A Remark on Herman's Theorem for Circle Diffeomorphisms

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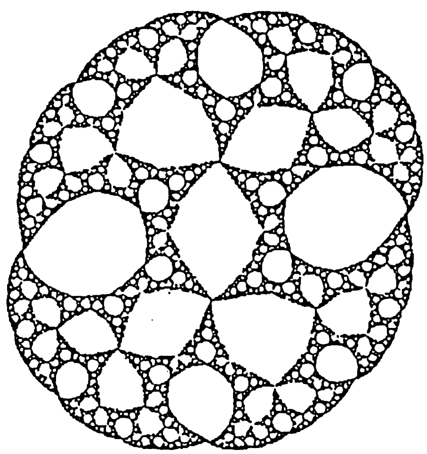

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A REMARK ON HERMAN'S THEOREM FOR CIRCLE DIFFEOMORPHISMS

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ABSTRACT

We define a class of real numbers that has full measure and is contained in the set of Roth numbers. We prove the C^1–part of Herman's theorem: if f is a C^3 diffeomorphism of the circle to itself with a rotation number ω in this class, then f is C^1–conjugate to a rotation by ω. As a result of restricting the class of admissible rotation numbers, our proof is substantially shorter than Yoccoz' proof.

1. INTRODUCTION

Recall Herman's theorem as it is stated and proved by Yoccoz [1984].

Herman's theorem: Let f be a $C^{2+\alpha}$ circle diffeomorphism ($\alpha > 0$), with an irrational rotation number ω which is Diophantine of order β (see section 3). Then for every $\varepsilon > 0$, f is $C^{1+\alpha-\beta-\varepsilon}$–conjugate to the rotation by ω.

For $\omega \in \mathbb{R}$ we denote the integer coefficients of its continued fraction expansion by $a_i(\omega)$ and the continued fraction approximants by $p_i(\omega)/q_i(\omega)$, so that

$$p_i(\omega) = a_i(\omega)p_{i-1}(\omega) + p_{i-2}(\omega) \ .$$
$$q_i(\omega) = a_i(\omega)q_{i-1}(\omega) + q_{i-2}(\omega) \ .$$

In this note we prove the C^1–part of Herman's theorem for all rotation numbers of sub–exponential growth. More precisely, we prove theorem 1.1.

Theorem 1.1: If the integers $a_i(\omega)$ have sub–exponential growth,

$$\limsup_i \sqrt[i]{a_i(\omega)} = 1,$$

then any C^3 circle diffeomorphisms with rotation number ω is C^1– conjugate to the rotation by ω.

For this more geometrically characterized (compared to Diophantine) class of rotation numbers, the proof we give is substantially shorter than Yoccoz' proof of the analogous result for rotation numbers satisfying a Diophantine condition. Moreover, the class of rotation numbers for which the assumption in theorem 1.1 holds is large.

Theorem 1.2: The set of ω for which the integers $a_i(\omega)$ have sub–exponential growth has full measure.

Definition 1.3: Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$. We say that an irrational number ω is ψ–renormalizable if there is a constant $C > 0$ such that for all i

$$a_i(\omega) < \psi(i + C) \ .$$

The set of ψ–renormalizable numbers will be denoted by R_ψ.

In particular, those numbers that are usually called of constant type (such as real roots of quadratic equations with integer coefficients) are constant–renormalizable.

For fixed $\lambda > 1$, the set R_{λ^i} consists of numbers ω for which the sequence $a_i(\omega)$ satisfies

$$a_i(\omega) < \text{const } \lambda^i.$$

That such a set has full measure follows from the more general proposition 1.4. This proposition as well as its proof is similar to a theorem by Khintchine [1963].

Proposition 1.4: Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be such that $\Sigma_{\mathbb{N}} \frac{\varphi(a)}{a^2} < \infty$ and φ is invertible with inverse φ^{-1}. Then the set of φ^{-1}–renormalizable numbers has full measure.

Remark: From the proof in the next section it will be clear that proposition 1.4 can be easily generalized to non–invertible φ.

Theorem 1.2. now follows easily.

