda \subset C, \Lambda \neq C$ then $\partial\mathcal{C}$ is the union of the Nielsen regions of the top and the bottom sides and these form a single component.

The connected components of the convex hull boundary are in one-to one correspondence with the components of Ω. Each component of $\partial\mathcal{C}$ is identified with the component of Ω^* it "faces", where the meaning of "faces" is made precise as follows. If X is any closed convex subset of $\mathbb{H}^3 \cup \hat{\mathbb{C}}$, there is (see [**14**, **5**]) a canonical retraction map r from $\mathbb{H}^3 \cup \hat{\mathbb{C}}$ to X whose restriction to $\hat{\mathbb{C}}$ is defined as follows: If $x \in X$, define $r(x) = x$. If $x \in \hat{\mathbb{C}} \setminus X$, find the largest horoball tangent to $\hat{\mathbb{C}}$ at x, whose interior is disjoint from X, and set $r(x)$ to be the unique point of contact. Clearly, in this case, $r(x) \in \partial X$. Now for a component Ω^* of Ω, we define $\partial\mathcal{C}^*$ to be the image of Ω^* under the retraction onto $\mathcal{C}$. We denote the quotient $\partial\mathcal{C}^*/\operatorname{Stab}(\Omega^*)$ by $S = S(G)$. As explained in more detail in Section 3.6, the quotients $\Omega^*/\operatorname{Stab}(\Omega^*)$ and S are surfaces of the same topological type.

REMARK. It is important to note that, although the two surfaces are homeomorphic, the retraction map itself is not a homeomorphism, nor are the two quotient surfaces conformally the same. A theorem of Sullivan, proved in detail in [**5**], is the assertion that if Ω^* is simply connected, the two surfaces with their natural Poincaré metrics are Lipschitz equivalent by a map with universally bounded Lipschitz constant. Sullivan's theorem may fail, however, if Ω^* is not simply connected. For present purposes we simply need to know that the surfaces have the same topology.

In this paper we study the geometry of $S(G)$ as G varies holomorphically with respect to some parameter $\mu \in \mathbb{C}$. More precisely, we suppose that $D \subset \mathbb{C}$ is a connected domain such that for each $\mu \in D$, we have a Kleinian group G_μ. We suppose that $\mu_0 \in D$ is some base point and that for each $\mu \in D$ we have a quasiconformal homeomorphism $i_\mu \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that induces a type preserving isomorphism $\phi_\mu \colon G_{\mu_0} \to G_\mu$; that is, ϕ_μ maps parabolic and loxodromic elements to elements of the same type. We assume that the maps i_μ depend holomorphically on μ. This implies, in particular, that for each $g_{\mu_0} \in G_{\mu_0}$, the coefficients of $g_\mu = \phi(g_{\mu_0})$ vary holomorphically with μ. Such a family of groups is called a *holomorphic family of Kleinian groups.* Given a component $\Omega^*(\mu_0)$ and $\mu \in D$, we use the quasiconformal conjugacy to determine $\Omega^*(\mu)$.

To describe the geometry of S_μ, we need to discuss geodesic laminations and pleated surfaces. This we do in the next section.

3. Geodesic laminations and pleated surfaces.

3.1. Geodesic laminations. Let S be a complete oriented hyperbolic surface of finite area.

A *geodesic lamination* on S is is a closed subset L of the surface consisting of a disjoint union of simple geodesics. The components of the lamination are called its *leaves.*

A *transverse measure* ν on L is an assignment of a regular countably additive measure to every interval transversal to L. If an interval is a subinterval of another, the assigned measure agrees with the restricted measure. Moreover, these measures are preserved by any isotopy mapping one transversal to another and preserving the leaves of the lamination. We call the pair (L, ν) a *measured lamination.* By abuse of terminology we usually refer to ν as a measured lamination and write $|\nu|$ for the underlying point set L.

If γ is a simple closed geodesic on S then we denote by δ_γ the measured lamination whose leaves consist of the geodesic γ and whose measure is an atomic unit mass on γ. We denote by $\mathcal{ML}(S)$ the space of measured laminations on S. The weak topology on measures gives a natural topology on $\mathcal{ML}(S)$: a sequence $\nu_n \in \mathcal{ML}(S)$ converges to $\nu \in \mathcal{ML}(S)$ if $\int_I f d\nu_n$ converges to $\int_I f d\nu$ for any open interval I transversal to all the $|\nu_n|$ and $|\nu|$ and for any continuous function f.

There is another way to describe the topology on $\mathcal{ML}(S)$ that is more convenient for our purposes. Let $\mathcal{S}$ denote the set of free homotopy classes of simple closed curves on S. In analogy with the construction of the embedding of measured foliations into $(\mathbb{R}^+)^{\mathcal{S}}$ described in [**6**], one can define an embedding of $\mathcal{ML}(S)$ into $(\mathbb{R}^+)^{\mathcal{S}}$ as follows. Fix $\nu \in \mathcal{ML}(S)$, and let $[\omega]$ be a free homotopy class of closed curves on S. Define

$$\hat{\nu}([\omega]) = \inf_{\omega \in [\omega]} \{\nu(\omega)\}.$$

Here, the infimum is taken over curves ω consisting of arcs which are either transversal to $|\nu|$ or which run along leaves of $|\nu|$. (The ν-measure of any arc contained in a leaf of $|\nu|$ is defined to be zero.) By suitably adapting the argument in [**6**] (Exposé 6, V.1-V.) one shows that the map of $\mathcal{ML}(S)$ to $(\mathbb{R}^+)^{\mathcal{S}}$ given by $\nu \mapsto \hat{\nu}([\omega])$ is an embedding.

3.2. The length of a lamination. Suppose for the moment that S is a compact surface and that $\nu \in \mathcal{ML}(S)$. The *length* of ν, $\ell(\nu)$, is the total mass of the measure on S that is locally the product of the measure ν on transversals to $|\nu|$ and hyperbolic distance along the leaves of $|\nu|$. Note that if γ is a simple closed geodesic then $\ell(\delta_\gamma)$ is exactly the hyperbolic length in the usual sense. It is known that $\ell \colon \mathcal{ML}(S) \to \mathbb{R}^+$ is continuous. We prove a stronger version of this fact in theorem 4.5.

For our applications in [**7**] and [**8**], it will be important to allow the case in which the surface S has cusps. Thus in order to discuss the length of a lamination, we should restrict ourselves to $\mathcal{ML}_0(S)$, the set of measured laminations none of whose leaves go out to the cusp. Now it is well known that for each cusp on S there is a horocyclic neighborhood of definite size about the cusp such that any simple geodesic that enters this neighborhood goes out to infinity in the cusp. Therefore if $\nu \in \mathcal{ML}_0(S)$, its support is contained in a compact subset of S, and if T is transverse to ν, then the lengths of the leaves of $|\nu|$ between successive intersections with T are uniformly bounded, so the above definition of $\ell(\nu)$ still makes sense. It will follow from theorem 4.5 that ℓ is continuous on $\mathcal{ML}_0(S)$.

