New examples of manifolds with completely integrable geodesic flows

G. P. Paternain
Department of Mathematics
SUNY Stony Brook
Stony Brook, NY 11794

R. J. Spatzier *
Department of Mathematics
University of Michigan
Ann Arbor, MI 48103

Dedicated to the memory of M. Strake

Abstract

We construct Riemannian manifolds with completely integrable geodesic flows, in particular various nonhomogeneous examples. The methods employed are a modification of Thimm's method, Riemannian submersions and connected sums.

1 *Introduction*

A flow g_t on a symplectic $2n$-dimensional manifold M is *completely integrable* if there are n Poisson-commuting, g_t-invariant C^∞-functions $f_1, \ldots, f_n$ whose differentials are independent a.e. in M. Poincaré realized that complete integrability is an exceptional phenomenon. Indeed, it was not until the past two decades that a large number of examples was discovered. In this paper, we explore the narrower realm of geodesic flows. Until now, very few examples of completely integrable geodesic flows were known. The classical examples are the flat tori, surfaces of revolution (Clairaut), n-dimensional ellipsoids with different principal axes (Jacobi) and $SO(3)$ with a left invariant metric (Euler). More recent examples are semisimple Lie groups with certain left invariant metrics due to Mishchenko and Fomenko [6]. Then Thimm devised a new method for constructing first integrals in involution on homogeneous spaces [15]. In particular, he proved the complete integrability of the geodesic flow on real and complex Grassmannians. Guillemin and Sternberg conceptualized this method and found further examples [9]. The spaces obtainable in this way have essentially been classified by Krämer [13]. For results concerning the complete integrability of the geodesic flows of other symmetric spaces of the compact type we refer to [5].

In this paper, we exhibit several new examples of Riemannian manifolds with completely integrable geodesic flows, and in particular various nonhomogeneous examples. We use several new techniques in these constructions.

The first construction is a simple variation of Thimm's method. In his method, the moment map of a Lie group action is used to pull back a family of Poisson-commuting

*Partially supported by the NSF and the Stony Brook Institute for the Mathematical Sciences, Sloan Foundation Fellow and AMS Centennial Fellow

functions from the Lie algebra to the symplectic space in question. While Thimm considered the case of an action of a group by isometries on a homogeneous space, one can instead use the isometry group together with the geodesic flow. This generalizes the construction of integrals on surfaces of revolution. As simple as it is, this variation of Thimm's method already yields several new examples such as the Wallach manifold $SU(3)/T^2$ where T^2 is a maximal torus in $SU(3)$. We explore this in Section 2.

In Section 3, we study the symplectic structure of a Riemannian submersion. When the submersion is given by an action of a Lie group G, the tangent bundle of the base space is symplectomorphic with the Marsden-Weinstein reduction of the tangent bundle of the total space with respect to the action of the group by derivatives. In particular, we see that G-invariant Poisson-commuting functions descend to Poisson-commuting functions. Thus the base space of the submersion has completely integrable geodesic flow if enough G- invariant Poisson-commuting functions descend to independent functions. This is the essence of the *submersion method*.

In Section 4 we construct various examples using the submersion method. To apply it, we observe that sometimes the integrals arising ¿from the Thimm method are invariant under the action of a subgroup of the isometry group. Then we show in the examples that enough Poisson-commuting functions descend to independent functions on the quotient space. Unfortunately, the independence of the functions on the base space is far from automatic. However we show that in general if X has completely integrable geodesic flow and admits an S^1-action that leaves the integrals invariant and N is a surface of revolution then the geodesic flow of $X \times_{S^1} N$ is completely integrable. Particular examples are $\mathbf{CP}^n\#-\mathbf{CP}^n$ and surface bundles over the Eschenburg examples. In the latter the base space is a quotient space of $SU(3)$ by S^1-action acting both from the left and the right. Some of these surface bundles are known to be strongly inhomogeneous, that is, they do not have the homotopy type of a compact homogeneous space [14]. Next, we show that certain Eschenburg examples themselves have completely integrable geodesic flows. Again most of these spaces are strongly inhomogeneous. They do not fall under the general submersion example above. Rather, we use the submersion method directly, and establish independence of sufficiently many functions by explicit computation. Note also that while the isometry groups of the Eschenburg manifolds are non-trivial, they are not big enough for the Thimm method to apply, due to dimensional reasons. Finally, we show that the geodesic flow of the exotic sphere used by Gromoll and Meyer in [7] is completely integrable. Here the integrals come both from a Thimm construction combined with the submersion method as well as from the isometry group. Let us remark that the geodesic flows of certain Kervaire spheres also admit a complete set of integrals on an open dense subset of the tangent bundle. It is not clear however whether these integrals extend to the full tangent bundle.

In Section 5, we use a glueing technique to construct metrics with completely integrable geodesic flows on $\mathbf{CP}^n\#\mathbf{CP}^n$ for n odd.

The second author is grateful to M. Strake who had introduced him to the Eschenburg examples. It also was first in discussions with him that the possibility of the complete integrability of the geodesic flows of the Eschenburg examples arose. We also thank D. Gromoll and B. Kasper for helpful comments and discussions.

2 *Variations on the Thimm method.*

First we recall Thimm's construction as modified by Guillemin and Sternberg [8, 9, 15]. We refer to [9] for more details.

Let N be a symplectic space with a Hamiltonian action of a Lie group G. Such an action is called *multiplicity free* if the algebra of the G-invariant functions on N is commutative under the Poisson bracket [8, p. 361]. Let $\Phi : N \to g^*$ denote the moment map of the action. Let $\{1\} = G_l \subset G_{l-1} \subset ... \subset G_1 = G$ be an ascending chain of Lie subgroups of G, and denote their Lie algebras by g_i. Furnish each coadjoint orbit of G_i in g_i^* with the Kostant-Kirillov symplectic structure. Then each subgroup G_{i+1} acts on each orbit in g_i^* in a Hamiltonian way. The moment maps are just the restrictions of the dual maps $j_i : g_i^* \to g_{i+1}^*$ to the coadjoint orbits. We will call the chain G_i *multiplicity- free* if the actions of the G_{i+1} on the coadjoint orbits of G_i on g_i^* are multiplicity free. This is quite a strong condition on the chain G_i. For compact groups it forces the G_i to be locally isomorphic to $SO(n)$, $SU(n)$, tori or products of these [11, 12].

If the G_i are a multiplicity-free chain and the action of G on N is multiplicity-free, then any G-invariant Hamiltionian on N is completely integrable [8, p. 366]. This is the essence of the *Thimm method*. This setup was studied in detail in [15, 9, 10]. If N is the cotangent bundle of a manifold M and G acts by derivatives then M is a homogeneous space G/K [10]. In this case, one calls (G, K) a *Gelfand pair*. They have been classified by Kramer in [13].

We observe that a variation of the Thimm method also gives complete integrability of some geodesic flows on homogeneous spaces G/K even when the pair (G, K) is not a Gelfand pair. First let $\mathrm{ind}\, G$ denote the index of G. It is defined as the codimension of a generic orbit of the coadjoint action of G on l^* (if G is semisimple $\mathrm{ind}\, G$=$\mathrm{rk}\, G$). Consider now a homogeneous space G/K that verifies the following conditions:

(1) $\dim G = 2\, \mathrm{d}im\, K \;+\; indG \;+\; 2$
(2) the isotropy group of G at some $v \in T(G/K)$ has dimension zero.