Proof of theorem 1.2: One observes that the set of numbers ω whose sequence of integers $\{a_i(\omega)\}$ has sub–exponential growth, coincides with the set $\bigcap_{\lambda>1} R_{\lambda^i}$. Proposition 1.4 implies, by taking $\varphi(a) = \frac{\ln a}{\ln \lambda}$, that for each $\lambda > 1$, R_{λ^i} has full measure. This implies that $\bigcap_{\lambda>1} R_{\lambda^i}$ has full measure, because this set is a countable intersection of sets of full measure (take the λ's to be rational). □

In section 2, we prove theorem 1.1 and proposition 1.4. In section 3 we compare the sets

R_{λ^i} with Diophantine numbers and Roth numbers.

2 MAIN RESULTS

First we prove proposition 1.4 .

Proof of proposition 1.4: (After Deligne [1976].) Let $T : [0,1) \to [0,1)$ be defined as follows:

$$T(\omega) = \mathrm{frac}(\frac{1}{\omega}) \ .$$

Let ν be the probability measure given by

$$d\nu = \frac{1}{\ln 2} \frac{d\omega}{1+\omega} \ .$$

Then ν is T–invariant and T is ergodic with respect to ν [Khintchine, 1963]. The coefficient $a_n(\omega)$ for an irrational number ω can now be calculated as follows:

$$a_n(\omega) = \mathrm{int}\{[T^{n-1}(\omega)]^{-1}\} \ .$$

The probability that $a_n(\omega) = a$ is given by

$$\int_{\mathrm{int}\{\omega^{-1}\}=a} d\nu \cong \frac{1}{a^2}$$

for ν almost all ω (which is the same as Lebesgue almost all ω). The probability P_C that a_i lies above the curve $a = \varphi^{-1}(i+C)$ or $i = \varphi(a)-C$ (see figure 2.1) satisfies

$$P_C \cong \Sigma_{\mathbb{N}} \frac{\max\{0, \varphi(a)-C\}}{a^2} < \infty \ .$$

This tends to zero as C tends to infinity. Thus the complement of R_ψ can be made to have

Figure 2.1

arbitrarily small measure. Since R_ψ is a T–invariant set and T is ergodic, it follows that R_ψ has full measure. Any irrational number ω in this set satisfies that there is a $C > 0$ *with*

$$a_i(\omega) < \varphi^{-1}(i+C) \ .$$

□

Theorem 1.1 is implied by the four lemmas listed below.

Lemma 2.1: Let f be a circle diffeomorphism with irrational rotation number such that lnDf has bounded variation and set $M_n = \max_x |x - f^{q_n}(x)|$. Then $\{M_n\}$ converges to zero at least exponentially fast.

Lemma 2.2: Let f be a C^3 circle diffeomorphism, with an irrational rotation number ω . Then $\max_x |\ln Df^{q_n}(x)| \leq \text{const } M_n^{1/2}$.

Lemma 2.3: Let f be a C^3 circle diffeomorphism, with an irrational rotation number ω contained in $\bigcap_{\lambda>1} R_{\lambda^i}$. Then $\sup_n \max_x |\ln Df^n(x)|$ is bounded.

Lemma 2.4 (Gottschalk and Hedlund): Let f be a circle diffeomorphism with irrational rotation number. The following statements are equivalent:

i) There is an orbit $\{x_i\}$ of f with

$$\sup_n |\Sigma_{i=0}^n \ln Df(x_i)| = \sup_n |\ln Df^n(x_0)| < \infty \ .$$

ii) There is a continuous function μ such that

$$\mu \circ f + \ln Df = \mu \ .$$

For the proofs of lemmas 2.1 , 2.2 , and 2.4 we refer to Yoccoz [1984]. The simplification comes about in the proof of lemma 2.3 , where it suffices to employ a standard number theoretical device (see for example proposition 1.6 of chapter 9 in Herman [1979]). This replaces the complicated estimate of Yoccoz [1984, sections 6 and 7] by the following reasoning:

Proof of lemma 2.3: We can decompose every $n \in \mathbb{N}$ in terms of $q_i(\omega)$

$$n = \Sigma_{i=1}^{k} b_i q_i \ ,$$

such that the b_i are bounded by the a_i :

$$b_i \leq a_i \ .$$

Then
$$\|\ln Df^n\| \leq \Sigma_{i=1}^{k} \|\ln Df^{b_i q_i}\| \leq \Sigma_{i=1}^{k} b_i \|\ln Df^{q_i}\|$$
$$\leq \Sigma_{i=1}^{k} a_i M_i^{1/2} \ .$$

By lemma 2.1 the M_i converge exponentially fast to zero. Since $\omega \in \bigcap_{\lambda>0} R_{\lambda^i}$, the a_i grow slower than λ^i for any λ. So the sum is bounded. □

Proof of theorem 1.1: Denote by h a conjugacy between f and and the rotation by ω:

$$h \circ f(x) = h(x) + \omega.$$

If h were differentiable then

$$\mu(x) = \ln Dh(x)$$

would satisfy the equation in lemma 2.4 ii. Since the rotation number ω is in $\bigcap_{\lambda > 1} R_{\lambda^i}$ lemma 2.3 applies. Therefore lemma 2.4 i holds, and we conclude that the equation in lemma 2.4 ii has a continuous solution μ. Such a solution is unique up to an additive constant. Choosing this constant suitably and integrating $\exp(\mu)$ one finds a conjugacy h, which is then C^1. □

Remarks: i) In the proof of lemma 2.1, the rate at which $M_i^{1/2}$ converges to zero depends only on the total non–linearity $\int_{S^1} |f''/f'| \, dx$. If a bound on the non–linearity is known then theorem 1.1 holds for exponentially renormalizable numbers with small enough exponent.

ii) On the other hand, with a little more work than lemmas 2.1 to 2.4, Yoccoz shows that M_i decreases faster than $(2/3)^i$ (Yoccoz [1984, section 6]).

3 RELATED RESULTS

If $\beta \geq 0$, one says that a real number ω is Diophantine of order β if there exists a C such that for all rational p/q

$$|\omega - \frac{p}{q}| \geq \frac{C}{q^{2+\beta}} .$$

Let Dio_β be the set of diophantine numbers of order β. Then the set of Roth numbers is defined as:

$$Roth \equiv \bigcap_{\beta>0} Dio_\beta .$$

(A number which is not Diophantine of any order is called Liouville.) The first lemma concerns a standard result (see Herman [1979, chapter 5]).

Lemma 3.1: i) $\omega \in Dio_\beta \Leftrightarrow$ there is a $K \geq 1$ with $a_{n+1}(\omega) < Kq_n(\omega)^\beta$.

ii) $\omega \in Roth \Leftrightarrow$ for all $\beta > 0$ there is a K with $a_{n+1}(\omega) < Kq_n(\omega)^\beta$.

iii) $\omega \in Roth \Leftrightarrow$ for all $\beta > 0$ $\Sigma_{\mathbb{N}}\, a_{n+1}(\omega)q_n(\omega)^{-\beta} < \infty$.

Now let γ denote the golden mean

$$\gamma = 1 + \cfrac{1}{1 + \cfrac{1}{1 + 1\ldots}} ,$$

and recall that for any number $\omega \in \mathbb{R}\backslash\mathbb{Q}$

$$q_n(\omega) > \gamma^n .$$

Proposition 3.2: i) $R_{\lambda^i} \subseteq Dio_\beta$, if $\lambda \leq \gamma^\beta$.

ii) $\bigcap_{\lambda>1} R_{\lambda^i} \subseteq Roth$.

Proof: To prove i), suppose that $\omega \notin Dio_\beta$. We have to prove that for all $\lambda \leq \gamma^\beta$, $\omega \notin R_{\lambda^i}$.

By assumption we have that for all $K \geq 1$, there is an n such that

$$a_{n+1}(\omega) > Kq_n(\omega)^\beta > K\gamma^{\beta n} > \gamma^{\beta n+\beta \ln K/\ln\gamma} \geq \lambda^{n+1+\ln K/\ln\gamma-1} = \lambda^{n+1+C} .$$

Therefore for all C there is an n such that

$$a_{n+1}(\omega) > \lambda^{n+1+C} .$$

The second statement is proved similarly. If $\omega \notin$ Roth, then there is an ε such that for all K, there is an n with

$$a_{n+1}(\omega) > Kq_n(\omega)^{\varepsilon} > K\gamma^{\varepsilon n} ,$$

which proves that there a subsequence of $\{a_n(\omega)\}$ which grows exponentially fast. □

In particular, the first part of this proposition implies that Herman's theorem also holds for exponentially renormalizable numbers as long β is taken to be $\ln\lambda/\ln\gamma$.