If $\{G_\mu\}$ is a holomorphic family of Kleinian groups, then we write ℓ_μ for the length function on $\mathcal{ML}_0(S_\mu)$. One of our main results is that, for $\nu \in \mathcal{ML}_0(S_\mu)$, $\mu \mapsto \ell_\mu(\nu)$ is continuous.

3.3. Pleated surfaces.

DEFINITION. A *pleated surface* in a hyperbolic 3-manifold N is a complete hyperbolic surface S together with a map $f : S \to N$ with the following property:

> Every point in S is in the interior of some geodesic arc which is mapped to a geodesic arc in N.

REMARK. If S is orientable and if $f(S)$ is contained in a single hyperbolic plane in N we follow the convention that the plane is two sided and that $f(S)$ lies on the side determined by the orientation.

DEFINITION. The *pleating locus* of a pleated surface (S, f) is the set of points in S that are contained in the interior of exactly one geodesic arc which is mapped by f to a geodesic arc. We shall often abuse language and identify both the pleated surface and the pleating locus with their images in N. The pleating locus of a connected pleated surface is always a geodesic lamination by [**3**, Lemma 5.1.4.]).

Let G be as in Section 2. Since each component $\Omega^*(G)$ is invariant under its stabilizer in G, it follows that $\partial\mathcal{C}^*/\operatorname{Stab}(\Omega^*(G))$ is a connected pleated surface with respect to its intrinsic hyperbolic metric. (See [**5**, **13**] and Section 3.5.)

3.4. Support planes and the bending measure. The pleating locus of a pleated surface has a naturally associated transverse measure, the bending measure. This is described in detail in [**5**] and [**14**] (Section 8.6). Here, we recall briefly the construction. For simplicity and ease of notation, we confine our discussion to the special case of the convex hull boundary $\partial\mathcal{C}$ of a Kleinian group G.

DEFINITION. A support plane P for $\partial\mathcal{C}$ at a point $x \in \partial\mathcal{C}$ is a hyperbolic plane P containing x such that $\partial\mathcal{C}$ is contained entirely in one of the two (closed) half spaces defined by P.

By [**5**, Corollary 1.6.3], a support plane for the limit set Λ intersects $\partial\mathcal{C}$ in either

1. part of a geodesic plane bounded by geodesics all of whose endpoints are in Λ or,
2. in a geodesic both of whose endpoints are in Λ.

Intersections of the first kind are called *flat pieces* and those of the second are called *bending lines.* Each point of $\partial\mathcal{C}$ belongs either to a flat piece or to a bending line. The pleating locus of $\partial\mathcal{C}$ consists exactly of the set of bending lines, see [**5**].

A support plane is oriented by the normal pointing outwards from $\partial\mathcal{C}$. The *bending angle* between two intersecting support planes P, Q is the absolute value of the angle $\theta(P,Q)$ between these outward normals.

Let $\pi(x)$ denote the set of oriented support planes at $x \in \partial\mathcal{C}$ and let

$$Z = Z(\mathcal{C}) = Z(G) = \{(x, P(x)) | x \in \partial\mathcal{C}, P(x) \in \pi(x)\}.$$

The topology on Z is induced from that of $\mathbb{H}^3 \times G_2(\mathbb{H}^3)$, where $G_2(\mathbb{H}^3)$ is the Grassmanian of 2-planes in $\mathbb{H}^3$.

Any path ω in Z projects to a path in $\partial\mathcal{C}$. Conversely we can extend any path on $\partial\mathcal{C}$ to a path on Z as follows: either $\pi(x)$ consists of a unique point, in which case there is nothing to do, or, following

[5, Definition 1.6.4] one can define the left and right extreme support planes P and Q at x and add to the path an arc in which the first coordinate x is fixed but the second moves continuously on the line in $G_2(\mathbb{H}^3)$ from P to Q. From now on we will assume that all paths on $\partial\mathcal{C}$ have been extended in this way to paths on Z.

Let $\omega\colon [0,1] \to Z$ be a path on $\partial\mathcal{C}$ as above. A *polygonal approximation* to ω is a sequence

$$\mathcal{P} = \{\omega(t_i) = (x_i, P_i)\} \in Z, \quad 0 = t_0 < t_1 < \ldots < t_n = 1,$$

such that $P_i \cap P_{i+1} \neq \emptyset \ \forall\ i = 0, \ldots, n-1$. By [5, Lemma 1.8.3], polygonal approximations always exist.

Let $\theta_i = \theta(P_{i-1}, P_i)$ be the bending angle between P_{i-1} and P_i, $i = 1, \ldots, n$.

DEFINITION. The *bending* $\beta(\omega)$ along ω is

$$\beta(\omega) = \inf_{\mathcal{P}} \sum_{i=1}^{n} \theta_i$$

where $\mathcal{P}$ runs over all polygonal approximations to ω.

Epstein and Marden show in [5, Section 1.11] that β defines a transverse measure on the pleating locus of $\partial\mathcal{C}$. By projection, it induces a transverse measure, also denoted β, on the pleating locus of $S = \partial\mathcal{C}^* / \operatorname{Stab}(\Omega^*(G))$ for any connected component $\partial\mathcal{C}^*$ of $\partial\mathcal{C}$.

If $\{G_\mu\}$, $\mu \in D$, is a holomorphic family of Kleinian groups we use the notation β_μ to denote the dependence of β on μ.

3.5. The intrinsic metric. As mentioned above, in [5] it is shown that there is an intrinsic metric on $\partial\mathcal{C}$ with respect to which it is a complete hyperbolic surface. Briefly, this is defined as follows.

Let ω be any path on S and let $\mathcal{P} = \{(x_i, P_i)\}$ be a polygonal approximation to ω. Let d_i be the hyperbolic length of the shortest path from x_{i-1} to x_i in the planes $P_{i-1} \cup P_i$.

DEFINITION. The *length* $\ell(\omega)$ of ω is

$$\ell(\omega) = \inf_{\mathcal{P}} \sum_{i=1}^{n} d_i$$

where again $\mathcal{P}$ runs over all polynomial approximations to ω.

The distance between two points on $\partial\mathcal{C}$ is the length of the shortest path joining them. This induces a metric on S in the obvious way.

If ω is a simple closed curve on S, denote by $\hat{\ell}([\omega])$ the length in this metric of the shortest curve in the free homotopy class $[\omega]$ of ω.

We have deliberately used the same notation ℓ for both the length on the surface S and for the lamination length. This is because if ω is a simple closed geodesic on S, the two definitions coincide. Namely, the length $\hat{\ell}([\omega])$ on S is equal to the length $\ell(\delta_\omega)$ of the measured lamination δ_ω whose support is ω and whose transverse measure is the atomic measure.

As usual we use the notation ℓ_μ to denote the dependence of ℓ on $\mu \in D$.

3.6. Transferring paths. In this section we give a proof of the well known fact that the surface $S_\mu = \partial\mathcal{C}/G_\mu$ is homeomorphic to the surface $\Omega^*(\mu)/\operatorname{Stab}(\Omega^*(\mu))$ for all groups G_μ in the holomorphic family $\{G_\mu\}$ based at $\mu_0 \in D$. We then show how to transfer paths together with their polygonal approximations from Z_0 to Z_μ.