Denote by R the Hamiltonian action generated by the geodesic flow of some left invariant metric. Let $\hat{G} = G \times R$. Clearly $\hat{G}$ acts by Hamiltonian transformations and leaves the quadratic form associated with the metric invariant. Suppose the left invariant metric on G/K has a geodesic which is not the orbit of a 1-parameter subgroup of G. Then a.e. the isotropy of $\hat{G}$ has dimension zero and $\dim \hat{G}$+$\mathrm{ind}\, \hat{G}$=$\dim T(G/K)$. By a dimension count, we deduce that the isotropy groups of the coadjoint action of $\hat{G}$ act transitively on the regular level surfaces of the moment map of $\hat{G}$. This implies that the action of $\hat{G}$ is multiplicity free by the equivalences stated in [9]. Therefore the Thimm method can be applied whenever a multiplicity free chain can be constructed for G.

Now, let us rewrite condition (2). Assume G is compact and denote by (,) some bi-invariant metric. Let k denote the Lie algebra of K and $k^\perp$ the orthogonal complement with respect to (,). Then it is easy to check that (2) is equivalent to:

(3) For some $X \in k^\perp$ we have $\mathrm{d}im\, [X, k] = \mathrm{d}im\, k$.

Let us see some examples:

Example 2.1 Consider the homogeneous space $SU(3)/T^2$ where T^2 is a maximal torus in $SU(3)$. This manifold can be also considered as the space of flags in $\mathbf{CP^2}$. Since $\dim SU(3) = 8$ and $\operatorname{ind} SU(3) = 2$ condition (1) is clearly verified. We will check condition (3).

The Lie algebra of $SU(3)$ consists of all the skew hermitian matrices with trace zero. In this case k consists of all the matrices Y of the form:

$$Y = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$$

where α, β and γ are purely imaginary and their sum is zero.

Consider the Killing metric on $SU(3)$ i.e. $(X,Y) = -\frac{1}{2} Re\, tr(XY)$. With respect to this product $k^{\perp}$ is the subset of $su(3)$ given by the matrices with zero entries on the diagonal. Let $X \in k^{\perp}$ be given by :

$$X = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Take $Y \in k$ as before and compute $[X,Y]$. We get:

$$[X,Y] = \begin{pmatrix} 0 & \beta-\alpha & \gamma-\alpha \\ \beta-\alpha & 0 & 0 \\ \gamma-\alpha & 0 & 0 \end{pmatrix}$$

Then we clearly have $\dim [X,k]{=}\dim k{=}2$ and condition (3) is verified.

Therefore the geodesic flow of a left invariant metric on $SU(3)/T^2$ is completely integrable provided that not every geodesic is the orbit of a 1-parameter subgroup of $SU(3)$.

Exactly the same arguments can be applied to other spaces. In particular, the geodesic flow of a left invariant metric on $SO(n+1)/SO(n-1)$ is completely integrable provided that not every geodesic is the orbit of a 1-parameter subgroup of $SO(n+1)$. The complete integrability of the geodesic flow of the normal homogeneous metrics on $SO(n{+}1)/SO(n{-}1)$ was obtained by Thimm [15, Proposition 5.3]. Here the original Thimm method works since the natural action of $SO(n+1) \times SO(2)$ on the tangent bundle of $SO(n+1)/SO(n-1)$ is multiplicity free.

3 *Submersion metrics and reduced spaces*

An especially nice class of Riemannian submersions is that given by isometric group actions. Their main symplectic feature, as we will see, is that their tangent bundles are Marsden-Weinstein reductions of the tangent bundles of the total spaces. This and other basic symplectic properties are fundamental to the examples studied in the remaining sections. We refer to [2, ch. 9] and [8, section 26] for all the basic definitions.

Let a Lie group G of dimension m act on a Riemannian manifold M with metric $\langle\, , \rangle_M$ by isometries. We endow the tangent bundle TM of M with the symplectic structure $\bar{\omega}$ obtained by pulling back the canonical symplectic structure on the cotangent bundle T^*M by the metric. Then G acts symplectically on TM. Let g denote the Lie algebra of G.

Lemma 3.1 *The moment map $\Phi : TM \to g^*$ is given by*

$$\Phi(v)(\xi) = \langle v, \xi(\beta(v))\rangle$$

for $v \in TM$ and $\xi \in g$ where β maps a tangent vector to its foot point.

Proof: Recall the formula for the moment map on the cotangent bundle, namely $\Phi^*(v) = v(\xi(\beta(v)))$ for $v \in T^*M$ and $\xi \in g$ [8, p. 222]. This readily implies the claim since the symplectic structure on TM is the pullback under the Riemannian structure. $\diamond$

Now suppose that G acts on M without isotropy. Set $B = M/G$ and endow B with the submersion metric. Denote by $\pi : M \to B$ the quotient map.

Lemma 3.2 *The moment map intersects the trivial coadjoint orbit $\{0\}$ in g^* cleanly, i.e. $\Phi^{-1}(0)$ is a submanifold of TM and at each point $x \in \Phi^{-1}(0)$ we have $T_x(\Phi^{-1}(0)) = d\Phi^{-1}(T_0(0)) = d\Phi^{-1}(0)$. Moreover, $\Phi^{-1}(0)$ is the set of all horizontal vectors.*

Proof: First note that $\Phi^{-1}(0) = \{v \in TM \mid \text{for all } \xi \in g\ \langle v, \xi(\beta(v))\rangle = 0\}$ is the set of all horizontal vectors, and thus a manifold.

For $w = (w_1, w_2) \in T_x(\Phi^{-1}(0))$ let $\{p_t\} \subset M$ and $\{v_t\} \subset TM$ be C^1-paths such that $w_1 = \frac{d}{dt}\Big|_{t=0} p_t$ and $w_2 = \frac{d}{dt}\Big|_{t=0} v_t$. Let $\tilde{\xi_1}, \ldots, \tilde{\xi_m}$ be a basis for g. Choose a coordinate system for M about x such that the first m coordinates are given by $\xi_1 = \tilde{\xi_1}(y), \ldots, \xi_m = \tilde{\xi_m}(y)$. In this coordinate system we may write $v_t = h_t + \eta + t\xi = h_t + \tilde{\eta}(p_t) + t\tilde{\xi}(p_t)$ where the h_t are horizontal and $\tilde{\eta}$ and $\tilde{\xi} \in g$. Since g^* is a vector space, T^*g is canonically identified with g^* and we have

$$\left(\frac{d}{dt}\Big|_{t=0} c_t\right)(\zeta) \stackrel{\text{def}}{=} \frac{d}{dt}\Big|_{t=0} (c_t(\zeta))$$

for $\zeta \in g^*$ and c_t a C^1-path in g^*. Since the h_t are horizontal we get

$$d\Phi_x(w)(\xi) = \frac{d}{dt}\Big|_{t=0} \langle v_t, \xi(p_t)\rangle = \frac{d}{dt}\Big|_{t=0} \langle h_t + \eta + t\xi(p_t), \xi(p_t)\rangle = \langle \xi(p_t), \xi(p_t)\rangle.$$

Thus $d\Phi_x(w) = 0$ if and only if $\xi = 0$ or equivalently if $w \in T_x(\Phi^{-1}(0))$. $\diamond$

By the last lemma the Marsden-Weinstein reduced space $TM//G$ of TM with respect to the $\{0\}$-coadjoint orbit is defined. Recall that it can be identified with the reduced space $\Phi^{-1}(0)/G$ [8, p. 192].