Proposition 3.3: For any λ

i) $R_{\lambda^i} \not\subset$ Roth .

ii) Roth $\not\subset R_{\lambda^i}$.

Proof: We prove i) for integer values of λ only. Let ℓ and m be two integers greater than one, to be chosen later. Let ω be the number in R_{ℓ^i} defined by ($q_0(\omega) = q_1(\omega) = 1$):

$$a_i(\omega) = 1 \text{ if } i \neq m^j \text{ for } j \in \mathbb{N} ,$$

$$a_{m^i}(\omega) = \psi(m^i) = \ell^{m^i} .$$

Since most of the a_i are equal to one, we have that if

$$k = \text{int}[\ln n/\ln m] ,$$

$$q_n(\omega) < \gamma^n \Pi^k(a_{m^i}(\omega) + 1) = \gamma^n \ell^{\Sigma^k m^i} \Pi^k(1 + \ell^{-m^i}) .$$

The latter product is convergent, and so there is a K with

$$q_{m^{k+1}-1}(\omega) < K\gamma^{m^{k+1}} \ell^{m^{k+1}(1-m^{-k}-1)/(m-1)} .$$

Therefore there is a $\varepsilon > 0$ such that

$$a_{m^{k+1}}(\omega) = \ell^{m^{k+1}} > K[q_{m^{k+1}-1}(\omega)]^{\varepsilon} ,$$

for all $k \in \mathbb{N}$. Thus ω cannot be Roth.

To prove ii), we construct a different number: The number ω be determined by $q_0(\omega) = q_1(\omega) = 1$ and

$$a_n(\omega) = \mathrm{int}[e^{n^2}]$$

is not exponentially renormalizable. However, because there is a C such that

$$q_n(\omega) = \mathrm{int}[e^{n^2}]q_{n-1}(\omega) + q_{n-2}(\omega) > e^{n^2-1/2}q_{n-1}(\omega) + q_{n-2}(\omega) \ ,$$

we also have

$$q_n(\omega) > e^{\Sigma^n(i^2)-n/2} > e^{n^3/3} \ .$$

Therefore, for all $\varepsilon > 0$, there is a $K > 0$ such that

$$a_{n+1}(\omega) < Kq_n(\omega)^{\varepsilon}$$

which is equivalent to ω being a Roth number. □

Proposition 3.4: Let $\psi(i) = e^{(1+\beta)^i}$. Then $\mathrm{Dio}_\beta \subseteq R_\psi$.

Proof: Assume $\omega \in \mathrm{Dio}_\beta$. Then there is a $K \geq 1$ with

$$a_1(\omega) < K \ ,$$

and $$a_{n+1}(\omega) < Kq_n(\omega)^\beta < K\,\Pi^n a_i(\omega)^\beta(1+\frac{1}{a_i(\omega)})^\beta < K\,2^{n\beta}\,\Pi^n a_i(\omega)^\beta \ . \qquad (*)$$

Now define $\vartheta : \mathbb{N} \to \mathbb{N}$ $$\vartheta(1) \equiv K \ ,$$

$$\vartheta(n+1) \equiv K\,2^{n\beta}\,\Pi^n\,\vartheta(i)^\beta \ .$$

Thus $$\vartheta(n+1) = 2^\beta\,\vartheta(n)^{1+\beta} \ .$$

One obtains

$$\vartheta(n) = \frac{1}{2}(2K)^{(1+\beta)^{n-1}} > e^{(1+\beta)^{n+1+C}} = \psi(n+1+C) \ ,$$

for appropriately chosen C . Since

$$a_1 < \vartheta(1) \ ,$$

one proves recursively, using (*), that

$$a_{n+1} < 2^{n\beta}\,\Pi^n\vartheta(i)^\beta = \vartheta(n+1) = \psi(n+1+C) \ . \qquad □$$

The last three results are summarized in the Venn–diagram of figure 3.1 .