PROPOSITION 3.1. *The surfaces*

$$S_\mu = \partial\mathcal{C}/G_\mu \text{ and } \Omega^*(\mu)/\operatorname{Stab}(\Omega^*(\mu))$$

are homeomorphic for all $\mu \in D$.

Proof. The retraction map $r\colon \Omega^* \to \partial\mathcal{C}^*$ defined in Section 2 is injective except on the inverse image of points in $\partial\mathcal{C}^*$ at which there is more than one support plane. Such points all lie on bending lines of $\partial\mathcal{C}^*$. Suppose that γ is such a bending line. The region $r^{-1}(\gamma)$ is found as follows. Let Q_1 and Q_2 be planes perpendicular to the extreme support planes along γ, and let the half plane determined by Q_i and not containing $\mathcal{C}(\Lambda)$ meet $\hat{\mathbb{C}}$ in a disk Δ_i, $i = 1, 2$. Then $\Delta_1 \cap \Delta_2$ is a lens shaped region in $\hat{\mathbb{C}}$ bounded by the circular arcs β_1 and β_2 that meet at the endpoints of γ on $\hat{\mathbb{C}}$. By convexity, the region between these arcs is entirely contained in Ω^* and it is not hard to see that it is exactly $r^{-1}(\gamma)$. We note that for distinct γ, these regions are disjoint.

Now let $\Omega^*/\sim_r$ be the quotient of Ω^* obtained by identifying points with the same image under r. Clearly $r^{-1}(\gamma)/\sim_r$ is a single open arc connecting the two endpoints of γ. Thus, by Moore's theorem, [**12**, Thm. 22], $\Omega^*/\sim_r$ is homeomorphic to Ω^*. Furthermore, r induces a homeomorphism from $\Omega^*/\sim_r$ to $\partial\mathcal{C}^*$. Since all of these operations are equivariant with respect to the action of G, it follows that $\Omega^*/\operatorname{Stab}(\Omega^*)$ and S are homeomorphic. □

An easy consequence of Proposition 3.1 is that the surfaces S_μ are homeomorphic for all μ in D. This follows since by assumption the groups G_μ are quasiconformally conjugate so that there is a quasiconformal homeomorphism between any two of the quotients $\Omega^*(\mu)/\operatorname{Stab}(\Omega^*(\mu))$ and $\Omega^*(\mu')/\operatorname{Stab}(\Omega^*(\mu'))$.

Next, if i_μ is the quasiconformal homeomorphism of $\hat{\mathbb{C}}$ that induces the isomorphism $\phi_\mu\colon G_{\mu_0} \to G_\mu$, then using Proposition 3.1 we see that it also induces a homeomorphism $S_{\mu_0} \to S_\mu$ and an isomorphism $\pi_1(S_{\mu_0}) \to \pi_1(S_\mu)$.

We also note that the inclusion $\Omega^*(\mu) \hookrightarrow \mathbb{H}^3 \cup \Omega^*(\mu)$ induces a homomorphism

$$j_\mu\colon \pi_1\left(\Omega^*(\mu)/\operatorname{Stab}(\Omega^*(\mu))\right) \cong \pi_1(S_\mu) \to \pi_1\left(\mathbb{H}^3 \cup \Omega^*(\mu)/G_\mu\right) \cong G_\mu.$$

This map is injective if and only if Ω^* is simply connected and surjective if and only if Ω^* is G-invariant. (See, for example, [**11**].)

The following proposition shows us how to transfer paths.

PROPOSITION 3.2. *There is a homeomorphism $R_\mu\colon Z_0 \to Z_\mu$ that induces an isomorphism $R_{\mu*}\colon \pi_1(S_{\mu_0}) \to \pi_1(S_\mu)$ such that R_μ is compatible with j_μ in the sense that $\phi_\mu j_{\mu_0} = j_\mu R_{\mu*}$.*

Proof. Note that the retraction map $r_\mu\colon \Omega^*(\mu) \to \partial\mathcal{C}_\mu$ extends naturally to a map $\hat{r}_\mu\colon \Omega^*(\mu) \to Z(\mathcal{C}_\mu)$ taking $z \in \Omega^*(\mu)$ to the pair $(r_\mu(z), P_r(z))$, where $P_r(z)$ is the support plane to $\partial\mathcal{C}_\mu$ tangent to the horosphere through z and $r_\mu(z)$ at $r_\mu(z)$. As remarked in [**5**], (p. 144), $\hat{r}_\mu$ is a homeomorphism. (We shall prove a more precise version of this result in Lemma 4.10 below.) Set $R_\mu = \hat{r}_\mu i_\mu \hat{r}_{\mu_0}^{-1}\colon Z_0 \to Z_\mu$. By its definition, R_μ is obviously a homeomorphism conjugating the actions of G_0 on Z_0 and G_μ on Z_μ. Moreover, using the remarks in Section 3.4 about extending paths on $\partial\mathcal{C}(\mu)$ to Z_μ, and noting that this extension obviously respects homotopies, we see that R_μ induces an isomorphism $R_{\mu*}\colon \pi_1(S_{\mu_0}) \to \pi_1(S_\mu)$.

To see that R_μ is compatible with j_μ, let ω be a path in $\Omega^*(\mu_0)/\operatorname{Stab}(\Omega^*(\mu_0))$ which maps to a non-trivial element $g_0 \in G_0$ under j_μ. Then ω lifts to a g_0-invariant path in $\Omega^*(\mu_0)$ whose ends limit at the fixed points $g_0^{\pm}$ of g_0. (If g_0 is parabolic these points coincide.) Clearly then $R_\mu(\omega)$ lifts to a g_μ-invariant path in $\Omega^*(\mu)$ whose ends limit at the points $i_\mu(g_0^\pm) = g_\mu^\pm$ giving the compatibility. $\square$

3.7. Transferring laminations. Our aim in this section is to show that the space of measured laminations on the surface $S_\mu = \partial\mathcal{C}/G_\mu$ does not depend on μ so that we are justified in talking about the space $\mathcal{ML}(S)$ without referring to μ. We do this using the map R_μ defined above. If ν is a family of pairwise disjoint geodesics on S_0, then we may use R_μ to define a family of pairwise disjoint curves on S_μ. It is standard, (see e.g. [**4**]), that this family consists of quasi-geodesics on S_μ, and hence may be replaced by a family of homotopic geodesics on S_μ. Since a transverse measure may be thought of simply as a measure on the space of geodesics, (see [**2**]), we may talk about the space of measured laminations $\mathcal{ML}(S)$ independently of μ.

4. The Theorems. In this section we state and then prove our results. We keep the notation of Sections 2 and 3.

THEOREM 4.3. (Geodesic Length is Continuous). *Let $\{G_\mu\}_{\mu\in D}$ be a holomorphic family of Kleinian groups with connected component $\Omega^*(\mu)$. Let $\partial\mathcal{C}^*(\mu)$ be the component of the convex hull boundary of G_μ facing $\Omega^*(\mu)$, and let*

$$S_\mu = \partial\mathcal{C}^*(\mu)/\operatorname{Stab}(\Omega^*(\mu)).$$

Then for each homotopy class $[\omega] \in \pi_1(S)$, the length function

$$\ell\colon \mu \mapsto \ell_\mu([\omega])$$

is continuous.