Proposition 3.3 *The Marsden-Weinstein reduced space $TM//G$ with respect to the $\{0\}$-coadjoint orbit is symplectomorphic with TB.*

Proof: Since $\Phi^{-1}(0)$ is the set of all horizontal vectors, $TM//G$ is diffeomorphic with TB. We will identify the two henceforth.

The canonical symplectic structure on T^*B pulled back to TB by the Riemannian metric $\langle\, , \rangle_B$ is the 2-form $\omega = -d\theta$ where θ is the 1-form defined by $\theta_x(v) = \langle x, d(\beta)_x v\rangle_B$

for $x \in TB$ and $v \in T_x TB$. Similarly denote by $\bar{\theta}$ the 1-form on TM, with $\bar{\omega} = -d\bar{\theta}$. Let $\bar{x} \in \Phi^{-1}(0)$ and $\bar{v} \in T_{\bar{x}}\Phi^{-1}(0)$. Then $\bar{x}$ and $d(\beta)_{\bar{x}}\bar{v}$ are horizontal vectors, and therefore

$$\bar{\theta}_{\bar{x}}(\bar{v}) = \langle \bar{x}, d(\beta)_{\bar{x}}\bar{v}\rangle_M = \langle d\pi(\bar{x}), d\pi(d(\beta)_{\pi\bar{x}}\bar{v})\rangle_B = \theta_{\pi\bar{x}}(d\pi\bar{v}).$$

If we restrict $d\pi$ to $\Phi^{-1}(0)$, we see that $d\pi^*\theta = \bar{\theta} \mid_{\Phi^{-1}(0)}$. Hence $\bar{\omega} \mid_{\Phi^{-1}(0)} = d\pi^*\omega$. By definition, the symplectic form ω_r on the reduced space satisfies $d\pi^*\omega_r = \bar{\omega} \mid_{\Phi^{-1}(0)}$. Since $d\pi$ is surjective, we get $\omega = \omega_r$. ◇

For a C^1-function f on a symplectic manifold X we denote by ξ_f the associated *Hamiltonian vector field*. Suppose G acts on X in a Hamiltonian way with moment map $\Phi : X \to g^*$ that intersects $\{0\}$ in g^* cleanly. If f is G-invariant then ξ_f is tangent to $\Phi^{-1}(0)$. Let ρ be the projection from $\Phi^{-1}(0)$ to the reduced space $Y = \Phi^{-1}(0)/G$. If $f' : Y \to \mathbf{R}$ denotes the induced function then $\xi_{f'} = d\rho(\xi_f)$ [1, Appendix 5C].

Lemma 3.4 *Let $f, g : X \to \mathbf{R}$ be G-invariant functions on X and $f', g' : Y \to \mathbf{R}$ the induced functions. Then we have*

$$\{f', g'\} \circ \rho = \{f, g\} \mid_{\Phi^{-1}(0)} .$$

In particular, Poisson-commuting G-invariant functions descend to Poisson-commuting functions.

Proof: This follows from the discussion before the lemma and

$$\xi_{\{f', g'\}} = [\xi_{f'}, \xi_{g'}] = d\rho\,([\xi_f, \xi_g].$$

◇

4 *Submersion examples.*

The idea of the constructions below is that sometimes the integrals that arise from the Thimm method are invariant under a subgroup of the isometry group. Then one can construct integrals for the quotient space by this subgroup endowed with the submersion metric. Of course, the main problem is to show independence of the integrals thus obtained.

We need to describe the Thimm method in more detail to understand the invariance properties of the Thimm integrals. First one finds a maximal family of functions on g^* in involution which are functionally independent. Their construction is inductive: one pulls back a family of such functions already constructed on g^*_{i+1} by j_i and appends a maximal number of functionally independent G_i-invariant functions on g_i. Now one can pull back this family of functions on g^* to the symplectic space N using the moment map Φ. Under our hypothesis, we get $n := \dim N/2$ many functions $f_1, ..., f_n$ in involution on N which almost everywhere are functionally independent. Furthermore they commute with any G-invariant Hamiltonian on N.

Next we observe some invariance properties of these integrals.

Lemma 4.1 *Let $j : g' \to g$ be a Lie subalgebra corresponding to the subgroup G' of G. Let $\phi' : g'^* \to \mathbf{R}$ be a function invariant under the coadjoint action of G'. Let $\phi = j^*(\phi') = \phi' \circ j^*$.*

a) *If τ is in the centralizer of g' in G then ϕ is invariant under τ.*

b) *If $\tau \in G'$ then ϕ is invariant under τ.*

Proof: Suppose τ is in the normalizer of g' in G. Then

$$\tau(\phi) = \phi \circ \text{ Ad}^*(\tau) = \phi' \circ j^* \circ \text{ Ad}^*(\tau) = \phi' \circ \text{ Ad}^*(\tau) \circ j^*.$$

If τ centralizes g' then $\text{Ad}^*(\tau) = id\,|_{g'}$, and **a)** follows. For **b)** recall that ϕ' is G'-invariant. $\diamond$

In particular we see that any function f_i constructed above is G_{l-1}-invariant. In fact, since the moment map is equivariant, it suffices to see this for the functions $\phi_i : l^* \to \mathbf{R}$ constructed above. Since G_{l-1} is contained in all the other subalgebras (except $\{1\}$) this follows from Lemma 4.1 b).

In the remainder of this section, we combine these observations with the results of the previous sections. We first describe a general construction of new manifolds supporting integrable geodesic flows from known ones.

Let X be a complete Riemannian manifold of dimension n whose geodesic flow is completely integrable. Suppose X admits a free S^1 action by isometries that leaves the integrals invariant. Let N be a complete surface of revolution and consider the diagonal action by isometries on $X \times N$. Since this action is free we can consider the quotient manifold $M = X \times_{S^1} N$ and endow it with the submersion metric. Then we have:

Proposition 4.2 *The geodesic flow on M is completely integrable.*

Proof: Denote by $f_1, ..., f_n$ the integrals coming from X and by g the metric on N. These functions extend to $T(X \times N) = TX \times TN$ in the obvious way and their extensions will be denoted by $\hat{f}_1, ..., \hat{f}_n, \hat{g}$. They clearly are integrals of the geodesic flow given by the product metric on $X \times N$. Since all these integrals are invariant under S^1 by hypothesis, they descend to $\Phi^{-1}(0)/S^1 = TM$ where Φ is the moment map corresponding to the diagonal action of S^1 on $X \times N$. Hence we get $n+1$ integrals in involution for the geodesic flow on M (cf. Proposition 3.3 and Lemma 3.4). We only need to show that they are functionally independent almost everywhere.

For this consider a point $(p_1, p_2) \in X \times N$. Set $H = \Phi^{-1}(0) \cap T_{(p_1,p_2)}(X \times N)$ which is nothing but the set of horizontal vectors at (p_1, p_2). Let $\tau : H \to T_{p_1}X$ be the restriction of the projection map. Clearly τ is onto if S^1 does not fix p_2. For a.e. p_1 and a.e. $v_1 \in T_{p_1}X$ the vector fields $\xi_{f_1}, ..., \xi_{f_n}$ at v_1 are linearly independent by assumption. Take such a v_1 and let v_2 be a non-zero vector on the projection of $\tau^{-1}(v_1)$ over $T_{p_2}N$. Moving v_1 a little if necessary, we can choose v_2 so that the geodesic through v_2 is not an orbit of the S^1 action. We will now check independence at the projection of (v_1, v_2) to $\Phi^{-1}(0)/S^1$.