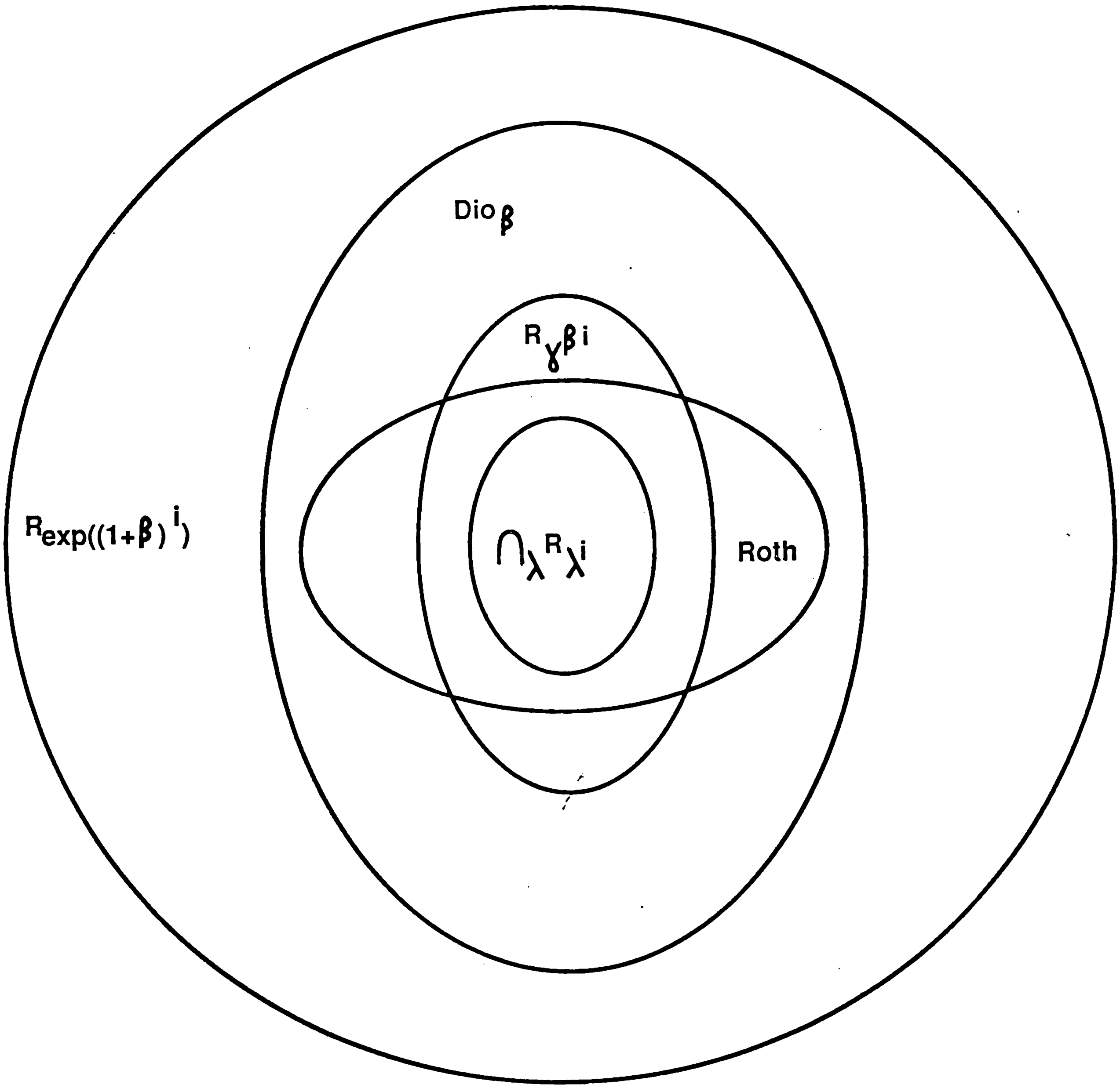

figure 3.1

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