COROLLARY 4.4. (Hyperbolic Structure is Continuous). *For a holomorphic family of Kleinian groups, the hyperbolic structure of S_μ varies continuously with μ.*

Proof. This is immediate from the theorem since the Teichmüller space of a hyperbolic surface S is embedded in $(\mathbb{R}^+)^{\mathcal{S}}$ by the map $\mu \mapsto \{\ell_\mu([\omega])\}_{[\omega]\in\mathcal{S}}$. $\square$

Note that in this embedding a point is determined by finitely many lengths [**6**, Exp. 7].

THEOREM 4.5. (Lamination Length is Continuous). *For a holomorphic family of Kleinian groups the map*

$$D \times \mathcal{ML}_0(S) \to \mathbb{R}^+$$

defined by $(\mu, \nu) \mapsto \ell_\mu(\nu)$ *is jointly continuous.*

THEOREM 4.6. (Bending is Continuous). *For a holomorphic family of Kleinian groups the map*

$$D \to \mathcal{ML}(S)$$

given by $\mu \mapsto \beta_\mu$ *is continuous.*

Theorem 4.5, in consequence of the remarks at the end of Section 3.5, can be regarded as a generalization of Theorem 4.3. The proof of joint continuity, however, requires some care. This we do at the end of the paper.

REMARK. It is important to note that the lengths we are concerned with here are defined by the hyperbolic structure of the convex hull boundary and not by the hyperbolic structure induced from the conformal structure of the sphere at infinity.

REMARK. Suppose that $\Omega^*(\mu)$ is not simply connected. Then certain closed curves on S_μ lift to closed curves on $\Omega^*(\mu)$. Such curves can never be in the pleating locus of $\partial\mathcal{C}^*$; that is, in the image of D under the map β in theorem 4.6. For by [**5**, Section 1.6.3], the bending lines of $\partial\mathcal{C}$ are geodesics with both endpoints in Λ, and hence they cannot correspond to closed curves in Ω^*.

Proofs of the Theorems.

The proofs of Theorems 4.3 and 4.6 are essentially the same so we prove them together. Note that the crux of the proofs lies in Propositions 4.8 and 4.12.

Suppose that $\{G_\mu\}_{\mu \in D}$ is a holomorphic family of Kleinian groups based at μ_0 with a particular component $\Omega^*(\mu)$ chosen as in Section 2. We saw in Section 3.6, that there is a well defined correspondence between homotopy classes of curves on S_μ and $S_{\mu'}$. If ω_μ is a curve on S_μ, we write $[\omega_\mu]$ for its homotopy class and $[\omega_{\mu'}]$ for the corresponding class in $\pi_1(S_{\mu'})$.

What we have to prove is that for each homotopy class $[\omega]$, the functions $\mu \mapsto \hat{\beta}_\mu([\omega_\mu])$ and $\mu \mapsto \hat{\ell}_\mu([\omega_\mu])$ are continuous in μ. Clearly, it will be enough to prove this as μ varies in a compact subset $\mathcal{K}$ of D. The results will follow directly from the following lemma:

LEMMA 4.7. *Given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\hat{\beta}_{\mu'}([\omega_{\mu'}]) < \hat{\beta}_\mu([\omega_\mu]) + \epsilon \tag{1}$$

and

$$\hat{\ell}_{\mu'}([\omega_{\mu'}]) < \hat{\ell}_\mu([\omega_\mu]) + \epsilon \tag{2}$$

whenever $\mu, \mu' \in \mathcal{K}$ and $|\mu - \mu'| < \delta$.

The constant δ depends on $\mathcal{K}$ and $[\omega]$ but not on μ. The *uniformity* of this estimate in μ is essential; otherwise we would only get semi-continuity of the functions ℓ_μ and β_μ. We are therefore obliged to work to obtain a good uniform estimate on the error involved in approximating the bending along a path ω by summing along a polygonal approximation to ω.

We begin by finding error estimates on approximations to the bending and length for a fixed group $G = G_\mu$. Later we discuss the problem of varying the group.

DEFINITION. A polygonal approximation $\mathcal{P} = \{(x_i, P_i)\}$ to a path ω in Z is an *(α, s)-approximation* if

$$\max_{1 \le i \le n} \theta(P_{i-1}, P_i) < \alpha$$

and

$$\max d_\omega(x_{i-1}, x_i) < s$$

where d_ω is distance along ω measured in the intrinsic metric on $\partial\mathcal{C}^*$.

PROPOSITION 4.8. (Error Estimate). *There is a universal constant K, and a function $s(\alpha)$, $0 < s(\alpha) < 1$, such that if $\mathcal{P}$ is an $(\alpha, s(\alpha))$-approximation to a path ω in Z, where $\alpha < \pi/2$, then*

$$\left|\sum_{\mathcal{P}} d_i - \ell(\omega)\right| < K\alpha\ell(\omega) \tag{3}$$

and

$$\left|\sum_{\mathcal{P}} \theta_i - \beta(\omega)\right| < K\alpha\ell(\omega) \tag{4}$$

where, as above, $\ell(\omega)$ is the length of ω in the intrinsic metric on $\partial\mathcal{C}^*$.

Proof. We begin with the first estimate, inequality 3. This is similar to the estimates in [**5**, p. 135] and [**13**, p. 257]. Let $\mathcal{P} = \{(x_i, P_i)\}$ be a polygonal approximation to ω. We shall estimate the contribution to inequality 3 for the segment of path between x_{i-1} and x_i.

We can choose a hyperbolic plane H through x_{i-1} and x_i, such that the shortest path from x_{i-1} to x_i in the planes $P_{i-1} \cup P_i$ is contained in the intersections of these planes with H. Let the segments of this path in P_{i-1} and P_i have lengths a_1 and a_2 respectively and let b be the length of the geodesic γ in $\mathbb{H}^3$ from x_{i-1} to x_i. Thus $a_1 + a_2 - b$ is an upper bound for the error. By assumption the angle θ_i between the planes P_{i-1} and P_i is less than α.

Let $A = P_{i-1} \cap P_i \cap H$ and let B be the foot of the perpendicular from A to γ. Let b_1 and b_2 be the distances from x_{i-1} and x_i to B respectively, so that $b = b_1 + b_2$. Adapting the argument in ([**13**, p. 258]) to the hyperbolic situation, we see that provided $\alpha < \pi/2$, since the acute angles of the triangle $Ax_{i-1}x_i$ are less than α, we have

$$\tanh a_i < \tanh b_i \sec\alpha, \; i = 1, 2.$$

Also note that $b_i < a_i < b \leq d_\omega(x_{i-1}, x_i)$.

We can choose $s = s(\alpha)$, $0 < s(\alpha) < 1$, such that if $d_\omega(x_{i-1}, x_i) < s$ then $\tanh a_i > (1-\alpha)a_i$, $i = 1, 2$. Using $\tanh b_i \leq b_i$ we find

$$a_1 + a_2 < (b_1 + b_2)\sec\alpha + \alpha(a_1 + a_2)$$

and hence

$$a_1 + a_2 - b < K\alpha d_\omega(x_{i-1}, x_i)$$

as required.