It is clear that $\xi_{\hat{f}_1}, ..., \xi_{\hat{f}_n}, \xi_{\hat{g}}$ are linearly independent at (v_1, v_2). Therefore we need to show that the tangent vector field W to the orbits of S^1 on $\Phi^{-1}(0)$ at (v_1, v_2) does not belong to the vector space spanned by $\xi_{\hat{f}_1}, ..., \xi_{\hat{f}_n}, \xi_{\hat{g}}$. Let W_1 denote the tangent field to the orbits of S^1 on TX and W_2 the corresponding tangent field to the orbits of S^1 on TN. Then W can be written as $(-W_1, W_2)$ where $W_1(v_1) \in T_{v_1}TX$ and $W_2(v_2) \in T_{v_2}TN$. Observe now that $\xi_{\hat{f}_i} \in TTX$ and $\xi_{\hat{g}} \in TTN$. Hence if W belongs to the space spanned by the $\xi_{\hat{f}_i}\,'s$ and $\xi_{\hat{g}}$ at (v_1, v_2) we deduce that W_2 and $\xi_{\hat{g}}$ are collinear at v_2. This implies that the geodesic through v_2 is an orbit of the S^1-action. This is a contradiction to the choice of v_2. $\diamond$

Let us see some applications of the previous proposition.

Example 4.3 Consider $SO(n)$ endowed with its standard bi-invariant metric. Then $SO(n)$ can be viewed as a symmetric space of $SO(n) \times SO(n)$ in the usual way. Consider the ascending chain of subgoups:

$$\{1\} \subset SO(2) \times \{1\} \subset ... \subset SO(n) \times \{1\} \subset ... \subset SO(n) \times SO(n)$$

This chain as well as the action of $SO(n) \times SO(n)$ on $TSO(n)$ are multiplicity free and the hypotheses of the Thimm method hold. Thus we recover the well-known fact that the geodesic flow on $SO(n)$ is completely integrable. By Lemma 4.1 the integrals are all invariant under $SO(2) \times \{1\} = S^1$. Now the proposition implies that $M = SO(n) \times_{S^1} N$ endowed with the submersion metric supports a completely integrable geodesic flow. Similar arguments apply to the case of $SU(n)$.

Let us describe the manifolds we get for the special case of $SU(2) = Spin(3) = S^3$ and $M^4 = SU(2) \times_{S^1} N^2$. If N is the 2-sphere then M is the non-trivial S^2-bundle over S^2 which is diffeomorphic to $\mathbf{CP^2} \# -\mathbf{CP^2}$. It is known that M is not diffeomorphic to any homogeneous space (cf. [3]). If N is euclidean 2-space, then M is the normal bundle of $\mathbf{CP^1}$ in $\mathbf{CP^2}$. Moreover if we consider S^1 acting on the plane by rotating n times then the corresponding M_n give all the line bundles over $\mathbf{CP^1}$. Finally we want to point out that the metric on $SU(2)$ does not need to be bi-invariant. For any left invariant metric the arguments work because the action of $\hat{G} = SU(2) \times R$ as in Section 2 is multiplicity free.

Example 4.4 Next we will construct a large class of metrics on $\mathbf{CP^n} \# -\mathbf{CP^n}$ with completely integrable geodesic flows, generalizing the last example.

Consider the Hopf fibration $S^1 \to S^{2n+1} \to \mathbf{CP^n}$. Denote by g_t the metric on S^{2n+1} which is obtained from the standard metric by multiplying with t^2 in the directions tangent to the S^1-orbits. The canonical action of the group $SU(n+1)$ on S^{2n+1} is by isometries and commutes with the S^1 action. Hence the group $SU(n+1) \times S^1$ acts on S^{2n+1} by isometries. It is known that (S^{2n+1}, g_t) can be viewed as distance spheres on $\mathbf{CP^{n+1}}$ with the metric induced by the Fubini-Study metric. For $t \leq \frac{n+1}{2n}$ they are called Berger spheres. We refer to [17] for details. The action of $SU(n+1) \times S^1$ on TS^{2n+1} is multiplicity free. Choosing a suitable chain it follows from the Thimm method that the geodesic flow on (S^{2n+1}, g_t) is completely integrable. By Lemma 4.1 the integrals are all invariant under the S^1-action.

Then Proposition 4.1 shows that the geodesic flow on $M = S^{2n+1} \times_{S^1} N$ is completely integrable for all real t. If we take $N = S^2$ then the corresponding M is diffeomorphic to $\mathbf{CP^{n+1}} \# - \mathbf{CP^{n+1}}$. If N euclidean 2-space then M is the normal bundle of $\mathbf{CP^n}$ in $\mathbf{CP^{n+1}}$.

Example 4.5 Let $G_{n-1,2}(\mathbf{R}) = SO(n+1)/SO(n-1) \times SO(2)$ denote the Grassmannian of 2-planes in n+1-space. Consider the fibration $S^1 \to SO(n+1)/SO(n-1) \to G_{n-1,2}(\mathbf{R})$, where S^1 acts on $SO(n+1)/SO(n-1)$ by right translations. As mentioned at the end of Section 2, the action of $SO(n+1) \times S^1$ on the tangent bundle of $SO(n+1)/SO(n-1)$ is multiplicity free. Consider metrics g_t on $SO(n+1)/SO(n-1)$ obtained from the normal homogeneous metric by multiplying with t^2 in the directions tangent to the S^1-orbits. Thus we can argue as in Example 4.4 to deduce that the geodesic flow on $M = SO(n+1)/SO(n-1) \times_{S^1} N$ is completely integrable for all real t. If N is the 2-sphere, M is a sphere bundle over the Grassmannian $G_{n-1,2}(\mathbf{R})$.

Example 4.6 Next consider surface bundles over the so called Eschenburg examples [4] (we will discuss the Eschenburg examples themselves below). Consider the group $SU(3)$ with its standard bi-invariant metric and let $SU(3) \times SU(3)$ act on $SU(3)$ by $(g_1, g_2)x = g_1 x g_2^{-1}$. Let k, l, p, q be a set of relatively prime integers. Define a one-parameter subgroup of $SU(3) \times SU(3)$ by

$$U_{klpq} = \{\exp 2\pi i t(\operatorname{diag}(k, l, -k-l), \operatorname{diag}(p, q, -p-q)) \mid t \in \mathbf{R}\}.$$

For certain choices of k, l, p and q the action of U_{klpq} on $SU(3)$ is fixed point free, in particular for the quadruple $(1, -1, 2m, 2m)$ [4, Proposition 21].

Consider the ascending chain of subgroups: $\{1\} \times U(1) \subset U(1) \times U(1) \subset U(1) \times U(2) \subset U(2) \times U(2) \subset U(2) \times SU(3) \subset SU(3) \times SU(3)$ where $U(1)$ and $U(2)$ are embedded into $SU(3)$ by adjusting the $(3,3)$-entry in the matrix in the obvious way. Note that $(1, \exp 2\pi i t \operatorname{diag}(2m, 2m, -4m))$ and $(\exp 2\pi i t \operatorname{diag}(1, -1, 0), 1)$ either belong to or centralize any subgroup in this chain. Thus all the first integrals on $TSU(3)$ are invariant under these one-parameter subgroup and thus under $U_{1,-1,2m,2m}$.