For the second estimate, inequality 4, we consider how the approximating sum changes as we repeatedly refine $\mathcal{P}$. As before, we shall verify the estimate on the section of the path between x_{i-1} and

x_i. Let $(z, P) = \omega_i(t)$, $t_{i-1} < t < t_i$ be an intermediate partition point on this section.

We use the following fact, proved in [**5**, Section 1.10]: there is either a unique horosphere E through x_{i-1} and orthogonal to each of the three planes, P_{i-1}, P, P_i or a there is a unique hyperbolic plane E orthogonal to each of them.

Call the lines in which these planes intersect E, l_{i-1}, l, l_i respectively. The angles between these lines are exactly the dihedral angles between the corresponding planes. When we add the point (z, P) to the partition, the angle sum $\sum_{\mathcal{P}} \theta_i$ is decreased by

$$\theta(l_{i-1}, l_i) - \theta(l_{i-1}, l) - \theta(l, l_i). \tag{5}$$

If E is a horosphere, the lines form a Euclidean triangle in E, and the sum in expression (5) is zero. If E is a hyperbolic plane, the lines form a hyperbolic triangle Δ in E and the expression in (5) is just the hyperbolic area of Δ.

In figure 1 we see the triangle Δ. The points x'_{i-1}, z', x'_i are the projections onto E of the points x_{i-1}, z, x_i.

Since we do not change the error between the actual bending and the bending measured along the polynomial approximation by adding a point to $\mathcal{P}$ for which E is a horosphere, we proceed adding points until we find a point (z, P) for which E is hyperbolic.

We claim that the total error in the approximating sum along the segment from x_{i-1} to x_i is bounded by the area of the region R in E enclosed by the lines l_{i-1}, l_i and the curve $\partial\mathcal{C}^* \cap E$. (Note that $\partial\mathcal{C}^* \cap E \neq \emptyset$ by convexity.)

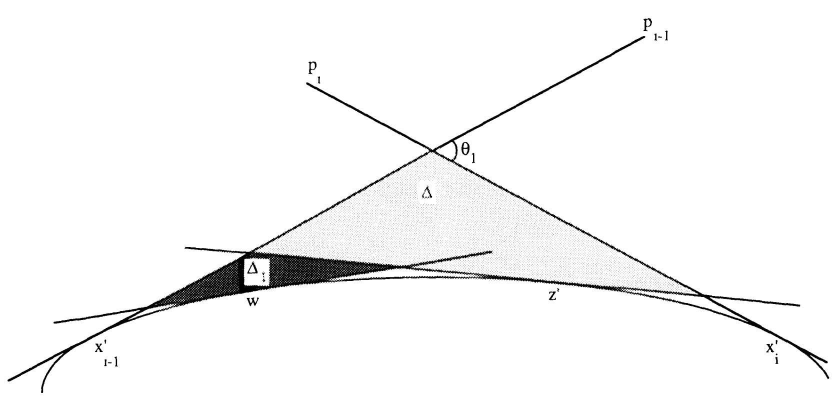

FIGURE 1. The region R.

Consider the triangle Δ_1 introduced by adding a new partition point $(w, Q) \in \omega$ between x_{i-1} and z. The sides of Δ_1 are the intersections of the planes P_{i-1}, P_i and Q with E. Arguing as above, the change in the approximating sum is measured in the plane E_1 determined by the planes P_{i-1}, Q, P as above; it is either zero or the hyperbolic area of Δ_1' formed by the intersection of the planes P_{i-1}, Q, P with E_1. In the latter case, since orthogonal projection decreases area, we have $area(\Delta_1') \leq area(\Delta_1)$. An inductive argument now proves the claim.

We now bound the hyperbolic area of R by comparing with Euclidean area in the disk model $\mathbb{D}$. It is an easy computation to see that a Euclidean triangle with one exterior angle α and opposite side of length $h < 1$ has area bounded by $h\alpha$. If we map the plane E to the disk $\mathbb{D}$ placing the vertex $l_{i-1} \cap l_i$ at the origin, then by convexity, the region R is contained in the Euclidean triangle, $0, x_{i-1}', x_i'$ (here we have identified x_{i-1}' and x_i' with their images in D).

By assumption $d_\omega(x_{i-1}, x_i) < s$ and hence the hyperbolic distance in $\mathbb{H}^3$ satisfies $d(x_{i-1}, x_i) < s$. Since x_{i-1}' and x_i' are perpendicular projections onto E of x_{i-1} and x_i, $d(x_{i-1}', x_i') < s$, where this is in measured in E. Since $\alpha < \pi/2$, the triangle $0, x_{i-1}', x_i'$ in the hyperbolic plane E is entirely contained in a circle of bounded Euclidean radius. Hence there is a bounded comparison between Euclidean and hyperbolic area. Choosing s to ensure that $h < 1$ in the above, the result follows. □

We now consider the effect of varying the group G_μ. The key point is to show that the maps R_μ defined in Section 3.6 depend continuously on μ. This result is Proposition 4.12. We begin with an easy observation about support planes. (See Section 3.4.)

LEMMA 4.9. (Support Planes). *Let $C \subset \hat{\mathbb{C}}$ be a circle containing at least two points of Λ and such that* $\operatorname{int}(C) \subset \Omega^*$ *(here* $\operatorname{int}(C)$ *denotes one of the complementary components of C in $\hat{\mathbb{C}}$). Then C is a support plane of $\partial\mathcal{C}$.*

Proof. Let $P(C)$ denote the plane in $\mathbb{H}^3$ that meets $\hat{\mathbb{C}}$ in C and let z be the center of $\operatorname{int}(C)$. No hemisphere centered at z whose (Euclidean) radius is less than the radius of C can be a support plane of $\partial\mathcal{C}$ because $\Lambda \cap \operatorname{int}(C) = \emptyset$; on the other hand, $P(C)$ certainly meets $\partial\mathcal{C}$. Thus, $P(C)$ is a support plane. □

In what follows we work in a neighborhood in D of our base point μ_0. We shall, however, take care that all our estimates are independent of the base point. First, however, we need some notation.

We denote by $d(*,*)$ Euclidean distance in $\mathbb{H}^3 \cup \hat{\mathbb{C}}$, and by $\mathrm{dist}(*;*)$ the natural metric on $\mathbb{H}^3 \times G_2(\mathbb{H}^3)$ which induces d on the first factor and the dihedral angle metric on the second. For $\zeta \in \mathbb{H}^3$, $\mathrm{ht}(\zeta)$ denotes the Euclidean height of ζ above $\mathbb{C}$ in the upper half space model of $\mathbb{H}^3$.

In the next lemma we prove that the modulus of continuity of the retraction $\hat{r}$ defined in Section 3.6 depends only on the Euclidean height in $\mathbb{H}^3$ of the image under r.

LEMMA 4.10. *Let $\Lambda \subset \mathbb{C}$ be an arbitrary closed set with diameter greater than some constant $c > 0$ and let K be a closed bounded convex subset of a connected component of $\mathbb{C} - \Lambda$. Let r be the retraction of $\mathbb{H}^3 \cap \hat{\mathbb{C}}$ onto the convex hull C of Λ. Then the map $\hat{r}$ is uniformly continuous on K, where the modulus of continuity depends only on $a = d(K, \Lambda)$, $b = \sup_{z \in K} d(z, \Lambda)$ and c.*

Proof. Let $z, w \in K$ and let $\hat{r}(z) = (\xi, P_\xi), \hat{r}(w) = (\eta, P_\eta)$ where P_ξ and P_η are support planes to C at ξ and η respectively. Since z and w lie in the same component of $\hat{\mathbb{C}} - \Lambda$, the hemispheres bounded by P_ξ and P_η, and containing z and w cannot be nested. Since K is compact in $\mathbb{C}$ there are clearly constants a' and b', depending only on a, b and c, such that

$$0 < a' \le (\mathrm{ht}(\xi), \mathrm{ht}(\eta)) \le b',$$

for all $z, w \in K$ and such that the diameters of the horospheres tangent to the support planes at $r(\xi)$ and $r(\eta)$ are also bounded between a' and b'. Taking account of these bounds we see the following:

1. If $P_\xi = P_\eta$ then $d(\xi, \eta)$ depends only on z, w and is uniformly small as $|z - w| \to 0$, with constants depending only on a' and b'.

2. If $\xi = \eta$, $P_\xi \ne P_\eta$, then $|z - w|$ is small if and only if $\mathrm{dist}(\xi, P_\xi; \eta, P_\eta)$ is small, again uniformly with constants depending only on a' and b'.

3. If $\xi \ne \eta$, $P_\xi \ne P_\eta$, but $P_\xi \cap P_\eta \ne \emptyset$, we can find, since K is convex, a point $\zeta \in P_\xi \cap P_\eta$ such that all points in the preimage $r^{-1}(\zeta)$

are in K. Choosing any point in the preimage and applying cases 1 and 2 to the resulting pairs, $(z, \hat{r}^{-1}(\zeta, P_\xi))$, $(\hat{r}^{-1}(\zeta, P_\xi), \hat{r}^{-1}(\zeta, P_\eta))$ and $(\hat{r}^{-1}(\zeta, P_\eta), w)$ in turn, we see that $|z-w|$ is small if and only if $\operatorname{dist}(\xi, P_\xi; \eta, P_\eta)$ is small, again uniformly with constants depending only on a' and b'.

4. If $P_\xi \cap P_\eta = \emptyset$ then, because $\operatorname{ht}(\xi)$ and $\operatorname{ht}(\eta)$ are bounded below, $|z-w|$ is uniformly bounded below, so this case does not need to be considered.

Now combining these observations, we obtain estimates that imply the uniform continuity. □

Next we estimate the change in $\hat{r}_\mu(z)$ as we vary μ.

LEMMA 4.11. *Let $\{G_\mu\}, \mu \in D$ be a holomorphic family of Kleinian groups based at μ_0. Assume, without loss of generality, that the groups are normalized so that the chosen component $\Omega^*(\mu)$ is bounded in $\mathbb{C}$. Let $K \subset \Omega^*(\mu_0)$ be convex and compact. Then, given $\epsilon > 0$, there exists $\delta > 0$, depending only on ϵ and on $d(K, \Lambda(\mu_0))$ such that*

1. *$K \subset \Omega^*(\mu)$ whenever $|\mu - \mu_0| < \delta$, and*
2. *$\operatorname{dist}(\hat{r}_\mu(z); \hat{r}_0(z)) < \epsilon$ whenever $|\mu - \mu_0| < \delta$ and $z \in K$.*

Proof. 1. Let $a = d(K, \Lambda(\mu_0))$ and let c be a lower bound on the diameter of $\Lambda(\mu_0)$. Using the uniform continuity of $i_\mu(z)$ (see Section 3.6) we see that there exists $\delta_0 > 0$ such that $d(K, \Lambda(\mu)) > a/2$ and the diameter of $\Lambda(\mu)$ is greater than $c/2$ whenever $|\mu - \mu_0| < \delta_0$. Since $d(K, w) \geq d(K, \Lambda(\mu))$ for any $w \notin \Omega^*(\mu)$, we see that K is contained in $\Omega^*(\mu)$ for all such μ.

2. Assume the diameter of $\Lambda(\mu)$ is bounded below by c. Pick $z \in K$. Since $d(K, \Lambda(\mu)) > a/2$ for $|\mu - \mu_0| < \delta_0$, we have a constant a'' independent of z so that $\operatorname{ht} r_\mu(z) \geq a''$. Let $\hat{r}_0(z) = (\xi, P_0)$, so that P_0 is a support plane to $\partial\mathcal{C}(\Lambda(\mu_0))$ at ξ. Consider the family of horospheres tangent to $\mathbb{C}$ at z and let γ be the hyperbolic geodesic through ξ and z. We parametrize this family as H_t, where t denotes signed Euclidean distance along γ from ξ to H_t. (The distance is negative on the segment between z and ξ.) The horosphere H_0 is tangent to P_0. Let S_t be the solid closed hemisphere in $\mathbb{H}^3 \cup \hat{\mathbb{C}}$ (Euclideanly) concentric to P_0 through the point $H_t \cap \gamma$ and containing z.

By a straightforward computation[1] we see that given $\epsilon > 0$, we can find $\epsilon_1 > 0$ such that if $|t| < \epsilon_1$ and if $u \in H_t \cap (\mathbb{H}^3 - S_{-\epsilon_1})$, then

$$\text{(6)} \qquad \operatorname{dist}(\xi, P_0; u, P_u) < \epsilon.$$

where P_u denotes the hyperbolic plane tangent to H_t at u. Note that the choice of ϵ_1 is independent of z since $\operatorname{ht}(\xi) > a''$, $\forall z \in K$.

By construction, $S_{-\epsilon_1} \cap \mathcal{C}(\mu_0) = \emptyset$ so there are no points of $\Lambda(\mu_0)$ in $S_{-\epsilon_1}$. Using the uniform continuity of i_μ we can find $\delta_1 > 0$ such that $|\mu - \mu_0| < \delta_1$ implies there are no points of $\Lambda(\mu)$ inside $S_{-\epsilon_1}$. Thus, $r_\mu(z) \notin H_t$ for $t < -\epsilon_1$, and $r_\mu(z) \in \mathbb{H}^3 - S_{-\epsilon_1}$.

Suppose that $P_0 \cap \partial\mathcal{C}(\mu_0)$ is not a bending line. Then we can find points $z_i \in \Lambda(\mu_0)$, $i = 1, 2, 3$, such that ξ is in the hyperbolic convex hull of $\{z_1, z_2, z_3\}$. Again by the uniform continuity of i_μ, $|i_\mu(z_i) - z_i| \to 0$ as $\mu \to \mu_0$, and hence we can find points $\xi_\mu \in \mathcal{C}(\mu)$ arbitrarily close to ξ for μ close to μ_0. If $P_0 \cap \partial\mathcal{C}(\mu_0)$ is a bending line it suffices to use its endpoints z_1, z_2. Now an open ball with center ξ and radius ϵ_1 intersects all the horospheres H_t, $|t| < \epsilon_1$, so we can find δ_2 such that $|\mu - \mu_0| < \delta_2$ implies $\xi_\mu \in H_t$ for some $|t| < \epsilon_1$. Once again the choice of δ_2 is uniform in μ since $\operatorname{ht}(\xi) > a''$.