¿From the last proposition we deduce that $M_m = SU(3) \times_{U_{1,-1,2m,2m}} N$ endowed with the submersion metric supports a completely integrable geodesic flow . If N is the 2-sphere, M_m is a sphere bundle over Eschenburg's strongly inhomogeneous 7-manifold $SU(3)/U_{1,-1,2m,2m}$. These spaces where studied in [14]. Metrically they have higher rank and topologically are strongly inhomogeneous and irreducible ([14, Proposition 4.2 and 4.6]).

Finally let us study some submersions that do not have the product type used in Proposition 4.2. The observations concerning the invariance of the Thimm integrals from the beginning of this section however are still crucial. Unfortunately, the calculations necessary become much more complicated.

Example 4.7 Here we will study the Eschenburg examples themselves. Let U_{klpq} be the one-parameter subgroup of $SU(3) \times SU(3)$ from Example 4.6 and endow $SU(3)$ with a bi-invariant metric. We will show below that for all m, the geodesic flow of the Eschenburg manifold $E_m \stackrel{\text{def}}{=} SU(3)/U_{1,-1,2m,2m}$ endowed with the submersion metric is completely integrable. As Eschenburg showed, this is another example of a strongly inhomogeneous manifold [4]. Also notice that for $m = 0$ we obtain a Wallach manifold [16].

For simplicity, set $U = U_{1,-1,2m,2m}$. Denote by $\Phi : TSU(3) \to su(3) + su(3)$ the moment map of the action of $SU(3) \times SU(3)$ on the tangent bundle $TSU(3)$. As in Example 4.6, we use the ascending chain of subgroups $\{1\} \times U(1) \subset U(1) \times U(1) \subset U(1) \times U(2) \subset U(2) \times U(2) \subset U(2) \times SU(3) \subset SU(3) \times SU(3)$. Let pr_1 and pr_2 be the projections of $su(3) + su(3)$ onto the first and second factor respectively. Denote by $pr_{u(i)}$ the orthogonal projection of $su(3)$ to $u(i)$. Further identify $su(3)^*$ with $su(3)$ via the Cartan-Killing form as usual. Then the Thimm functions on $TSU(3)$ are the pull backs under the moment map of the following functions on $su(3) + su(3)$:

$$\begin{array}{ll}
f_1 = i\,\mathrm{tr}(\xi) \circ pr_{u(1)} \circ pr_1 & f_5 = i\,\mathrm{tr}(\xi^3) \circ pr_1 \\
f_2 = i\,\mathrm{tr}(\xi) \circ pr_{u(2)} \circ pr_1 & f_6 = i\,\mathrm{tr}(\xi) \circ pr_{u(1)} \circ pr_2 \\
f_3 = \mathrm{tr}(\xi^2) \circ pr_{u(2)} \circ pr_1 & f_7 = i\,\mathrm{tr}(\xi) \circ pr_{u(2)} \circ pr_2 \\
f_4 = \mathrm{tr}(\xi^2) \circ pr_1 & f_8 = \mathrm{tr}(\xi^2) \circ pr_{u(2)} \circ pr_2 .
\end{array}$$

As in Example 4.6, all the Thimm integrals $f_i \circ \Phi$ on $TSU(3)$ are invariant under U, and thus induce Poisson-commuting functions $\tilde{f}_i$ on TE_m.

Let us now show the independence of seven of these functions, namely $\tilde{f}_2, \ldots, \tilde{f}_8$. First note that by real analyticity we only need to establish the independence of these functions at one point.

Let $\mathcal{H}$ denote the set of horizontal vectors on $TSU(3)$. Recall that $\mathcal{H} = \Phi_U^{-1}(0)$ where Φ_U is the moment map of the action of U on $TSU(3)$.

First we will reduce the problem to a calculation in the Lie algebra. Suppose that the $\tilde{f}_i$, $i = 2, \ldots, 8$, are dependent at the projection v of a vector $\hat{v} \in \mathcal{H}$ via some relation $\sum_{i=2}^{8} c_i d\tilde{f}_i = 0$ on $T_v E_m$. Then the 1-form $\sum_{i=2}^{8} c_i d(f_i \circ \Phi)$ is 0 on horizontal lifts of double tangent vectors. Since the functions $f_i \circ \Phi$ are U-invariant, $\sum_{i=2}^{8} c_i d(f_i \circ \Phi)$ is also 0 on tangent vectors to the U-orbit of $\hat{v}$. This implies that $\sum_{i=2}^{8} c_i d(f_i \circ \Phi) = 0$ on $T_{\hat{v}}\mathcal{H}$. Now suppose that $\Phi(\mathcal{H})$ is a manifold in a neighborhood of $\Phi(\hat{v})$ and that $\Phi(\hat{v})$ is a regular value of $\Phi : \mathcal{H} \to \Phi(\mathcal{H})$. Then a dependence of the restrictions of the $f_i \circ \Phi$ to $\mathcal{H}$ at $\hat{v}$ implies a dependence of the restrictions of the f_i to the image of $\mathcal{H}$ under Φ at $\Phi(\hat{v})$. This is the reduction to a calculation in the Lie algebra.

Next we need to determine $\Phi(\mathcal{H})$. First let us describe Φ itself. As usual identify $T_1 SU(3)$ with the orthogonal complement (with respect to the Cartan-Killing form) of the diagonal embedding $\Delta su(3)$ of $su(3)$ into $su(3) \times su(3)$, that is with $\{(X, -X) \mid X \in su(3)\}$. Then we have the following formula for the value of the moment map at a translate of a vector $(X, -X)$ in $T_1 SU(3)$

$$\Phi((g_1, g_2)_*(X, -X)) = (\operatorname{Ad} g_1(X), -\operatorname{Ad} g_2(X)).$$

Thus $\mathcal{R} \stackrel{\text{def}}{=} \Phi(TSU(3))$ is given by

$$\mathcal{R} = \{(X, Y) \mid X \text{ is conjugate to } -Y \text{ in } su(3)\}.$$

Next note that $\Phi_U = i^* \circ \Phi$ where $i : u \to su(3) + su(3)$ is the embedding of the Lie algebra u of U into $su(3) + su(3)$. Thus the horizontal vectors in $TSU(3)$ are the preimage $\mathcal{H} = \Phi_U^{-1}(0) = \Phi^{-1}(u^\perp)$. Hence the image of $\mathcal{H}$ under Φ is

$$\Phi(\mathcal{H}) = \mathcal{R} \cap u^\perp.$$

Let us now give an outline of the calculations that show the independence of the restrictions of $f_2, \ldots, f_8$ to $\Phi(\mathcal{H})$ at the point $p \in \Phi(\mathcal{H})$ given by $p = (P, -P)$ where

$$P = \begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

One easily shows that $\mathcal{R} \cap u^\perp$ is a manifold in a neighborhood of p, and that p is a regular value of Φ. Suppose that on $T_p(\mathcal{R} \cap u^\perp)$ we have

$$(*) \qquad \sum_{i=2}^{8} c_i df_i = 0.$$

We will exhibit several tangent vectors in $T_p(\mathcal{R} \cap u^\perp)$ which force various relations between the coefficients c_i, forcing them to be 0 eventually.

1. Set $p_1^t = (P_1^t, -P_1^t)$ where

$$P_1^t = \begin{pmatrix} 0 & 2 & 1+t \\ -2 & 0 & 0 \\ -1-t & 0 & 0 \end{pmatrix}.$$

   Then $p_1^t \in \mathcal{R} \cap u^\perp$. Since the nontrivial projections $pr_{u(i)}$ of P_1^t are all constant, only df_4 and df_5 can be nonzero on $v_1 = \frac{d}{dt}\Big|_{t=0} p_1^t$. The eigenvalues of P_1^t are $0, \sqrt{-4-(1+t)^2}$ and $-\sqrt{-4-(1+t)^2}$. Therefore we get $df_5(v_1) = 0$, $df_4(v_1) \neq 0$, and thus $c_4 = 0$.

2. Set $p_2^t = (P_2^t, -P_2^t)$ where

$$P_2^t = \begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & it \\ -1 & it & 0 \end{pmatrix}.$$

   Then $p_2^t \in \mathcal{R} \cap u^\perp$. The eigenvalues of P_2^t satisfy the equation

$$-\lambda^3 - \lambda(t^2+5) - 4it = 0.$$

   Hence $f_5(p_2^t) = -i(tr(P_2^t)(t^2+5) - 12it) = -12t$. As above and since $c_4 = 0$ we conclude that $c_5 = 0$.

3. Let $s(t)=\sqrt{(t+2)^2-3}$ and set $p_3^t=(P_3^t,-Q_3^t)$ where

$$P_3^t=\begin{pmatrix}0 & 2+t & 1\\ -2-t & 0 & 0\\ -1 & 0 & 0\end{pmatrix} \text{ and } Q_3^t=\begin{pmatrix}0 & 2 & s(t)\\ -2 & 0 & 0\\ -s(t) & 0 & 0\end{pmatrix}.$$

A calculation of the eigenvalues shows that $p_3^t\in\mathcal{R}\cap u^\perp$ and that only df_3 gives a nonzero contribution in (*) when applied to $\frac{d}{dt}\Big|_{t=0}p_3^t$. Therefore we get $c_3=0$.

4. Considering $p_4^t=(Q_3^t,-P_3^t)$ we find that $c_8=0$.

5. Set $p_5^t=(P,-\operatorname{Ad}(\exp tA)(P))$ where

$$A=\begin{pmatrix}0 & i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Clearly, p_5^t lies in $\mathcal{R}\cap u^\perp$ since $id\times A$ commutes with u. Since

$$[A,P]=\begin{pmatrix}-4i & 0 & 0\\ 0 & 4i & i\\ 0 & i & 0\end{pmatrix},$$

we get $df_2(\frac{d}{dt}\Big|_{t=0}p_5^t)=df_7(\frac{d}{dt}\Big|_{t=0}p_5^t)=0$ while $df_6(\frac{d}{dt}\Big|_{t=0}p_5^t)\neq 0$. This implies $c_6=0$.

6. Let $p_6^t=(P_6^t,-P_6^t)$ where

$$p_6^t=\begin{pmatrix}t & 2 & 1\\ -2 & -\frac{6m-1}{6m+1}t & 0\\ -1 & 0 & \frac{-2}{6m+1}t\end{pmatrix}.$$

One sees easily that $p_6^t\in\mathcal{R}\cap u^\perp$, and then that $c_2=c_7$.

7. Suppose that $c_2=c_7\neq 0$. Then $df_2=-df_7$ on $T_p(\mathcal{R}\cap u^\perp)$. Note that

$$\left(\exp t\begin{pmatrix}0 & i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}P,-P\right)\in\mathcal{R}.$$

The tangent vector v to this curve at 0 is given by

$$\left(\left[\begin{pmatrix}0 & i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix},P\right],0\right)=\left(\begin{pmatrix}-4i & 0 & 0\\ 0 & 4i & i\\ 0 & i & 0\end{pmatrix},0\right).$$

Thus v is not perpendicular to u while $df_2(v)=df_7(v)=0$. Hence $df_2=-df_7$ on $T_p(\mathcal{R}\cap u^\perp)+\mathbf{R}v=T_p\mathcal{R}$.

On the other hand, consider the curve in $\mathcal{R}$ given by

$$\left(\exp t \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} P, -P\right).$$

Its tangent vector w at 0 is

$$\left(\left[\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, P\right], 0\right) = \left(\begin{pmatrix} -2i & 0 & 0 \\ 0 & 0 & 2i \\ 0 & 2i & 2i \end{pmatrix}, 0\right).$$

Clearly we have $df_2(w) \neq df_7(w)$, a contradiction. Therefore we get $c_2 = c_7 = 0$, and $f_2, \ldots, f_8$ are a.e. independent.

As a final application of the submersion method we construct a Riemannian metric with completely integrable geodesic flow on an exotic sphere. Again the submersion in question does not have the product type. The integrals themselves arise both from the submersion method combined with a Thimm construction as well as from the isometry group of this exotic sphere.

Example 4.8 Consider the exotic 7-sphere Σ constructed by Gromoll and Meyer in [7]. It arises as a biquotient of $Sp(2)$ by the following action of $Sp(1)$. For $q \in Sp(1)$ and $Q \in Sp(2)$ set

$$(q, Q) \mapsto \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} Q \begin{pmatrix} \bar{q} & 0 \\ 0 & \bar{q} \end{pmatrix}$$

where $\bar{q}$ denotes the complex conjugate of q. This also defines an embedding U of $Sp(1)$ into $Sp(2) \times Sp(2)$. Note that the canonical $O(2)$ in $Sp(2)$ commutes with the right action of $Sp(1)$ while an obvious $Sp(1)$ commutes with the left action. We give Σ the submersion metric determined by the biinvariant metric on $Sp(2)$.

The basic argument is much the same as in Example 4.7. Again, let $\Phi : TSp(2) \to sp(2) + sp(2)$ denote the moment map of the action of $Sp(2) \times Sp(2)$ on the tangent bundle $TSp(2)$. Let pr_1 and pr_2 denote the projections of $sp(2) + sp(2)$ onto the first and second factor respectively. Further we denote the orthogonal projection to a subalgebra $h \subset sp(2)$ by pr_h. We embed $sp(2)$ into $u(4)$ canonically. Then we define the following functions on $sp(2) + sp(2)$ using complex valued traces:

$$\begin{aligned}
f_1 &= \mathrm{tr}(\xi^2) \circ pr_{sp(1)\times 1} \circ pr_1 & f_5 &= \mathrm{tr}(\xi^4) \circ pr_1 \\
f_2 &= \mathrm{tr}(\xi^2) \circ pr_{so(2)} \circ pr_2 & f_6 &= \mathrm{tr}(\xi^2) \circ pr_{1\times l} \circ pr_1 \\
f_3 &= \mathrm{tr}(\xi^4) \circ pr_{sp(1)\times sp(1)} \circ pr_1 & f_7 &= \mathrm{tr}(\xi^2) \circ pr_{sp(1)\times sp(1)} \circ pr_2 \\
f_4 &= \mathrm{tr}(\xi^2) \circ pr_1 & &
\end{aligned}$$

where $so(2)$ refers to the Lie algebra of the canonical $O(2)$ above while l refers to the subalgebra of $sp(1)$ generated by