Now let $\delta = \min(\delta_0, \delta_1, \delta_2)$. Then for $|\mu - \mu_0| < \delta$ there are points $\xi_\mu \in \mathcal{C}(\mu)$ in a ball of radius ϵ_1 about ξ. It follows from the above that the expanding horospheres can't hit $\mathcal{C}(\mu)$ for $t < -\epsilon_1$ but they must hit by the time $t = \epsilon_1$ since by then they have enclosed some point ξ_μ. The first time H_t hits $\mathcal{C}(\mu)$ it does so at a point η of $\partial\mathcal{C}(\mu)$; moreover, the tangent plane P_η to this H_t is a support plane. Therefore $\hat{r}_\mu(z) = (\eta, P_\eta)$. By inequality 6 we see that $\operatorname{dist}(\eta, P_\eta; \xi, P_\xi) < \epsilon$ as required. □

Now we can make the statement of continuity that we need.

PROPOSITION 4.12. *Let G_μ be a holomorphic family of groups as above. Then the map*

$$R_\mu = \hat{r}_\mu i_\mu \hat{r}_0^{-1} \colon Z(G_0) \times D \to Z(G_\mu)$$

is jointly continuous.

[1]Do the computation in the case ξ lies directly above z and conjugate remembering that although conjugation doesn't preserve Euclidean distance, the Euclidean and hyperbolic metrics are equivalent in a bounded region of $\mathbb{H}^3$.

Proof. First note that the map is well defined in a neighborhood of μ_0 by Lemma 4.11, part 1. Given $w \in Z(G_0)$ we have

$$\begin{aligned}\operatorname{dist}&\left(w; \hat{r}_\mu i_\mu \hat{r}_0^{-1}(w)\right)\\ &\le \operatorname{dist}\left(w; \hat{r}_0 i_\mu \hat{r}_0^{-1}(w)\right) + \operatorname{dist}\left(\hat{r}_0 i_\mu \hat{r}_0^{-1}(w); \hat{r}_\mu i_\mu \hat{r}_0^{-1}(w)\right).\end{aligned}$$

The first term is small by the continuity of $\hat{r}_0$ and i_μ, independently of the choice of μ_0 by Lemma 4.10. The second term is uniformly small by Lemma 4.11, part 2, with $z = i_\mu \hat{r}_0^{-1}(w)$. □

We are ready now to prove our main theorems. We want to prove that the lengths and the bending measure change continuously as we vary the group. Suppose we are given a path ω_0 in $Z(G_0)$ that projects to the homotopy class of some curve γ_0 on $S_0 = S_{\mu_0}$. We will need a bound on the lengths of the paths $\omega_\mu = R_\mu(\omega_0)$ for nearby μ. This will enable us to control the number of partition points in the estimates and to complete the proof of the main theorem.

LEMMA 4.13. (Bound on Path Lengths). *Suppose that μ varies in a relatively compact set $\mathcal{K} \subset D$, $\mu_0 \in \mathcal{K}$, and that ω_0 is a rectifiable path in $Z(G_0)$. Then there exists $M > 0$ such that for all $\mu \in \mathcal{K}$ the path $\omega_\mu = R_\mu(\omega_0) \in Z(G_\mu)$ satisfies $\ell_\mu(\omega_\mu) \le M$ and $\beta_\mu(\omega_\mu) \le M$.*

Proof. Let $\mathcal{P}_0 = \{(x_i, P_i) \in Z(G_0), i = 0, \dots, n\}$ be a polygonal approximation to ω_0. The points $R_\mu(x_i, P_i), i = 0, \dots, n$ form a polygonal approximation to $w_\mu = R_\mu(\omega_0)$. From Lemma 4.11 and Proposition 4.12 we see that the $\operatorname{dist}(x_i, P_i; R_\mu(x_i, P_i))$ varies continuously with μ (independently of μ_0) provided that ω_μ stays in a compact part of $\mathbb{H}^3$. Clearly lengths and angles measured along polygonal approximations are upper bounds for the same quantities measured along the paths ω_μ themselves. Therefore, since $\mathcal{K}$ is compact the bounds exist. □

Proofs of Theorems 4.3 *and* 4.6. We are finally in a position to complete the proof of Theorems 4.3 and 4.6. We carry out the proof for the bending angle β; the proof for the length function is entirely similar. We always suppose we are working in a compact set $\mathcal{K} \subset D$, and that we are given $\mu_0 \in \mathcal{K}$, a homotopy class $[\omega_0]$ of paths on S_0, and $\epsilon > 0$. Write β_0 for β_{μ_0}. Let M be an upper bound for lengths and total bending of paths ω_μ in the homotopy class of $[\omega_\mu]$, chosen as in Lemma 4.13. We may assume that ω_0 has been chosen so that $\beta_0(\omega_0) < \hat{\beta}_0([\omega_0]) + \epsilon$ and that $\ell_0(\omega_0) \leq M$, $\beta_0(\omega_0) \leq M$.

By subdividing the bending angle at bending lines if necessary, we may find an $(\epsilon, s(\epsilon)/2)$-partition $\mathcal{P}$ with $|\mathcal{P}| \leq \frac{2M^2}{s(\epsilon)\epsilon} = N$, where $|\mathcal{P}|$ denotes the number of points in the partition $\mathcal{P}$. (Here $s(\epsilon)$ is chosen as in Proposition 4.8 and we assume $\epsilon < \pi/2$.)

Applying Proposition 4.8 we find

$$\left|\beta_0(\omega_0) - \sum_{\mathcal{P}} \theta_i\right| < K\epsilon M.$$

Now using Lemma 4.11, given $\eta > 0$, we can find a neighborhood U_0 of $\mu_0 \in \mathcal{K}$ such that for all $\mu \in U_0$ we have an $(\epsilon + \eta, s(\epsilon + \eta))$-polygonal approximation $\mathcal{P}_\mu = \{(x_i', P_i')\}_{i=0}^N$ to a path $\omega_\mu \in [\omega_\mu]$ such that

$$\left|\sum_{\mathcal{P}_\mu} \theta_i' - \sum_{\mathcal{P}_0} \theta_i\right| < N\eta.$$

Applying Proposition 4.8 to $\mathcal{P}_\mu$, we find

$$\left|\beta_\mu(\omega_\mu) - \sum_{\mathcal{P}_\mu} \theta_i'\right| < K(\epsilon + \eta)M.$$

Choosing $\eta = \epsilon^2 s(\epsilon)$ we get

$$|\beta_0(\omega_0) - \beta_\mu(\omega_\mu)| < KM\left(2\epsilon + \epsilon^2 s(\epsilon)\right) + 2M^2\epsilon.$$

This proves inequality 1 of Lemma 4.7 and hence completes the proof of Theorem 4.6. $\square$

Proof of Theorem 4.5. In Section 3.6 we used the function R_μ to transfer laminations from S_{μ_0} to S_μ. Here it will be more convenient to pass to the universal cover to obtain this correspondence. For each μ, there is a normalized Fuchsian group Γ_μ acting on $\mathbb{D}$, where $\mathbb{D}$ is the unit disk, that uniformizes the hyperbolic surface S_μ. Since S_μ is compact with at most finitely many punctures, $\Lambda(\Gamma_\mu) = \partial\mathbb{D}$. By Corollary 4.4 the hyperbolic structures of the surfaces S_μ and hence the Fuchsian groups Γ_μ depend continuously on μ; that is, the point Γ_μ determines in the Teichmüller space $T(\Gamma_{\mu_0})$ depends continuously on μ.