$$\begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}.$$

These functions are all invariant functions on some subalgebra pulled back to $sp(2)+sp(2)$. These subalgebras are either contained in each other as in Thimm's argument or commute with each other. It easily follows that they all Poisson commute. Note that f_2 and f_6 are just first integrals coming from the the isometry group. Also note that all these functions are invariant under the adjoint action of U. Hence their pullbacks to $TSp(2)$ under Φ are invariant under the action of $Sp(1)$ on $Sp(2)$, and thus they descend to functions $\tilde{f}_i$, $i = 1, \ldots, 7$ on $T\Sigma$. As in Example 4.7 the independence of the $\tilde{f}_i$ at the projection of a horizontal vector $\hat{v}$ is equivalent to the independence of the restrictions of $f_1, \ldots, f_7$ to $\mathcal{R} \cap u^\perp$ near $\Phi(\hat{v})$ where u is the Lie algebra of U and $\mathcal{R} = \{(X, Y) \mid X \text{ is conjugate to } -Y\}$. We assume here that $\Phi(\hat{v})$ is a regular value of $\Phi : \mathcal{H} \to \Phi(\mathcal{H})$ and that $\mathcal{R} \cap u^\perp$ is a manifold near $\Phi(\hat{v})$.

Next we will indicate a point p in $\mathcal{R} \cap u^\perp$ and tangent vectors in $T_p(\mathcal{R} \cap u^\perp)$ that show the independence of $f_1, \ldots, f_7$.

Let

$$F(t) \overset{\text{def}}{=} \begin{pmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{pmatrix}$$

and

$$G(t) \overset{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & \cos \pi t + j \sin \pi t \end{pmatrix}$$

where $1, i, j$ and $k = ij$ are the standard basis of the quaternions. Set $Q \overset{\text{def}}{=} F(\frac{1}{3})G(\frac{1}{4})$ and

$$P \overset{\text{def}}{=} \begin{pmatrix} 2i - 2j - \frac{149+18\sqrt{2}\sqrt{3}}{9}k & 1 + 3i + 2j - 3k \\ -1 + 3i + 2j - 3k & 5i + (6 + \sqrt{2}\sqrt{3})j + \frac{26}{3}k \end{pmatrix}.$$

Define $R = QPQ^*$ where Q^* is the conjugate transpose of Q, and set $p = (R, P)$. Then $p \in \mathcal{R} \cap u^\perp$. One can check that p is a regular value of Φ, that $\mathcal{R}$ and $u^\perp$ intersect transversally at p and that $\mathcal{R} \cap u^\perp$ is a manifold near p.

Next we will list the relevant tangent vectors in $T_p(\mathcal{R} \cap u^\perp)$. We need the following matrices:

$$D_1 \overset{\text{def}}{=} \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \quad D_2 \overset{\text{def}}{=} \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \quad D_3 \overset{\text{def}}{=} \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \quad D_4 \overset{\text{def}}{=} \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$$

$$D_5 \overset{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad D_6 \overset{\text{def}}{=} \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} \quad D_7 \overset{\text{def}}{=} \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}.$$

Then the tangent vectors are:

1. $v_1 = (D_5, -Q^* D_5 Q)$
2. $v_2 = (R, P)$
3. $v_3 = (0, [D_5, P])$
4. $v_4 = (-Q(\sqrt{2}\sqrt{3}D_7 + D_3)Q^*, \sqrt{2}\sqrt{3}D_7 + D_3)$

5. $v_5 = (-QD_1Q^*, D_1 + [\frac{3}{80}D_4 - \frac{9}{80}D_6, P])$
6. $v_6 = ([D_7, R], 0)$
7. $v_7 = (-D_2, Q^*D2Q + [\frac{1}{48}D_3 + \frac{1}{40}D_4 - \frac{3}{40}D_6, P])$.

As in Example 4.7, evaluating a relation of the df_i's on $T_p(\mathcal{R} \cap u^\perp)$ on these seven tangent vectors forces this relation to be trivial. This long calculation as well as finding the vectors above was done by computer using Mathematica.

5 *Connected Sums.*

In this section we will combine the submersion technique of the last section with a glueing trick to construct metrics on $\mathbf{CP^{n+1}}\#\mathbf{CP^{n+1}}$ for n even with completely integrable geodesic flows. Topologically these spaces are obtained from two copies of $S^{2n+1} \times_{S^1} D^2$ where D^2 is the 2-disk and S^1 acts diagonally, glued along their boundary $S^{2n+1} \times_{S^1} S^1 = S^{2n+1}$ by an orientation reversing map. The metrics that we will use were already considered in [3]. Let us describe them.

Consider the Hopf fibration $S^1 \to S^{2n+1} \to \mathbf{CP^n}$ and endow S^{2n+1} with the metric g_t as in Example 4.4. Now equip $\mathbf{R}^2$ with a metric h_t ($t^2 \neq 1$) given in polar coordinates by :

$$h_t(\partial/\partial r, \partial/\partial r) = 1 \quad h_t(\partial/\partial r, \partial/\partial\theta) = 0 \quad h_t(\partial/\partial\theta, \partial/\partial\theta) = f_t^2(r)$$

where $f_t(r)$ is a smooth function with the properties $f_t(0) = 1$, $f_t'(0) = 1$ and $f_t(r) \equiv 2\pi t^2/\sqrt{t^4 - 1}$ for sufficently big $r > R$.

Set $\eta = S^{2n+1} \times_{S^1} \mathbf{R}^2$ with the submersion metric. If we restrict to the disk bundle $D_{\bar{R}}(\eta)$ with $\bar{R} > R$, then an annular neighborhood of the boundary splits isometrically as $\partial D_{\bar{R}}(\eta) \times I$ where I denotes an interval. In fact, $A = \{X \in \mathbf{R}^2 \mid R < \| X \| < \bar{R}\}$ splits isometrically as $S^1 \times I$ and S^1 acts trivially on I. Then

$$S^{2n+1} \times_{S^1} A = S^{2n+1} \times_{S^1} (S^1 \times I) = (S^{2n+1} \times_{S^1} S^1) \times I = S^{2n+1} \times I$$

and $S^{2n+1} = \partial D_{\bar{R}}(\eta)$ gets back the metric of constant curvature. Since the metric splits as a product $S^{2n+1} \times I$ near the boundary, by glueing two such disk bundles we get a smooth metric on $\mathbf{CP^{n+1}}\#\mathbf{CP^{n+1}}$.

According to Example 4.4 the metric on the disk bundle $D_{\bar{R}}(\eta)$ is completely integrable with first integrals $f_1, ..., f_{2n+1}, f_{2n+2}$. In fact $f_1, ..., f_{2n+1}$ are induced by the Thimm integrals on the tangent bundle of (S^{2n+1}, g_t) and f_{2n+2} is induced by the metric h_t (cf. Proposition 4.2). All the f_i *'s* are invariant under derivatives of translations on I. Therefore they will fit together smoothly with the integrals on the second $D_{\bar{R}}(\eta)$ if they happen to be invariant under the derivative of the orientation reversing map that we use for the glueing.

As a glueing map on the boundary S^{2n+1} we will take the complex conjugation τ i.e. the restriction to $S^{2n+1} \subset \mathbf{C^{n+1}}$ of the map:

$$(z_1, ..., z_{n+1}) \to (\bar{z}_1, ..., \bar{z}_{n+1})$$

This map is orientation reversing for n even (for n odd, one rediscovers Example 4.4).

As we will see below some of the functions f_i are not invariant under $d\tau$. Thus a small modification will be needed.

Denote by π the projection map $\pi : S^{2n+1} \times S^1 \to S^{2n+1} \times_{S^1} S^1$ and by σ the map $\sigma : S^{2n+1} \times S^1 \to S^{2n+1} \times S^1$ given by $\sigma(z_1, ..., z_{n+1}, e^{i\theta}) = (\bar{z}_1, ..., \bar{z}_{n+1}, e^{-i\theta})$. Note that σ takes S^1-orbits into S^1-orbits since

$$e^{i\varphi}.\sigma(z_1, ..., z_{n+1}, e^{i\theta}) = \sigma(e^{-i\varphi}.(z_1, ..., z_{n+1}, e^{i\theta}))$$

Hence σ descends to a map $\hat{\sigma} : S^{2n+1} \times_{S^1} S^1 \to S^{2n+1} \times_{S^1} S^1$. Observe that under the natural diffeomorphism $\psi : S^{2n+1} \to S^{2n+1} \times_{S^1} S^1$ given by $\psi(z_1, ..., z_{n+1}) = \pi(z_1, ..., z_{n+1}, 1)$, the map $\hat{\sigma}$ is complex conjugation, i.e. $\psi^{-1} \circ \hat{\sigma} \circ \psi = \tau$.

Since h_t is invariant under the map $(r, \theta) \to (r, -\theta)$ we deduce that the integral f_{2n+2} will be invariant under the derivative of τ.

Therefore we need to find integrals on (S^{2n+1}, g_t) which are invariant under $d\tau$ and under the S^1-action. In view of the previous arguments this automatically implies that the induced integrals on $S^{2n+1} \times_{S^1} S^1$ are also invariant under $d\tau$ and that we will be able to fit them smoothly.

Recall that the integrals we have on (S^{2n+1}, g_t) were obtained by the Thimm method using the action of the group $SU(n+1) \times S^1$. Let f_{2n+1} denote the integral induced by the S^1-action. Since $e^{i\varphi}.\tau(z_1, ..., z_{n+1}) = \tau(e^{-i\varphi}.(z_1, ..., z_{n+1}))$, we see that f_{2n+1} is not invariant under $d\tau$. But f_{2n+1}^2 is clearly invariant and still is a first integral. We will now use a similar trick for the integrals that arise from the $SU(n+1)$-action.

Identify S^{2n+1} with $SU(n+1)/SU(n)$ in the usual way, i.e. by means of the diffeomorphism $[A] \to A(1, 0, ..., 0)$ where $[A]$ denotes the equivalence class of a matrix $A \in SU(n+1)$. Since $\tau \circ A = \bar{A} \circ \tau$ it is easy to check that τ operates on $SU(n+1)/SU(n)$ as the map $[A] \to [\bar{A}]$. Decompose $su(n+1)$ as $su(n) \oplus m$ where m denotes the orthogonal complement of $su(n)$ in $su(n+1)$ with respect to the standard Killing form. The moment map ϕ of the action of $SU(n+1)$ on the tangent bundle of $SU(n+1)/SU(n)$ can be written as (cf. [15, Lemma 3.2]):

$$\phi(dL_A(B)) = \mathrm{Ad}_A(B)$$

where $A \in SU(n+1)$, $B \in m$ and L_A denotes the left translation on $SU(n+1)/SU(n)$.

Since the integrals arising from the Thimm method have the form $h \circ \phi$ where $h \in C^\infty(su(n+1))$, they are invariant under the derivative of $\tau[A] = [\bar{A}]$ if and only if for every $A \in SU(n+1)$ and $B \in m$ we have

$$h(\mathrm{Ad}_{\bar{A}}(\bar{B})) = h(\mathrm{Ad}_A(B)).$$

Therefore $h \circ \phi$ is invariant under $d\tau$ if h is invariant under conjugation on $su(n+1)$. If $B \in su(n+1)$ then $\bar{B} = -B^t$. Hence we need h such that $h(B) = h(-B^t)$.

Denote by $\pi_j : su(n+1) \to u(j)$ the map defined by:

$$su(n+1) \ni \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to \delta \in u(j)$$

The 2n-functions in involution on $su(n+1)$ that we get from the Thimm method are ([15, Proof of Theorem 7.4]):

$$h_j(B) = -\frac{1}{2} itr(\pi_j B) \quad j = 1, ..., n$$

$$h_{n+j-1}(B) = -\frac{1}{4} tr(\pi_j B)^2 \quad j = 2, ..., n+1$$

Clearly the h_{n+j-1} $'s$ are invariant under $B \to -B^t$, but the h_j $'s$ are not. Instead consider the functions:

$$h_j^2(B) \quad j = 1, ..., n$$

$$h_{n+j-1}(B) \quad j = 2, ..., n+1$$

Now they are all invariant under $B \to -B^t$, they are still in involution and they are functionally independent a.e. Hence the pull back of these functions by the moment map ϕ gives a set of 2n-functions that verifies all the necessary conditions.

References

[1] V. I. Arnold, Mathematical Methods of Classical Mechanics, Springer Verlag 1984.

[2] Besse, A. L., Einstein manifolds, Springer Verlag, Erg. der Math., 3. Folge, Bd. 10 (1987).

[3] J. Cheeger, Some examples of manifolds of nonnegative curvature, J. Diff. Geom. **8** (1972), 623–628.

[4] J.-H. Eschenburg, New examples of manifolds with strictly positive curvature, Invent. Math. **66** (1982), 469–480.

[5] A. T. Fomenko, Integrability and nonintegrability in geometry and dynamics, Kluwer Academic Publishers, Dordrecht, 1988.

[6] A. T. Fomenko, A. S. Mishchenko, Euler equations on finite-dimensional Lie groups, Izv. Akad. Nauk SSSR, Ser. Mat. **42** (1978), 396–415.

[7] D. Gromoll, W. Meyer, An exotic sphere with nonnegative sectional curvature, Ann. of Math. **100** (1974), 401–406.

[8] V. Guillemin, S. Sternberg, Symplectic techniques in physics, Cambridge University Press, Cambridge 1984.

[9] V. Guillemin, S. Sternberg, On collective complete integrability according to the method of Thimm, Ergod. Th. and Dyn. Syst. **3** (1983), 219–230.

[10] V. Guillemin, S. Sternberg, Multiplicity-free spaces, J. Diff. Geom. **19** (1984), 31–56.

[11] G. Heckman, Projections of orbits and asymptotic behavior of multiplicities for compact Lie groups, Invent. Math. **67** (1982), 333–356.

[12] M. Krämer, Multiplicity free subgroups of compact connected Lie groups, Arch. Math. **27** (1976), 28–36.

[13] M. Krämer, Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen, Compositio Mathematica **38** (1979), 129–153.

[14] R. J. Spatzier, M. Strake, Some examples of higher rank manifolds of nonnegative curvature, Comm. Math. Helv. **65** (1990), 299–317.

[15] A. Thimm, Integrable geodesic flows on homogeneous spaces, Ergod. Th. and Dyn. Syst. **1** (1981), 495–517.

[16] N. R. Wallach, Compact homogeneous Riemannian manifolds with strictly positive curvature, Ann. of Math. **96** (1972), 277–295.

[17] W. Ziller, The Jacobi equation on naturally reductive compact Riemannian homogeneous spaces, Comm. Math. Helv. **52** (1977), 573-590.