It is well known (see e.g. [**1**]) that all quasiconformal homeomorphisms $h_\mu\colon \mathbb{D} \to \mathbb{D}$ inducing the isomorphism $\psi_\mu\colon \Gamma_{\mu_0} \to \Gamma_\mu$ extend to $\partial\mathbb{D}$ and agree there. Such boundary maps are continuous and hence determined by their values at the fixed points of the hyperbolic elements of Γ_{μ_0}. By the measurable Riemann mapping theorem, [**1**], h_μ depends continuously on μ. Hence if γ is a geodesic in $\mathbb{D}$ with endpoints $\gamma^\pm$ on $\partial\mathbb{D}$, then $h_\mu(\gamma)$ has endpoints $h_\mu(\gamma^\pm)$ on $\partial\mathbb{D}$ and these depend continuously on μ.

Now we use the natural identification of geodesics in $\mathbb{D}$ with their endpoints on $\partial\mathbb{D}$. Thus geodesics are represented by points in $X = (\partial\mathbb{D} \times \partial\mathbb{D} - \{\text{diag}\})/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $\partial\mathbb{D} \times \partial\mathbb{D}$ by exchanging the factors. So if ν_0 is a measured lamination on S_{μ_0}, $|\nu_0|$ lifts to a $\pi_1(S)$ invariant subset of X and ν_0 corresponds to a $\pi_1(S)$ invariant measure on X, (see [**2**] for details). By the continuity of h_μ on $\partial\mathbb{D}$, the correspondence of measured laminations is continuous. By Proposition 3.2 this correspondence is compatible with the correspondence described in Section 3.7.

To obtain the joint continuity of $\ell_\mu(\nu)$ asserted in Theorem 4.5, we will show that $\ell_\mu(\nu)$ is equicontinuous in the variable μ for fixed ν. To do this, fix $(\mu_0, \nu_0) \in D \times \mathcal{ML}_0(S)$. We call a set $R \subset S$ a *regular flow box* for ν_0 if:

(i) R is a closed hyperbolic rectangle embedded in S,
(ii) the horizontal sides T, T' of R are either disjoint from $|\nu_0|$ or are transversal to $|\nu_0|$. If a leaf γ of $|\nu_0|$ intersects R then it intersects both T and T' and,
(iii) the vertical sides of R are disjoint from $|\nu_0|$.

It is clear that the family $\mathcal{F} = \mathcal{F}(\nu_0)$ of regular flow boxes for ν_0 form a semi-algebra of sets; that is, if $B, B' \in \mathcal{F}$ then $B \cap B' \in \mathcal{F}$ and $B - B'$ is a finite union of sets in $\mathcal{F}$. Thus we can find a cover of S by sets in $\mathcal{F}$ whose interiors are pairwise disjoint and so that a finite number, say $B_1, \ldots, B_n$ cover $|\nu_0|$. Recall from Section 3.2 that $|\nu_0|$ is contained in a compact part of S.

To obtain a suitable expression for the length of $|\nu_0|$ it is convenient to use the natural identification of geodesics in $\mathbb{D}$ with endpoints in the space X defined above.

If $(x, y) \in X$, let $\gamma(x, y)$ be the geodesic in $\mathbb{D}$ joining x to y. If $\tilde{B}$ is a lift to $\mathbb{D}$ of a regular flow box $B \in \mathcal{F}$ with $\gamma(x, y) \cap \tilde{B} \neq \emptyset$, denote by $f(\mu_0) = f_{\tilde{B}}(x, y)(\mu_0)$ the hyperbolic length of the arc

$\gamma(x,y) \cap \tilde{B}$. Clearly $f(\mu_0)$ is independent of the lift $\tilde{B}$ of B.

Let $\tilde{B}_1, \dots, \tilde{B}_n$ be a set of lifts of $B_1, \dots, B_n$. Then,

$$\ell_{\mu_0}(\nu_0) = \sum_{i=1}^{n} \int_{\tilde{B}_i} f_{\tilde{B}_i}(x,y)(\mu_0)\, d\nu_0(x,y). \tag{7}$$

Now let μ vary in D and fix $B \in \mathcal{F}(\nu_0)$ with lift $\tilde{B}$. The endpoints, $\xi_1, \dots, \xi_8$ on $\partial\mathbb{D}$ of the extended sides of $\tilde{B}$ vary continuously with μ under the maps h_μ as do the endpoints of the leaves of $\nu_0(\mu)$. Thus the geodesics joining the appropriate pairs of the points $h_\mu(\xi_j)$, $j = 1, \dots, 8$, define the lift of a regular flow box $\tilde{B}(\mu)$ for ν_0 for all $\mu \in D$.

The boxes $\tilde{B}_i(\mu)$ still form a disjoint cover for $|\nu_0|$ so that

$$\ell_\mu(\nu_0) = \sum_{i=1}^{n} \int_{\tilde{B}_i(\mu)} f_{\tilde{B}_i(\mu)}(h_\mu x, h_\mu y)\, d\nu_0(h_\mu x, h_\mu y). \tag{8}$$

Since the function $f_{\tilde{B}(\mu)}(h_\mu(x), h_\mu(y))(\mu)$ is also obviously continuous in μ, we obtain the continuous dependence of the function $\ell_\mu(\nu_0)$ on μ.

Now let us vary ν; let $\nu_k \in \mathcal{ML}_0(S), k = 1, \dots \infty$ be a sequence such that $\nu_k \to \nu_0$. Because the vertical sides of each flow box B do not intersect $|\nu_0|$, we see that the same is true for $|\nu_k|$ for k large enough, and by taking a subsequence if necessary, that the sets $B_1, \dots, B_n$ are also a cover of $|\nu_k|$ by regular flow boxes for ν_k.

By the definition of the topology on $\mathcal{ML}_0(S)$, the sum $\sum_{i=1}^{n} \nu_k(B_i)$ is bounded for all k. Thus the expression (8) holds with ν_k replacing ν_0, and we see that $\mu \to \ell_\mu(\nu)$ is equicontinuous in μ.

There is clearly an upper bound on $f_{\tilde{B}_i}(x,y)(\mu)$ for μ in a compact subset of D. Moreover, it follows from the definition of the topology on $\mathcal{ML}_0(S)$ and from formula (8) that $\nu \to \ell_\mu(\nu)$ is continuous on on some neighborhood of ν_0 for fixed μ.

Finally, the equicontinuity of $\ell_\mu(\nu)$ in μ for fixed ν and the continuity in ν for fixed μ is enough to imply the joint continuity as claimed.

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